LINZ 2012

33rd Linz Seminar on Fuzzy Set Theory

Abstracts

Enriched Category Theory and Related Topics

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Abstracts

Ulrich Höhle
Lawrence N. Stout
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Editors

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ENRICHED CATEGORY THEORY AND RELATED TOPICS

ABSTRACTS

Ulrich Höhle, Lawrence N. Stout, Erich Peter Klement
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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside of fuzzy set theory can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work.

LINZ 2012 will be the 33rd seminar carrying on this tradition and is devoted to the theme "Enriched Category Theory and Related Topics". The goal of the seminar is to present and to discuss recent advances in enriched category theory and its various applications in pure and applied mathematics.

A large number of highly interesting contributions were submitted for presentation at LINZ 2012. This volume contains the abstracts of this impressive collection. The regular contributions are complemented by four invited talks which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

Ulrich Höhle
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Contents

On the nature of correspondence between partial metrics and fuzzy equalities

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1 Introduction

The correspondence between partial metrics and fuzzy equalities was discovered in 2006 [1]. It was immediately apparent that there was a duality between metric and logical viewpoints, and so the question about the nature of correspondence between partial metrics and fuzzy equalities arose.

Initially, the authors of [1] suggested that we should talk about equivalence between partial metrics and fuzzy equalities up to the choice of dual notation. This suggestion was based on the notion that the duality between metric and logical viewpoints belonged to the metalevel and was a part of the mindset of the practitioners in the respective fields, but did not affect the mathematical structures involved. We refer to this suggestion as the **equivalence approach**.

The equivalence approach remains a legitimate way of viewing this correspondence. In particular, while there is a varierty of possible choices of allowed spaces and morphisms, in all cases studied so far there are (covariant) isomorphisms of the corresponding categories of partial metric spaces and spaces equipped with fuzzy equalities. The induced specialization orders on a partial metric space and the corresponding space equipped with a fuzzy equality also coincide. So, in this sense there seems to be no duality between partial metrics and fuzzy equalities themselves.

Later Mustafa Demirci suggested that the duality between metric and logical viewpoints should nevertheless be brought into formalization of this correspondence by explictly requiring that logical values and distances were respresented by dual structures. We refer to this suggestion as the **duality approach**.

It turns out that the duality approach to understanding this situation is preferrable. It allows to formally express a larger chunk of existing informal mathematical practice, and it allows to do so without explicitly considering the metalevel. Even more importantly, being closer to the respective intuitions of the practitioners in the related fields the duality approach makes it easier to develop applications.

Another aspect of the duality between logical values and distances is that the multiplicative notation is used on the logical side and the additive notation is used on the metric side. This suggests that it might be possible to bring some kind of **exponentiation** into play as well, potentially resulting in a more complicated correspondence and, perhaps, a genuine duality between partial metrics and fuzzy equalities. To the best of our knowledge, this has not been done so far and should be considered an open problem. (It should be noted here that it is not uncommon to start with a metric $d(x,y)$, to express the degree of similarity of x and y as $f(x,y) = e^{-d(x,y)}$, and to call the resulting $f(x,y)$ a fuzzy metric with the appropriate transformation of the axioms of a metric.)

2 Definitions

We provide informal sketches of definitions of quantale-valued partial metrics [3] and quantale-valued sets (sets equipped with quantale-valued fuzzy equalities) [2].

2.1 Quantale-valued Partial Metrics

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a + b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a + c \geq b\}$. Certain additional conditions are imposed.

Definition 1. *A V-partialpseudometric space is a set X equipped with a map $p : X \times X \to V$ (**partial pseudometric**) subject to the axioms*

- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + (p(y,z) \dot{-} p(y,y))$

2.2 Quantale-valued Sets

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

Definition 2. *An M-valued set is a set X equipped with a map $E : X \times X \to M$ (**fuzzy equality**) subject to the axioms*

- $E(x,y) \sqsubseteq E(x,x)$
- $E(x,y) = E(y,x)$
- $E(x,y) * (E(y,y) \Rightarrow E(y,z)) \sqsubseteq E(x,z)$

3 Equivalence approach

Whenever we have a quantale in the sense of section 2.1, we can equip it with a dual order, $\sqsubseteq = \geq$, and it becomes a quantale in the sense of section 2.2 (and vice versa in the opposite direction).

Define $*$ as $+$, $a \Rightarrow b$ as $b \dot{-} a$, 1 as 0 (and vice versa in the opposite direction).

Then partial pseudometrics and fuzzy equalities coincide as sets of functions. This justifies the equivalence approach.

4 Duality approach

However we found it convenient to press the duality approach as far as possible.

4.1 Partial Ultrametrics Valued in Browerian algebras

For example, consider Ω-sets valued in Heyting algebras. Following the duality approach, on the metric side of things we will talk about partial ultrametrics valued in dual Heyting algebras, but really pressing this approach as far as possible, we'll use the terminology ”partial ultrametrics valued in Browerian algebras”, and when Ω is actually the algebra of open sets of a topological space X, we will consider partial ultrametrics valued in the algebra of closed sets of the same space.

This helps to understand and establish the following result.

4.2 Sheaves of Sets as Co-sheaves of α-ultrametrics and Non-expansive Maps

Consider a complete Heyting algebra Ω. Consider a corresponding complete Browerian algebra α.

Then every separated pre-sheaf of sets over Ω can be thought of as a separated co-pre-sheaf of α-ultrametrics and non-expansive maps over α.

To develop the necessary intuition one should first consider the case when Ω and α are the algebras of, respectively, open and closed sets of a given topological space.

4.3 Partial Metrics into Non-negative Reals

In the logical situations (arising in domains for denotational semantics, and, in general, in connection with the specialization order on the space of distances) we typically have to flip the ray of non-negative reals, making 0 the top element.

If we press the duality approach as far as possible, the logical counter-part of the partial metrics into non-negative reals ought to be **fuzzy equalities valued in non-positive reals**. So instead of flipping the ray of non-negative reals we replace it with the symmetric ray of non-positive reals.

Partial ultrametrics correspond to idempotent logic (usually, to the ordinary intuitionistic logic). Partial metrics should typically correspond to linear logic, and we think about linear logic as the resource-sensitive logic. So, from the linear logic point of view, it is natural to think about the weight (self-distance) of an element as the work which still needs to be done to make it fully defined. This is the work to be done, something owed, hence negative.

4.4 Intuition Related to Relaxed Metrics

Relaxed metrics typically map (x,y) into an interval number $[l(x,y),u(x,y)]$, where u is usually a partial metric, and l is usually a symmetric function, such that $l(x,y) \leq u(x,y)$.

Function u yields an upper bound for the inequality between "true, underlying x and y"; essentially, "x and y differ no more that $u(x,y)$", while l yields a lower bound for that, essentially, "x and y differ at least by $l(x,y)$". There is an intimate relationship between l and negative information, and also between l and tolerances.

From the earlier logical considerations of relaxed metrics we know that u dualizes, but l does not. This means that on the logical side, U becames negative (non-positive, actually), but L remains non-negative.

So, while U represents a work still owed (a work to estimate distance better, actually), and hence negative, L represents a work done, and hence positive (on the logical side). Interestingly enough, the condition $l(x,y) \leq u(x,y)$ on the metric side becomes $L(x,y)+U(x,y) \leq 0$ on the logical side.

If the distance between elements, x and y, is precesely defined (often the case for maximal elements x and y), then $l(x,y) = u(x,y)$, or equivalently $L(x,y)+U(x,y) = 0$, expressing the fact that no further computations are owed.

In general the amount which expresses debt here is not $U(x,y)$, but $L(x,y)+U(x,y) = l(x,y)-u(x,y)$. (Note that $l(x,x)$ is always 0, so the self-distance is always fully owed.)

5 Conclusion

The correspondence between partial metrics and fuzzy equalities allows for the transfer of results and methods between these field, and helps in considering non-trivial interplay between metric and logical situations.

There is a long list of situations where this correspondence should be useful. We only name a few of them here.

It is particularly important to study metric counterparts of the logical research generalizing the fuzzy equalities to the non-commutative case and to categories, in particular results for sets valued in non-commutative quantales (Höhle and Kubiak) and results for sets valued in Grothendieck topologies (Higgs).

Weighted quasi-metrics are a remarkably effective instrument on the metric side, and their logical counterparts would probably be as useful as the global quantale-valued sets which are the logical counterparts of weighted metrics.

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A Stone-type adjunction for fixed-basis fuzzy topological spaces in abstract categories and its applications

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Abstract. In this study, fixed-basis fuzzy topological spaces are formulated on the basis of a certain object of an abstract category, and a Stone-type adjunction for them is established. Applications and consequences of this adjunction is discussed. As its particular consequence, it is shown that the category of B-categories (and so the category of quantale preordered sets) is dually adjoint to the category of base spaces.

1 Introduction and motivation

Since the inception of the fuzzy topological spaces (called the lattice-valued topological spaces or the many valued topological spaces in more recent terminology), their truth value structures (or their bases in the terminology of [8, 5, 12, 13]) have been extensively studied in the literature. The selection of bases varies from author to author and from paper to paper. Completely distributive complete lattices with order-reversing involutions [16], semiframes [11], frames [6], cl_∞-monoids [10], GL-monoids with square roots [7], complete groupoids [5], complete quasi-monoidal lattices [8, 12], semi-quantales [2, 13] and algebras in varieties [14] are known examples of such bases. The diversity of bases naturally brings the question of how the notion of fixed-basis fuzzy topological space can be defined on the basis of an object L of an abstract category **C**. Although a similar categorical question is also valid for other approaches to the notion of fuzzy topological spaces such as variable-basis fuzzy topological spaces [11–13] and generalized lattice-valued topological spaces [2], we focus on only the fixed-basis case in this study. Apart from fuzzy topologies, Stone-type adjunctions form an important theme of the order-theory (see [3] and the references therein). Among others, the adjunction between the category **Loc** of locales and the category **Top** of topological spaces is a well-known example of these adjunctions. The studies on Stone-type adjunctions give rise to a fundamental question: Is it possible to extend the adjunction between **Loc** and **Top** to an adjunction between an abstract category **C** and a category of spaces in some generalized sense? This question is tantamount to the formulation of Stone-type adjunctions for abstract categories. Its solution relies on the fixed-basis fuzzy topological spaces asked in the former question. The main aim of this study is to introduce the notion of **C**-$\mathcal{M}$-L-space as a categorical generalization of fixed-basis fuzzy topological space being an answer to the former question, and is to construct a dual adjunction between **C** and the category of **C**-$\mathcal{M}$-L-spaces providing an answer to the latter question.

2 $\mathbf{C}$-$\mathcal{M}$-L-spaces and their dual adjunction with $\mathbf{C}$

Let the category $\mathbf{C}$ have products, and let $\mathcal{M}$ be a class of monomorphisms in $\mathbf{C}$. Furthermore, let us fix a $\mathbf{C}$-object L.

