RELATIONSHIPS BETWEEN A GOULD TYPE
SET-VALUED INTEGRAL AND OTHER
SET-VALUED INTEGRALS

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1 Introduction

Gould [4] has defined an integral of a real function with respect to a finitely additive
vector-valued measure.

Precupanu and Croitoru [5,6] have extended the definition of Gould to the set-valued
case, obtaining a Gould type integral for a real function with respect to an additive set-
valued measure (called multimeasure) and in [3], Gavriluţ has studied a Gould type
integral of a real-valued function with respect to a multisubmeasure. In [8], Precupanu
and Satco introduced and studied an Aumann type integral in which the selectors are
vector-valued Gould integrals.

In Precupanu, Gavriluţ and Croitoru [7] we study a set-valued Gould type integral
of a real function with respect to an arbitrary set multifunction taking values in \( \mathcal{P}_f(X) \),
the family of all nonvoid closed subsets of a Banach space \( X \). In this paper we present
some properties of the integral presented in [7], pointing out some relationships with
Brooks integrability, a Dunford type integrability and Aumann-Gould integrability.

2 Terminology and notations

Let \( T \) be an abstract nonempty set, \( \mathcal{A} \) an algebra of subsets of \( T, X \) a real Banach space,
\( \mathcal{B}_0(X) \) the family of nonempty subsets of \( X, \mathcal{P}_f(X) \) the family of nonempty closed
subsets of \( X, \mathcal{P}_b(X) \) the family of nonempty closed bounded subsets of \( X, \mathcal{P}_{fc}(X) \)
the family of nonempty closed convex subsets of \( X, \mathcal{P}_{kfc}(X) \) the family of nonempty compact convex subsets of \( X \) and \( \mathcal{P}_{wkc}(X) \) the family of nonempty weakly compact convex subsets of \( X \).

By \( ^\cdot + ^\cdot \) we mean the Minkowski addition on \( \mathcal{B}_0(X) \), that is, \( M + N = \overline{M + N} \), for
every \( M, N \in \mathcal{B}_0(X) \), where \( M + N = \{ x + y | x \in M, y \in N \} \) and \( \overline{M + N} \) is the closure of
\( M + N \) with respect to the topology induced by the norm of \( X \).

For every \( M, N \in \mathcal{B}_0(X) \), we denote \( h(M,N) = \max\{ e(M,N), e(N,M) \} \), where
\( e(M,N) = \sup_{x \in M} d(x,N) \) is the excess of \( M \) over \( N \) and \( d(x,N) \) is the distance from \( x \)
to \( N \). It is known that \( h \) becomes an extended metric on \( \mathcal{P}_f(X) \) (i.e. is a metric which
can also take the value $+\infty$), called the Hausdorff pseudo-metric and $h$ becomes a metric on $\mathcal{P}_{bf}(X)$ and $(\mathcal{P}_{bf}(X), h)$ and $(\mathcal{P}_{m}(X), h)$ are complete metric spaces. We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_{0}(X)$, where 0 is the origin of $X$.

**Definition 1.** Let $\varphi : \mathcal{A} \to \mathcal{P}_{f}(X)$ be such that $\varphi(\emptyset) = \{0\}$. $\varphi$ is said to be:

(a) a fuzzy multimeasure if $\varphi(A) \subset \varphi(B)$, for every $A, B \in \mathcal{A}$, with $A \subset B$.

(b) a multisubmeasure if:

(i) $\varphi(A) \subset \varphi(B)$, for every $A, B \in \mathcal{A}$, with $A \subset B$ and

(ii) $\varphi(A \cup B) \subset \varphi(A) \Box \varphi(B)$, for every $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$.

(c) a multimeasure if $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ for every $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$.

(d) an $h$-multimeasure if $\lim_{n \to \infty} h(\varphi(A), \Box \sum_{k=1}^{n} \varphi(A_{k})) = 0$, for every sequence of mutual disjoint sets $(A_{n})_{n \in \mathbb{N}^{*}} \subset \mathcal{A}$, with $A = \bigcup_{n=1}^{\infty} A_{n} \in \mathcal{A}$.

**Definition 2.** We consider the following set functions associated to an arbitrary multi-valued set function $\varphi : \mathcal{A} \to \mathcal{P}_{0}(X)$, with $\varphi(\emptyset) = \{0\}$:

(a) $\bar{\varphi}$ (the variation of $\varphi$) defined, for every $A \in \mathcal{A}$, by

$$\bar{\varphi}(A) = \sup\{\sum_{i=1}^{n} |\varphi(A_{i})|; (A_{i})_{i=1}^{n} \subset \mathcal{A} \text{ is a partition of } A, n \in \mathbb{N}^{*}\}.$$ 

We say that $\varphi$ is of finite variation if $\bar{\varphi}(T) < \infty$.

