LINZ 2011

32nd Linz Seminar on Fuzzy Set Theory

Abstracts

Decision Theory: Qualitative and Quantitative Approaches

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Abstracts

Didier Dubois
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Editors

LINZ 2011

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DECISION THEORY: QUALITATIVE AND QUANTITATIVE APPROACHES

ABSTRACTS

Didier Dubois, Michel Grabisch, Radko Mesiar, Erich Peter Klement
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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2011 will be the 32nd seminar carrying on this tradition and is devoted to the theme "Decision Theory: Qualitative and Quantitative Approaches". The goal of the seminar is to present and to discuss recent advances in the theory of decision procedures and to concentrate on its applications in various areas.

A large number of highly interesting contributions were submitted for possible presentation at LINZ 2011. In order to maintain the traditional spirit of the Linz Seminars — no parallel sessions and enough room for discussions — we selected those twenty-eight submissions which, in our opinion, fitted best to the focus of this seminar. This volume contains the abstracts of this impressive selection. These regular contributions are complemented by six invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

Didier Dubois
Michel Grabisch
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Contents

Impatience and myopia through belief functions

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Abstract. Building upon Choquet's integral representation Theorem [3], we characterize several continuity properties of totally monotone capacities v defined on the Borel sets of a Polish space Ω, in terms of the specific properties of the related σ-additive Möbius transform. In the case of bounded and measurable income streams, we show that these continuity properties characterize myopic or impatient behaviors of decision-makers evaluating income streams x through the Choquet integral of x with respect to the belief function v.

1 Introduction

Real-valued set functions which are not necessarily additive are extensively used in decision theory. According to the interpretation they may represent a transferable utility cooperative game or else non-additive probabilities and belief functions. These functions also appear through Choquet integration as representing decision rules for multi-criteria decision problems and, in particular, multi-period and choice problems.

In multi-period choice problems, the use of totally monotone capacities (belief functions for short) v for ranking income streams x through the Choquet integral of x with respect to v, has been intensively proposed by several authors including the pioneer papers of [5] and [4]. It has also been recognized that in the countable time setting some continuity properties of totally monotone v enable to disentangle myopia from impatience (see for instance [1], and [2]).

The purpose of this paper is to allow an extension of the previous analyses to the general case of possibly continuous time. Therefore this paper mainly focuses on the characterization of several continuity properties of totally monotone capacities v defined on the Borel sets of a Polish space Ω. Building upon Choquet's integral representation Theorem [3], we show that some classical continuity properties are directly connected with the characterization of the related extreme belief functions, thus providing a way to compute the desired belief functions through the use of the related σ-additive Möbius transform. In particular, it will be shown that extreme points of the set of outer-continuous belief functions (resp. outer-continuous and $\mathcal{G}$-inner continuous, inner-continuous) are the σ-filter games (resp. unanimity games w.r.t. compact sets, unanimity games w.r.t. finite sets). These results enable us to derive integral representations for belief functions satisfying the various continuity properties. Finally we investigate the link between different notions of inter-temporal myopic or impatient behavior and continuity properties of belief functions (when the decision-maker evaluates income streams through the Choquet integral with respect to a belief function).

2 Preliminaries

Let Ω be a polish space. Denote $\mathcal{B} = \mathcal{B}(\Omega)$ the σ-algebra of borelian sets of Ω. Let $\mathcal{V}$ be the set of all games defined on $\mathcal{B}$ i.e. $\mathcal{V} = \{\nu : \mathcal{B} \to \mathbb{R}, \nu(\emptyset) = 0\}$. The term *capacity* will be used to designate elements ν of $\mathcal{V}$ which are *monotone* i.e. satisfy $[B_1 \subset B_2 \implies \nu(B_1) \leq \nu(B_2)]$ and *normalized* i.e. satisfy $\nu(\Omega) = 1$. A capacity $\nu : \mathcal{B} \to [0,1]$ on the σ-algebra $\mathcal{B}$ is called *totally monotone* or a *belief function* if ν is k-monotone for all $k \geq 2, k \in \mathbb{N}$ i.e. if for every family $(B_1, B_2, \ldots, B_k) \in \mathcal{B}^k$,

$$\nu\left(\bigcup_{j=1}^{k} B_j\right) + \sum_{J, \emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|}\, \nu\left(\bigcap_{j\in J} B_j\right) \geq 0. \tag{1}$$

Lemma 1. [3]
The extreme points of the set of belief functions defined on a measurable space $(\Omega, \mathcal{B})$ *are the filter games.*

A nonempty set p of elements of $\mathcal{B}$ is called a *filter* if
(i) $\forall A, B \in \mathcal{B}, [A \in p, A \subset B \implies B \in p]$,
(ii) $\forall A, B \in \mathcal{B}, [A, B \in p \implies A \cap B \in p]$.
The filter p is called *proper* if $\emptyset \notin p$. A game $\nu : \mathcal{B} \to \{0,1\}$ is called a *filter game* if the set $p := \{B \in \mathcal{B} : \nu(B) = 1\}$ is a filter. In this case ν is denoted u_p where u_p is obviously defined by $u_p(B) = 1$ if $B \in p$, and $u_p(B) = 0$ otherwise.

A game $\nu \in \mathcal{V}$ is called *outer-continuous* if it is outer-continuous at every $B \in \mathcal{B}$ i.e. if $\forall B \in \mathcal{B}, B_n \in \mathcal{B}, B_n \downarrow B \implies \nu(B_n) \underset{n\to\infty}{\longrightarrow} \nu(B)$.

A game $\nu \in \mathcal{V}$ is called *$\mathcal{G}$-inner-continuous at* $G \in \mathcal{G} := \{\text{open sets}\}$ if $G_n \in \mathcal{G}, G_n \uparrow G \implies \nu(G_n) \underset{n\to\infty}{\longrightarrow} \nu(G)$.

3 Some results

Proposition 1. *The set* $\text{ext}Bel_{E_o} := \{$*"extreme" outer-continuous belief functions*$\}$ *is the set of filter games* u_p *where* p *is a proper filter closed under countable intersection.*

Denote by Σ_o the σ-algebra on $\text{ext}Bel_{E_o}$ generated by the family $\{\tilde{B} : B \in \mathcal{B}, B \neq \emptyset\}$, where $\tilde{B} = \{u_p \in \text{ext}Bel_{E_o} : B \in p\}$.

Theorem 1. *For every outer-continuous belief function* v *there exists a* σ*-additive measure* μ_v *on* Σ_o *such that for all* $B \in \mathcal{B}$*,*

$$v(B) = \int_{\text{ext}Bel_{E_o}} u(B)\, \mathrm{d}\mu_v(u) = \mu_v(\tilde{B}). \tag{2}$$

Conversely, given a σ-additive measure μ on Σ_o, the expression above defines an outer-continuous belief function on $\mathcal{B}$.

Proposition 2. *The set*

$\text{ext}Bel_{E_{o,\mathcal{G}}} := \{$*"extreme" outer-continuous and $\mathcal{G}$-inner-continuous belief functions*$\}$ *is the set of unanimity games u_K for some nonempty compact subset K of Ω: $u_K(B) = 1$ if $B \supseteq K, u_K(B) = 0$ otherwise.*

Denote by $\Sigma_{o,\mathcal{G}}$ the σ-algebra on $\text{ext}Bel_{E_{o,\mathcal{G}}}$ generated by the family $\{\tilde{B} : B \in \mathcal{B}, B \neq \emptyset\}$, where $\tilde{B} = \{u_K : K \text{ compact}, \emptyset \neq K \subseteq B\}$.

Theorem 2. *For every outer-continous and $\mathcal{G}$-inner-continuous belief function v there exists a σ-additive measure μ_v on $\Sigma_{o,\mathcal{G}}$ such that for all $B \in \mathcal{B}$,*

$$v(B) = \int_{\text{ext}Bel_{E_{o,\mathcal{G}}}} u(B)\ \mathrm{d}\mu_v(u) = \mu_v\Big(\{u_K : K \text{ compact}, \emptyset \neq K \subseteq B\}\Big). \tag{3}$$

Conversely, given a σ-additive measure μ on $\Sigma_{o,\mathcal{G}}$, the expression above defines an outer-continuous and $\mathcal{G}$-inner-continuous belief function on $\mathcal{B}$.

4 Myopia and impatience in continuous time $(\Omega = \mathbb{R}_+)$

$L^\infty(\Omega)$ is the set of bounded real-valued measurable functions on $(\Omega, \mathcal{B})$ with $\mathcal{B}$ the set of borelians of Ω. Interpret $x \in L^\infty(\Omega)$ as a continuous stream of incomes. Let $\succsim$ be a weak order on L^∞ representable by a Choquet integral w.r.t. a belief function v on $(\Omega, \mathcal{B})$.

Definition 1. *$\succsim$ is myopic if for every $B_n, B_n \in \mathcal{B}$ such that $B_n \downarrow \emptyset$, then for every $x, y \in L^\infty$, $c \in \mathbb{R}$: $x \succ y$ implies $x \succ y + c\ 1_{B_n}$ for n large enough.*

Theorem 3. *$\succsim$ is myopic if and only if v is outer-continuous.*

Definition 2. *$\succsim$ is impatient if for every $x \in L^\infty$, and for every $\varepsilon > 0$, there exists a time $T_o(x, \varepsilon) := T_o \in \mathbb{R}_+$ such that $\forall T \geq T_o : (x + \varepsilon)1_{[0,T]} \succ x$.*

Theorem 4. *v is $\mathcal{G}$-inner-continuous $\Longrightarrow$ $\succsim$ is impatient.*

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Cooperative games with a hierarchically structured player set

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Abstract. A situation in which a finite set of players $N \subset \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair (N, v) where $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any *coalition* $S \subseteq N$, $v(S) \in \mathbb{R}$ is the *worth* of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate.

In a TU-game (N, v) there are no restrictions on the cooperation possibilities of the players, i.e. every coalition $S \subseteq N$ is feasible and can generate its worth. Various models with restrictions on coalition formation are discussed in the literature. One of the first game theoretic models with cooperation restrictions are the *games in coalition structure* in which the set of players is partitioned into disjoint sets which represent social groups such that for a particular player it is more easy to cooperate with players in its own group than to cooperate with players in other groups. Another model in which there are restrictions on the possibilities of cooperation are the *games with limited communication structure* where the edges of an undirected graph on the set of players represent binary communication links between the players such that players can cooperate if and only if they are connected. A coalition that is not connected can only earn the sum of the worths of its maximally connected subsets or components.

Games with a permission structure

In this presentation we mainly focus on models of restricted cooperation where the restrictions arise because the players belong to some hierarchical structure. One of the first models in this class are the *games with a permission structure* which describe situations in which the players in a TU-game are part of a hierarchical organization, refered to as a *permission structure*, such that there are players that need permission from other players before they are allowed to cooperate. Thus, the possibilities of coalition formation are determined by the positions of the players in the permission structure. Various assumptions can be made about how a permission structure affects the cooperation possibilities. In the *conjunctive approach*, it is assumed that every player needs permission from *all* its predecessors before it is allowed to cooperate. Consequently, a coalition is feasible if and only if for every player in the coalition it holds that all its predecessors belong to the coalition. Alternatively, in the *disjunctive approach* it is assumed that every player needs permission from *at least one* of its predecessors (if it has any) before it is allowed to cooperate with other players. This means that a coalition is feasible if and only if every player in the coalition (who has at least one predecessor in the permission structure) has at least one predecessor who also belongs to the coalition.

By union closedness of the set of feasible coalitions, every coalition has a unique largest feasible subset. To take account of the limited cooperation possibilities, for every game with a permission structure a restricted game is defined which assigns to every coalition the worth of its largest feasible subcoalition in the original game. The disjunctive and conjunctive approach yield different restricted games. A *solution* for games with a permission structure is a function that assigns to every such a game a payoff distribution over the individual players. Applying known solutions for TU-games to the restricted games yields solutions for games with a permission structure. For example, applying the *Shapley value* to the conjunctive, respectively disjunctive, restricted game, yields the *disjunctive* and *conjunctive Shapley permission values*. Similar, one can apply the *Banzhaf value*, *Nucleolus* or any other solution to the two restricted games. In this presentation we discuss comparable axiomatizations of the (conjunctive and disjunctive) Shapley- and Banzhaf permission values.

Some related models of restricted cooperation

After discussing several solutions for games with a permission structure, we mention some other (more general or specific) models and applications. A first generalization concerns games where the set of feasible coalitions is an *antimatroid*, i.e. sets of feasible coalitions that contain the empty set and are (i) *closed under union* (, i.e. the union of every pair of feasible coalitions is also feasible), and (ii) *accessible* (,i.e. for every feasible coalition there is a player such that without this player the coalition is still feasible). Further, the sets of feasible coalitions that can be the set of conjunctive feasible coalitions of some permission structure are characterized as those antimatroids that are *closed under intersection*. The sets of feasible coalitions that can be the set of disjunctive feasible coalitions of some permission structure are characterized as those antimatroids that satisfy the so-called *path property*. An example of antimatroids that cannot be obtained from permission structures are the feasible coalitions in *ordered partition voting* where there is an ordered partition of the player set, such that to activate players in a particular level, a qualified majority approval in every higher level is necessary.

Compared to the properties that define an antimatroid, it turns out that the sets of connected coalitions of players in an undirected (communication) graph are characterized by a weaker union property, but stronger accessibility property. The weaker union property is *union stability* (, i.e. the union of every pair of feasible coalitions that are not disjoint is also feasible). The stronger accessibility property is *2-accessibility* (, i.e. for every feasible coalition there are at least two players such that without any of these two players the remaining coalition is still feasible). Adding additional properties characterize special subclasses of undirected graphs. For example, adding closedness under intersection yields the sets of connected coalitions of some cycle-complete graph.

A further generalization of games on antimatroids are games with restricted cooperation where the set of feasible coalitions can be any set of players that is closed under union. An example which is not an antimatroid is a *majority cooperation situation* where the player set is partitioned in a coalition structure such that a coalition is feasible if, whenever it contains a player from an element of the partition, it contains a (qualified) majority of the players of that element.

Looking at applications of games with a permission structure, it is useful to know that *peer group games* are a special subclass of games with a permission structure. To be specific, a game with a permission structure is a peer group game if and only if the permission structure is a rooted tree and the game is additive (, i.e. every player has a weight, and the worth that can be generated by any coalition of players is just the sum of the weights of the players in the coalition). Although at first sight this seems a narrow class of games, it contains many applications such as auction games, dual airport games and polluted river games. Another class of applications are *hierarchically structured firms* where the permission structure is a rooted tree, and the game is a convex game defined on the 'lowest level' of the hierarchy (, i.e. the players that have no successor).

Finally, we mention that peer group games are also a special class of the so-called (weighted) *digraph games*. In these games every player has a weight, but to earn that weight it needs all its direct predecessors. A digraph game is a peer group game if and only if the digraph is transitive.

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Overlap functions, ignorance functions and bi-entropic functions in pairwise comparisons

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In the literature, two main approaches can be found to deal with preferences, which can probably be considered the most natural ways of considering such a problem. The first approach evaluates its preference individually, regardless of which the other preferences are. The second approach compares preferences to each other. This latter approach makes use of binary relations in order to express preferences in a qualitative way.

Recently, some methods based on pairwise comparison of preferences have been proved to be useful for problems of classification in the field of machine learning. In particular, Hüllermeier and Brinker([3]) propose a method for learning fuzzy preference relations that can be used to solve multi-class classification problems. This method is based on the use of t-norms and negations to model concepts such as incomparability or indifference.

In this talk we propose a different functional approach to model these two key concepts. The new approach is based on the concepts of overlap function([2]) and ignorance function ([1]). Overlap functions provide an analytical tool to measure up to what extent a given element can be considered to be part of two different classes. This is clearly connected to having data that simultaneously support two different alternatives. Overlap functions can be seen as way of generalizing t-norms in the previous modeling, by, in particular, dropping out associativity (although we are forced to impose positivity). Since associativity is not crucial for the construction of indifference relations, overlap functions allow to model this sort of relations. Nevertheless, although some of the most common used t-norms (including the minimum) are part of the class of overlap functions, there are t-norms that are not overlap functions as well as overlap functions that are not t-norms. Hence we are dealing with a class of functions different from that of the t-norms.

On the other hand, ignorance functions measure in an analytical way the lack of information that an expert suffers when trying to determine if a given element belongs to one class or another. This can also be seen as linked to missing evidence, in the sense that data do not support neither one nor the other alternative. In this sense, ignorance functions allow to build in a different way incomparability relations that do include but are not restricted to those given in terms of t-norms and negations.

Finally we introduce in this talk the concept of bi-entropic function as a unifying frame for the previous concepts. Bi-entropic functions can be understood as a measure of non-information, or as an extension of the concept of entropy when dealing with a preference comparison problem. Both overlap functions and ignorance functions can be recovered in a functional way from bi-entropic functions. Conversely, if appropriate overlap and ignorance functions are known that fit well for a given problem, they can be used to build a bi-entropic function that in some sense encompasses both of them. In this way, we are able to provide a theoretical framework which is different from the usual one to represent indifference and incomparability, to treat them as analytical concepts, to link the techniques that make use of these two concepts to techniques that are used in other fields, and to derive different properties interrelating all these concepts.

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Grouping functions for fuzzy modeling of pairwise comparisons

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A widely used approach dealing with preferences is to carry on pairwise comparisons in order to build a binary relation that allows to express the preferences in a qualitative way. Recently, some methods based on pairwise comparison of decision alternatives have been used for problems of classification in the field of machine learning. In particular, Hüllermeier and Brinker([3]) proposed a method for learning fuzzy preference relations from data that can be used to solve multi-class classification problems. This method makes use of the so-called indifference and incomparability relations, which are a very suitable tool to represent, respectively, two different types of uncertainty when it comes to predicting the class of a new instance: conflict, which appears if data provide evidence supporting simultaneously the two considered alternatives and ignorance, if none of them is supported by data. These two concepts are usually modeled by means of very well-known operators as t-norms and negations. For instance, an indifference I between alternative x_k and alternative x_l is usually built from a weak preference relation R as follows :

$$I_{kl} = T(R(x_k,x_l),R(x_l,x_k)) ,$$

with T a t-norm. On the other hand, incomparability J between alternative x_k and alternative x_l is modeled by means of a t-norm T and a negation N in this way:

$$J_{kl} = T(N(R(x_k,x_l)),N(R(x_l,x_k))) .$$

In this talk we propose a different analytical approach. Instead of modeling the amount of evidence supporting simultaneously both preferences, we introduce the concept of grouping function, that is, a symmetric, non-decreasing and continuous mapping G_G : $[0,1]^2 \to [0,1]$ that vanishes only at the point $(0,0)$ and that takes the value 1 if and only if one of its inputs is equal to one, as a functional measure of the amount of data that supports either one of the alternatives or the other. Following this line of reasoning, it can be related to the concept of overlap function (see [1]) in such a way that we arrive at a theoretical framework for fuzzy modeling of pairwise comparison that allows a new mathematical description of the concepts of indifference and incomparability, different from the one in terms of t-norms, but that also recovers the most important cases of the latter approach (in particular, the modeling by using the minimum t-norm). We will explain in the talk also the conceptual motivation behind this new approach, that allows to connect preference problems with other fields such as image processing. In particular, also the way in which the concept of strict preference can be rewritten in terms of grouping functions will be explained in the talk.

The efficiency of the new approach is proved by applying it to the particular case of the decision rule proposed in the context of multi-class classification by the authors of [2]:

$$x_{selection} = \text{arg max}_{k\in\{1,\ldots,n\}} \sum_{1\leq l\neq k\leq n} P_{kl} - \frac{1}{2} I_{kl} + \frac{N_k}{N_k + N_l} J_{kl}$$

where P_{kl} denotes the strict preference of x_k over x_l and N_k is the number of training examples belonging to class x_k. This formula is nothing but a generalization of the well-known weighted voting but it accepts further generalization in terms of grouping functions.

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Different models of fuzzy automata and their applications

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Determinization of nondeterministic finite automata is a well elaborated problem that plays an important role in text processing, natural language processing, compiler theory, system verification and testing, and many other areas of computer science, but also in fields outside of computer science like molecular biology. The standard determinization algorithm, known as *subset construction* or *powerset construction*, converts a nondeterministic automaton with n states into an equivalent deterministic automaton with up to 2^n states. Although in the worst case subset construction yields a deterministic automaton that is exponentially larger than the input nondeterministic automaton, which sometimes makes the construction impractical for large nondeterministic automata, this determinization algorithm is renowned for its good performance in practice.

Determinization of a fuzzy finite automaton is considered here as a procedure of its conversion into an equivalent *crisp-deterministic fuzzy automaton*, which can be viewed as being deterministic with possibly infinitely many states, but with fuzziness (vagueness) concentrated only in final states. This kind of determinism was first studied in [2], for fuzzy finite automata over a complete distributive lattice, and in [16], for fuzzy finite automata over a lattice-ordered monoid, where such algorithms were given which generalize the subset construction. Another algorithm, provided in [9], is also a generalization of the subset construction and for any input it generates a smaller crisp-deterministic fuzzy automaton than the algorithms from [2, 16]. This crisp-deterministic fuzzy automaton can be alternatively constructed by means of the Nerode right congruence of the original fuzzy automaton, and it was called in [10] the *Nerode automaton* of this fuzzy automaton. The Nerode automaton was constructed in [9] for fuzzy finite automata over a complete residuated lattice, but it was noted that the same construction can be also applied to fuzzy finite automata over a lattice-ordered monoid, and moreover, to weighted finite automata over a semiring. Nerode automata and some their improvements were recently studied within the framework of weighted finite automata over strong bimonoids [7, 11]. Note that strong bimonoids can be viewed as a semirings which might lack distributivity and include both semirings and lattices.

The above-mentioned determinization methods give crisp-deterministic fuzzy automata which are equivalent to the original fuzzy automata from the aspect of recognition of fuzzy languages. However, in addition to the fuzzy language recognized by a fuzzy automaton, there are also other important types of fuzzy languages associated with fuzzy automata. For instance, fuzzy languages generated by fuzzy automata play a very important role in the theory of fuzzy discrete-event systems. These languages can not be represented by means of crisp-deterministic fuzzy automata, because they require to keep fuzziness not only in the final state, but also in all other states. Here we introduce a general definition of an automaton with fuzzy states, crisp inputs, and the transition function which acts deterministically on fuzzy states and crisp inputs. Such an automaton is called just an *automaton with fuzzy states*. We show that an automaton with fuzzy states can be easily constructed starting from an arbitrary crisp-deterministic fuzzy automaton, and these two automata recognize the same fuzzy language. On the other hand, considering fuzzy states as crisp singletons, every automaton with fuzzy states can be transformed into a crisp-deterministic fuzzy automaton. We also show that two important types of crisp-deterministic fuzzy automata can be regarded as automata with fuzzy states. First, we show that Nerode automata can be considered as automata with fuzzy states, and we prove that they are equivalent to the original fuzzy automata both from the aspects of recognition and generation of fuzzy languages. Another important example of automata with fuzzy states is the *derivative automaton* associated with a fuzzy language. It was introduced in [10], and it was proved that it is a unique (up to an isomorphism) minimal crisp-deterministic fuzzy automaton recognizing the given fuzzy language. Here we prove that the derivative automaton can also be considered as an automaton with fuzzy states, that it is a minimal automaton with fuzzy states which recognizes the given fuzzy language, and that it generates the prefix-closure of this fuzzy language.

In the modeling of fuzzy discrete-event systems, in [3, 4, 21, 25] the classical fuzzy automata were used, but in [5, 14, 17–20, 23, 24] fuzzy discrete-event systems were modeled using automata with fuzzy states and fuzzy events, where fuzzy events are given by fuzzy matrices associated with input letters. These fuzzy matrices represent degrees to which inputs cause transitions between crisp states, and consequently, this kind of automata with fuzzy states is nothing but Nerode automata of fuzzy automata. Here we determine necessary and sufficient conditions under which an automaton with fuzzy states can be represented as the Nerode automaton of some fuzzy automaton. They are given in terms of solvability of some particular linear systems of fuzzy relation equations. We provide an example of a finite automaton with fuzzy states which can not be represented as the Nerode automaton of some fuzzy automaton. We also show that automata with fuzzy states are computationally more powerful than the fuzzy automata, in the sense that they generate a larger class of fuzzy languages. In other words, we prove that every fuzzy language generated by a fuzzy automaton is prefix-closed, that every prefix-closed fuzzy language is generated by an automaton with fuzzy states, and that there are fuzzy languages that are generated by automata with fuzzy states but are not prefix-closed. However, this is not true if we require automata to be finite, since we can provide an example of a fuzzy language which is generated by a fuzzy finite automaton, but it can not be generated by a finite automaton with fuzzy states.

It is well known that fuzzy automata are the basis for the study of multistage fuzzy decision processes, which was initiated in [1] (see also [12, 13, 15]). The automata involved are automata with fuzzy states, fuzzy inputs (represented by fuzzy subsets of the input alphabet), and the transition function which acts deterministically on fuzzy states and fuzzy inputs. Fuzzy inputs are used to represent fuzzy constraints, whereas fuzzy goals are represented by fuzzy states. Here we also consider such a model of fuzzy automata, its semantics, and relationships with other models. Especially, we discuss relationships with the above-mentioned model of automata with fuzzy states and fuzzy events, which was proposed in [19] as the best way to build a FDES decision model. It is worth noting that such a FDES decision model was applied in [19] to HIV/AIDS treatment planning.

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On the arity gap of aggregation functions

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Abstract. We briefly survey earlier and recent results concerning the arity gap of functions, and present an explicit classification of aggregation functions according to their arity gap which is shown to be either 1 or 2.

1 Introduction

The process of merging or combining sets of values (often real values) into a single one is usually achieved by so-called aggregation functions. Usually, an aggregation function on a closed real interval $[a,b] \subseteq \overline{\mathbb{R}}$ is a mapping $M\colon [a,b]^n \to [a,b]$ which is order-preserving and fulfills the boundary conditions $M(a,\ldots,a) = a$ and $M(b,\ldots,b) = b$. Classical examples of aggregation functions include weighted arithmetic means (discrete versions of Lebesgue integrals), as well as certain non-additive fuzzy integrals such as the Choquet integral [3] and the Sugeno integral [15,16]. For general background, see [1,11] and for a recent reference, see [10].

In this paper, we study the arity gap of order-preserving functions, in particular, of aggregation functions. Loosely speaking, the arity gap of a function f measures the minimum decrease in the number of essential variables when essential variables of f are identified.

Let A and B be arbitrary nonempty sets. A *function of several variables from A to B* is a map $f\colon A^n \to B$ for some integer $n \geq 1$ called the *arity* of f. If $A = B$, then we speak of *operations on A.* Operations on the two-element set $\{0,1\}$ are called *Boolean functions.*

We say that the i-th variable of $f\colon A^n \to B$ ($1 \leq i \leq n$) is *essential*, if there exist n-tuples $(a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_n),(a_1,\ldots,a_{i-1},a_i',a_{i+1},\ldots,a_n) \in A^n$ that only differ in the i-th position, such that

$$f(a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_n) \neq f(a_1,\ldots,a_{i-1},a_i',a_{i+1},\ldots,a_n).$$

If the i-th variable of f is not essential, then we say that it is *inessential.* The number of essential variables of f is called the *essential arity* of f and it is denoted by $\operatorname{ess} f$.

For $i,j \in \{1,\ldots,n\}$, $i \neq j$, the function $f_{i\leftarrow j}\colon A^n \to B$ given by the rule

$$f_{i\leftarrow j}(a_1,\ldots,a_n) = f(a_1,\ldots,a_{i-1},a_j,a_{i+1},\ldots,a_n),$$

for all $a_1,\ldots,a_n \in A$, is called a *variable identification minor* of f, obtained by identifying the i-th variable with the j-th variable. Note that the i-th variable of $f_{i\leftarrow j}$ is necessarily inessential. Two functions f and g are said to be *equivalent*, denoted $f \equiv g$, if each one can be obtained from the other by permutation of variables and addition or deletion of inessential variables. Clearly, if $f \equiv g$, then $\operatorname{ess} f = \operatorname{ess} g$.

The *arity gap* of $f \colon A^n \to B$ ($\operatorname{ess} f \geq 2$) is $\operatorname{gap} f := \min_{i \neq j}(\operatorname{ess} f - \operatorname{ess} f_{i \leftarrow j})$, where i and j range over the set of indices of essential variables of f. Note that, by definition, $1 \leq \operatorname{gap} f \leq \operatorname{ess} f$. Moreover, if $f \equiv g$, then $\operatorname{gap} f = \operatorname{gap} g$.

Example 1. Let F be an arbitrary field. Consider the polynomial function $f \colon F^3 \to F$ induced by $x_1 x_3 - x_2 x_3$. It is clear that all variables of f are essential, i.e., $\operatorname{ess} f = 3$. Looking at the various variable identification minors of f we see that $\operatorname{ess} f_{1 \leftarrow 2} = \operatorname{ess} f_{2 \leftarrow 1} = 0$ and $\operatorname{ess} f_{1 \leftarrow 3} = \operatorname{ess} f_{3 \leftarrow 1} = \operatorname{ess} f_{2 \leftarrow 3} = \operatorname{ess} f_{3 \leftarrow 2} = 2$. Hence $\operatorname{gap} f = 1$.

Example 2. Let A be a finite set with $k \geq 2$ elements, say, $A = \{1, \ldots, k\}$. Let $f \colon A^n \to A$, $2 \leq n \leq k$, be given by the rule: $f(a_1, \ldots, a_n)$ is 2 if $(a_1, \ldots, a_n) = (1, \ldots, n)$, and 1 otherwise. It is easy to see that all variables of f are essential, and for all $i \neq j$, the function $f_{i \leftarrow j}$ is identically 1. Hence $\operatorname{gap} f = n$.

As shown by the examples above, every positive integer is the arity gap of some function of several variables. Are all positive integers possible as the arity gaps of functions of several variables from A to B for a fixed domain A and codomain B? Does the size of the domain or the codomain have any influence on the set of possible arity gaps? Or even, could one hope to classify functions according to their arity gap? These questions have been raised and studied by several authors.

Salomaa [14] showed that the arity gap of any Boolean function is at most 2. This result was extended to functions defined on arbitrary finite domains by Willard [17], who showed that the same upper bound holds for the arity gap of any function $f \colon A^n \to B$, provided that $\operatorname{ess} f = n > \max(|A|, 3)$. In fact, he showed that if the arity gap of such a function f is 2, then f is totally symmetric. This line of research culminated into a complete classification of functions according to their arity gap originally presented in [5] in the setting of functions with finite domains; in [7] it was observed that this result holds for functions with arbitrary, possibly infinite domains.

Salomaa's [14] result on the upper bound for the arity gap of Boolean functions was strengthened in [4], where Boolean functions were completely classified according to their arity gap. Using tools provided by Berman and Kisielewicz [2] and Willard [17], in [5] a similar explicit classification was established for all pseudo-Boolean functions, i.e., functions $f \colon \{0,1\}^n \to \mathbb{R}$. As it turns out, this leads to analogous classifications of wider classes of functions. In [6], this result on pseudo-Boolean functions was the key step in showing that among polynomial functions of bounded distributive lattices (in particular, Sugeno integrals) only truncated ternary medians (ternary medians, respectively) have arity gap 2; all the others have arity gap 1. Using similar techniques, [8] presented explicit descriptions of the arity gap of well-known extensions of pseudo-Boolean functions to the whole real line, namely, Owen and Lovász extensions. As the latter subsume Choquet integrals, a complete classification of Choquet integrals according to their arity gap was also attained.

Both the Sugeno and Choquet integrals constitute particular examples of aggregation functions. Thus, it is natural to ask for extensions of these descriptions of the arity gap of aggregation functions. This question was considered and answered in [8] via a dichotomy theorem which completely classified the order-preserving functions $f\colon A^n \to B$, for “bidirected” partially ordered sets A and B, into those with arity gap 1 and those with arity gap 2.

In this paper, we present a corollary to this result when restricted to the case of chains A and B, which explicitly describes those order-preserving functions that have arity gap 1 and those that have arity gap 2. As a by-product we obtain similar descriptions for the class of aggregation functions, in particular, for the classes of Sugeno and Choquet integrals.

2 The arity gap of order-preserving functions

Let A and B be chains (totally ordered sets). A function $f\colon A^n \to B$ is said to be *order-preserving* if for all $\mathbf{a},\mathbf{b} \in A^n$, $f(\mathbf{a}) \leq_B f(\mathbf{b})$ whenever $\mathbf{a} \leq_A \mathbf{b}$, where $\mathbf{a} \leq_A \mathbf{b}$ denotes the componentwise ordering of tuples. An example is the ternary median function med given by $\mathrm{med}(a,b,c) := (a\wedge b)\vee(a\wedge c)\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)\wedge(b\vee c)$. The following result provides an explicit classification of order-preserving functions (on chains) according to their arity gap.

Theorem 1 ([8]). *Let A and B be chains, and $f\colon A^n \to B$ be an order-preserving function. Then* $\mathrm{gap}\, f = 2$ *if and only if* $n = 3$ *and* $f = \mathrm{med}\big(h(x_1),h(x_2),h(x_3)\big)$ *for some nonconstant order-preserving unary function $h\colon A \to B$ (here* med *denotes the median function on* $\mathrm{Im}\, h$*). Otherwise* $\mathrm{gap}\, f = 1$.

Choquet and Sugeno integrals are usually defined in terms of certain set functions. A *fuzzy measure* on $[n] = \{1,\ldots,n\}$ is any order-preserving map $v\colon 2^{[n]} \to [0,1]$ satisfying $v(\emptyset) = 0$ and $v([n]) = 1$. The *Choquet integral* of $\mathbf{x} \in \mathbb{R}^n$ with respect to v is defined by

$$\mathcal{C}_v(\mathbf{x}) = \sum_{i\in[n]} x_{(i)}\big(v(A_{(i)}) - v(A_{(i+1)})\big), \qquad (1)$$

where $(\cdot)$ indicates the permutation on $[n]$ such that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$, and $A_{(i)} = \{(i),\ldots,(n)\}$ and $A_{(n+1)} = \emptyset$. As it turns out [12], Choquet integrals coincide exactly with the Lovász extensions of those order-preserving pseudo-Boolean functions f that fulfill $f(c,\ldots,c) = c$ for $c \in \{0,1\}$; in fact, from (1) it follows that Choquet integrals are idempotent, i.e., $f(c,\ldots,c) = c$ for $c \in \mathbb{R}$. As an immediate consequence of Theorem 1, we get an explicit description of Choquet integrals with arity gap 2.

Corollary 1. *A Choquet integral $f\colon \mathbb{R}^n \to \mathbb{R}$ has arity gap* 2 *if and only if*

$$f \equiv (x_1\wedge x_2) + (x_1\wedge x_3) + (x_2\wedge x_3) - 2\cdot(x_1\wedge x_2\wedge x_3).$$

Any other Choquet integral has arity gap 1.

Proof. Clearly, the condition is sufficient. To see that it is also necessary just observe that the function h given by Theorem 1 must be the identity function since Choquet integrals are idempotent.

As observed in [9, 13], a convenient way to introduce Sugeno integrals is via lattice polynomial functions, that is, functions which can be obtained as compositions of the lattice operations and variables (projections) and constants. Sugeno integrals can then be viewed as idempotent lattice polynomial functions. Thus, by Theorem 1, we also have the following explicit classification of Sugeno integrals according to their arity gap.

Corollary 2. *A Sugeno integral* $f\colon A^n \to A$ *on a chain* A *has arity gap* 2 *if and only if* $f \equiv \operatorname{med}(x_1, x_2, x_3)$. *Any other Sugeno integral has arity gap* 1.

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On three properties of the discrete Choquet integral

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Abstract. Three important properties in aggregation theory are investigated, namely horizontal min-additivity, horizontal max-additivity, and comonotonic additivity, which are defined by certain relaxations of the Cauchy functional equation in several variables. We show that these properties are equivalent and we completely describe the functions characterized by them. By adding some regularity conditions, these functions coincide with the Lovász extensions vanishing at the origin, which subsume the discrete Choquet integrals.

1 Introduction

A noteworthy aggregation function is the so-called discrete Choquet integral, which has been widely investigated in aggregation theory, due to its many applications for instance in decision making. A convenient way to introduce the discrete Choquet integral is via the concept of Lovász extension. An n-place Lovász extension is a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each of the $n!$ subdomains

$$\mathbb{R}^n_\sigma = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\} \qquad (\sigma \in S_n)$$

is an affine function, where S_n denotes the set of permutations on $[n] = \{1,\ldots,n\}$. An n-place Choquet integral is simply a nondecreasing (in each variable) n-place Lovász extension which vanishes at the origin. For general background, see [5, §5.4].

In this paper we investigate three properties of the discrete Choquet integral, namely, comonotonic additivity, horizontal min-additivity, and horizontal max-additivity. After recalling the definitions of Lovász extensions and discrete Choquet integrals (Section 2), we show that the three properties above are actually equivalent. We describe the function class axiomatized by these properties and we show that, up to certain regularity conditions (based on those we usually add to the Cauchy functional equation to get linear solutions only), these properties completely characterize those n-place Lovász extensions which vanish at the origin. Nondecreasing monotonicity is then added to characterize the class of n-place Choquet integrals (Section 3).

We employ the following notation throughout the paper. Let $\mathbb{R}_+ = [0,\infty[$ and $\mathbb{R}_- =]-\infty,0]$. For every $A \subseteq [n]$, the symbol $\mathbf{1}_A$ denotes the n-tuple whose ith component is 1, if $i \in A$, and 0, otherwise. Let also $\mathbf{1} = \mathbf{1}_{[n]}$ and $\mathbf{0} = \mathbf{1}_\varnothing$. The symbols $\wedge$ and $\vee$ denote the minimum and maximum functions, respectively. For every function $f\colon \mathbb{R}^n \to \mathbb{R}$, we define its diagonal section $\delta_f\colon \mathbb{R} \to \mathbb{R}$ by $\delta_f(x) = f(x\mathbf{1})$. More generally, for every $A \subseteq [n]$, we define the function $\delta_f^A\colon \mathbb{R} \to \mathbb{R}$ by $\delta_f^A(x) = f(x\mathbf{1}_A)$.

It is important to notice that comonotonic additivity as well as horizontal min-additivity and horizontal max-additivity extend the classical additivity property defined by the Cauchy functional equation for n-place functions

$$f(\mathbf{x}+\mathbf{x}') = f(\mathbf{x}) + f(\mathbf{x}') \qquad (\mathbf{x},\mathbf{x}' \in \mathbb{R}^n). \tag{1}$$

In this regard, recall that the general solution $f\colon \mathbb{R}^n \to \mathbb{R}$ of the Cauchy equation (1) is given by $f(\mathbf{x}) = \sum_{k=1}^n f_k(x_k)$, where the $f_k\colon \mathbb{R} \to \mathbb{R}$ $(k \in [n])$ are arbitrary solutions of the basic Cauchy equation $f_k(x+x') = f_k(x) + f_k(x')$ (see [1, §2–4]). If f_k is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure, then f_k is necessarily a linear function ([1]).

2 Lovász extensions

Consider a *pseudo-Boolean function*, that is, a function $\phi\colon \{0,1\}^n \to \mathbb{R}$, and define the set function $v_\phi\colon 2^{[n]} \to \mathbb{R}$ by $v_\phi(A) = \phi(\mathbf{1}_A)$ for every $A \subseteq [n]$. Hammer and Rudeanu [6] showed that such a function has a unique representation as a multilinear polynomial of n variables

$$\phi(\mathbf{x}) = \sum_{A\subseteq[n]} a_\phi(A) \prod_{i\in A} x_i\,,$$

where the set function $a_\phi\colon 2^{[n]} \to \mathbb{R}$, called the *Möbius transform* of v_ϕ, is defined by

$$a_\phi(A) = \sum_{B\subseteq A} (-1)^{|A|-|B|}\, v_\phi(B).$$

The *Lovász extension* of a pseudo-Boolean function $\phi\colon \{0,1\}^n \to \mathbb{R}$ is the function $f_\phi\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each subdomain $\mathbb{R}^n_\sigma$ $(\sigma \in S_n)$ is the unique affine function which agrees with ϕ at the $n+1$ vertices of the n-simplex $[0,1]^n \cap \mathbb{R}^n_\sigma$ (see [7, 9]). We then have $f_\phi|_{\{0,1\}^n} = \phi$.

It can be shown (see [5, §5.4.2]) that the Lovász extension of a pseudo-Boolean function $\phi\colon \{0,1\}^n \to \mathbb{R}$ is the continuous function $f_\phi(\mathbf{x}) = \sum_{A\subseteq[n]} a_\phi(A) \bigwedge_{i\in A} x_i$. Its restriction to $\mathbb{R}^n_\sigma$ is the affine function

$$f_\phi(\mathbf{x}) = (1 - x_{\sigma(n)})\,\phi(\mathbf{0}) + x_{\sigma(1)}\, v_\phi(A^\uparrow_\sigma(1)) + \sum_{i=2}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)})\, v_\phi(A^\uparrow_\sigma(i)), \tag{2}$$

where $A^\uparrow_\sigma(i) = \{\sigma(i),\ldots,\sigma(n)\}$. We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a *Lovász extension* if there is a pseudo-Boolean function $\phi\colon \{0,1\}^n \to \mathbb{R}$ such that $f = f_\phi$.

An n-place *Choquet integral* is a nondecreasing Lovász extension $f_\phi\colon \mathbb{R}^n \to \mathbb{R}$ such that $f_\phi(\mathbf{0}) = 0$. It is easy to see that a Lovász extension $f\colon \mathbb{R}^n \to \mathbb{R}$ is an n-place Choquet integral if and only if its underlying pseudo-Boolean function $\phi = f|_{\{0,1\}^n}$ is nondecreasing and vanishes at the origin (see [5, §5.4]).

3 Axiomatizations of Lovász extensions and discrete Choquet integrals

Two n-tuples $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$ are said to be *comonotonic* if there exists $\sigma \in S_n$ such that $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n_\sigma$. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *comonotonically additive* if, for every comonotonic n-tuples $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$, we have

$$f(\mathbf{x}+\mathbf{x}') = f(\mathbf{x}) + f(\mathbf{x}'). \tag{3}$$

Given $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, let $[\![\mathbf{x}]\!]_c = \mathbf{x} - \mathbf{x} \wedge c$ and $[\![\mathbf{x}]\!]^c = \mathbf{x} - \mathbf{x} \vee c$. We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is

– *horizontally min-additive* if, for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) + f([\![\mathbf{x}]\!]_c). \tag{4}$$

– *horizontally max-additive* if, for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \vee c) + f([\![\mathbf{x}]\!]^c). \tag{5}$$

We now describe the function classes axiomatized by these two properties. To this extent, we let $A^{\downarrow}_\sigma(i) = \{\sigma(1),\ldots,\sigma(i)\}$.