Definition 1. *For a set X, we call the pair $\left(X, \tau \overset{m}{\to} L^X\right)$ a $\mathbf{C}$-$\mathcal{M}$-L-space, and $\tau \overset{m}{\to} L^X$ a $\mathbf{C}$-$\mathcal{M}$-L-topology on X iff $\tau \overset{m}{\to} L^X$ is an $\mathcal{M}$-morphism.*

Proposition 1. *Each function $f : X \to Y$ determines a unique $\mathbf{C}$-morphism $f_L^{\leftarrow} : L^Y \to L^X$ (called backward $\mathbf{C}$-L-power operator of f) such that $\pi_x \circ f_L^{\leftarrow} = \pi_{f(x)}$ for all $x \in X$.*

Definition 2. *Given $\mathbf{C}$-$\mathcal{M}$-L-spaces $\left(X, \tau \overset{m_1}{\to} L^X\right)$ and $\left(Y, \nu \overset{m_2}{\to} L^Y\right)$, a function $f : X \to Y$ is $\mathbf{C}$-$\mathcal{M}$-L-continuous iff there exists a $\mathbf{C}$-morphism $r_f : \nu \to \tau$ filling out the following commuting diagram:*

$$\begin{array}{ccc} L^Y & \xrightarrow{f_L^{\leftarrow}} & L^X \\ m_2 \uparrow & & \uparrow m_1 \\ \nu & \xrightarrow{r_f} & \tau \end{array}$$

$\mathbf{C}$-$\mathcal{M}$-L-spaces and $\mathbf{C}$-$\mathcal{M}$-L-continuous maps constitute a category that we denote by $\mathbf{C}$-$\mathcal{M}$-L-$\mathbf{SPC}$. By supplying examples, it will be justified that $\mathbf{C}$-$\mathcal{M}$-L-$\mathbf{SPC}$ is a categorical unification of many familiar categories of fixed-basis fuzzy topological spaces. As the central result of this study, we will establish a categorical generalization of the adjunction between **Loc** and **Top** in the following manner:

Theorem 1. *For $\mathcal{E} \subseteq Mor(\mathbf{C})$ and $\mathcal{M} \subseteq Mon(\mathbf{C})$, let $\mathbf{C}$ have $(\mathcal{E}, \mathcal{M})$-factorizations and the unique $(\mathcal{E}, \mathcal{M})$-diagonalization property in the sense of [1]. Then $\mathbf{C}$ is dually adjoint to $\mathbf{C}$-$\mathcal{M}$-L-**SPC**.*

Referring to the unit and co-unit of the adjunction $\mathbf{C}^{op} \dashv \mathbf{C}$-$\mathcal{M}$-$L$-$\mathbf{SPC}$, we define L-spatial $\mathbf{C}$-objects and L-sober $\mathbf{C}$-$\mathcal{M}$-L-spaces, and then point out in this study that the restriction of the adjunction in Theorem 1 to the full subcategory of $\mathbf{C}$ of all L-spatial objects and the full subcategory of $\mathbf{C}$-$\mathcal{M}$-L-$\mathbf{SPC}$ of all L-sober objects gives a dual equivalence between these subcategories. The adjunction $\mathbf{C}^{op} \dashv \mathbf{C}$-$\mathcal{M}$-$L$-$\mathbf{SPC}$ covers many known and new dual adjunctions between various kinds of ordered-structures and various kinds of generalized topological spaces. Because of practical purposes, we pay a special attention to the explicit determination of $\mathbf{C}$-$\mathcal{M}$-L-$\mathbf{SPC}$ for a concrete category $\mathbf{C}$. In particular, it will be proven in this talk to be an application of Theorem 1 that the category $\mathbf{Cat}(B)$ of B-categories [9, 15] (and so the category **p**-Q-**Set** of pre-Q-sets [9]) is dually adjoint to the category **BS** of base spaces [4].

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Enriched topological systems and variable-basis enriched functors

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This work is motivated in part by a question arising from programming: if bitstream x corresponds to bitstream y to some degree α, and if bitstream y satisfies predicate a to some degree β, then would it not be appropriate to model the possibility that bitstream x satisfies predicate a to at least some degree related to both α and β? The current multi-valued literature on topological and other systems, e.g., [1, 2, 4, 5, 9, 10], ultimately rooted from [11], does not address this question. Notions from enriched categories help us address this question and its consequences.

An *enriched category* $\mathcal{C}$ [7] over a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ [8] is a class of objects with the following data (C1, C2, C3) subject to the following axioms (D1, D2, D3), where in the latter the last applied compositions are from $\mathcal{M}$ and the other compositions come from (C3):

C1 $\forall a, b \in \mathcal{C}, \exists! \, \mathcal{C}(a,b) \in |\mathcal{M}|$ (*existence of hom-objects*).
C2 $\forall a \in \mathcal{C}, \exists \, id_a : I \to \mathcal{C}(a,a)$ (*identities*).
C3 $\forall a, b, c \in \mathcal{C}, \exists! \circ_{abc} : \mathcal{C}(b,c) \otimes \mathcal{C}(a,b) \to \mathcal{C}(a,c)$ (*composition of hom-objects*).
D1 $\forall a, b, c, d \in \mathcal{C}$,

$$(\circ_{abd}) \circ (\circ_{bcd} \otimes \mathbf{1}) \;=\; (\circ_{acd}) \circ (\mathbf{1} \otimes \circ_{abc}) \circ \alpha.$$

D2 $\forall a, b \in \mathcal{C}$,

$$\lambda \;=\; (\circ_{abb}) \circ (id_b \otimes \mathbf{1}).$$

D3 $\forall a, b \in \mathcal{C}$,

$$\rho \;=\; (\circ_{aab}) \circ (\mathbf{1} \otimes id_a).$$

It is also said that $\mathcal{C}$ is an *$\mathcal{M}$-enriched category*.

It is well-known that a meet semilattice L (a poset closed under finite meets), taken as a preordered category, is a (strict) monoidal category in which $\otimes$ is the binary meet, I is the top element $\top$, and the associator α and the unitors λ, ρ are all identities. Further, it can be shown:

Proposition 1. *A set X, replacing $\mathcal{C}$ above, is an L-enriched category if and only if there is an* equality relation *E on X such that:*

E1 $E : X \times X \to L$ *is a mapping (*degrees of correspondence*).*
E2 $\forall x \in X, E(x,x) = \top$ *(*total existence*).*
E3 $\forall x,y,z \in X, E(x,y) \wedge E(y,z) \le E(x,z)$ *(*transitivity*).*

It should be noted that each (C_i) corresponds precisely to each (E_i).

The consequent of the proposition is taken as the definition of (X,E) as an *L-enriched set*.

If (E2) were to be replaced by a symmetry condition ($\forall x,y \in X, E(x,y) = E(y,x)$), then the Fourman-Scott definition [3] of an *L-valued set* would result as cited by Höhle in [6].

For L-enriched set (X,E), $E(x,y)$ is interpreted as the degree to which bitstream x corresponds to bitstream y.

Finally, with an eye to variable-basis notions later, (X,E,L) is called an *enriched set*.

Example 1. Examples of enriched sets include the following:

1. Let X be a set and L be a meet semilattice L with $|L| \ge 2$. Choose $a \in L - \{\top\}$ and put $E : X \times X \to L$ by
$$E(x,y) = \begin{cases} a, & x \ne y \\ \top, & x = y \end{cases}.$$
Then (X,E,L) is an enriched set.
2. Let (X,d) be an ultrametric space bounded by 1, and put $E : X \times X \to L$ by
$$E(x,y) = 1 - d(x,y).$$
Then (X,E,L) is an enriched set.

Given $\mathcal{M}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$, then $F : \mathcal{C} \to \mathcal{D}$ is an *$\mathcal{M}$-enriched functor* [7] if the following hold:

F1 $\forall a \in \mathcal{C}, \exists! F(a) \in \mathcal{D}$.
F2 $\forall a,b \in \mathcal{C}, \exists! F_{ab} \in \mathcal{M}(\mathcal{C}(a,b), \mathcal{D}(F(a),F(b)))$.
F3 $\forall a \in \mathcal{C}, F_{aa} \circ id_a = id_{F(a)}$ (in $\mathcal{M}$).
F4 $\forall a,b,c \in \mathcal{C}$, in $\mathcal{M}$ it is the case that
$$F_{ac} \circ (\circ_{abc}) \;=\; \left(\circ_{F(a)F(b)F(c)}\right) \circ (F_{bc} \otimes F_{ab}).$$

Proposition 2. *Given L-enriched sets (X,E) and (Y,F), where L is a meet semilattice, it is the case that $f : (X,E) \to (Y,F)$ is an L-enriched functor if and only $f : X \to Y$ is a mapping such that $\forall x,y \in X$,*
$$E(x,y) \le F(f(x), f(y)).$$

The variable-basis extension of L-enriched functors makes use of monoidal functors as defined in [8]. Let $\mathcal{M},\mathcal{N}$ be monoidal categories and let $\mathcal{C}$ be an $\mathcal{M}$-enriched category and $\mathcal{D}$ be an $\mathcal{N}$-enriched category. Then $(F,\Psi):\mathcal{C}\to\mathcal{D}$ is a(n) (*variable-basis*) *enriched functor* if the following hold:

V0 $\Psi^{op}:\mathcal{M}\leftarrow\mathcal{N}$ is a monoidal functor as defined in [8].
V1 $\forall a\in\mathcal{C},\exists!F(a)\in\mathcal{D}$.
V2 $\forall a,b\in\mathcal{C},\exists!F_{ab}\in\mathcal{M}(\mathcal{C}(a,b),\Psi^{op}[\mathcal{D}(F(a),F(b))])$.
V3 $\forall a\in\mathcal{C},F_{aa}\circ id_a=\Psi^{op}\left(id_{F(a)}\right)$ (in $\mathcal{M}$).
V4 $\forall a,b,c\in\mathcal{C}$, in $\mathcal{M}$ it is the case that

$$F_{ac}\circ(\circ_{abc}) \;=\; \Psi^{op}\left(\circ_{F(a)F(b)F(c)}\right)\circ(F_{bc}\otimes F_{ab}).$$

The backward direction of, and notation for, Ψ^{op} are both motivated by topological systems and variable-basis topology and, in particular, enriched topological systems taken up below.

Proposition 3. *Given enriched sets (X,E,L) and (Y,F,M), where L,M are meet semilattices, it is the case that $(f,\psi):(X,E,L)\to(Y,F,M)$ is an enriched functor if and only $f:X\to Y$ is a mapping and $\psi^{op}:L\leftarrow M$ is a meet-semilattice morphism such that $\forall x,y\in X$,*

$$E(x,y)\le\psi^{op}[F(f(x),f(y))].$$

The proposition justifies the following definition:

Definition 1. *The category* **EnrSet** *comprises enriched sets (X,E,L) as objects and enriched functors (f,ψ) as morphisms; and in this setting, the latter are called* enriched mappings*. The full subcategory in which each L is a frame and each ψ is a localic morphism is denoted* $\mathbf{EnrSet_{Frm}}$.

It is straightforward that **EnrSet** and $\mathbf{EnrSet_{Frm}}$ are categories using the compositions and identities of **Set** and $\mathbf{SLat}(\wedge)$, the latter denoting the category of (finite) meet semilattices and (finite) meet preserving mappings.

Enriched topological systems, namely topological systems based upon enriched sets, can now be defined.

Definition 2. **EnrTopSys** *has ground category* $\mathbf{EnrSet_{Frm}}\times\mathbf{Loc}$ *and comprises the following data satisfying the following axioms:*

1. ***Objects:*** $((X,L,E),A,\vDash)$, *called* enriched topological systems.
   - (a) (X,L,E) *is an enriched set, A is a locale (*ground conditions*).*
   - (b) $\vDash$ *is an L-satisfaction relation on (X,A), i.e., $\vDash$ satisfies both arbitrary $\bigvee$ and finite $\wedge$ interchange laws (*topological system consition*).*
   - (c) *E and $\vDash$ are compatible, i.e., $\forall x,y\in X,\forall a\in A$, $E(x,y)\wedge\vDash(y,a)\;\le\;\vDash(x,a)$ (*compatibility condition*).*
2. ***Morphisms:*** $(f,\psi,\varphi):((X,L,E),A,\vDash)\to((Y,M,F),B,\vDash)$, *called* enriched continuous functions.

(a) $(f,\psi):(X,E,L)\to(Y,F,M)$ *is an enriched mapping,* $\varphi:A\to B$ *is a localic morphism (*ground conditions*).*

(b) $\forall x\in X,\forall b\in B,\models(x,\varphi^{op}(b))\le\psi^{op}(\models(f(x),b))$ *(*partial adjointness*).*

3. ***Composition and identities****: those of the ground* $\mathbf{EnrSet_{Frm}}\times\mathbf{Loc}$.

Both enriched topological systems and enriched continuous functions are in plentiful supply, with a number of example classes at hand, including the following example class.

Example 2. Each enriched set (X,L,E) with L a frame generates an enriched topological system. Given (X,L,E), put

$$\tau=\left\{u\in L^X:\forall x,y\in X,E(x,y)\wedge u(y)\le u(x)\right\}.$$

1. $\forall y\in X,E_y:X\to L$ by $E_y(x)=E(x,y)$. It follows that $\{E_y:y\in X\}\subset\tau$. It is important to note that the proof makes explicit use of the transitivity condition (E3) above.
2. The collection τ contains all constant L-subsets of X.
3. It follows from the infinite distributive law of L that τ is an L-topology on X and hence a stratified L-topology on X.

Since L is a frame, τ is a locale. Now put $\models\,:X\times\tau\to L$ by

$$\models(x,u)=u(x).$$

It can be checked that $\models$ satisfies the arbitrary $\bigvee$ and finite $\wedge$ interchange laws and that E and $\models$ are compatible. Hence $((X,E,L),\tau,\models)$ is an enriched topological system.

