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Mathematical Aspects of Non-Classical Logics and Fuzzy Inference

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June 13–17, 2000

Abstracts

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Editors

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Triangular Norms: What They Are and What They Are Good For[1]

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The history of triangular norms started with the paper "Statistical metrics" [39]. The main idea of Karl Menger was to construct metric spaces where probability distributions rather than numbers are used in order to describe the distance between two elements of the space in question. Triangular norms (t-norms for short) naturally came into the picture in the course of the generalization of the classical triangle inequality to this more general setting. The original set of axioms for t-norms was considerably weaker, including among others also the functions which are known today as triangular conorms.

Consequently, the first field where t-norms played a major role was the theory of probabilistic metric spaces (as statistical metric spaces were called after 1964). Berthold Schweizer and Abe Sklar in [45, 46, 47] provided the axioms of t-norms, as they are used today, and a redefinition of statistical metric spaces given in [50] led to a rapid development of the field. Many results concerning t-norms were obtained in the course of this development, most of which are summarized in the monograph [49].

Mathematically speaking, the theory of (continuous) t-norms has two rather independent roots, namely, the field of (specific) functional equations and the theory of (special topological) semigroups.

Concerning functional equations, t-norms are closely related to the equation of associativity (which is still unsolved in its most general form). The earliest source in this context seems to be [1], further results in this direction were obtained in [8, 11, 2, 27]. Especially János Aczél's monograph (both the German [3] and the English [4] version) had (and still has) a big impact on the development of t-norms. The main result based on this background was the full characterization of continuous Archimedean t-norms by means of additive generators in [34] (for the case of strict t-norms see [47]). Further significant contributions are due to a group of Spanish researchers around Enric Trillas and Claudi Alsina.

Another direction of research was the identification of several parameterized families of t-norms as solutions of some (more or less) natural functional equations. The perhaps most famous result in this context has been proven in [19], showing that the family of Frank t-norms and t-conorms (together with ordinal sums thereof) are the only solutions of the so-called Frank functional equation.

[1] Slightly modified introduction of the monograph "Triangular Norms" (Kluwer, in press) by E.P. Klement, R. Mesiar and E. Pap [32]

The study of a class of compact, irreducibly connected topological semigroups was initiated in [17], including a characterization of such semigroups, where the boundary points (at the same time annihilator and neutral element) are the only idempotent elements and where no nilpotent elements exist. In the language of t-norms, this provides a full representation of strict t-norms. In [40] all such semigroups, where the boundary points play the role of annihilator and neutral element, were characterized (see also [43]). Again in the language of t-norms, this provides a representation of all continuous t-norms [34].

Several construction methods from the theory of semigroups, such as (isomorphic) transformations (which are closely related to generators mentioned above) and ordinal sums [14, 13, 48], have been successfully applied to construct whole families of t-norms from a few given prototypical examples [48].

Summarizing, starting with only three t-norms, namely, the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$, it is possible to construct all continuous t-norms by means of isomorphic transformations and ordinal sums [34].

Many specific results, such as characterizations of the order or convergence theorems, are based on this general representation for continuous t-norms.

Non-continuous t-norms, such as the drastic product $T_{\mathbf{D}}$, have been considered from the very beginning [46]. In [34] even an additive generator for this t-norm was given. However, a general classification of non-continuous t-norms is still not known.

For the construction of not necessarily continuous t-norms, several methods, which are more or less related to those already mentioned, have been proposed recently.

Besides the already mentioned probabilistic metric spaces, t-norms and t-conorms proved to be useful or even indispensable in a number of additional areas:

- Based on the seminal work of Jan Łukasiewicz [35, 36, 37] and Kurt Gödel [20] in the twenties and thirties, an extensive theory of many-valued logics has been developed during the past few decades. The crucial role of t-norms in this context is presented in the monographs [24, 26, 12].

- Already in his first paper [60] on fuzzy sets, Lotfi A. Zadeh suggested to use the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and, in a restricted sense, the Łukasiewicz t-conorm $S_{\mathbf{L}}$. Very early traces of (some slight variations of) t-norms and t-conorms in the context of integration of fuzzy sets can be found in [52], first concepts for a unified theory of fuzzy sets (based on $T_{\mathbf{M}}$ and $S_{\mathbf{M}}$) were presented in [41] and [21, 22, 23]. The use of general t-norms and t-conorms for modeling the intersection and the union of fuzzy sets (see, e.g., [33, 42]) apparently goes back to some seminars held by Enric Trillas and to suggestions given by Ulrich Höhle during some conferences in the late seventies. The first papers using general t-norms and t-conorms for operations on fuzzy sets were [6, 5, 15, 29, 30] (see also [16]). A full characterization of strong negations as models of the complement of fuzzy sets can be found in [56]. Fuzzy sets recently found many practical applications, in particular in connection with intelligent control (see [38, 55] and [54, 53]).

- A very fast developing field is that of general (not necessarily associative) aggregation operators [61, 58, 18, 25], some of which have a close relationship with t-norms, e.g., uninorms [59, 31] and nullnorms [10].

- In the context of generalized measures and integrals, t-norms and t-conorms can be used to generalize the standard set operations [9], on the one hand, or standard arithmetic operations, on the other hand [28, 52, 57, 44] (compare also [51]).

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Compositional Rule of Inference as Generalized Modus Ponens

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Compositional rule of inference is defined to describe the inference process in fuzzy environment and its general form is

$$B'(y) = \sup_{x \in X} T(A'(x), R(x,y)), \quad y \in Y\,, \tag{1}$$

where T is a left continuous conjunctor (monotone extension of the Boolean conjunction with neutral element 1, mostly some continuous t-norm) and R is a fuzzy relation defined on $X \times Y$. We restrict our considerations to the case when $R(x,y) = r(A(x), B(y))$ with $r : [0,1]^2 \to [0,1]$ and $A \in [0,1]^X, B \in [0,1]^Y$ with $Ran\,A = Ran\,B = [0,1]$. We will discuss under which conditions the compositional rule of inference yields a generalized modus ponens, i.e., if $A' = A$ then (1) results in $B' = B$, or equivalently,

$$\sup_{x \in X} T(A(x), r(A(x), B(y))) = B(y)\ , \quad y \in Y\,. \tag{2}$$

Under the above requirements, (2) can be rewritten into the form

$$\sup_{u \in [0,1]} T(u, r(u,v)) = v, \quad v \in [0,1]\,. \tag{3}$$

It is immediate that (3) requires $r \le I_T$, where I_T is the residual operator adjoined to T via

$$I_T(u,v) = \sup\left(w \in [0,1] | T(u,w) \le v\right)\ ,$$

$r(1,1) = 1$ and $r(1,0) = 0$.
Note that $r \le I_T$ ensures

$$\sup_{u \in [0,1]} T(u, r(u,v)) \le v, \quad v \in [0,1]\,.$$

Supposing the left continuity of $r(.,v)$, $v \in [0,1]$, r solves (3) if and only if $r \le I_T$ and for all $v \in [0,1]$ there is some u_v that $T(u_v, r(u_v, v)) = v$. This holds, e.g., if either (i) or (ii) holds true, where

(i) $r(1,v) = v$ for all $v \in [0,1]$,
(ii) $r(v,v) = 1$ for all $v \in [0,1]$.
Note that the validity either of the condition (i) or (ii) ensures

$$\sup_{u\in[0,1]} T(u, r(u,v)) \geq v, \quad v \in [0,1] .$$

EXAMPLES.