Theorem 1. *A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is horizontally min-additive if and only if there exists $g\colon \mathbb{R}^n \to \mathbb{R}$, with δ_g and $\delta_g^A|_{\mathbb{R}_+}$ additive for every $A \subseteq [n]$, such that, for every $\sigma \in S_n$,*

$$f(\mathbf{x}) = \delta_g(x_{\sigma(1)}) + \sum_{i=2}^{n} \delta_g^{A^{\uparrow}_\sigma(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma). \tag{6}$$

In this case, we can choose $g = f$.

Theorem 2. *A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is horizontally max-additive if and only if there exists $h\colon \mathbb{R}^n \to \mathbb{R}$, with δ_h and $\delta_h^A|_{\mathbb{R}_-}$ additive for every $A \subseteq [n]$, such that, for every $\sigma \in S_n$,*

$$f(\mathbf{x}) = \delta_h(x_{\sigma(n)}) + \sum_{i=1}^{n-1} \delta_h^{A^{\downarrow}_\sigma(i)}(x_{\sigma(i)} - x_{\sigma(i+1)}) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma).$$

In this case, we can choose $h = f$.

Using Theorems 1 and 2, one can show that each of the two horizontal additivity properties is equivalent to comonotonic additivity. Thus we have the following result.

Theorem 3. *For any function $f\colon \mathbb{R}^n \to \mathbb{R}$, the following assertions are equivalent.*

(*i*) *f is comonotonically additive.*
(*ii*) *f is horizontally min-additive.*
(*iii*) *f is horizontally max-additive.*

If any of these conditions is fulfilled, then δ_f, $\delta_f^A|_{\mathbb{R}_+}$, and $\delta_f^A|_{\mathbb{R}_-}$ are additive $\forall\, A \subseteq [n]$.

We now axiomatize the class of n-place Lovász extensions. To this extent, a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *positively homogeneous of degree one* if $f(c\mathbf{x}) = c\,f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ and every $c > 0$.

Theorem 4. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function and let $f_0 = f - f(\mathbf{0})$. Then f is a Lovász extension if and only if the following conditions hold:*

(i) *f_0 is comonotonically additive or horizontally min-additive or horizontally max-additive.*

(ii) *Each of the maps δ_{f_0} and $\delta_{f_0}^A|_{\mathbb{R}_+}$ $(A \subseteq [n])$ is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure.*

The set $\mathbb{R}_+$ can be replaced by $\mathbb{R}_-$ in (ii). Condition (ii) holds whenever Condition (i) holds and $\delta_{f_0}^A$ is positively homogeneous of degree one for every $A \subseteq [n]$.

Remark 1. (a) Since any Lovász extension vanishing at the origin is positively homogeneous of degree one, Condition (ii) of Theorem 4 can be replaced by the stronger condition: f_0 is positively homogeneous of degree one.

(b) Axiomatizations of the class of n-place Choquet integrals can be immediately derived from Theorem 4 by adding nondecreasing monotonicity. Similar axiomatizations using comonotone additivity (resp. horizontal min-additivity) were obtained by de Campos and Bolaños [3] (resp. by Benvenuti et al. [2, §2.5]).

(c) The concept of comonotonic additivity appeared first in Dellacherie [4] and then in Schmeidler [8]. The concept of horizontal min-additivity was previously considered by Šipoš [10] and then by Benvenuti et al. [2, §2.3] where it was called "horizontal additivity".

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Multivalued relations over lattices and semirings and their applications

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A *multivalued relation* among sets A and B is any function from $A \times B$ to V, where V is a set of values with $|V| \geqslant 2$. If $B = A$ we talk about a multivalued relation on A. The most studied type of multivalued relations are *fuzzy relations*. In Zadeh's original definition of a fuzzy relation [11] values were taken from the real unit interval $[0,1]$, whereas Goguen [7] proposed the study of fuzzy sets and relations with values in an arbitrary lattice. Another important type of multivalued relations are multivalued relations among finite sets with values in a field, ring, or a semiring. They are well known as *matrices*.

Distributive lattices and related lattice-ordered structures, such as residuated lattices, lattice-ordered monoids and others, represent an excellent framework for the study of multivalued relations. Namely, ordering and certain good properties of these structures, such as idempotency of the supremum and distributivity of the infimum or multiplication over the supremum, enable many important properties of the classical two-valued relations to be transferred to multivalued relations. For example, it is possible to define transitivity, fuzzy equivalences and fuzzy quasi-orders (or fuzzy preorders, in some sources), to effectively solve fuzzy relation equations and inequalities, and so on. In our research, fuzzy equivalences and fuzzy quasi-orders were used in [5, 10] to reduce the number of states of fuzzy automata, and it has been shown that they give better results than crisp relations, which were used for this purpose before. Moreover, the main role in the study of bisimulations for fuzzy automata conducted in [4] had *uniform fuzzy relations*, which have been introduced in [3] as a kind of fuzzy equivalences that relate elements of two possible different sets.

As far as matrices over fields, rings, and semirings are concerned, they were usually studied in terms of solving systems of equations and inequalities, and were not considered as a generalization of two-valued relations. The reason for this probably lies in the fact that, unlike the ordered structures that are used in the theory of fuzzy sets, semirings are not required to be ordered, and also, the set $\{0,1\}$, which consists of the zero and the unit of a semiring, does not necessarily form a subsemiring, and matrices with entries in $\{0,1\}$ can not be considered as two-valued relations.

If the methods based on fuzzy relations, developed within the theory of fuzzy automata, we try to apply to weighted automata over semirings, we naturally encounter the problem: for which type of semirings matrices over them behave like fuzzy relations or classical two-valued relations. We show that a very important and a quite wide class of semirings, the class of *additively idempotent semirings*, has this property. We examine basic properties of multivalued relations with values in an additively idempotent semiring, and in particular, we define and study multivalued quasi-orders, equivalences, uniform relations, and so on. We also consider various applications of these multivalued relations, including applications in the study of weighted automata over additively idempotent semirings.

It is worth noting that additively idempotent semirings include many very important semirings, such as the well-known tropical semirings, arctic semirings, Viterbi semiring, Boolean semiring, and others. Additively idempotent semirings have significant applications in many areas of mathematics, computer science, and operations research, e.g., in the theory of automata and formal languages, optimization theory, idempotent analysis, theory of programming languages, data analysis, discrete event systems theory, algebraic modeling of fuzziness and uncertainty, algebra of formal processes, etc (cf. [1, 6, 8]). In particular, applications of additively idempotent semirings include solution of a wide variety of optimal path problems in graphs, extensions of classical algorithms for shortest path problems to a whole class of nonclassical path-finding problems (such as shortest paths with time constraints, shortest paths with time-dependent lengths on the arcs, etc.), solution of various nonlinear partial differential equations, such as Hamilton-Jacobi, and Bürgers equations, the importance of which is well-known in physics, etc.

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Weighted quasi-arithmetic means as conjugate means

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Abstract. Let $I \subseteq \mathbb{R}$ be a nonempty open interval and let $n \geq 3$ be a fixed natural number. In this paper we characterize those conjugate means of n variables generated by a weighted arithmetic mean, and which are weighted quasi-arithmetic means themselves.

1 Introduction

Throughout this paper let $I \subseteq \mathbb{R}$ be a nonempty open interval and let $n \geq 2$ be a natural number. A function $M : I^n \to I$ is called a *mean* on I if the following properties hold:

1. $min\{x_1, \dots, x_n\} \leq M(x_1, \dots, x_n) \leq max\{x_1, \dots, x_n\}$ for every $x_1, \dots, x_n \in I$,
2. M is continuous on I^n.

In order to introduce the so-called *conjugate means* we will need the following notion. Denote by K_n the set of n-tuples $(p_1, \dots, p_n)$ such that

$$min\{x_i\} \leq \sum_{i=1}^{n} p_i x_i + (1 - \sum_{i=1}^{n} p_i)M \leq max\{x_i\}$$

whenever

$$min\{x_i\} \leq M \leq max\{x_i\} \quad (i = 1, \dots, n),$$

holds for the real numbers $x_1, \dots, x_n, M$.

In [2] Daróczy and Páles provided the following characterization of the sets K_n:

Theorem 1. *$(p_1, \dots, p_n) \in K_n$ ($n \geq 2$) if and only if*

$$p_j \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i - p_j \leq 1 \quad (j = 1, \dots, n).$$

Let $\mathcal{CM}(I)$ denote the class of continuous and strictly monotone real valued functions defined on the interval I, and let M be a mean of n variables on I. Then for any $\varphi \in \mathcal{CM}(I)$ and $x_1, \dots, x_n \in I$

$$min\{\varphi(x_i)\} \leq \varphi(M(x_1, \dots, x_n)) \leq max\{\varphi(x_i)\}.$$

Hence by Theorem 1 we have that

$$min\{\varphi(x_i)\} \leq \sum_{i=1}^{n} p_i\varphi(x_i) + (1-\sum_{i=1}^{n} p_i)\varphi(M(x_1,\ldots,x_n)) \leq max\{\varphi(x_i)\}$$

holds for all $(p_1,\ldots,p_n) \in K_n$ and $x_1,\ldots,x_n \in I$.

Moreover, we have that

$$M_\varphi^{p_1,\ldots,p_n}(x_1,\ldots,x_n) := \varphi^{-1}\left(\sum_{i=1}^{n} p_i\varphi(x_i) + (1-\sum_{i=1}^{n} p_i)\varphi(M(x_1,\ldots,x_n))\right)$$

is always between $min\{x_i\}$ and $max\{x_i\}$, that is, $M_\varphi^{p_1,\ldots,p_n} : I^n \to I$ is a mean. We call this mean *the conjugate mean generated by M with weights* $p_1,\ldots,p_n$.

This class of means includes the weighted quasi-arithmetic means, in particular, the quasi-arithmetic means. For instance, if $n=2$ we get the following mean:

$$M_\varphi^{(p_1,p_2)}(x,y) := \varphi^{-1}(p_1\varphi(x) + p_2\varphi(y) + (1-p_1-p_2)\varphi(M(x,y))) \quad (x,y \in I) \qquad (1)$$

where $(p_1,p_2) \in [0,1]^2$ (see Theorem 1). If $p_1+p_2=1$ in (1) we get a weighted quasi-arithmetic mean, and if $p_1=p_2=\frac{1}{2}$ we get a quasi-arithmetic mean.

Let now M be the n-variable weighted arithmetic mean with fixed weights $\alpha_1,\ldots,\alpha_n$, that is, $M(x_1,\ldots,x_n) := \sum_{i=1}^{n}\alpha_i x_i$ where $\alpha_1+\ldots+\alpha_n=1$ and $\alpha_i>0$, $i=1,\ldots,n$. We are interested in those conjugate means of n variables generated by the weighted arithmetic mean M, and which are weighted quasi-arithmetic means themselves. In other words, we seek functions $\varphi,\psi \in \mathcal{CM}(I)$ and parameters $p_1,\ldots,p_n$, $q_1,\ldots,q_n$ such that

$$\varphi^{-1}\left(\sum_{i=1}^{n} p_i\varphi(x_i) + (1-\sum_{i=1}^{n} p_i)\varphi\left(\sum_{i=1}^{n}\alpha_i x_i\right)\right) = \psi^{-1}\left(\sum_{i=1}^{n} q_i\psi(x_i)\right)$$

holds for all $x_1,\ldots,x_n \in I$, where $(p_1,\ldots,p_n) \in K_n$ and

$$\sum_{i=1}^{n} q_i = \sum_{i=1}^{n}\alpha_i = 1, \quad q_i,\alpha_i > 0 \ (i=1,\ldots,n).$$

This problem has been solved in the special case $n=2$ and when M is the arithmetic mean by Daróczy and Dascăl [4]. In the next section we provide the solutions of this functional equation for $n \geq 3$.

2 Main result

In order to state our main result we need the following notation. Let $\varphi,\psi \in \mathcal{CM}(I)$. If there exist $a \neq 0$ and b such that for every $x \in I$

$$\psi(x) = a\varphi(x) + b$$

then we say that φ is *equivalent* to ψ on I and denote it by $\varphi(x) \sim \psi(x)$ for every $x \in I$.

Theorem 2. *Let $n \geq 3$ be a fixed natural number, $(p_1,\ldots,p_n) \in K_n$, $q_1+\ldots+q_n = \alpha_1+\ldots+\alpha_n = 1$, $q_i > 0$, $\alpha_i > 0$ $(i = 1,\ldots,n)$. If the functions $\varphi,\psi \in \mathcal{CM}(I)$ are solutions of the functional equation*

$$\varphi^{-1}\left(\sum_{i=1}^{n} p_i\varphi(x_i) + (1-\sum_{i=1}^{n} p_i)\varphi(\sum_{i=1}^{n}\alpha_i x_i)\right) = \psi^{-1}\left(\sum_{i=1}^{n} q_i\psi(x_i)\right) \tag{2}$$

$(x_1,\ldots,x_n \in I)$ then

$$q_j - p_j = \alpha_j(1-\sum_{i=1}^{n} p_i) \quad (j = 1,\ldots,n)$$

and the following cases are possible

- *if $\sum_{i=1}^{n} p_i = 0$ then $\psi(x) \sim x$ on I, φ is arbitrary;*
- *if $\sum_{i=1}^{n} p_i = 1$ then if $p_i > 0$ for $i = 1,\ldots,n$ then $\varphi(x) \sim \psi(x)$ $(x,y \in I)$, otherwise there are no solutions;*
- *if $\sum_{i=1}^{n} p_i \neq 0,1$ then $\varphi(x) \sim x$, $\psi(x) \sim x$ $(x,y \in I)$.*

Conversely, the functions given in the cases above are solutions of the functional equation (2)*.*

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The God-Einstein-Oppenheimer dice puzzle

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We use the God-Einstein-Oppenheimer dice puzzle to introduce the notion of non-transitive dice and point out the connection with the ancient game of Rock-Paper-Scissors. We then introduce the notion of winning probabilities between the components of a real-valued random vector. Assembling these winning probabilities into a reciprocal relation facilitates the study of their structural properties, which can be neatly expressed in the cycle-transitivity framework. This framework encompasses numerous existing types of transitivity for reciprocal relations, including, inter alia, different types of so-called stochastic transitivity and Tanino's multiplicative transitivity.

Cycle-transitivity depends upon the choice of a so-called upper bound function. When using as upper bound function the well-known probabilistic sum t-conorm with different order statistics as inputs, we unveil truly stochastic types of transitivity, which can be linked with the frequency of so-called product triangles. Two important realizations are weak product transitivity (also called dice-transitivity), the type of transitivity characterizing winning probabilities between independent random variables, and moderate product transitivity, a type of transitivity that is weaker than mutual rank transitivity, the type of transitivity exhibited by the mutual rank probabilities between the elements of a poset. In the latter context, we establish a connection with proportional stochastic transitivity and linear extension majority cycles.

Time permitting, we discuss a generalization of winning probabilities between comonotone random variables, called proportional expected differences, and show how they lead to a layered alternative to the popular notion of stochastic dominance, thereby alleviating a number of shortcomings.

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Formal Concept Analysis and lattice-valued interchange systems

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1 Introduction

Formal concept analysis (FCA) consists of many well known methods which may be used for data analysis and knowledge representation. FCA is developed via Galois connections which are defined between powersets and which are determined by relations between the underlying sets. The underlying sets include a set of objects and a set of attributes or properties which the objects may have. FCA clusters objects in the powerset of objects and clusters properties in the powerset of properties, and these clusters are paired by the Galois connection. This pairing is natural with respect to a given relation between the sets of objects and properties.

Definition 1. *A formal context is an ordered triple (G,M,R) where G is the set of objects, M is the set of attributes, and R is a relation from G to M, i.e., $R \subset G \times M$.*

Definition 2. *A Galois connection is an ordered quadruple $(f,(P,\leq),(Q,\sqsubseteq),g)$ such that $(P,\leq)$ and $(Q,\sqsubseteq)$ are partially ordered sets, and $f : P \to Q$ and $g : Q \to P$ are order-reversing functions such that for each $p \in P$, $p \leq gf(p)$ and for each $q \in Q$, $q \sqsubseteq fg(q)$.*

Definition 3. *(Alternate Definition) A Galois connection is an ordered quadruple $(f,$ $(P,\leq),(Q,\sqsubseteq),g)$ such that $(P,\leq)$ and $(Q,\sqsubseteq)$ are partially ordered sets, and for each $p \in P$ and $q \in Q$, $p \leq g(q)$ if and only if $q \sqsubseteq f(p)$.*

Galois connections may be defined with order-reversing or order-preserving functions. They were originally defined by O. Ore [6] with order-reversing functions. Seemingly, the first mention of order-preserving functions in Galois connections was by J. Schmidt in [7]. For the work described in this abstract, we use order-reversing functions.

Sometimes for brevity, we write (f,g) instead of $(f,(P,\leq),(Q,\sqsubseteq),g)$ for a Galois connection.

The following proposition is well known; see, for example, [3] and [5].

Proposition 1. *Let* $(f,(P,\leq),(Q,\sqsubseteq),g)$ *be a Galois connection.*

1. $g^{\rightarrow}(Q)$ *and* $f^{\rightarrow}(P)$ *are anti-isomorphic partially ordered sets, and* $f|_{g^{\rightarrow}(Q)}^{f^{\rightarrow}(P)} : g^{\rightarrow}(Q) \rightarrow f^{\rightarrow}(P)$ *and* $g|_{f^{\rightarrow}(P)}^{g^{\rightarrow}(Q)} : f^{\rightarrow}(P) \rightarrow g^{\rightarrow}(Q)$ *are order-reversing bijections. In fact,* $f|_{g^{\rightarrow}(Q)}^{f^{\rightarrow}(P)}$ *and* $g|_{f^{\rightarrow}(P)}^{g^{\rightarrow}(Q)}$ *are anti-isomorphic inverses of each other.*
2. *P and Q are naturally organized or structured by the fibers of* f *and g, respectively. Each fiber of* f *contains exactly one point of* $g^{\rightarrow}(Q)$*, and each fiber of g contains exactly one point of* $f^{\rightarrow}(P)$*. The image point in each fiber is the largest element of the fiber.*
3. *The partition of fibers of P has the same partially ordered structure as* $g^{\rightarrow}(Q)$*, and the partition of fibers of Q has the same partially ordered structure as* $f^{\rightarrow}(P)$*. If* E_1 *and* E_2 *are two fibers or equivalence classes, for example, in P, then* $E_1 \leq E_2$ *if and only if there exist* $p_1 \in E_1$ *and* $p_2 \in E_2$ *such that* $p_1 \leq p_2$*. Thus, since* $g^{\rightarrow}(Q)$ *and* $f^{\rightarrow}(P)$ *are anti-isomorphic partially ordered sets, then the set of fibers in P and the set of fibers in Q are anti-isomorphic partially ordered sets.*
4. *The image points are called fixed points.* $p \in g^{\rightarrow}(Q)$ *if and only if* $p = gf(p)$*. Likewise,* $q \in f^{\rightarrow}(P)$ *if and only if* $q = fg(q)$*.*
5. $fgf = f$ *and* $gfg = g$*.*
6. *If P or Q is a [complete] lattice, then so are* $g^{\rightarrow}(Q)$ *and* $f^{\rightarrow}(P)$*. However,* $g^{\rightarrow}(Q)$ *and* $f^{\rightarrow}(P)$ *may not be sublattices of P and Q, respectively.*

The following result is from Birkhoff [1] with the terminology from Ore [6]. The expression "Galois connection" was essentially first used by Ore; Ore called the construction a "Galois connexion." Birkhoff called his construction, which is defined between powersets, a polarity.

Proposition 2. *Let G and M be arbitrary sets, and let* $R \subset G \times M$ *be a relation. Define* $H : \wp(G) \rightarrow \wp(M)$ *and* $K : \wp(M) \rightarrow \wp(G)$ *by*

$$\text{for } S \subset G, H(S) = \{m \in M | gRm \forall g \in S\}$$

$$\text{for } T \subset M, K(T) = \{g \in G | gRm \forall m \in T\}$$

$(H,\wp(X),\wp(Y),K)$ *is a Galois connection where the orderings on both* $\wp(X)$ *and* $\wp(Y)$ *are the subset orderings.*

Definition 4. *Let* (G,M,R) *be a formal context. A formal concept of the formal context is an ordered pair* (A,B) *with* $A \subset G$ *and* $B \subset M$ *such that* $H(A) = B$ *and* $K(B) = A$*. If* (A,B) *and* (C,D) *are formal concepts of* (G,M,R)*, then* $(A,B) \leq (C,D)$ *if* $A \subset C$ *or equivalently, if* $D \subset B$*.*

Definition 5. *Let* $\mathcal{K} = (G,M,R)$ *be a formal context. The set of all formal concepts of* $\mathcal{K}$ *is called the concept lattice of* $\mathcal{K}$*.*

Theorem 1. *Let $\mathcal{K} = (G, M, R)$ be a formal context, and let $(H, \wp(X), \wp(Y), K)$ be the associated Galois connection. The concept lattice of $\mathcal{K}$ is a complete lattice, and it is isomorphic to $g^{\rightarrow}(Q)$ and anti-isomorphic to $f^{\rightarrow}(P)$*

Though not a standard definition in FCA, we find the following definition useful.

Definition 6. *Let (G, M, R) be a formal context. A formal pre-concept of the formal context is an ordered pair (C, D) with $C \subset G$ and $D \subset M$ such that $KH(C) = K(D)$ or equivalently, $HK(D) = H(C)$.*

Proposition 3. *Let (G, M, R) be a formal context, and let (H, K) be the associated Galois connection. A formal pre-concept of the formal context is an ordered pair (C, D) with $C \subset G$ and $D \subset M$ such that C and D are elements of anti-isomorphic fibers of H and K, respectively.*

Proposition 4. *For a formal context (G, M, R), (C, D) is a formal pre-concept if and only if $(K(D), H(C))$ is a formal concept.*

2 A Category of Formal Contexts

To facilitate additional mathematical investigations in FCA, we want to define a category whose objects are formal contexts. Questions which immediately come to mind include what are the morphisms of such a category and what properties does the category have. For example, does the category have a base category with a natural forgetful functor.

In the previous paragraph, we phrased the questions as if there is only one possible category. Of course, there may be several natural and useful categories which have formal contexts as their objects.

As we address the first and most immediate question which is what are the morphisms of this category. We ask ourselves what properties or characteristics do formal contexts have. We want to know what properties the morphisms should preserve. Interestingly, though a formal context is defined in terms of two sets and a relation between them, the important characteristics of a formal context are characteristics of the associated Galois connection.

Thus, given two formal contexts $\mathcal{K}_1 = (G_1, M_1, R_1)$ and $\mathcal{K}_2 = (G_2, M_2, R_2)$, a morphism from $\mathcal{K}_1$ to $\mathcal{K}_2$ needs to respect the Galois connections $(H_1, \wp(G_1), \wp(M_1), K_1)$ and $(H_2, \wp(G_2), \wp(M_2), K_2)$, determined by $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. One way of defining morphisms from $\mathcal{K}_1$ to $\mathcal{K}_2$ is to define the morphisms as pairs of functions (f, g) such that $(f, g) : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ if

$$f : \wp(G_1) \rightarrow \wp(G_2) \text{ and } g : \wp(M_1) \rightarrow \wp(M_2)$$

with

$$H_2 \circ f = g \circ H_1 \text{ and } f \circ K_1 = K_2 \circ g.$$

This category is essentially defined in [4]. Let $(f,g) : \mathcal{K}_1 \to \mathcal{K}_2$. [4] considers the case where f and g may be arbitrary functions and the case where they are order-preserving functions. We first assume that f and g are arbitrary functions. From [4], we get the following:

- If g_1 and g_1' are in the same equivalence class of G_1, then $f(g_1)$ and $f(g_1')$ must be in the same equivalence class of G_2. Similarly, if m_1 and m_1' are in the same equivalence class of M_1, then $g(m_1)$ and $g(m_1')$ must be in the same equivalence class in M_2.
- f and g take fixed points to fixed points.
- Formal pre-concepts in $\mathcal{K}_1$ are mapped to formal pre-concepts in $\mathcal{K}_2$, i.e., if (C,D) is a formal pre-concept in $\mathcal{K}_1$, then $(f(C),g(D))$ is a formal pre-concept in $\mathcal{K}_2$.

A powerset has a natural ordering, the subset ordering, and this ordering is important in FCA. Thus, in addition to the above conditions on f and g, we require that f and g be order-preserving.

In this paper, we are, however, interested in a different category of formal concepts. For reasons which will become clear later, we want the second function in an ordered pair of a morphism $(f,g) : \mathcal{K}_1 \to \mathcal{K}_2$ to be defined from $\wp(M_2)$ to $\wp(M_1)$, i.e., we want $g : \wp(M_2) \to \wp(M_1)$.

As above, we will require that f and g be order-preserving; that they respect the equivalence classes, i.e., elements in one equivalence class must be mapped into the same equivalence class; and that f and g map fixed points to fixed points. Further, we require when $C_1 \in \wp(G_1)$ and $D_2 \in \wp(M_2)$, then $(C_1, g(D_2))$ is a formal pre-concept of $\mathcal{K}_1$ if and only if $(f(C_1), D_2)$ is a formal pre-concept of $\mathcal{K}_2$.

Thus, our category of choice which we denote by **FCI** comprises objects which are formal contexts $\mathcal{K} = (G,M,R)$ and morphisms $(f,\varphi) : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$H_1 = \varphi^{op} \circ H_2 \circ f \text{ and } K_2 = f \circ K_1 \circ \varphi^{op}$$

where $(H_1, \wp(G_1), \wp(M_1), K_1)$ and $(H_2, \wp(G_2), \wp(M_2), K_2)$ are the Galois connections determined by $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively.

We have replaced the $g : \wp(M_2) \to \wp(M_1)$ by $\varphi^{op} : \wp(M_2) \to \wp(M_1)$ where φ is a morphism in $\mathbf{Set^{op}}$.

The base category for **FCI** is $\mathbf{Set} \times \mathbf{Set^{op}}$ where the forgetful functor applied to $\mathcal{K}$, when $\mathcal{K} = (G,M,R)$, yields $(\wp(G), \wp(M))$.

3 Interchange systems

An interchange system is a recently defined concept [2] which uses a relation to relate objects in two sets. Interestingly, an interchange system is a generalization of a topological system in which the second set is a collection of properties, and the relation then matched objects of the first set with their properties in the second set, which in a topological system is a frame or locale [8]. The next definition and the following proposition are from [2].

Definition 7. *An interchange system is a triple $(X,A,\vDash)$, where $(X,A) \in |\mathbf{Set} \times \mathbf{Set}^{op}|$ and $\vDash$ is a satisfaction relation from X to A, i.e., $\vDash\ \subset X \times A$ is a relation from X to A. The set A is said to be the set of predicates. Interchange morphisms between interchange systems are ordered pairs*

$$(f,\varphi) : (X,A,\vDash_1) \to (Y,B,\vDash_2)$$

with $(f,\varphi) \in \mathbf{Set} \times \mathbf{Set}^{op}$, $f : X \to Y$ a set function, and $\varphi : A \to B$ a $\mathbf{Set}^{op}$ morphism satisfying the morphism interchange property that for all $x \in X$ and all $b \in B$,

$$f(x) \vDash_2 b \text{ if and only if } x \vDash_1 \varphi^{op}(b).$$

The category **IntSys** *comprises all interchange systems and interchange morphisms, along with the compositions and identities inherited from $\mathbf{Set} \times \mathbf{Set}^{op}$. In the above, we refer to $\mathbf{Set} \times \mathbf{Set}^{op}$ as the ground category for* **IntSys***.*

Closely associated with interchange systems and interchange morphisms are "interchange spaces" and "interchange-continuous" mappings. Given an interchange system $(X,A,\vDash)$, there is a mapping $ext : A \to \wp(X)$ defined by

$$ext(a) = \{x \in X : x \vDash a\},$$

along with the interchange space $(X, ext^{\to}(A))$.

Proposition 5. *If $(f,\varphi) : (X,A,\vDash_1) \to (Y,B,\vDash_2)$ is an* **IntSys** *morphism, then $f : (X, ext^{\to}(A)) \to (Y, ext^{\to}(B))$ has the property that*

$$\forall V \in ext^{\to}(B),\ f^{\leftarrow}(V) \in ext^{\to}(A).$$

The proposition justifies saying that the map f is interchange-continuous.

Theorem 2. *Let $(X,A,\vDash_1)$ and $(Y,B,\vDash_2)$ be interchange systems, and let $(f,\varphi) : (X,A,\vDash_1) \to (Y,B,\vDash_2)$ be an interchange morphism. $(X,A,\vDash_1)$ and $(Y,B,\vDash_2)$ are formal contexts, and if f and φ^{op} are surjective, then*

$$(f^{\to}, ((\varphi^{op})^{\to})^{op}) : (X,A,\vDash_1) \to (Y,B,\vDash_2)$$

*has all the characteristics of a morphism in **FCI**, except $f^{\to}$ and $\varphi^{op\to}$ may not take fixed points to fixed points. Hence,*

$$(f^{\Rightarrow}, ((\varphi^{op})^{\Rightarrow})^{op}) : (X,A,\vDash_1) \to (Y,B,\vDash_2)$$

*is a morphism in **FCI** when $f^{\Rightarrow} = K_2 \circ H_2 \circ f^{\to}$ and $(\varphi^{op})^{\Rightarrow} = H_1 \circ K_1 \circ (\varphi^{op})^{\to}$, where (H_1,K_1) is the Galois connection induced by $(X,A,\vDash_1)$ and (H_2,K_2) is the Galois connection induced by $(Y,B,\vDash_2)$.*

The functions f and φ^{op} need to be surjective because the relations $\vDash_1$ and $\vDash_2$ may involve elements in Y and A, respectively, which are not in the images of arbitrary functions f and φ^{op}, respectively.

Let $\mathcal{K} = (G,M,R)$ be a formal context. Define $I\mathcal{K} = (\wp(G), \wp(M), \vDash)$, where $\vDash$ is the relation from $\wp(G)$ to $\wp(M)$, i.e., $\vDash \subset \wp(G) \times \wp(M)$, such that $C \vDash D$ if and only if (C,D) is a formal pre-concept. Additionally, let $\mathcal{K}_1 = (G_1,M_1,R_1)$ and $\mathcal{K}_2 = (G_2,M_2,R_2)$ be formal contexts with $(f,\varphi) : \mathcal{K}_1 \to \mathcal{K}_2$ a formal context morphism. Then $(f,\varphi) : I\mathcal{K}_1 \to I\mathcal{K}_2$ is an interchange morphism. To show that (f,φ) is an interchange morphism, let $C_1 \in \wp(G_1)$ and $D_2 \in \wp(M_2)$. $(f(C_1),D_2)$ is a formal pre-concept in $\mathcal{K}_2$ if and only if $(C_1,\varphi^{op}(D_2))$ is a formal pre-concept in $\mathcal{K}_1$. Thus, $f(C_1) \vDash_2 D_2$ if and only if $C_1 \vDash_1 \varphi^{op}(D_2)$.

Thus, we have a functor $I : \mathbf{FCI} \to \mathbf{IntSys}$ such that

$$I(\mathcal{K}) = I\mathcal{K}$$

and

$$I((f,\varphi) : \mathcal{K}_1 \to \mathcal{K}_2) = (f,\varphi) : I\mathcal{K}_1 \to I\mathcal{K}_2.$$

In fact, we have the following.

Theorem 3. $I : \mathbf{FCI} \to \mathbf{IntSys}$ *is an embedding.*

In [2], lattice-valued extensions to interchange systems are introduced. The current work goes on to investigate analogous lattice-valued extensions of both formal contexts and concepts and the consequences of lattice-valued extensions of **FCI** for lattice-valued interchange systems and for FCA. Additionally, much of the motivation for lattice-valued interchange systems comes from predicate transformer semantics. Thus, the current work will also include possible applications of lattice-valued FCA to programming semantics.

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Multiplicative utility function and fuzzy operators

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Our starting point is the multiplicative utility function which is extensively used in the theory of multicriteria decision making. Its associativity is shown and as its generalization a fuzzy operator class is introduced with fine and useful properties. As special cases it reduces to well-known operators of fuzzy theory: min/max, product, Einstein, Hamacher, Dombi and drastic. As a consequence, we generalize the addition of velocities in Einstein's special relativity theory to multiple moving objects. Also, a new form of the Hamacher operator is given, which makes multi-argument calculations easier. We examined the De Morgan identity which connects the conjunctive and disjunctive operators by a negation. It is shown that in some special cases (min/max, drastic and Dombi) the operator class forms a De Morgan triple with any involutive negation.

The Multiplicative Utility Function

In their seminal treatment of multiattribute utility (MAU) theory, Keeney and Raiffa show how certain conditions of independence among attributes yield the so called multiplicative multiattribute utility function

$$u_M(z) = \frac{1}{k}\left(\prod_{i=1}^{n}(1+kk_iu_i(z_i)) - 1\right) \tag{1}$$

where $z = (z_1, \ldots, z_n)$, $u_i : \mathbb{R} \to [0,1]$ are utility functions, z_i are evaluations, k_i are weights of the ith criteria, and k is a scaling constant. The formula can also be expanded as

$$\begin{aligned} u_M(z) = &\sum_{i=1}^{n} k_iu_i(z_i) + k\sum_{i<j} k_ik_ju_i(z_i)u_j(z_j) + \\ &+ k^2\sum k_ik_jk_lu_i(z_i)u_j(z_j)u_l(z_l) + \ldots \\ &+ k^{n-1}k_1k_2\ldots k_nu_1(z_1)\ldots u_n(z_n). \end{aligned} \tag{2}$$

allowing also for $k = 0$.

Lemma 1. *If* $k = 0$ *then*

$$u_M(z) = \sum_{i=1}^{n} k_iu_i(z_i). \tag{3}$$

Proof. By substituting $k = 0$ into the expanded formula of u_M (2) we get the result.

The utility function is normal if $u_M(z) = 0$ whether $u_i(z_i) = 0$, and $u_M(z) = 1$ whether $u_i(z_i) = 1$ for all $i \in \{1,\dots,n\}$. A normal $u_M(z)$ implies

$$1+k = \prod_{i=1}^{n}(1+kk_i), \tag{4}$$

i.e. assuming the normality of u_M, k is determined only by the weights k_i.

The Associativity of the Multiplicative Utility Function

Let us substitute $x_i := k_i u_i(z_i)$ in the formula (1). Then the transformed multiplicative utility function is

$$u_M^*(x) = \frac{1}{k}\left(\prod_{i=1}^{n}(1+kx_i) - 1\right). \tag{5}$$

Theorem 1. *The function u_M^* is associative.*

Proof. The proof is based on the representation theorem of Aczél. It can be easily verified, that (5) can also be written in the form $F(x,y) = f^{-1}(f(x)+f(y))$, by putting

$$f(x) = \ln(1+kx), \tag{6}$$

and

$$f^{-1}(x) = \frac{1}{k}(e^x - 1). \tag{7}$$

Logical operators and the Multiplicative Utility Function

Let $g : [0,1] \to [0,\infty]$ be a generator function of a strict operator. Let

$$f(x) = \ln(1+\gamma g(x)), \tag{8}$$

and so

$$f^{-1}(x) = g^{-1}\left(\frac{1}{\gamma}e^x - 1\right). \tag{9}$$

Note, that for all $\gamma \in (0,\infty)$, f fulfills the requirements of a generator function of a strict operator. By Aczél's theorem, the associative operator $o : [0,1]^n \to [0,1]$ generated by f is

$$o(x_1,\dots,x_n) = g^{-1}\left(\frac{1}{\gamma}\left(\prod(1+\gamma g(x_i)) - 1\right)\right). \tag{10}$$

Similarly to (2), by first expanding the argument of g^{-1} to

$$\sum_{i=1}^{n} g(x_i) + \gamma \sum_{i<j} g(x_i)g(x_j) + \\ + \gamma^2 \sum g(x_i)g(x_j)g(x_l) + \dots \\ + \gamma^{n-1} g(x_1)\dots g(x_n),$$

we can put

$$o(x_1,\dots,x_n)|_{\gamma=0} = g^{-1}\left(\sum g(x_i)\right), \tag{11}$$

thus the case $\gamma = 0$ also results in a strict operator. Next, we will show that different types of operators fit into the framework depending on the choice of function f. From now on, let us assume

$$g(x) = \left(\frac{1-x}{x}\right)^{\alpha},$$

the generator function of the Dombi operator.

The Generalized Dombi operator

Definition 1. *The generator functions of the Generalized Dombi operator are*

$$f_c(x) = \ln\left(1 + \gamma_c \left(\frac{1-x}{x}\right)^{\alpha}\right), \qquad \alpha > 0 \tag{12}$$

$$f_d(x) = \ln\left(1 + \gamma_d \left(\frac{1-x}{x}\right)^{\alpha}\right), \qquad \alpha < 0 \tag{13}$$

where $\gamma_c, \gamma_d \in [0,\infty]$. *From*

$$c(x) = f_c^{-1}\left(\sum_{i=1}^{n} f_c(x_i)\right),$$

$$d(x) = f_d^{-1}\left(\sum_{i=1}^{n} f_d(x_i)\right),$$

and

$$f_c^{-1}(x) = \frac{1}{1+\left(\frac{1}{\gamma_c}\left(e^x - 1\right)\right)^{1/\alpha}}, \qquad \alpha > 0 \tag{14}$$

$$f_d^{-1}(x) = \frac{1}{1+\left(\frac{1}{\gamma_d}\left(e^x - 1\right)\right)^{1/\alpha}}, \qquad \alpha < 0 \tag{15}$$

the operators are

$$c^{(\alpha)}_{GD,\gamma_c}(x) = \frac{1}{1+D_{\gamma_c}(x)}, \qquad \alpha > 0 \tag{16}$$

$$d^{(\alpha)}_{GD,\gamma_d}(x) = \frac{1}{1+D_{\gamma_d}(x)}, \qquad \alpha < 0 \tag{17}$$

where $\gamma_c, \gamma_d \in [0, \infty]$ *and*

$$D_\gamma(x) = \left(\frac{1}{\gamma} \left(\prod_{i=1}^{n} \left(1 + \gamma \left(\frac{1 - x_i}{x_i} \right)^\alpha \right) - 1 \right) \right)^{1/\alpha}. \tag{18}$$

Equations (16) and (17) differ only in the sign of α and so the Generalized Dombi operator class is:

$$o_{GD,\gamma}^{(\alpha)}(x) = \frac{1}{1 + \left(\frac{1}{\gamma} \left(\prod_{i=1}^{n} \left(1 + \gamma \left(\frac{1-x_i}{x_i} \right)^\alpha \right) - 1 \right) \right)^{1/\alpha}} \tag{19}$$

In the forthcoming sections, we will show that $o_{GD,\gamma}^{(\alpha)}$ is a strict operator for $\alpha \in (-\infty, \infty)$ and $\gamma \in (0, \infty)$.

The Dombi operator case

The Dombi operator has the form

$$o_D^{(\alpha)}(x) = \frac{1}{1 + \left(\sum_{i=1}^{n} \left(\frac{1-x_i}{x_i} \right)^\alpha \right)^{1/\alpha}} \tag{20}$$

and if $\alpha > 0$ then the operator is conjunctive and if $\alpha < 0$ then the operator is disjunctive. The next corollary follows from lemma 1, by the substitution $k = \gamma$.

Corollary 1. *The Dombi operator is a special case of the Generalized Dombi operator, i.e. if* $\gamma_c = \gamma_d = 0$ *then*

$$c_{GD,0}^{(\alpha)}(x) = c_D^{(\alpha)}(x), \tag{21}$$

$$d_{GD,0}^{(\alpha)}(x) = d_D^{(\alpha)}(x). \tag{22}$$

Conclusions

In this lecture we have

1. proved the associativity of the multiplicative utility function,
2. introduced the generalized operator:

$$\frac{1}{1 + \left(\frac{1}{\gamma} \left(\prod_{i=1}^{n} \left(1 + \gamma \left(\frac{1-x_i}{x_i} \right)^\alpha \right) - 1 \right) \right)^{1/\alpha}}$$

3. presented new forms of rational involutive negations:

$$n_{\nu_*}(x) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\left(\frac{1-x}{x}\right)^{-1}}$$
$$n_{\nu,\nu_0}(x) = \frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}}$$

4. proved that the new operator connectives form a De Morgan triple with a negation iff
$$\frac{\gamma_d}{\gamma_c} = \left(\frac{1-\nu_0}{\nu_0}\cdot\frac{1-\nu}{\nu}\right)^{\alpha}$$
5. proved that the Dombi operators form a De Morgan triple with any rational involutive negation
6. showed that the generalized operator has the following limits

Type of operator	Value of γ	Value of α	
		conj.	disj.
Dombi	0	$0<\alpha$	$\alpha<0$
Product	1	1	-1
Einstein	2	1	-1
Hamacher	$\gamma\in(0,\infty)$	1	-1
Drastic	∞	$0<\alpha$	$\alpha<0$
Min-max	0	∞	$-\infty$

7. introduced new forms of the Hamacher operators

$$o_H^{(\alpha)}(x) = \frac{1}{1+\left(\frac{1}{\gamma_d}\left(\prod_{i=1}^n\left(1+\gamma_d\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}}$$

8. presented new forms of the Einstein operators

$$o_{GD,2}^{(\alpha)}(x) = \frac{1}{1+2\left(\prod_{i=1}^n\left(1+2\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)^{1/\alpha}}$$

9. showed that the addition of several velocities in the framework of special relativity is:
$$v = \frac{c}{1+2\left(\prod_{i=1}^n\left(1+2\frac{v_i}{c-v_i}\right)-1\right)^{-1}}.$$

This new parametric operator family has some useful applications. The two parameters offer more freedom in the sense that by adopting two, instead of just one parameter, the operator can be made to fit the problem in question better. Because we have two parameters to play with instead of one.

Lexicographic refinements of fuzzy measures, Sugeno integrals and qualitative bipolar decision criteria

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In decision applications, especially multicriteria decision-making, numerical approaches are often questionable because it is hard to elicit numerical values quantifying preference, criteria importance or uncertainty. More often than not, multicriteria decision-making methods come down to number-crunching recipes with debatable foundations. One way out of this difficulty is to adopt a qualitative approach where only maximum and minimum operations are used. Such methods enjoy a property of scale invariance that insures their robustness. One of the most sophisticated aggregation operation making sense on qualitative scales is Sugeno integral. It is qualitative, hence robust to elicitation, and it assumes commensurability between preference intensity and criteria importance or similarly, utility and uncertainty. However, since absolute qualitative value scales must have few levels so as to remain cognitively plausible, there are not more classes of equivalent decisions than value levels in the scale. Hence this approach suffers from a lack of discrimination power. In particular, qualitative aggregations such as Sugeno integrals cannot be strictly increasing and violate the strict Pareto property.

