Introducing fuzziness in monads: cases of the powerobject monad and the term monad

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Fuzzy mathematics often starts by taking a piece of classical mathematics and introducing fuzziness to existing mathematical concepts, and then proceeds to a more essential adoption of a fuzzy perspective. Our paper explores this progression for two important monads: the powerset monad and its generalizations to fuzzy powerobjects, and the term monad and its generalization to using fuzzy sets of constants. This brings together two lines of research previously discussed at Linz.The powerset and its fuzzy analogs are important in the development of topology in a fuzzy world and the term monad and its fuzzy analogs are vital in understanding fuzzy computer science.

We work with fuzzy sets with values in a completely distributive lattice L equipped with a semigroup operator $\star$ which distributes over both $\bigwedge$ and $\bigvee$. Such a lattice will have residuation for both $\star$ and $\wedge$, we write $a \rightarrow -$ for the right adjoint to $a \star -$ and $a \Rightarrow -$ for the right adjoint to $a \wedge -$. This generalizes the setting of working over the unit interval with a continuous t-norm.

1 Categories of increasing fuzziness

Most of mathematics is done in the category Set whose objects are sets and morphisms are mappings (functions). One step in fuzzifying this is to replace subsets of A with functions from A to L. This still lives in the category Set: L^A is an object of Set and L-sets $\alpha\colon A \to L$ are just elements of L^A. To increase incorporate fuzziness from the start we can work in the category $\mathsf{Set}(L)$ introduced by Goguen in [6]. Others have worked in categories allowing multiple lattices or in categories generalizing the category of sets with relations instead of functions.

The category $\mathsf{Set}(L)$ has as objects pairs (A,α) where $\alpha\colon A \to L$ and as morphisms $f : (A,\alpha) \to (B,\beta)$ mappings $f\colon A \to B$ such that $\beta(f(a) \geq \alpha(a)$ for all $a \in A$.

It is known that $\mathsf{Set}(L)$ is topological over Set and has a monoidal structure using $(A,\alpha) \otimes (B,\beta) = (A \times B, \alpha \star \beta)$ and products using $(A,\alpha) \times (B,\beta) = (A \times B, \alpha \wedge \beta)$. Pultr [8] showed how to get exponentials for both. Because $\mathsf{Set}(L)$ is topological, it has all limits and colimits [1]. The category $\mathsf{Set}(L)$ is also a quasitopos, but not a topos. As pointed out in [9] the logic studied in fuzzy set theory is the logic of unbalanced subobjects (those with underlying map the identity) rather than the logic in the quasitopos structure. This observation informs our choice of fuzzy powerobject functor.

There is a lifting functor $\mathbf{C}\colon \mathsf{Set} \to \mathsf{Set}(L)$ taking A to the crisp fuzzy set $(A,\top)$. On functions this functor is the identity.

There are three natural functors from $\mathsf{Set}(L)$ to Set to consider:

1. the underlying set functor $\mathbf{U}$ taking (A,α) to A and $f:(A,\alpha)\to(B,\beta)$ to f
2. the full members functor $\mathbf{F}$ taking (A,α) to $\{a|\alpha(a)=\top\}$. A function $f:(A,\alpha)\to(B,\beta)$ takes full members to full members, so the action of this functor on maps is restriction to the full members.
3. the support functor $\mathbf{S}$ taking (A,α) to $\{a|\alpha(a)>\bot\}$. Again maps $f:(A,\alpha)\to(B,\beta)$ restrict to functions on the supports since we have $\beta(f(a))\geq\alpha(a)>\bot$.

We get $\mathbf{UC}=\mathbf{FC}=\mathbf{SC}=\mathrm{id}$ and $\mathbf{U}\dashv\mathbf{C}\dashv\mathbf{F}$.

2 Monads with increasing fuzziness

2.1 Powerobject monads

As mentioned in Section 1, it is possible to replace a set A by L^A and still work in Set. The powerset monad $\mathbf{L}_{id}=(L_{id},\eta,\mu)$, from Manes [7], can be seen as the first step to introduce fuzziness in a categorical setting: The covariant powerset functor $L_{id}\colon \mathsf{Set}\to\mathsf{Set}$ is obtained by $L_{id}A=L^A$, and, following [5], for a morphism $f\colon A\to B$ in Set, by defining for all $y\in B$, $L_{id}f(\alpha)(y)=\bigvee_{f(x)=y}\alpha(x)$. The natural transformations $\eta_A\colon A\Longrightarrow L_{id}A$ by $\eta_A(x)(x')=\top$ if $x=x'$ and $\bot$ otherwise, and $\mu_A: L_{id}L_{id}A\Longrightarrow L_{id}A$ by $\mu_A(\mathcal{A})(x)=\bigvee_{\beta\in L_{id}A}\alpha(x)\wedge\mathcal{A}(\beta)$. In [7] it was shown that $\mathbf{L}_{id}=(L_{id},\eta,\mu)$ is a monad.