(1) For any conjunctor T^*, especially for any t-norm T^*, we have

$$T^* \leq I_T \quad \text{and} \quad T^*(1,v) = v, \ v \in [0,1].$$

The case $r = T = \min$ leads just to the Mamdani model, and $T = \min$, $r = T_P$ is the Larsen model.

(2) For any left-continuous conjunctor $T^* \geq T, I_{T^*} \leq I_T$ and $I_{T^*}(1,v) = v, I_{T^*}(v,v) = v$. Especially, Gödel implication $I_{\min} = r$ can be combined with any t-norm T, and $I_T = r$ can be combined with T.

(3) Any S-implication $I_S(u,v) = S(n(u),v)$ fulfills $I_S(1,v) = v, v \in [0,1]$, so we have only to check the relation $I_S \leq I_T$, to be allowed to combine $r = I_S$ and T. So, e.g., Kleene–Dienes implication $I_{\max}(u,v) = \max(1-u,v)$ can be combined with any T.

(4) Rescher-Gaines implication

$$I_{R,S}(u,v) = \begin{cases} 1 & \text{if } u \leq v \\ 0 & \text{else,} \end{cases}$$

fulfills $I_{R,S}(v,v) = 1, v \in [0,1]$, and it can be combined with any T.

(5) Any fuzzy equivalence relation r on $[0,1]$ fulfills $r(v,v) = 1, v \in [0,1]$, and hence only the inequality $r \leq I_T$ should be checked.

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A New Approach for Interpolation and Extrapolation of Compact Fuzzy Quantities
—
The One-Dimensional Case

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1 Introduction

Fuzzy set theory is a formal framework for modeling input-output relations, for which only vague/linguistic information is available to describe them. In this talk we deal with the problem of rule interpolation and rule extrapolation for fuzzy and possibilistic systems. Such systems are used for representing and processing vague **If-Then**-rules, and recently, they have been increasingly applied above all in the field of control engineering, pattern recognition and expert systems. The methodology of rule interpolation is required for deducing plausible conclusions from sparse (incomplete) rule bases. The methods proposed so far in the literature for rule interpolation are mainly conceived for the application to fuzzy control and miss certain logical characteristics of an inference.

Rule interpolation may be considered as an "inference technique" for fuzzy rule bases for which the premises do not cover the whole input space. Of course, any inference mechanism has to satisfy certain logical criterions. It may be said that the methods proposed so far in the literature are mainly conceived for the application to fuzzy control purposes and that they are not appropriate for applications where the logical aspects of approximate reasoning are intrinsically important. All this serves as a motivation for looking for a more flexible method which is superior to the proposed ones above all with respect to its general applicability to fuzzy as well as to possibilistic systems.

2 Conditions on rule interpolation/extrapolation

[I0] Validity of the conclusion
The conclusion should be a fuzzy subset of the universe $\mathcal{Y}$ with a valid membership function. Usually, further conditions for validity are required too: For example, when a method is restricted to the use of e.g. trapezoidal membership functions, then the result may be expected to be trapezoidal too. The normality of the quantities is frequently supposed too.

[I1] Compatibility with the rule-base
For all $i \in \{1, \dots, N\}$ and all $A \in \mathcal{F}(\mathcal{X})$ it follows from $A = A_i$ that $\mathcal{I}(A) = B_i$. This condition is the *modus ponens* rule in logic.

[I2] Monotonicity condition
If $A^* \in \mathcal{F}(\mathcal{X})$ is more specific than $A \in \mathcal{F}(\mathcal{X})$, then $\mathcal{I}(A^*)$ is more specific than $\mathcal{I}(A)$, i.e., for all $A, A^* \in \mathcal{F}(\mathcal{X})$ the inequality $A^* \sqsubset A$ implies the inequality $\mathcal{I}(A^*) \sqsubset \mathcal{I}(A)$.

[I3] Continuity condition
For $\varepsilon > 0$ there exists $\delta > 0$ such that if $A, A^* \in \mathcal{F}(\mathcal{X})$, and $d_{\mathcal{X}}(A, A^*) \leq \delta$ then for the corresponding conclusions we have $d_{\mathcal{Y}}(\mathcal{I}(A), \mathcal{I}(A^*)) \leq \varepsilon$.

[I4] $\mathcal{I}(A \cap_{\mathcal{X}} A^*) = \mathcal{I}(A) \cap_{\mathcal{Y}} \mathcal{I}(A^*)$, whenever $A \cap_{\mathcal{X}} A^*$ has valid membership function.

[I5] $\mathcal{I}(A \cup_{\mathcal{X}} A^*) \sqsupset \mathcal{I}(A) \cup_{\mathcal{Y}} \mathcal{I}(A^*)$ whenever $A \cup_{\mathcal{X}} A^*$ has valid membership function.

[I6] Identity principle
If $\mathcal{X} = \mathcal{Y}$, $\mathcal{I}(A_1) = A_1$ and $\mathcal{I}(A_2) = A_2$ then for any observation A for which the antecedents of the basis of interpolation/extrapolation are A_1 and A_2 we should have $\mathcal{I}(A) = A$.

[I6*] Linearity principle
If $\mathcal{X} = \mathcal{Y} = \mathrm{I\!R}$, $\mathcal{I}(A_1) = A_1 + c$ and $\mathcal{I}(A_2) = A_2 + c$ then for any observation A for which the antecedents of the basis of interpolation/extrapolation are A_1 and A_2 we should have $\mathcal{I}(A) = A + c$.

[I7] Preserving "in between"
If the observation A is in between A_i and A_j (resp. A_j is in between A and A_i) then $\mathcal{I}(A)$ should be in between B_i and B_j (resp. B_j should be in between $\mathcal{I}(A)$ and B_i).

Definition 1. We call a mapping $\mathcal{I} : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$ *interpolation/extrapolation* if it satisfies axioms **[I0]**–**[I5]** and *linear interpolation/extrapolation* if it satisfies axioms **[I0]**–**[I6]** (in case $\mathcal{X} = \mathcal{Y} = \mathrm{I\!R}$ also **[I6*]** is required).

3 Flank-representation of compact fuzzy quantities

Let A be a compact fuzzy quantity. Denote l_A and r_A the left and the right endpoints of its support, respectively. Let p_A (called the *reference point* of A) be a point in the kernel of A. In particular, we may use e.g.

> $p_A = c_A$ where c_A denotes the center of the kernel of A; it exists since the support is compact and therefore the kernel is also compact, or we will use
>
> $p_A = m_A$ where m_A denotes the midpoint of A which is the center of its support (given by $(l_A + r_A)/2$) when A is balanced: We call a compact fuzzy quantity A *balanced* if $\mu_A(m_A) = 1$.

The left and the right side-functions of A are defined by $\mu_A|_{[l_A,[A]_1^-]}$ and $\mu_A|_{[[A]_1^+,r_A]}$, respectively.