Returning to the definition of an enriched topological system, certain comments should be made. First, the compatibility condition addresses the question posed at the beginning of this abstract. Second, it should be noted that partial adjointness is a significant weakening of the adjointness condition of Vickers [11] and the associated systems literature, but it should also be noted that the inequality retained above from Vickers' adjointness is a natural and important one from the standpoint of programming. These considerations motivate weakening the adjointness condition for the morphisms of the important category **Loc-TopSys** [2, 9, 10] to partial adjointness as formally stated in the above definition, thereby forming the category **Loc-TopSys**$(\le)$.

Theorem 1. **EnrTopSys** *maps functorially into* **Loc-TopSys**$(\le)$.

This theorem (with its proof) indicates that with respect to objects, traditional topological systems in the sense of **Loc-TopSys** already accommodate enriched topological systems; but with respect to morphisms, **Loc-TopSys** must be generalized to **Loc-TopSys**$(\le)$ to accommodate enriched continuous functions between enriched topological systems.

Finally, enriched topological systems afford new links to lattice-valued topology and L-topological spaces in particular. For example, let $((X,L,E),A,\models)$ be an enriched topological system. In addition to the already known frame map $ext_L : A \to L^X$ and the attendant L-topological space $(X, ext_L^{\to}(A))$, there is the frame map

$$ext_{(E,L)} : A \to L^{X \times X} \quad \text{by} \quad ext_{(E,L)}(a)(x,y) = E(x,y) \wedge \models (y,a)$$

as well as, for fixed $y \in Y$, the frame map

$$ext_{(E,L,y)} : A \to L^X \quad \text{by} \quad ext_{(E,L,y)}(a)(x) = E(x,y) \wedge \models (y,a).$$

Theorem 2. *Let $((X,L,E),A,\models)$ be an enriched topological system. The following hold:*

1. *$\forall y \in Y$, $ext_{(E,L,y)}^{\to}(A) \prec ext_L^{\to}(A)$, i.e., the former L-topology is a refinement of the latter L-topology with respect to the ordering of L^X.*
2. *Within $L^{(L^X)}$, it is the case that*
$$ext_L^{\to}(A) \subset \bigvee_{y \in X} ext_{(E,L,y)}^{\to}(A) \equiv \left\langle \left\langle \bigcup_{y \in X} ext_{(E,L,y)}^{\to}(A) \right\rangle \right\rangle.$$
3. *$(X, ext_L^{\to}(A))$ L-homeomorphically embeds into $\left(X \times X, ext_{(E,L)}^{\to}(A)\right)$, namely the former is L-homeomorphic to the subspace $\left(\Delta(X \times X), \left[ext_{(E,L)}^{\to}(A)\right]_{|\Delta(X \times X)}\right)$.*

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Ontology $<$ Logic or Ontology $=$ Logic ?

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An equivalent formulation of the title question is "*Are terms important?*" Yet another isomorphic formulation is "*Is concept (of) a type?*"

Quantales are nice. Rosenthal [19] defined a (unital) quantale $(Q, \bullet, 1, \vee)$ to be a monoid $(Q, \bullet, 1)$ and a complete semilattice $(Q, \vee)$, so that $\bullet$ distributes over $\vee$. This abstract was partly inspired by the LINZ2012 call for papers text "quantales and its applications to theoretical computer science", yet, this is not an abstract about "quantales and its applications to theoretical computer science". What is this abstract then about? It is about logic, it is about fuzzy and uncertainty representation, but in particular it is also, but not only, and very much in particular not only, about truth values.

We are bold enough to say, that this abstract is not given justice, until the reader is eventually at a saturated understanding about the main claim of this abstract (not saying the reader has to agree with the authors on the claim), in fact being one important main claims of our work during the past decade, ever since the underlying ideas behind compositions involving the term monad [4] was presented at LINZ2000.

This main claim is stated in the following

Theorem 1. *Yes, terms are important!*

Initially we want to say something informally also about the other questions. It will be clear that

Proposition 1. *"Ontology $<$ Logic" iff "Concept is a type", and "Ontology = Logic" iff "Concept is of a type".*

Corollary 1. *"Being of a type" and "Being a type" is mutually exclusive.*

1 Logic and fuzzy logic

Logic is not only computing with truth values. For propositional calculus, yes, but as soon as we involve sentences with content as provided by terms, in turn building upon an underlying signature, logic computation involves much more than mere manipulation of truth values.

Fuzzy logic is in a simple view extensions of whatever is crisp. Traditionally, fuzzy logic is extending crisp truth values to fuzzy truth values. Most of the fuzzy logic literature indeed does not go beyond fuzzification of anything else but truth values. Moreover, approach like Hajek's BL [14] do go on into predicates, but terms inside predicates are left as crisp objects so that e.g. substitution still is the very traditional and crisp one.

The situation 'Ontology $<$ Logic' appears typically in description logic, which assumes concepts to be atomic, i.e. description logic appears more like a propositional calculus than a predicate calculus. In fact, the underlying assumption seems very much like having one single type `concept`, and having often a huge number of atomic concepts, like e.g. seen in the medical vocabulary SNOMED CT, that like OWL/RDF has adopted EL++ as a variant of description logic for its ontological purposes. The simplicity of description logic is certainly intentional, as the motivation of using such a 'partial logic' is given the need to capture vocabulary, terminology and thesauri more than explicitely reasoning with these concepts and structures.

However, were we to become interested in fuzzy ontology there is a risk that fuzzy ontology in this narrow sense takes routes that even moves away from logical thinking. Such fuzzy ontologies may later appear in fuzzy reasoning, and then it is not clear that fuzzy approaches in fuzzyfying ontologies correlates with fuzzification of the logical machineries.

This calls for using terms, and indeed assigning an important role to terms and their semantics. Clearly, we also strongly speak in favour of terms in the wider sense, in particular concerning uncertainty modelling of terms and and not just involving terms.

2 Terms in the wider sense

Terms are not interesting as such. Terms are interesting as part of sentences, and not to forget, terms are interesting as part of other terms, the latter interest obviously leading to *substitution*.

Terms are defined by a corresponding term monad, means that substitutions are morphisms in the Kleisli category related to that particular term monad.

In [5, 6] we pointed a number of paradigms capturing different ways of modelling uncertainty in these respects. These paradigms make a clear distinction between 'operating with fuzzy' and 'fuzzy operation'. The underlying term monad for the former is the composition of the fuzzy powerset monad with the traditional term monad, and doing all this over `Set`. The underlying term monad for the latter builds upon an endofunctor over $\mathtt{Set}(Q)$, where in principle Q could be a quantale, or could be something else, yet appropriate. This gives us the basis for the "fully fuzzy" situation which has it's starting point in considerations for terms and substituting with terms. Note that truth values of sentences have not yet entered the scene at all. Notably, one might even allow oneself to have a crisp logic with "fully fuzzy" terms. In fact, in real life applications, this is indeed what happens mostly, i.e. observations and assessments of data and information are fuzzy, but decision-making, like in health care for interventions, must in the end be crisp.

It should also be remarked that a shift from one-sorted to many-sorted is far from trivial, even if folklore literature claims otherwise. Algebraic considerations need also be precisely handled, as pointed out in [6].

Such terms then as included in sentences provides leads again to question and notions about fuzzy sentences, and so on and so forth. The entire logic machinery all the way down to inference calculi can be nicely described e.g. in the framework of Meseguer's general logic [16]. Moreover, general logic can be further generalized from the viewpoint of Theorem 1, namely, that a substitution oriented generalized general logic indeed is more than feasible, not to say very desirable [7, 8].

3 Type theory

Whereas for terms, informal definitions of the term set mostly correspond to the formal definitions of terms, so that ambiguities are avoided. Concerning λ-terms, the situation with informal definitions about what is and isn't λ-terms is less obvious, in particular in the typed case. In [8] we make this situation explicit by considering levels of signatures, i.e. being very observant about where particular types and related operators reside especially before and after λ-abstraction. Type constructors also need to be handled formally, and their respective algebras must be identified with utmost care.

In this abstract we will not provide detail. However, we may say that starting from a usual signature $\Sigma = (S, \Omega)$, identifying the underlying primitive operations, we have the term monad $\mathbf{T}_\Sigma$, over $\mathtt{Set}$, or fully fuzzy over $\mathtt{Set}(Q)$. This situation is *signatures, terms and algebras at level one*.

Then we may create a new signature $\mathtt{S}_\Sigma = (\{\mathtt{type}\}, \Omega')$, on *signature level two*, with $\mathtt{type}$ as the only sort, and operators in Ω' to be understood as type constructors. Interesting on level two is the algebra of $\mathtt{type}$, namely, $\mathfrak{A}(\mathtt{type})$ is the underlying category of your choice.

Now we can make $\mathsf{T}_{\mathtt{S}_\Sigma}\varnothing$ the sort set for *signature level three*, and the interesting part is defining some operators into this signature.

In this separation of levels it is very transparent how e.g. operators at level one are shifted over to level three. The most important observation at this stage is that λ is not a 'term transformer' but an 'operator mover' between level one and level three.

All this notions can be made precise, and we are able to show e.g. how problems with variable renaming can be avoided. This is fully developed in [8].

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Closure operators on modules over quantaloids: applications to algebraic logic

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Motivated by the study of the sentential logics and π-institutions, we introduce the notion of closure operator on modules over quantaloids, and always driven by the search of the solution of the Isomorphism Problem, as we explain below, we introduce the notions of interpretability and representability between closure operators. This yields a very rich theory with many nice properties: We prove that there exists a duality in the categories of modules over quantaloids, that they are strongly complete and strongly cocomplete, that they are (Epi, Mono)-structured regular categories, that they have enough injectives and projectives, and that they satisfy the strong amalgamation property, among others. Some of these results are generalizations of the same results obtained by Solovyov for categories of modules over quantales (see [10]). We characterize monos and epis in the categories of modules over quantaloids, and furthermore prove that every epi is induced by a closure operator on its domain.

We also study the notions of closure system on a module over a quantaloid, and prove that they are exactly the submodules of the dual module, and that the standard correspondence between closure operators and closure systems on a set extends to a natural isomorphism. We prove that the set of closure operators that are interpretable by a given morphism τ is a principal filter of the lattice of closure operators on its domain. As a consequence, we obtain that every extension of an interpretable closure operator is also interpretable by the same morphism. One instantiation of this result is the well-known fact (see Theorem 2.15 of [4]) that if a sentential logic has an algebraic semantics, then every extension of it also has an algebraic semantics and with the same defining equations.

The Problem of the Isomorphism has its origin in the work of Blok and Jónsson, who in order to study the property of algebraizability for sentential logics, and the equivalence between deductive systems in general, introduced the notion of *equivalence* between *structural closure operators* on a set X acted on by a monoid M, or an *M-set* (see [1]). As usual, given a monoid $(M,\cdot,1)$, an M-set consists of a set X and a monoid action $\star : M \times X \to X$, where $1 \star x = x$ and $a \star (b \star x) = (a \cdot b) \star x$, for all $a, b \in M$ and $x \in X$. While the use of closure operators to encode entailment relations is very well known, the action of the monoid is introduced to formalize the notion of structurality, that is, “entailments are preserved by uniform substitutions,” a property usually required for logics.

Given an M-set $\langle X, \cdot \rangle$, a closure operator C on X is *structural* on $\langle X, \cdot \rangle$ if and only if it satisfies the following property: for every $\sigma \in M$, and every $\Gamma \subseteq X$, $\sigma \cdot C\Gamma \subseteq C(\sigma \cdot \Gamma)$, where $\sigma \cdot \Gamma = \{\sigma \cdot \varphi : \varphi \in \Gamma\}$. This can be shortly written as follows:

$$\forall \sigma \in M, \quad \sigma C \leqslant C\sigma. \tag{Str}$$

This is known as the *structurality property* for C, since it takes the following form, when expressed in terms of $\vdash_C$, the closure relation on X associated with the closure operator C (defined by $\varphi \in C\Gamma$ iff $\Gamma \vdash_C \varphi$): for every $\Gamma \subseteq X$, every $\varphi \in X$, and every $\sigma \in M$,

$$\Gamma \vdash_C \varphi \Rightarrow \sigma \cdot \Gamma \vdash_C \sigma \cdot \varphi.$$

For every $\sigma \in M$, a unary operation $C\sigma$ on $\mathbf{Cl}(C) = \langle \mathrm{Cl}(C), \subseteq \rangle$, the lattice of *theories* or *closed sets* of C, is defined in the following way: $C\sigma(\Gamma) = C(\sigma \cdot \Gamma)$. The *expanded lattice of theories* of a structural closure operator C is defined as the structure $\langle \mathbf{Cl}(C), (C\sigma)_{\sigma \in M} \rangle$.