Another variant of this monad uses a $\star$ in the definition of μ:

$$\mu_X^\star(\mathcal{A})(x)=\bigvee_{A\in L_{id}X} A(x)\star\mathcal{A}(A)$$

This can be made more fully fuzzy by considering the internal fuzzy powerobject functor from $\mathsf{Set}(L)$ to $\mathsf{Set}(L)$ which has action on objects given by

$$\mathcal{U}^\star(A,\alpha)=(L^A,\pi_{(A,\alpha)})\text{ where }\pi_{(A,\alpha)}(f)=\bigwedge_{a\in A}(f(a)\to\alpha(a))$$

Notice that one of the options for $\star$ is $\wedge$, in which case we write $\Rightarrow$ for the residuation and we get

$$\mathcal{U}^\wedge(A,\alpha)=(L^A,\xi_{(A,\alpha)})\text{ where }\xi_{(A,\alpha)}(f)=\bigwedge_{a\in A}(f(a)\Rightarrow\alpha(a))$$

There are three functors $\mathcal{U}^\star\colon \mathsf{Set}(L)\to\mathsf{Set}(L)$ giving unbalanced powerobjects as objects of $\mathsf{Set}(L)$: one contravariant (inverse image) and its covariant right adjoint and (corresponding to direct image) left adjoint taking $f:(A,\alpha)\to(B,\beta)$ to $\exists_f$ where

$$\exists_f(A,\alpha')(b)=\bigvee_{f(a)=b}\alpha'(a)$$

The covariant internal unbalanced powerobject monad uses the functor $\mathcal{U}^\star$ with $\exists_f$ for its action on maps.

The monadic structure comes from

$$\eta_{(A,\alpha)} : (A,\alpha) \Longrightarrow \mathcal{U}^\star(A,\alpha)$$

where

$$\eta_{(A,\alpha)}(a)(t) = \begin{cases} \top & \text{if } t = a \\ \bot & \text{otherwise} \end{cases}$$

Notice that the degree of membership of $\eta_{(A,\alpha)}(a)$ in $\mathcal{U}(A,\alpha)$ is

$$\bigwedge_t (\eta_{(A,\alpha)}(a)(t) \to \alpha(t)) = \top \to \alpha(a) = \alpha(a)$$

so $\eta_{(A,\alpha)}$ is a map in $\mathsf{Set}(L)$.

The union is given by the natural transformation

$$\mu^\star_{(A,\alpha)}(\mathcal{U}^\star)^2(A,\alpha) \Longrightarrow \mathcal{U}^\star(A,\alpha)$$

with

$$\mu^\star_A(L^A,\tau)(a) = \bigvee_f (\tau(f) \star f(a))$$

Proposition 1. $(\mathcal{U}^\star,\eta,\mu^\star)$ *is a monad.*

We can use the crisp functor to relate these monads on Set and $\mathsf{Set}(L)$. We get $\mathbf{C}(\nu_A) = \eta_{\mathbf{C}(A)}$ and $\mu^\star\colon \mathcal{U}^\star(\mathcal{U}^\star(\mathbf{C}(A)) = \mathbf{C}\left(L^{L^A}\right) \to \mathcal{U}^\star(\mathbf{C}(A)) = \mathbf{C}\left(L^A\right)$ is $\mathbf{C}(\mu^\star : L^{L^A} \to L^A)$.

2.2 Term monads

It is useful to adopt a more functorial presentation of the set of terms, as opposed to using the conventional inductive definition of terms, where we bind ourselves to certain styles of proofs. The term monad over Set is used, for example, in [4, 5]. In [7] it was shown that $\mathbf{T}_\Omega = (T_\Omega, \eta^{T_\Omega}, \mu^{T_\Omega})$ is a monad.

The first step to generalize term monad to fuzzy terms was to compose the monads $\mathbf{L}_{id}$ and $\mathbf{T}_\Omega$. This requires a distributive law in the form of a natural transformation $\sigma : \mathbf{T}_\Omega \mathbf{L}_{id} \Longrightarrow \mathbf{L}_{id} \mathbf{T}_\Omega$.