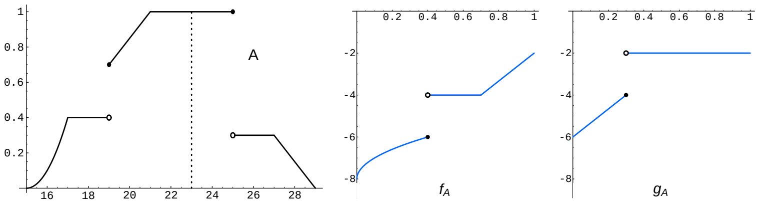

Figure 1: An example membership function (left), its first (center) and second (right) flank-functions

Definition 2. We represent compact fuzzy quantities by *representing triples* of the form (p, f, g) where p is a real number, f and g are *flank-functions*, that is, non-decreasing and left-continuous functions mapping from $[0, 1]$ to $]-\infty, 0]$. For a compact fuzzy quantity A equipped with a reference point p_A its representing triple (p_A, f_A, g_A) is defined by $f_A(x) = \left(\mu_A|_{[l_A, p_A]} \circ (. + p_A)\right)^{(-1)}(x)$ and $g_A(x) = \left(\mu_A|_{[p_A, r_A]} \circ (p_A - .)\right)^{(-1)}(x)$, $(x \in [0, 1])$. $f^{(-1)}$ stand for the pseudo-inverse. For a visualization, see Figure 1.

The *normalized* flank-functions of A (of type $[0, 1] \to]-\infty, 0]$) are defined by $\tilde{f}_A(x) = \begin{cases} f_A(x) & \text{if } f_A(0) = 0 \\ -\frac{f_A(x)}{f_A(0)} & \text{otherwise} \end{cases}$, $\quad \tilde{g}_A(x) = \begin{cases} g_A(x) & \text{if } g_A(0) = 0 \\ -\frac{g_A(x)}{g_A(0)} & \text{otherwise} \end{cases}$.

4 The new method – description

In out method both the input and the output space will be specified to the set of the reals; that is, $\mathcal{X} = \mathcal{Y} = \mathbb{R}$. Further, we will work with compact fuzzy quantities. Suppose that our knowledge base consists a finite number of *rules* of the following form: **If** $X = A_i$ **Then** $Y = B_i$ $(i = 1, 2, \ldots, n)$ where each A_i and B_i is a compact fuzzy quantity. Further, suppose that our observation is the compact fuzzy quantity A (represented by (p_A, f_A, g_A)). We would like to determine the conclusion B which corresponds to A.

Which rules to choose? In the first step the determination of the two rules which are going to be used for the linear interpolation is performed. We consider the set $\{p_{A_i} \mid i = 1, 2, \ldots, n\}$ which divides $\mathbb{R}$ into $n-1$ intervals and two half-lines. Since we are dealing with interpolation we may suppose that p_A lies in one of the intervals of the partition and not in one of the half-lines. In case p_A lies in one of the half-lines linear extrapolation is required. Choose those indices ζ, ξ where p_A is in the interval $[p_{A_\zeta}, p_{A_\xi}]$. (If p_A coincides with one of the points, say with p_{A_ζ} then we choose ζ and another ξ for which p_{A_ξ} is next to p_{A_ζ}.) Without loss of generality we may assume that $p_A \in [p_{A_1}, p_{A_2}]$ and so A lies 'in between' A_1 and A_2.

Linear averages: In the second step A' – a linear average of A_1 and A_2 – is computed in the following way: We determine the coefficient r which produces p_A as the linear combination of p_{A_1} and p_{A_2}; that is, let $r = \frac{p_{A_2} - p_A}{p_{A_2} - p_{A_1}}$. We have $r \in [0, 1]$ and $p_A = r \cdot p_{A_1} + (1 - r) \cdot p_{A_2}$. The triplet $(p_{A'}, f_{A'}, g_{A'})$ representing the linear average of A_1 and A_2 is defined by

$$\begin{aligned} p_{A'} &= p_A, \\ f_{A'}(x) &= r \cdot f_{A_1}(x) + (1 - r) \cdot f_{A_2}(x), \\ g_{A'}(x) &= r \cdot g_{A_1}(x) + (1 - r) \cdot g_{A_2}(x). \end{aligned} \tag{1}$$

The triplet $(p_{B'}, f_{B'}, g_{B'})$ representing the "same" linear average of B_1 and B_2 is computed analogously.

$$\begin{aligned} p_{B'} &= r \cdot p_{B_1} + (1-r) \cdot p_{B_2}, \\ f_{B'}(x) &= r \cdot f_{B_1}(x) + (1-r) \cdot f_{B_2}(x), \\ g_{B'}(x) &= r \cdot g_{B_1}(x) + (1-r) \cdot g_{B_2}(x). \end{aligned} \tag{2}$$

For a visualization, see Figure 2 where $p_A = c_A$ is used.

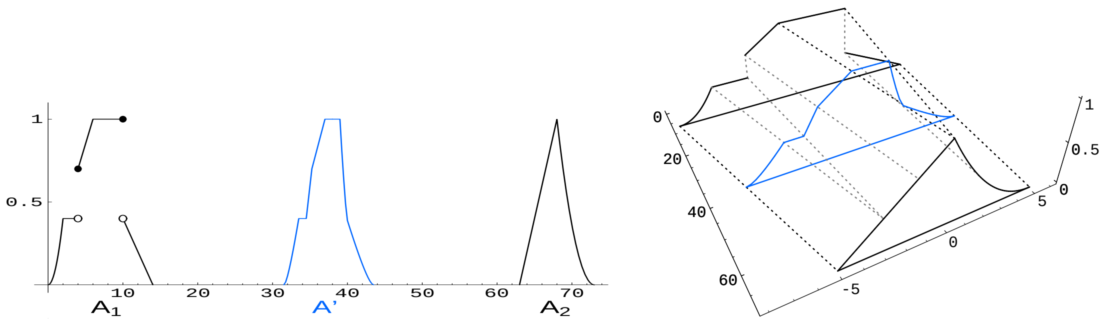

Figure 2: Computation of the linear average A' (note that the reference point of A' is determined by the reference point of the observation)

Measuring the "difference" between the observation and the linear average: At this point given are A, A' and B' by their triplets (p_A, f_A, g_A), $(p_{A'}, f_{A'}, g_{A'})$ and $(p_{B'}, f_{B'}, g_{B'})$ respectively. Before B is computed it is precisely "measured" how A arises from A'. The main principle is:

'B should be derived from B' in the same manner as A has been derived from A''.

To reach this goal first this "measurement" is presented: It is done by means of two transformations; one of them transforms $f_{A'}$ into f_A the other one transforms $g_{A'}$ into g_A. That is, we need two transformations $t_{A,f}$ and $t_{A,g}$ such that

$$\begin{aligned} f_A &= t_{A,f} \circ f_{A'}, \\ g_A &= t_{A,g} \circ g_{A'}. \end{aligned} \tag{3}$$

This means that $t_{A,f}$ should be a function of type $[0, f_{A'}(0)] \to [0, f_A(0)]$ and similarly, $t_{A,g}$ should be a function of type $[0, g_{A'}(0)] \to [0, g_A(0)]$ defined by

$$\begin{aligned} t_{A,f} &= f_A \circ f_{A'}^{(-1)}, \\ t_{A,g} &= g_A \circ g_{A'}^{(-1)}. \end{aligned} \tag{4}$$