In this talk, we report results obtained when trying to increase the discrimination power of Sugeno integrals, generalizing known refinements of the minimum and maximum such as leximin and leximax. The representation of leximin and leximax by sums of numbers of different orders of magnitude (forming a super-increasing sequence) can be generalized to weighted max and min (yielding a "big-stepped" weighted average) and Sugeno integral (yielding a "big-stepped" Choquet integral). This methodology also requires the fuzzy measure (monotonic set-function) involved to be lexicographically refined. We show this is possible by means of qualitative Moebius transforms introduced by Michel Grabisch. Such refined fuzzy measures can be represented by numerical set-functions, and we show they can always be belief or plausibility functions in the sense of Shafer. Lexicographic refinements can also be applied to the case of bipolar bivariate evaluations, thus bridging the gap with cumulative prospect theory.

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Ordinal sums and shuffles of copulas

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1 Introduction

The investigation about the dependence among random variables is one of the main topic in applied probability and statistics. In fact, properties of the joint probability law of a random vector $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ could be crucial interest when one wants to predict the behaviour of multivariate systems.

Nowadays, *copulas* represent the building block of the modern theory of multivariate distributions. In fact, it has been recognized by *Sklar's Theorem* [10] that any multivariate distribution functions associated with a random vector $\mathbf{X}$ can be constructed and fitted (to some available data) by means of a two-step procedure: first, the marginals are chosen; then, the dependence is modelled by means of a suitable copula. For more details, see for instance [4].

This decomposition allows practitioners to match any set of individual distributions to a specified dependence structure. Hence, for a given set of random variables, different dependency structures can be imposed on the variables by specifying different copulas. For instance, copulas having tail dependence can be applied to capture the observation that large losses from different risk types tend to strike simultaneously during stress situations.

Especially in a financial context, the selection of a suitable copula associated to a multivariate stochastic model (representing, e.g., a market/credit portfolio) is essential to derive some relevant quantities of the model (like *value-at-risk*), which could affect the possible choices of a risk-manager.

In order to provide a variety of examples of copulas to be used in practice, a number of families and constructions have been developed during the years (see [3] and the references therein). However, most of these constructions are of analytical nature and do not have a genuine probabilistic interpretation.

In this talk, we will revisit two constructions of copulas already known, namely *ordinal sums* and *shuffles*, by presenting their possible stochastic representation.

The concept of ordinal sum was introduced in the algebraic framework of semigroups and, hence, it was used in the theory of triangular norms (see [9, 5] and the references therein). It applies equally well to bivariate copulas and has been recently extended to higher-dimensional copulas in [6].

The concept of shuffles of copulas was introduced by [8], restricting to the case when the copula coincides with the comonotonicity copula M_d (for an extension, see also [7]). It is grounded on the fact that one can generate new copulas by means of a suitable rearrangement of the mass distribution of a given starting copula. Recently, a measure-theoretic interpretation of shuffles has been presented in [2].

By using the new approach by [1], we will present a new method for constructing copulas that encompasses both the multivariate shuffles and the ordinal sum construction. Such a method is also used in order to provide an approximation of any copulas. In order to reach our goal, we rely on measure-theoretic techniques that are grounded on the well-known one-to-one correspondence between copulas and special probability measures.

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A discrete Choquet integral for ordered systems

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1 Introduction

We focus on Choquet integrals with respect to a finite universe N.

While the classical approach almost always assumes the family of measurable subsets of N to form an algebra, many practical situations (*e.g.*, cooperative games, multicriteria decision making) require a more general setting with only the members of a certain subfamily $\mathcal{F} \subseteq 2^N$ being feasible and no particular "nice" algebraic structure apparent.

In such a general situation, the classical definition of the Choquet integral is no longer easily utilizable: Many functions become non-measurable in the sense that their level sets do not necessarily belong to the family $\mathcal{F}$.

It is the purpose of the present paper, to extend the notion of a Choquet integral to arbitrary families $\mathcal{F}$ of subsets in such a way that functions can be integrated with respect to general set functions (and capacities being a particular case). To do so, we consider $\mathcal{F}$ as an ordered system (whose order relation may arise from a particular application model under consideration).

2 Fundamental notions

An *ordered system* is a pair $(\mathcal{F}, \preceq)$, where $\mathcal{F}$ is a family of non-empty subsets of some set N with $n := |N| < \infty$ that covers all elements of N, *i.e.*,

$$\bigcup_{F \in \mathcal{F}} F = N,$$

(partially) ordered by the precedence relation $\preceq$ (e.g., set inclusion). We set $m := |\mathcal{F}|$ and, for notational convenience, arrange (index) the members of $\mathcal{F} = \{F_1, \ldots, F_m\}$ in a monotonically decreasing fashion, *i.e.*, such that

$$F_i \succeq F_j \quad \Longrightarrow \quad i \leq j \quad (1 \leq i, j \leq m). \tag{1}$$

A *valuation* on $\mathcal{F}$ is a function $v : \mathcal{F} \to \mathbb{R}$. Setting $\mathcal{F}_0 := \mathcal{F} \cup \{\emptyset\}$ and $v(\emptyset) := 0$, valuations are usually called *games* defined on a subfamily of 2^N. If in addition v is non-negative and *isotone* (or *monotone*) w.r.t. $\preceq$ (*i.e.*, $v(F) \leq v(G)$ whenever $F \preceq G$), we call v a *capacity* or a *fuzzy measure*.

Given a valuation v, we define its Möbius transform β as the unique solution of the system

$$v(F) = \sum_{G \preceq F, G \in \mathcal{F}} \beta_G, \forall F \in \mathcal{F}.$$

If $m \geq 0$, then v is called a *belief function*. For any $F \in \mathcal{F}$, we define the *unanimity game* (or *simple (belief) function* ζ^F by $\zeta^F(G) = 1$ iff $G \succeq F$, and 0 otherwise. We have for any valuation $v = \sum_{F \in \mathcal{F}} \beta_F \zeta^F$. Hence we associate to v the belief functions

$$v^+ := \sum_{\beta_F \geq 0} \beta_F \zeta^F \quad \text{and} \quad v^- := \sum_{\beta_F \leq 0} (-\beta_F) \zeta^F$$

and thus obtain the natural representation $v = v^+ - v^-$.

Assume $(\mathcal{F}_0, \preceq) = (2^N, \subseteq)$ and let $v : \mathcal{F}_0 \to \mathbb{R}$ be a game. For any non-negative vector $f \in \mathbb{R}^n_+$, the *(classical) Choquet integral* [1] w.r.t. v is defined by

$$\int f\, dv := \int_0^\infty v(\{i \in N \mid f_i \geq \alpha\}) d\alpha. \tag{2}$$

It is well known that, using the Möbius transform β,

$$\int f\, dv = \sum_{F \subseteq N} \beta_F \min_{i \in F} f_i. \tag{3}$$

Proposition 1. *(Lovász, 1983) The functional $f \mapsto \int f\, dv$ is concave if and only if v is supermodular, i.e., if v satisfies the inequality $v(F \cup G) + v(F \cap G) \geq v(F) + v(G)$, for all $F, G \subseteq N$.*

3 Integrals

We now construct the discrete Choquet integral for an ordered system $(\mathcal{F}, \preceq)$ in several steps and first consider belief functions. An *upper integral* for the belief function v is a functional $[v] : \mathbb{R}^N \to \mathbb{R}_+$ such that

(i) $[v](\lambda f) = \lambda [v](f) \geq 0$ for all scalars $\lambda \geq 0$.
(ii) $[v](f + g) \geq [v](f) + [v](g)$ for all $f, g \in \mathbb{R}^N_+$.
(iii) $[v](\mathbf{1}_F) \geq v(F)$ for all $F \in \mathcal{F}$.

The key observation is that the class of upper integrals of v possesses a unique lower envelope v^*.

Lemma 1. *For any belief function v, there is a unique upper integral v^* that provides a lower bound for all upper integrals $[v]$ in the sense $v^*(f) \leq [v](f)$ for all $f \in \mathbb{R}^N_+$. Moreover, one has*

$$v^*(f) = \max\Big\{ \langle v, y \rangle \mid y \in \mathbb{R}^{\mathcal{F}}_+, \sum_{F \in \mathcal{F}} y_F \mathbf{1}_F \leq f \Big\}.$$

The same approach has been taken by Lehrer, who calls it the *concave integral* [3, 4], with the difference that $\mathcal{F} = 2^N$ and that v can be any capacity. We call the upper integral v^* established in Lemma 1 the *Choquet integral* of the belief function v and henceforth use the notation

$$\int_{\mathcal{F}} f\, dv := v^*(f).$$

We extend the Choquet integral to arbitrary valuations v via

$$\int_{\mathcal{F}} f\, dv := \int_{\mathcal{F}} f\, dv^+ - \int_{\mathcal{F}} f\, dv^- \quad \text{for all } f \in \mathbb{R}^N_+.$$

Note that the Choquet integral is positively homogeneous for any valuation.

We now present a heuristic algorithm for the computation of the Choquet integral relative to the ordered system $(\mathcal{F}, \preceq)$, which generalizes the well-known *north-west corner rule* for the solution of assignment problems. As usual, we denote the empty string by $\Box$. Also, we set $\mathcal{F}(X) := \{F \in \mathcal{F} \mid F \subseteq X\}$ for all $X \subseteq N$.

Given the non-negative weighting $f \in \mathbb{R}^N_+$, consider the following procedure (Monge Algorithm (MA)):

(M0) Initialize: $X \leftarrow N$, $\mathcal{M} \leftarrow \emptyset$, $c \leftarrow f$, $y \leftarrow 0$, $\pi \leftarrow \Box$;
(M1) Let $M = F_i \in \mathcal{F}(X)$ be the set with minimal index i and choose an element $p \in M$ of minimal weight $c_p = \min_{j \in M} c_j$;
(M2) Update: $X \leftarrow X \setminus \{p\}$, $\mathcal{M} \leftarrow \mathcal{M} \cup \{M\}$, $y_M \leftarrow c_p$, $c \leftarrow (c - c_p \mathbf{1}_M)$, $\pi \leftarrow (\pi p)$;
(M3) If $\mathcal{F}(X) = \emptyset$, Stop and Output $(\mathcal{M}, y, \pi)$. Else goto (M1);

Given any valuation v, associate with the output (y, π) of MA the quantity

$$[f](v) := \langle v, y \rangle = \sum_{F \in \mathcal{F}} y_F v(F).$$

Since (y, π) does not depend on v, it is clear that $v \mapsto [f](v)$ is a linear functional on the set of valuations.

Theorem 1. *Assume y is the output of MA for f. The following are equivalent:*

(a) $\langle y, \zeta^F \rangle = \int_{\mathcal{F}} f\, d\zeta^F$ *for all* $F \in \mathcal{F}$.
(b) $\langle y, v \rangle = \int_{\mathcal{F}} f\, dv$ *for all set functions* v.

Corollary 1. *Assume that the Monge algorithm computes the Choquet integral for all simple functions ζ^F. Then we have*

$$\int_{\mathcal{F}} f\, dv = \sum_{F \in \mathcal{F}} \beta(F) \int_{\mathcal{F}} f\, d\zeta^F \quad \textit{for all set functions } v = \sum_{F \in \mathcal{F}} \beta(F) \zeta^F.$$

4 Ordering by containment

We investigate in this section systems under the set-theoretic containment order relation $\subseteq$ and consider the system $(\mathcal{F}, \subseteq)$.

Lemma 2. *Let* $(\mathcal{F},\subseteq)$ *be arbitrary and* $f : N \to \mathbb{R}_+$*. Then for any* $F \in \mathcal{F}$*,*

$$\int f\,d\zeta^F = \min_{j\in F} f_j.$$

Assume that $\mathcal{F}$ is *weakly union-closed* in the sense $F \cap G \neq \emptyset$ implies $F \cup G \in \mathcal{F}$ for all $F, G \in \mathcal{F}$.

Theorem 2. *Let* $(\mathcal{F},\subseteq)$ *be a weakly union-closed system. For all* $f \in \mathbb{R}^n_+$*, all valuations* v *with* $v = \sum_{F\in\mathcal{F}} \beta(F)\zeta^F$*, we have*

$$\langle y, v\rangle = \int_{\mathcal{F}} f\,dv = \sum_{F\in\mathcal{F}} \beta_F \int f\,d\zeta^F = \sum_{F\in\mathcal{F}} \beta_F \min_{i\in F} f_i.$$

Corollary 2. *Let* $(\mathcal{F},\subseteq)$ *be weakly union-closed and* $f \in \mathbb{R}^N_+$*. Then*

$$\int_{\mathcal{F}} f\,dv = \int f\,d\hat{v} \quad \textit{holds for all valuations } v,$$

and $\hat{v}$ *is determined by* $\hat{v}(S) = \int_{\mathcal{F}} \mathbf{1}_S\,dv = \sum_{F \text{ maximal in } \mathcal{F}(S)} v(F), \quad \forall S \in 2^N.$

Remark 1. (i) Corollary 2 shows that the Choquet integral on a weakly union-closed family essentially equals the classical Choquet integral, and therefore inherits all its properties (in particular, comonotonic additivity).

(ii) It extends the classical Choquet integral in the sense that if f is $\mathcal{F}$-measurable then $\int_{\mathcal{F}} f\,dv = \int f\,dv$.

(iii) A capacity v on $(\mathcal{F},\subseteq)$ may not yield $\hat{v}$ as a capacity on $(2^N,\subseteq)$. Therefore, the Choquet integral is not necessarily monotone if v is a capacity.

From Proposition 1, we immediately see:

Corollary 3. *Let* $(\mathcal{F},\subseteq)$ *be weakly union-closed and* v *an arbitrary valuation. Then the following are equivalent:*

(i) The operator $f \mapsto \int_{\mathcal{F}} f\,dv$ *is superadditive on* $\mathbb{R}^N_+$*.*
(ii) The extension $\hat{v} : 2^N \to \mathbb{R}$ *of* v *is supermodular.*

An *algebra* is a collection $\mathcal{A}$ of subsets of N that is closed under set union and set complementation with $\emptyset, N \in \mathcal{A}$. In particular, $\mathcal{F} = \mathcal{A} \setminus \{\emptyset\}$ is a weakly union-closed family. Lehrer [2] (see also Teper [5]) has introduced a discrete integral relative to the algebra $\mathcal{A}$ as follows. Given a probability measure P on $\mathcal{A}$ and a non-negative function $f \in \mathbb{R}^N_+$, define

$$\int_{\mathcal{L}} f\,dP_{\mathcal{A}} := \sup_{\lambda\geq 0}\Big\{\sum_{S\in\mathcal{A}} \lambda_S P(S) \mid \sum_{S\in\mathcal{A}} \lambda_S \mathbf{1}_S \leq f\Big\}.$$

Lehrer shows that the functional $f \mapsto \int_{\mathcal{L}} f\,dP_{\mathcal{A}}$ is a concave operator on $\mathbb{R}^N_+$. Let us exhibit Lehrer's integral as a special case of our general Choquet integral.

Proposition 2. *Let* $\mathcal{A}$ *be an algebra and* P *a probability measure on* $\mathcal{A}$*. Setting* $\mathcal{F} = \mathcal{A} \setminus \{\emptyset\}$*, one then has*

$$\int_{\mathcal{L}} f\,dP_{\mathcal{A}} = \int_{\mathcal{F}} f\,dP \quad \textit{for all } f \in \mathbb{R}^N_+.$$

In particular, Lehrer's integral can be computed with the Monge algorithm.

5 Supermodularity and superadditivity

Theorem 3. *(Generalization of Lovász' result (Proposition 1)) Assume that* $\mathcal{F}$ *is union-closed and* v *a capacity on* $(\mathcal{F}, \subseteq)$. *Then the following statements are equivalent:*

(i) $\int_{\mathcal{F}} f\,dv = \max\left\{\langle v,y\rangle \mid y \in \mathbb{R}_+^{\mathcal{F}}, \sum_{F\in\mathcal{F}} y_F \mathbf{1}_F \leq f\right\}$ *for all* $f \in \mathbb{R}_+^n$.

(ii) The functional $f \mapsto \int_{\mathcal{F}} f\,dv$ *is superadditive on* $\mathbb{R}_+^N$.

(iii) v *is supermodular.*

Corollary 4. *Let* $\mathcal{F}$ *be a union-closed and* v *a capacity with extension* $\hat{v}$ *on* $(\mathcal{F}, \subseteq)$. *Then the following statements are equivalent:*

(i) $v : \mathcal{F} \to \mathbb{R}$ *is supermodular on* $(\mathcal{F}, \subseteq)$.

(ii) $\hat{v} : 2^N \to \mathbb{R}$ *is supermodular on* $(2^N, \subseteq)$.

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Local and relativized local finiteness in t-norm-based structures

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1 Preliminaries

An algebraic structure is *locally finite* iff each of its finite subset generates a finite subalgebra only.

In this paper we look at this property of local finiteness for t-norm based structures. By $T_{\mathsf{L}}, T_{\mathsf{P}}, T_{\mathsf{G}}$ we denote the basic t-norms, i.e. the Łukasiewicz, the product, and the Gödel t-norm, respectively. Furthermore $I_{\mathsf{L}}, I_{\mathsf{P}}, I_{\mathsf{G}}$ shall be their residuation operations, and $N_{\mathsf{L}}, N_{\mathsf{P}}, N_{\mathsf{G}}$ the corresponding standard negation functions defined via $N_\alpha(x) = I_\alpha(x,0)$ in all these cases (and yielding $N_{\mathsf{P}} = N_{\mathsf{G}}$).

Generally, given a left continuous t-norm T and its residuation operation I_T, we denote by N_T the corresponding standard negation function given as $N_T(x) = I_T(x,0)$. And S_T shall be the t-conorm related to the t-norm T in the standard way.

2 Results for t-norm-monoids

Proposition 1. *The Gödel monoid* $([0,1], T_{\mathsf{G}}, 1)$ *is locally finite, and so is its negation-extended version* $([0,1], T_{\mathsf{G}}, N_{\mathsf{G}}, 1)$.

Proposition 2. *The product monoid* $([0,1], T_{\mathsf{P}}, 1)$ *is not locally finite, and so is its negation-extended version* $([0,1], T_{\mathsf{P}}, N_{\mathsf{P}}, 1)$.

Proof: Any $a \in (0,1)$ generates an infinite submonoid of $([0,1], T_{\mathsf{P}}, 1)$.

Proposition 3. *The Łukasiewicz monoid* $([0,1], T_{\mathsf{L}}, 1)$ *is locally finite.*

Theorem 1. *A t-norm monoid* $([0,1], T, 1)$ *with a continuous t-norm* T *is locally finite if and only if* T *does only have locally finite summands in its representation as ordinal sum of archimedean summands.*

Corollary 1. *A t-norm monoid* $([0,1], T, 1)$ *with a continuous t-norm* T *is locally finite if and only if* T *does not have a product-norm isomorphic summand in its representation as ordinal sum of archimedean summands.*

Proposition 4. *If a continous t-norm* T *has a product-isomorphic summand in its ordinal sum representation then any extension of the t-norm monoid* $([0,1], T, 1)$ *is not locally finite.*

Proposition 5. *The t-norm monoid* $([0,1], T_{\mathsf{nM}}, 1)$ *based upon the nilpotent minimum*

$$T_{\mathsf{nM}}(x,y) = \begin{cases} \min\{x,y\}, & \text{if } u+v>1 \\ 0 & \text{otherwise} \end{cases}$$

is locally finite.

3 Results for extended t-norm-monoids

Proposition 6. *The residuation-extended Gödel monoid* $([0,1], T_{\mathsf{G}}, I_{\mathsf{G}}, 1)$ *is locally finite.*

Proposition 7. *The residuation-extended product monoid* $([0,1], T_{\mathsf{P}}, I_{\mathsf{P}}, 1)$ *is not locally finite.*

Proposition 8. *The residuation-extended Łukasiewicz monoid* $([0,1], T_{\mathsf{L}}, I_{\mathsf{L}}, 1)$ *is not locally finite.*

Theorem 2. *A residuation-extended t-norm-monoid* $([0,1], T, I_T, 1)$ *with a continuous t-norm* T *is locally finite if and only if* T *does only have locally finite summands in its representation as ordinal sum of archimedean summands.*

This theorem immediately yields the following corollary.

Corollary 2. *A residuation-extended t-norm-monoid* $([0,1], T, I_T, 1)$ *with a continuous t-norm* T *is locally finite if and only if it is based upon the Gödel monoid, i.e. iff* $T = T_{\mathsf{G}}$.

Proposition 9. *The negation-extended Łukasiewicz monoid* $([0,1], T_{\mathsf{L}}, N_{\mathsf{L}}, 1)$ *is not locally finite.*

The problem here really comes from the irrational numbers.

Proposition 10. *The negation-extended rational Łukasiewicz monoid* $([0,1]\cap\mathbb{Q}, T_{\mathsf{L}}, N_{\mathsf{L}}, 1)$ *is locally finite.*

These results can also be extended to the corresponding residuated lattices.

Proposition 11. *The Gödel-algebra* $([0,1], \wedge, \vee, T_{\mathsf{G}}, I_{\mathsf{G}}, 0)$ *is locally finite.*

Proposition 12. *The product-algebra* $([0,1], \wedge, \vee, T_{\mathsf{P}}, I_{\mathsf{P}}, 0)$ *is not locally finite.*

Proposition 13. *The Łukasiewicz-algebra* $([0,1], \wedge, \vee, T_{\mathsf{L}}, I_{\mathsf{L}}, 0)$ *is not locally finite.*

4 Results for t-norm-bimonoids

Proposition 14. *A t-conorm monoid* $([0,1],S_T,0)$ *is locally finite iff its corresponding t-norm monoid* $([0,1],T,1)$ *is.*

We are also interested in the t-norm based bimonoids $([0,1],T,S_T,1,0)$. In general, a bimonoid is an algebraic structure $\mathfrak{A} = (A,*_1,*_2,e_1,e_2)$ such that both $(A,*_1,e_1)$ and $(A,*_2,e_2)$ are monoids.

Proposition 15. *The Gödel-bimonoid* $([0,1],T_{\mathsf{G}},S_{\mathsf{G}},1,0)$ *is locally finite.*

Proposition 16. *The product-bimonoid* $([0,1],T_{\mathsf{P}},S_{\mathsf{P}},1,0)$ *is not locally finite.*

Proposition 17. *The Łukasiewicz-bimonoid* $([0,1],T_{\mathsf{L}},S_{\mathsf{L}},1,0)$ *is not locally finite.*

Proposition 18. *The rational Łukasiewicz-bimonoid* $([0,1]\cap\mathbb{Q},T_{\mathsf{L}},S_{\mathsf{L}},1,0)$ *is locally finite.*

Theorem 3. *Suppose that* T *is a continuous t-norm with ordinal sum representation* $T = \sum_{i\in I}([l_i,r_i],T_i,\varphi_i)$ *without product-isomorphic summands. Assume furthermore that for each Łukasiewicz summand* $([l_k,r_k],T_{\mathsf{L}},\varphi_k)$ *the interval* $[1-r_k,1-l_k]$ *does not overlap with any domain interval* $[l_i,r_i]$ *for a Łukasiewicz summand* $([l_i,r_i],T_{\mathsf{L}},\varphi_i)$*,* $i\in I$*. Then the t-norm bimonoid* $([0,1],T,S_T,1,0)$ *is locally finite.*

Example: The t-norm bimonoid $([0,1],T^*,S_{T^*},1,0)$ with the continuous t-norm

$$T^* = \sum_{i\in\{1\}}([\frac{1}{2},1],T_{\mathsf{L}},\varphi^*) \tag{1}$$

and the order isomorphism $\varphi^* : [\frac{1}{2},1] \to [0,1]$ given by $\varphi^*(x) = 2x-1$ is locally finite.

By the way, the particular choice of the order isomorphism φ^* is unimportant here.

Proposition 19. *The* T_{nM}*-bimonoid, based upon the nilpotent minimum* T_{nM}*, is locally finite.*

5 Relativized local finiteness

Definition 1. *A t-norm based algebraic structure* $\mathfrak{A}$ *over the unit interval is rationally locally finite iff each finite set* $G\subseteq[0,1]\cap\mathbb{Q}$ *generates only a finite substructure of* $\mathfrak{A}$*.*

Proposition 20. *A t-norm monoid* $([0,1],T,1)$ *is rationally locally finite iff its corresponding t-conorm monoid* $([0,1],S_T,0)$ *is rationally locally finite.*

Proposition 21. *Suppose that* T *is a continuous t-norm with an ordinal sum representation* $T = \sum_{i\in I}([l_i,r_i],T_{\mathsf{L}},\varphi_i)$ *which has only Łukasiewicz-isomorphic summands. If the order isomorphisms in the* T*-summands map rationals to rationals then the residuation-extended t-norm-monoid* $([0,1],T,I_T,1)$ *is rationally locally finite.*

Corollary 3. *Suppose that T is a continuous t-norm with an ordinal sum representation $T = \sum_{i \in I}([l_i, r_i], T_{\text{L}}, \varphi_i)$ which has only Łukasiewicz-isomorphic summands. If the order isomorphisms in the T-summands are rational functions then the residuation-extended t-norm-monoid $([0,1], T, I_T, 1)$ is rationally locally finite.*

It is a routine matter to include the lattice structure of $[0,1]$ into these considerations.

Proposition 22. *The Łukasiewicz-algebra $([0,1], \wedge, \vee, T_{\text{L}}, I_{\text{L}}, 0)$ is rationally locally finite.*

Proposition 23. *Suppose that T is a continuous t-norm with an ordinal sum representation $T = \sum_{i \in I}([l_i, r_i], T_{\text{L}}, \varphi_i)$ which has only Łukasiewicz-isomorphic summands. If all the order isomorphisms in the T-summands are rational functions then the t-algebra $([0,1], \wedge, \vee, T, I_T, 0)$ rationally locally finite.*

Remark: This notion of rational local finiteness is, of course, only a particular case of a more general notion of relative local finiteness which might be defined in the following way.

Definition 2. *Let $\mathfrak{A}$ be an algebraic structure and $M \subseteq |\mathfrak{A}|$. Then $\mathfrak{A}$ is M-*locally finite *iff for each finite $G \subseteq M$ one has that the substructure $\langle G \rangle_{\mathfrak{A}}$ has a finite carrier.*

Actually it is not clear what will be the importance of this more general notion. However, it seems particularly with respect to computer science topics that the particular case $M = \mathbb{Q}$, i.e. the case of rational local finiteness, might be the most important one: internally all numbers used in a computer are rational ones.

It is an obvious fact that for $M_1 \subseteq M_2$ the M_2-local finiteness of an algebraic structure $\mathfrak{A}$ implies its M_1-local finiteness.

Obviously, this relativized local finiteness is also transferred back and forth between t-norm based and t-conorm based monoids.

Proposition 24. *A t-norm monoid $([0,1], T, 1)$ is M-locally finite iff its corresponding t-conorm monoid $([0,1], S_T, 0)$ is $(1-M)$-locally finite.*

Generalizing again the Choquet integral: the profile dependent Choquet integral

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In decision analysis, and especially in multiple criteria decision analysis (for an updated survey see [9]), several non additive integrals have been introduced [11, 12]. Among them we remember the Choquet integral [6], the Sugeno integral [27], the Shilkret integral [26]. Recently the bipolar Choquet integral [13, 14] (see also [17]), the level dependent Choquet integral [16], the level dependent Sugeno [23], the level dependent Shilkret integral [4] and the bipolar level dependent Choquet [16] integral have been introduced. Very recently, on the basis of a minimal set of axioms, one concept of universal integral giving a common framework to many of the above integrals have been proposed [20, 21]. In the same line, in this paper we try to provide a generalization of one of the above integrals, the Choquet integral, in order to find the above integrals and other aggregation functions as its special cases, at least under some specific conditions. In fact, one of above integrals, the level dependent Choquet integral, already has very interesting good properties in this sense, because it contains as particular cases the Choquet integral and the Sugeno integral [17]. The idea of the level dependent Choquet integral is to consider a capacity that depends also on the level of evaluations to be aggregated. The further generalization of the Choquet integral that we propose in this paper, the profile dependent Choquet integral, extends this idea considering a capacity which depends on the whole vector of evaluations to be aggregated.

After remembering the main aggregation functions and non additive integrals already introduced in the literature, and after introducing some others new non addtive integrals and aggregation functions (the bipolar Sugeno integral, the bipolar Shilkret integral, the bipolar level dependent Sugeno integral, the bipolar level dependent Shilkret integral, the bipolar cumulative utility) we introduce and give a characterization of the profile dependent Choquet integral. We show also how it can be used to represent other aggregation functions and non additive integrals. Some results related to some aggregation functions and non additive integrals, either already knew or introduced in this paper, have an autonomous interest. More in detail, to the best of our knowledge, the following results are original:

- the characterization of level dependent Choquet integral integral and cumulative utility in terms of comonotone modularity;
- the characterization of bipolar level dependent Choquet integral integral in terms of bipolar cardinal tail independence;
- the characterization of bipolar level dependent Choquet integral integral and bipolar cumulative utility in terms of bipolar comonotone modularity;

– the characterization of level dependent Shilkret integral, bipolar Sugeno integral, bipolar Shilkret integral, bipolar level dependent Sugeno integral and bipolar level dependent Shilkret integral;
– the representation of bipolar Sugeno integral in terms of level dependent bipolar Choquet integral.

Let us consider a set of criteria $N = \{1,\ldots,n\}$. In general an aggregation function is a function $G : (\alpha,\beta)^n \to (\alpha,\beta)$, $(\alpha,\beta) \subseteq \mathbf{R}$, where (α,β) means one of the intervals

$$[\alpha,\beta],]\alpha,\beta],[\alpha,\beta[,]\alpha,\beta[,$$

and possibly also

$$]-\infty,\beta[,]-\infty,\beta],[\alpha,+\infty[,]\alpha,+\infty[,]-\infty,+\infty[,$$

such that

1. $G(\alpha,\ldots,\alpha)=\alpha$ if $\alpha \in (\alpha,\beta)$ and $lim_{x\to\alpha^+} G(x,\ldots,x)=\alpha$ if $\alpha \notin (\alpha,\beta)$, and $G(\beta,\ldots,\beta)=\beta$ if $\beta \in (\alpha,\beta)$ and $lim_{x\to\beta^-} G(x,\ldots,x)=\beta$ if $\beta \notin (\alpha,\beta)$,
2. for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$,

$$\mathbf{x} \geq \mathbf{y} \Rightarrow G(\mathbf{x}) \geq G(\mathbf{y}).$$

The following properties of an aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$, $(\alpha,\beta) \subseteq \mathbf{R}$, are useful to characterize some of the aggregation functions we shall discuss in the following:

– idempotency: for all $\mathbf{a} \in (\alpha,\beta)^n$ such that $\mathbf{a} = [a,\ldots,a]$, $G(\mathbf{a}) = a$;
– homogeneity: for all $\mathbf{x} \in (\alpha,\beta)^n$ and $c > 0$ such that $c\mathbf{x} \in (\alpha,\beta)^n$, $G(c\mathbf{x}) = cG(\mathbf{x})$;
– stable for minimum: for all $\mathbf{x} \in (\alpha,\beta)^n$ and $c \in (\alpha,\beta)$, $G(\mathbf{x} \wedge [c,\ldots,c]) = \min(G(\mathbf{x}), c)$, where, for any $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, $\mathbf{x} \wedge \mathbf{y} = \mathbf{z}$ with $z_i = \min(x_i,y_i), i = 1,\ldots,n$ (in case $\mathbf{y} \in (\alpha,\beta)^n$ is a constant, i.e. $y_i = h, i = 1,\ldots,n$, then we can write $\mathbf{x} \wedge h$);
– additivity: for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, $G(\mathbf{x}+\mathbf{y}) = G(\mathbf{x}) + G(\mathbf{y})$;
– maxitivity: for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, $G(\mathbf{x} \vee \mathbf{y}) = \max(G(\mathbf{x}),G(\mathbf{y}))$, where, for any $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, $\mathbf{x} \vee \mathbf{y} = \mathbf{z}$ with $z_i = \max(x_i,y_i), i = 1,\ldots,n$;
– modularity: for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, $G(\mathbf{x} \vee \mathbf{y}) + G(\mathbf{x} \wedge \mathbf{y}) = \max G(\mathbf{x}) + G(\mathbf{y})$;
– comonotonic additivity: for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$ being comonotone, i.e. such that for all $i,j \in N$ $(x_i - x_j)(y_i - y_j) \geq 0$, $G(\mathbf{x}+\mathbf{y}) = G(\mathbf{x}) + G(\mathbf{y})$;
– comonotonic maxitivity: for all comonotone $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$,

$$G(\mathbf{x} \vee \mathbf{y})) = \max(G(\mathbf{x}),G(\mathbf{y}));$$

– comonotonic modularity: for all comonotone $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$,

$$G(\mathbf{x} \vee \mathbf{y}) + G(\mathbf{x} \wedge \mathbf{y}) = G(\mathbf{x}) + G(\mathbf{y});$$

– bipolar comonotonic additivity: for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$ being bipolar comonotone, i.e. such that for all $i,j \in N$ $(|x_i| - |x_j|)(|y_i| - |y_j|) \geq 0$ and $x_i x_j \geq 0$, $G(\mathbf{x}+\mathbf{y}) = G(\mathbf{x}) + G(\mathbf{y})$;

– bipolar comonotonic maxitivity: for all bipolar comonotone $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, $G(\mathbf{x} \vee_{bip} \mathbf{y}) = \max^{bip}(G(\mathbf{x}),G(\mathbf{y}))$, where for all $a,b \in \mathbf{R}$ $\max^{bip}(a,b) = a$ if $|a| > |b|$, $\max^{bip}(a,b) = b$ if $|a| < |b|$ and $\max^{bip}(a,b) = 0$ if $|a| = |b|$, and for $\mathbf{w},\mathbf{z} \in (\alpha,\beta)^n$, $\mathbf{w} \vee \mathbf{z} = \mathbf{h}$, with $h_i = \max^{bip}(w_i,z_i)$, $i = 1,\ldots,n$;
– bipolar comonotonic modularity: for all bipolar comonotone $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, $G(\mathbf{x} \vee^{bip} \mathbf{y}) + G(\mathbf{x} \wedge^{bip} \mathbf{y}) = G(\mathbf{x}) + G(\mathbf{y})$, where $\mathbf{w} \wedge \mathbf{z} = \mathbf{h}$, with $h_i = \min^{bip}(w_i,z_i)$, $i = 1,\ldots,n$ and $\min^{bip}(a,b) = a$ if $|a| < |b|$, $\min^{bip}(a,b) = b$ if $|a| > |b|$ and $\min^{bip}(a,b) = 0$ if $|a| = |b|$;
– bipolar stability of the sign: for all $r,s \in (\alpha,\beta)$ such that $r > s > 0$ and $-r,-s \in (\alpha,\beta)$, and for all $A,B \subseteq N$ with $A \cap B = \emptyset$, $G(r\mathbf{1}_{A,B})G(s\mathbf{1}_{A,B}) > 0$ or $G(r\mathbf{1}_{A,B}) = G(s\mathbf{1}_{A,B}) = 0$, where $1_{A,B}$ is the vector with the $i-th$ component equalt to 1 if $i \in A$, equal to -1 if $i \in B$ and 0 otherwise; in simple words, $G(r\mathbf{1}_{A,B})$ and $G(s\mathbf{1}_{A,B})$ have the same sign;
– bipolar stability with respect to the minimum: for all $r,s \in (\alpha,\beta)$ such that $r > s > 0$ and $-r,-s \in (\alpha,\beta)$, and for all $A,B \subseteq N$ with $A \cap B = \emptyset$, if $|G(r\mathbf{1}_{A,B})| > |G(s\mathbf{1}_{A,B})|$, then $|G(s\mathbf{1}_{A,B})| = s$;
– cardinal tail independence: for all $\mathbf{x},\mathbf{y},\mathbf{w},\mathbf{z} \in (\alpha,\beta)^n$ and $A \subset N$ such that, for all $i \in A$ and $j \in N-A$;

$$x_i \geq w_j, x_i \geq z_j, y_i \geq w_j, y_i \geq z_j$$

we have

$$G(\mathbf{x}_A,\mathbf{w}_{-A}) - G(\mathbf{y}_A,\mathbf{w}_{-A}) = G(\mathbf{x}_A,\mathbf{z}_{-A}) - G(\mathbf{y}_A,\mathbf{z}_{-A})$$

where, for all $\mathbf{h},\mathbf{k} \in (\alpha,\beta)^n$, $\mathbf{m} = (\mathbf{h}_A,\mathbf{k}_{-A})$ is defined in such a way that, for all $i \in N$, $m_i = h_i$ if $i \in A$ and $m_i = k_i$ if $i \in N-A$;
– bipolar cardinal tail independence: for all $\mathbf{x},\mathbf{y},\mathbf{w},\mathbf{z} \in (\alpha,\beta)^n$ and $A \subset N$ such that, for all $i \in A$ and $j \in N-A$,

$$x_i \geq |w_j|, x_i \geq |z_j|, y_i \geq |w_j|, y_i \geq |z_j|$$

we have

$$G(\mathbf{x}_A,\mathbf{w}_{-A}) - G(\mathbf{y}_A,\mathbf{w}_{-A}) = G(\mathbf{x}_A,\mathbf{z}_{-A}) - G(\mathbf{y}_A,\mathbf{z}_{-A});$$

– regularity: for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$ such that there exist $t,r \in (\alpha,\beta)$ with $r > t$, for which $x_i = y_i$ if $x_i \leq t$ and $x_i = r$ and $y_i = t$ if $x_i \geq t$, then $G(\mathbf{x}) - G(\mathbf{y}) \leq r-t$.

An aggregation function is an *additive utility* if there exists function $f_i : (\alpha,\beta) \to R$, $i = 1,\ldots,n$ such that

$$G(\mathbf{x}) = AU(\mathbf{x}) = f_1(x_1) + \ldots + f_n(x_n).$$

Theorem 1. [3, 15] *An aggregation function is an additive utility if and only if it is modular.*

An aggregation function is a *weighted average* if there exists a vector of weights $\mathbf{w} = [w_1,\ldots,w_n], 0 \leq w_i \leq 1, i = 1,\ldots,n$, and $w_1 + \ldots + w_n = 1$, such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,

$$G(\mathbf{x}) = WA(\mathbf{x},\mathbf{w}) = w_1x_1 + \ldots + w_nx_n.$$

Theorem 2. *An aggregation function is a weighted average if and only if it is additive and idempotent.*

An aggregation function is a *weighted maxmin* if there exists a vector of weights $\mathbf{w} = [w_1,\dots,w_n], 0 \le w_i \le 1, i = 1,\dots,n$, and $\max_i w_i = 1$, such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,

$$G(\mathbf{x}) = WMaxMin(\mathbf{x},\mathbf{w}) = \max_i(\min(w_i,x_i)).$$

The weighted maxmin was proposed by Dubois and Prade under the name of weighted maximum [8]. Here we prefer to call it weighted maxmin in order to underline that it takes the maximum after the evaluations $x_i, i = 1,\dots,n$, are "weighted" by the corresponding w_i using the minimum operator. In this way we reserve the term weighted maximum to another aggregation function which takes the maximum after the evaluations $x_i, i = 1,\dots,n$, are weighted by the corresponding w_i using the usual product. Let remark also that Marichal calls weithed maxmin another aggregation function which, in fact, corresponds to Sugeno integral [22].

Theorem 3. *An aggregation function is a weighted maxmin if and only it is idempotent, maxitive and stable with respect to minimum (maximum).*

An aggregation function is a *weighted maximum* if there exists a vector of weights $\mathbf{w} = [w_1,\dots,w_n], 0 \le w_i \le 1, i = 1,\dots,n$, and $\max_i w_i = 1$, such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,

$$G(\mathbf{x}) = WMax(\mathbf{x},\mathbf{w}) = \max_i w_i x_i.$$

Theorem 4. *An aggregation function is a weighted maximum if and only it is idempotent, maxitive and homogeneous.*

A capacity is function $\mu : 2^N \to [0,1]$ satisfying the following properties:

1. $\mu(\emptyset) = 0, \mu(N) = 1$,
2. for all $A \subseteq B \subseteq N, \mu(A) \le \mu(B)$.

The Choquet integral [6] of a vector of evaluations $\mathbf{x} = [x_1,\dots,x_n] \in (\alpha,\beta)^n$ with respect to the capacity μ is given by

$$Ch(\mathbf{x},\mu) = \int_{\min_i x_i}^{\max_i x_i} \mu(\{i \in N : x_i \ge t\}dt + \min_i x_i$$

Observe that if we consider $\mathbf{x} \in (\alpha,\beta)^n \cap \mathbf{R}^n_+$ the Choquet integral can be written as

$$Ch(\mathbf{x},\mu) = \int_0^{+\infty} \mu(\{i \in N : x_i \ge t\})dt$$

Theorem 5. [25] *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and comonotone additive if and only if there exists a capacity μ such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\mathbf{x}) = Ch(\mathbf{x},\mu).$$

A measure on N with a scale (α,β) is any function $\nu : 2^N \to (\alpha,\beta)$ such that:

1. $\nu(\emptyset) = \alpha, \nu(N) = \beta$,
2. for all $A \subseteq B \subseteq N, \nu(A) \leq \nu(B)$.

The Sugeno integral [27] of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in \alpha,\beta)^n$ with respect to the measure ν on N with scale (α,β) is given by

$$Su(\mathbf{x},\nu) = \max_{A\subseteq N} \min(\nu(A), \min_{i\in A} x_i).$$

Theorem 6. [7] *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent, maxitive and stable with respect to the minimum if and only if there exists a measure ν on N with a scale (α,β) such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\mathbf{x}) = Su(\mathbf{x},\nu).$$

The Shilkret integral [26] with respect to a capacity μ of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in \mathbf{R}^n$ is given by

$$Sh(\mathbf{x},\mu) = \max_{i\in N}[x_i\mu(\{j \in N : x_j \geq x_i\}].$$

Theorem 7. [26] *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent, comonote maxitive and homogeneous if and only if there exists a capacity μ on N such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\mathbf{x}) = Sh(\mathbf{x},\mu).$$

A level dependent capacity is a function $\mu_{LD} : 2^N \times (\alpha,\beta) \to [0,1]$ satisfying the following properties:

1. for all $t \in (\alpha,\beta)$, $\mu_{LD}(\emptyset,t) = 0, \mu_{LD}(N,t) = 1$,
2. for all $t \in (\alpha,\beta)$ and for all $A \subseteq B \subseteq N, \mu(A,t) \leq \mu(B,t)$,
3. for all $A \subseteq N$, $\mu_{LD}(A,t)$ considered as a function with respect to t is Lebesgue measurable.