In their approximation, Blok and Jónsson define two structural closure operators on two M-sets to be equivalent if their expanded lattices of theories are isomorphic. Later, they prove that under certain hypotheses (the existence of basis), this is equivalent to the existence of conservative and mutually inverse interpretations, which is the original idea of equivalence between deductive systems emerging from the work of Blok and Pigozzi. This equivalence between the lattice-theoretic property of having isomorphic expanded lattices of theories, and the semantic property of being mutually interpretable is known by the name of the *Isomorphism Theorem*. And the problem of determining in which situations there exists an Isomorphism Theorem is called the *Isomorphism Problem*.

The first Isomorphism Theorem was proved by Blok and Pigozzi in [2] for algebraizable sentential logics, and later it was obtained for k-dimensional deductive systems by them in [3] and for Gentzen systems by Rebagliato and Verd in [9]. But there is not a general Isomorphism Theorem for structural closure operators on M-sets, as there are counterexamples for that (see [8]).

In turn, Voutsadakis studied in [11] the notion of equivalence of π-institutions at different levels (quasi-equivalence and deductive equivalence) and identified term π-institutions, for which a certain kind of Isomorphism Theorem also holds. The notion of π-institution was introduced by Fiadeiro and Sernadas in their article [5] and can be viewed as a generalization of deductive systems allowing multiple sorts. They constitute a very wide categorical framework embracing sentential logics, Gentzen systems, etc., as they include structural closure operators on M-sets as a particular case. Therefore, a general Isomorphism Theorem for π-institutions is not possible (see [7]).

Sufficient conditions for the existence of an Isomorphism Theorem were provided in [8] and [7] for structural closure operators on M-sets (and graduated M-sets), and π-institutions that encompass all the previous known cases. The first complete solution of the Isomorphism Problem was found for closure operators on modules over residuated complete lattices, or *quantales* (see [6]). In this article, the modules providing an Isomorphism Theorem are identified as the projective modules. In particular, cyclic projective modules are characterized in several ways, from which the Isomorphism Theorem for k-deductive systems follows, and also for Gentzen systems, using that coproducts of projectives are projective. The Isomorphism Problem for π-institutions remained open.

One of our main results, as an application of the theory of closure operators on modules over quantaloids to Algebraic Logic, is the following:

Theorem 1. *If $\mathcal{Q}$ is a quantaloid, then a $\mathcal{Q}$-module P is projective if and only if every representation of a closure operator on P into another closure operator is induced.*

This is the key result to establish that every equivalence between two closure operators on projective modules is induced by mutually inverse interpretations. That is the general solution for the Isomorphism Problem in the setting of modules over quantaloids.

We also explain how every π-institution induces a closure operator on a module over a quantaloid, and every translation between π-institutions induces a morphism in the fibered category of all modules over quantaloids. Thus, we show how the theory of closure operators on modules over quantaloids is a generalization of the theory of interpretations and representations of π-institutions.

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Categories of fuzzy sets and relations

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We define a category whose objects are fuzzy sets and whose maps are relations subject to certain natural conditions. We enrich this category with additional structure coming from t-norms and negations on the unit interval. We develop the basic properties of this category and consider its relation to other familiar categories.

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Sheaves on involutive quantales: Grothendieck quantales

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1 Introduction

It is well known that sheaves on a topological space (X, O) give rise to a category called a Grothendieck topos [7], which can be seen as a constructive universe of sets. The local sections of a sheaf F exist locally at opens of X, such that the subobjects of F form a complete Heyting algebra (or frame), not necessarily a Boolean algebra. This sketches a rough idea of the link between logic and geometry, which is so fruitfully exploited in topos theory.

Another important field is non-commutative geometry [2], in which geometry is dealt with implicitly through the study of non-commutative algebras, like C^*-algebras. Attempts to make the hidden non-commutative topology more explicit have led to several formalisms, including the theory of (involutive) quantales [8, 11]. Frames, like the lattice of opens of a topological space, are commutative idempotent quantales (with a trivial involution). It is not a surprise that people started thinking about sheaves on quantales.

This idea sounds very natural, but there is a certain risk involved: are quantales really good candidates for non-commutative topology and can we find a definition of sheaves on a quantale that encapsulates C^*-algebras? Unfortunately, this is still a matter of discussion, after almost thirty years of research.

Although older definitions of sheaves on quantales (e.g., [9]) may diverge, more recent versions are based on the observation that sheaves on a locale (frame) O can be presented in the form idempotent symmetric matrices with values in O [3]. The indices of the matrix represent the local sections and the values of the matrix give the regions in which pairs of local sections agree. By replacing the frame by an involutive quantale Q, we obtain Q-valued sets. Many more references can be found in the recent paper of Resende [10].

2 Enrichment over involutive quantaloids

The matrix approach is elegant, but problems emerge when one tries to conceptualize the sheafification of Q-valued sets. By considering Q-valued sets as enriched categories [1], we obtain more insight in these matters. They resemble metric spaces, which can be considered as categories enriched over the quantale of positive real numbers (extended with infinity). Some caution is in order: Q-valued sets are not categories enriched over Q, but rather over an involutive quantaloid Q_E, obtained by splitting a certain class E of idempotents of Q. Alternatively phrased, Q-valued sets are rather *reflexive*, transitive and symmetric matrices with values in Q_E (i.e., symmetric monads or equivalence relations). Having settled this, the sheafification of Q-valued sets may be defined as the Cauchy completion of Q_E-categories ([14] is an early example). Many elements of enriched category theory contribute to sheaf theory (distributors [12], limits, etc.). On the other hand, sheaves on an involutive quantale Q can be cast in the form of modules over Q [13, 6, 5]. The more lattice theoretic oriented module theory has several advantages.

3 Grothendieck quantales

The sheaves on a locale give a localic Grothendieck topos. What about non-localic Grothendieck toposes? We will show that every Grothendieck topos can be seen as the category (allegory [4]) of sheaves on what we call a *Grothendieck quantale*. A plausible definition of a Grothendieck quantale might be: an involutive quantale such that the category of sheaves on it is a topos (this definition is slightly simplified). The main result of the talk is a simple axiomatization of Grothendieck quantales [5]. If there is time left, I would like to address some of the questions raised in the introduction.

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Topology based on premultiplicative quantaloids: a common basis for many-valued and non-commutative topology

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Abstract. The purpose of this talk is to explain that topological spaces can be formulated in any framework of premultiplicative quantaloid. In particular, the following results are obtained in a cooperation with Tomasz Kubiak (Poznań, Poland) during summer 2011.

Let $\mathcal{Q}$ be a quantaloid ([2]). First we recall the cocompletion of $\mathcal{Q}$-enriched categories (so-called $\mathcal{Q}$-categories (cf. [3])) and specify the power $\mathcal{Q}$-category monad $\mathbf{T}_{\mathbb{P}}$ which is hidden behind the concept of cocompletion. Then we have the following theorems.

Theorem 1. *Let* $\mathbb{P}(\mathbb{X}) \xrightarrow{\xi} \mathbb{X}$ *be a* $\mathcal{Q}$*-functor. Then the following assertions are equivalent:*

(i) $(\mathbb{X},\xi)$ *satisfies the first algebra axiom — i.e.* $\xi \cdot \eta_{\mathbb{X}} = 1_{\mathbb{X}}$.
(ii) $(\mathbb{X},\xi)$ *is a* $\mathbf{T}_{\mathbb{P}}$*-algebra.*

Theorem 2. *Let* $\mathbb{X}$ *be a skeletal* $\mathcal{Q}$*-category and* $\mathbb{P}(\mathbb{X})$ *be the power* $\mathcal{Q}$*-category. Then the following assertions are equivalent:*

(i) $\mathbb{X}$ *is cocomplete.*
(ii) *There exists a* $\mathcal{Q}$*-functor* $\mathbb{P}(\mathbb{X}) \xrightarrow{\xi} \mathbb{X}$ *satisfying the first algebra axiom w.r.t. the power* $\mathcal{Q}$*-category monad.*

After these preparations we introduce the concept of premultiplicative quantaloids.

Definition 1. *A quantaloid* $\mathcal{Q}$ *is called* premultiplicative *if every hom-set* $\mathcal{Q}(a,b)$ *has an binary operation* $\odot$ *satisfying the following conditions:*

(pm1) $\odot$ *is distributive over* non empty *joins in both variables,*
(pm2) $\odot$ *is* subdistributive *over the* composition *in both variables — i.e. for all* $a,b,c \in \mathrm{obj}(\mathcal{Q})$ *and* $\alpha,\beta \in \mathcal{Q}(a,b)$ *the subsequent relations are valid:*

$$\gamma \cdot (\alpha \odot \beta) \ \leq\ (\gamma \cdot \alpha) \odot (\gamma \cdot \beta), \qquad \gamma \in \mathcal{Q}(b,c)$$
$$(\alpha \odot \beta) \cdot \gamma \ \leq\ (\alpha \cdot \gamma) \odot (\beta \cdot \gamma), \qquad \gamma \in \mathcal{Q}(c,a).$$

In this context $\odot$ *is a called a* premultiplication. $\square$

Example 1. Let $[0,1]$ be the real unit interval equipped with the usual ordering and with Łukasiewicz' arithmetic conjunction $*$ — i.e.

$$\alpha * \beta \;=\; \max(\alpha+\beta-1,0), \quad \alpha,\beta \in [0,1].$$

Obviously, $([0,1],*)$ is a unital quantale. Further, let Q be the quantaloid with one object determined by $([0,1],*)$. Then Q is a premultiplicative quantaloid w.r.t. the *binary minimum* as well as w.r.t. the *binary arithmetic mean*. $\square$

Example 2. Let $(L,')$ be a complete De Morgan algebra — this means that L is a complete (not necessarily distributive) lattice provided with an order reversing involution $'$. In particular, the universal upper (resp. lower) bound in L is denoted by $\top$ (resp. $\bot$). Then we construct a quantaloid Q as follows. The set of objects of Q is given by L enlarged by a further element ω — i.e.

$$\mathrm{obj}(Q) \;=\; L \cup \{\omega\}.$$

The hom-sets of Q with their respective partial orderings are given by:

- $Q(a,a)$ is the two-point lattice for all $a \in L \cup \{\omega\}$.
- $Q(a,b)$ is a singleton, if $a,b \in L$ with $a \neq b$.
- $Q(\omega,b) \;=\; \{\lambda \in L \mid \lambda \leq b\}$ with the ordering from L, if $\omega \neq b$.
- $Q(a,\omega) = \{\lambda \in L \mid a' \leq \lambda\}$ with the ordering from L^{op}, if $a \neq \omega$.

Then there exists a unique composition law satisfying the following properties:

- The composition preserves arbitrary joins in each variable separately.
- On $Q(a,a)$ the composition is the meet of the two-point lattice.
- If $a \neq b$ and $b \neq c$, then the composition attaches the universal lower bound of $Q(a,c)$ to all $(\lambda_1,\lambda_2) \in Q(a,b) \times Q(b,c)$.

Finally, the multiplicative structure on Q is determined as follows: On $Q(a,a)$ we use again the meet, while on hom-sets consisting of a unique morphism the binary operation is evident. In order to complete the situation we have only to define binary operations on $Q(\omega,b)$ and $Q(a,\omega)$ with $a,b \in L$:

$$\lambda_1 \odot_{\omega}^{b} \lambda_2 \;=\; \begin{cases} \lambda_1, & \lambda_2 \neq \bot, \\ \bot, & \lambda_2 = \bot. \end{cases} \qquad \lambda_1 \odot_{a}^{\omega} \lambda_2 \;=\; \begin{cases} \lambda_2, & \lambda_1 \neq \top, \\ \top, & \lambda_2 = \top. \end{cases}$$

All this shows that Q is a premultiplicative quantaloid. $\square$

Let Q be a premultiplicative quantaloid with the local premultiplication $\odot$ and $\mathbf{Cat}(Q$ be the category of Q-categories and Q-functors. We fix a Q-category $\mathbb{X} = (X,e,d)$ and consider the Q-functor $\boxdot : \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{X}) \to \mathbb{P}(\mathbb{X})$ induced by $\odot$. Further, let $\mathbf{1} \xrightarrow{\top_{\mathbb{X}}} \mathbb{P}(\mathbb{X})$ be a Q-functor defined by:

$$\top_{\mathbb{X}}(a) \;=\; (a, f_a^{\top}), \quad f_a^{\top}(x) \;=\; \bigvee Q(a,e(x)), \quad x \in X, \qquad a \in \mathrm{obj}(Q).$$

An extremal subobject $\mathbb{U} \xrightarrow{\iota} \mathbb{P}(\mathbb{X})$ of $\mathbb{P}(\mathbb{X})$ is called a *topology* on $\mathbb{X}$ iff ι satisfies the following axioms:

(T1) $\top_{\mathbb{X}}$ factors through ι.
(T2) $\boxdot \cdot (\iota \times \iota)$ factors through ι.
(T3) $\mu_{\mathbb{X}} \cdot \mathbb{P}(\iota)$ factors through ι.