Computation of the conclusion: We can not apply $t_{A,f}$ directly to obtain f_B from $f_{B'}$, that is $f_B = t_{A,f} \circ f_{B'}$ doesn't work, since the range of $f_{B'}$ may be different from the domain of $t_{A,f}$. The same applies to $t_{A,g}$. Therefore we first "linearly rescale" $f_{B'}$ (resp. $g_{B'}$) to fit $t_{A,f}$ (resp. to fit $t_{A,g}$) by the help of a linear transformation. Then we apply $t_{A,f}$ (resp. $t_{A,g}$) and finally we apply the inverse of this "rescaling". Let φ_f

(resp. φ_g) be a function of type $[0, f_{B'}(0)] \to [0, \infty[$ (resp. $[0, g_{B'}(0)] \to [0, \infty[$) given by

$$\begin{aligned} \varphi_f(x) &= \frac{f_{A'}(0)}{f_{B'}(0)} \cdot x, \\ \varphi_g(x) &= \frac{g_{A'}(0)}{g_{B'}(0)} \cdot x. \end{aligned} \tag{5}$$

Denote $t_{B,f} = \varphi_f^{-1} \circ t_{A,f} \circ \varphi_f$ (resp. $t_{B,g} = \varphi_g^{-1} \circ t_{A,g} \circ \varphi_g$). Then B is defined by the representing triple (p_B, f_B, g_B) via

$$\begin{aligned} p_B &= p_{B'} \\ f_B &= t_{B,f} \circ f_{B'}, \\ g_B &= t_{B,g} \circ g_{B'}. \end{aligned} \tag{6}$$

Note the analogy between (6) and (3). In a more detailed form, $f_B = \varphi_f^{-1} \circ f_A \circ f_{A'}^{(-1)} \circ \varphi_f \circ f_{B'}$ and $g_B = \varphi_g^{-1} \circ g_A \circ g_{A'}^{(-1)} \circ \varphi_g \circ g_{B'}$ and straightforward computation shows that defining $\psi_f : [0,1] \to [0, \infty[$, (resp. $\psi_g : [0,1] \to [0, \infty[$) by $\psi_f(x) = -\frac{f_{B'}(0) \cdot f_A(0)}{f_{A'}(0)} \cdot x$ (resp. $\psi_g(x) = -\frac{g_{B'}(0) \cdot g_A(0)}{g_{A'}(0)} \cdot x$) we have

$$\begin{aligned} f_B &= \psi_f \circ \tilde{f}_A \circ \tilde{f}_{A'}^{(-1)} \circ \tilde{f}_{B'}, \\ g_B &= \psi_g \circ \tilde{g}_A \circ \tilde{g}_{A'}^{(-1)} \circ \tilde{g}_{B'}. \end{aligned} \tag{7}$$

Theorem 3. *Suppose that our rule-base consist of n rules of the form **If** $X = A_i$ **Then** $Y = B_i$ where A_i and B_i are compact fuzzy quantities of $\mathbb{R}$ for all $1 \le i \le n$.*

a.) *If their reference points are the center of their kernels, respectively, then the method presented here is a linear interpolation in the sense of Definition 1 if and only if*

i. the A_i's are modeled by continuous membership functions

ii. the constant parts in $\mu_{A_i}|_{Ker(A_i)}$ fit to the constant parts in $\mu_{B_i}|_{Ker(B_i)}$, that is, for each $i = 1, \ldots, n$, $Ran(\tilde{f}_{B_i}) \subseteq Ran(\tilde{f}_{A_i})$.

iii. the observations (A's) are modeled by fuzzy peaks and

b.) *The method presented here is a linear interpolation in the sense of Definition 1 which in addition satisfies* **[I7]** *if and only if the A_i's, the B_i's and the observations are balanced, their reference points are the center of their supports, respectively, and the above-listed three conditions hold.*

Remark 4. Any other choice is as well possible in Theorem 3/**a** provided that $D(A, A^*) < \delta$ ensures $|p_A - p_{A^*}| < constant \cdot \delta$.

The presented method is suitable for linear *extrapolation* of compact fuzzy quantities as well. If p_A lies in one of the half-lines of the partition defined by $\{p_{A_i} \mid i = 1, 2, \ldots, n\}$ then we take those indices ζ, ξ for which p_{A_ζ} is the end-point of the half-line in question and p_{A_ξ} lies next to p_{A_ζ}. Without loss of generality we may assume $\zeta = 1$ and $\xi = 2$; that is, $p_{A_1} \in [p_{A_2}, p_A]$. By defining r as in the case of the interpolation we have now $r < 0$. Therefore the definitions in (1) and (2) may not yield valid representing triples since the functions in question may violate monotonicity and non-positivity. In this case the linear averages A' and B' are not computable. Two solutions are proposed: 1. Shifting the "extreme" rule, 2. Careful construction of the knowledge-base

Since for the fuzzy quantities reference points are assigned to, all the results of the classical interpolation theory can be adapted to this method. For instance, a polinomial can be constructed for a finite number of pre-described points of the form (p_{A_i}, p_{B_i}) with the help of Lagrange interpolation. Then the linear average is computed accordingly: In general, the linear average of the antecedents is computed form *different* rules than the linear average for the consequences.

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Hierarchical Fuzzy Rule Bases and Interpolation

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One of the most important objectives of both symbolic and subsymbolic intelligent computational methods is to find an acceptable tradeoff between approximation error and computational complexity when dealing with very complex and analytically unknown systems modelling, control and reasoning.

The inclusion of subsymbolic information as fuzzy membership functions, NN activation functions, etc, and later the introduction of rule interpolation in fuzzy rule based systems did help with reducing the complexity compared to earlier symbolic and dense models. On the other hand, the hierarchical structured models of certain well structured systems with high state space dimensionality contributed to more drastic complexity reduction as locally the number of state variables could be decreased.

However, in general, no method is available, which effectively reduces the dimensionality in the case of a more general class of systems. Our approach proposes a combination of the structured hierarchical approach with interpolation, in this case *among* the sub-rule bases themselves in addition to the rule interpolation *within* the sub-rule bases. The essential point in this approach is that this former type of interpolation is done in the projection subspaces rather than in the cylindric extensions forming the common superspaces of the sub-rule bases in question.

The problem of model identification within such a model will be also touched upon.

A Double Dualization in Categorical Logic and Some of its Applications

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Starting from a duality between distributive lattices and posets, we describe a double dual endofunctor on the category of distributive lattices (and some of its enrichments). This functor is used to obtain completeness and conservativeness theorems for some non-classical logics (including some modal logics) in a unified manner. Furthermore strong amalgamation ('pushouts of monos are monos') is proved for some enrichments of the category of distributive lattices. As a consequence, a form of the interpolation lemma for the corresponding calculi is obtained.

The lifting of this functor from distributive lattices to coherent categories turns out to be the Makkai's topos of types. This topos is used to prove some completeness as well as conservativeness results in first-order modal logic.