The Choquet integral of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in (\alpha,\beta)^n$ with respect to the level dependent capacity μ_{LD} [16] is given by

$$Ch_{LD}(\mathbf{x}) = \int_{\min_i x_i}^{\max_i x_i} \mu_{LD}(\{i \in N : x_i \geq t\},t)dt + \min_i x_i$$

Theorem 8. [16] *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and cardinal tail independent if and only if there exists a level dependent capacity μ_{LD} such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\mathbf{x}) = Ch_{LD}(\mathbf{x},\mu_{LD}).$$

Theorem 9. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent, comonotone modular and regular if and only if there exists a level dependent capacity μ_{LD} such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\mathbf{x}) = Ch_{LD}(\mathbf{x},\mu_{LD}).$$

A level dependent measure on N with a scale (α,β) is any function $\nu_{LD} : 2^N \times (\alpha,\beta) \to (\alpha,\beta)$ such that:

1. for all $t \in (\alpha,\beta)\nu_{LD}(\emptyset,t) = \alpha$ and $\nu_{LD}(N,t) = t$,
2. for all $t,r \in (\alpha,\beta)$ such that $t \le r$, and $A \subseteq B \subseteq N, \nu_{LD}(A,t) \le \nu(B,r)$.

An aggregation function is a *cumulative utility* [5] if there exists a level dependent measure ν_{LD}, such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,

$$\begin{aligned} G(\mathbf{x}) &= CU(\mathbf{x},\nu_{LD}) \\ &= \sum_{i\in N}(\nu_{LD}(\{j \in N : x_j \ge x_{(i)}\},x_{(i)}) - \nu_{LD}(\{j \in N : x_j \ge x_{(i)}\},x_{(i-1)})), \end{aligned}$$

where $(\cdot)$ is a permutation of the indices of criteria such that $x_{(i)} \le x_{(i+1)}$, $i = 1,\dots,n-1$, and $x_{(0)} = \alpha$.

Theorem 10. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and comonote modular if and only if there exists a level dependent measure ν_{LD} on N with a scale (α,β) such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\boldsymbol{x}) = CU(\boldsymbol{x},\nu_{LD}).$$

The level dependent Sugeno integral [23] of a vector $\mathbf{x} = [x_1,\dots,x_n] \in (\alpha,\beta)^n$ is given by

$$Su_{LD}(\mathbf{x},\nu_{LD}) = \max_{i\in N}\nu_{LD}(\{j \in N : x_j \ge x_i\},x_i).$$

Theorem 11. [23] *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and comonote maxitive if and only if there exists a level dependent measure ν_{LD} on N with a scale (α,β) such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\boldsymbol{x}) = Su_{LD}(\boldsymbol{x},\nu_{LD}).$$

A level dependent capacity μ_{LD} is said Shilkret compatible if for for all $t,r \in (\alpha,\beta)$ such that $t \le r$, and $A \subseteq N, t\mu_{LD}(A,t) \le r\mu_{LD}(B,r)$.

The level dependent Shilkret integral [4] with respect to a level dependent capacity Shilkret compatible μ_{LD} of a vector $\mathbf{x} = [x_1,\dots,x_n] \in \mathbf{R}^n$ is given by

$$Sh_{LD}(\mathbf{x},\mu_{LD}) = \max_{i\in N}[x_i\mu_{LD}(\{j \in N : x_j \ge x_i\},x_i)].$$

Theorem 12. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and comonote maxitive if and only if there exists a level dependent capacity Shilkret compatible μ_{LD} on N such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\boldsymbol{x}) = Sh_{LD}(\boldsymbol{x},\mu_{LD}).$$

Corollary 1. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is a level dependent Sugeno integral with respect to a level dependent measure ν_{LD} on N with a scale (α,β) if and only if it is a level dependent Shilkret integral with respect to a Shilkret compatible level dependent capacity μ_{LD}. More precisely, for all $t \in (\alpha,\beta)$ and $A \subseteq N$,*

$$\nu_{LD}(A,t) = t\mu_{LD}(A).$$

Let us consider the set $Q = \{(A,B) : A,B \subseteq N, A \cap B = \emptyset\}$. A bicapacity is function $\mu_b : 2^N \to [0,1]$ satisfying the following properties:

1. $\mu_b(\emptyset,\emptyset) = 0$,
2. $\mu_b(N,\emptyset) = 1$, $\mu_b(\emptyset,N) = -1$,
3. for all $(A,B),(C,D) \in Q$ such that $A \subseteq C$ and $B \supseteq C$, $\mu_b(A,B) \leq \mu(C,D)$.

The bipolar Choquet integral [13, 14] (see also [17]) of a vector of evaluations $\mathbf{x} = [x_1,\ldots,x_n] \in \mathbf{R}^n$ with respect to the bicapacity μ_b is given by

$$Ch_b(\mathbf{x}) = \int_0^{\max_i |x_i|} \mu_b(\{i \in N : x_i \geq t\}, \{i \in N : x_i \leq -t\})dt$$

Theorem 13. [17] *An aggregation function* $G : (\alpha,\beta)^n \to (\alpha,\beta)$ *is idempotent and bipolar comonotonic additive if and only if there exists a bicapacity* μ_b *such that, for all* $\boldsymbol{x} \in (\alpha,\beta)^n$,

$$G(\boldsymbol{x}) = Ch_b(\boldsymbol{x},\mu_b).$$

A bipolar measure on N with a scale (α,β), $\alpha < 0 < \beta$, is any function $\nu_b : Q \to (\alpha,\beta)$ satisfying the following properties:

1. $\nu_b(\emptyset,\emptyset) = 0$,
2. $\nu_b(N,\emptyset) = \beta$, $\nu_b(\emptyset,N) = \alpha$,
3. for all $(A,B),(C,D) \in Q$ such that $A \subseteq C$ and $B \supseteq D$, $\mu_b(A,B) \leq \mu(C,D)$.

The bipolar Sugeno integral of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in \mathbf{R}^n$ with respect to the bipolar measure ν_b on N with scale (α,β) is given by

$$u_b(\mathbf{x},\nu_b) =$$
$$\max_{i\in N}^{bip}(sign(\nu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})$$
$$\min(|\nu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})|, |x_i|).$$

Theorem 14. *An aggregation function* $G : (\alpha,\beta)^n \to (\alpha,\beta)$ *is idempotent, bipolar comonotone maxitive, bipolar stable with respect to the sign and bipolar stable with respect to the minimum if and only if there exists a bipolar measure* ν_b *on* N *with a scale* (α,β) *such that, for all* $\boldsymbol{x} \in (\alpha,\beta)^n$,

$$G(\boldsymbol{x}) = Su_b(\boldsymbol{x},\nu).$$

The bipolar Shilkret integral of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in \mathbf{R}^n$ is given by

$$Sh_b(\mathbf{x},\mu_b) = \max_{i\in N}^{bip}[x_i\mu_b(\mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\}].$$

Theorem 15. *An aggregation function* $G : (\alpha,\beta)^n \to (\alpha,\beta)$ *is idempotent, bipolar comonotone maxitive and homogeneous if and only if there exists a bicapacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in (\alpha,\beta)^n$,

$$G(\boldsymbol{x}) = Sh_b(\boldsymbol{x},\mu_b).$$

A bipolar level dependent measure on N with a scale (α,β), $\alpha < 0 < \beta$, is any function $\nu_{bLD} \times (\alpha,\beta) : Q \to (\alpha,\beta)$ satisfying the following properties:

1. $\nu_{bLD}(\emptyset,\emptyset,t) = 0$ for all $t \in (\alpha,\beta)$,
2. $\nu_{bLD}(N,\emptyset,t) = \beta$, $\nu_{bLD}(\emptyset,N,t) = \alpha$ for all $t \in (\alpha,\beta)$,
3. for all $(A,B),(C,D) \in Q$ such that $A \subseteq C$ and $B \supseteq D$, and for all $t \in (\alpha,\beta)$, $\mu_b(A,B,t) \leq \mu(C,D,t)$.

The bipolar level dependent Sugeno integral of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in (\alpha,\beta)^n$ is given by

$$Su_{bLD}(\mathbf{x},\nu_{LD}) = \max_{i\in N}^{bip} \nu_{bLD}(\{j \in N : x_j \geq |x_i|\},\{j \in N : x_j \leq -|x_i|\},x_i).$$

Theorem 16. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and bipolar comonote maxitive if and only if there exists a bipolar level dependent measure ν_{bLD} on N with a scale (α,β) such that, for all $\boldsymbol{x} \in (\alpha,\beta)^n$,*

$$G(\boldsymbol{x}) = Su_{bLD}(\boldsymbol{x},\nu_{bLD}).$$

The bipolar level dependent Shilkret integral of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in (\alpha,\beta)^n$ is given by

$$Sh_{bLD}(\mathbf{x},\mu_{bLD}) = \max_{i\in N}^{bip}[x_i\mu_{bLD}(\{j \in N : x_j \geq |x_i|\},\{j \in N : x_j \leq -|x_i|\},|x_i|)].$$

Theorem 17. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and bipolar comonotone maxitive if and only if there exists a level dependent capacity μ_{bLD} on N such that, for all $\boldsymbol{x} \in (\alpha,\beta)^n$,*

$$G(\boldsymbol{x}) = Sh_{bLD}(\boldsymbol{x},\mu_{bLD}).$$

Corollary 2. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is a bipolar level dependent Sugeno integral with respect to a bipolar level dependent measure ν_{bLD} on N with a scale (α,β) if and only if it is a bipolar level dependent Shilkret integral with respect to a Shilkret compatible bipolar level dependent capacity μ_{bLD}. More precisely, for all $t \in (\alpha,\beta)$ and $(A,B) \in Q$,*

$$\nu_{bLD}(A,B,t) = t\mu_{bLD}(A,B).$$

A bipolar level dependent bicapacity [16] is a function $\mu_{bLD} : Q_{LD} \times [(\alpha,\beta) \cap \mathbf{R}_+] \to [0,1]$ satisfying the following properties:

1. for all $t \in (\alpha,\beta) \cap \mathbf{R}_+$, $\mu_{bLD}(\emptyset,\emptyset,t) = 0, \mu_{bLD}(N,\emptyset,t) = 1, \mu_{bLD}(\emptyset,N,t) = -1$
2. for all $(A,B,t),(C,D,t) \in Q_{LD}$, $A \subseteq C, B \supseteq D, \mu_{bLD}(A,B,t) \leq \mu_{bLD}(C,D,t)$,
3. for all $(A,B,t) \subseteq Q_{LD}$, $\mu_{bLD}(A,B,t)$ considered as a function with respect to t is Lebesgue measurable.

The Choquet integral of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in (\alpha,\beta)^n$ with respect to the level dependent bicapacity μ_{bLD} [16] is given by

$$Ch_{bLD}(\mathbf{x}) = \int_0^{\max_i |x_i|} \mu_{bLD}(\{i \in N : x_i \geq t\},\{i \in N : x_i \leq -t\},t)dt$$

Theorem 18. [16] *An aggregation function* $G : (\alpha,\beta)^n \to (\alpha,\beta)$ *is idempotent and bipolar cardinal tail independent if and only if there exists a bipolar level dependent bicapacity* μ_{bLD} *such that, for all* $\boldsymbol{x} \in (\alpha,\beta)^n$,

$$G(\boldsymbol{x}) = Ch_{bLD}(\boldsymbol{x}, \mu_{bLD}).$$

Theorem 19. *An aggregation function* $G : (\alpha,\beta)^n \to (\alpha,\beta)$ *is idempotent, bipolar comonotone modular and regular if and only if there exists a bipolar level dependent capacity* μ_{bLD} *such that, for all* $\boldsymbol{x} \in (\alpha,\beta)^n$,

$$G(\boldsymbol{x}) = Ch_{bLD}(\boldsymbol{x}, \mu_{bLD}).$$

The following result shows the relatioship between the bipolar level dependent Choquet integral and the bipolar Sugeno integral. We define a level dependent bicapacity μ_{bLD} Sugeno compatible if for all $(A,B) \in Q$ and $t \in (\alpha,\beta)$ with $t \geq 0$, $\mu_{bLD,t}(A,B) = 1$ or $\mu_{bLD}(A,B,t) = -1$ of $\mu_{bLD}(A,B,t) = 0$, and $|\mu_{bLD}(A,B,t)| = 1$ if $|\mu_{bLD}(A,B,r)| = 1$ for some $r > t$.

Theorem 20. *If for all* $(A,B),(C,D) \in Q$ *such that* $A \subseteq C$ *and* $B \subseteq D$, $G(\boldsymbol{1}_{A,B})G(\boldsymbol{1}_{C,D}) \geq 0$, *then there exists a level dependent bicapacity Sugeno compatible* μ_{bLD} *and a bipolar measure* ν_b *such that for all* $\boldsymbol{x} \in (\alpha,\beta)^n$

$$Su_b(\boldsymbol{x}, \nu_b) = Ch_{bLD}(\boldsymbol{x}, \mu_{bLD}).$$

Moreover, and for all $(A,B) \in Q$ *and* $t \in [0, \max(|\alpha|,|\beta|)]$

- $\mu_{bLD}(A,B,t) = -1$ *if* $\nu_{bLD}(A,B) \leq -t < 0$,
- $\mu_{bLD}(A,B,t) = 1$ *if* $\nu_{bLD}(A,B) \geq t > 0$,
- $\mu_{bLD}(A,B,t) = 0$ *otherwise.*

An aggregation function is a *bipolar cumulative utility* if there exists a bipolar level dependent measure ν_{LD}, such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,

$$G(\mathbf{x}) = CU_b(\mathbf{x}, \nu_{bLD}) = \sum_{i \in N} (CU^*_{[i]}(\mathbf{x}, \nu_{bLD}) - CU^{**}_{[i]}(\mathbf{x}, \nu_{bLD}))$$

where

$$CU^*_i(\mathbf{x}, \nu_{bLD}) = \nu_{bLD}(\{j \in N : x_j \geq x_{[i]}\}, \{j \in N : x_j \leq -|x_{[i]}|\}, |x_{[i]}|),$$

$$CU^{**}_i(\mathbf{x}, \nu_{bLD})) = \nu_{bLD}(\{j \in N : x_j \geq x_{[i]}\}, \{j \in N : x_j \leq -|x_{[i]}|\}, |x_{[i-1]}|)),$$

and $[\cdot]$ is a permutation of the indices of criteria such that $|x_{[i]}| \leq |x_{[i+1]}|$, $i = 1, \ldots, n-1$, and $x_{[0]} = 0$.

Theorem 21. *An aggregation function* $G : (\alpha,\beta)^n \to (\alpha,\beta)$ *is idempotent and bipolar comonote modular if and only if there exists a bipolar level dependent measure* ν_{bLD} *on* N *with a scale* (α,β) *such that, for all* $\boldsymbol{x} \in (\alpha,\beta)^n$,

$$G(\boldsymbol{x}) = CU_b(\boldsymbol{x}, \nu_{bLD}).$$

Let us consider the following set $W = \{(A,(x_1,...,x_n),t) \in 2^N \times (\alpha,\beta)^{n+1} : A \subseteq \{i \in N : x_i \geq t\}\}$. A profile dependent capacity is a function $\mu_{PD} : W \to [0,1]$ satisfying the following properties:

1. for all $\mathbf{x} \in (\alpha,\beta)^n$ and $t \in (\alpha,\beta)$, $\mu_{PD}(\emptyset,\mathbf{x},t) = 0$,
2. for all $\mathbf{x} \in (\alpha,\beta)^n$ and $t \in (\alpha,\beta)$ such that $\min x_i \geq t$, $\mu_{LD}(N,\mathbf{x},t) = 1$,
3. for all $\mathbf{x} \in (\alpha,\beta)^n$ and $t \in (\alpha,\beta)$ such that $A \subseteq B \subseteq \{i \in N : x_i \geq t\}, \mu(A,\mathbf{x},t) \leq \mu(B,\mathbf{x},t)$,
4. for all $(A,\mathbf{x},t) \in W$, $\mu_{PD}(A,\mathbf{x},t) \leq 1$,
5. for all $(A,\mathbf{x},t),(A,\mathbf{y},t) \in W$ such that $\mathbf{x} \wedge t = \mathbf{y} \wedge t$, $\mu_{PD}(A,\mathbf{x},t) = \mu_{PD}(A,\mathbf{y},t)$,
6. for all $\mathbf{x} \in (\alpha,\beta)^n$, $\mu_{PD}(\{i \in N : x_i \geq t\},\mathbf{x},t)$ considered as a function with respect to t is Lebesgue measurable,
7. for all $\mathbf{x},\mathbf{y} \in (\alpha,\beta)^n$, if $\mathbf{x} \geq \mathbf{y}$ then

$$\int_{\min_i x_i}^{\max_i x_i} \mu_{PD}(\{i \in N : x_i \geq t\},\mathbf{x},t)dt + \min_i x_i \geq$$
$$\int_{\min_i y_i}^{\max_i y_i} \mu_{PD}(\{i \in N : y_i \geq t\},\mathbf{y},t)dt + \min_i y_i.$$

The Choquet integral of a vector $\mathbf{x} = [x_1,\ldots,x_n] \in (\alpha,\beta)^n$ with respect to the profile dependent capacity μ_{PD} is given by

$$Ch_{PD}(\mathbf{x}) = \int_{\min_i x_i}^{\max_i x_i} \mu_{PD}(\{i \in N : x_i \geq t\},\mathbf{x},t)dt + \min_i x_i$$

Theorem 22. *An aggregation function $G : (\alpha,\beta)^n \to (\alpha,\beta)$ is idempotent and regular if and only if there exists a profile dependent capacity μ_{PD} such that, for all $\mathbf{x} \in (\alpha,\beta)^n$,*

$$G(\mathbf{x}) = Ch_{PD}(\mathbf{x},\mu_{PD}).$$

Let us consider an aggregation function $G : [0,1]^n \to [0,1]$ which is representable as a profile dependent Choquet integral, but is not representable as a level dependent Choquet integral:

$$G(x_i,\ldots,x_n) = (\max_i x_i - \min_i x_i) \cdot \min_i x_i + \min_i x_i.$$

Aggregation function G can be represented as profile dependent Choquet integral whose profile dependent capacity is defined as follow: for all $(A,\mathbf{x},t) \in W$, $\mu_{PD}(A,\mathbf{x},t) = 1$ if $t \leq \min_i x_i$, and $\mu_{PD}(A,\mathbf{x},t) = \min_i x_i$ if $t > \min_i x_i$.

For any $(A,\mathbf{x},t) \in W$ with $t < \beta$, let us define, if the limit exists and it is finite,

$$G'_t(A,\mathbf{x},t) = lim_{h\to 0^+} \frac{G((\mathbf{x} \wedge t) + h\mathbf{1}_A) - G(\mathbf{x} \wedge t)}{h}.$$

Theorem 23. *If G is dempotent and regular, then there exists a profile dependent capacity μ_{PD} such that for all $(A,\mathbf{x},t) \in W$*

$$\mu_{PD}(A,\mathbf{x},t) = G'_t(A,\mathbf{x},t)$$

and

$$G(\boldsymbol{x}) = \int_{\min_i x_i}^{\max_i x_i} \mu_{PD}(\{i \in N : x_i \geq t\}, \boldsymbol{x}, t)dt + \min_i x_i,$$

i.e.

$$G(\boldsymbol{x}) = \int_{\min_i x_i}^{\max_i x_i} G'_t(A, \boldsymbol{x} \wedge t)dt + \min_i x_i.$$

The following results describe relationships between different generalizations of Choquet integral.

Theorem 24. –

$$Ch(\boldsymbol{x}n, \mu) = Ch_{LD}(\boldsymbol{x}, \mu_{LD})$$

for any $\boldsymbol{x} \in (\alpha, \beta)^n$ *if and only if for all* $A \subseteq N$ $\mu(A) = \mu_{LD}(A, t)$ *almost everywhere with respect to* $t \in (\alpha, \beta)$ [16]*;*

–

$$Ch_{LD}(\boldsymbol{x}, \mu_{LD}) = Ch_{PD}(\boldsymbol{x}, \mu_{PD})$$

for any $\boldsymbol{x} \in (\alpha, \beta)^n$ *if and only if for all* $(A, (x_1, ..., x_n), t) \in W$ $\mu_{LD}(A, t) = \mu_{PD}(A, \boldsymbol{x}, t)$ *almost everywhere with respect to* $t \in (\alpha, \beta)$*;*

–

$$Ch_b(\boldsymbol{x}, \mu_b) = Ch_{bLD}(\boldsymbol{x}, \mu_{bLD})$$

for any $\boldsymbol{x} \in (\alpha, \beta)^n$ *if and only if for all* $(A, B) \in Q$ $\mu_b(A, B) = \mu_{bLD}(A, B, t)$ *almost everywhere with respect to* $t \in (\alpha, \beta)$ *with* $t > 0$ [16]*;*

–

$$Ch_{bLD}(\boldsymbol{x}, \mu_{bLD}) = Ch_{PD}(\boldsymbol{x}, \mu_{PD})$$

for any $\boldsymbol{x} \in (\alpha, \beta)^n$ *if and only if for all* $(A, \boldsymbol{x}, t) \in W$

$$\mu_{PD}(A, \boldsymbol{x}, t) = 1 + \mu_{bLD}(\emptyset, N - A, -t) \;\; \text{if } t < 0,$$

$$\mu_{PD}(A, \boldsymbol{x}, t) = \mu_b(A, B, t) - \mu_b(\emptyset, B, t) \;\; \text{if } 0 \leq t \leq |\min_i x_i| \;\; \text{and} \;\; \min_i x_i < 0,$$

$$\mu_{PD}(A, \boldsymbol{x}, t) = \mu_b(A, \emptyset, t) \text{ if } \min_i x_i \geq 0$$

almost everywhere with respect to $t \in (\alpha, \beta)$.

We show now how some of the aggregation functions we considered can be represented as a Choquet integral or a generalization of the Choquet integral.

Theorem 25. – *An aggregation function G is an additive utility if and only if there exists a level dependent capacity* μ_{LD} *such that for all* $A, B \subseteq N$ *with* $A \cap B = \emptyset$ *and for all* $t \in (\alpha, \beta)$ $\mu_{LD}(A \cup B, t) = \mu_{LD}(A, t) + \mu_{LD}(B, t)$ *and*

$$G(\boldsymbol{x}) = Ch_{LD}(\boldsymbol{x}, \mu_{LD})$$

for all $\boldsymbol{x} \in (\alpha, \beta)^n$*; in this case,* $f_i(x) = \mu_{LD}(\{i\}, x)$ *for all* $i = 1, \ldots, n$ *and* $x \in (\alpha, \beta)$*;*

– *an aggregation function G is a weighted average if and only if there exists a capacity μ such that for all $A, B \subseteq N$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$ and*

$$G(\mathbf{x}) = Ch(\mathbf{x}, \mu)$$

for all $\mathbf{x} \in (\alpha, \beta)^n$; in this case, $w_i = \mu(\{i\})$ for all $i = 1, \ldots, n$;

– *an aggregation function G is a weighted maxmin if and only if there exists a level dependent capacity μ_{LD} being Boolean, i.e. for all $A \subseteq N$ and for all $t \in (\alpha, \beta)$ $\mu_{LD}(A,t) = 0$ or $\mu_{LD}(A,t) = 1$, antitone with respect to t, i.e. for all $r, t \in (\alpha, \beta)$ with $r > t$ and for all $A \subseteq N$ $\mu_{LD}(a,r) \leq \mu_{LD}(a,t)$, and maxitive, i.e. for all $A, B \subseteq N$ and for all $t \in (\alpha, \beta)$ $\mu(A \cup B, t) = \max(\mu(A,t), \mu(B,t))$ and*

$$G(\mathbf{x}) = Ch_{LD}(\mathbf{x}, \mu_{LD})$$

for all $\mathbf{x} \in (\alpha, \beta)^n$; in this case, $w_i = sup\{t \in (\alpha, \beta) : \mu_{LD}(\{i\}, t) = 1\}$ for all $i = 1, \ldots, n$;

– *if an aggregation function G is a weighted maximum with respect to weights $w_i, i = 1, \ldots, n$, then there exists a profile dependent capacity μ_{PD} for which $\mu_{PD}(A, \mathbf{x}, t) = w_{i^*}$, with $w_{i^*} x_{i^*} = \max\{w_i x_i : i \in N\}$ for all $(A, \mathbf{x}, t) \in W$, such that for all $\mathbf{x} \in (\alpha, \beta)^n$*

$$G(\mathbf{x}) = Ch_{PD}(\mathbf{x}, \mu_{PD});$$

– *an aggregation function G is a Sugeno integral if and only if there exists a Boolean and antitone with respect to t level dependent capacity μ_{LD} such that*

$$G(\mathbf{x}) = Ch_{LD}(\mathbf{x}, \mu)$$

for all $\mathbf{x} \in (\alpha, \beta)^n$; in this case $\nu(A) = sup\{t \in (\alpha, \beta) : \mu_{LD}(A,t) = 1\}$ for all $A \subseteq N$ [16]*;*

– *if a regular aggregation function G is a level dependent Sugeno integral with respect to a level dependent measure ν_{LD} on N, then there exists a profile dependent capacity μ_{PD} for which for all $(A, \mathbf{x}, t) \in W$,*

$$\mu_{PD}(A, \mathbf{x}, t) = \frac{d\nu_{LD}(A^*, t^*)}{dt}$$

where $t^* = \min_{i \in A^*} x_i$ and $A \subseteq A^* \subseteq N$ with

$$\nu_{LD}(A^*, t^*) = \max\{\nu_{LD}(B, \min_{i \in B} x_i) : B \supseteq A\},$$

such that for all $\mathbf{x} \in (\alpha, \beta)^n$

$$G(\mathbf{x}) = Ch_{PD}(\mathbf{x}, \mu_{PD});$$

– *if a regular aggregation function G is a level dependent Shilkret integral with respect to a level dependent capacity μ_{LD} on N, then there exists a profile dependent capacity μ_{PD} for which for all $(A, \mathbf{x}, t) \in W$, $\mu_{PD}(A, \mathbf{x}, t) = \mu_{LD}(A^*, t^*)$ where $t^* = \min_{i \in A^*} x_i$ and $A \subseteq A^* \subseteq N$ with*

$$t^* \mu_{LD}(A^*, t^*) = \max\{r \mu_{LD}(B, \min_{i \in B} x_i) : B \supseteq A\},$$

such that for all $\boldsymbol{x} \in (\alpha, \beta)^n$

$$G(\boldsymbol{x}) = Ch_{PD}(\boldsymbol{x}, \mu_{PD}).$$

The applications of above integrals in real decision problems needs the determination of corresponding capacities and, usually, this is a quite complex task. In these cases, a very useful approach is the robust ordinal regression [19], *first proposed for additive utility functions* [18, 10], *and then introduced also in the use of the nonadditive integrals* [1, 2].

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Bichains

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1 Introduction

This paper is a study of a variety of algebras that arise in the investigation of the truth value algebra of type-2 fuzzy sets [5]. The variety generated by the truth value algebra of type-2 fuzzy sets with only its two semilattice operations in its type is generated by a four-element algebra that has a particularly simple form, which we call a bichain. Our initial goal is to understand the equational properties of this particular bichain, and thus of the truth value algebra of type-2 fuzzy sets. We outline the progress on this goal, and on our study of bichains in general.

2 The Algebra of Fuzzy Truth Values of Type-2 Fuzzy Sets

The underlying set of the algebra of truth values of type-2 fuzzy sets is $M = Map([0,1], [0,1])$, the set of all functions from the unit interval into itself. The operations imposed are certain convolutions of operations on $[0,1]$.

The two binary operations $\sqcap$ and $\sqcup$ corresponding to meet and join satisfy the following equations. The details may be found in [4].

Corollary 1. *Let $f,\ g,\ h \in M$. The following equations hold in $(M, \sqcap, \sqcup)$.*

1. $f \sqcup f = f$; $f \sqcap f = f$
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
4. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$

It is not the case that the list of equations above is an equational basis for the variety generated by $(M, \sqcup, \sqcap)$. Whether there is a finite basis for this variety remains open.

3 Bichains in the Variety of Birkhoff Systems

Definition 1. *An algebra $(A, \sqcap, \sqcup)$ with two binary operations is called a **bisemilattice** if it satisfies equations (1)–(3) above, and a **Birkhoff system** if it satisfies equations (1)-(4) above.*

In any bisemilattice, each of the operations $\sqcap$ and $\sqcup$ induces a partial order on the underlying set of elements.

Definition 2. *If an algebra $(A, \sqcap, \sqcup)$ is a bisemilattice and the partial orders induced by the two operations $\sqcap$ and $\sqcup$ are chains, then $(A, \sqcap, \sqcup)$ is a* ***bichain****.*

Theorem 1. *A bichain is a Birkhoff system.*

Thus the variety generated by bichains is contained in the variety of Birkhoff systems. However, this containment is strict.

We only consider finite bichains. When describing a bichain $\{1, 2, ..., n\}$ with n elements, we assume the $\sqcap$-order is $1 < 2 < \cdots < n$ and then just give the $\sqcup$-order. Any permutation φ of $1, 2, .., n$ for the $\sqcup$-order gives a bichain, so up to isomorphism there are $n!$ n-element bichains. We will generally depict bichains in the following manner.

$$\begin{array}{cc} n & \varphi(n) \\ \vdots & \vdots \\ 2 & \varphi(2) \\ 1 & \varphi(1) \\ \sqcap & \sqcup \end{array}$$

We list below three bichains that play a big role in what we do. We list only the column giving the ordering induced by $\sqcup$.

$$\begin{array}{ccc} & & 4 \\ 1 & 2 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \\ \sqcup & \sqcup & \sqcup \\ \mathbb{A}_4 & \mathbb{A}_5 & \mathbb{B} \end{array}$$

We were led to the consideration of bichains by the fact that the variety generated by $(M, \sqcap, \sqcup)$ is generated by the 4-element bichain $\mathbb{B}$ [2, 3].

4 Projective and Subdirectly Irreducible Algebras

We begin with two concepts that play a central role in our investigation.skip

Definition 3. *An algebra $\mathbb{P}$ is* ***weakly projective*** *in a variety $\mathcal{V}$ if for every homomorphism $f : \mathbb{P} \to \mathbb{E}$ and every onto homomorphism $g : \mathbb{A} \twoheadrightarrow \mathbb{E}$, there is a homomorphism $h : \mathbb{P} \to \mathbb{A}$ with $gh = f$.*

We refer to *weakly projective* simply as *projective*. The algebra $\mathbb{A}_4$ is the only three-element bichain that is not projective.

Definition 4. *An algebra $\mathbb{A}$ is* ***subdirectly irreducible*** *in a variety $\mathcal{V}$ if whenever it is a subalgebra of a product, then at least one of the projections into a component is one-to-one.*

For an algebra $\mathbb{P}$ in a variety $\mathcal{V}$ define $\mathcal{V}_{\mathbb{P}} = \{\mathbb{A} \in \mathcal{V} : \mathbb{P} \not\hookrightarrow \mathbb{A}\}$. Here $\mathbb{P} \not\hookrightarrow \mathbb{A}$ means $\mathbb{P}$ is not isomorphic to a subalgebra of $\mathbb{A}$. Denote by $\mathcal{V}(\mathbb{P})$ the variety generated by $\mathbb{P}$.

Proposition 1. *[1] If $\mathbb{P}$ is projective in $\mathcal{V}$ and subdirectly irreducible, then $\mathcal{V}_{\mathbb{P}}$ is a variety, and is the largest subvariety of $\mathcal{V}$ that does not contain $\mathbb{P}$.*

The situation in the proposition is sometimes referred to as a splitting, as it splits the lattice of subvarieties of $\mathcal{V}$ into two parts, those that contain the variety $\mathcal{V}(\mathbb{P})$, and those that are contained in $\mathcal{V}_{\mathbb{P}}$. Further, such a splitting comes equipped with an equation, called the splitting equation, defining the variety $\mathcal{V}_{\mathbb{P}}$ relative to the equations defining $\mathcal{V}$.

The algebra $\mathbb{A}_5$ is projective and subdirectly irreducible in the variety of Birkhoff systems, and so is subdirectly irreducible and projective in the variety $\mathcal{V}$ generated by bichains. The equation $[x(y+z)][xy+xz] = [x(y+z)]+[xy+xz]$ is a splitting equation for $\mathbb{A}_5$, in other words the equation defining $\mathcal{V}_{\mathbb{A}_5}$ within the variety generated by bichains. It is interesting to compare this equation to the usual distributive law.

Proposition 2. *The splitting variety $\mathcal{V}_{\mathbb{A}_5}$ contains $\mathcal{V}(\mathbb{B})$.*

We conjecture that in the variety generated by bichains, $\mathcal{V}_{\mathbb{A}_5} = \mathcal{V}(\mathbb{B})$. To lend credence to this, we have shown a bichain belongs to $\mathcal{V}_{\mathbb{A}_5}$ if and only if it belongs to $\mathcal{V}(\mathbb{B})$. But this remains an open problem. If this conjecture turns out to be true, then an equational basis for $\mathcal{V}(\mathbb{B})$, and hence for $\mathcal{V}(M,\sqcap,\sqcup)$, is one for the variety generated by bichains plus the splitting equation.

In investigating bichains in general, one fundamental problem is determining which ones are projective. As we have seen, projectivity is connected with splitting and hence with equational bases.

In examining the four-element bichains, the projective ones turned out to be exactly the ones that did not contain $\mathbb{A}_4$ as a subalgebra. This led to the conjecture that a finite bichain is projective if and only if it does not contain $\mathbb{A}_4$. This is indeed the case. The following two definitions are key concepts.

Definition 5. *A permutation φ of $\{1,2,...,n\}$ is **special** if it maps the initial segment $\{1,2,...,\varphi^{-1}(n)-1)\}$ onto itself.*

The significance of this is that bichains not having $\mathbb{A}_4$ as a subalgebra are all given by special permutations.

Definition 6. *Suppose $\mathbb{C}$ is a finite bichain of length n given by the permutation φ, and let $x_1,\ldots,x_n$ be generators of the free Birkhoff system on n generators. Form elements $x_1^0,\ldots,x_n^0$ by setting $x_i^0 = x_n \sqcap \cdots \sqcap x_i$. Then define for each $p \geq 0$*

$$x_{\varphi(i)}^{2p+1} = x_{\varphi(1)}^{2p} \sqcup \cdots \sqcup x_{\varphi(i)}^{2p}$$
$$x_i^{2p+2} = x_n^{2p+1} \sqcap \cdots \sqcap x_i^{2p+1}$$

*We say $\mathbb{C}$ is **left-right-projective** (and write LR-projective) if there is a p with $x_i^p = x_i^{p+1}$ for each $i = 1,\ldots,n$. The least such p is called the LR-**length** of $\mathbb{C}$.*

Being LR-projective is equivalent to having the above sequence of terms stabilizing after p steps for any elements $x_1,\ldots,x_n$ in any Birkhoff system A. Note that LR-projective is perhaps stronger than projective. One of our main theorems is the following.

Theorem 2. *Any finite bichain that does not contain $\mathbb{A}_4$ is* LR*-projective.*

The proof of this is rather long, and involves a number of lemmas.

One way to prove that a bichain $\mathbb{C}$ containing $\mathbb{A}_4$ is not projective is to construct an onto homomorphism $\mathbb{A} \twoheadrightarrow \mathbb{C}$, with $\mathbb{A}$ in the variety of Birkhoff systems, such that there does not exist a homomorphism $\mathbb{C} \to \mathbb{A}$ with the composition $\mathbb{C} \to \mathbb{A} \twoheadrightarrow \mathbb{C}$ being the identity map. Using some rather elaborate constructions and a sequence of lemmas, we have the proved the following.

Theorem 3. *A finite bichain is projective in the variety of Birkhoff systems if and only if it does not contain a copy of the three-element bichain* $\mathbb{A}_4$.

In particular, LR-projective coincides with projective.

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Weakly linear systems of fuzzy relation inequalities and equations

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Systems of fuzzy relation equations and inequalities emerged from the study aimed at medical applications [10, 12], and since they have found a much wider field of applications, and have been applied in fuzzy control, discrete dynamic systems, knowledge engineering, identification of fuzzy systems, prediction of fuzzy systems, decision-making, fuzzy information retrieval, fuzzy pattern recognition, image compression and reconstruction, and in other areas.

The most studied systems were *linear systems* of fuzzy relation equations and inequalities, by which we mean systems of the form $U \circ V_i = W_i$ $(i \in I)$, or the dual systems $V_i \circ U = W_i$ $(i \in I)$, or systems that are obtained from them by replacing equalities with inequalities. Here U denotes an unknown fuzzy relation, V_i and W_i are either given fuzzy relations or given fuzzy sets, and $\circ$ denotes the composition operation on fuzzy relations, or between fuzzy sets and fuzzy relations. These systems were first studied by Sanchez [10–13], who discussed linear systems over the Gödel structure, but here we consider them in a more general context, over a complete residuated lattice. It is known that each linear system of inequalities $U \circ V_i \leqslant W_i$ $(i \in I)$ has a solution, and also, it has the greatest one, but the opposite system $W_i \leqslant U \circ V_i$ $(i \in I)$ may not have a solution in general (cf. [8, 9]). Consequently, a linear system of equations $U \circ V_i = W_i$ $(i \in I)$ also need not be solvable, but if it is solvable, then it has the greatest solution, which is the same as the greatest solution to $U \circ V_i \leqslant W_i$ $(i \in I)$ and it was described by Sanchez using fuzzy implication (cf. [11, 13]). In particular, linear systems of equations with $V_i = W_i$, for every i, are solvable and have the greatest solutions.

Here we consider some more complex non-linear systems. In the monopartite case we deal with a single non-empty set A, given fuzzy relations V_i $(i \in I)$ and Z on A, and an unknown fuzzy relation U on A, and we discuss the systems

(M1) $U \circ V_i \leqslant V_i \circ U$ $(i \in I)$, $U \leqslant Z$;
(M2) $V_i \circ U \leqslant U \circ V_i$ $(i \in I)$, $U \leqslant Z$;
(M3) $U \circ V_i = V_i \circ U$ $(i \in I)$, $U \leqslant Z$;
(M4) $U \circ V_i \leqslant V_i \circ U$ $(i \in I)$, $U^{-1} \circ V_i \leqslant V_i \circ U^{-1}$ $(i \in I)$,$U \leqslant Z$;
(M5) $V_i \circ U \leqslant U \circ V_i$ $(i \in I)$, $V_i \circ U^{-1} \leqslant U^{-1} \circ V_i$ $(i \in I)$, $U \leqslant Z$;
(M6) $U \circ V_i = V_i \circ U$ $(i \in I)$, $U^{-1} \circ V_i = V_i \circ U^{-1}$ $(i \in I)$, $U \leqslant Z$.

In the bipartite case we deal with two possibly different non-empty sets A and B, given fuzzy relations V_i $(i \in I)$ on A and W_i $(i \in I)$ on B, a given fuzzy relation Z between A and B, and an unknown fuzzy relation U between A and B, and we discuss the systems:

(B1) $U^{-1} \circ V_i \leqslant W_i \circ U^{-1}$ $(i \in I)$, $U \circ W_i \leqslant V_i \circ U$ $(i \in I)$, $U \leqslant Z$;
(B2) $V_i \circ U \leqslant U \circ W_i$ $(i \in I)$, $W_i \circ U^{-1} \leqslant U^{-1} \circ V_i$ $(i \in I)$, $U \leqslant Z$;
(B3) $U^{-1} \circ V_i = W_i \circ U^{-1}$ $(i \in I)$, $U \leqslant Z$;
(B4) $V_i \circ U = U \circ W_i$ $(i \in I)$, $U \leqslant Z$.

All these systems we call the *weakly linear systems*. The inequality $U \leqslant Z$ is included in all these systems because in many situations we have a task to find solutions contained in a given fuzzy relation.

First we present the main results concerning the monopartite case, which was studied in [5]. We show that each of the systems (M1)–(M6) possesses the greatest solution, and besides, if Z is a fuzzy quasi-order, then the greatest solutions to (M1)–(M3) are fuzzy quasi-orders, and if Z is a fuzzy equivalence, then the greatest solutions to (M4)–(M6) are fuzzy equivalences. The problem of computing the greatest solutions to systems (M1)–(M6) we reduce to the problem of computing the greatest post-fixed points of particular isotone functions on the lattice of fuzzy quasi-orders or the lattice of fuzzy equivalences. For each of the systems (M1)–(M6) we define a suitable isotone function and a descending chain of fuzzy relations which corresponds to this function and this system. If the underlying structure of truth values is a locally finite residuated lattice, then this chain must be finite and its smallest element is the greatest solution we are looking for. But, if this structure is not locally finite, then the chain may not be finite and its infimum may not be equal to the greatest solution to the considered system. We determine some sufficient conditions for the finiteness of the descending chains of the systems (M1)–(M6), as well as some sufficient conditions under which the infima of these chains are equal to the greatest solutions to the systems. It is worth noting that in the iterative procedure for computing the greatest solution to any of the systems (M1)–(M6), every single step may be viewed as the solving a particular linear system, and just for that reason we call these systems weakly linear. The algorithm for computing the greatest solution to any of the systems (M1)–(M6) can be modified so that it computes the greatest crisp solution to this system, and this algorithm works when the underlying structure of truth values is an arbitrary complete residuated lattice. However, the greatest crisp solution can be strictly less, and even have a strictly greater index, than the greatest fuzzy solution to the system.