The axiom (T1) means that 'the whole space is open. (T2) is the intersection axiom and (T3) means that ι is closed under internal joins — i.e. ι is cocontinuous.

If $\mathbb{X}$ is provided with a topology ι, then $(\mathbb{X}, \iota)$ is called a *topological space in the sense of the quantaloid $\mathcal{Q}$.*

Topological spaces in the sense of $\mathcal{Q}$ form a category which is topological over **Cat**$(\mathcal{Q})$.

In the case of Example 1 topological spaces are many valued topological spaces (cf. [1]), while in the case of Example 2 we obtain non-commutative topological spaces provided the underlying De Morgan algebra is given by the lattice of all closed subspaces of an arbitrary Hilbert space.

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Two new classification theorems on residuated monoids

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Abstract. We present two very recent Mostert-Shields style classification theorems on residuated *l*-monoids along with some related results in substructural logics.

1 Introduction

Residuated lattices have been introduced in the 30s of the last century by Ward and Dilworth [30] to investigate ideal theory of commutative rings with unit. Examples of residuated lattices include Boolean algebras, Heyting algebras [6], MV-algebras [3], basic logic algebras, [8] and lattice-ordered groups; a variety of other algebraic structures can be rendered as residuated lattices. The topic did not become a leading trend on its own right back then. Nowadays the investigation of residuated lattices (that is, residuated monoids on lattices) has got a new impetus and has been staying in the focus of strong international attention. The reason is that residuated lattices turned out to be algebraic counterparts of substructural logics [27, 26]. Applications of substructural logics and residuated lattices span across proof theory, algebra, and computer science. An extensive monograph discussing residuated lattices went to print in 2007 [7]. Substructural logics encompass among many others, classical logic, intuitionistic logic, relevance logics, many-valued logics, t-norm-based logics, linear logic and their non-commutative versions. These logics had different motivations, different methodology, and have mainly been investigated by isolated groups of researchers. The theory of substructural logics has put all these logics, along with many others, under the same motivational and methodological umbrella. Residuated lattices themselves have been the key component in this remarkable unification.

Residuated lattices on the real unit interval $[0,1]$ are of particular interest. On $[0,1]$, FL_e-monoids (see Definition 1) are referred to as uninorms, integral FL_e-monoids are referred to as t-norms. Because they are residuated, those uninorms and t-norms $*$ are left-continuous, as two-place functions. The residuum $\to$ is given by $x \to y = \sup\{z : z * x \leq y\}$. They determine both a substructural logic (obtained by interpreting conjunction as $*$ and implication as $\to$) and a variety of commutative, integral and bounded residuated lattices, see [7]. It follows that left-continuous uninorms originate a substructural logic, which may lack not only contraction, but also weakening.

Both for left-continuous t-norms and for left-continuous uninorms, those with an involutive negation are of special interest. (Note that for t-norms negation is defined by $\neg x = x \to 0$, while for uninorms negation is defined as $\neg x = x \to f$, where f is a fixed, but arbitrary element of $[0,1]$, and stands for falsum just like 0 does in case of a t-norm). Involutive t-norms and uninorms have very interesting symmetry properties [11, 14, 10, 24] and, as a consequence, for involutive t-norms and uninorms we have beautiful geometric constructions which are lacking for general left-continuous t-norms and uninorms [12, 20, 23]. Furthermore, not only involutive t-norms and uninorms have very interesting symmetry properties, but their logical calculi have important symmetry properties too: Both sides of a sequent may contain more than one formula, while (hyper)sequent calculi for their non-involutive counterparts admit at most one formula on the right.

A particularly interesting question is whether the variety of algebras of a certain logic are generated by only the algebras on $[0,1]$ which are called *standard algebras*. If the answer is yes, we say that the logic in question admits standard completeness. For the logics **BL** and **MTL** this problem has been solved in [2] and [17], respectively.

As for the *classification problem* of residuated lattices, this task seems to be possible only by posing additional conditions. The first result in this direction is due to Mostert and Shields who investigated certain topological semigroups on compact manifolds with connected, regular boundary in [28]. Being topological means that the monoid operation of the residuated lattice is continuous with respect to the underlying topology. They proved that such semigroups are ordinal sums in the sense of Clifford [4] of product, Boolean, and Łukasieticz summands.

Next, the dropping of the topologically connected property of the underlying chain can successfully be compensated by assuming the divisibility condition (which is, in fact, the dual notion of the well-known naturally ordered property). The divisibility condition is the algebraic analogue of the Intermediate Value Theorem in real analysis, and it can be considered a stronger version of continuity of the monoidal operation: Indeed, on a finite chain the order topology is the discrete one, so every operation is continuous and hence does not necessarily obey the divisibility condition. Under the assumption of divisibility, residuated chains, that is BL-chains, have been classified, again, as ordinal sums with product, Boolean, and Łukasieticz summands. The divisibility condition proved to be strong enough for the classification of residuated lattices over arbitrary lattices too [22]. Fodor has classified those uninorms which have continuous underlying t-norm and t-conorm [5]. But divisibility aside, no classification seemed to be likely to exist due to the richness of residuated structures.

In this paper a first step is made in this direction: In one of the two classification theorems of ours we do not assume divisibility nor even the slightest version of continuity.

First of all, we classify strongly involutive uninorms algebras (SIU-algebras), that is bounded, representable, sharp, involutive FL_e-monoids over arbitrary lattices for which their cone operations are dually isomorphic. Let us remark that assuming the duality condition proved to be equivalent to assuming the divisibility condition *only* for the positive and negative cones of such algebras.

Second, we classify sharp involutive FL_e-monoids on complete, order-dense, semi-separable chains. Here neither divisibility nor even the weakest form of continuity is assumed. Surprisingly, the restriction of those monoids to their negative cone is necessarily continuous with respect to the order topology of their underlying chain. The result seems only to hold under the condition $t = f$, and hence a classification for involutive FL_e-monoids is still lacking, but in any case the result is very surprising, as involutive *integral* monoids may have discontinuities even below the fixed point of their negation. While for involutive integral monoids (and even for involutive t-norms) a classification is still lacking, for sharp involutive FL_e-monoids on complete, order-dense, semi-separable chains we can present a classification. Since $[0,1]$ is a complete, order-dense, semi-separable chain, our result provides with the classification of sharp, involutive uninorms too. Remarkably, the adding of the involution condition to residuated *integral* monoids does not bring us *any closer* to the solution of the related classification problem: As revealed by the rotation construction [12], every residuated integral monoid can arise as a subsemigroup of an involutive residuated integral monoid.

Third, from the logical point of view, we want to solve some standard completeness problems. Since uninorm logics are algebraizable in the sense of Blok and Pigozzi [1], we can state the standard completeness problem in an algebraic way, recalling that valid equations correspond to theorems of the associated logic and valid quasiequations correspond to provable consequence relations. Now the question is if there is an equation (resp., a quasiequation) of sharp, involutive, representable[4] FL_e-monoids which is valid in all sharp, involutive FL_e-monoids on $[0,1]$ but not in all sharp, representable involutive FL_e-monoids? When such an equation (resp., quasiequation) does not exist, the corresponding logic is standard complete (resp., finitely strongly standard complete). In [25], it is shown that the logic of uninorm algebras is standard complete, and the problem has been left open for the logic of involutive uninorm algebras (aka. bounded, representable, sharp, involutive FL_e-algebras). We prove that the logic of sharp, involutive uninorm algebras is not standard complete and that the logic of involutive uninorm algebras is not finitely strongly standard complete. In addition, we axiomatize the logic of SIU-algebras and prove that it is finitely strongly complete with respect to the class of standard SIU-algebras, it is not strongly complete with respect to the class of all standard SIU-algebras, and that tautologicity and consequence relation in it are co-NP complete.

2 Preliminaries

As said in the introduction, uninorms are commutative, isotone monoids on $[0,1]$. On general universe, however, we shall refer to them as FL_e-monoids:

Definition 1. Call $\mathcal{U} = \langle X, \circledast, \leq, t, f \rangle$ and as well its monoidal operation $\circledast$ an *FL_e-monoid* if $\mathcal{C} = \langle X, \leq \rangle$ is a poset and $\circledast$ is a commutative, residuated monoid over $\mathcal{C}$ with neutral element t. Define the positive and the negative cone of $\mathcal{U}$ by $X^+ = \{x \in$

[4] An FL_e-monoid is representable if it is subdirect product of chains.

$X \mid x \geq t\}$ and $X^- = \{x \in X \mid x \leq t\}$, respectively. Call an FL_e-monoid $\mathcal{U}$ *involutive*, if for $x \in X$, $(x')' = x$ holds, where $x' = x \to_{\circledast} f$. Call an involutive FL_e-monoid $\mathcal{U}$ *sharp*, if $t = f$. Call a sharp, involutive FL_e-monoid a SIU-algebra, if for $x, y \in X^-$, $x' \circledast y' = (x \circledast y)'$ holds.

Standing notation: For an FL_e-monoid $\langle X, \circledast, \leq, t, f \rangle$, throughout the paper we denote the negative and the positive cone operation of $\circledast$, by $\otimes$ and $\oplus$, respectively.

Let $\mathcal{U}$ be an FL_e-monoid. The algebra $\mathcal{U}$, and as well $\circledast$ is called *conic* if every element of X is comparable with t, that is, if $X = X^+ \cup X^-$. $\mathcal{U}$ is called *finite* if X is a finite set, $\mathcal{U}$ is called *bounded* if X has top $\top$ and bottom $\bot$ element. If X is linearly ordered, we speak about FL_e*-chains*. Since $\circledast$ is residuated, it is as well partially-ordered (isotone), and therefore, $' : X \to X$ is an order-reversing involution. A partially-ordered monoid is called integral (resp. dually integral) if the underlying poset has its greatest (resp. least) element and it coincides with the neutral element of the monoid. It is not difficult to see that $\circledast$ restricted to X^- (resp. X^+) is integral (resp. dually integral).

3 Two new Mostert-Shields type classification theorems

In [20] the authors give a structural description of conic, involutive FL_e-monoids by proving that the cone operations of any involutive, conic FL_e-monoid uniquely determine the FL_e-monoid via, what is called twin rotation:

Theorem 1. [20] **(Conic Representation Theorem)** *For any conic, involutive FL_e-monoid it holds true that*

$$x \circledast y = \begin{cases} x \oplus y & \text{if } x, y \in X^+ \\ x \otimes y & \text{if } x, y \in X^- \\ (x \to_{\oplus} y')' & \text{if } x \in X^+,\, y \in X^-,\, \text{and } x \leq y' \\ (y \to_{\otimes} x')' & \text{if } x \in X^+,\, y \in X^-,\, \text{and } x \not\leq y' \\ (y \to_{\oplus} x')' & \text{if } x \in X^-,\, y \in X^+,\, \text{and } x \leq y' \\ (x \to_{\otimes} y')' & \text{if } x \in X^-,\, y \in X^+,\, \text{and } x \not\leq y' \end{cases} . \tag{1}$$

In [15] SIU-algebras on $[0,1]$ have been classified. This result has been generalized in [18], where we classify SIU-algebras over arbitrary lattices:

Theorem 2. ([18]) *$\mathcal{U} = \langle X, \circledast, \leq, t, f \rangle$ is a SIU-algebra if and only if its negative cone is a BL-algebra with components which are either product or minimum components, $\oplus$ is the dual of $\otimes$, and $\circledast$ is given by (1).*

Then, in [19] the authors can even weaken the quite usual continuity condition, which was posed for the cone operators in SIU-algebras, and classify a subclass of sharp, involutive FL_e-monoids on $[0,1]$ as follows:

Definition 2. ([19])A chain $\langle X, \leq \rangle$ is called semi-separable if there exists $Y \subset X$ such that Y is dense in X and the cardinality of Y is smaller than the cardinality of X.

Definition 3. For an involutive FL$_e$-monoid $\mathcal{U} = \langle X, \circledast, \leq, t, f \rangle$ on a complete poset let

$$Sk(x) = \begin{cases} \max\{u \in X^+ \mid u \oplus x = x\}, & \text{if } x \in X^+ \\ (\inf\{u \in X^- \mid u \otimes x = x\})', & \text{if } x \in X^- \end{cases}$$

and call it the skeleton of $\circledast$ (c.f. [21]).