(b) $\tilde{\varphi}$ defined, for every $A \subset T$, by

$$\tilde{\varphi}(A) = \inf\{\bar{\varphi}(B); A \subset B, B \in \mathcal{A}\}.$$ 

**Remark 1.** I. $\bar{\varphi}$ is monotone, that is, $\bar{\varphi}(A) \leq \bar{\varphi}(B)$, for every $A, B \in \mathcal{A}$, with $A \subset B$. Also, $\tilde{\varphi}$ is monotone and $\bar{\varphi}(A) = \tilde{\varphi}(A)$, for every $A \in \mathcal{A}$.

II. If $\varphi$ is subadditive, then $\tilde{\varphi}$ is finitely additive.

III. If $\varphi : \mathcal{A} \to \mathcal{P}_{f}(X)$ is of finite variation, then $\varphi$ takes its values in $\mathcal{P}_{bf}(X)$.

### 3 The fuzzy Gould type integral

In what follows, we suppose $\varphi : \mathcal{A} \to \mathcal{P}_{f}(X)$ is a fuzzy multimeasure of finite variation and $f : T \to \mathbb{R}$ is a bounded function.

**Definition 3.** (a) A partition of $T$ is a finite family $P = \{A_{i}\}_{i=\overline{1,m}} \subset \mathcal{A}$ such that $A_{i} \cap A_{j} = \emptyset, i \neq j$ and $\bigcup_{i=1}^{n} A_{i} = T$.

(b) Let $P = \{A_{i}\}_{i=\overline{1,n}}$ and $P' = \{B_{j}\}_{j=\overline{1,m}}$ be two partitions of $T$. $P'$ is said to be finer than $P$, denoted $P \leq P'$ or $P' \geq P$, if for every $j = \overline{1,m}$, there exists $i_{j} = \overline{1,n}$ so that $B_{j} \subset A_{i_{j}}$.

**Definition 4.** (i) $f$ is said to be $\tilde{\bar{\varphi}}$- totally-measurable on $T$ if for every $\varepsilon > 0$ there exists a partition $P_{\varepsilon} = \{A_{i}\}_{i=\overline{1,n}}$ of $T$ such that:

(i) $\tilde{\bar{\varphi}}(A_{0}) < \varepsilon$ and

(ii) $\sup_{i,s \in A_{i}} |f(t) - f(s)| = \text{osc}(f, A_{i}) < \varepsilon$, for every $i \in \{1, \ldots, n\}$. 
Remark 2. II) If \( \varphi : \mathcal{A} \to \mathcal{P}(\mathbb{R}) \) is a multisubmeasure and \( f \) is \( \varphi \)-totally-measurable, then \( f \) is Gould \( \varphi \)-integrable [4].

Definition 5. If \( f : T \to \mathbb{R} \) is a bounded function, we denote \( \sigma_{f,\varphi}(P) = \sum_{i=1}^{n} f(t_i)\varphi(A_i) \), for every partition \( P = \{A_i\}_{i=1}^{n} \) of \( T \) and every \( t_i \in A_i \).

(a) \( f \) is said to be \( \varphi \)-integrable on \( T \) if the net \( (\sigma_{f,\varphi}(P))_{P \in (\mathcal{P}(\mathcal{P}_{\text{fin}}),\leq)} \) is convergent in \( (\mathcal{P}_{\text{fin}}(X),h) \), where \( \mathcal{P} \), the set of all partitions of \( T \), is ordered by the relation \( \leq \) given in Definition 3.1-(b).

If \( (\sigma_{f,\varphi}(P))_{P \in (\mathcal{P}(\mathcal{P}_{\text{fin}}),\leq)} \) is convergent, then its limit is called the integral of \( f \) on \( T \) with respect to \( \varphi \), denoted by \( \int_{T} f \, d\varphi \).

Remark 3. 1. if \( f \) is \( \varphi \)-integrable on \( T \) and if only if there exists a set \( I \in \mathcal{P}_{\text{fin}}(X) \) such that for every \( \varepsilon > 0 \), there exists a partition \( P_{k} \) of \( T \), so that for every other partition of \( T \), \( P = \{A_{i}\}_{i=1}^{n} \), with \( P \geq P_{k} \) and every choice of points \( t_{i} \in A_{i} \), \( i = 1, \ldots, n \), we have \( |f|_{P} \leq \varepsilon \).

II. If \( \varphi : \mathcal{A} \to \mathcal{P}_{\text{fin}}(X) \), then \( \int_{T} f \, d\varphi \in \mathcal{P}_{\text{fin}}(X) \).