To introduce the term monad in $\mathsf{Set}(L)$ we set $id^0(A,\alpha) = (\{\varnothing\},\top)$ and $id^n(A,\alpha) = (id^n A, id^n(\alpha)$, where $id^n(\alpha)(a_1,\ldots,a_n) = \bigwedge_{i=1}^n \alpha(a_i)$. When we need the monoidal structure we replace $\wedge$ by $\star$. A constant $\mathsf{Set}(L)$- covariant functor $(A,\alpha)_{\mathsf{Set}(L)}$ assings any (X,ξ) to (A,α) and all morphisms $f\colon (X,\xi) \to (Y,\upsilon)$ to the identity morphism $id_{(A,\alpha)}$. If $\{(A_i,\alpha_i) \mid i \in I\}$ is a family of L-sets then the coproduct is $\coprod_{i\in I}(A_i,\alpha_i)$.

Let k be a cardinal number and $\{(\Omega_n,\vartheta_n) \mid n \le k\}$ be a family of L-sets. We have

$$\coprod_{n\le k} (\Omega_n,\vartheta_n)_{\mathsf{Set}(L)} \times id^n(X,\xi) = \big(\bigcup_{n\le k} \{n\} \times \Omega_n \times id^n X, \alpha\big), \tag{1}$$

where $\alpha(n,\omega,(x)_{i\leq n}) = \vartheta_n(\omega) \wedge id^n(\xi)((x_i)_{i\leq n})$, $\omega \in \Omega_n$ and $(x_i)_{i\leq n} \in X^n$.

Consider $(\Omega,\vartheta) = \bigsqcup_{n\leq k}(\Omega_n,\vartheta_n)$ as a fuzzy operator domain. The term functor over $\mathsf{Set}(L)$ can now be defined by transfinite induction. Let $T^0_{(\Omega,\vartheta)} = id$ and $T^1(X,\xi)$ be the right side of the equation 1. Define

$$T^{\iota}_{(\Omega,\vartheta)}(X,\xi) = \bigsqcup_{n\leq k}(\Omega_n,\vartheta_n)_{\mathsf{Set}(L)} \times id^n \bigvee_{0<\kappa<\iota} T^{\kappa}_{(\Omega,\vartheta)}(X,\xi)$$

for each positive ordinal ι. Finally, let $T_{(\Omega,\vartheta)}(X,\xi) = \bigsqcup\{T^0(X,\xi), \bigvee_{0<\iota<\bar{k}} T^{\iota}_{(\Omega,\vartheta)}(X,\xi)\}$, where $\bar{k}$ is the least cardinal greater than k and $\aleph_0$. Notice that $T^{\iota}_{(\Omega,\vartheta)}, T_{(\Omega,\vartheta)} : \mathsf{Set}(L) \to \mathsf{Set}(L)$ and $\bigvee_{0<\iota<\bar{k}} T^{\iota}_{(\Omega,\vartheta)}(X,\xi)$ denotes the colimit for the family $\{T^{\iota}_{(\Omega,\vartheta)}(X,\xi) \mid 0 < \iota < \bar{k}\}$.

Lemma 1. *For each positive ordinal there exists a unique* α_ι*, such that* $T^{\iota}_{(\Omega,\vartheta)}(X,\xi) = (T^{\iota}_{\Omega}X,\alpha_\iota)$*, and there exists a unique* α *such that* $T_{(\Omega,\vartheta)}(X,\xi) = (T_\Omega X,\alpha)$.

Using Lemma 1 we can easily see that $T_{(\Omega,\vartheta)}$ indeed is a functor. Further, we can extend the functor to a monad, since $T_{(\Omega,\vartheta)}$ can be shown to be idempotent. Once we have the fuzzy term monad on $\mathsf{Set}(L)$ we consider the composition with the unbalanced powerobject monad by constructing a natural transformation $\sigma^\star : \mathbf{T}_{(\Omega,\vartheta)}\mathcal{U}^\star \Longrightarrow \mathcal{U}^\star\mathbf{T}_{(\Omega,\vartheta)}$ as a prerequisite for the monad composition of $\mathcal{U}^\star$ and $\mathbf{T}_{(\Omega,\vartheta)}$ using distributive laws [2].

This understanding of the term monad in a fuzzy setting gives a start to a well founded non-classical logic programming. What remains is a similar understanding of unification.

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