In the bipartite case, any of the systems (B1)–(B4) does not necessarily have a non-trivial solution (different than the empty relation), but if it has, then it has the greatest solution. For any of the systems (B1) and (B2), if Z is a partial fuzzy function, then this greatest solution is also a partial fuzzy function. Recall that a *partial fuzzy function* is defined as a fuzzy relation R between A and B satisfying $R \circ R^{-1} \circ R \leqslant R$, and a partial fuzzy function which is a surjective $\mathcal{L}$-function is called a *uniform fuzzy relation* [2, 3, 6]. If H is a non-trivial solution to (B1), then $H \circ H^{-1}$ is a solution to $U \circ V_i \leqslant V_i \circ U$ $(i \in I)$, $U \leqslant Z \circ Z^{-1}$, and $H^{-1} \circ H$ is a solution to $U \circ W_i \leqslant W_i \circ U$ $(i \in I)$, $U \leqslant Z^{-1} \circ Z$. Moreover, both $H \circ H^{-1}$ and $H^{-1} \circ H$ are symmetric and transitive fuzzy relations, but they may not be reflexive. We have that $H \circ H^{-1}$ and $H^{-1} \circ H$ are reflexive if and only if H is a uniform fuzzy relation. The most important problems we are dealing with is the examination of the existence of a uniform solution to the system (B1), a solution that is a uniform fuzzy relation, and construction of the greatest uniform solution to (B1), which is also the greatest solution to the system (B1) overall. We show that the existence and construction of the greatest uniform solution are given in terms of certain relationships between the greatest solutions to the corresponding weakly linear systems on A and B.

It is worth noting that weakly linear systems emerged from the fuzzy automata theory, from research aimed at state reduction, bisimulation and equivalence of fuzzy automata, but we show that they also have important applications in other fields, e.g. in the concurrency theory and social network analysis.

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Open questions concerning different kinds of fuzzy orderings

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When considering different kinds of comparison or orderings on a set, several classical kinds of binary relations (e.g.: total preorders, interval orders, semi-orders, acyclic binary relations) appear in a natural way. Needless to say that in all theses classes of relations, when we analyze if an element *a* is related to another element *b*, the relationship is either VOID (empty $= 0$) or TOTAL ($= 1$): Either they are NOT related, or they are. No intermediate situation is allowed.

However, it is typical in many models (e.g.: in Economics, Decision Making, ...) to consider comparisons or binary relations (e.g. "preferences") that are "GRADED", in order, say, to describe an "intensity" in the relationship between two given elements. In this case, two elements could be related "at any level between 0 (empty-void relation) and 1 (totally related)". Of course, in this case, the binary relation becomes FUZZY.

Typical kinds of binary relations established for the crisp setting should be extended to the fuzzy setting, in some appropriate way. However, it is well known that many equivalent definitions that appear in the crisp setting (e.g.: when defining a total preorder, an interval order, a semi-order...) fail to be equivalent when extended (in a natural way) to the fuzzy setting. In this case two natural questions arise: the first one is, despite they are not equivalent, are the different definitions somehow connected? The second question is which one of the alternatives should be considered as the right definition for the fuzzy notion?

We have already worked on the first question in some particular cases as for preorders [2], interval orders [1] or semi-orders [3]. However, there are more definitions than the ones we have considered in these contributions. In addition to this, the definition for some fuzzy concept usually involves a t-norm and if we change the t-norm we get a different definition. Therefore, there are still many open problems related to the first question above.

Concerning the second question, obviously there will not be a unique best definition. On the contrary, usually each definition verifies properties than other definitions do not. Thus, the "right" definition will be related to the context, it will depend on the properties we consider the most important in each case.
As it is known, one good property of any definition for a fuzzy relation is that the definition is preserved when considering α-cuts, i. e., it is very practical to handle definitions

such that the fuzzy relation satisfies the property if and only if their α-cuts verifies the same property (for crisp relations). We will study if different definitions of semi-order have a good behavior in this respect.

In addition, in the crisp setting a typical question is that of converting a given qualitative scale (say a certain kind of ordering or preference –understood as a total preorder, interval order and so on–) by means of a suitable quantitative scale or numerical representation (through, to put an example, a utility function). This question of numerical representability has not been translated yet (in a general and systematic way) to the fuzzy setting, and to the study of graded preferences.

Which could be the difficulties that would immediately arise if we try to do so?

We will analyze these question and related items, trying to introduce the main open problems that will be in order.

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The concave integral for capacities and its applications

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In many economic activities individuals often face risks and uncertainties concerning future events. The probabilities of these events are rarely known, and individuals are left to act on their subjective beliefs. Since the work of Ellsberg (1961), the conventional theory based on (additive) expected utility has become somewhat controversial, both on descriptive and normative grounds. There is a cumulative indication that individuals often do not use regular (additive) subjective probability. Rather, they exhibit what is referred to as an uncertainty aversion.

Schmeidler (1989) proposed an alternative theory to that of additive subjective probabilities. In Schmeidler's model, individuals make assessments that fail to be additive across disjoint events. The expected value of utility with respect to a non-additive probability distribution is defined according to the Choquet integral. The decision maker chooses the act that maximizes the expected utility. Following Choquet, a non-additive probability is referred to as a capacity.

The central theme of my talk is a new integral for capacities, defined in a fashion similar to Lebesgue integral. The key feature of this integral is concavity, interpreted in the context of decision making, as uncertainty aversion.

The talk will be divided into six short parts.

First part – axiomatization. It turns out that four axioms characterize the concave integral. Beyond concavity three more axioms are needed. The first requires that when the underlying probability space consists of one point, the integral coincides with the conventional integral. The second is an axiom of monotonicity with respect to capacities. It states that an additive capacity P assigns to every subset a value which is greater than or equal to that assigned by v, if and only if the integral of any non-negative function with respect to P is greater than or equal to the integral taken with respect to v.

The last axiom states that when integrating an indicator of a set S, the integral depends only on the values that the capacity takes on the subsets of S. In other words, the integral of an indicator of S does not depend on the values that the capacity ascribes to any event outside of S.

Second part – properties of the concave integral. This part is devoted to some essential properties of the concave integral. A particularly important question is to identify the capacities for which the integral coincides with the minimum of the capacity's core members. It turns out that these capacities are those having a large core (Sharkey, 1982).

Third part – integral for fuzzy capacities. Fuzzy capacities assign subjective expected values to some, but not all, random variables (e.g., portfolios). In particular, a fuzzy capacity may assign subjective probabilities only to some events and not to all. An integral w.r.t. fuzzy capacities that aggregates all available information is introduced. The definition of fuzzy capacities enables one to define the integral of a partially-specified capacity. This is essential to the case where the underlying probability is additive but the decision maker is not fully informed of it.

The integral w.r.t. fuzzy capacities is inspired by Azrieli and Lehrer (2007) who used the operational technique (concavification and alike) extensively and employed it to investigate cooperative population games.

Forth part – large spaces. The definition of concave integral for capacities is applied to large spaces. The notion of *loose extendability* is introduced and its relation to the concave integral is studied. Some convergence theorems are given.

Fifth part – Choquet and the concave integral under one roof. A general scheme that generalizes Choquet and the concave integrals is introduced.

Sixth part – applications. Pricing rules determined by the concave integral are discussed.

Collective transitivity in majorities based on difference in support

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1 Introduction

According to *simple majority*, x defeats y when the number of individuals who prefer x to y is greater than the number of individuals who prefer y to x. Since simple majority requires very poor support for declaring an alternative as a winner, other majorities have been introduced and studied in the literature (see Fishburn [2, chapter 6], Ferejohn and Grether [1], Saari [15, pp. 122-123], and García-Lapresta and Llamazares [4], among others).

In order to avoid some drawbacks of simple and absolute majorities, and other voting systems, in García-Lapresta and Llamazares [4] are introduced and analyzed M_k *majorities*, a class of voting systems based on difference of votes. Given two alternatives, x and y, for M_k, x is collectively preferred to y, when the number of individuals who prefer x to y exceeds the number of individuals who prefer y to x by at least a fixed integer k from 0 to $m-1$, where m is the number of voters. We note that M_k majorities are located between simple majority and unanimity, in the extreme cases of $k=0$ and $k=m-1$, respectively. Subsequently, M_k majorities have been characterized axiomatically by Llamazares [9] and Houy [7].

A feature of simple majority, and other classic voting systems, is that they require individuals to declare dichotomous preferences: they can only declare if an alternative is preferred to another, or if they are indifferent. All kinds of preference modalities are identified and voters' opinions are misrepresented.

The importance of considering intensities of preference in the design of appropriate voting systems has been advocated by Nurmi [14]. In this way, García-Lapresta and Llamazares [3] provide some axiomatic characterizations of several decision rules that aggregate fuzzy preferences through different kind of means. Additionally, in [3, Prop. 2], simple majority has been obtained as a specific case of the mentioned decision rules. Likewise, another kind of majorities can be obtained through operators that aggregate fuzzy preferences (on this, see Llamazares and García-Lapresta [11, 12] and Llamazares [8, 10]).

In García-Lapresta and Llamazares [5], majorities based on difference in support are introduced and characterized by means of some independent axioms. These majorities extend majorities based on difference of votes by allowing individuals to show their intensities of preference among alternatives.

In this paper we analyze when majorities based on difference in support provide transitive collective preference relations for every profile of individual preferences satisfying some transitivity conditions.

2 Preliminaries

Consider m voters, $V = \{1,\ldots,m\}$, with $m \geq 2$, showing the intensity of their preferences on n alternatives, $X = \{x_1,\ldots,x_n\}$, with $n \geq 2$, through *reciprocal* preference relations $R_v : X \times X \longrightarrow [0,1]$, for $v = 1,\ldots,m$, i.e., $R_v(x_i,x_j) + R_v(x_j,x_i) = 1$ for all $x_i,x_j \in X$. So, voters can show intensities of preference by means of numbers between 0 and 1: $R_v(x_i,x_j) = 0$, when v prefers absolutely x_j to x_i; $R_v(x_i,x_j) = 0.5$, when v is indifferent between x_i and x_j; $R_v(x_i,x_j) = 1$, when v prefers absolutely x_i to x_j; and, whatever number different to 0, 0.5 and 1, for not extreme preferences, nor for indifference, in the sense that the closer the number is to 1, the more x_i is preferred to x_j (see Nurmi [13] and García-Lapresta and Llamazares [3]). With $\mathcal{R}(X)$ we denote the set of reciprocal preference relations on X.

A *profile* is a vector $(R_1,\ldots,R_m)$ containing the individual reciprocal preferences. Accordingly, the set of profiles is denoted by $\mathcal{R}(X)^m$.

We assume that individual preferences are consistent with respect to a kind of transitivity conditions in the framework of reciprocal preferences (see García-Lapresta and Meneses [6]).

Definition 1. *Given an increasing monotonic function* $g : [0.5,1]^2 \longrightarrow [0.5,1]$, *henceforth a* monotonic operator*,* $R \in \mathcal{R}(X)$ *is* g-transitive *if for all* $x_i,x_j,x_l \in X$*, when* $R(x_i,x_j) > 0.5$ *and* $R(x_j,x_l) > 0.5$*, it holds* $R(x_i,x_l) > 0.5$ *and*

$$R(x_i,x_l) \geq g(R(x_i,x_j),R(x_j,x_l)).$$

With T_g we denote the set of all g-transitive reciprocal preference relations.

Notice that if f and g are two monotonic operators such that $f \leq g$, then $T_g \subseteq T_f$.

In our analysis we have considered the following cases:

1. R is *min-transitive* if R is g-transitive being $g(a,b) = \min\{a,b\}$ for all $(a,b) \in [0.5,1]^2$.
2. R is *am-transitive* if R is g-transitive being $g(a,b) = \frac{a+b}{2}$ for all $(a,b) \in [0.5,1]^2$.
3. R is 1*-transitive* if R is g-transitive being $g(a,b) = 1$ for all $(a,b) \in [0.5,1]^2$.

We denote with $T_{\min}$, T_{am} and T_1 the sets of all min-transitive, am-transitive and 1-transitive reciprocal preference relations, respectively. Clearly, $T_1 \subset T_{am} \subset T_{\min}$.

An *ordinary preference relation* on X is an *asymmetric* binary relation on X: if $x_i P x_j$, then does not happen $x_j P x_i$. With $\mathcal{P}(X)$ we denote the set of ordinary preference relations on X.

$P \in \mathcal{P}(X)$ is *transitive* if for all $x_i, x_j, x_l \in X$ it holds that if $x_i \, P \, x_j$ and $x_j \, P \, x_l$, then it also holds $x_i \, P \, x_l$.

We now introduce the class of *majorities based on difference in support* (García-Lapresta and Llamazares [5]).

Given a threshold $k \in [0,m)$, the $\widetilde{M}_k$ *majority* is the mapping

$$\widetilde{M}_k : \mathcal{R}(X)^m \longrightarrow \mathcal{P}(X)$$

defined by $\widetilde{M}_k(R_1,\ldots,R_m) = P_k$, where

$$x_i \, P_k \, x_j \; \Leftrightarrow \; \sum_{v=1}^{m} R_v(x_i,x_j) > \sum_{v=1}^{m} R_v(x_j,x_i) + k.$$

It is easy to see (García-Lapresta and Llamazares [5]) that $\widetilde{M}_k$ can be defined through the average of the individual intensities of preference:

$$x_i \, P_k \, x_j \; \Leftrightarrow \; \frac{1}{m} \sum_{v=1}^{m} R_v(x_i,x_j) > 0.5 + \frac{k}{2m}.$$

3 The results

We now present necessary and sufficient conditions on thresholds k for ensuring that majorities based on difference in support provide transitive collective preferences P_k for every profile of several types of individual reciprocal preference relations. To be more concrete, we have obtained results for profiles of g-transitive individual preferences such that $g \leq \min$ or $g \geq \mathrm{ma}$.

With K_g we denote the set of thresholds $k \in [0,m)$ such that P_k is transitive for any profile $(R^1,\ldots,R^m) \in T_g^m$. Then, the complement of K_g with respect to $[0,m)$, $(K_g)^c$, is the set of thresholds $k \in [0,m)$ such that P_k is not transitive for some profile $(R^1,\ldots,R^m) \in T_g^m$. Notice that if f and g are two monotonic operators such that $f \leq g$, then $K_f \subseteq K_g$ and, consequently, $(K_g)^c \subseteq (K_f)^c$.

Proposition 1. *There is no* $k \in [0,m)$ *such that* P_k *is transitive for every profile of individual preferences* $(R^1,\ldots,R^m) \in T_{\min}^m$*; so,* $K_{\min} = \emptyset$.

Corollary 1. *For each monotonic operator* $g \leq \min$*, there does not exist* $k \in [0,m)$ *such that* P_k *is transitive for every profile of individual preferences* $(R^1,\ldots,R^m) \in T_g^m$*; in other words,* $K_g = \emptyset$.

Proposition 2. *If* $k \in [m-1,m)$*, then* P_k *is transitive for every profile of individual preferences* $(R^1,\ldots,R^m) \in T_{\mathrm{ma}}^m$*; in other words,* $[m-1,m) \subseteq K_{\mathrm{ma}}$.

Proposition 3. *If* $k \in [0,m-1)$*, then there exists some profile of individual preferences* $(R^1,\ldots,R^m) \in T_1^m$ *such that* P_k *is not transitive; in other words,* $[0,m-1) \subseteq (K_1)^c$.

Corollary 2. *For each monotonic operator* $g \geq \mathrm{ma}$*,* P_k *is transitive for every profile of individual preferences* $(R^1,\ldots,R^m) \in T_g^m$ *if and only if* $k \in [m-1,m)$*; in other words,* $K_g = [m-1,m)$.

4 Further research

Our goal is to get results for any type of individual reciprocal preference relation; in such sense, what remains to be shown is what happens for reciprocal relations between min-transitive and *am*-transitive ones.

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Weighted Banzhaf power and interaction indexes through weighted approximations of games

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Abstract. In cooperative game theory, various kinds of power indexes are used to measure the influence that a given player has on the outcome of the game or to define a way of sharing the benefits of the game among the players. The best known power indexes are due to Shapley [15, 16] and Banzhaf [1, 5] and there are many other examples of such indexes in the literature.
When one is concerned by the analysis of the behavior of players in a game, the information provided by power indexes might be far insufficient, for instance due to the lack of information on how the players interact within the game. The notion of *interaction index* was then introduced to measure an interaction degree among players in coalitions; see [13, 12, 7, 8, 14, 10, 6] for the definitions and axiomatic characterizations of the Shapley and Banzhaf interaction indexes as well as many others.
In addition to the axiomatic characterizations the Shapley power index and the Banzhaf power and interaction indexes were shown to be solutions of simple least squares approximation problems (see [2] for the Shapley index, [11] for the Banzhaf power index and [9] for the Banzhaf interaction index).
We generalize the non-weighted approach of [11, 9] by adding a weighted, probabilistic viewpoint: A weight $w(S)$ is assigned to every coalition S of players that represents the probability that coalition S forms. The solution of the weighted least squares problem associated with the probability distribution w was given in [3, 4] in the special case when the players behave independently of each other to form coalitions.
In this particular setting we introduce a weighted Banzhaf interaction index associated with w by considering, as in [11, 9], the leading coefficients of the approximations of the game by polynomials of specified degrees. We then study the most important properties of these weighted indexes and their relations with the classical Banzhaf and Shapley indexes.

A *cooperative game* on a finite set of players $N = \{1,\dots,n\}$ is a set function $v\colon 2^N \to \mathbb{R}$ which assigns to each coalition S of players a real number $v(S)$ representing the *worth* of S.[1] Identifying the subsets of N with the elements of $\{0,1\}^n$, we see that a game $v\colon 2^N \to \mathbb{R}$ corresponds to a pseudo-Boolean function $f\colon \{0,1\}^n \to \mathbb{R}$ (the correspondence is given by $v(S) = f(\mathbf{1}_S)$, where $\mathbf{1}_S$ denotes the characteristic vector of

[1] Usually, the condition $v(\varnothing) = 0$ is required for v to define a game. However, we do not need this restriction in the present work.

S in $\{0,1\}^n$). We will henceforth use the same symbol to denote both a given pseudo-Boolean function and its underlying set function (game).

Every pseudo-Boolean function $f\colon \{0,1\}^n \to \mathbb{R}$ can be represented by a multilinear polynomial of degree at most n of the form

$$f(\mathbf{x}) = \sum_{S\subseteq N} a(S) \prod_{i\in S} x_i\,,$$

where the set function $a\colon 2^N \to \mathbb{R}$ is the *Möbius transform* of f.

Let $\mathcal{G}^N$ denote the set of games on N. A *power index* [15] on N is a function $\phi\colon \mathcal{G}^N \times N \to \mathbb{R}$ that assigns to every player $i \in N$ in a game $f \in \mathcal{G}^N$ his/her prospect $\phi(f,i)$ from playing the game. An *interaction index* [10] on N is a function $I\colon \mathcal{G}^N \times 2^N \to \mathbb{R}$ that measures in a game $f \in \mathcal{G}^N$ the interaction degree among the players of a coalition $S \subseteq N$.

For instance, the *Banzhaf interaction index* [10] of a coalition $S \subseteq N$ in a game $f \in \mathcal{G}^N$ is defined by

$$I_{\mathrm{B}}(f,S) = \sum_{T\supseteq S} \left(\frac{1}{2}\right)^{|T|-|S|} a(T) = \frac{1}{2^{n-|S|}} \sum_{T\subseteq N\setminus S} (\Delta^S f)(T), \tag{1}$$

where the *S-difference* $\Delta^S f$ is defined inductively by $\Delta^{\varnothing} f = f$ and $\Delta^S f = \Delta^{\{i\}}\Delta^{S\setminus\{i\}} f$ for $i \in S$, with $\Delta^{\{i\}} f(\mathbf{x}) = f(\mathbf{x} \mid x_i = 1) - f(\mathbf{x} \mid x_i = 0)$. The *Banzhaf power index* [5] of a player $i \in N$ in a game $f \in \mathcal{G}^N$ is then given by $\phi_{\mathrm{B}}(f,i) = I_{\mathrm{B}}(f,\{i\})$.

Let us now introduce a weighted least squares approximation problem which generalizes the one considered in [11,9]. For $k \in \{0,\dots,n\}$, denote by V_k the set of all multilinear polynomials $g\colon \{0,1\}^n \to \mathbb{R}$ of degree at most k, that is of the form

$$g(\mathbf{x}) = \sum_{\substack{S\subseteq N\\ |S|\leqslant k}} c(S) \prod_{i\in S} x_i, \qquad c(S) \in \mathbb{R}.$$

We also consider a weight function $w\colon \{0,1\}^n \to \,]0,\infty[$. For every pseudo-Boolean function $f\colon \{0,1\}^n \to \mathbb{R}$, we define the *best kth approximation of f* as the unique multilinear polynomial $f_k \in V_k$ that minimizes the squared distance

$$\sum_{\mathbf{x}\in\{0,1\}^n} w(\mathbf{x})\big(f(\mathbf{x}) - g(\mathbf{x})\big)^2 = \sum_{S\subseteq N} w(S)\big(f(S) - g(S)\big)^2 \tag{2}$$

among all functions $g \in V_k$.

Clearly, we can assume without loss of generality that the weights $w(S)$ are (multiplicatively) normalized so that $\sum_{S\subseteq N} w(S) = 1$. We then immediately see that the weights define a probability distribution over 2^N and we can interpret $w(S)$ as the probability that coalition S forms, that is, $w(S) = \Pr(C = S)$, where C denotes a random coalition.

In the special case of equiprobability, the approximation above reduces to standard least squares, and a closed form expression of the approximation f_k of f was given in [11,9] and it was shown that, writing

$$f_k(\mathbf{x}) = \sum_{\substack{S\subseteq N\\ |S|\leqslant k}} a_k(S) \prod_{i\in S} x_i, \tag{3}$$

we have

$$I_{\mathrm{B}}(f,S) = a_{|S|}(S). \tag{4}$$

Thus $I_{\mathrm{B}}(f,S)$ is exactly the coefficient of the monomial $\prod_{i\in S} x_i$ in the best approximation of f by a multilinear polynomial of degree at most $|S|$.

Now, suppose that the players behave independently of each other to form coalitions, which means that the events $(C \ni i)$, for $i \in N$, are independent. Under this assumption, the weight function w is completely determined by the vector $\mathbf{p} = (p_1,\ldots,p_n)$, where $p_i = \Pr(C \ni i) = \sum_{S\ni i} w(S)$ (we assume $0 < p_i < 1$), by the formula

$$w(S) = \prod_{i\in S} p_i \prod_{i\in N\setminus S} (1-p_i).$$

In this particular setting, the weighted approximation problem was presented and solved in [3] and [4, Theorem 4] by noticing that the distance in (2) is the natural L^2-distance associated with the measure w, with respect to the inner product

$$\langle f,g\rangle = \sum_{\mathbf{x}\in\{0,1\}^n} w(\mathbf{x}) f(\mathbf{x}) g(\mathbf{x}),$$

and that the functions

$$v_S\colon \{0,1\}^n \to \mathbb{R}\colon \mathbf{x} \mapsto \prod_{i\in S} \frac{x_i - p_i}{\sqrt{p_i(1-p_i)}}$$

form an orthonormal basis of the vector space of pseudo-Boolean functions.

Using these functions, we immediately obtain that f_k is of the form (3) where

$$a_k(S) = \sum_{\substack{T\supseteq S\\ |T|\leqslant k}} \frac{\prod_{i\in T\setminus S}(-p_i)}{\prod_{i\in T}\sqrt{p_i(1-p_i)}} \langle f, v_T\rangle.$$

Using this solution, we define the index by analogy with (4).

Definition 1. *The weighted Banzhaf interaction index associated to w is*

$$I_{\mathrm{B},\mathbf{p}}\colon \mathcal{G}^N \times 2^N \to \mathbb{R} : (f,S) \mapsto I_{\mathrm{B},\mathbf{p}}(f,S) = a_{|S|}(S) = \frac{\langle f, v_S\rangle}{\prod_{i\in S}\sqrt{p_i(1-p_i)}}.$$

Then we show that most of the properties of the standard Banzhaf index can be generalized to the weighted index. For instance, Formula (1) is a particular case of

$$I_{\mathrm{B},\mathbf{p}}(f,S) = \sum_{T\supseteq S} a(T) \prod_{i\in T\setminus S} p_i = \sum_{T\subseteq N\setminus S} p_T^S\,(\Delta^S f)(T),$$

where $p_T^S = \Pr(T \subseteq C \subseteq S\cup T) = \prod_{i\in T} p_i \prod_{i\in(N\setminus S)\setminus T}(1-p_i)$.

This shows that the weighted Banzhaf interaction index belongs to the class of probabilistic interaction indexes introduced in [6], and we can moreover provide a nice interpretation of the probabilities p_T^S as conditional probabilities.

We then analyze the behaviour of the index with respect to null or dummy players or more generally to dummy coalitions, and we show how to compute the weighted Banzhaf index in terms of Owen's multilinear extension $\overline{f}$ of the game f. We also provide conversion formulas between the indexes corresponding to different weights, and show how to recover f from the weighted Banzhaf index.

Finally, we show that the standard Banzhaf index is the average of the weighted Banzhaf indexes over all the possible weights and that the Shapley index is the average of the weighted Banzhaf indexes over all possible symmetric weights.

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Local and global classification of aggregation functions

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Classification of objects considered in any domain is an important tool for the transparentness, better understanding of the considered domain, but also for construction and application of discussed objects. As an example, recall conjunction operators in many–valued logics. They are characterized by the boolean conjunction of propositions "i–th input is greater or equal to the output". Similarly, disjunction operators are characterized by the boolean conjunction of propositions "i–th input is smaller or equal to the output". The aim of this contribution is to open the topic of classification of aggregation functions acting on bounded posets (covering, among others, conjunction and disjunction operators in many–valued logics). In the area of aggregation functions acting on real intervals, such a classification was proposed by Dubois and Prade at AGOP'2001 conference in Oviedo, see also [12]. In Dubois – Prade approach, conjunctive, disjunctive, averaging and remaining aggregation functions were considered, defined by their relationship to *Min* and *Max* functions. The class $\mathcal{C}$ of all (n–ary) conjunctive functions (acting on a real interval $[a,b]$) is characterized by the inequality $A \leq Min$, while the inequality $A \geq Max$ is characteristic for the disjunctive aggregation functions. Concerning the averaging aggregation functions, they should satisfy $Min \leq A \leq Max$. To exclude the trivial overlapping of conjunctive and averaging (disjunctive and averaging) aggregation functions, the class $\mathcal{P}$ of pure averaging aggregation functions consists of all averaging aggregation functions up to *Min* and *Max*. Denoting $\mathcal{A}$ the class of all aggregation functions (n–ary, on real interval $[a,b]$), $\mathcal{R} = \mathcal{A} \setminus (\mathcal{C} \cup \mathcal{P})$ consists of all remaining aggregation functions, which are neither conjunctive, nor disjunctive nor averaging. Thus this standard classification $(\mathcal{C}, \mathcal{D}, \mathcal{P}, \mathcal{R})$ forms a partition of the class $\mathcal{A}$. In several domains we need to classify the aggregation of more complex objects, which rarely form a chain, but they can be considered as elements of some (bounded) lattice or poset (we will use this abbreviation for a partially ordered set throughout this paper). This is, for example, the case of aggregation of fuzzy sets (intersection, union), of distribution functions (convolution), etc. However, such a classification of aggregation functions on posets is missing in the literature so far. Obviously, we cannot repeat the approach of Dubois and Prade once *Min* and *Max* are not defined.

Consider a poset $(P, \leq, \mathbf{0}, \mathbf{1})$ and a non–decreasing mapping $A : P^n \to P$ satisfying $A(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}, A(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$. Then A is called an (n–ary) aggregation function on P, and we denote by $\mathcal{A}$ the class of all such mappings.

For a given aggregation function $A : P^n \to P$ and $\mathbf{x} = (x_1, \ldots, x_n) \in P^n$ we denote:

$$g_A(\mathbf{x}) = card\{i | x_i \geq A(\mathbf{x})\},$$

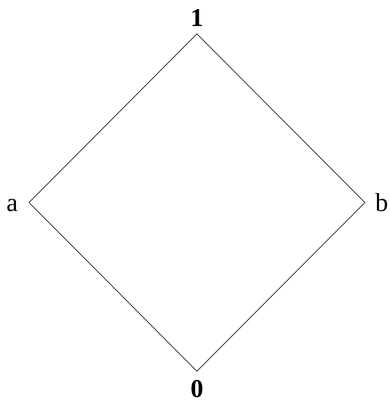

Fig. 1. Hasse diagram of the diamond lattice D

and

$$s_A(\mathbf{x}) = card\{i|x_i \le A(\mathbf{x})\}.$$

For any subdomain $E \subseteq P^n$, we define mappings $\gamma^E, \sigma^E : \mathcal{A} \to \{0,1,\ldots,n\}$ by

$$\gamma^E(A) = inf\{g_A(\mathbf{x})|\mathbf{x} \in E\},$$

$$\sigma^E(A) = inf\{s_A(\mathbf{x})|\mathbf{x} \in E\}.$$

Finally, we abbreviate $\gamma^{P^n} = \gamma, \sigma^{P^n} = \sigma$.

Functions γ^E and σ^E allow to introduce a classification of aggregation functions from $\mathcal{A}$.

Proposition 1. *$\mathcal{C}^E = \{\mathcal{C}_0^E, \mathcal{C}_1^E, \ldots, \mathcal{C}_n^E\}$ and $\mathcal{D}^E = \{\mathcal{D}_0^E, \mathcal{D}_1^E, \ldots, \mathcal{D}_n^E\}$ given by $\mathcal{C}_i^E = (\gamma^E)^{-1}(\{i\})$ and $\mathcal{D}_i^E = (\sigma^E)^{-1}(\{i\})$, $i = 0,1,\ldots,n$, are partitions of $\mathcal{A}$.*

Classifications $\mathcal{C}^E, \mathcal{D}^E$ based on $E \neq P^n$ will be called local, while classifications $\mathcal{C}^{P^n} = \mathcal{C}, \mathcal{D}^{P^n} = \mathcal{D}$ will be called global.

Example 1. **(i)** Consider, for example, the diamond lattice $D = \{\mathbf{0}, a, b, \mathbf{1}\}$ visualised in Figure 1. Then a mapping $A : D \to D$ is a unary aggregation function on D if and only if $A(\mathbf{0}) = \mathbf{0}$ and $A(\mathbf{1}) = \mathbf{1}$ (i.e., the values $A(a)$ and $A(b)$ can be chosen arbitrarily). Moreover $A \in \mathcal{C}_1^D \cap \mathcal{D}_1^D$ if and only if $A(a) = a$, $A(b) = b$.

Define mapping $B : D^2 \to D$ as follows (for $x, y \in D$):

$$B(x,y) = \mathbf{0} \text{ if } \mathbf{0} \in \{x,y\},$$

$$B(x,y) = \mathbf{1} \text{ if } \mathbf{1} \in \{x,y\} \text{ and } \mathbf{0} \notin \{x,y\},$$

$$B(x,x) = x \text{ if } x \in D,$$

$$B(a,b) = B(b,a).$$

Then B is well defined and it is a binary aggregation function on D. Moreover, if $B(a,b) = \mathbf{0}$, then $B \in \mathcal{C}_1^{D^2} \cap \mathcal{D}_0^{D^2}$. If $B(a,b) = \mathbf{1}$ then $B \in \mathcal{C}_0^{D^2} \cap \mathcal{D}_1^{D^2}$. Finally, if $B(a,b) \in \{a,b\}$ then $B \in \mathcal{C}_1^{D^2} \cap \mathcal{D}_1^{D^2}$.
Finally, we introduce a ternary aggregation function $C : D^3 \to D$ as follows: $C(x,y,z) = u$ whenever the triple $\{x,y,z\}$ contains at least two times u; $C(x,y,z) = \mathbf{0}$ whenever $\{x,y,z\} = \{\mathbf{0},a,b\}$ and $C(x,y,z) = \mathbf{1}$ whenever $\{x,y,z\} = \{\mathbf{1},a,b\}$. Then $C \in \mathcal{C}_1^{D^3} \cap \mathcal{D}_1^{D^3}$.

(ii) Consider the product $\Pi : [0,\infty]^n \to [0,\infty]$. Then $\Pi \in \mathcal{C}_n^{[0,1]^n}, \Pi \in \mathcal{D}_n^{[1,\infty]^n}, \Pi \in \mathcal{C}_0 \cap \mathcal{D}_0$.

All next considerations for local and global classifications are similar and thus we will discuss global classifications only.

Proposition 2. *Let $A : P^n \to P$ be a fixed aggregation function. Then $\gamma(A) + \sigma(A) \leq n+1$, and if P is not a chain, then $\gamma(A) = n$ implies $\sigma(A) = 0$, and $\sigma(A) = n$ implies $\gamma(A) = 0$.*

An aggregation function $A : P^n \to P$ belongs to the class $\mathcal{C}_n$ ($\mathcal{D}_n$) if and only if for all $\mathbf{x} = (x_1, \ldots, x_n) \in P^n$ it holds $x_i \geq A(\mathbf{x})$ ($x_i \leq A(\mathbf{x})$) for each $i \in \{1, \ldots, n\}$. In the case of standard aggregation functions on $[0,1]$ (or any real interval) this means that A is conjunctive (disjunctive). Therefore, aggregation functions from the class $\mathcal{C}_n$ will be called *strongly conjunctive*, and we identify $\mathcal{C}_n = \mathcal{C}_s$. Moreover, the aggregation functions from $\mathcal{C}_w = \bigcup_{i=1}^{n-1} \mathcal{C}_i$ will be called *weakly conjunctive*. Finally, $\mathcal{C}_a = \mathcal{C}_0$ is the class of aggregation functions admitting the existence of $\mathbf{x} \in P^n$ such that for each $i \in \{1, \ldots, n\}$, either $x_i < A(\mathbf{x})$ or $x_i \perp A(\mathbf{x})$ (x_i is incomparable to $A(\mathbf{x})$); these aggregation functions will be called *anticonjunctive*. Similarly, the classes $\mathcal{D}_s, \mathcal{D}_w$ and $\mathcal{D}_a$ of strongly disjunctive, weakly disjunctive and antidisjunctive aggregation functions can be introduced.

Definition 1. *Let an aggregation function $A : P^n \to P$ be given. Then*

1. *A is called strongly averaging whenever it is both weakly conjunctive and weakly disjunctive, $A \in \mathcal{A}_s = \mathcal{C}_w \cap \mathcal{D}_w$;*
2. *A is called weakly averaging whenever it is either weakly conjunctive or weakly disjunctive, $A \in \mathcal{A}_w = \mathcal{C}_w \cup \mathcal{D}_w$.*

Definition 2.
1. *The partition $\{\mathcal{C}_s,\ \mathcal{A}_w,\ \mathcal{D}_s,\ \mathcal{C}_0 \cap \mathcal{D}_0\}$ will be called a weak classification of the class $\mathcal{A}$ of all n–ary aggregation functions on a fixed poset $(P, \leq, \mathbf{0}, \mathbf{1})$.*
2. *The partition $\{\mathcal{C}_s,\ \mathcal{A}_s,\ \mathcal{D}_s,\ \mathcal{C}_w \setminus \mathcal{D}_w,\ \mathcal{D}_w \setminus \mathcal{C}_w,\ \mathcal{C}_0 \cap \mathcal{D}_0\}$ will be called a strong classification of the class $\mathcal{A}$ of all n–ary aggregation functions on a fixed poset $(P, \leq, \mathbf{0}, \mathbf{1})$.*

In the case of bounded lattices which are not chains we have three different general classifications of aggregation functions:

- weak classification $\{\mathcal{C}_s, \mathcal{D}_s, \mathcal{A}_w, \mathcal{C}_0 \cap \mathcal{D}_0\}$
- strong classification $\{\mathcal{C}_s, \mathcal{D}_s, \mathcal{A}_s, \mathcal{C}_w \setminus \mathcal{A}_s, \mathcal{D}_w \setminus \mathcal{A}_s, \mathcal{C}_0 \cap \mathcal{D}_0\}$
- lattice classification $\{\mathcal{C}_s, \mathcal{D}_s, \mathcal{A}_l, \mathcal{R}\}$, where $\mathcal{R} = (\mathcal{C}_s \cup \mathcal{D}_s \cup \mathcal{A}_l)$ and $A \in \mathcal{A}_l$ if and only if $Min \leq A \leq Max$, and $A \notin \{Min, Max\}$.

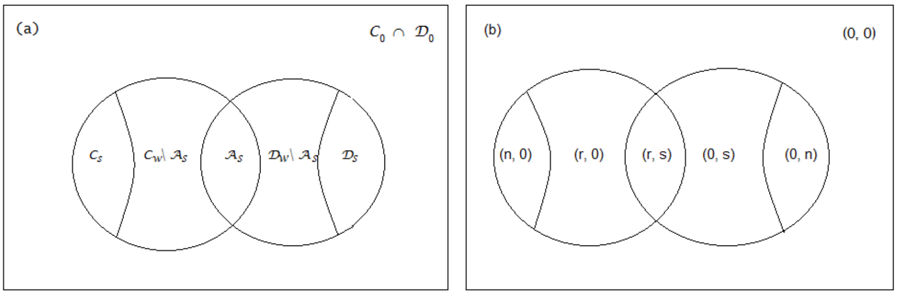

Fig. 2. Strong classification on the class $\mathcal{A}$ described by (a) classes symbols, (b) values of $(\gamma(A), \sigma(A))$, here r, s are arbitrary values from $\{1, \ldots, n-1\}$

Note that each strongly conjunctive aggregation function A satisfies $A(\mathbf{x}) \leq x_i$ (for each i and each $\mathbf{x}$) and thus $A \leq Min$, and thus $\mathcal{C}_S \cap \mathcal{A}_l = \emptyset$. Similarly, $\mathcal{D}_S \cap \mathcal{A}_l = \emptyset$.

Note that there are some sufficient conditions ensuring the belongingness of considered aggregation functions into a relevant class. So, for example, let $A : P^n \to P$ be an aggregation function with neutral element **1** (**0**). Then necessarily A is strongly conjunctive (strongly disjunctive). Internality of A (i.e., $A(\mathbf{x}) \in \{x_1, \ldots, x_n\}$ for all $\mathbf{x} \in P^n$) forces $\gamma(A) \geq 1$ and $\sigma(A) \geq 1$, thus if P is not a chain, A is necessarily strongly averaging. Observe also that any kind of averaging we have introduced ensures the idempotency of A. It is well known that if P is a chain then also the reverse claim is valid (note that then all introduced concepts of averaging coincide). In general, if P is a lattice, the idempotency is equivalent to the lattice – averaging concept (but neither to the strong averaging nor to the weak averaging). Moreover, if P is a lattice, then the only idempotent strongly conjunctive (strongly disjunctive) aggregation function is Min (Max). This is not true on a general poset, where we can have several strongly conjunctive (strongly disjunctive) idempotent aggregation functions.

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Min-product semiring of transition bistochastic matrices and mobility measures in social sciences

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1 Introduction

Mobility indices play important role in social sciences. Modeling mobility indices Shorrocks [24] defined them on a set of transition matrices $\mathcal{T}$ as a continuous function $M : \mathcal{T} \to \mathbb{R}$. He defines the mobility measure M for some transition matrix $P \in \mathcal{T}$ as a bounded function $M(I) \leqslant M(P) \leqslant M(Q)$, where I represents a unit matrix to which the minimal mobility value $M(I) = 0$ is assigned, and Q is a transition matrix which has all identical rows and the maximal mobility $M(Q) = 1$. In the investigation of mobility it is clear which matrices have to have the minimal and maximal value of the mobility index, but the question how we should rank matrices which have the value of the mobility index between 0 and 1 remains open. Different mobility measures induce different orderings of transition matrices [5], while the choice of mobility measure depends on the kind of investigation. Dardanoni [9, 10] introduced a partial ordering on a restricting the domain of transition matrices on monotone matrices. Aebi, Neusser and Steiner [1] define a total quasi-ordering on a set of transition matrices by introducing so called 2-decreasing mobility functional. Some other type of partial orderings on the domain of transition matrices are also given in [8, 23].

Square matrices over a semiring gebnerate also a semiring, see [15]. In this paper we investigate the ordering on a set of monotone bistochastic (doubly stochastic) transition matrices by forming a semiring in which mobility measure induces an ordering in the Shorrocks' sense. The proofs and further results are contained in the paper [12, 20]. We shall use the notions an results related to Markov chains and transition matrices [22], nonnegative matrices [13, 18, 22], as well as semiring theory [7, 11, 15, 17, 21].

2 Monotone transition matrices

In the mobility theory there is a need for a restriction of the domain of transition matrices (see [24]). Many authors propose for that purpose the class of monotone transition matrices which play important role in intergenerational mobility ([6, 9, 16]).

Example 1. Let X and Y be father's and son's socio-economic status, respectively, with n possible values, which correspond to socio-economic classes ordered from the worst to the best. The corresponding discrete Markov chains are given by the equation $p'_y = p'_x P$, where P denotes the $n \times n$ transition matrix with transition probabilities p_{ij}, i.e., probabilities that the son is in the class j, if the father is in the class i, and p_x, p_y are marginal distributions of father's and son's social status. Each row i, of the transition matrix P, represents the probability distribution of the son whose father belongs to the social class i. The sons whose fathers have a higher social status have an advantage in the relation to the sons whose fathers are of a lower social status.

Therefore, in the intergenerational mobility there are considered monotone matrices [13]: A transition matrix $P = [p_{ij}]_{n\times n}$ of discrete Markov chain with ordered state is *monotone* if each row stochastically dominates the row above it, i.e.,

$$\sum_{j=1}^{l} p_{(i+1)j} \geqslant \sum_{j=1}^{l} p_{ij} \text{ for all } i = 1,2,\ldots,n-1 \text{ and } l = 1,2,\ldots,n-1.$$

3 The mobility measure in Shorrocks' sense

Definition 1. A function $M : \mathcal{T} \to \mathbb{R}$ is *mobility measure in Shorrocks' sense*, on a domain of transition matrices $\mathcal{T}$, if it satisfies the following conditions

(N) Normalization: $0 \leqslant M(P) \leqslant 1$, for all $P \in \mathcal{T}$.

(M) Monotonicity: Mobility index reflects the change of increase in the matrix off-diagonal elements at the expense of diagonal elements. We write $P \succ_s P'$ when

$$\mathbf{min}(P,P') = P' \text{ if } p_{ij} \geqslant p'_{ij} \text{ for all } i \neq j \text{ and } p_{ij} > p'_{ij} \text{ for some } i \neq j \qquad (1)$$

holds, and then $P \succ_s P'$ implies $M(P) > M(P')$.

(I) Immobility: $M(I) = 0$, where I is the unit matrix.

(PM) Perfect mobility: Matrices with identical rows have the mobility index 1.

Shorrocks [24] has given a counterexample for the monotonicity axiom and the perfect mobility axiom. Shorrocks assumes that a perfectly mobile structure is given by the maximal value of the mobility measure and that the precise ranking is insignificant, so the main conflict remains between the monotonicity axiom and the perfect mobility axiom. As one of the way out of this conflict Shorrocks proposed adapting the monotonicity condition by replacing the condition $M(P) > M(P')$ by a weaker one.