Example 1. I) Let \( \varphi(A) = \{\mu(A)\} \), for every \( A \in \mathcal{A} \), where \( \mu : \mathcal{A} \to \mathbb{R}_{+} \) is a finitely additive measure of finite variation and \( f : T \to \mathbb{R} \). Then \( f \) is \( \varphi \)-integrable if and only if \( f \) is Gould \( \mu \)-integrable [4]. In this case, \( \int_{A} f \, d\varphi = \{(G) \int_{A} f \, d\mu\} \), for every \( A \in \mathcal{A} \), where \( (G) \int_{A} f \, d\mu \) is the Gould integral [4] of \( f \) with respect to \( \mu \).

II) Let \( X = \mathbb{R} \), \( \varphi(A) = \{\mu_{1}(A), \mu_{2}(A)\} \), for every \( A \in \mathcal{A} \), where \( \mu_{1}, \mu_{2} \) are real set functions of finite variation with \( \mu_{1}(\emptyset) = \mu_{2}(\emptyset) = 0 \), such that \( \mu_{1} \leq \mu_{2} \) and let \( f : T \to \mathbb{R} \) be a bounded function. Then \( f \) is \( \varphi \)-integrable if and only if \( f \) is \( \mu_{1} \)-integrable and \( \mu_{2} \)-integrable.

In this case, \( \int_{A} f \, d\varphi = \int_{A} f \, d\mu_{1}, \int_{A} f \, d\mu_{2} \), for every \( A \in \mathcal{A} \).

III) (Gavriliuţ [3]) Let \( T = \{1,2\}, \mathcal{A} = \mathcal{P}(T) \), \( f : T \to \mathbb{R} \), \( f(t) = \begin{cases} 1, & t = 1 \\ 0, & t = 2 \end{cases} \) and \( \varphi : \mathcal{A} \to \mathcal{P}_{\text{fin}}(\mathbb{R}) \) defined by: \( \varphi(A) = \begin{cases} \{0\}, & A = \emptyset \\ \{0,1\}, & A = \{1\} \cup A = \{2\} \end{cases} \). Then \( f \) is \( \varphi \)-integrable and \( \int_{T} f \, d\varphi = [-1,1] \).

IV) (Gavriliuţ [3]) Let \( \mu : \mathcal{A} \to \mathbb{R}_{+} \) be a submeasure of finite variation, \( \varphi(A) = [0,\mu(A)] \) for every \( A \in \mathcal{A} \) and \( f : T \to \mathbb{R} \), a bounded \( \varphi \)-totally-measurable function. Then \( f \) is \( \varphi \)-integrable and \( \int_{T} f \, d\varphi = [0,(G) \int_{T} f \, d\mu] \).

Theorem 1. Suppose \( \varphi : \mathcal{A} \to \mathcal{P}_{\text{fin}}(X) \) is a multisubmeasure and \( f : T \to \mathbb{R} \) is a bounded function. If \( f \) is Gould \( \varphi \)-integrable [4], then \( f \) is Dunford \( \varphi \)-integrable [2].

Theorem 2. Let \( \varphi : \mathcal{A} \to \mathcal{P}_{\text{fin}}(X) \) be a \( h \)-multimeasure of finite variation and let \( f : T \to \mathbb{R} \) be a Brooks \( \varphi \)-integrable [1] bounded function such that:

(i) there exists \( (f_{n})_{n \in \mathbb{N}} \) an uniformly bounded positive defining sequence of \( f \) and

(ii) for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that for every \( n \in \mathbb{N}^{*} \), \( x_{i}, y_{i} \in \mathbb{R} \) with \( |x_{i} - y_{i}| < \delta \), \( \forall i \in \{1,2,\ldots,n\} \) and every \( \{B_{i}\}_{i=1}^{n} \subset \mathcal{A} \) we have

\[
\sum_{i=1}^{n} |x_{i} - y_{i}| \cdot |\varphi(B_{i})| < \varepsilon.
\]
Then $f$ is $\varphi$-integrable and $\int_T f \, d\varphi = (B) \int_T f \, d\varphi$, where $(B) \int_T f \, d\varphi$ is the Brooks integral [1] of $f$ with respect to $\varphi$.

In [8], Precupanu and Satco have obtained the following result concerning the relationships between the $\varphi$-integrability and Aumann-Gould integrability:

**Theorem 3.** Suppose $\varphi : A \rightarrow \mathcal{P}_{wkc}(X)$ is a multimeasure, $(T, \mathcal{A}, \bar{\varphi})$ is complete and $X$ has the Radon-Nikodym property. Then any $\varphi$-integrable function is Aumann-Gould integrable and $\int_E f \, d\varphi = (AG) \int_E f \, d\varphi$, $\forall E \in \mathcal{A}$, where the integral in the right side is the Aumann-Gould integral.

For the fuzzy multimeasures case, we are going to study the relationships between the fuzzy Gould type integral [7], a Dunford type set-valued integral and the respective Aumann-Gould integral.

**References**