On generalized continuous and left-continuous t-norms
over chains:
a survey

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Abstract

Continuous t-norms, as operations over \([0,1]\), have been deeply studied (see [14, 3] for general surveys). The prominent examples are Łukasiewicz \((x *_L y = \max(0, x + y - 1))\), Minimum and product t-norms and the decomposition theorem for continuous t-norm states roughly speaking that any continuous t-norm is an ordinal sum of these three basic t-norms [18, 15]. In the last decade (mainly from logical studies as we explain later), the case of left-continuous t-norms has also deserved increasing attention, see for example [8–12, 17, 18]. Unfortunately, in this latter case we do not have a decomposition theorem as in the case for continuous t-norms and only partial results are known.

From a logical point of view, there have also been many interesting advances in the recent past regarding fuzzy logical systems based on t-norms. Namely, the logic of continuous t-norms and their residua, called BL, introduced in [7], and the logic of left continuous t-norm and their residua, called MTL (Monoidal t-norm based logic) [6], have been defined and deeply studied. Both logics as well as all their axiomatic extensions are complete with respect to the class of linearly ordered algebras (chains) of their corresponding varieties and, in many cases, also with respect to the subclass of chains over the real unit interval, the so-called standard chains, which are defined by a t-norm and its residuum.

In parallel, the algebraic counterpart of the logics BL, MTL and other related t-norm based fuzzy logics, varieties of MTL-algebras and their classes of chains, have been also deeply studied, see for instance [7, 5] for BL-chains and [19] for MTL-chains. In this line it has been proved a new decomposition theorem of BL-chains (and in particular of continuous t-norms) as ordinal sums of Wajsberg hoops, where the notion of ordinal sum is slightly different than the usual one when dealing with t-norms (see [4, 2, 1] for basic notions and results). Also for MTL chains it has been proven a decomposition as ordinal sum of prelinear semihoops [19]. Nevertheless, as in the case of left continuous t-norms, this decomposition does not help much since so far it is unknown a characterization of the indecomposable linearly ordered semihoops. This is the main reason that this paper restricts itself mainly to the continuous t-norms and their generalization, the BL-chains.
The aims of this paper are:

- To provide a summary of the recent results on BL and MTL chains (and its potential interest for researchers on t-norms). We will give the definition of Hoop and its basic types, the notion of ordinal sum of hoops and the decomposition theorem of any BL-chains as ordinal sum of Wajsberg hoops.
- To motivate that a proper generalization of continuous and left-continuous t-norms over other domains than the unit interval is the one provided by BL and MTL chains (or algebras)
- To study and characterize these “generalized” continuous and some left-continuous t-norms over general chains, and specially over the rational unit interval.

We believe that the results surveyed in this paper may serve for a better understanding of Zadeh’s fuzzy sets [20] when asking values over the rational unit interval, over a finite chain, or more generally even over not necessarily linearly ordered lattices.

References

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