Fuzzy Sets and Non-Monotonic Reasoning: The Possibilistic Connection

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While many researchers in fuzzy set theory, especially on the engineering side have focused on the relationship between fuzzy logic and neural nets, another, less visible trend has devoted efforts towards bridging the gap between fuzzy sets and commonsense exception-tolerant reasoning. The key idea is to exploit the properties of qualitative possibility and necessity measures. A possibility distribution basically encodes a plausibility ordering over a set of states of nature and represents a cognitive state. The plausible entailment of a proposition A from a proposition B in the framework of a plausibility ordering, called "possibilistic entaiment" means that B is true in all most plausible states where A is true. The plausibility ordering is then said to sanction the rule "if A then B". This entailment is non monotonic and its characteristic properties have been laid bare. It can also be described using the notion of conditional necessity function. This entailment is a good model of a rule tainted with exceptions and it captures the logic of jumping to plausible conclusions. Conversely a rule "if A then B" is interpreted as a constraint that delimits the set of possibility distributions where the plausible entailment of B from A takes place. Inference of a new rule from a set of such rules means that the entailment expressed by this new rule holds for all possibility distributions sanctioning the set of rules. This kind of entailment, called preferential entailment, has been studied in the literature under various guises. A syntactic calculus of rules which is sound and complete with respect to the above semantics has been devised by Kraus Lehmann and Magidor. A rule can also be interpreted as a conditional event, introduced by De Finetti, as studied by Nguyen, Goodman, Walker, and Calabrese. Probabilistic semantics of preferential inference also exist, either using infinitesimal probabilities, or restricting to a special class of probabilities called "big-stepped probabilities". However preferential inference is very cautious and it is more useful to select a particular possibility distribution among the feasible ones sanctioning the rule base, actually the least informative one in some sense. It comes down to computing a complete preordering on the rule base, attaching priorities to rules. The set of rules sanctioned by that particular plausibility ordering is called the "rational closure" of the rule-base. Rational closure can actually be computed by replacing rules by their material implication counterparts, keeping the ordering and applying possibilistic logic to the obtained ordered set of classical logic propositions. This leads to computerized exception tolerant plausible inference tools.

Preferential entailment is monotonic with respect to the addition of new rules in the rule base, but rational entailment is not. In particular, it can be proved that rules contained in the rational closure of a rule base, but not in the preferential one, can be canceled, or, even, the converse rule can be derived, by simple addition of suitable rules to the rule-base. It enable rule bases which provide counterintuitive conclusions to be repaired through the addition of missing relevant information.

Properties of the Lattice Change Functor

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Work in the foundations of fuzzy sets allows multiple targets for the truth functions: we may consider several diffferent discretizations of the unit interval, several different t-norms on [0,1], or several different complete lattice ordered semigroups. In practice we may want to allow for the needed level of fuzziness to change with further information about the system we are modeling or trying to control (allowing for richer lattices of truth values) and we may need to make decisions by defuzzifying. Such a change in the underlying structure where the truth values lie gives us a systematic change in the fuzzy sets. What properties that systematic change will have depends on the preservation properties of the change of lattice.

First we can consider how the propositional fuzzy logic changes: a function $f : L_1 \to L_2$ transforms an L_1-fuzzy subset of A to an L_2-fuzzy subset of A by composition with f. If f is order preserving this will induce a functor from the category of L_1-fuzzy subsets of A to the category of L_2-fuzzy subsets. Additional preservation properties will be needed for this to preserve the additional structure we use in fuzzy set theory.

At a somewhat richer level we can look at the properties of two functors $f^* : \mathbf{Set}(L_2) \to \mathbf{Set}(L_1)$ and $\Sigma_f : \mathbf{Set}(L_1) \to \mathbf{Set}(L_2)$ induced by pullback along f and compositon with f. Since these categories of fuzzy sets are quasitopoi with a second closed structure and weak representation of unbalanced subobjects, we can ask what preservation properties result of these functors result from properties of f.

We can also consider a much larger category whose objects are triples $(A, L, \alpha : A \to L)$ and whose morphisms from $(A, L, \alpha : A \to L)$ to $(B, L', \beta : B \to L')$ are pairs $(f : A \to B, \xi : L \to L')$ with $\beta(f(a)) \geq \xi(\alpha(a))$ for all $a \in A$. For fixed L this is a fibration over $\mathbf{Sets}$. For a fixed set A it is fibred over the appropriate category of lattices.

Monad Compositions and Generalized Terms

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We compose set functors with the term monad and show how such compositions can be extended to monads. Composition of monads thus provides a method for extending the notion of terms. Variable substitutions, viewed as morphisms in the corresponding Kleisli categories over composed monads, can then be seen more generally as variables assigned e.g. to (many-valued) sets of terms. We discuss some examples of powerset and double powerset functors composed with the term functor. Further we will provide more general results on constructing new monads from given ones, in particular concerning composition of submonads.

Fuzzy t-Norm Predicate Logic is Hard

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Recent development of mathematical fuzzy logic will be briefly surveyed. In particular, the Basic predicate logic BL∀ and its three strengthenings Ł∀ (Lukasiewicz predicate logic), G∀ (Gödel predicate logic) and Π∀ (product predicate logic) will be recalled. Each of has a natural axiom system which is complete with respect to all "sound" interpretations over any algebra from the respective class of algebras (BL-algebras, MV-algebras, G-algebras and Π-algebras.) Ł∀, G∀ and Π∀ have each its own *standard* semantics, the ordered interval $0, 1]$ with the corresponding continuous t-norm and its residuum (Lukasiewicz, Gödel and product t-norm). A standard tautology of such a logic is a formula true in all interpretations of the language over the standard semantics. It is known that the set of all standard tautologies of G∀ is recursively axiomatizable, whereas the corresponding sets of Ł∀ and Π∀ are not (see my *Metamathematics of fuzzy logic*, Kluwer 1998, for details). Analogously we may define a standard tautology of BL∀ to be a formula true in all interpretations over all t-algebras, i.e. $[0, 1]$ with any continuous t-norm and its residuum. The following is the main result: the set of all formulas of predicate logic that are tautologies with respect to all continuous t-norms (standard BL∀-tautologies) is heavily non-recursive (Π_2-hard).

Fuzzy Sets in a Heyting Valued Model

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The aim of this talk is to present a new and natural interpretation of fuzzy sets, fuzzy relations, and fuzzy mappings in a Heyting valued model for intuitionistic set thoery. By this interpretation we can consistently obtain various notions and properties of fuzzy sets, relations and mappings. Though the model is essentially the same as that of Takeuti-Titani etc., the interpretation is original and unique.

We interpret fuzzy sets and relations in a cumulative Heyting valued model for intuitionistic set thoery. In the model we can easily define basic notions and operations of sets and relations. By the interpretation with the canonical embedding we can get most of the standard defining equations of basic notions and operations of fuzzy sets and relations. There are several definitions of fuzzy mapping in the literature, one of which identifies fuzzy mapping with fuzzy relation. Our interpretation of fuzzy mappings seems to be unique but quite natural and it would make clear the meaning of Zadeh's extension principle.

Let H be a complete Heyting algebra and $V^{(H)}$ be the H-valued model in [1]. The Heyting value $\|\varphi\|$ is defined for every sentence φ of $V^{(H)}$. The canonical embedding corresponds each crisp set x to its check set Check$x$. Basic operations such as intersection, union, and complement of sets, and composition and inverse of relations are naturally defined in the model.

Every set A in $V^{(H)}$ is called an *H-fuzzy set*, and for a crisp set X every subset in $V^{(H)}$ of the check set Check$X$ is called an *H-fuzzy subset of X*. The mapping $\mu_A : X \longrightarrow H;\ x \longmapsto \|\,\mathrm{Check}x \in A\,\|$ is called the *membership function of A on X*. There is a natural correspondence between H-fuzzy subsets of X and mappings from X to H, which preserves order and basic set operations. Namely, if A and B are H-fuzzy subsets of X, then $A \subseteq B$ in $V^{(H)}$ iff $\mu_A \le \mu_B$, and we have $\mu_{A\cap B} = \mu_A \wedge \mu_B$, $\mu_{A\cup B} = \mu_A \vee \mu_B$, and $\mu_{\mathrm{Check}X\setminus A} = \neg\,\mu_A$.

