Comparing Localic and Fixed-Basis Topological Products

Fatma Bayoumi, Stephen E. Rodabaugh

Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt; College of Science, Technology, Engineering, Mathematics (STEM), Youngstown State University, Youngstown, OH, USA 44555-3609

Products play a crucial role in point-set topology and increasingly a correspondingly crucial role in point-set lattice-theoretic (poslat) topology. In 1976, Dowker & Papert [1] constructed the coproduct of frames, a deep construction which furnishes the product for point-free topology based on the notion of a frame, a product repackaged by Johnstone [5] in 1982 using sites and coverages, and then repackaged recently and more accessibly by Pultr [8] using the quotient of a frame by a binary relation. Unless stated otherwise, $L$ in the sequel is a frame.

Question 1. The fundamental question is the following: how does the notion of localic product compare with corresponding notions for traditional and lattice-valued topology? This question can take various forms, in each of which the localic product is denoted by $\bigoplus$ and the product topology by $\bigotimes$:

1. Given a family $\{(X_\gamma, T_\gamma) : \gamma \in \Gamma\} \subset |\text{Top}|$, the corresponding product space $\left( \prod_{\gamma \in \Gamma} X_\gamma \bigotimes_{\gamma \in \Gamma} T_\gamma \right)$, and the localic product $\bigoplus_{\gamma \in \Gamma} T_\gamma$ and $\bigoplus_{\gamma \in \Gamma} T_\gamma$ compare as frames; e.g., are they isomorphic?
2. Given a family $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset |\text{L-Top}|$, the corresponding $L$-topological product space $\left( \prod_{\gamma \in \Gamma} X_\gamma \bigotimes_{\gamma \in \Gamma} \tau_\gamma \right)$, and the localic product $\bigoplus_{\gamma \in \Gamma} \tau_\gamma$ and $\bigoplus_{\gamma \in \Gamma} \tau_\gamma$ compare as frames; e.g., are they isomorphic?
3. Given a family $\{(X_\gamma, L_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset |\text{Loc-Top}| [16]$, the corresponding product space $\left( \prod_{\gamma \in \Gamma} X_\gamma \bigotimes_{\gamma \in \Gamma} L_\gamma, \bigotimes_{\gamma \in \Gamma} \tau_\gamma \right)$, and the localic product $\bigoplus_{\gamma \in \Gamma} \tau_\gamma$ and $\bigoplus_{\gamma \in \Gamma} \tau_\gamma$ compare as frames; e.g., are they isomorphic?

Questions 1(1, 2) concern fixed-basis topology and are the primary focus of this note, while Question 1(3) concerns a form of variable-basis topology and is the subject of future work, as is also the investigation of these product questions when the underlying lattices of membership values are allowed to be quantales or other extensions of frames. We note that preliminary discussions on Question 1(2) are given in [11, ?] and the present work may be viewed as an (overdue) update and expansion of those discussions.

A key tool for making comparisons in Question 1(2) is the notion of $L$-spatiality, which is part of the $L$-spectrum adjunction $L\Omega \dashv LPT$ studied in [2,
3, 6, 7, 9, 10, 11, 12, 13, 15, 17]. Restricting \( L \)-spatiality to the cases when \( L \) is a frame for ease in this abstract, a locale \( A \) is \( L \)-\textit{spatial} if the “comparison” map \( \Phi_L : A \to L^{Lpt(A)} \) is injective, where

\[
Lpt(A) = \text{ Frm}(A, L), \quad \Phi_L(a)(p) = p(a).
\]

Since the \( L \)-spectrum \( LPT(A) \equiv (Lpt(A), \Phi_L^{-1}(A)) \) is an \( L \)-topological space, then an \( L \)-spatial locale is order-isomorphic to the \( L \)-topology of its \( L \)-spectrum; and if we choose \( L = 2 \), then \( L \)-spatiality, via the categorical isomorphism \( G_L : \text{ Top} \to L\text{-Top} \), reduces to the traditional spatiality of [4, 5].