Definition 2 (Weak monotonicity (WM)). We have that $P \succeq P'$ implies $M(P) \geqslant M(P')$, where the condition $P \succeq P'$ is related to the operation given by (1).

Definition 3. Let $\mathcal{P}$ be the set of all bistochastic transition $n \times n$ matrices with the unit matrix I. We say that a matrix $P \in \mathcal{P}$ is *more mobile in the Shorrocks' sense* than a matrix $Q \in \mathcal{P}$, in the notation $P \succeq Q$, if it holds $M(P) \geqslant M(Q)$, where M is a mobility measure which satisfies axioms (I), (WM), (PM).

In our investigations of the mobility index and the corresponding order we have used special semirings of transition matrices. One of the result, see [12], is contained in the following theorem.

Theorem 1. *Let $\mathcal{P}'$ be the set of all primitive irreducible bistochastic transition $n \times n$ matrices corresponding to homogeneous Markov chain, with the unit matrix I, endowed with the (idempotent) operation* $\mathbf{min} : \mathcal{P}'^2 \to \mathcal{P}'$ *defined for each two matrices P_i, P_j from $\mathcal{P}'$ in the following way:*

$$\mathbf{min}(P_i, P_j) = P_i \quad \textit{if} \quad M(P_j) \geqslant M(P_i),$$

which induces the ordering on $\mathcal{P}'$ in the sense of Definition 3. Then $(\mathcal{P}', \mathbf{min}, *)$, where $*$ is the usual operation of the matrix multiplication, is a semiring with a neutral and an unit element.

Many authors have offered an outline of mobility measures and properties that meet them ([2, 5, 16, 25]). Therefore it was natural to investigate whether some of known mobility measures satisfy the conditions of Theorem 1, and then the properties of the ordering induced by this mobility measure. It turns out ([12]) that Bartholomew's index [4], which is an average number of income classes crossed by individuals, satisfies the required conditions.

Proposition 1. *The normalized Bartholomew's index $M_{NB} : \mathcal{P}^2 \to \mathbb{R}$ given by*

$$M_{NB}(P) = \frac{3}{n^2 - 1} \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} |i - j|,$$

satisfies the immobility axiom, the monotonicity axiom and the perfect mobility axiom on the set $\mathcal{P}'$. It induces in the semiring $(\mathcal{P}', \mathbf{min}, *)$ a partial ordering.

D'Agostino and Dardanoni [8] have introduced *Spearman's footrule* as a mobility function for a permutation matrix P given by

$$M_S(P) = \sum_{(i,j) \in S(P)} |i - j|,$$

where $S(P) = \{(i, j) \mid p_{ij} = 1\}$ is the characteristic set of P.

Proposition 2. *The normalized Bartholomew's index gives the same ordering of permutation matrices in the semiring $(\mathcal{P}', \mathbf{min}, *)$ as the Spearman's footrule.*

Monotone matrices occurring in the investigation of the intergenerational mobility are mostly doubly stochastic, see [25]. Namely, a special attention is paid to mobility indices that reflect equality of life chances, usually called equilibrium mobility indices. They satisfy the perfect mobility axiom, and therefore the mobility index reflects the equality of the sons' life chances irrelevant of their fathers' social class. On the other side, mobility indices have to reflect a greater mobility in the case when sons go far from their fathers' social class, and therefore it is desirable that they satisfy also the monotonicity axiom. Therefore, we have transferred, in [12], the previous results to a special class of monotone doubly stochastic transition matrices, and we prove that the corresponding ordering is compatible with restrictions of some well-known orderings to this set, e.g., as Dardanoni's ordering [9] and an ordering given by Aebi, Neusser and Steiner [1].

We shall investigate in the future the extension of the whole theory to continuous state Markov processes, since the theory of stochastic monotonicity is recently well developed there, e.g., [3, 14].

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Conjoint measurement and valued relations

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Conjoint measurement studies relations defined on cartesian products. In multiple-criteria decision making, preference relations on a set of alternatives described by their multi-dimensional vector of attributes can often be represented by means of a conjoint measurement model. Another application field is decision making under uncertainty where a preference relation compares alternatives evaluated on different states of nature. Many other situations in decision theory are amenable to such models, which justifies their study.

The first part of our talk presents a classical result of this theory, namely a characterization of relations that can be represented by an additive value function. The relations characterized in such a manner are very particular since they are weak orders (complete transitive relations) on the elements of the cartesian product. We show how the axiomatic analysis of the relations that are representable by an additive value function offers clues for the elicitation of these functions. In practice, indeed, the preference relations on the cartesian product are unknown and it is the aim of the decision aiding process to reveal and represent them. Hence, as long as we may consider the Decision Maker's preference as compatible with the axioms of the model, we may in principle obtain information from the DM that allow us to construct the additive value function. Note that the necessary information is obtained from the decision maker through questions formulated in terms of preference only. The preference relation normally is the only observable in such models.

In the second part of this talk, we introduce and discuss models that have been developed more recently, yet in the same spirit, and are more directly connected with the interests of the Fuzzy Sets Community. While the classical additive value functions theory is concerned with preferences that are transitive and complete relations, Denis Bouyssou and myself have been working on more general models that encompass wide categories of binary relations. Two main types of models have been analyzed, namely these based on traces on differences and those based on traces on levels [1–4].

Building on this work, we show how to characterize a general class of valued relations on a cartesian product. In this model, the value associated with each pair of alternatives has an ordinal character. In other words, such a valued relation is equivalent to a chain of binary relations, which is the set of cuts of the valued relation. Such a model may be viewed as an ordinal aggregation procedure that takes the attribute vectors of any pair of alternatives as input and returns the "value" associated with this pair in the preference relation. We discuss some examples of aggregation procedures used in practice which fit this model.

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Superlinear extensions of exact games on σ-algebras
A probabilistic representation

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1 Exact Functionals and Exact Games

Let $(\Omega,\mathscr{F})$ be a measurable space, where $\mathscr{F}$ is a σ-algebra of subsets of a nonempty set Ω. Denote by $B(\Omega,\mathscr{F})$ the vector space of bounded measurable functions on Ω with the sup norm and by $ba(\Omega,\mathscr{F})$ the vector space of finitely additive bounded set functions on $\mathscr{F}$, which is the dual space of $B(\Omega,\mathscr{F})$, with the corresponding duality $\langle \mu, X\rangle$ for $X \in B(\Omega,\mathscr{F})$ and $\mu \in ba(\Omega,\mathscr{F})$ given by the integration. Let $ca(\Omega,\mathscr{F})$ be the vector subspace of $ba(\Omega,\mathscr{F})$ consisting of all countably additive bounded set functions on $\mathscr{F}$.

Let $(\Omega,\mathscr{F},P)$ be a complete probability space. Denote by $L^\infty(\Omega,\mathscr{F},P)$ the vector space of P-essentially bounded functions on Ω with the sup norm. The norm dual of $L^\infty(\Omega,\mathscr{F},P)$ is the vector subspace $ba(\Omega,\mathscr{F},P)$ of $ba(\Omega,\mathscr{F})$ consisting of all finitely additive bounded set functions on $\mathscr{F}$ which vanish at every $A\in\mathscr{F}$ with $P(A)=0$. Let $ca(\Omega,\mathscr{F},P)$ be the vector subspace of $ba(\Omega,\mathscr{F},P)$ consisting of all countably additive bounded set functions on $\mathscr{F}$. Denote by $\mathscr{P}(\Omega,\mathscr{F},P)$ the set of probability measures in $ca(\Omega,\mathscr{F},P)$.

Definition 1. *A functional $\Gamma : B(\Omega,\mathscr{F}) \to \mathbb{R}$ is exact if the following conditions are satisfied.*

(Upper semicontinuity): Γ *is upper semicontinuous.*
(Superadditivity): $\Gamma(X+Y) \ge \Gamma(X)+\Gamma(Y)$ *for every* $X,Y \in B(\Omega,\mathscr{F})$.
(Positive homogeneity): $\Gamma(\alpha X) = \alpha\Gamma(X)$ *for every* $X \in B(\Omega,\mathscr{F})$ *and* $\alpha \ge 0$.
(Translation invariance): $\Gamma(X+\alpha) = \Gamma(X)+\alpha\Gamma(1)$ *for every* $X \in B(\Omega,\mathscr{F})$ *and* $\alpha\in\mathbb{R}$.

A functional satisfying superadditivity and positive homogeneity is said to be *superlinear*.

Theorem 1. *A functional $\Gamma : B(\Omega,\mathscr{F}) \to \mathbb{R}$ is exact if and only if there exists a unique weak*-compact, convex subset $\mathscr{C}$ of $ba(\Omega,\mathscr{F})$ such that $\Gamma(X) = \min_{\mu\in\mathscr{C}}\langle \mu, X\rangle$ for every $X \in B(\Omega,\mathscr{F})$, where $\mathscr{C}$ is of the form*

$$\mathscr{C} = \{\mu \in ba(\Omega,\mathscr{F}) \mid \Gamma(X) \le \langle \mu, X\rangle \ \forall X \in B(\Omega,\mathscr{F}),\ \mu(\Omega)=\Gamma(1)\}.$$

A *set function* is a real-valued function on $\mathscr{F}$. A set function $\nu : \mathscr{F} \to \mathbb{R}$ with $\nu(\emptyset)=0$ is a *game*. The *core* $\mathscr{C}(\nu)$ of a game ν is defined by

$$\mathscr{C}(\nu) = \{\mu \in ba(\Omega,\mathscr{F}) \mid \nu \le \mu \text{ and } \mu(\Omega)=\nu(\Omega)\}.$$

A game is *balanced* if its core is nonempty. A balanced game $\nu : \mathscr{F} \to \mathbb{R}$ is *exact* if $\nu(A) = \min_{\mu\in\mathscr{C}(\nu)} \mu(A)$ for every $A \in \mathscr{F}$.

Definition 2. *A functional $\Gamma : B(\Omega,\mathscr{F}) \to \mathbb{R}$ is an exact extension of a game $\nu : \mathscr{F} \to \mathbb{R}$ to $B(\Omega,\mathscr{F})$ if it is exact and $\Gamma(\chi_A) = \nu(A)$ for every $A \in \mathscr{F}$. The minimal exact extension Γ_ν of ν is an exact extension of ν such that $\Gamma_\nu \le \Gamma$ for every exact extension Γ of ν.*

Recall that *superdifferential* $\partial\Gamma(X)$ of Γ at $X \in B(\Omega,\mathscr{F})$ is given by:

$$\partial\Gamma(X) = \{\mu \in ba(\Omega,\mathscr{F}) \mid \Gamma(Y) - \Gamma(X) \le \langle \mu, Y - X\rangle \ \forall Y \in B(\Omega,\mathscr{F})\},$$

where an element in $\partial\Gamma(X)$ is called a *supergradient* of Γ at X. If Γ is exact, then $\partial\Gamma(X)$ is nonempty for every $X \in B(\Omega,\mathscr{F})$.

Theorem 2. *Let $\nu : \mathscr{F} \to \mathbb{R}$ be a balanced game. Then the following conditions are equivalent.*

(i) ν *is exact.*
(ii) ν *has a minimal exact extension to $\Gamma_\nu : B(\Omega,\mathscr{F}) \to \mathbb{R}$ given by:*

$$\Gamma_\nu(X) = \min_{\mu\in\mathscr{C}(\nu)} \langle \mu, X\rangle.$$

(iii) ν *has an exact extension $\Gamma : B(\Omega,\mathscr{F}) \to \mathbb{R}$ with $\mathscr{C}(\nu) = \partial\Gamma(0) = \partial\Gamma(1)$.*

2 Exact Functionals on L^∞-Spaces

Let $(\Omega,\mathscr{F},P)$ be a nonatomic probability space. The (*probability*) *law* (or *distribution*) of a random variable X is a probability measure $P \circ X^{-1}$ on the Borel space $(\mathbb{R},\mathscr{B})$. When a random variable X has the same law with Y, we denote $X \sim Y$. The *distribution function* of X is given by $F_X(x) = P(X \le x)$. The (*upper*) *quantile function* $q_X : [0,1) \to \mathbb{R} \cup \{-\infty\}$ of X is defined by $q_X(t) = \inf\{x \in \mathbb{R} \mid F_X(x) > t\}$, which is nondecreasing and right-continuous, and satisfies $q_{-X}(t) = q_X(1-t)$ a.e. $t \in (0,1)$. For each $\alpha \in (0,1]$, define the functional $q_\alpha : L^\infty(\Omega,\mathscr{F},P) \to \mathbb{R}$ by $q_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha q_X(t)dt$, and for $\alpha = 0$, let $q_0(X) := \operatorname{ess.inf} X$. Then, $\alpha \mapsto q_\alpha(X)$ is a nondecreasing continuous function on $[0,1]$ for every $X \in L^\infty(\Omega,\mathscr{F},P)$.

Definition 3. *A functional $\Gamma : L^\infty(\Omega,\mathscr{F},P) \to \mathbb{R}$ is law invariant if $\Gamma(X) = \Gamma(Y)$ whenever $X \sim Y$.*

Definition 4. *(i) A subset C of $L^1(\Omega,\mathscr{F},P)$ is law invariant whenever $Y \in C$ and $\tilde{Y} \sim Y$ implies $\tilde{Y} \in C$.*
(ii) A subset $\mathscr{C}$ of $ca(\Omega,\mathscr{F},P)$ is law invariant if the set $C = \{\frac{d\mu}{dP} \in L^1(\Omega,\mathscr{F},P) \mid \mu \in \mathscr{C}\}$ is law invariant.

Definition 5. *A functional $\Gamma : L^\infty(\Omega,\mathscr{F},P) \to \mathbb{R}$ has the Lebesgue property whenever $X_n \to X$ a.e. with $\sup_n \|X_n\|_\infty < \infty$ implies $\lim_n \Gamma(X_n) = \Gamma(X)$.*

Theorem 3. *Let $\Gamma : L^\infty(\Omega, \mathscr{F}, P) \to \mathbb{R}$ be a functional. Then, the following conditions are equivalent.*

(i) *Γ is a law invariant exact functional with the Lebesgue property.*
(ii) *There exists a unique, law invariant, weak*-compact, convex set $\mathscr{C} \subset ca(\Omega, \mathscr{F}, P)$ such that $\Gamma(X) = \min_{\mu \in \mathscr{C}} \langle \mu, X \rangle$ for every $X \in L^\infty(\Omega, \mathscr{F}, P)$.*
(iii) *Γ is superadditive and there exist a subset $\mathscr{M}$ of $ca([0,1])$ and a family $\{v_m \mid m \in \mathscr{M}\}$ of law invariant, weak*-continuous, linear functionals on $L^\infty(\Omega, \mathscr{F}, P)$ such that $\Gamma(X) = \inf_{m \in \mathscr{M}} [\int_0^1 q_\alpha(X) dm(\alpha) + v_m(X)]$ for every $X \in L^\infty(\Omega, \mathscr{F}, P)$.*

Definition 6. *A functional $\Gamma_\varphi : L^\infty(\Omega, \mathscr{F}, P) \to \mathbb{R}$ is the Choquet functional of a function $\varphi : [0,1] \to \mathbb{R}$ of bounded variation with $\varphi(0) = 0$ if it is of the form*

$$\Gamma_\varphi(X) = \int_0^\infty \varphi(P(X \geq t))dt + \int_{-\infty}^0 [\varphi(P(X \geq t)) - \varphi(1)]dt.$$

Theorem 4. *A functional $\Gamma : L^\infty(\Omega, \mathscr{F}, P) \to \mathbb{R}$ is a law invariant exact functional with the Lebesgue property if and only if it is superadditive and there exists a family Π of functions $\varphi : [0,1] \to \mathbb{R}$ with $\varphi(0) = 0$ which are written as a difference of two non-decreasing concave functions on $[0,1]$ vanishing at 0, and a family $\{w_\varphi \mid \varphi \in \Pi\}$ of law invariant, weak*-continuous linear functionals on $L^\infty(\Omega, \mathscr{F}, P)$ such that $\Gamma(X) = \inf_{\varphi \in \Pi} [-\Gamma_\varphi(-X) + w_\varphi(X)]$ for every $X \in L^\infty(\Omega, \mathscr{F}, P)$.*

3 Anonymous Exact Games

Let $(\Omega, \mathscr{F}, P)$ be a probability space. A transformation $\tau : \Omega \to \Omega$ is *bi-measurable* if it is a bijection such that both τ and τ^{-1} are measurable mappings. A transformation $\tau : \Omega \to \Omega$ is *measure-preserving* if it is a measurable mapping such that $P \circ \tau^{-1} = P$. If τ is a bi-measurable, measure-preserving transformation, then τ^{-1} is automatically measure-preserving: $P \circ \tau = P$. Denote by $\mathbb{T}(\Omega, \mathscr{F}, P)$ the space of bi-measurable, measure-preserving transformations on $(\Omega, \mathscr{F}, P)$.

Definition 7. *Let $\nu : \mathscr{F} \to \mathbb{R}$ be a balanced game.*

(i) *ν is anonymous if there exists a nonatomic control measure P for $\mathscr{C}(\nu)$ such that $\nu \circ \tau = \nu$ for every $\tau \in \mathbb{T}(\Omega, \mathscr{F}, P)$.*
(ii) *ν has the anonymous core if there exists a nonatomic control measure P for $\mathscr{C}(\nu)$ such that $\mu \circ \tau \in \mathscr{C}(\nu)$ for every $\mu \in \mathscr{C}(\nu)$ and $\tau \in \mathbb{T}(\Omega, \mathscr{F}, P)$.*

Assumption 1 *$\mathscr{F}$ is countably generated.*

A probability measure P satisfying

$$\lim_{P(A) \to 0} \sup_{\mu \in \mathscr{C}(\nu)} \mu(A) = 0$$

is called a *control measure* for $\mathscr{C}(\nu)$, with respect to which every element in $\mathscr{C}(\nu)$ is uniformly absolutely continuous.

Theorem 5. *Let $\nu : \mathscr{F} \to \mathbb{R}$ be an exact game. Then, the following conditions are equivalent.*

(i) ν *is anonymous.*
(ii) ν *has the anonymous core.*
(iii) ν *has a nonatomic control measure P for $\mathscr{C}(\nu)$ such that its minimal exact extension to $L^\infty(\Omega, \mathscr{F}, P)$ is a law invariant functional with the Lebesgue property.*

Theorem 6. *For every continuous, anonymous, exact game $\nu : \mathscr{F} \to \mathbb{R}$, there exist a nonatomic probability measure P and a unique continuous function $\varphi : [0,1] \to \mathbb{R}$ such that $\nu = \varphi \circ P$.*

Theorem 7. *A bounded continuous game $\nu : \mathscr{F} \to \mathbb{R}$ is anonymous and convex if and only if there exist a nonatomic probability measure P and a unique, continuous, convex function $\varphi : [0,1] \to \mathbb{R}$ such that $\nu = \varphi \circ P$.*

Let P be a nonatomic probability measure on $(\Omega, \mathscr{F})$. For an arbitrarily given $A \in \mathscr{F}$ and $t \in [0,1]$, we define the family $\mathscr{K}_t^P(A)$ of measurable subsets of A by:

$$\mathscr{K}_t^P(A) = \{E \in \mathscr{F} \mid E \subset A \text{ and } P(E) = tP(A)\}.$$

For an arbitrarily given $A, B \in \mathscr{F}$ and $t \in [0,1]$, we denote by $\mathscr{K}_t^P(A,B)$ the family of sets $C \in \mathscr{F}$ such that C is a union of two disjoint sets $E \in \mathscr{K}_t^P(A)$ and $F \in \mathscr{K}_{1-t}^P(B)$. It can be shown that $\mathscr{K}_t^P(A,B)$ is nonempty for every $A, B \in \mathscr{F}$ and $t \in [0,1]$ (see [6]).

Definition 8. *A game $\nu : \mathscr{F} \to \mathbb{R}$ is P-convex if for every $A, B \in \mathscr{F}$ and $t \in [0,1]$, we have $\nu(C) \le t\nu(A) + (1-t)\nu(B)$ for every $C \in \mathscr{K}_t^P(A,B)$.*

Proposition 1 ([3]). *A continuous game $\nu : \mathscr{F} \to \mathbb{R}$ is P-convex if and only if there exists a unique, continuous, convex function $\varphi : [0,1] \to \mathbb{R}$ such that $\nu = \varphi \circ P$.*

Corollary 1. *A bounded continuous game $\nu : \mathscr{F} \to \mathbb{R}$ is anonymous and convex if and only if there exist a nonatomic probability measure P such that ν is P-convex.*

Corollary 2. *A bounded, continuous, P-convex game is anonymous and exact.*

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Interaction indices revisited

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The intention of this paper is to introduce new axioms for interaction indices, new interaction indices and new ideas to characterize these indices.
If time is left we will point out how to generalize these results to fuzzy interaction indices working in Sobolev spaces.

In the Grabisch-Roubens characterizations of the Shapley-interaction index [2] 5 axioms were used : linearity , dummy player axiom , symmetry, efficiency and a recursivity axiom. The first 4 axioms are rather "natural" axioms, but the recursivity axiom is rather technical and will not be considered by all as a "natural" axiom. In the Fujimoto-Kojadinovic-Marichal paper [1] a new set of axioms were proposed : Linearity was replaced by additivity , monotonicity and k-monotonicity , whereas dummy player axiom was exchanged by a dummy partnership axiom. Moreover symmetry and efficiency were overtaken and the complicated recursivity axiom was replaced by a "recursion free " consistency property for "reduced " partnerships.

(a) Thus it would be desirable to have (instead of [efficiency and recursivity] or instead of [efficiency and reduced partnership]) one generalized "natural" efficiency axiom for indices. In this paper we propose such a natural coalition-efficiency axiom .

(b) This coalition-efficiency axiom together with linearity, dummy partnership (or dummy player) axiom and symmetry leads to a new characterization of the Shapley interaction index.

(c) We introduce a new random interaction index which can be characterized by linearity , dummy partnership (or dummy player) axiom and coalition efficiency (that is , no symmetry is needed).

(d) We show the connection between the random interaction index and the chaining interaction index.

(e) We unify the Banzhaf, chaining , internal and external interaction index (see [1]) to a more general Sincov interaction index .

We now give some details. Let U be an infinite set, the universe of players.
A game on U is a set function $v : 2^U \to \mathbb{R}$ with $v(\emptyset) = 0$.
The members of 2^U are coalitions. A set $N \subset U$ is carrier of a game v if $v(S) = v(S \cap N)$

for all $S \subset U$. A finite game is a game which has finite support. Moreover we set $\mathcal{U} = 2^U \setminus (\emptyset)$ and let $\mathcal{G}$ and $\mathcal{G}^N$ be the set of finite games and the set of games with finite carrier $N \subset U$, respectively.
We denote by Π_N all permutations of the finite set N and by $\mathcal{M}_N$ the set of all maximal chains in the powerset of N. Then there is a bijection f between Π_N and $\mathcal{M}_N$ given by $f(\pi) := m_\pi = \{\emptyset, \{\pi(1)\}, \{\pi(1), \pi_(2)\}, ..., \{\pi(1), ..., \pi(n)\}\}$.
Let us denote by m_π^S the minimal coalition belonging to the maximal chain m_π that contains S. In the special case $S = \{\pi(k)\}$ we have that $m_\pi^{\pi(k)}$ is the set of all precedents of $\pi(k)$ in the set $\pi(N) = \{\pi(1), ..., \pi(n)\}$.

For any function $\varphi : \mathcal{G} \times \mathcal{U} \to \mathbb{R}$ we call for fixed $v \in \mathcal{G}^N$ the mapping $\varphi(v, \cdot) : \mathcal{U} \to \mathbb{R}$ an interaction value and for fixed $v \in \mathcal{G}^N$ and for fixed $S \subset N$ we call $\varphi(v,S)$ an interaction index (or i-index , for short) of the coalition S in the game v.
In order to characterize interaction indices we use 3 characteristic expressions :

(I) the discrete derivation $\Delta_S v(T) = \sum_{L \subset S} (-1)^{s-l} v(L \cup T)$ as a measure for the marginal interaction among the players of the coalition S in the presence of the coalition T $(S \subset N, T \subset N \setminus S, |S| = s \geq 2)$,

(II) the s-th order derivative of v at S with $\delta_S v(T \cup S) = \Delta_S v(T)$ and

(III) the well-known Möbiustransform $m(v,S) = \sum_{T \subset S} (-1)^{s-t} v(T)$ together with the co- Möbiustransform $m^*(v,S) = \sum_{T \supset S} m(v,T)$.

For characterization theorems it is important to choose an appropriate basis for $\mathcal{G}^N$. Weber [3] proposed to take $\{w_T^N : \emptyset \neq T \subset N\}$ (where $w_T^N(S) = 1$ for $T = S \cap N$ and $w_T^N(S) = 0$ otherwise). We here propose $\{u_T : \emptyset \neq T \subset N\}$ where $u_T(S) = 1$ if $S \supset T$ and $u_T(S) = 0$ otherwise. To see the difference let us consider all interaction indices satisfying linearity and dummy partnership axiom. In the first case we get the well known probabilistic interaction indices

(1) $\varphi(v,S) = \sum_{T \subset N \setminus S} p_T^S(N) \Delta_S v(T)$, $\sum_{T \subset N \setminus S} p_T^S(N) = 1$, $v \in \mathcal{G}^N$.

Here $p_T^S(N)$ are real constants, $N \subset U$ is finite and $\emptyset \neq S \subset N$. But if $S \not\subset N$ then $\varphi(v,S) = 0$. Moreover, the expression in (1) is dependent upon N.

In the second case we obtain “normed” interaction indices

(2) $\varphi(v,S) = \sum_{T \supset S} \beta_T^S \, m(v,T)$, $\beta_S^S = 1$, $S \subset U$, S finite , $v \in \mathcal{G}$.

Here again β_T^S are real constants, but we have no dependence upon the carrier N since $m(v,S) = 0$ if $S \not\subset N$. The proof for (2) is easy : Linearity implies
$\varphi(v,S) = \sum_{\emptyset \neq T \subset N} \beta_T^S \, m(v,T)$, where $\beta_T^S = \varphi(u_T, S)$.
The dummy player axiom implies $\beta_T^S = \varphi(u_T, S) = 0$ if $S \not\subset T, T$ finite. And the dummy

partnership axiom implies $\beta_S^S = 1$.

It is more complicated to prove (1). The reason is that linearity implies that $\varphi(v,S) = \sum_{\emptyset \neq T \subset N} \alpha_T^S(N)\, v(T)$ where $\alpha_T^S(N) = \varphi(w_T^N, S)$. But then - going over to (1) - new real constant will be introduced, namely $p_T^S(N) = \alpha_{T\cup S}(N)$ (which are now the "correct" constants).
Let us remark that the connection between β_T^S and $p_T^S(N)$ is given by

$\beta_L^S = \sum_{T: L\setminus S \subset T \subset N\setminus S}\ p_T^S(N)$ for $L \supset S$ so that $\beta_S^S = \sum_{T\subset N\setminus S}\ p_T^S(N) = 1$.

If the coefficients $p_T^S(N)$ in (1) depend only upon the cardinalities of T, S and N then $\varphi(v,S)$ is called a cardinal-probabilistic interaction index. It is known that this is the case iff $\varphi(v,S)$ satisfies linearity (in the first argument), dummy partnership axiom and symmetry.

The same result is true for (2) but the proof for this fact is now nearly trivial. For example, if $T_1 \supset S$ and $T_2 \supset S$ have the same finite cardinality choose a permutation π of U with $\pi(T_1) = \pi(T_2)$ and $\pi(S) = S$. Then we get
$\beta_{T_1}^S = \varphi(u_{T_1}, S) = \varphi(\pi u_{T_1}, \pi S) = \varphi(u_{\pi T_1}, S) = \varphi(u_{T_2}, S) = \beta_{T_2}^S$.

Note that a concrete cardinal-probabilistic interaction index must be independent upon the carrier of N. The following result contains a sufficient condition.

For each family of nonnegative numbers
$\{p_t^s(n) : 1 \leq n < \infty, 1 \leq s \leq n, 0 \leq t \leq n-s\}$ satisfying

(3) $\sum_{t=0}^{n-s} \binom{n-s}{t}\, p_t^s(n) = 1$ and $p_t^s(n) = p_t^s(n+1) + p_{t+1}^s(n+1)$

we have for $v \in \mathcal{G}$ and for every two nonempty carriers $M, N \subset U$
$\sum_{T\subset N\setminus S} p_t^s(n)\Delta_S v(T) = \sum_{T\subset M\setminus S} p_t^s(m)\Delta_S v(T)$.

Note that all known cardinal-probabilistic interaction indices satisfy (3) :

- the Shapley i-index where $p_t^s(n) = \frac{1}{n-s+1}\binom{n-s}{t}^{-1}$ and $\beta_t^s = \frac{1}{t-s+1}$,

- the chaining i-index where $p_t^s(n) = \frac{s}{s+t}\binom{n}{s+t}^{-1}$ and $\beta_t^s = \frac{s}{t}$ and the generalized

- Banzhaf i-index where $p_t^s(n) = c^t(1-c)^{n-t-s}$ and $\beta_t^s = c^{t-s}$ where $c \in [0,1]$.

Note that $c = \frac{1}{2}$ leads to the Banzhaf index, c = 0 gives the well known Möbius-transform or internal i-index $m(v,S)$ whereas $c = 1$ gives
the co- Möbiustransform or external i-index $m^*(v,S) = \sum_{T\supset S} m(v,T)$.

Let us here introduce the Sincov i-index (a more general i-index than the generalized Banzhaf i-index), to see the characterization results in [1] concerning the Bazhaf, chaining, internal and external i-index under a unified viewpoint:

(4) $\varphi(v,P) = \sum_{T \supset S} \beta_t^s \, m(v,T)$, $\beta_t^s \cdot \beta_s^r = \beta_t^r$, $\beta_t^s > 0$, $T \supset S \supset R$.

Note that that the β_t^s satisfy a Sincov functional equation which implies immediately that $\beta_s^s = 1$ (put $t = r = s$) and $\beta_t^r = \frac{\beta_t^1}{\beta_r^1}$.
Thus every Sincov i-index satisfies the dummy partnership axiom and results in [1] imply that each cardinal probabilistic i-index satisfying the partnership allocation axiom (which implies $\beta_t^s \cdot \beta_s^1 = \beta_t^1$) is a Sincov i-index.

Let us now present the efficiency axiom : we have
for each finite, nonempty $N \subset U$, for each $v \in \mathcal{G}^N$ and for each $S \subset N, S \neq N$

(5) $\sum_{i \in N \setminus S} \varphi(v, S \cup i) = \Delta_S v(N)$.

First note that $\Delta_S v(N) = \Delta_S v(N \setminus S)$ is the marginal contribution of the players of S to the coalition $N \setminus S$.
On the left hand each player $i \in N \setminus S$ is ready to contribute his expected gain $\varphi(v, S \cup i)$ if he joins the coalition S. The sum of these expected gains $\sum_{i \in N \setminus S} \varphi(v, S \cup i)$ is equal to the total contribution $\Delta_S v(N \setminus S)$ which the players of $N \setminus S$ are ready to pay if S joins the coalition $N \setminus S$.

For $S = \emptyset$ we get in (5) the usual efficiency axiom $\sum_{i \in N} \varphi(v,i) = v(N)$.

Now we can show in one line that the Shapley i-index is the only i-index satisfying linearity , dummy player (or dummy partnership) axiom, symmetry and efficiency : For arbitrary $j \in T \setminus S$ we get

$1 = \Delta_S u_T(T) = \sum_{i \in T \setminus S} \beta_T^{S \cup i} = (t-s)\beta_T^{S \cup j}$. Thus $\beta_T^{S \cup j} = \frac{1}{t-s}$ or $\beta_T^S = \frac{1}{t-s-1}$.

We also remark that in this characterization of the Shapley interaction index the dummy player axiom can be replaced by the weaker “finiteness” axiom :
For each infinite $S \subset U$ and for each $v \in \mathcal{G}$ we require $\varphi(v,S) = 0$.
Now we define the random order interaction index :

(6) $\varphi(v,S) = \sum_{\pi \in \Pi_N} r_\pi \, \delta_S v(m_\pi^S)$, $\sum_{\pi \in \Pi_N} r_\pi = 1$.

We see immediately that for $S = \{i\}$ we get the random order index introduced by Weber [3] and that (6) is a generalization of the chaining interaction index ($r_\pi = \frac{1}{n!}$).
Moreover (6) is a probabilistic interaction index :
let us put $m_\pi^S = S \cup T$ with $T \subset N \setminus S$ to obtain :

$$\varphi(v,S) = \sum_{T \subset N \setminus S} \sum_{\pi \in \Pi_N, m_\pi^S = S \cup T} r_\pi \, \delta_S v(T \cup S) =$$

$= \sum_{T \subset N \setminus S} \sum_{\pi \in \Pi_N, m_\pi^S = S \cup T} r_\pi \, \Delta_S v(T).$

The statements follow because of $\sum_{T \subset N \setminus S} p_T^S = 1$ with $p_T^S = \sum_{\pi \in \Pi_N, m_\pi^S = S \cup T} r_\pi$.

The random order interaction index is of interest because of the following two characterization theorems (in analogy to results of Weber [3]) :
Let φ be a probabilistic i-index.

1. Then φ satisfies the coalition efficiency axiom iff

(7) $\sum_{i \in N \setminus S} p_{N \setminus (S \cup i)}^{S \cup i} = 1$ and $\sum_{i \in T} p_{T \setminus i}^{S \cup i} = \sum_{i \notin T \cup S} p_T^{S \cup i}$, $T \subset N \setminus S$.

2. φ satisfies the coalition efficiency axiom iff φ is a random order i-index.

Using this result we can give a further proof that the Shapley interaction index is characterized by linearity , Dummy axiom, coalition efficiency and symmetry. Using the symmetry we get from (7)

$t \, p_{t-1}^{s+1} = (n-t-s) \, p_t^{s+1}$ and $(n-s) \, p_{n-s-1}^{s+1} = 1$ so that

$a_t := \binom{n-s-1}{t} \, p_t^{s+1} = \binom{n-s-1}{t-1} \, p_{t-1}^{s+1} = a_{t-1} = \ldots = a_{n-s-1} = p_{n-s-1}^{s+1} = \frac{1}{n-s}.$

Thus $p_t^{s+1} = \frac{1}{n-s} \binom{n-s-1}{t-1}^{-1}$, or $p_t^s = \frac{1}{n-s+1} \binom{n-s}{t}^{-1}, 0 \le t \le n-s.$

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Variable-basis categorically-algebraic dualities

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1 Introduction

Recently, we introduced in [34] a new setting for topological structures motivated by *point-set lattice-theoretic* (*poslat*) *topology* of S. E. Rodabaugh [27] successfully developed in the framework of powerset theories with the underlying algebraic structures for topology being *semi-quantales* [31, 30]. Based in category theory and universal algebra, the new approach is called *categorically-algebraic* (*catalg*) *topology* to underline its motivating theories and to distinguish it from the above-mentioned poslat setting. The underlying idea is to replace semi-quantales with algebras from an arbitrary variety, and to consider an abstract category as the ground for topology. Simple as it is, that provides the new setting with a high flexibility. For example, the case of set-induced ground category, called *variety-based topology*, allows one not only to extend the classical concepts of fixed- and variable-basis topology [29], but also to introduce a new *multi-basis* one, thereby incorporating the most important topological theories currently popular in the fuzzy community, e.g., (variable-basis) lattice-valued approach of S. E. Rodabaugh [30], (fixed-basis) (L,M)-fuzzy topological spaces of C. Guido, U. Höhle, T. Kubiak and A. Šostak [15, 17, 21], as well as (multi-basis) generalized topology of M. Demirci [11]. Moreover, in some cases the border between crisp and many-valued developments gets ultimately erased. In particular, many-valued framework of S. E. Rodabaugh appears to be “crisp” (goes in line with the crisp categorically-theoretic machinery), whereas the framework of C. Guido *et al.* is a truly fuzzy setting (requires *lattice-valued catalg topology*). At the moment, the new theory is rapidly progressing in several directions influencing each other significantly, e.g., catalg spaces [34], catalg systems [35, 37], catalg dualities [38, 39], catalg powerset operators [40], catalg attachment [12, 43], lattice-valued catalg topology [36]. It is the purpose of this paper to show one of the most promising developments, namely, the extension of the natural duality theory.

The theory of natural dualities was motivated by numerous topological representation theorems for algebraic structures from the last century. In particular, M. Stone represented both Boolean algebras [44] and distributive lattices [45], whereas L. Pontrjagin considered abelian groups [24]. The real push, however, was given by the famous representation of distributive lattices of H. Priestley [25], which immediately initiated a plethora of parallel results. The above theorems translate algebraic problems, usually stated in an abstract symbolic language, into dual, topological problems, where geometric intuition comes to our help. Induced by the advantage, in the last quarter of the 20th century *natural duality theory* began to appear, developed by D. Clark, B. Davey, M. Haviar, H. Priestley, *etc.* [4, 7, 9, 10, 26], which provided a general machinery (based in category theory and universal algebra) for obtaining topological representations of algebraic structures, (partially) incorporating the cited examples as particular subcases.

Being classically motivated, the theory of natural dualities relies explicitly on crisp topology. On the other hand, already in 1992 S. E. Rodabaugh [28] has come out with a poslat generalization of the Stone representation theorems. Inspired by his ideas, in [41, 42] we partially generalized the results of [28] for (variable-basis) variety-based topology and turned our attention to fuzzification of the Priestley representation theorem, which appeared to be more difficult to attack. Finally, in [39] we have managed to break through, providing a fixed-basis variety-based generalization of the result. Almost immediately, we saw an opening for a much broader theory, i.e., that of *categorically-algebraic dualities* extending the classical natural dualities. With the idea in mind, we presented a fixed-basis variety-based version of the new theory in [38]. The desired shift from fixed-basis to variable-basis turned out to be not so easy, but not unmanageable. The sticking point was to avoid the truncated variable-basis representation framework of S. E. Rodabaugh, restricted to isomorphisms between the underlying lattices of the spaces. It is the main goal of this paper to present a variable-basis modification of catalg dualities. The achievement serves as yet another proof of the fruitfulness of catalg framework, urging the shift from poslat to catalg in the modern topological theories, the latter being a more convenient tool for successful development of fuzzy mathematics.

The paper uses both category theory and universal algebra, relying more on the former. The necessary categorical background can be found in [1, 22, 23]. For the notions of universal algebra [3, 6, 14] are recommended. Although we tried to make the paper as much self-contained as possible, some details are still omitted and left to the reader.

2 Algebraic preliminaries

For convenience of the reader, we begin with those algebraic and categorical preliminaries, which are crucial for the fruitful perusal of the paper.

Definition 1. *Let* $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ *be a (possibly proper) class of cardinal numbers. An* Ω-algebra *is a pair* $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$ *comprising a set A and a family of maps* $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ *(*n_λ-ary primitive operations *on A). An* Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda\in\Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda\in\Lambda})$ *is a map* $A \xrightarrow{\varphi} B$ *such that* $f \circ \omega_\lambda^A = \omega_\lambda^B \circ f^{n_\lambda}$ *for every* $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ *is the construct of* Ω*-algebras and* Ω*-homomorphisms.*

From now on, every concrete category comes equipped with the underlying functor $|-|$.

Definition 2. *Let* $\mathcal{M}$ *(resp.* $\mathcal{E}$*) be the class of* Ω*-homomorphisms with injective (resp. surjective) underlying maps. A* variety of Ω-algebras *is a full subcategory of* $\mathbf{Alg}(\Omega)$ *closed under the formation of products,* $\mathcal{M}$*-subobjects and* $\mathcal{E}$*-quotients. The objects (resp. morphisms) of a variety are called* algebras *(resp.* homomorphisms*).*

Definition 3. *Given a variety* $\mathbf{A}$*, a* reduct *of* $\mathbf{A}$ *is a pair* $(\|-\|, \mathbf{B})$*, where* $\mathbf{B}$ *is a variety such that* $\Omega_{\mathbf{B}} \subseteq \Omega_{\mathbf{A}}$ *and* $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ *is a concrete functor.*

The categorical dual of a variety $\mathbf{A}$ is denoted $\mathbf{LoA}$, whose objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Given a homomorphism φ, the corresponding localic one is denoted φ^{op} and vice versa. Every localic $\mathbf{A}$-algebra A provides the subcategory $\mathbf{S}_A$ of $\mathbf{LoA}$, with the only morphism the identity $A \xrightarrow{1_A} A$.

The reader should notice a significant deviation from the framework of universal algebra [3, 6, 14], where the algebras have a set of finitary operations.

3 Categorically-algebraic topology

This section serves as a brief introduction into the theory of *categorically-algebraic topology*, the subsequent results of the manuscript are based upon. The reader is advised to recall powerset theories of S. E. Rodabaugh [30, 31].

Every set map $X \xrightarrow{f} Y$ gives rise to the following two operators: *image operator* $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$, $f^{\rightarrow}(S) = \{f(x) \mid x \in S\}$ and *preimage operator* $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$, $f^{\leftarrow}(T) = \{x \mid f(x) \in T\}$. Preimage operators can be extended to a more general setting.

Definition 4. *A* categorically-algebraic backward powerset theory *(*cabp-theory*) in a category* $\mathbf{X}$ *(*ground category *of the theory) is a functor* $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$*, where* $\mathbf{A}$ *is a variety.*

Example 1. Let $\mathbf{Set}$ be the category of sets and maps. Given a variety $\mathbf{A}$, every subcategory $\mathbf{C}$ of $\mathbf{LoA}$ induces a functor $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{S}=(-)^{\leftarrow}} \mathbf{LoA}$, $((X,A) \xrightarrow{(f,\varphi)} (Y,B))^{\leftarrow} = A^X \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} B^Y$, $(f,\varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$. The functor $\mathbf{Set} \times \mathbf{S}_A \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$ is denoted $\mathcal{S}_A = (-)^{\leftarrow}_A$ and is called *fixed-basis approach*. The case $\mathbf{C} \neq \mathbf{S}_A$ is called *variable-basis approach*. In particular, the functor $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^{\leftarrow}_{\mathbf{2}}} \mathbf{LoCBAlg}$ (complete Boolean algebras), where $\mathbf{2} = \{\bot, \top\}$, provides the above-mentioned preimage operator.