An H-fuzzy subset R of $X \times Y$ is called an *H-fuzzy relation from X to Y*. If R and S are H-fuzzy relations from X to Y and from Y to Z respectively, then the composition $S\circ R$ is also an H-fuzzy relation from X to Z, and $\mu_{S\circ R}\langle xz\rangle = \bigvee_{y\in Y}(\mu_R\langle xy\rangle \wedge \mu_S\langle yz\rangle)$ for all $x \in X, z \in Z$. If R is an H-fuzzy relation from X to Y, then the inverse relation R^{-1} is an H-fuzzy relation from Y to X, and $\mu_{R^{-1}}\langle yx\rangle = \mu_R\langle xy\rangle$ for all $x \in X, y \in Y$. For an H-fuzzy relation R on X, R is reflexive (in $V^{(H)}$) iff $\mu_R\langle xx\rangle = \mathbf{1}$ $(\forall x \in X)$, symmetric iff $\mu_R\langle xy\rangle = \mu_R\langle yx\rangle$ $(\forall x, y \in X)$, and transitive iff $\mu_R\langle xy\rangle \wedge \mu_R\langle yz\rangle \le \mu_R\langle xz\rangle$ $(\forall x, y, z \in X)$.

An *H-fuzzy mapping f from X to Y* is a mapping from Check$X$ to Check$Y$ in $V^{(H)}$. For an H-fuzzy relation f from X to Y, f is an H-fuzzy mapping from X to Y iff its

membership function μ_f on $X \times Y$ satisfies the following two conditions:
(1) $\bigvee_{y \in Y} \mu_f\langle xy \rangle = \mathbf{1} \quad (\forall x \in X)$, (2) $\mu_f\langle xy \rangle \wedge \mu_f\langle xz \rangle > \mathbf{0}$ implies $y = z \quad (\forall x \in X, \forall y, z \in Y)$.

Images and inverse images of relations and mappings are also naturally defined in the model. For an H-fuzzy mapping f from X to Y and an H-fuzzy set A, the image $f(A)$ is an H-fuzzy subset of Y. For every crisp mapping $\varphi : X \longrightarrow Y$, its check set Check$\varphi$ is an H-fuzzy mapping from X to Y, and then for all H-fuzzy set A, Check$\varphi(A)$ is an H-fuzzy subset of Y and its membership function satisfies the equation in the famous Zadeh's extension principle.

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Normal Forms for Fuzzy Logic

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In this paper, we examine and compare De Morgan-, Kleene-, and Boolean-disjunctive and conjunctive normal forms and consider their role in fuzzy settings. In particular, we show that there are normal forms and truth tables for classical fuzzy propositional logic and interval-valued fuzzy propositional logic that are completely analogous to those for Boolean propositional logic. Thus, determining logical equivalence of two expressions in fuzzy propositional logic is a finite problem, and similarly for the interval-valued case. Algorithms for getting normal forms from truth tables for the classical fuzzy case and for the interval-valued case are provided. This also sheds some mathematical light on Turksen's work on "interval valued fuzzy sets".

Opening and Closure Operators of Fuzzy Relations: Basic Properties and Applications to Fuzzy Rule-Based Systems

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Images of fuzzy sets under fuzzy relations have been investigated mainly in two contexts: On the one hand, mostly under the term "full image" [5], they can be regarded as very general tools for fuzzy inference, leading to the so-called "compositional rule of inference" [1, 5]. On the other hand, under the term "extensional hull", the image of a fuzzy set under a fuzzy equivalence relation yields the smallest superset which is "closed" under the relation, where this property is usually called "extensionality" [6].

In the first part of this contribution, after recalling some basic definitions and properties, we propose a new generalized concept of closedness under a fuzzy relation (let us call it "congruence") which naturally extends the notion of extensionality to arbitrary binary fuzzy relations. Based on these considerations, under the assumption of T-transitivity, we are able to give explicit formulae for the congruent opening, i.e. the largest congruent subset, and the congruent closing, i.e. the smallest congruent superset, of a fuzzy set. It will turn out that this directly leads to full images—as already known for fuzzy equivalence relations.

The second part is devoted to a new view on images of fuzzy sets under fuzzy relations—by integration of the results on congruence and the inference-based interpretation of images under fuzzy relations, we are able to provide a new framework for defining linguistic modifiers, both ordering-based ones like "at least", "at most", or "between" and usual weakening and intensifying ones like "more or less", "roughly", or "very".

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Inequalities in Strict De Morgan Systems

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The methods for rewriting a Boolean, Kleene, or De Morgan expression to a normal form use equational properties of the variety of Boolean, Kleene, or De Morgan algebras, respectively. For De Morgan systems, where there is no practical hope for normal forms, we take a broader point of view and examine some of the inequalities (equalities) that hold. In previous work, we used families of inequalities to separate the varieties generated by strict De Morgan systems—algebras consisting of the unit interval with its natural order, a strict t-norm, and a strong negation. In that work we showed, in fact, that the family of varieties generated by nonisomorphic strict De Morgan systems forms an antichain. This is a continuation of that investigation.

Fuzzy Control and Fuzzy Functions

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Abstract

In many applications of fuzzy sets, especially in fuzzy control, the notions of fuzzy points and fuzzy functions play an important role. Nevertheless, these concepts are usually understood and used on a very intuitive basis without providing a formal definition. In this paper we discuss an proach to clarify and formalise these notions and show some consequences for fuzzy control and fuzzy interpolation.

Our basic framework for fuzzy sets is the unit interval L = [0, 1] endowed with some continuous t-norm, denoted by $*$. This t-norm induces a residuated implication and a biimplication, denoted by $\rightarrow$ and $\leftrightarrow$, respectively.

The very fundamental notion that we need, is the concept of an equality relation that allows to introduce the notion (fuzzy) singletons, which is crucial for the definition of fuzzy functions.

In approximate reasoning if-then rules of the form

$$\text{If } \xi \text{ is A, then } \eta \text{ is B} \tag{1}$$

are very common where ξ and η are variables with domains X and Y, respectively. A and B are linguistic terms like *positive big* or *approximately zero*. These linguistic terms are usually modelled by suitable fuzzy sets, say $\mu_A \in L^X$ and $\mu_B \in L^Y$. In addition to such general rules one has specific information like

$$\xi \text{ is } A' \tag{2}$$

where A' is represented by the fuzzy set $\mu_{A'} \in L^X$ (or simply by $\mu \in L^X$). The application of a single rule of the form (1) to the information (2) is usually formalised on the basis of a computing scheme of the following form. The rule is encoded as a fuzzy relation of the form

$$\rho(x,y) = \rho_{\odot}(x,y) = \mu_A(x) \odot \mu_B(y) \tag{3}$$

where $\odot$ is either the operation $*$ or the residuated implication $\rightarrow$. For a given input information in the form of the fuzzy set $\mu_{A'} \in L^X$, the 'output' fuzzy set $\nu_{\text{conclusion}}$ is computed as the composition of the fuzzy relation $\rho_{\odot}$ and the fuzzy set $\mu_{A'}$, i.e.

$$\rho[\mu_{A'}](y) = \bigvee_{x \in X} \{\mu_{A'}(x) * \rho(x,y)\} \tag{4}$$

for all $y \in Y$. This scheme is called sup-$*$-inference. In fuzzy control, for instance, usually $* = \min = \odot$ is chosen.

