Weak implication on m–generalized Łukasiewicz algebras of order n

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Abstract. J. Vaz De Carvalho and T. Almada in *A generalization of the Łukasiewicz algebras*, Studia Logica 69 (2001), 329-338 introduced the variety $\mathcal{L}_n^m$, $m \geq 1$, $n \geq 2$, of m–generalized Łukasiewicz algebras of order n as a generalization of Łukasiewicz algebras of order n and a particular case of Ockham algebras. In this note, we define an implication operation on m–generalized Łukasiewicz algebras of order n, called weak implication, from which we obtain a new description of the congruences on these algebras. This last result allows us to show that $\mathcal{L}_n^m$ is semisimple and locally finite in a different way from that describe in the above mentioned paper.

1 Introduction and preliminaries

In 1977, generalizing De Morgan algebras by omitting the polarity condition (i.e.: the law of double negation), J. Berman ([1]) began the study of which he called distributive lattices with an additional unary operation. Two years latter, A. Urquhart in [8] introduced the name Ockham lattices with the justification that the so–called De Morgan laws are due, at least, in the case of propositional logic to William Ockham. These algebras are the algebraic counterpart of logics provided with a negation operator which satisfies De Morgan laws. Then recall that

An Ockham algebra is an algebra $\langle L, \wedge, \vee, f, 0, 1\rangle$ of type $(2,2,1,0,0)$ where the reduct $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and f is a unary operation satisfying the following conditions:

(O1) $f(0) = 1$,
(O2) $f(1) = 0$,
(O3) $f(x \wedge y) = f(x) \vee f(y)$,
(O4) $f(x \vee y) = f(x) \wedge f(y)$.

The name Ockham algebras has become classical and from that moment on, many articles have been published about this class of algebras. Many of the results obtained have been reproduced in the important book by T. Blyth and J. Varlet ([3]), which may be consulted by any reader interested in broadening their knowledge on the topic.

In 1940, Gr.C. Moisil introduced Łukasiewicz algebras of order 3 and 4 in order to obtain the algebraic counterpart of the corresponding Łukasiewicz logics. A year later, this author generalized these notions by defining Łukasiewicz algebras of order n (or L_n–algebras) ([5, 6, 2]) and he studied them from an algebraic point of view. On the other hand, in 1969, R. Cignoli ([4]) indicated an equational definition of these algebras equivalent to that given by Moisil. Henceforth, we will denote by $\mathcal{L}_n$ the variety of L_n–algebras.

It is well known that the most important example of an L_n–algebra is the chain of n rational fractions $C_n = \{\frac{j}{n-1}, 0 \leq j \leq n-1\}$ endowed with the natural lattice structure and the unary operations $\sim$ and D_i, defined as follows: $f(\frac{j}{n-1}) = 1 - \frac{j}{n-1}$ while $D_i(\frac{j}{n-1}) = 0$ if $i + j < n$ and $D_i(\frac{j}{n-1}) = 1$ in the other cases.

Ockham algebras which are more closely related to De Morgan algebras are the ones that satisfy the identity $f^{2m}(x) = x$, for some $m \geq 1$. The variety of these algebras will be denoted by $\mathcal{K}_{m,0}$. As Łukasiewicz algebras of order n have a reduct which is a De Morgan algebra, T. Almada and J. Vaz de Carvalho in [9] generalized them by considering algebras of the same type which have a reduct in $\mathcal{K}_{m,0}$. Hence, they introduced the variety $\mathcal{L}_n^m$of m–generalized Łukasiewicz algebras of order n which were defined as follows:

An m–generalized Łukasiewicz algebras of order n (or L_n^m–algebra) is an algebra $\langle A, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 0, 1\rangle$ of type $(2, 2, 1, \ldots, 1, 0, 0)$ such that

(GL_1) $\langle A, \vee, \wedge, f, 0, 1\rangle$ is a bounded distributive lattice and f is a dual endomorphism satisfying the identity $f^{2m}(x) = x$,
(GL_2) $D_i(x \wedge \bigvee_{p=0}^{m-1} f^{2p}(y)) = D_i(x) \wedge D_i(\bigvee_{p=0}^{m-1} f^{2p}(y)), \;\; 1 \leq i \leq n-1$,
(GL_3) $D_i(x) \wedge D_j(x) = D_j(x), \;\; 1 \leq i \leq j \leq n-1$,
(GL_4) $D_i(x) \vee f(D_i(x)) = 1, \;\; 1 \leq i \leq n-1$,
(GL_5) $D_i(f(\bigvee_{p=0}^{m-1} f^{2p}(x))) = f(D_{n-i}(\bigvee_{p=0}^{m-1} f^{2p}(x))), \;\; 1 \leq i \leq n-1$,
(GL_6) $D_i(D_j(x)) = D_j(x), \;\; 1 \leq i \leq n-1$,
(GL_7) $x \vee D_1(x) = D_1(x)$,
(GL_8) $D_i(x) = D_i(\bigvee_{p=0}^{m-1} f^{2p}(x)), \;\; 1 \leq i \leq n-1$,
(GL_9) $(x \wedge f(x)) \vee y \vee f(y) = y \vee f(y)$,
(GL_{10}) $\bigvee_{p=0}^{m-1} f^{2p}(x) \leq \bigvee_{p=0}^{m-1} f^{2p}(y) \vee f(D_i(\bigvee_{p=0}^{m-1} f^{2p}(y))) \vee D_{i+1}(\bigvee_{p=0}^{m-1} f^{2p}(x)), \;\; 1 \leq i \leq n-2$.