Theorem 3. ([19]) *On a complete, order-dense, semi-separable chain, $\mathcal{U}$ is a sharp, involutive FL$_e$-monoid satisfying*

$$\text{for } x \in X^-, \; Sk(x)' \circledast x = x \tag{2}$$

if and only if the negative cone of $\mathcal{U}$ is a BL-chain without Łukasievicz components, its positive cone is the dual of its negative cone with respect to $'$, and $\circledast$ is given by (1).

We remark that due to the well-known Mostert–Shields classification theorem, a BL-chain without Łukasievicz components is exactly an ordinal sum of Boolean and product summands in the sense of Clifford [4].

4 Applications in Substructural Logic

4.1 The logic of SIU-algebras: axiomatization and standard completeness

Substructural fuzzy logics on a countable propositional language with formulas built inductively as usual from a set of propositional variables, binary connectives $\bigodot$, $\to$, $\wedge$, $\vee$, and constants $\bot$, $\top$, $\mathbf{f}$, $\mathbf{t}$, with defined connectives:

$$\begin{aligned} \neg A &=_{def} A \to \mathbf{f} \\ A \bigoplus B &=_{def} \neg(\neg A \bigodot \neg B) \\ A \leftrightarrow B &=_{def} (A \to B) \wedge (B \to A) \end{aligned}$$

Definition 4. **MAILL** (which is equivalent to $\mathbf{FL}_e$ with $\bot$ and $\top$) is the substructural logic consisting of the following axioms and rules:

(L1) $A \to A$
(L2) $(A \to B) \to ((B \to C) \to (A \to C))$
(L3) $(A \to (B \to C)) \to (B \to (A \to C))$
(L4) $((A \bigodot B) \to C) \leftrightarrow (A \to (B \to C))$
(L5) $(A \wedge B) \to A$
(L6) $(A \wedge B) \to B$
(L7) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(L8) $A \to (A \vee B)$
(L9) $B \to (A \vee B)$
(L10) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(L11) $A \leftrightarrow (\mathbf{t} \to A)$
(L12) $\bot \to A$
(L13) $A \to \top$

$$\frac{A \quad A \to B}{B} \; (mp) \qquad \frac{A \quad B}{A \wedge B} \; (adj)$$

Definition 5. Uninorm logic **UL** and involutive uninorm logic **IUL** are **MAILL** plus (PRL) $(A \to B) \wedge \mathbf{t}) \vee ((B \to A) \wedge \mathbf{t})$ and **UL** plus (INV) $\neg\neg A \to A$, respectively. Strongly involutive uninorm logic **SIUL** is **IUL** plus $f \to e$ and $(\phi \odot \psi)' \to (\phi' \odot \psi')$.

It turns out that **SIUL** is algebraizable in the sense of [1], and its equivalent algebraic semantics is constituted by the variety of SIU-algebras.

Theorem 4. ([18]) *(1)* **SIUL** *is finitely strongly complete with respect to the class of standard SIU-algebras. (2)* **SIUL** *is not strongly complete with respect to the class of all standard SIU-algebras. (3) Tautologicity and consequence relation in* **SIUL** *are co-NP complete.*

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Level dependent capacities and integrals

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Level dependent capacities have been proposed in [1] during the 28$^{\text{th}}$ Linz Seminar in 2007 (see also [2, 4]). An axiomatic approach to universal integral based on standard capacities was given in [3]. We discuss the axiomatization of universal integrals based on level dependent capacities.

Given a measurable space $(X,\mathcal{A})$, the set of all measurable functions from X to $[0,1]$ is denoted by $\mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$, and the set of all capacities on $(X,\mathcal{A})$ by $\mathcal{M}_1^{(X,\mathcal{A})}$. A level dependent capacity on $(X,\mathcal{A})$ is a family $(m_t)_{t\in]0,1]}$ of set functions $m_t\colon \mathcal{A}\to[0,1]$, where each m_t is a capacity on $(X,\mathcal{A})$, and for the set of all level dependent capacities on $(X,\mathcal{A})$ we write $\mathfrak{M}_1^{(X,\mathcal{A})}$. If $M_1=(m_{t,1})_{t\in]0,1]}$ and $M_2=(m_{t,2})_{t\in]0,1]}$ are two level dependent capacities then we say that M_1 is *smaller* than M_2 (in symbols $M_1\le M_2$) if $M_1,M_2\in\mathfrak{M}_1^{(X,\mathcal{A})}$ for some measurable space $(X,\mathcal{A})$, and $m_{t,1}(A)\le m_{t,2}(A)$ for all $t\in\,]0,1]$ and $A\in\mathcal{A}$. For a fixed $M\in\mathfrak{M}_1^{(X,\mathcal{A})}$, a function $f\in\mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ is called *M-$\mathcal{A}$-measurable* if the function $h_{M,f}\colon\,]0,1]\to[0,1]$ given by

$$h_{M,f}(t)=m_t(\{f\ge t\})$$

is Borel measurable. The set of all M-$\mathcal{A}$-measurable functions in $\mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ will be denoted by $\mathcal{F}_{[0,1]}^{(X,\mathcal{A},M)}$. Moreover, we put

$$\mathcal{L}_{[0,1]}=\bigcup_{(X,\mathcal{A})\in\mathcal{S}}\left(\bigcup_{M\in\mathfrak{M}_1^{(X,\mathcal{A})}}\left(M\times\mathcal{F}_{[0,1]}^{(X,\mathcal{A},M)}\right)\right),$$

where $\mathcal{S}$ is the class of all measurable spaces. Similarly, we put

$$\mathcal{D}_{[0,1]}=\bigcup_{(X,\mathcal{A})\in\mathcal{S}}\left(\mathcal{M}_1^{(X,\mathcal{A})}\times\mathcal{F}_{[0,1]}^{(X,\mathcal{A})}\right).$$

Definition 1. A function $\mathbf{L}\colon \mathcal{L}_{[0,1]} \to [0,1]$ is called a *level dependent capacity-based universal integral* if the following axioms hold:

(L1) $\mathbf{L}$ is nondecreasing in each component, i.e., for each measurable space $(X,\mathcal{A})$, for all level dependent capacities $M_1, M_2 \in \mathfrak{M}_1^{(X,\mathcal{A})}$ satisfying $M_1 \leq M_2$, and for all functions $f_1 \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A},M_1)}$, $f_2 \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A},M_2)}$ with $f_1 \leq f_2$ we have

$$\mathbf{L}(M_1, f_1) \leq \mathbf{L}(M_2, f_2),$$

(L2) there is a universal integral $\mathbf{I}\colon \mathcal{D}_{[0,1]} \to [0,1]$ such that for each measurable space $(X,\mathcal{A})$, for each capacity $m \in \mathcal{M}_1^{(X,\mathcal{A})}$, for each $f \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A},M)}$, and for each level dependent capacity $M = (m_t)_{t\in]0,1]} \in \mathfrak{M}_1^{(X,\mathcal{A})}$ satisfying $m_t = m$ for all $t \in [\inf f, \sup f] \cap]0,1]$ we have

$$\mathbf{L}(M, f) = \mathbf{I}(m, f),$$

(L3) for all pairs $(M_1, f_1), (M_2, f_2) \in \mathcal{L}_{[0,1]}$ with $h_{M_1,f_1} = h_{M_2,f_2}$ we have

$$\mathbf{L}(M_1, f_1) = \mathbf{L}(M_2, f_2).$$

Observe that, because of axiom (L2), each level dependent capacity-based universal integral $\mathbf{L}$ is an extension of some universal integral $\mathbf{I}$.

Remark 1. (i) The Choquet integral with respect to level dependent capacities (introduced in [2], see also [1]) is a special case of Definition 1 in the sense that the universal integral $\mathbf{I}$ in axiom (L2) is the classical Choquet integral.

(ii) The Sugeno integral based on level dependent capacities (studied in [4]) is another special case of Definition 1: here the universal integral $\mathbf{I}$ in axiom (L2) is the classical Sugeno integral.

Because of axiom (L3), for each level dependent capacity-based universal integral $\mathbf{L}$ and for each pair $(M, f) \in \mathcal{L}_{[0,1]}$, the value $\mathbf{L}(M, f)$ depends only on the function $h_{M,f}$ which is Borel measurable. Denote by $\mathcal{V}$ the set of all Borel measurable functions from $]0,1]$ to $[0,1]$.

Theorem 1. *A function* $\mathbf{L}\colon \mathcal{L}_{[0,1]} \to [0,1]$ *is a level dependent capacity-based universal integral if and only if there is a semicopula* $\otimes\colon [0,1]^2 \to [0,1]$ *and a function* $V\colon \mathcal{V} \to [0,1]$ *satisfying the following conditions:*

(V1) V *is nondecreasing,*

(V2) $V(d \cdot \mathbf{1}_{]0,c]}) = c \otimes d$ *for all* $c, d \in [0,1]$,

(V3) $\mathbf{L}(M, f) = V(h_{M,f})$ *for all* $(M, f) \in \mathcal{L}_{[0,1]}$.

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Globalization of Cauchy complete preordered sets valued in a divisible quantale

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A unital quantale $(Q,\&)$ is divisible if whenever $a \leq b$ in Q, there are $c_1, c_2 \in Q$ such that $a = c_1\&b = b\&c_2$. Given a divisible unital quantale $(Q,\&)$, it is possible to construct different quantaloids from it. In this note, for each divisible quantale $(Q,\&)$ we consider two special quantaloids, $\mathbf{Q}$ and $\underset{\sim}{Q}$. $\mathbf{Q}$ has only one object which is identified with the top element $1 \in Q$, and $\mathbf{Q}(1,1) = Q$ with composition given by $\alpha \circ \beta = \alpha\&\beta$. The quantaloid $\underset{\sim}{Q}$ is constructed as in [3],

- objects: elements $a \in Q$.
- morphisms: $\underset{\sim}{Q}(a,b) = \{\alpha \in Q : \alpha \leq a \wedge b\}$.
- composition: $\beta \circ \alpha = (\beta \swarrow b)\&\alpha = \beta\&(b \searrow \alpha)$ for all $\alpha \in \underset{\sim}{Q}(a,b), \beta \in \underset{\sim}{Q}(b,c)$.
- the unit 1_a of $\underset{\sim}{Q}(a,a)$ is a.
- the partial order on $\underset{\sim}{Q}(a,b)$ is inherited from Q.

A $\mathbf{Q}$-category $\mathbb{A}$ is a set A equipped with a map $\mathbb{A} : A \times A \longrightarrow Q$ such that

(1) $\forall x \in A, \mathbb{A}(x,x) = 1$;
(2) $\forall x,y,z \in A, \mathbb{A}(y,z)\&\mathbb{A}(x,y) \leq \mathbb{A}(x,z)$.

$\mathbf{Q}$-categories are a special case of categories enriched in a monoidal closed category [2], and have been studied both as quantitative domains [9] and as sets endowed with fuzzy orders [1].

A $\underset{\sim}{Q}$-category $\mathbb{A}$ is a set A equipped with a map $\mathbb{A} : A \times A \longrightarrow Q$ satisfying:

(1) $\mathbb{A}(x,y) \leq \mathbb{A}(x,x) \wedge \mathbb{A}(y,y)$ for all $x,y \in X$;
(2) $\mathbb{A}(y,z)\&(\mathbb{A}(y,y) \searrow \mathbb{A}(x,y)) \leq \mathbb{A}(x,z)$ for all $x,y,z \in A$.

$\underset{\sim}{Q}$-categories are examples of categories enriched in a bicategory [8, 10], and can be studied as Q-subsets with quantale-valued preorders [7].

We are concerned with the relationship between the $\mathbf{Q}$-categories and $\underset{\sim}{Q}$-categories. This problem belongs to the change-base issue in the theory of enriched categories [4]. We consider three lax functors $\mathfrak{G}_{\mathfrak{b}}, \mathfrak{G}_{\mathfrak{f}}$ and $\mathfrak{G}$ from $\underset{\sim}{Q}$ to $\mathbf{Q}$, given by $\mathfrak{G}_{\mathfrak{b}}\alpha = b \searrow \alpha$, $\mathfrak{G}_{\mathfrak{f}}\alpha = a \swarrow \alpha$ and $\mathfrak{G}\alpha = (b \searrow \alpha) \wedge (\alpha \swarrow a)$ for all $\alpha \in \underset{\sim}{Q}(a,b)$. These lax functors give rise to three functors:

- $\mathfrak{G}_{\mathfrak{b}} : \underset{\sim}{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Q}\text{-}\mathbf{Cat}$, the backward globalization functor;
- $\mathfrak{G}_{\mathfrak{f}} : \underset{\sim}{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Q}\text{-}\mathbf{Cat}$, the forward globalization functor;
- $\mathfrak{G} : \underset{\sim}{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Q}\text{-}\mathbf{Cat}$, the globalization functor.