For a collection of if-then rules of the form

$$\text{If } \xi \text{ is } A_i, \text{ then } \eta \text{ is } B_i, \ (i \in I), \tag{5}$$

where the linguistic terms A_i and B_i are modelled by the fuzzy set $\mu_{A_i} \in L^X$ and $\mu_{B_i} \in L^Y$. The output fuzzy set for a given 'input fuzzy set' $\mu \in L^X$ is usually computed either by

$$\bigwedge_{i \in I} \rho_i[\mu], \tag{6}$$

when the fuzzy relations ρ_i are of the type $\rho_{\rightarrow}$ or by

$$\bigvee_{i \in I} \rho_i[\mu], \tag{7}$$

when the fuzzy relations ρ_i are of the type ρ_*. In other words, we associate with the collection (5) of if-then-rules either the fuzzy relation

$$\rho_U(x, y) = \bigwedge_{i \in I} \rho_{\rightarrow}(x, y) = \bigwedge_{i \in I} (\mu_i(x) \rightarrow \nu_i(y)) \tag{8}$$

or the fuzzy relation

$$\rho_U(x, y) = \bigvee_{i \in I} \rho_*(x, y) = \bigvee_{i \in I} (\mu_i(x) * \nu_i(y)) \tag{9}$$

Considering (5) as the system of fuzzy relational equations

$$\rho[\mu_i] = \nu_i \text{ for } i \in I$$

where the fuzzy sets μ_i and ν_i are given and solution of the system in the form of a fuzzy relation ρ has to be constructed, then it is well known that ρ_U is always a solution if there exists a solution at all. In this case, ρ_U is the greatest solution. ρ might not be a solution, even if there exists a solution of the system.

In some applications of approximate reasoning, especially in fuzzy control, the if-then-rules (5) are intended to describe a functional dependence between the variable ξ and η. In this case we would expect the fuzzy relation ρ_U or ρ_L constructed from these rules to behave like a fuzzy function. But what do we mean by a fuzzy function? A fundamental property of a function is that it is a relation that does never assign two or more elements from the codomain to one element of the domain. Thus we need the concept of a one-element (fuzzy) set again which means that we have to assume suitable equality relations on the domains X of the variable ξ and Y of the variable η. In our considerations we will usually choose special equality relations on X respectively Y that are induced by the fuzzy sets appearing in the rules. But for the moment we do not need this assumption, we just have to assume that there is an equality relation E on the set X and an equality relation F on the set Y. We require a similar extensionality property from a fuzzy relation as from fuzzy sets.

In this framework, we can provide various results that connect the notion of a fuzzy function with fuzzy control and approximate reasoning schemes.

Let us now return to the fuzzy relations ρ_U and ρ_L that are induced by the collection of if-then-rules (5). The following theorem shows that they are extensional when the considered fuzzy sets are extensional.

Theorem 1. *Let the fuzzy sets μ_i and ν_i appearing in the if-then-rules (5) be normal. (A fuzzy set jt is normal if there exists an x such that $\mu(x) = 1$ holds.) The fuzzy relation ρ_L defined in Equation (9) is a solution to the system of fuzzy relational equations $\rho[\mu_i] = \nu_i (i \in I)$ if and only if*

$$(\forall i, j \in I) \left(\bigvee_{x \in X} (\mu_i(x) * \mu_j(x)) \leq \bigvee_{y \in Y} (\nu_i(y) \leftrightarrow \nu_j(y)) \right) \tag{10}$$

holds.

Theorem 2. *Let E and F be equality relations on X and Y, respectively, such that the fuzzy sets μ_i and ν_j correspond to the extensional hulls of the points x_i and y_j, respectively ($i \in I$). If the ordinary partial function $f(x_i) = y_j$ is extensional w.r.t. E and F, then the fuzzy relation ρ_L is a partial fuzzy function and $\rho_L = \rho_f$ holds.*

In the same context the fuzzy relation ρ_U is a fully defined fuzzy relation, but usually not a partial fuzzy function. The proof is obvious from the definition of ρ_U.

Corollary 3. *Let E and F be equality relations on X and Y, respectively, such that the fuzzy sets μ_i and ν_i correspond to the extensional hulls of the points x_i and y_j, respectively ($i \in I$). Then*

$$\bigvee_{y \in y} \rho_U(x, y) = 1$$

holds.

Theorem 4. *Let E and F be equality relations on X and Y, respectively, and let $f : X \to Y$ be an (ordinary) extensional function. Let $\{x_i | i \in I\} \subseteq X$ be a set of elements of X and let f_I denote the (ordinary) partial function defined by $f_I(x_i) = f(x_i)$ for $i \in I$. Let μ_i denote the extensional hull of the point x_i w.r.t. E, and let ν_i denote the extensional hull of the point $f(x_i)$ w.r.t. F. Then*

$$\rho_L = \rho_{f_I} \leq \rho_f \leq \rho_U$$

holds.

We can interpret Theorem 3 in the context of fuzzy control in the following way. Fuzzy control aims at determining an (unknown) control function

$$f : X \to Y.$$

This function is described by if-then-rules of the form (5). The fuzzy sets μ_i and ν_i appearing in the rules are considered as extensional hulls of single points x_i and y_i. Thus the rules specify the partial control function f_I. Of course, the underlying equality relations must be related to the control function in the sense that the control function

f is extensional. This means simply that we choose narrower fuzzy sets where exact values are quite important for a good control and wider fuzzy sets where even a rough controller output provides a reasonable control. Since the partial control function f_I does not specify controller outputs for all inputs, we have to take the information encoded in the fuzzy sets (or in the equality relations) into account. Therefore we consider the extensional hull of the partial control function f_I that is equal to the fuzzy relation ρ_L, in other words, to the Mamdani-type fuzzy control scheme. This provides a lower approximation for the extensional hull of the (unknown) control function f. The fuzzy relation ρ_U can be seen as an upper approximation of the extensional hull of f.

Corollary 5. *Let E and F be equality relations on X and Y, respectively, such that the fuzzy sets μ_i and ν_i correspond to the extensional hulls of the points x_i and y_i, respectively ($i \in I$). Let the ordinary partial function $f(x_i) = y_i$ be extensional w.r.t. E and F. If*

$$\rho_L = \rho_U$$

holds, then ρ_L and ρ_U are fuzzy functions.

Inference models

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In this paper a consistent model of the generalized modus ponenses is discussed in the approximate reasoning. Inputs are implication rules (the so-called: IF-THEN structures) and statements. Output is a statement (conclusion). Approximate reasoning allows fuzzy inputs, fuzzy antecedents, fuzzy consequents, or combinations of these. In general in the approximate reasoning the continous logic is used instead of classical logic.

One of the most successful area of fuzzy theory is the fuzzy control based on inference, although from logical point of view only few theoretical papers deal with it. The problem is, that there are several operator systems in fuzzy logic (conjunction, disjunction, negation) and there are a lot of implication. The so-called modifiers also play an important role and they still no have any axiomatic bases. The inference is an algorithm to calculate the unknown values. But what kind of inference is good?