2 Weak implication in $\mathbf{L_n^m}$–algebras

In this section, we describe the congruences on L_n^m–algebras by means of certain subsets of them. Besides, we introduce an implication operation, called weak implication, which allows us to characaterize the subdirectly irreducible algebras in this variety by a different reasoning to that established in [9].

Definition 1. *Let* $A \in \mathcal{L}_n^m$ *and let* F *be a filter of* A*. We say that* F *is*

(i) *a g-filter if for all* $x \in F$ *there is* $b \in F \cap S(A)$ *such that* $b \leq x$*, where* $S(A) = \{x \in A : f^2(x) = x\}$*,*

(ii) *an m-filter if the hypothesis* $x \in F$ *implies* $fD_1 f^{2m-1}x \in F$.

It is worth noting that the notion of g-filter generalizes that of Stone filter in L_n–algebras ([2, 4]).

We will denote by $\mathcal{F}(A)$, $\mathcal{F}_g(A)$ and $\mathcal{F}_m(A)$ the set of all filters, g-filters and m-filters of A respectively.

Proposition 1. *Let* $A \in \mathcal{L}_n^m$. *Then* $F \in \mathcal{F}_g(A)$ *if and only if* $F \in \mathcal{F}_m(A)$.

Proposition 2. *Let* $A \in \mathcal{L}_n^m$. *Then there is an isomorphism between* $\mathcal{F}_g(A)$ *and* $\mathcal{F}(S(A))$ *under the correspondences* $F \longmapsto F \cap S(A)$ *and* $F^* \longmapsto F = \{x \in L : b \leq x \text{ for } b \in F^*\}$, *which are inverse to one another.*

Theorem 1. *Let* $A \in \mathcal{L}_n^m$ *with more than one element and let* $Con(A)$ *be the lattice of all congruences of A. Then*

(i) $Con(A) = \{R(F) : F \in \mathcal{F}_m(A)\}$, *where* $R(F) = \{(x,y) \in A \times A :$ *exists* $w \in F$ *such that* $x \wedge D_{n-1}w = y \wedge D_{n-1}w\}$,

(ii) *the lattices* $Con(A)$ *and* $\mathcal{F}_m(A)$ *are isomorphic considering the applications* $\theta \longmapsto [1]_\theta$ *and* $F \longmapsto R(F)$ *which are mutually inverse.*

Theorem 2. *Every non–trivial* L_n^m*–algebra has the congruence extension property.*

Next, with the aim of characterizing subdirectly irreducible L_n^m–algebras we introduce a new binary operation $\rightarrow$ which we call *weak implication*. In $m = 2$ case, this operation coincides with the one considered by R. Cignoli for L_n–algebras. Let $A \in \mathcal{L}_n^m$, then for all $x, y \in A$ we define

$$x \rightarrow y = D_1 f^{2m-1}x \vee y.$$

Proposition 3. *Let* $A \in \mathcal{L}_n^m$. *Then the following properties are verified:*

(I_1) $x \rightarrow 1 = 1$,
(I_2) $x \rightarrow x = 1$,
(I_3) $1 \rightarrow x = x$,
(I_4) $x \rightarrow (y \rightarrow x) = 1$,
(I_5) $x \leq y$ *implies* $x \rightarrow y = 1$,
(I_6) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$,
(I_7) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$,
(I_8) $((x \rightarrow y) \rightarrow x) \rightarrow x = 1$,
(I_9) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I_{10}) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,
(I_{11}) $x \rightarrow (x \wedge y) = x \rightarrow y$,
(I_{12}) $x \rightarrow (x \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
(I_{13}) $(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
(I_{14}) $D_i x \rightarrow D_i y = fD_i x \vee D_i y$,
(I_{15}) $D_i x \rightarrow D_i y = 1$ *if and only if* $D_i x \leq D_i y$,
(I_{16}) $x \rightarrow fD_1 f^{2m-1}x = 1$.

Definition 2. *Let* $A \in \mathcal{L}_n^m$. *We say that* $D \subseteq L$ *is a deductive system* (*d.s.*) *of* A *if it verifies:* (D_1) $1 \in D$ *and* (D_2) $x, x \to y \in D$ *imply* $y \in D$.

We will denote by $\mathcal{D}(A)$ the set of all deductive systems of A.

Theorem 3. *Let* $A \in \mathcal{L}_n^m$. *Then* $Con(A) = \{R(D) : D \in \mathcal{D}(A)\}$, *where* $R(D) = \{(x,y) \in A \times A : D_i x, D_i x \to D_i y \in D\}$.

From Proposition 3, Theorem 3 and the results established in [7] we prove that

Theorem 4. *Every non–trivial* L_n^m*–algebra is semisimple.*

Proposition 4. *Let* $A \in \mathcal{L}_n^m$ *and* $F \subseteq A$. *Then* F *is an m-filter if and only if* F *is a d.s..*

Corollary 1. *Let* $A \in \mathcal{L}_n^m$. *Then there is an isomorphism between* $\mathcal{D}(A)$ *and* $\mathcal{F}(S(A))$, *both ordered by set inclusion.*

Proposition 5. *Let* $A \in \mathcal{L}_n^m$ *and* $D \in \mathcal{D}(A)$. *Then the following conditions are equivalent:*

(i) *D is maximal,*
(ii) *D is irreducible,*
(iii) *D is completely irreducible.*

Theorem 5. *Let* $A \in \mathcal{L}_n^m$. *Then the following conditions are equivalent:*

(i) *A is subdirectly irreducible,*
(ii) $S(A)$ *is a subalgebra of* C_n,
(iii) *A is simple.*

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