Theorem 1. *Suppose $\mathbb{A}$ is a Cauchy complete $\mathcal{Q}$-category. Then both the forward globalization $\mathfrak{G}_{\mathfrak{f}}\mathbb{A}$ and the backward globalization $\mathfrak{G}_{\mathfrak{b}}\mathbb{A}$ are Cauchy complete* **Q**-*categories.*

But, whether the functor $\mathfrak{G}$ preserves Cauchy completeness remains open.

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Discrete partial metric spaces

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A partial metric space is a generalisation of a metric space introducing non zero self distance. Originally motivated by the need to model computable partially defined information such as the asymmetric topological spaces of Scott domain theory in Computer Science, it now falls short in an important respect. The *present cost* of computing information, such as processor time or memory used, is rarely expressible in domain theory. In contrast contemporary algorithms incorporate tight control over the cost of computing resources. Complexity theory in Computer Science has dramatically advanced through the understanding of algorithms over discrete totally defined data structures such as directed graphs, and without the need of partially defined information. And so we have an unfortunate longstanding separation of partial metric spaces for modelling partially defined computable information from the highly advanced complexity theory of algorithms for costing totally defined computable information. It is thus reasonable to propose that a theory of cost for partial metric spaces must be possible to help bridge the separation of domain theory and complexity theory. Today's talk will present our research into understanding and resolving the issues of introducing a complexity theory style notion of cost to partial metric spaces. As working examples we consider the cost of computing a double negation $\neg\neg p$ in two-valued propositional logic, the cost of computing *negation as failure* in logic programming, and a cost model for the *hiaton* time delay proposed by Wadge. The importance of our research is to keep pushing forward from an earlier world of classical domain theory modelling computability of partially defined information to the contemporary reality of Computer Science being a world of dynamic, adaptive, intelligent, & biocomputing systems. "*Building better minds together ... No challenge today is more important than creating beneficial artificial general intelligence (AGI), with broad capabilities at the human level and ultimately beyond*"[4]. Given then a fuzzy set $(A, m : A \to [0,1])$ so useful in modelling such sophisticated systems it is necessary to ask what is the cost of computing $m(x)$ for any $x \in A$? More precisely, how can the definition of m be constrained to always incorporate an appropriate notion of cost? While we are a long way from being able to answer this fascinating question there is a relevant role model for how category theory has already enriched computation. The introduction of *monads* by Moggi[5] to computation and later

[4] Open Cog Foundation opencog.org
[5] Notions of computation and monads, Eugenio Moggi, Information and Computation 93(1)

functional programming in Haskell[6] is being used to formalise our understanding of how to introduce cost to partial metric spaces. Why? Functional programming offers a λ-calculus based model of what can be defined in a logic of computation, which can then be enriched with monads to provide a behavioural model of how efficiently a functional program is being used. From this programming experience of the complexity of computation we work to extrapolate a theory & practice of *discrete partial metric spaces*.

[6] www.haskell.org

Categories isomorphic to L-fuzzy closure system spaces

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1 Introduction and preliminaries

One purpose of this paper is to propose a new kind of L-fuzzy closure operators which is equivalent to L-fuzzy closure systems. Besides, some other characterizations of L-fuzzy closure systems will be presented.

Throughout this paper, $(L, \vee, \wedge, ')$ denotes a completely distributive De Morgan algebra. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. The set of nonzero coprimes in L is denoted by $J(L)$. For $a,\ b \in L$, we say "a is wedge below b" in symbol $a \prec b$ if for every subset $D \subseteq L, \bigvee D \geqslant b$ implies $a \leqslant d$ for some $d \in D$.

For a nonempty set X, L^X denotes the set of all L-fuzzy subsets on X. The set of nonzero coprimes in L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X)$ is exactly the set of all fuzzy points x_λ $(\lambda \in J(L))$. The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively.

Definition 1 ([7]). *A mapping $\varphi : L^X \to L$ is called an L-fuzzy closure system on X if it satisfies the following conditions:*

(S1) $\varphi(\underline{\perp}) = \top$; *(S2)* $\varphi(\bigwedge_{i\in I} A_i) \geqslant \bigwedge_{i\in I} \varphi(A_i)$.

The pair (X, φ) is called an L-fuzzy closure system space if φ is an L-fuzzy closure system on X.
A mapping $f : X \to Y$ between two L-fuzzy closure system spaces (X, φ_X) and (Y, φ_Y) is called continuous if $\forall A \in L^Y$, $\varphi_X(f^\leftarrow(A)) \geqslant \varphi_Y(A)$, where $f^\leftarrow$ is defined by $f^\leftarrow(A)(x) = A(f(x))$ [18].

It is easy to check that L-fuzzy closure system spaces and their continuous mappings form a category, denoted by L-**FCS**.

Definition 2 ([8]). *A mapping $\tau : L^X \to L$ is called an L-fuzzy pretopology on X if it satisfies the following conditions:*

(LFPT1) $\tau(\underline{\top}) = \top$; *(LFPT2)* $\tau(\bigvee_{i\in I} A_i) \geqslant \bigwedge_{i\in I} \tau(A_i)$.

The pair (X, τ) is called an L-fuzzy pretopological space if τ is an L-fuzzy pretopology on X. A mapping $f : X \to Y$ between two L-fuzzy pretopological spaces (X, τ_X) and (Y, τ_Y) is called continuous if $\forall A \in L^Y$, $\tau_X(f^\leftarrow(A)) \geqslant \tau_Y(A)$.

It is easy to check that L-fuzzy pretopological spaces and their continuous mappings form a category, denoted by L-**FPTOP**.

Theorem 1. *L-**FCS** is isomorphic to L-**FPTOP**.*

2 L-fuzzy closure systems characterized by L-fuzzy closure operators

Definition 3. *An L-fuzzy closure operator on X is a mapping $\mathcal{C} : L^X \to L^{J(L^X)}$ satisfying the following conditions:*

(C1) $\mathcal{C}(\underline{\perp})(x_\lambda) = \perp$ *for any* $x_\lambda \in J(L^X)$;
(C2) $\mathcal{C}(A)(x_\lambda) = \top$ *for any* $x_\lambda \leqslant A$;
(C3) $A \leqslant B \Rightarrow \mathcal{C}(A) \leqslant \mathcal{C}(B)$;
(C4) $\mathcal{C}(A)(x_\lambda) = \bigwedge_{x_\lambda \nleqslant B \geqslant A} \bigvee_{y_\mu \nleqslant B} \mathcal{C}(B)(y_\mu)$.

A set X equipped with an L-fuzzy closure operator $\mathcal{C}$, denoted by $(X, \mathcal{C})$, is called an L-fuzzy closure space. A mapping $f : X \to Y$ between two L-fuzzy closure spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ is called continuous if $\forall x_\lambda \in J(L^X)$, $\forall A \in L^X$, $\mathcal{C}_X(A)(x_\lambda) \leqslant \mathcal{C}_Y(f^{\to}(A))(f(x)_\lambda)$.

It is easy to check that L-fuzzy closure spaces and their continuous mappings form a category, denoted by L-**FC**.

Theorem 2. *A mapping $f : X \to Y$ between two L-fuzzy closure spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ is continuous if and only if $\forall x_\lambda \in J(L^X)$, $\forall B \in L^Y$, $\mathcal{C}_X(f^{\leftarrow}(B))(x_\lambda) \leqslant \mathcal{C}_Y(B)(f(x)_\lambda)$.*

Theorem 3. *If φ is an L-fuzzy closure system on X, define $\mathcal{C}_\varphi : L^X \to L^{J(L^X)}$ as follows,*

$$\forall x_\lambda \in J(L^X),\ \forall A \in L^X,\ \ \mathcal{C}_\varphi(A)(x_\lambda) = \bigwedge_{x_\lambda \nleqslant B \geqslant A} \varphi(B)',$$

then $\mathcal{C}_\varphi$ is an L-fuzzy closure operator on X.

Theorem 4. *If $f : (X, \varphi_X) \to (Y, \varphi_Y)$ is continuous with respect to L-fuzzy closure systems φ_X and φ_Y, then $f : (X, \mathcal{C}_{\varphi_X}) \to (Y, \mathcal{C}_{\varphi_Y})$ is continuous with respect to L-fuzzy closure operators $\mathcal{C}_{\varphi_X}$ and $\mathcal{C}_{\varphi_Y}$.*

Theorem 5. *Let $\mathcal{C}$ be an L-fuzzy closure operator on X. Define $\varphi_{\mathcal{C}} : L^X \to L$ by*

$$\forall A \in L^X,\ \ \varphi_{\mathcal{C}}(A) = \bigwedge_{x_\lambda \nleqslant A} (\mathcal{C}(A)(x_\lambda))'.$$

Then $\varphi_{\mathcal{C}}$ is an L-fuzzy closure system on X.

Theorem 6. *If $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is continuous with respect to L-fuzzy closure operators $\mathcal{C}_X$ and $\mathcal{C}_Y$, then $f : (X, \varphi_{\mathcal{C}_X}) \to (Y, \varphi_{\mathcal{C}_Y})$ is continuous with respect to L-fuzzy closure systems $\varphi_{\mathcal{C}_X}$ and $\varphi_{\mathcal{C}_Y}$.*

Theorem 7. (1) *If $\mathcal{C}$ is an L-fuzzy closure operator, then $\mathcal{C}_{\varphi_{\mathcal{C}}} = \mathcal{C}$.*
(2) *If φ is an L-fuzzy closure system, then $\varphi_{\mathcal{C}_\varphi} = \varphi$.*

Theorem 8. *L-**FCS** is isomorphic to L-**FC**.*

3 The other characterizations of L-fuzzy closure systems

Definition 4. *An L-fuzzy interior operator on X is a mapping* $I : L^X \to L^{J(L^X)}$ *satisfying the following conditions:*

(I1) $I(\underline{\top})(x_\lambda) = \top$ *for any* $x_\lambda \in J(L^X)$*;*
(I2) $I(A)(x_\lambda) = \bot$ *for any* $x_\lambda \nleqslant A$*;*
(I3) $A \leqslant B \Rightarrow I(A) \leqslant I(B)$*;*
(I4) $I(A)(x_\lambda) = \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} I(B)(y_\mu)$.

A set X equipped with an L-fuzzy interior operator I, denoted by (X, I)*, is called an L-fuzzy interior space. A mapping* $f : X \to Y$ *between two L-fuzzy interior spaces* (X, I_X) *and* (Y, I_Y) *is called continuous if* $\forall x_\lambda \in J(L^X),\ \forall B \in L^Y,\ I_X(f^{\leftarrow}(B))(x_\lambda) \geqslant I_Y(B)(f(x)_\lambda)$.

It is easy to check that L-fuzzy interior spaces and their continuous mappings form a category, denoted by L-**FI**.

Definition 5. *An L-fuzzy neighborhood system on X is defined to be a set* $N = \{N_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ *of mappings* $N_{x_\lambda} : L^X \to L$ *satisfying the following conditions:*

(LN1) $N_{x_\lambda}(\underline{\top}) = \top,\ N_{x_\lambda}(\underline{\bot}) = \bot$*;*
(LN2) $N_{x_\lambda}(A) = \bot$ *for any* $x_\lambda \nleqslant A$*;*
(LN3) $A \leqslant B \Rightarrow N_{x_\lambda}(A) \leqslant N_{x_\lambda}(B)$*;*
(LN4) $N_{x_\lambda}(A) = \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} N_{y_\mu}(B)$.

A set X equipped with an L-fuzzy neighborhood system $N = \{N_{x_\lambda} \mid x_\lambda \in J(L^X)\}$*, denoted by* (X,N)*, is called an L-fuzzy neighborhood space. A mapping* $f : X \to Y$ *between two L-fuzzy neighborhood spaces* (X,N_X) *and* (Y,N_Y) *is called continuous if* $\forall x_\lambda \in J(L^X),\ \forall B \in L^Y,\ (N_X)_{x_\lambda}(f^{\leftarrow}(B)) \geqslant (N_Y)_{f(x)_\lambda}(B)$.

The category of L-fuzzy neighborhood spaces with their continuous mappings is denoted by L-**FN**.