Fuzzy theory deals with uncertain information, but in control the input and the result are not uncertain. (So the membership function should have a different meaning, etc.)

As in control area the only strict monotonously increasing operator is useful. We are concentrating on only those operators that come from this class. The theoretical basis of the modifiers is also given. The crucial point is choosing the implicaton, because the residual implication is non-continuous.

Our solution is using the classical extension with approximate properties. We use different models for the inference: solving equations, optimization and surface approximation. We show that all of them are consistent with the classical inference.

Axiomatizing t-Norm-Based Many-Valued Logics

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The ŁUKASIEWICZ systems, the GÖDEL systems, and the product logic all are systems of many-valued logic which are (or can be) t-norm based in the sense that one may start the semantic considerations with a particular (even continuous) t-norm in the real unit interval, which serves a truth degree function for some conjunction connective. In a uniform way one can defines all the other connectives on this basis.

So one has a kind of standard semantic approach toward a large class of many-valued logics. The problem then is to look for adequate axiomatizations for these t-norm based systems. Because of the diversity of t-norms the most suitable way to reach this goal seems to be to look for axiomatizations which cover a wide class of such systems – in the sense that the particular t-norm based systems then can be axiomatized by some extension of the "core" system.

There is, however, an important difference between the standard approach toward all these particular systems, and also toward the POST systems, and the present situation to look for some core system for the t-norm based systems in general: for all these particular systems S some single, "standard" semantical approach is constitutive which is determined by some standard logical matrix for S.

For some core system for the class of all t-norm based systems in general (it seems that) one does not have such a standard logical matrix. Therefore another type of semantical characterization is needed, which is provided by some suitable class of algebraic structures, similarly to the semantical characterizations of the ŁUKASIEWICZ systems by MV-algebras.

Looking at the actual approaches using t-norms, one recognizes that there is an important restriction – to left continuous (or even to continuous) t-norms, because one (usually) likes to have available an R-implication connective in each one of these t-norm based systems of many-valued logic.

In any case it is structurally important for the t-norm based systems that one has on the one hand that the basic t-norm constitutes a commutative semigroup with a neutral element, on the other hand it is important that the usual ordering (of the reals of the unit interval) is a (lattice) ordering which has a universal lower bound and a universal upper bound. And it is also important that both structures "fit together" in the sense that the semigroup operation $\mathbf{t}$ is non-decreasing w.r.t. this lattice ordering.

Furthermore one usually likes to have the t-norm combined with a corresponding R-implication operation, which algebraically means that the lattice-ordered monoid constituted by the truth degree structure should even be a residuated one.

Based on this type of algebraic structure one has given adequate axiomatizations of the logics which are characterized by such classes of algebraic structures. They cover the classes of t-norm based many-valued logics.

These approaches shall be discussed, some recent results explained, and some open problems mentioned.

A Bicategorical View of M-Valued Sets

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Galois connections were introduced by Ore [1]. Recall that a Galois connection is established between two posets P and Q whenever there exist mappings $f : P \to Q$ and $g : Q \to P$) such that

$$f(p_1) \subseteq f(p_2) \text{ whenever } p_1 \supset p_2$$

$$g(q_1) \subseteq g(q_2) \text{ whenever } q_1 \supset q_2$$

and

$$\forall_{p\in P} gf(p) \supseteq p, \forall_{q\in Q} fg(q) \supseteq q$$

Höhle [2] points out that residuation in a commutative po-monoid is equivalent to the existence of a Galois connection between the underlying poset and its dual, i.e., in a commutative po-monoid M with underlying poset $(L, \leq)$, if

$$\beta * \alpha \leq \gamma \Leftrightarrow \alpha \leq (\beta \to \gamma), \forall \alpha, \beta, \gamma \in L \text{ (residuation)}$$

then

$$(\beta \to \circ \beta * \alpha) \geq \alpha \text{ and } (\beta * ????? \circ \beta \to \gamma) \leq \gamma \text{ (Galois connection)}$$

where $\beta \to$ and $\beta *$ denote (the obvious) maps from L to L for fixed β, and $\circ$ denotes composition.

This equivalence is most clearly expressed (though not in so many words) in [1] (p. 500), where Ore shows that the Galois correspondence between a (complete) poset P and its dual is given by

$$p \to (\overline{p})^{\alpha}, p \to (p^*)^{\alpha^{-1}}$$

where $\overline{p}$ is the least element containing p in a substructure of P (P_1) with all (finite and infinite) intersections matching and p^* is the greatest element containing p in an isomorphic substructure of P (Q_1, to P_1) with all unions matching (those of P), and α is an isomorphism between P_1 and Q_1. This form of "auto- connection" makes clear the "implication-like" character of Galois connections. Thus, the existence of a Galois connection between a poset P and its dual guarantees that each element of P is bounded above and below by a "closed" element of P. It is apparent that, w.r.t. the connection between residuation and Galois connections noted by Höhle, $\beta \to$ generates the P_1 substructure and $\beta *$ generates the Q_1 substructure needed to establish the Galois connection.

The connection between Galois connections and (binary) relations has also been noted, both for the crisp [1] and the fuzzy case [3]. It is known, for instance, that ([1], Th. 10), any Galois connection between the structures of all subsets of two sets can be defined by means of a binary relation between the two sets. To prove similar results in the fuzzy case it is necessary, first of all, to motivate a notion of subsethood for fuzzy subsets: this is done (in [3], following [4]) by setting the subsethood degree of A_1 in A_2 (in universe X) to be

$$\bigwedge_{x \in X} (A_1(x) \to A_2(x)).$$

Once this has been done, a fuzzy Galois connection can be defined in a manner analogous to the crisp case.

It is also known from [2] that the singletons of an M-valued set (X, E) and morphisms from (X, E) to these singletons in the category **M-SET** (M-valued sets and structure-preserving maps between them) yield an algebraic theory (in clone form) in this same category. From this it follows that there exists a T-algebra $((X, E), \xi)$; the monadic character of the set of all such T-algebras is apparent (see [5], p. 137 for a relevant comment). Höhle ([2], see esp. p. 60 and p.56) also points out that every (strict and extensional) L-fuzzy subset of (X, E) corresponds to a morphism (of a certain type) in **M-SET**.

Koslowski [6] notes that pre-ordered sets can be viewed as monads in **rel** (the bicategory of sets, relations, and inclusions), and proceeds from this insight to arrive at a deeper understanding of the fact that the bicategory **idl** of pre-ordered sets, order-ideals, and inclusions inherits properties from **rel**. It is the intent of this paper to motivate and discuss the properties of another bicategory which I shall call **mbi**. This bicategory consists of M-valued sets as 0-cells, strict and extensional L-fuzzy subsets as 1-cells, and inclusions as 2-cells. It will be shown that **mbi** is a sub-bicategory of **kar** (the Cauchy-completion of **rel**). It will be seen that this bi-categorical view serves to unify (to a degree) the various properties and interrelationships adumbrated above (residuation, Galois connections, (fuzzy) binary relations). In fact, M- valued sets will be seen to be fuzzy information systems in the sense of [7], i.e., fuzzy "interpolads," [6], and the link from **mbi** to **rel** will be seen to be mediated by these fuzzy interpolads just as the link from **idl** to **rel** is mediated by the monadic status of pre-ordered sets.

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