Definition 5. *Let* $\mathbf{X}$ *be a category and let* $\mathcal{T}_I = ((P_i, (\|-\|_i, \mathbf{B}_i)))_{i \in I}$ *be a set-indexed family such that* $\mathbf{X} \xrightarrow{P_i} \mathbf{LoA}_i$ *is a cabp-theory in* $\mathbf{X}$ *and* $(\|-\|_i, \mathbf{B}_i)$ *is a reduct of* $\mathbf{A}_i$ *for every* $i \in I$*. A* composite categorically-algebraic topological theory *(*ccat-theory*)* in $\mathbf{X}$ induced by $\mathcal{T}_I$ *is the functor* $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoB}_i$*, given by the equality* $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoB}_i \xrightarrow{\Gamma_j} \mathbf{LoB}_j = \mathbf{X} \xrightarrow{P_j} \mathbf{LoA}_j \xrightarrow{\|-\|_j^{op}} \mathbf{LoB}_j$ *for every* $j \in I$*, where* Γ_j *is the respective projection functor. A ccat-theory induced by a singleton family is denoted* T*.*

Definition 6. *Let* T_I *be a ccat-theory in a category* $\mathbf{X}$*.* $\mathbf{CTop}(T_I)$ *is the concrete category over* $\mathbf{X}$*, whose objects (*composite categorically-algebraic topological spaces *or* T_I-spaces*) are pairs* $(X, (\tau_i)_{i \in I})$*, where* X *is an* $\mathbf{X}$*-object and* τ_i *is a subalgebra of* $T_i(X)$ *for every* $i \in I$ *(*$(\tau_i)_{i \in I}$ *is called* composite categorically-algebraic topology *or* T_I-topology *on* X*), and whose morphisms* $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ *are* $\mathbf{X}$*-morphisms* $X \xrightarrow{f} Y$ *such that* $((T_i f)^{op})^{\rightarrow}(\sigma_i) \subseteq \tau_i$ *for every* $i \in I$ *(*composite categorically-algebraic continuity *or* T_I-continuity*). The category* $\mathbf{CTop}(T)$ *is denoted* $\mathbf{Top}(T)$*.*

Example 2. The case of the ground category $\mathbf{X} = \mathbf{Set} \times \mathbf{C}$ is called *variety-based topology*. In particular, $\mathbf{Top}((\mathcal{S}_Q, \mathbf{B}))$ provides the category $Q_{\mathbf{B}}$-**Top**, which is the framework for *fixed-basis variety-based topology*, whereas $\mathbf{Top}((\mathcal{S}, \mathbf{B}))$ gives the category $(\mathbf{C}, \mathbf{B})$-**Top**, which is the framework for *variable-basis variety-based topology* (the case $\mathbf{A} = \mathbf{B}$ is denoted **C**-**Top**). More specific, $\mathbf{Top}((\mathcal{P}, \mathbf{Frm}))$ (*frames* [2, 18]) is isomorphic to the classical category **Top**, whereas $\mathbf{CTop}(((\mathcal{P}, \mathbf{Frm}))_{i\in\{1,2\}})$ is isomorphic to the category **BiTop** of bitopological spaces and bicontinous maps of J. C. Kelly [19].

Notice that the framework of S. E. Rodabaugh [30] passes directly from powerset theories to topological spaces, never introducing the intermediate step of topological theories, which in our case is motivated by the observation that the standard powerset theory $\mathcal{P}$ is based in Boolean algebras, whereas the category **Top** relies on frames (dropping a part of the algebraic structure). Another crucial point is that the non-variety-based catalg framework obliterates the concept of *basis* for a topological space, going back to the notion of *base set* of powerset of J. A. Goguen [13]. In particular, the numerous debates in the fuzzy community on the advantage of either fixed- or variable-basis setting over its rival [29] are redundant in case of an arbitrary ground category **X**.

Two important properties of catalg topology will be indispensable in the forthcoming developments. The first one generalizes the classical result of general topology, stating that continuity of a map can be checked on the elements of a subbase, which has already been extended to poslat topology by S. E. Rodabaugh [29, Theorem 3.2.6].

Definition 7. *Let* **A** *be a variety and let* $\Omega \subseteq \Omega_{\mathbf{A}}$. *Given an algebra* A *and a subset* $S \subseteq A$, $\langle S \rangle_{\Omega}$ *stands for the smallest* Ω*-subreduct of* A *containing* S *(*$\langle S \rangle_{\Omega_{\mathbf{A}}}$ *is shortened to* $\langle S \rangle$*). Given a cat-theory* $\mathbf{X} \xrightarrow{T} \mathbf{LoB}$, *a subclass* $\Omega \subseteq \Omega_{\mathbf{B}}$ *and a* T*-space* (X, τ), *a subset* $S \subseteq T(X)$ *is an* Ω-base *of* τ *provided that* $\tau = \langle S \rangle_{\Omega}$. $\Omega_{\mathbf{B}}$*-bases are called* subbases.

Lemma 1. *Let* T_I *be a ccat-theory in a category* **X** *and let* $(X, (\tau_i)_{i\in I})$, $(Y, (\sigma_i)_{i\in I})$ *be* T_I*-spaces such that* $\sigma_i = \langle S_i \rangle_{\Omega_i}$ *for every* $i \in I$. *An* **X***-morphism* $X \xrightarrow{f} Y$ *is* T_I*-continuous iff* $((T_i f)^{op})^{\rightarrow}(S_i) \subseteq \tau_i$ *for every* $i \in I$.

The second property extends the standard construction of products of topological spaces.

Lemma 2. *Let* T_I *be a ccat-theory in a category* **X**. *If* **X** *has products, then the category* $\mathbf{CTop}(T_I)$ *has concrete products.*

Proof. Given a set-indexed family $((X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$ of T_I-spaces, the respective product is $((\prod_{k\in J} X_k, (\prod_{k\in J} \tau_{k_i})_{i\in I}) \xrightarrow{\pi_j} (X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$, where $(\prod_{k\in J} X_k \xrightarrow{\pi_j} X_j)_{j\in J}$ is an **X**-product of $(X_j)_{j\in J}$ and $\prod_{k\in J} \tau_{k_i} = \langle \bigcup_{j\in J} ((T_i \pi_j)^{op})^{\rightarrow}(\tau_{j_i}) \rangle$ for every $i \in I$. □

Corollary 1. $Q_{\mathbf{B}}$-**Top** *has concrete products.* $(\mathbf{C}, \mathbf{B})$-**Top** *has concrete products provided that the category* **C** *has products.*

Notice that products in **C** are actually coproducts of the respective algebras.

4 Categorically-algebraic dualities

This section applies variety-based topology to extend the theory of natural dualities. We start with a modification of the fixed-basis approach of [38] (an unnecessary tight condition is relaxed), which the variable-basis case relies upon.

4.1 Fixed-basis approach

The theory of natural dualities is based in the concept of *schizophrenic object*, i.e., a finite set M equipped with two structures: algebraic (providing an algebra M_A), and topological (assumed to be discrete) with additional enrichment consisting of finitary total and partial operations as well as finitary relations (providing a structured topological space M_T). The setting constructs a kind of dual equivalence (*natural duality*) between the quasi-variety (closure under isomorphic images, subalgebras and products) $\boldsymbol{\mathcal{A}}$ generated by the algebra M_A, and the topological quasi-variety (closure under isomorphic images, closed subspaces and non-empty products) $\mathfrak{X}$ generated by the space M_T.

Variety-based framework modifies the setting as follows: (1) the category **Top** is replaced with the category $Q_{\mathbf{B}}$-**Top**; (2) every requirement of finiteness on the structures in question is dropped; (3) topological enrichment is reduced to relations, incorporating both total and partial operations as their particular kinds; (4) arbitrary topologies on the set M are allowed; (5) (topological) quasi-variety is substituted by the notion of (*sobriety*) *spatiality* in the sense of P. T. Johnstone [18]; (6) an equivalence between the categories of *sober spaces* and *spatial localic algebras* is established.

We begin by developing the framework of enriched topological spaces (the reader should recall Definitions 1, 2). For the sake of convenience, the prefix "$Q_{\mathbf{B}}$" is added to the respective topological stuff, e.g., "$Q_{\mathbf{B}}$-space", "$Q_{\mathbf{B}}$-topology", "$Q_{\mathbf{B}}$-continuity", *etc.*

Definition 8. *Let $\Sigma = (m_\upsilon)_{\upsilon\in\Upsilon}$ be a (possibly proper) class of cardinal numbers. A* Σ-structure *is a pair $(R, (\varpi^R_\upsilon)_{\upsilon\in\Upsilon})$ comprising a set R and a family of subsets $\varpi^R_\upsilon \subseteq R^{m_\upsilon}$ (m_υ*-ary primitive relations *on R). A* Σ-homomorphism $(R, (\varpi^R_\upsilon)_{\upsilon\in\Upsilon}) \xrightarrow{f} (S, (\varpi^S_\upsilon)_{\upsilon\in\Upsilon})$ *is a map $R \xrightarrow{f} S$ such that $(f^{m_\upsilon})^{\to}(\varpi^R_\upsilon) \subseteq \varpi^S_\upsilon$ for every $\upsilon \in \Upsilon$.* $\mathbf{Rel}(\Sigma)$ *is the construct of Σ-structures and Σ-homomorphisms.*

Definition 9. *Let $\mathcal{R}$ be the class of Σ-homomorphisms $R \xrightarrow{f} S$ such that for every $\upsilon \in \Upsilon$ and every $\langle r_i\rangle_{m_\upsilon} \in R^{m_\upsilon}$, $\langle f(r_i)\rangle_{m_\upsilon} \in \varpi^S_\upsilon$ implies $\langle r_i\rangle_{m_\upsilon} \in \varpi^R_\upsilon$. Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the subclass of $\mathcal{R}$ of those Σ-homomorphisms which have injective (resp. surjective) underlying maps. A* variety of Σ-structures *is a full subcategory of* $\mathbf{Rel}(\Sigma)$ *closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients. The objects (resp. morphisms) of a variety are called* structures *(resp.* homomorphisms*).*

To make a distinction from varieties of algebras, the prefix "r" is added to the respective relational stuff, e.g., "r-variety", "r-structure", "r-homomorphism", *etc.*

Example 3. The construct **BPos** of bounded partially ordered sets (*posets*) and bound- as well as order-preserving maps (*b-order-preserving maps*) is an r-variety induced by the category $\mathbf{Rel}(2,1,1)$, based in a single binary relation and two unary relations.

Definition 10. *Given an r-variety* $\mathbf{R}$*,* $Q_{\mathbf{B}}$-$\mathbf{RTop}$ *is the concrete category over* $Q_{\mathbf{B}}$-$\mathbf{Top}$*, whose objects (*r-$Q_{\mathbf{B}}$-spaces*) are pairs* (R,τ)*, where* R *is an r-structure and* $(|R|,\tau)$ *is a* $Q_{\mathbf{B}}$*-space, and whose morphisms (*r-$Q_{\mathbf{B}}$-morphisms*)* $(R,\tau)\xrightarrow{f}(S,\sigma)$ *are* $Q_{\mathbf{B}}$*-continuous maps* $(|R|,\tau)\xrightarrow{f}(|S|,\sigma)$ *such that* $R\xrightarrow{f}S$ *is an r-homomorphism.*

Example 4. The category **Top** enriched in the r-variety **BPos** provides the category **BPosTop** of bounded potopological spaces and b-order-preserving continuous maps.

The machinery of variety-based dualities relies on 3 steps: (1) replace $\mathfrak{X}$ (resp. $\mathcal{A}$) with $Q_{\mathbf{B}}$-$\mathbf{RTop}$ (resp. a variety $\mathbf{E}$); (2) construct two functors $Q_{\mathbf{B}}\text{-}\mathbf{RTop}\xrightarrow{H}\mathbf{LoE}$ and $\mathbf{LoE}\xrightarrow{G}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ such that G is a right adjoint to H; (3) single out subcategories of $Q_{\mathbf{B}}$-$\mathbf{RTop}$ (resp. $\mathbf{LoE}$), the restriction to which of the adjunction gives an equivalence.

The first step being already made, we proceed to the second one. To obtain the functors in question, we introduce several additional notions.

Definition 11. *Suppose* $\mathbf{C}$ *is a subcategory of* $\mathbf{E}$*. An* r-reduct *of* $\mathbf{C}$ *is a pair* $(\|-\|,\mathbf{S})$*, where* $\mathbf{S}$ *is an r-variety and* $\mathbf{C}\xrightarrow{\|-\|}\mathbf{S}$ *is a concrete functor. An r-reduct is called* algebraic *provided that for every* $\mathbf{C}$*-object* C *and every* $\upsilon\in\Upsilon_{\mathbf{S}}$*,* $\varpi_{\upsilon}^{\|C\|}$ *is a subalgebra of* $C^{m_{\upsilon}}$*.*

Example 5. The functor $\mathbf{Quant}\xrightarrow{\|-\|}\mathbf{LBPos}$ (*quantales* [20,32] and lower-bounded posets) defined by $\|A\xrightarrow{\varphi}B\|=(A,\leqslant,\bot)\xrightarrow{\varphi}(B,\leqslant,\bot)$ provides an algebraic r-reduct of **Quant**, whereas $(\|-\|,\mathbf{BPos})$ gives a non-algebraic r-reduct of **Quant**.

Definition 12. *Given a variety* $\mathbf{D}$*, a* $Q_{\mathbf{B}}$-topological $\mathbf{D}$-algebra *is a pair* (D,τ)*, where* D *is a* $\mathbf{D}$*-algebra,* $(|D|,\tau)$ *is a* $Q_{\mathbf{B}}$*-space, and every primitive* $\mathbf{D}$*-operation* $|D|^{n_\lambda}\xrightarrow{\omega_\lambda^D}|D|$ *on* D *is* $Q_{\mathbf{B}}$*-continuous.*

See Examples 8, 9 for a concrete illustration of the concept. Notice that [38,39] use the term $Q_{\mathbf{B}}$-continuous instead of $Q_{\mathbf{B}}$-topological. The change of this paper was motivated by our wish to be in line with the already existing terminology.

The preliminaries in hand, we proceed to the variety-based version of schizophrenic object, which is the cornerstone of the desired duality.

Definition 13. *A* variety-based schizophrenic object *(*vbs-object*) is a pair* $(\mathbb{E},\boldsymbol{\delta})$*, where* $\mathbb{E}$ *is an* $\mathbf{E}$*-algebra and* $\boldsymbol{\delta}$ *is a* $Q_{\mathbf{B}}$*-topology on* $|\mathbb{E}|$*.*

Fix a vbs-object $(\mathbb{E},\boldsymbol{\delta})$ and introduce the following two requirements:

($\mathcal{R}$) $\mathbf{R}$ is an algebraic r-reduct of $\mathbf{S}_{\mathbb{E}}$.
($\mathcal{C}$) $(\mathbb{E},\boldsymbol{\delta})$ is a $Q_{\mathbf{B}}$-topological algebra.

It should be underlined that [38,39] demanded $\mathbf{R}$ to be an algebraic r-reduct of $\mathbf{E}$ instead of $\mathbf{S}_{\mathbb{E}}$. The setting of this paper relaxes the requirement (preserving all results) to boost the flexibility of the framework.

Lemma 3. *If* $(\mathcal{R})$, $(\mathcal{C})$ *hold, then there exists a functor* $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{H} \mathbf{LoE}$ *given by* $H((R,\tau) \xrightarrow{f} (S,\sigma)) = Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{E}\|) \xrightarrow{(f_{\mathbb{E}}^{\leftarrow})^{op}} Q_{\mathbf{B}}\text{-}\mathbf{RTop}(S,\|\mathbb{E}\|)$.

Proof. As an example, show that $Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{E}\|)$ is a subalgebra of $\mathbb{E}^{|R|}$. Given $\lambda \in \Lambda_{\mathbf{E}}$ and $\alpha_i \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{E}\|)$ for every $i \in n_\lambda$, check that $\omega_\lambda^{\mathbb{E}^{|R|}}(\langle\alpha_i\rangle_{n_\lambda})$ is an r-$Q_{\mathbf{B}}$-morphism. Start with the case of being an r-homomorphism. Given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j\rangle_{m_\upsilon} \in \varpi_\upsilon^R$, $\langle\alpha_i(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{E}\|}$ for every $i \in n_\lambda$, that implies $\langle(\omega_\lambda^{\mathbb{E}^{|R|}}(\langle\alpha_i\rangle_{n_\lambda}))(r_j)\rangle_{m_\upsilon} = \langle\omega_\lambda^{\mathbb{E}}(\langle\alpha_i(r_j)\rangle_{n_\lambda})\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{E}\|}$ by $(\mathcal{R})$. To show $Q_{\mathbf{B}}$-continuity, notice that products of $Q_{\mathbf{B}}$-spaces provide the $Q_{\mathbf{B}}$-continuous map $R \xrightarrow{\alpha} |\mathbb{E}|^{n_\lambda}$ with $\pi_i \circ \alpha = \alpha_i$ (π_i being the respective projection map) for every $i \in n_\lambda$. Since $\omega_\lambda^{\mathbb{E}^{|R|}}(\langle\alpha_i\rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{E}} \circ \alpha$ and $\omega_\lambda^{\mathbb{E}}$ is $Q_{\mathbf{B}}$-continuous by $(\mathcal{C})$, $\omega_\lambda^{\mathbb{E}^{|R|}}(\langle\alpha_i\rangle_{n_\lambda})$ must be as well. □

Lemma 4. *If* $(\mathcal{R})$, $(\mathcal{C})$ *hold, then the functor* $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{H} \mathbf{LoE}$ *has a right adjoint.*

Proof. Show that every E in $\mathbf{LoE}$ has an H-co-universal arrow $HG(E) \xrightarrow{\varepsilon_E{}^{op}} E$. Let the underlying set of $G(E)$ be $\mathbf{E}(E,\mathbb{E})$. For $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle\varphi_j\rangle_{m_\upsilon} \in (\mathbf{E}(E,\mathbb{E}))^{m_\upsilon}$, let $\langle\varphi_j\rangle_{m_\upsilon} \in \varpi_\upsilon^{\mathbf{E}(E,\mathbb{E})}$ iff $\langle\varphi_j(e)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{E}\|}$ for every $e \in E$ (pointwise structure). For $e \in E$ and $\alpha \in \boldsymbol{\delta}$, let $\mathbf{E}(E,\mathbb{E}) \xrightarrow{t_{e\alpha}} Q$, $t_{e\alpha}(\varphi) = ev_e((\varphi_Q^{\leftarrow})(\alpha)) = \alpha \circ \varphi(e)$ and set $\tau = \langle\{t_{e\alpha} \,|\, e \in E,\ \alpha \in \boldsymbol{\delta}\}\rangle$. Define the map $E \xrightarrow{\varepsilon_E} (HG(E) = Q_{\mathbf{B}}\text{-}\mathbf{RTop}(\mathbf{E}(E,\mathbb{E}),\|\mathbb{E}\|))$ by $\varepsilon_E(e) = ev_e$. □

Corollary 2. *If* $(\mathcal{R})$, $(\mathcal{C})$ *hold, then there exists an adjoint situation* $(\eta,\varepsilon) : H \dashv G :$ $\mathbf{LoE} \to Q_{\mathbf{B}}\text{-}\mathbf{RTop}$.

Proof. Given a localic homomorphism $E_1 \xrightarrow{\varphi^{op}} E_2$, $G(E_1 \xrightarrow{\varphi^{op}} E_2) = G(E_1) \xrightarrow{G\varphi^{op}} G(E_2)$ with $G\varphi^{op} = \varphi_{\mathbb{E}}^{\leftarrow}$. Given an r-$Q_{\mathbf{B}}$-space R, $R \xrightarrow{\eta_R} (GH(R) = \mathbf{E}(Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{E}\|),\mathbb{E}))$ is defined by $(\eta_R(r))(f) = f(r)$. □

Having completed the second step, we turn to the last one, singling out the subcategories to get an equivalence between. Start by recalling some categorical preliminaries.

Lemma 5. *Let* $(\eta,\varepsilon) : F \dashv G : \mathbf{A} \to \mathbf{X}$ *be an adjoint situation. Let* $\overline{\mathbf{A}}$ *(resp.* $\overline{\mathbf{X}}$*) be the full subcategory of* $\mathbf{A}$ *(resp.* $\mathbf{X}$*) of the objects* A *(resp.* X*) such that* $FG(A) \xrightarrow{\varepsilon_A} A$ *(resp.* $X \xrightarrow{\eta_X} GF(X)$*) is an isomorphism in* $\mathbf{A}$ *(resp.* $\mathbf{X}$*). There exists the restriction* $(\overline{\eta},\overline{\varepsilon}) : \overline{F} \dashv \overline{G} : \overline{\mathbf{A}} \to \overline{\mathbf{X}}$ *which is an equivalence, maximal in the sense that every other equivalence* $(\overline{\overline{\eta}},\overline{\overline{\varepsilon}}) : \overline{\overline{F}} \dashv \overline{\overline{G}} : \overline{\overline{\mathbf{A}}} \to \overline{\overline{\mathbf{X}}}$ *gives subcategories* $\overline{\overline{\mathbf{A}}}$ *(resp.* $\overline{\overline{\mathbf{X}}}$*) of* $\overline{\mathbf{A}}$ *(resp.* $\overline{\mathbf{X}}$*).*

The following applies Lemma 5 to the adjunction of Corollary 2 and characterizes the category $\overline{\mathbf{LoE}}$ (resp. $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$).

Definition 14. *An r-*$Q_{\mathbf{B}}$*-space* (R,τ) *is called*

1. $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-$T_0$ *provided that (a) every distinct* $r_1, r_2 \in R$ *have an r-*$Q_{\mathbf{B}}$*-morphism* $R \xrightarrow{f} \|\mathbb{E}\|$ *such that* $f(r_1) \neq f(r_2)$*; (b) given* $\upsilon \in \Upsilon_{\mathbf{R}}$ *and* $\langle r_j\rangle_{m_\upsilon} \in R^{m_\upsilon}$*, if* $\langle f(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{E}\|}$ *for every r-*$Q_{\mathbf{B}}$*-morphism* $R \xrightarrow{f} \|\mathbb{E}\|$*, then* $\langle r_j\rangle_{m_\upsilon} \in \varpi_\upsilon^R$*;*

2. $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-$S_0$ *provided that (a) every homomorphism* $Q_{\mathbf{B}}$-$\mathbf{RTop}(R,\|\mathbb{E}\|) \xrightarrow{\varphi} \mathbb{E}$ *has some* $r \in R$ *such that* $\varphi(f) = f(r)$ *for every r-*$Q_{\mathbf{B}}$*-morphism* $R \xrightarrow{f} \|\mathbb{E}\|$*; (b)* $\tau = \langle\{ f_Q^{\leftarrow}(\alpha) \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{E}\|),\ \alpha \in \boldsymbol{\delta}\}\rangle$*;*
3. $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-sober *provided that it is both* $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-$T_0$ *and* $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-$S_0$.

Basically, an r-$Q_{\mathbf{B}}$-space is $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-sober iff it is a “closed” (the meaning should be clarified) r-$Q_{\mathbf{B}}$-substructure of some power of $(\|\mathbb{E}\|,\boldsymbol{\delta})$.

The next definition (the dual version of [8, Section 11.20]) and example give the intuition for the new notions.

Definition 15. *A* **BPosTop**-*space* $(X,\leqslant,\bot,\top,\tau)$ *is called* totally order-disconnected *provided that for every* $x,y \in X$ *such that* $x \nleqslant y$*, there exists a* clopen *(closed and open)* up-set $U \subseteq X$ *(*$z \in U$ *and* $z \leqslant w$ *yield* $w \in U$*) such that* $x \in U$ *and* $y \notin U$*.*

Example 6. Given the lattice **2** of the variety **Lat** (lattices), $(\|-\|,\mathbf{BPos})$ is an algebraic r-reduct of $\mathbf{S_2}$. Equipped with the discrete topology $\tau^d = \{\varnothing,\{\bot\},\{\top\},\mathbf{2}\}$, the lattice provides a topological algebra (Example 8). A **BPosTop**-space X is $r_{(\mathbf{2},\tau^d)}$-T_0 iff X is totally order-disconnected.

Definition 16. *A* **LoE**-*object* E *is called* $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-spatial *provided that (a) every distinct* $e_1,e_2 \in E$ *have a homomorphism* $E \xrightarrow{\varphi} \mathbb{E}$ *with* $\varphi(e_1) \neq \varphi(e_2)$*; (b) every r-*$Q_{\mathbf{B}}$*-morphism* $\mathbf{E}(E,\mathbb{E}) \xrightarrow{f} \|\mathbb{E}\|$ *has* $e \in E$ *with* $f(\varphi) = \varphi(e)$ *for every homomorphism* $E \xrightarrow{\varphi} \mathbb{E}$*.*

Briefly speaking, a **LoE**-object is $r_{(\mathbb{E},\boldsymbol{\delta})}$-$Q_{\mathbf{B}}$-spatial iff it is a “closed” (the meaning should be clarified) subalgebra of some power of $\mathbb{E}$.

The preliminaries in hand, the desired characterization is a matter of technique.

Lemma 6. *An r-*$Q_{\mathbf{B}}$*-space* R *is* $r_{(\mathbb{E},\boldsymbol{\delta})}$*-*$Q_{\mathbf{B}}$*-sober iff* η_R *is an isomorphism. A* **LoE**-*object* E *is* $r_{(\mathbb{E},\boldsymbol{\delta})}$*-*$Q_{\mathbf{B}}$*-spatial iff* ε_E *is an isomorphism.*

Corollary 3. $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$ *is the full subcategory* $Q_{\mathbf{B}}$-**(**$\mathbb{E}$**,**$\boldsymbol{\delta}$**)RSob** *of* $Q_{\mathbf{B}}$-**RTop** *comprising precisely the* $r_{(\mathbb{E},\boldsymbol{\delta})}$*-*$Q_{\mathbf{B}}$*-sober r-*$Q_{\mathbf{B}}$*-spaces.* $\overline{\mathbf{LoE}}$ *is the full subcategory* $Q_{\mathbf{B}}$-**(**$\mathbb{E}$**,**$\boldsymbol{\delta}$**)RSpat** *of* **LoE** *comprising precisely the* $r_{(\mathbb{E},\boldsymbol{\delta})}$*-*$Q_{\mathbf{B}}$*-spatial localic algebras.*

The main theorem of this subsection is, thus, as follows.

Theorem 1. *If* $(\mathcal{R})$ *and* $(\mathcal{C})$ *hold, then there exists the equivalence* $(\overline{\eta},\overline{\varepsilon}) : \overline{H} \dashv \overline{G} :$ $Q_{\mathbf{B}}$-**(**$\mathbb{E}$**,**$\boldsymbol{\delta}$**)RSpat** $\to$ $Q_{\mathbf{B}}$-**(**$\mathbb{E}$**,**$\boldsymbol{\delta}$**)RSob**.

4.2 Variable-basis approach

This subsection extends the results of the previous one to the variable-basis world, replacing the category $Q_{\mathbf{B}}$-**Top** with $(\mathbf{C},\mathbf{B})$-**Top**. The main difference from the framework of S. E. Rodabaugh [28] is the lack of truncation of the setting to isomorphisms between the underlying algebras of the spaces.

We begin by developing the framework of variable-basis enrichment.

Definition 17. *Given a subcategory* $\mathbf{C}$ *of* $\mathbf{LoA}$ *and an r-variety* $\mathbf{R}$*,* $(\mathbf{C},\mathbf{B})$-$\mathbf{RTop}$ *is the concrete category over* $(\mathbf{C},\mathbf{B})$-$\mathbf{Top}$*, whose objects (*r-$(\mathbf{C},\mathbf{B})$-spaces*) are triples* (R,C,τ) *such that* R *is an r-structure and* $(|R|,C,\tau)$ *is a* $(\mathbf{C},\mathbf{B})$*-space, and whose morphisms (*r-$(\mathbf{C},\mathbf{B})$-morphisms*)* $(R_1,C_1,\tau_1) \xrightarrow{(f,\varphi)} (R_2,C_2,\tau_2)$ *are* $(\mathbf{C},\mathbf{B})$*-continuous morphisms* $(|R_1|,C_1,\tau_1) \xrightarrow{(f,\varphi)} (|R_2|,C_2,\tau_2)$ *such that* $R_1 \xrightarrow{f} R_2$ *is an r-homomorphism.*

Example 7. The category $\mathbf{C}$-$\mathbf{Top}$ with $\mathbf{A} = \mathbf{USQuant}$ (unital semi-quantales), enriched in the r-variety $\mathbf{BPos}$, gives the category $\mathbf{C}$-$\mathbf{BPosTop}$ of variable-basis poslat bounded potopological spaces and b-order-preserving continuous morphisms.

From now on, the category $\mathbf{C}$ used in the definition of $(\mathbf{C},\mathbf{B})$-$\mathbf{Top}$ is supposed to have powers of objects (we never require the existence of products).

The following introduces the conditions for obtaining an adjunction between the categories $(\mathbf{C},\mathbf{B})$-$\mathbf{Top}$ and $\mathbf{LoE}$. The machinery translates the developments from the previous subsection into variable-basis language, adding some new requirements.

Definition 18. *Given a variety* $\mathbf{D}$*, a* $(\mathbf{C},\mathbf{B})$-topological $\mathbf{D}$-algebra *is a triple* (D,C,τ)*, where* D *is a* $\mathbf{D}$*-algebra,* $(|D|,C,\tau)$ *is a* $(\mathbf{C},\mathbf{B})$*-space, and for every primitive* $\mathbf{D}$*-operation* $|D|^{n_\lambda} \xrightarrow{\omega_\lambda^D} |D|$ *on* D *and every* $i \in n_\lambda$*, the* $\mathbf{Set} \times \mathbf{C}$*-morphism* $|(|D|,C,\tau)^{n_\lambda}| \xrightarrow{(\omega_\lambda^D,\pi_i)} |(|D|,C,\tau)|$*, where* $C^{n_\lambda} \xrightarrow{\pi_i} C$ *is the respective projection map, is* $(\mathbf{C},\mathbf{B})$*-continuous.*

The next two lemmas provide important examples of the new notion.

Lemma 7. *Let* $\mathbf{D}$ *be a finitary variety and let* C *be a* $\mathbf{C}$*-object. Suppose that* $\Omega_\mathbf{B}$ *induces the structure of semi-frame on* $\|C\|$*, and the projections of every power of* C *in* $\mathbf{C}$ *are sections. For every* $\mathbf{D}$*-algebra* D*, the pair* $(|D|,C)$ *with the* discrete $(\mathbf{C},\mathbf{B})$-topology $\tau^d = C^{|D|}$ *provides a* $(\mathbf{C},\mathbf{B})$*-topological algebra* (D,C,τ^d)*.*

Example 8. The lattice $\mathbf{2}$, being equipped with the standard discrete topology $\tau^d = \{\varnothing,\{\bot\},\{\top\},\mathbf{2}\}$, provides a topological algebra.

Lemma 8. *Let* $\mathbf{D}$ *be a variety such that* $\Omega_\mathbf{D} \subseteq \Omega_\mathbf{B}$*, let* D *be a* $\mathbf{D}$*-algebra, let* C *be a* $\mathbf{C}$*-object and let* $|D| \xrightarrow{\varphi} |C|$ *be a* $\mathbf{D}$*-homomorphism such that* $D^{n_\lambda} \xrightarrow{(\pi_{j_1}^C)^{op}\circ\varphi\circ\pi_i^D} C^{n_\lambda} = D^{n_\lambda} \xrightarrow{(\pi_{j_2}^C)^{op}\circ\varphi\circ\pi_i^D} C^{n_\lambda}$ *for every* $\lambda \in \Lambda_\mathbf{D}$ *and every* $i, j_1, j_2 \in n_\lambda$*. The* Sierpinski $(\mathbf{C},\mathbf{B})$-topology $\tau^s = \langle\varphi\rangle$ *on* $(|D|,C)$ *provides a* $(\mathbf{C},\mathbf{B})$*-topological algebra* (D,C,τ^s)*.*

Example 9. The frame $\mathbf{2}$, being equipped with the classical Sierpinski topology $\tau^s = \{\varnothing,\{\top\},\mathbf{2}\}$, provides a topological algebra.

Definition 19. *Let* $\mathbf{D}$ *be a variety, let* $\mathbf{E}$ *be its reduct and let* $\mathbf{C}$ *be a subcategory of* $\mathbf{D}$*, which has copowers of objects. A* $\mathbf{C}$*-object* C *is called (a)* $\mathbf{E}$-p-entropic *provided that for every* $\mathbf{C}$*-object* C'*, every* $\lambda \in \Lambda_\mathbf{E}$ *and every family* $(C' \xrightarrow{\varphi_i} C)_{i \in n_\lambda}$ *of* $\mathbf{C}$*-morphisms, the composition* $C' \xrightarrow{[\varphi_i]_{n_\lambda}} C^{n_\lambda} \xrightarrow{\omega_\lambda^{\|C\|}} C$ *is a* $\mathbf{C}$*-morphism, where* $C' \xrightarrow{[\varphi_i]_{n_\lambda}} C^{n_\lambda} \xrightarrow{\pi_i} C = C' \xrightarrow{\varphi_i} C$ *for every* $i \in n_\lambda$*; (b)* $\mathbf{E}$-c-idempotent *provided that for every* $\mathbf{E}$*-algebra* E*, every*

$\lambda \in \Lambda_{\mathbf{E}}$ *and every* $\langle e_i \rangle_{n_\lambda} \in E^{n_\lambda}$, $\omega_\lambda^{\|(^{|E|}C)^{|C|}\|}(\langle \mu_{e_i} \rangle_{n_\lambda}) = \mu_{\omega_\lambda^E(\langle e_i \rangle_{n_\lambda})}$, *where* $C \xrightarrow{\mu_e} {}^{|E|}C$ *is the respective copower injection; (c) an* **E**-p-c-mode *provided that* C *is both* **E**-*p-entropic and* **E**-*c-idempotent.* **C** *is called* **E**-p-entropic *(***E**-c-idempotent*,* the category of **E**-p-c-modes*) provided that every* **C**-*object is* **E**-p-entropic *(***E**-c-idempotent, an **E**-p-c-mode*).*

The terminology of Definition 19 is motivated by the theory of *modes* (idempotent, entropic algebras) [33]. In particular, every **E**-c-idempotent **C**-algebra C provides the idempotent **E**-algebra $\|C\|$, whereas every entropic **D**-algebra is **D**-p-entropic w.r.t. every full subcategory of **D**. Further examples follow.

Example 10. If $\mathbf{D} = \mathbf{E} = \mathbf{Set}$, then **D** is the category of **E**-p-c-modes.

Example 11. If $\mathbf{D} = \mathbf{E} = \mathbf{CSLat}(\bigvee)$ ($\bigvee$-semilattices), then **D** is **E**-p-entropic, but not **E**-c-idempotent.

Example 12. If D is a **D**-algebra, then $\mathbf{S}_D$ is the category of **E**-p-c-modes iff $\|D\|$ is an idempotent **E**-algebra. If **E** has nullary operations, then $\mathbf{S}_D$ is the category of **E**-p-c-modes iff D is a singleton algebra.

The preliminaries in hand, we proceed to the definition of a variable-basis analogue of variety-based schizophrenic object.

Definition 20. *A* variable-basis variety-based schizophrenic object *(*vvbs-object*) is a triple* $(\mathbb{E},\mathbb{C},\boldsymbol{\delta})$ *with* $\mathbb{E}$ *an* **E**-*algebra,* $\mathbb{C}$ *a* **C**-*object, and* $\boldsymbol{\delta}$ *a* $(\mathbf{C},\mathbf{B})$-*topology on* $(|\mathbb{E}|,\mathbb{C})$.

To obtain the required adjunction, we fix a vvbs-object $(\mathbb{E},\mathbb{C},\boldsymbol{\delta})$ and consider the following set of requirements, for the sake of shortness denoted $(\mathcal{REQ})$:

$(\mathcal{R}_1)$ **E** is a reduct of **B**.
$(\mathcal{R}_2)$ **R** is an algebraic r-reduct of $\mathbf{S}_{\mathbb{E}}$.
$(\mathcal{C})$ $(\mathbb{E},\mathbb{C},\boldsymbol{\delta})$ is a $(\mathbf{C},\mathbf{B})$-topological algebra.
$(\mathcal{M})$ **LoC** is a category of **E**-p-c-modes.

Lemma 9. *If* $(\mathcal{REQ})$ *hold, then there exists a functor* $(\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop} \xrightarrow{H} \mathbf{LoE}$ *given by* $H((R_1,C_1,\tau_1) \xrightarrow{(f,\varphi)} (R_2,C_2,\tau_2)) = (\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}((R_1,C_1,\tau_1),(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})) \xrightarrow{(f_{\mathbb{E}}^{\leftarrow},\varphi_{\mathbb{C}}^{\leftarrow})^{op}} (\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}((R_2,C_2,\tau_2),(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta}))$.

Proof. To show that $(\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}((R,C,\tau),(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta}))$ is a subalgebra of $\mathbb{E}^{|R|} \times \|C^{|\mathbb{C}|}\|$, use **E**-p-entropicity of $(\mathcal{M})$ to verify that given $\lambda \in \Lambda_{\mathbf{E}}$ and $(f_i,\varphi_i) \in H(R,C,\tau)$ for every $i \in n_\lambda$, the map $\mathbb{C} \xrightarrow{\omega_\lambda^{\|C^{|\mathbb{C}|}\|}(\langle \varphi_i^{op} \rangle_{n_\lambda})} C = \mathbb{C} \xrightarrow{[\varphi_i^{op}]_{n_\lambda}} C^{n_\lambda} \xrightarrow{\omega_\lambda^{\|C\|}} C$ is a **LoC**-morphism. □

Lemma 10. *If* $(\mathcal{REQ})$ *hold, then the functor* $(\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop} \xrightarrow{H} \mathbf{LoE}$ *has a right adjoint.*

Proof. Show that every **LoE**-object E has an H-co-universal arrow $HG(E) \xrightarrow{\varepsilon_E^{op}} E$. The underlying pointwise r-structure of $G(E)$ is already described in Lemma 4, whereas the algebraic basis of $G(E)$ is the power $\mathbb{C}^{|E|}$. Given $e \in E$ and $\alpha \in \boldsymbol{\delta}$, let $\mathbf{E}(E,\mathbb{E}) \xrightarrow{s_{e\alpha}} \mathbb{C}^{|E|}$, $s_{e\alpha}(\varphi) = ev_e((\varphi,\pi_e^{\mathbb{C}})^{\leftarrow}(\alpha)) = (\pi_e^{\mathbb{C}})^{op} \circ \alpha \circ \varphi(e)$ and put $\sigma = \langle \{s_{e\alpha} \mid e \in E,\ \alpha \in \boldsymbol{\delta}\} \rangle$. Define the map $E \xrightarrow{\varepsilon_E} (HG(E) = (\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}((\mathbf{E}(E,\mathbb{E}),\mathbb{C}^{|E|},\sigma),(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})))$ by $\varepsilon_E(e) = (ev_e,\pi_e^{\mathbb{C}})$ and use **E**-c-idempotency of $(\mathcal{M})$ to show that ε_E is an **E**-homomorphism. □

Corollary 4. *If* $(\mathcal{REQ})$ *hold, then there exists an adjoint situation* $(\eta,\varepsilon): H \dashv G$: $\mathbf{LoE} \to (\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}$.

Proof. Given a localic homomorphism $E_1 \xrightarrow{\varphi^{op}} E_2$, $G(E_1 \xrightarrow{\varphi^{op}} E_2) = G(E_1) \xrightarrow{G\varphi^{op}} G(E_2)$ with $G\varphi^{op} = (\varphi^{\leftarrow}_{\mathbb{E}}, \psi)$, where ψ is defined by $\mathbb{C}^{|E_1|} \xrightarrow{\psi} \mathbb{C}^{|E_2|} \xrightarrow{\pi_e^{E_2}} \mathbb{C} = \mathbb{C}^{|E_1|} \xrightarrow{\pi^{E_1}_{\varphi(e)}} \mathbb{C}$ for every $e \in E_2$. Given an r-$(\mathbf{C},\mathbf{B})$-space (R,C,τ), $(R,C,\tau) \xrightarrow{\eta_{(R,C,\tau)}=(g,\phi)} (GH(R,C,\tau) = (\mathbf{E}((\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}((R,C,\tau),(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})),\mathbb{E}),\mathbb{C}^{H(R,C,\tau)},\sigma))$ is defined by $(g(r))(f,\varphi) = f(r)$, whereas ϕ is the $\mathbf{C}$-morphism $[\varphi]_{H(R,C,\tau)}$, provided by the equality $C \xrightarrow{[\varphi]_{H(R,C,\tau)}} \mathbb{C}^{H(R,C,\tau)} \xrightarrow{\pi^{\mathbb{C}}_{(f,\varphi)}} \mathbb{C} = C \xrightarrow{\varphi} \mathbb{C}$ for every $(f,\varphi) \in H(R,C,\tau)$. □

The adjunction obtained, we single out the subcategories for variable-basis duality.

Definition 21. *An r-*$(\mathbf{C},\mathbf{B})$*-space* (R,C,τ) *is called*

1. $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$-$T_0$ *provided that (a) every distinct* $r_1, r_2 \in R$ *have an r-*$(\mathbf{C},\mathbf{B})$*-morphism* $(R,C,\tau) \xrightarrow{(f,\varphi)} (\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})$ *with* $f(r_1) \neq f(r_2)$*; (b) given* $\upsilon \in \Upsilon_{\mathbf{R}}$ *and* $\langle r_j\rangle_{m_\upsilon} \in R^{m_\upsilon}$*, if* $\langle f(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{E}\|}$ *for every r-*$(\mathbf{C},\mathbf{B})$*-morphism* $(R,C,\tau) \xrightarrow{(f,\varphi)} (\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})$*, then* $\langle r_j\rangle_{m_\upsilon} \in \varpi_\upsilon^R$*; (c) the homomorphism* $\mathbb{C}^{H(R,C,\tau)} \xrightarrow{([\varphi]_{H(R,C,\tau)})^{op}} C$ *is surjective;*
2. $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$-$S_0$ *provided that (a) every homomorphism* $(\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}((R,C,\tau),(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})) \xrightarrow{\varphi} \mathbb{E}$ *has an* $r \in R$ *such that* $\varphi(g,\psi) = g(r)$ *for every r-*$(\mathbf{C},\mathbf{B})$*-morphism* $(R,C,\tau) \xrightarrow{(g,\psi)} (\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})$*; (b)* $\tau = \langle\{(f,\varphi)^{\leftarrow}(\alpha) \,|\, (f,\varphi) \in (\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}((R,C,\tau),(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})), \alpha \in \boldsymbol{\delta}\}\rangle$*; (c) the homomorphism* $\mathbb{C}^{H(R,C,\tau)} \xrightarrow{([\varphi]_{H(R,C,\tau)})^{op}} C$ *is injective;*
3. $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$-sober *provided that it is* $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$-$T_0$ *and* $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$-$S_0$.

