# Constructing t-norms from a given behaviour on join-irreducible elements

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Triangular norms (t-norms for short) were originally studied in the theory of probabilistic metric spaces in order to generalize the classical triangle inequality to this field [7, 12]. Later on, they played an important role as interpretation of the conjunction in many-valued logics [4], in particular in fuzzy logics [8]. An important class of t-norms is the class of sup-preserving t-norms which play a major role, particularly in residuated lattices [4]. Triangular conorms are introduced as dual notion of triangular norms [10]. There have been some construction methods of t-norms on various classes of lattices [11, 5, 9, 6]. Some lattices can be generated by a class of elements: join- or meet-irreducible elements and some others [1, 2]. In this contribution, we focus on constructing t-norms on complete lattices from a given behavior on join-irreducible elements. We present the sup-extension method to describe the behavior of a t-norm on a finite distributive lattice by means of join-irreducible elements by the following theorem:

**Theorem.** Let $L$ a finite distributive lattice and $J(L)^* = J(L) \cup \{0,1\}$. If $T$ is a t-norm on $J(L)^*$ , then the function $\widetilde{T}$ defined as follows:

$$\widetilde{T}(x,y) = \bigvee_{j \in \eta(x)} \bigvee_{k \in \eta(y)} T(j,k)$$

where $\eta(x)$=$\{i \in\ J(L)^* | i \leq x\}$=$J(L)^* \cap \downarrow x$ for all $x \in\ L$ , is a t-norm on $L$.

We provide a method to construct t-conorms by carrying out the dual method on meet-irreducible elements. We also obtain some inf-preserving t-conorms on principle ideals of a lattice from given inf-preserving t-conorms. We show that if $T$ is a t-norm on a complete lattice $L$ and every join-irreducible element of $L$ is idempotent, then $T = \wedge$. We give a method to construct t-norms on a product of distributive lattices $L = L_1 \times L_2 \times \ldots \times L_n$. Giving a partititon of the set of join-irreducible elements of $L = L_1 \times L_2 \times \ldots \times L_n$, we show that for a given t-norm on join-irreducible elements, the restriction to each set that forms the partition is not necessarily a t-norm, but a t-subnorm. Moreover, we partition the set of join-irreducible elements $J(L^{[n]})$ of a power of chains $L^{[n]}$, given in [13], into $n$ sets such that

$$\begin{aligned} J_1 &= \{(0,0,\ldots,0,a) \mid a \in L\} \\ J_2 &= \{(0,0,\ldots,a,a) \mid a \in L\} \\ &\vdots \\ J_{n-1} &= \{(0,a,\ldots,a,a) \mid a \in L\} \\ J_n &= \{(a,a,\ldots,a,a) \mid a \in L\}. \end{aligned}$$

and provide a method to construct a t-norm from given t-norms on the parts of this partition. We show that if $L$ is finite, then the constructed t-norm is sup-preserving. In an interval valued lattice $L$, the set $D_L = \{[x,x] \mid x \in L\}$ is called the diagonal of $L$ [3]. Our last result on the characterization of sup-preserving t-norms on $L^{[n]}$ extends result of [3]: Any sup-preserving t-norm $T$ on $L^{[n]}$ can be characterized by its behaviour on $J_n$ and $T((0,\ldots,0,1),(0,\ldots,0,1))$ if it is closed on the diagonal.

## References


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