One could hope that the localic product of \( L \)-topologies is \( L \)-spatial—and hence order-isomorphic to its spectrum \( L \)-topology—if and only if it is order-isomorphic to the product \( L \)-topology of \( L \)-topologies; restated, the localic product should be (up to order-isomorphism) the product topology precisely when it is a particular topology. Thus possible answers to Questions 1(1) and 1(2) above can be proposed, respectively, as follows:

**Proposition 1 (localic and traditional products).** \( \bigoplus_{\gamma \in \Gamma} \tau_\gamma \cong \bigotimes_{\gamma \in \Gamma} \tau_\gamma \) if and only if \( \bigoplus_{\gamma \in \Gamma} \tau_\gamma \) is spatial.

**Conjecture 1 (localic and fuzzy products).** \( \bigoplus_{\gamma \in \Gamma} \tau_\gamma \cong \bigotimes_{\gamma \in \Gamma} \tau_\gamma \) if and only if \( \bigoplus_{\gamma \in \Gamma} \tau_\gamma \) is \( L \)-spatial.

Proposition 1 is in fact a theorem: its statement appears in [1] without proof, and its proof is decidedly nontrivial and deserves to be in the literature. It is not known whether Conjecture 1 is true—necessity is always true and we conjecture sufficiency to be false; but it becomes a theorem under conditions each of which includes Proposition 1 when \( L = 2 \). Examples of such results are the following:

**Theorem 1.** Necessity in Conjecture 1 always holds.

The following alternative to Conjecture 1 is true:

**Theorem 2.** \( \bigoplus_{\gamma \in \Gamma} \tau_\gamma \cong \bigotimes_{\gamma \in \Gamma} \Phi_L^{-1}(\tau_\gamma) \) if and only if \( \bigoplus_{\gamma \in \Gamma} \tau_\gamma \) is \( L \)-spatial.

Sufficiency in Conjecture 1 holds under certain conditions in the next two theorems which are always valid for traditional topological spaces:

**Theorem 3.** Let \( L \) be spatial and \( \{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \) be a family of product-sum separated \( L \)-topological spaces. Then sufficiency in the Conjecture holds.

**Remark 1.** The property of “product-sum separated spaces” is a technical condition which always holds for traditional spaces. In the case of two traditional spaces \( (X, T), (Y, S) \), this condition says that if \( A, B \neq \emptyset \), then

\[
A \times B \subseteq C + D \implies A \subseteq C \text{ or } B \subseteq D,
\]

where

\[
C + D = (C \times Y) \cup (X \times D).
\]
The fuzzy analogue of this property—given \((X, \tau), (Y, \sigma)\) with \(a, b \neq \perp\),
\[
a \otimes b \leq c \oplus d \Rightarrow a \leq c \text{ or } b \leq d,
\]
where
\[
(a \otimes b) (x, y) = a (x) \wedge b (y), \quad (c \oplus d) (x, y) = c (x) \vee d (y),
\]
—generally does not hold for \(L\)-topological spaces (the \(L\)-topological product is rather “messy” as compared with the traditional product). But there are not only non-trivial spaces one can manufacture which as a family have such a property, there are in fact important classes of spaces which form such families: the fuzzy real lines and fuzzy unit intervals for \(L\) a complete DeMorgan algebra; the alternative fuzzy real lines and fuzzy unit intervals for \(L\) any semiframe; and the \(L\)-soberifications of any \(2\)-topological spaces for any \(L\) a semiframe.

**Theorem 4.** If \(\{ (X_\gamma, \tau_\gamma) : \gamma \in \Gamma \} \) is a family of join-separated \(L\)-topological spaces, then sufficiency in the Conjecture holds.

A weaker result than sufficiency in Conjecture 1 is given by the next theorem:

**Theorem 5.** \(\otimes_{\gamma \in \Gamma} \tau_\gamma\) is a sublocale of \(\bigoplus_{\gamma \in \Gamma} \tau_\gamma\).

**References**

11. S. E. Rodabaugh, *A point set lattice-theoretic framework* \( T \) *which contains* \( \text{Loc} \) *as a subcategory of singleton spaces and in which there are general classes of Stone representation and compactification theorems*, first draft February 1986 / second draft April 1987, Youngstown State University Central Printing Office (Youngstown, Ohio).