Definition 22. *A* $\mathbf{LoE}$*-object* E *is called* $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$-spatial *provided that (a) every distinct* $e_1, e_2 \in E$ *have either a homomorphism* $E \xrightarrow{\varphi} \mathbb{E}$ *with* $\varphi(e_1) \neq \varphi(e_2)$*, or a* $c \in \mathbb{C}$ *with* $(\pi^{\mathbb{C}}_{e_1})^{op}(c) \neq (\pi^{\mathbb{C}}_{e_2})^{op}(c)$*; (b) every r-*$(\mathbf{C},\mathbf{B})$*-morphism* $(\mathbf{E}(E,\mathbb{E}),\mathbb{C}^{|E|},\tau) \xrightarrow{(f,\varphi)} (\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})$ *has an* $e \in E$ *such that* $f(\psi) = \psi(e)$ *for every* $\psi \in \mathbf{E}(E,\mathbb{E})$*, and* $\varphi = \pi^{\mathbb{C}}_e$.

The preliminaries in hand, the desired characterization is straightforward.

Lemma 11. *A space* (R,C,τ) *is* $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$*-sober iff* η_R *is an isomorphism. A* $\mathbf{LoE}$*-object* E *is* $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$*-spatial iff* ε_E *is an isomorphism.*

Corollary 5. $\overline{(\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}}$ *is the full subcategory* $(\mathbf{C},\mathbf{B})$**-(**$\mathbb{E}$**,**$\mathbb{C}$**,**$\boldsymbol{\delta}$**)RSob** *of* $(\mathbf{C},\mathbf{B})\text{-}\mathbf{RTop}$ *comprising precisely the* $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$*-sober spaces, whereas* $\overline{\mathbf{LoE}}$ *is the full subcategory* $(\mathbf{C},\mathbf{B})$**-(**$\mathbb{E}$**,**$\mathbb{C}$**,**$\boldsymbol{\delta}$**)RSpat** *of* $\mathbf{LoE}$ *comprising precisely the* $r_{(\mathbb{E},\mathbb{C},\boldsymbol{\delta})}$-$(\mathbf{C},\mathbf{B})$*-spatial localic algebras.*

We are now ready to state the main result of this subsection.

Theorem 2. *If requirements* $(\mathcal{REQ})$ *hold, then there exists the equivalence* $(\overline{\eta},\overline{\varepsilon}) : \overline{H} \dashv \overline{G}$: $(\mathbf{C},\mathbf{B})$**-(**$\mathbb{E}$**,**$\mathbb{C}$**,**$\boldsymbol{\delta}$**)RSpat** $\to$ $(\mathbf{C},\mathbf{B})$**-(**$\mathbb{E}$**,**$\mathbb{C}$**,**$\boldsymbol{\delta}$**)RSob**.

4.3 Examples

Below we include the representation theorem(s) of H. Priestley (resp. M. Stone) for distributive lattices (resp. bounded distributive lattices and Boolean algebras) into our variable- (resp. fixed-) basis setting.

Priestley representation theorem. The cabp-theory $\mathcal{P}$ of Example 1 provides the category $(\mathbf{S_2}, \mathbf{Frm})$-**Top** isomorphic (and thus, shortened) to the category **Top** of crisp topological spaces. Consider the category **BPosTop** of Example 4 and let $\mathbf{E} = \mathbf{Lat}$, $\mathbb{E} = \mathbf{2}$. Equip $(|\mathbf{2}|, \mathbf{2})$ with the discrete topology τ^d and get the continuous algebra $(\mathbb{E}, \mathbb{C}, \boldsymbol{\delta}) = (\mathbf{2}, \mathbf{2}, \tau^d)$. Satisfaction of $(\mathcal{REQ})$ (Example 12) provides the adjunction $(\eta, \varepsilon) : H \dashv G$: $\mathbf{LoLat} \to \mathbf{BPosTop}$, where (1) $\mathbf{BPosTop} \xrightarrow{H} \mathbf{LoLat}$, $H(X) = (\mathcal{PCOU}(X), \cap, \cup)$, $Hf = (f^{\leftarrow})^{op}$ with $\mathcal{PCOU}(X)$ the proper (non-empty) clopen up-sets of X; (2) $\mathbf{LoLat} \xrightarrow{G} \mathbf{BPosTop}$, $G(E) = (\mathcal{PF}(E), \subseteq, \tau)$, $G\varphi = (\varphi^{op})^{\leftarrow}$ with $\mathcal{PF}(E)$ the prime filters of E (including $\varnothing$, E) and $\tau = \langle \{\rho_e \,|\, e \in E\} \cup \{\hat{\rho}_e \,|\, e \in E\} \rangle$, where $F \in \rho_e$ (resp. $F \in \hat{\rho}_e$) iff $e \in F$ (resp. $e \notin F$); (3) $E \xrightarrow{\varepsilon_E} HG(E)$, $\varepsilon_E(e) = \rho_e$; (4) $X \xrightarrow{\eta_X} GH(X)$, $\eta_X(x) = \{U \in \mathcal{PCOU}(X) \,|\, x \in U\}$. The obtained framework is that of H. Priestley, except for the target categories: **Lat** (resp. **BPosTop**) in place of **DLat** (distributive lattices) (resp. **BPrSpc** (bounded Priestley spaces)). Theorem 2 gives rise to the equivalence $(\overline{\eta}, \overline{\varepsilon}) : \overline{H} \dashv \overline{G}$: $(\mathbf{2}, \mathbf{2}, \tau^d)\mathbf{Spat} \to (\mathbf{2}, \mathbf{2}, \tau^d)\mathbf{Sob}$.

Lemma 12. *A lattice E is $r_{(\mathbf{2},\mathbf{2},\tau^d)}$-spatial iff it is distributive. A bounded potopological space X is $r_{(\mathbf{2},\mathbf{2},\tau^d)}$-sober iff it is a bounded Priestley space.*

Proof. As an example, prove the first statement. Necessity: for an $r_{(\mathbf{2},\mathbf{2},\tau^d)}$-spatial lattice E, $E \cong HG(E) = \mathcal{PCOU}(\mathcal{PF}(E))$, the latter lattice being distributive as a sublattice of $\mathcal{P}(\mathcal{PF}(E))$. Sufficiency: use the Priestley duality [4] to get that the H-co-universal arrow ε_E is an isomorphism and then apply Lemma 11. □

Theorem 3 (Priestley duality). *There exists the equivalence* **LoDLat** $\sim$ **BPrSpc**.

Stone representation theorems. Enrich **Top** in **Set** instead of **BPos** and let $\mathbf{E} = \mathbf{Frm}$, $\mathbb{E} = \mathbf{2}$. Equip $|\mathbf{2}|$ with the Sierpinski topology τ^s and get the continuous algebra $(\mathbb{E}, \boldsymbol{\delta}) = (\mathbf{2}, \tau^s)$. Satisfaction of $(\mathcal{R})$, $(\mathcal{C})$ gives the adjoint situation $(\eta, \varepsilon) : H \dashv G$: $\mathbf{Loc} \to \mathbf{Top}$, where (1) $\mathbf{Top} \xrightarrow{H} \mathbf{Loc}$, $H(X) = (\tau, \cap, \bigcup)$, $Hf = (f^{\leftarrow})^{op}$; (2) $\mathbf{Loc} \xrightarrow{G} \mathbf{Top}$, $G(E) = (\mathcal{CPF}(E), \subseteq, \tau)$, $G\varphi = (\varphi^{op})^{\leftarrow}$ with $\mathcal{CPF}(E)$ the completely prime filters of E and $\tau = \{\rho_e \,|\, e \in E\}$; (3) $E \xrightarrow{\varepsilon_E} HG(E)$, $\varepsilon_E(e) = \rho_e$; (4) $X \xrightarrow{\eta_X} GH(X)$, $\eta_X(x) = \{U \in \tau_X \,|\, x \in U\}$. The setting is precisely that of P. T. Johnstone, providing the equivalence $(\overline{\eta}, \overline{\varepsilon}) : \overline{H} \dashv \overline{G} : (\mathbf{2}, \tau^s)\mathbf{Spat} \to (\mathbf{2}, \tau^s)\mathbf{Sob}$ by Theorem 1.

Lemma 13 (P. T. Johnstone). LoBDLat *(bounded distributive lattices) is isomorphic to the subcategory* **CohLoc** *of* $(\mathbf{2}, \tau^s)$**Spat** *of coherent locales and coherent maps.*

Theorem 4 (Stone duality I). *With* **CohSpc** *(coherent spaces) being the preimage of* **CohLoc** *under $\overline{H}$, there exists the equivalence* **LoBDLat** $\sim$ **CohSpc**.

Since **BAlg** is a subcategory of **BDLat**, the restriction of the above duality gives

Theorem 5 (Stone duality II). *There exists the equivalence* **LoBAlg** $\sim$ **StSpc** *(Stone spaces).*

Notice that since the frame **2** is not a singleton, Example 12 does not allow the extension of the machinery to the variable-basis case.

5 Conclusion

Motivated by the new categorically-algebraic framework for doing topology, we introduced a variable-basis variety-based generalization of the classical theory of natural dualities, replacing the standard crisp topology with the variety-based one. The advantages of the new developments in the fixed-basis case have already been discussed in [38, 39]. The shift to variable-basis not only extends the theory, but also brings several new problems, some of which are discussed below.

As was already mentioned, our variable-basis approach stems from the idea of S. E. Rodabaugh [28, 29], but is essentially different from it. The new setting is strictly richer than its poslat counterpart, truncated to isomorphisms between the underlying lattices of the spaces (a rather heavy restriction, cutting off the benefit of variable-basis). On the other hand, in our framework the fixed-basis case is not a particular instance of the variable-basis case (see the examples of the previous section), and that is the most crucial distinction. Meta-mathematically restated, the standard definition of the fixed-basis approach of the fuzzy community is too weak to accommodate the demands of variety-based dualities. The next problem then springs into mind at once.

Problem 1. What will be the extension of the notion of fixed-basis to bring it in line with the variable-basis concept of categorically-algebraic dualities?

Briefly speaking, allowing something apart from isomorphisms in the variable-basis case, one should allow other morphisms than identity in the fixed-basis one.

To continue the topic of the previous paragraph, recall that non-variety-based catalg setting disguises the notion of basis for topology in an abstract category **X**. With the idea in mind, the second (and more challenging) problem is as follows.

Problem 2. What will be the generalization of the theory of natural dualities to the non-variety-based categorically-algebraic framework?

The recent paper of D. Hofmann and I. Stubbe [16] on "Stone duality" for topological theories (in the monadic sense of the authors) could shed some light on the topic.

The classical natural duality theory relies on the notion of (topological) quasi-variety generated by algebraic (resp. topological) side of schizophrenic object. Variety-based framework replaces quasi-varieties with the concepts of sobriety and spatiality. The remarks after Definitions 14, 16, however, show that even the modified approach is potentially expressible in the language of varieties. The last problem is then immediate.

Problem 3. Can one define the categories $Q_{\mathbf{B}}$-**($\mathbb{C}$,$\boldsymbol{\delta}$)RSob** (resp. $(\mathbf{C},\mathbf{B})$-**($\mathbb{E}$,$\mathbb{C}$,$\boldsymbol{\delta}$)RSob**) and $Q_{\mathbf{B}}$-**($\mathbb{C}$,$\boldsymbol{\delta}$)RSpat** (resp. $(\mathbf{C},\mathbf{B})$-**($\mathbb{E}$,$\mathbb{C}$,$\boldsymbol{\delta}$)RSpat**) as particular "(quasi-)varieties" (the meaning is to clarify) generated by $(\|\mathbb{C}\|,\boldsymbol{\delta})$ (resp. $(\|\mathbb{E}\|,\mathbb{C},\boldsymbol{\delta})$) and $\mathbb{C}$ (resp. $(\mathbb{E},\mathbb{C})$)?

The answer can be found in a close inspection of the *topological quasi-varieties* of [5].

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Relations between risk aversion and notions of ageing: use of semi-copulas

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We start by considering relevant families of utility functions (in a scalar variable), characterized in terms of *absolute local measure of risk aversion*, and families of univariate survival functions, possessing different properties of *ageing*.

In a first part of the talk we then discuss several relations and analogies between such two types of families.

The economic-probabilistic meaning of such relations will also be analyzed.

Within our discussion we will, in particular, make use of the notion of semi-copula and of the representation, in terms of "dependence" properties of appropriate semi-copulas, of notions of ageing.

In the second part of the talk we shall point out some specific aspects related with the extension of the above study to the analysis of multivariate utility functions.

The above developments are related with topics treated in the references listed below.

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Why it is important to make decompositions of information divergences?

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Dedication: This paper is devoted to the Igor Vajda, my great colleague, who died unexpectedly in May 2010.

What is the optimal statistical decision? And how it is related to the statistical information theory?

By trying to answer these difficult questions, we will illustrate the necessity of understanding of structure of information divergences. This may be understand in particular through deconvolutions, leading to an optimal statistical inference. We will illustrate deconvolution of information divergence in the exponential family, which will gave us an optimal tests (optimal in the sense of Bahadur (see [1, 2]).

0.1 Deconvolution of information divergence and optimal testing

Consider a statistical model with N independent observations $y_1, \dots, y_N$ which are distributed according to gamma densities

$$f(y_i|\vartheta) = \begin{cases} \gamma_i(\vartheta)^{v_i} \dfrac{y_i^{v_i-1}}{\Gamma(v_i)} \exp(-\gamma_i(\vartheta) y_i), & \text{for } y_i > 0, \\ 0, & \text{for } y_i \leq 0. \end{cases} \tag{1}$$

Here $\vartheta := (\vartheta_1, \dots, \vartheta_p)$ is vector of unknown scale parameters, which are the parameters of interest and $v = (v_1, ..., v_N)$ is the vector of known shape parameters. The parameter space Θ is an open subset of $\mathbf{R}^p$, $\gamma_i \in C^2(\Theta)$ and matrix of first order derivatives of the mapping $\gamma := (\gamma_1, \dots, \gamma_N)$ has full rank on Θ.

This model is motivated e.g. by a situation when we observe time intervals between $(N+1)$ succesive random events in a Poisson process. In this case the parameters $\gamma_i(\vartheta)$ are equal to the (usually parametrized) intensity γ and the shape parameters are equal identically 1.

Model (1) is a regular exponential family (see [3]), the sufficient statistics for the canonical parameter $\gamma \in \Gamma$ has the form $t(y) = -y$ and $\Gamma = \{(\gamma_1, \dots, \gamma_N), \gamma_i > 0;\ i = 1, \dots, N\}$. The ”covering” property

$$\{t(y) : y \in Y\} \subseteq \{E_\gamma[t(y)] : \gamma \in \Gamma\}$$

(see [4]) together with the relation

$$E_\gamma[t(y)] = \frac{\partial\kappa(\gamma)}{\partial\gamma},$$

where $\kappa(\gamma) = N\ln(\Gamma(v)) - v\sum_{i=1}^{N}\ln(\gamma_i)$, enables us to associate with each value of $t(y)$ a value $\hat{\gamma}_y \in \Gamma$ which satisfies

$$\frac{\partial\kappa(\gamma)}{\partial\gamma}|_{\gamma=\hat{\gamma}_y} = t(y). \tag{2}$$

It follows from (16) that $\hat{\gamma}_y$ is the MLE of the canonical parameter γ in the family (1). By the use of (16) we can define the I-divergence of the observed vector y in the sense of [4]:

$$I_N(y,\gamma) := I(\hat{\gamma}_y,\gamma).$$

Here $I(\gamma^\star,\gamma)$ is the Kullback-Leibler divergence between the parameters $\gamma^\star$ and γ. The I-divergence has nice geometrical properties, let us mention only the Pythagorean relation

$$I(\overline{\gamma},\gamma) = I(\overline{\gamma},\gamma^\star) + I(\gamma^\star,\gamma)$$

for every $\gamma,\overline{\gamma},\gamma^\star \in int(\Gamma)$ such that $(E_{\overline{\gamma}}(t) - E_{\gamma^\star}(t))^T(\gamma^\star - \gamma) = 0$. Here $int(\Gamma)$ denotes the interior of the set Γ. The Pythagorean relation can be used for construction of the MLE density in regular exponential family, see [5] for details.

The I-divergence has nice statistical consequences. Let us consider the likelihood ratio (LR) λ_1 of the test of the hypothesis (2) and the LR λ_2 of the test of the homogeneity hypothesis $H_0 : \gamma_1 = ... = \gamma_N$ in the family (1). Then we have the following interesting relation for every vector of canonical parameters $(\gamma_0,..,\gamma_0) \in \Gamma^N$:

$$I_N(y,(\gamma_0,..,\gamma_0)) = -\ln\lambda_1 + (-\ln\lambda_2|\gamma_1 = ... = \gamma_N). \tag{3}$$

Here the variables $-\ln\lambda_1$ and $-\ln\lambda_2|\gamma_1 = ... = \gamma_N$, i.e. the $-\ln\lambda_2$ under the condition $H_0 : \gamma_1 = ... = \gamma_N$, are independent. The deconvolution (3) of I_N is the consequence of the Theorem 4 in [6]. Both tests are asymptotically optimal in the Bahadur sense ([7, 7]).

0.2 Generally on relation between the ϕ-divergences and statistical information

After demonstrating the importance of studying decompositions of I-divergences, we will discuss relation between f-divergences $D_f(P,Q)$ and statistical informations $I_\pi(P, Q) \equiv I_\pi(P,Q)$ (differences $B_\pi - B_\pi(P,Q)$ between the prior and posterior Bayes risks). This relationship has been established by [9]).

We discuss generalization of this relationship to the *alternative* ϕ-*divergences* $\mathcal{D}_\phi(P_1, P_2,...,P_n)$ and general *statistical informations* $I_{\pi_1,\pi_2,...,\pi_{n-1}}(P_1,P_2,...,P_n)$ of [10, 11].

Here the *alternative* ϕ-*divergence* $\mathcal{D}_\phi(P_1,P_2,...,P_n)$ means the integral

$$\int_{\mathcal{X}} \phi(p_1,p_2,...,p_n)\mathrm{d}\mu \quad \text{for} \quad p_i = \frac{\mathrm{d}P_i}{\mathrm{d}\mu}, \;\; \mu \gg \{P_1,P_2,...,P_n\} \tag{4}$$

where $\phi : [0,\infty)^n \longrightarrow (-\infty,\infty]$ is convex, continuous and homogeneous in the sense

$$\phi(\alpha t_1, \alpha t_2, ..., \alpha t_n) = \alpha\phi(t_1, t_2, ..., t_n) \quad \text{for all } \alpha \geq 0. \tag{5}$$

These ϕ-divergences were introduced by [12]. Igor Vajda extend the definition of ϕ-divergences by:

(i) admitting in (4) convex functions $\phi : [0,\infty)^n \longrightarrow (-\infty,\infty]$ which are finite on $(0,\infty)^n$ and possibly infinite at the boundary,

(ii) replacing the continuity by the lower semicontinuity, and

(iii) assuming strict convexity at $(t_1, t_2, ..., t_n) = (1, 1, ..., 1)$ with $\phi(1, ..., 1) = 0$.

The last assumption guaratnees that $\mathcal{D}_\phi(P_1, P_2, ..., P_n)$ is nonnegative, equal zero if and only if all probability measures $P_1, P_2, ..., P_n$ coincide.

The *statistical information* $I_{\pi_1,\pi_2,...,\pi_{n-1}}(P_1, P_2, ..., P_n)$ is the difference between the classical prior Bayes risk $B_{\pi_1,\pi_2,...,\pi_{n-1}}$ and the posterior Bayes risk $B_{\pi_1,\pi_2,...,\pi_{n-1}}(P_1, P_2, ..., P_n)$ in the statistical decision model with conditional probability measures $P_1, P_2, ..., P_n$ on an observation space $\mathcal{X}$ which is equipped with a σ-algebra and a dominating σ-finite measure μ leading to the densities considered in (4). These probability measures are assumed to govern observations with prior probabilities $\pi_1, \pi_2, ..., \pi_n$ where $\pi_1, \pi_2, ..., \pi_{n-1}$ are from the open simplex

$$S_{n-1} = \left\{ \pi_i > 0, \sum_{i=1}^{n-1} \pi_i < 1 \right\} \subset \mathbb{R}^{n-1} \quad \text{and} \quad \pi_n = 1 - \sum_{i=1}^{n-1} \pi_i.$$

Integral (4) is well defined (but possibly infinite) which follows from the inequality between $\phi(t_1, t_2, ..., t_n)$ and its support plane at the point $(t_1, t_2, ..., t_n) = (1, 1, ..., 1)$.

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Soft computing-based risk management - fuzzy, multilevel structured decision making system

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Abstract. Decision making in risk management is basically a complex system usually with uncertain input factors and approximate reasoning principle. Based on the strength of those attributes it is a reasonable way for fast and human-like decision to group the factors or decision rules, and to use the fuzzy approach in the risk level calculation modeling. The paper points out existing approaches, and presents an additional advantage of this model-structure: the possibility to gain the different factor-group's impact in the system or in the decision making process, and the multilevel construction of the decision process. As example a possible crisis monitoring application is presented.

1 Introduction

Risk management is a complex, multi-criteria and multi-parametrical system full of uncertainties and vagueness. Considering all those conditions fuzzy set theory helps manage complexity and uncertainties and gives a user-friendly visualization of the system construction and working model.

The fuzzy-based risk management models assume that the risk factors are fuzzified (because of their uncertainties or linguistic representation), furthermore the risk management and risk level calculation statements are represented in the form of *if premises then conclusion* rule forms, and the risk factor calculation or output decision (summarized output) is obtained using fuzzy approximate reasoning methods.

Considering fuzzy set theory and system theory results, there is a further possibility to extend the fuzzy-based risk management models with the hierarchical or multilevel construction of the decision process, grouping the risk factors or rules. This approach supports the possibility of gaining some risk factors' groups or rule subsystems, depending on their importance or other significant environment characteristics or by laying emphasis on risk management actors'. Other possibilities are the extension of the modeling with type 2 fuzzy sets, representing the level of the uncertainties of the membership values, or using of special, problem-oriented types of operators in the fuzzy decision making process.

The relationship between risk factors, risks and their consequences are represented in different forms, but in [1] a well-structured solution, suitable for the fuzzy approach is given. A risk management system can be built up as a multilevel or hierarchical system of the risk factors (inputs), risk management actions (decision making system) and direction or directions for the next level of risk situation solving algorithm. A possible preliminary system construction of the risk management principle can be given based on this structured risk factor classification and on the fact, that some risk factor groups, risk factors or management actions have a weighted role in the system operation. The system parameters are represented with the fuzzy sets, and the grouped risk factors values give intermediate result. Considering some system input parameters, which determine the risk factors role in the decision making system, intermediate results can be weighted and forwarded to the next level of the reasoning process [2]. Actually outputs of previous decision making level are risk factors for the action on the next level of the risk management process. Risk factors in a complex system are grouped according to the risk relevant events or decision steps. Actions or decision steps are described by the 'if … then' type rules. With the output those components frame one unit in the hole risk management system, where the items are usually grouped according to the principle of the time-scheduling, significance or other criteria (Fig. 1 shows a global system construction). Input Risk Factors (RF) grouped and assigned to the current action are described by the Fuzzy Risk Measure Sets (FRMS), and can include the fuzziness of their measured or detected membership.

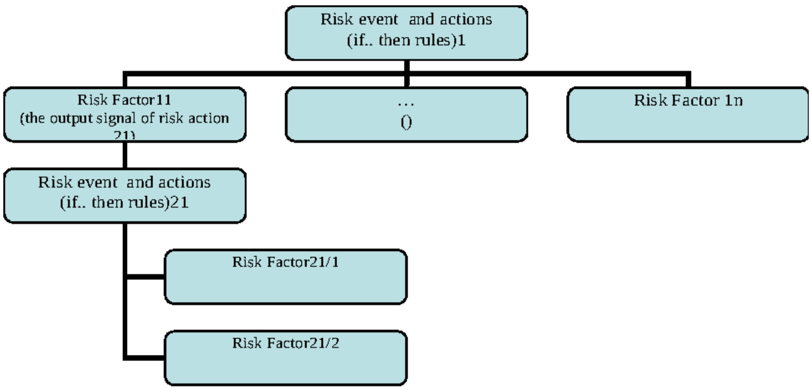

Figure 1. The hierarchical constructed risk management system

2 Example

Crisis or disaster event monitoring provides basic information for many decisions in today's social life. The disaster recovery strategies of the country, the financial investment plans of investors, or the level of the tourism and traveling activities all depend on different groups of disaster or crisis factors.

The disaster can be defined as an unforeseen event that causes great damage, destruction and human suffering, evolved from a natural or man-made event that negatively affects life, property, livelihood or industry. A disaster is the start of a crisis, and often results in permanent changes to human societies, the ecosystem and environment.

Based on the experts' observations [3], [4], the risk factors, which (prejudice) predict disaster situation can be classified in the group of natural disasters and man-made disasters. Furthermore, a risk management system was constructed in Matlab fuzzy environment, based on the mentioned principle, with fuzzified risk factor inputs and hierarchically constructed rule base system, shown in Fig. 2. The risk or disaster factors, as the inputs of one subsystem of the global fuzzy decision making system, give outputs for the next level of decision, where the main natural and man-made disaster classes result the total impact of this risk category.

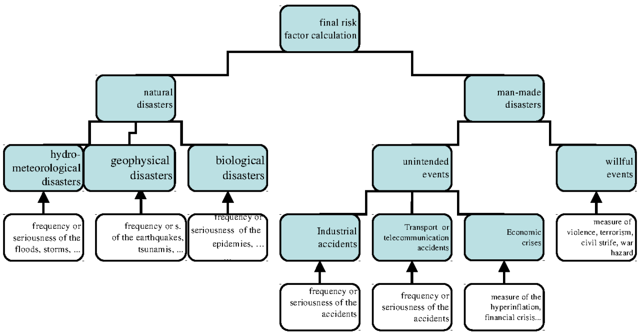

Figure 2. Hierarchically constructed decision system

This approach allows additional possibilities to handle the set of risk factors. It is easy to add one factor to a factors-subset; the complexity of the rule base system has been changed only in the affected subsystem. In different seasons, environmental situations, and so on, some of the risk groups are more important for the global conclusion than others, so they can be achieved with an importance factor (a number from the $[0,1]$).

The man-made disasters have an element of human intent or negligence. However, some of those events can also occur as the result of a natural disaster. The man-made factors and disasters can be structured in a similar way, as the natural risks, events. One of the possible classifications of the basic man-made risk factors or disaster events (applied in our example) is as follows:

1. Unintended events (industrial accidents, chemical spills, collapses of industrial infrastructures); transport or telecommunication accidents (by air, rail, road or water means of transport); economic crises (growth collapse, hyperinflation, and financial crisis).
2. Willful events (violence, terrorism, civil strife, riots, and war).

The effects of man-made disasters as the inputs in the decision making process are represented with their relative frequency, and the premises of the related fuzzy rules are very often represented with the membership functions: never, rarely, frequently, etc.

The final traveling risk level in a country depending on both disasters as risk factors' groups is shown in Fig. 3.

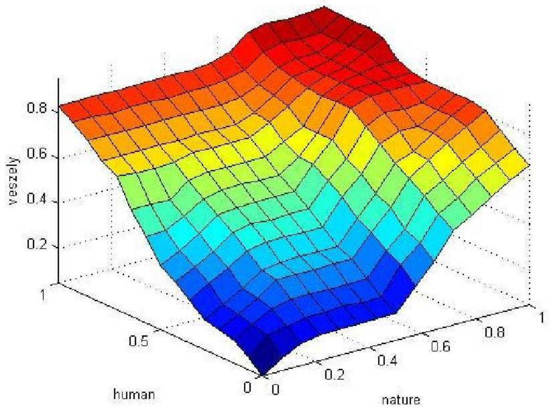

Figure 3. The final conclusion about traveling risk in a country based on both disasters' as risk factors' groups

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Resource allocation problems in hierarchical models based on multistep Choquet integrals

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1 Introduction

The Choquet integral is known to be among the most powerful tools in the Multicriteria Decision Making Theory [6]. Nevertheless, the dynamic aspect of the fuzzy measure-based modelling has been very scarcely if at all researched to the present moment. To the author's best knowledge, the only papers discussing the problem of the Choquet integral maximization are [3], and to some extent [2]. Intuitively, the motivation for such problems can be given in the following way. Assume one is engaged in resource distribution planning for some complex system. In applications, such as risk management, portfolio optimization, research planning, information security, etc, the system can usually be represented as a hierarchical structure. The latter allows to depict a taxonomy of the problem subfactors(criteria) and to analyse logical links between them. The relative weights of criteria and their interactions can be modelled by means of a capacity with aggregation process relying on the Choquet integral. Now let the values of the criteria $(x_1,\ldots,x_n)$ to be not just static constants but functions $(x_1 = f_1(z_1),\ldots,x_n = f_n(z_n))$ dependant on some variables and possibly non-linear. These variables are used to control the system parameters, e.g. the amount of hours spent on research in some area or perhaps the volume of the investment made to improve some part of a complex system. We would like to obtain the best strategy for resource distribution under a budget constraint.

The paper is focused on three main points. These are single-branch optimization, propagation of optimal values along the hierarchy and capacity identification coupled with robust solution search. We now go through them in detail.

2 Single-branch Optimization

We look at the following problem.

$$C_\nu(f_1(z_1),\ldots,f_n(z_n)) \uparrow max$$
$$\mathbf{s.t.} \begin{cases} \sum z_i = B \\ z_i \geqslant 0 \end{cases}$$

Where C_ν is the Choquet integral w.r.t. capacity ν of evaluating function $F : X \to \mathbb{R}$, which values $x_i = f_i(z_i)$ are concave and smooth functions (for justification in applications see e.g. [4]). We analyse the influence of the capacity properties on the problem, starting by extending the Lovasz convexity theorem [7] to a non-linear case. For k-monotone capacity ($k \geqslant 2$) the objective function is shown to be concave (though non-differentiable), and therefore, might be easily optimized. However, the general case requires some further elaboration. We first prove the problem complexity to ensure that its special structure does not lead to a polynomial time algorithm. This is done by reducing to quadratic maximization on a non-convex set, which is known to be NP-hard [8].

We then introduce an algorithm which allows to deduce a shortest disjunctive decomposition of an arbitrary capacity to a set of totally monotone measures(i.e. belief functions). The original Choquet integral is represented as

$$\int F d\nu = \max_{\nu = \max Bel_i} \int F dBel_i$$

where Bel_i are totally monotone and the number of disjuncts is minimal. The result is achieved by starting from the totally monotone core, introduced in [1], and further elaboration on bijective correspondences between sets of maximal chains, elements of 2^X where X is the set of criteria, and partitions of the feasible area simplex. The decomposition allows to obtain the global optimum by solving several concave problems, while the minimality ensures that the algorithm is optimal. We also propose a local search algorithm based on convexification of the objective function. It is known that the Choquet integral w.r.t. any capacity can be represented by a difference of two integrals with respect to totally monotone measures [3]

$$\int F d\nu = \int F d\nu^+ - \int F d\nu^-$$

In the optimization context, the class of functions, allowing such decomposition is called D.C. (difference of convex). Convexification is then performed by substituting $\int F d\nu^-$ with its linear approximation.

3 Optimal Value Propagation

We next analyse the propagation of optimal values along a hierarchical structure. In the decision making context such models are known by the name of multistep Choquet integrals [9]. Capacity properties are once again employed for the analysis of solution stability and behaviour of the optimal value and optimal point functions. The main research object is the following parametric function

$$C^*_\nu(z, B) = \max_z C_\nu(z), \quad \sum z = B$$

It is shown that k-monotone ($k \geqslant 2$) capacity produces a concave optimal value function and, therefore, the whole tree can be represented as a single concave optimization problem. In the general case, the optimal function is shown to be quasi-concave, hence the methods employed for single-branch optimization are not directly applicable. We propose some approximation approaches to perform multistep propagation and discuss how the disjunctive decomposition obtained earlier can be employed.

4 Capacity Identification and Robust Optimization

The majority of known up-to-date capacity identification methods [5] employ some sort of an approximation scheme, requiring the decision maker(DM) to evaluate several learning samples. Unfortunately, this is not always possible in some modelling applications among listed above. Another aspect of the problem is imprecision in the assessment of the criteria weights and their interaction character. These factors require for some robust mechanisms to be introduced in the model. We analyse the following problem

$$\begin{aligned} & \max_{\nu}(C^*(\nu) - C(\nu, z_r)) \qquad \downarrow \min_{z_r} \\ \mathbf{s.t.} \quad & \nu \in \mathcal{U} \\ & \sum z_r = B \\ & z_r \in [0; B] \end{aligned}$$

where $C^*(\nu) = \max_z C_\nu(z)$ and $\mathcal{U}$ is the feasible set defined by the DM preferences on criteria weights and interaction. In other words, our purpose is to find the solution which minimizes the maximum possible deviation from optimum for all capacities compliant with the information provided by the DM. Known methods, such as the Shapley value and the interaction index [5] are modified to account for potential errors in the DM evaluations. This is done by surrounding the nominal values obtained during the initial evaluation with some confidence intervals.

We again discern between the convex and non-convex cases and analyse the dependence between the capacity properties, solution stability and computability. Due to linearity of the Choquet integral in measure, it becomes possible to reduce the semi-infinite problem above to a finite one. However, the dimensionality of the problem still does not allow to obtain a precise solution. We therefore proceed with the approximation of the upper bound. Notice that for some $\nu_{\min}$ and $\nu_{\max}$, such that $\nu_{\min} \leqslant \nu \leqslant \nu_{\max}$, for all $\nu \in \mathcal{U}$ the following holds for a fixed $F > 0$

$$\int F d\nu_{\min} \leqslant \int F d\nu \leqslant \int F d\nu_{\max}$$

The upper bound can then be obtained by solving

$$\begin{aligned} & C^*(\nu_{\max}) - C(\nu_{\min}, z_r) \qquad \downarrow \min_{z_r} \\ \mathbf{s.t.} \quad & \nu_{\max}, \nu_{\min} \in \mathcal{U} \\ & \sum z_r = B \\ & z_r \in [0; B] \end{aligned}$$

where $\nu_{\max}$ and $\nu_{\min}$ are not generally unique but belong to some Pareto-optimal set.

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Logics for arguing pro and contra

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1 Introduction

Medical decision support aims at facilitating the process of deciding on a patient's treatment on the basis of the available information. In particular, from electronically documented signs and symptoms of a patient, possibly present diseases are expected to be determined in an automatic way.

In this contribution, we take into account a characteristic feature of the decision-making process in medicine that suggests the usage of certain non-classical logics. Namely, arguments supporting a conjecture are typically not treated as of proving character, but just as an indication for a certain possibility. As a particular consequence, arguments in favour of a conjecture and arguments against a conjecture are collected independently. A medical decision support system should in fact not just provide a list of diseases whose presence is possible. It should rather inform that certain facts indicate the presence and others indicate the absence of a disease. Moreover, contradiction should be allowed rather than result in an error message.

We consider in the sequel a couple of logics that emulate this kind of reasoning. We stay at the propositional level and, in contrast to most formalisms to deal with uncertainty [2], we address qualitative aspects only.

Among the considered logics, one is apparently new, the other ones are well-known. In case of the latter, the achievement is that they are viewed from a common perspective. Moreover, there have been heated debates around the question if certain non-classical logics are meaningful. In the present context, Belnap's four-valued logic, among others, turns up in a natural way and on well-defined grounds.

2 Reasoning about not directly testable facts

Let us once more consider the example of medical decision support. We deal on the one hand with a patient's signs and symptoms; they represent the available information. On the other hand, we deal with the question if a patient has a certain disease; we may assume that the presence of the disease can in general not be directly tested. Indeed, decision support would otherwise not really make sense. Thus the available facts are in general not sufficient to decide the question under consideration.

According to this observation our logics are designed. Formally, we start as usual with a set of worlds. The worlds are meant to vary over the available facts; a single world is meant to a represent a possible situation, characterised by the detectable information. Properties that are not directly testable, in whose clarification we might anyhow be interested do themselves not appear in our model. In fact, we do not model a property by the set of worlds in which it holds. We rather associate with each property those worlds in which the available information is sufficient to decide it positively or negatively, or in which the available information at least suggests its truth or falsity.

Thus a world represents information that may give an indication of a property. This does not imply that the property in question actually holds in this world; if the property holds or not is left open. It is understood that the property itself can in general not be reconstructed from the available information, not even partially. We deal with arguments in favour of or against some conjecture, not more. We understand our logics as a formal tool to interchange arguments pro and contra.

With any property we associate a set of worlds, and this set contains those worlds that reflect testable facts speaking in favour of the property. This means that the set associated with a property and the set associated with its negation are not necessarily set-theoretic complements. The relationship between facts in favour of and facts against a conjecture is not fixed. We will review five possibilities how this relationship can look like and indicate the corresponding logics.

3 The general framework

The logics have the following specifications in common. The language comprises a countable set $\varphi_1, \varphi_2, \ldots$ of variables and the constants $\top$ and $\bot$. *Lattice formulas*, or *formulas* for short, are built up by means of the connectives $\wedge$, $\vee$, and $\neg$. An *implicational formula*, or *implication* for short, is a pair α, β of lattice formulas, written $\alpha \to \beta$.

On the semantical side, we have a pair $(W, \mathcal{B})$ of a non-empty set W, whose elements are called *worlds*, and a subset $\mathcal{B}$ of $\mathcal{P}W$ containing $\emptyset$ and W and closed under $\cap$, $\cup$. Formulas are modelled in $\mathcal{B}$, the constants being assigned $\emptyset$ and W, respectively, and $\wedge$, $\vee$ being interpreted by $\cap$, $\cup$, respectively. If v is an evaluation of the formulas, then an implication $\alpha \to \beta$ is satisfied if $v(\alpha) \subseteq v(\beta)$.

The five logics below differ in their interpretation of $\neg$. In addition, W might be endowed with additional structure.

4 The negation interpreted in a constructive way

We have to determine the way the negation is handled in our logics. Consider a set of worlds W; let φ be a variable assigned the set of worlds $v(\varphi)$. We understand φ as representing a yes-no property and we understand $v(\varphi)$ as containing the worlds reflecting those facts speaking in favour of φ. Furthermore, we understand $\neg\varphi$ as representing the negated property φ. We have to specify which subset $\neg\varphi$ is assigned. In fact, the question how the interpretations of a property φ and its negation $\neg\varphi$ are interrelated is open.

Let us first consider the possibility that $v(\neg\varphi)$ depends on $v(\varphi)$ in a constructive way; this means that $v(\neg\varphi)$ is derivable from $v(\varphi)$ on the basis of the structure of the set of worlds. A reasonable principle seems to be that $v(\neg\varphi)$ contains the worlds that speak against φ because they are, in a sense to be made precise, separated from all worlds that speak in favour of φ.

The simplest way to construct $v(\neg\varphi)$ from $v(\varphi)$ is to set

$$v(\neg\varphi) \;=\; v(\varphi)^c.$$

To use the set-theoretic complement amounts to say that φ can be told to hold or not to hold in all worlds. The result is classical propositional logic, restricted to (what we call) implications.

This procedure might be reasonable if W is finite; in general we guess that classical logic is in the present context of little interest.

If W represents a continuum of possibilities, a sharp boundary between $v(\varphi)$ and $v(\neg\varphi)$ is usually inappropriate. In such cases, we should require that worlds speaking in favour of φ and against φ are separated from each other by some form of neighbourhood.

A modest approach to realise this idea is to endow W with a topology and to call two worlds separated if they possess disjoint open neighbourhoods. We request that each formula is interpreted by an open set and that a world separated by any world speaking in favour of φ speaks against φ. This leads to the definition

$$v(\neg\varphi) \;=\; (v(\varphi)^-)^c,$$

where A^- denotes the closure of $A \subseteq W$. We are led to intuitionistic logic, endowed with its residual negation as an extra connective and then restricted to implications. An axiomatisation can be found in [6].

A more application-friendly procedure is to endow W with a metric and call two worlds separated if their distance is larger than or equal to a given threshold $s > 0$. We note that our setting has then some resemblance with the setting of Williamson's Logic of Clarity [8]. Proceeding analogously as before, we define

$$v(\neg\varphi) \;=\; (U_s(v(\varphi))^c,$$

where $U_s(A)$ is the open s-neighbourhood of $A \subseteq W$. For an axiomatisation, the key fact to be used is that our models are distributive lattices endowed with a lower-semicomplement function $\neg$ such that $a \leq \neg\neg a$.

5 The negation interpreted in an independent way

Alternatively, we may want the set $v(\neg\varphi)$ to contain those worlds that actually speak against φ. In this case the situation is symmetric with respect to φ and $\neg\varphi$ and we are required to let the interpretations of φ and $\neg\varphi$ "float freely", that is, we do not assume that $v(\neg\varphi)$ is derivable from $v(\varphi)$ inside $(W, \mathcal{B})$.

In this case we may reasonably assume that both the interpretation of a property and the interpretation of its negation determine this property uniquely. Consequently, we can assume that $v(\varphi)$ and $v(\neg\varphi)$ determine each other mutually, so that there is an order-reversing, involutive operation mapping $v(\varphi)$ to $v(\neg\varphi)$. We are led to De Morgan algebras as associated to our calculi.

There are now two ways to go. We may first assume that the information that a world provides with respect to an unknown fact φ is never contradictory: either it speaks in favour of φ or against φ or neither of these two possibilities applies. The interpretations of φ and ¬φ should then have an empty intersection. The resulting logic is Kleene's three-valued logic.

Second, we may allow that facts speak both in favour of and against a property. Only in this case our initial requirement to allow contradictions is fulfilled; the setting is actually in best accordance with our introductory example, cf. [1]. The resulting logic is Belnap's four-valued logic [3].

6 Gradedness

We have considered the situation that facts either speak in favour of some unknown property or against it. Needless to say, in applications such statements typically turn up in graded form. In medicine we might want to specify the degree to which we find the presence of a disease plausible.

The last three logics considered above allow generalisations in this respect. The logic with fixed distance between φ and ¬φ can be modified to a logic with a continuous transition. As an appropriate formal setting, the Logic of Approximate Reasoning [4] may, e.g., serve. In case of the logics where properties and their negations are interpreted independently, we may replace the crisp sets modelling φ and ¬φ by fuzzy sets, whose support is optionally requested not to overlap. For approaches in this direction see [5, 7].

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