Definition 6. *An L-fuzzy quasi-coincident neighborhood system on X is defined to be a set* $Q = \{Q_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ *of mappings* $Q_{x_\lambda} : L^X \to L$ *satisfying the following conditions:*

(QN1) $Q_{x_\lambda}(\underline{\bot}) = \bot,\ Q_{x_\lambda}(\underline{\top}) = \top$*;*
(QN2) $Q_{x_\lambda}(A) \neq \bot \Rightarrow x_\lambda \nleqslant A'$*;*
(QN3) $A \leqslant B \Rightarrow Q_{x_\lambda}(A) \leqslant Q_{x_\lambda}(B)$*;*
(QN4) $Q_{x_\lambda}(A) = \bigvee_{x_\lambda \nleqslant B \geqslant A'} \bigwedge_{y_\mu \nleqslant B} Q_{y_\mu}(B')$.

A set X equipped with an L-fuzzy quasi-coincident neighborhood system $Q = \{Q_{x_\lambda} \mid x_\lambda \in J(L^X)\}$*, denoted by* (X,Q)*, is called an L-fuzzy quasi-coincident neighborhood space. A mapping* $f : X \to Y$ *between two L-fuzzy quasi-coincident neighborhood spaces* (X,Q_X) *and* (Y,Q_Y) *is called continuous if* $\forall x_\lambda \in J(L^X),\ \forall B \in L^Y,\ (Q_X)_{x_\lambda}(f^{\leftarrow}(B)) \geqslant (Q_Y)_{f(x)_\lambda}(B)$.

The category of L-fuzzy quasi-coincident neighborhood spaces with their continuous mappings is denoted by L-**FQN**.

Theorem 9. L-**FCS**, L-**FPTOP**, L-**FC**, L-**FI**, L-**FN** *and* L-**FQN** *are all isomorphic.*

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Variable range categories of approximate systems

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1 Introduction and motivation

In our paper [14] the concept of an $\mathbb{M}$-approximate system where $\mathbb{M}$ is a fixed complete lattice was introduced and basic properties of the category of $\mathbb{M}$-approximate systems were studied. We regard the concept of an $\mathbb{M}$-approximate system and the corresponding category as the framework for a unified approach to various categories related to (fuzzy) (bi)toplogical spaces ([2], [5], [3], [4], [13], [8], [11], [12], etc) and to (fuzzy) rough sets ([10], [1], etc). Although the attempts to study the relations between fuzzy topological space and fuzzy rough sets and to introduce a context allowing to give a unified view on these notions were undertaken also by other authors, see e.g. [6], [7], [15], [16], the approach presented in [14] is essentially different. In this work we continue the research of $\mathbb{M}$-approximate systems. However, as different from our previous work here we consider the case of *a variable range* $\mathbb{M}$, that is allow to change lattice $\mathbb{M}$. In particular this alllows to include also the category of *LM*-topological spaces with varied lattice *M* in the scope of our research. In our work two lattices will play the fundamental role. The first one is an infinitely distributive lattice, that is a complete lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee)$, satisfying the infinite distributivity laws $a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \wedge b_i)$ and $a \vee (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \vee b_i)$ for all $a \in \mathbb{L}$, $\{b_i \mid i \in I\} \subseteq \mathbb{L}$. Its top and bottom elements are $1_\mathbb{L}$ and $0_\mathbb{L}$ respectively. Sometimes we assume that the lattice $\mathbb{L}$ is equipped with an order reversing involution $^c : \mathbb{L} \to \mathbb{L}$. In particular, if $\mathbb{L}$ is enriched with a binary operation $\mathbb{L} * \mathbb{L} \to \mathbb{L}$ such that $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, *)$, is Girard monoid, in particular an MV-algebra then involution is naturally defined by $a^c = (a \mapsto 0) \mapsto 0$. The second lattice belonging to the context of our work is $\mathbb{M}$. At the moment we assume only its completeness, however in applications to *LM*-fuzzy topology we need to assume that it is complete distributive. Its bottom and top elements are $0_\mathbb{M}$ and $1_\mathbb{M}$ resp., $0_\mathbb{M} \neq 1_\mathbb{M}$. that is $\mathbb{M}$ contains at least two elements. For the categories of complete lattices, complete infinitely distributive lattices and of complete infinitely distributive lattices with an order reversing involution will be denoted **CLAT, IDL** and **IDLC** respectively.

2 Basic definitions

Definition 1. *An upper* $\mathbb{M}$*-approximate operator on* $\mathbb{L}$ *is a mapping* $u : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ *s.t.*

(1u) $u(0_\mathbb{L}, \alpha) = 0_\mathbb{L}\ \forall \alpha \in \mathbb{M}$;

(2u) $a \leq u(a,\alpha)\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(3u) $u(a \vee b,\alpha) = u(a,\alpha) \vee u(b,\alpha)\ \forall a,b \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(4u) $u(u(a,\alpha),\alpha) = u(a,\alpha)\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(5u) $\alpha \leq \beta, \alpha,\beta \in \mathbb{M} \Longrightarrow u(a,\alpha) \leq u(a,\beta)\ \forall a \in \mathbb{L}$.

Definition 2. *A lower $\mathbb{M}$-approximate operator on $\mathbb{L}$ is a mapping $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ s. t.*

(1l) $l(1_{\mathbb{L}},\alpha) = 1_{\mathbb{L}}\ \forall \alpha \in \mathbb{M}$;
(2l) $a \geq l(a,\alpha)\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(3l) $l(a \wedge b,\alpha) = l(a,\alpha) \wedge l(b,\alpha)\ \forall a,b \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(4l) $l(l(a,\alpha),\alpha) = l(a,\alpha)\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(5l) $\alpha \leq \beta, \alpha,\beta \in \mathbb{M} \Longrightarrow l(a,\alpha) \geq l(a,\beta)\ \forall a \in \mathbb{L}$.

Definition 3. *A quadraple $(\mathbb{L},\mathbb{M},u,l)$, where $u : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ and $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ are upper and lower $\mathbb{M}$-approximate operators on $\mathbb{L}$, is called an $\mathbb{M}$-approximate system on $\mathbb{L}$ or just an approximate system. An approximate system is called*

(T) tight, if $u(a,0_{\mathbb{M}}) = l(a,0_{\mathbb{M}}) = a\ \forall a \in \mathbb{L}$;
(SA) semicontinuous from above if
$u(a,\bigwedge_{i\in I}\alpha_i) = \bigwedge_{i\in I} u(a,\alpha_i), l(a,\bigwedge_{i\in I}\alpha_i) = \bigvee_{i\in I} l(a,\alpha_i)$;
(WA) weakly semicontinuous from above if
$u(a,\alpha_i) = a\ \forall i \in I \Longrightarrow u(a,\bigwedge_{i\in I}\alpha_i) = a$ *and* $l(a,\alpha_i) = a\ \forall i \in I \Longrightarrow l(a,\bigvee_{i\in I}\alpha_i) = a$.

If X is a set, L is a lattice, $\mathbb{L} = L^X$ and $(\mathbb{L},\mathbb{M},u,l)$ is an approximate system, the tuple $(X,L,\mathbb{M},u,l)$ is called an approximate space.

3 Lattice of $\mathbb{M}$-approximate systems on a fixed lattice $\mathbb{L}$

Let $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ stand for the family of all $\mathbb{M}$-approximate systems $(\mathbb{L},\mathbb{M},u,l)$ where $\mathbb{L}$ and $\mathbb{M}$ are fixed. Further, let T-$\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$, D-$\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$, SA-$\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$, WA-$\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ stand on the subfamilies of $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ consisting respectively of tight, self-dual, semicontinuous from above, and weakly semicontinuous from above $\mathbb{M}$-approximate systems on $\mathbb{L}$, respectively. We introduce a partial order $\preceq$ on the family $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ by setting

$$(\mathbb{L},u_1,l_1) \preceq (\mathbb{L},u_2,l_2) \text{ iff } u_1 \geq u_2 \text{ and } l_1 \leq l_2.$$

Theorem 1. *$(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}),\preceq)$ is a complete lattice. Its top element is $(\mathbb{L},u_\top,l_\top)$ where $u_\top(a,\alpha) = l_\top(a,\alpha) = a$ for every $a \in \mathbb{L}$ and every $\alpha \in \mathbb{M}$ and its bottom element is $(\mathbb{L},u_\perp,l_\perp)$ where*

$$u_\perp(a,\alpha) = \begin{cases} 1_{\mathbb{L}} & \text{if } a \neq 0_{\mathbb{L}} \\ 0_{\mathbb{L}}, & \text{if } a = 0_{\mathbb{L}} \end{cases}$$

$$l_\perp(a,\alpha) = \begin{cases} 0_{\mathbb{L}} & \text{if } a \neq 1_{\mathbb{L}} \\ 1_{\mathbb{L}}, & \text{if } a = 1_{\mathbb{L}} \end{cases}$$

Theorem 2. *$(\text{T-}\mathcal{AS}^{\mathbb{M}}(\mathbb{L}),\preceq)$ is a complete lattice whose top element is the same as in $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}),\preceq)$, that is $(\mathbb{L},\mathbb{M},u_\top,l_\top)$, and whose bottom element is $(\mathbb{L},\mathbb{M},u^t_\perp,l^t_\perp)$, where*

$$u^t_\perp(a,\alpha)=\begin{cases}1_{\mathbb{L}} & \text{if } a\neq 0_{\mathbb{L}} \text{ and } \alpha\neq 0_{\mathbb{M}}\\ 0_{\mathbb{L}}, & \text{if } a=0_{\mathbb{L}}\\ a, & \text{if } \alpha=0_{\mathbb{M}}\end{cases}$$

$$l^t_\perp(a,\alpha)=\begin{cases}0_{\mathbb{L}} & \text{if } a\neq 1_{\mathbb{L}} \text{ and } \alpha\neq 0_{\mathbb{M}}\\ 1_{\mathbb{L}}, & \text{if } a=1_{\mathbb{L}}\\ a, & \text{if } \alpha=0_{\mathbb{M}}\end{cases}$$

Theorem 3. *The family $(\text{WA-}\mathcal{AS}^{\mathbb{M}}(\mathbb{L}),\preceq)$ of weakly semicontinuous from above $\mathbb{M}$-approximate systems is a complete sublattice of the lattice $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}),\preceq)$.*

Theorem 4. *Let $\mathrm{D}\in Ob(\mathbf{IDLC})$. Then the family $\text{D-}\mathcal{AS}^{\mathbb{M}}(\mathbb{L}),\preceq)$ of self-dual approximate systems is a complete sublattice of the lattice $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}),\preceq)$.*

4 Category AS of approximate systems

Let **AS** be the family of all approximate systems $(\mathbb{L},\mathbb{M},u,l)$. To consider **AS** as a category whose class of objects are all $\mathbb{M}$-approximate systems $(\mathbb{L},\mathbb{M},u,l)$ where $\mathbb{L}\in Ob(\mathbf{IDL})$ and $\mathbb{M}\in Ob(\mathbf{CLAT})$ we have to specify its morphisms. Given two approximate systems $(\mathbb{L}_1,\mathbb{M}_1,u_1,l_1),(\mathbb{L}_2,\mathbb{M}_2,u_2,l_2)\in Ob(\mathbf{AS})$ by a morphism

$$F:(\mathbb{L}_1,\mathbb{M}_1,u_1,l_1)\to(\mathbb{L}_2,\mathbb{M}_2,u_2,l_2)$$

we call a pair $F=(f,\varphi)$ such that

(1m) $f:\mathbb{L}_1\to\mathbb{L}_2$ is a morphism in the category $\mathbf{IDL}^{op}$;
(2m) $\varphi:\mathbb{M}_1\to\mathbb{M}_2$ is a morphism in the category $\mathbf{CLAT}^{op}$;
(3m) $u_1(f(b),\varphi(\beta))\leq f(u_2(b,\beta))\ \forall b\in\mathbb{L}_2,\ \forall\beta\in\mathbb{M}_2$;
(4m) $f(l_2(b,\beta))\leq l_1(f(b),\varphi(\beta))\ \forall b\in\mathbb{L}_2,\forall\beta\in\mathbb{M}_2$

Remark 1. The category $\mathbf{AS}^{\mathbb{M}}$, where $\mathbb{M}$ is a fixed lattice, which was studied in [14] can be identified with a subcategory of the category **T-AS** having $\mathbb{M}$-approximate systems $(\mathbb{L},\mathbb{M},u,l)$ as objects and pairs $F=(f,id_{\mathbb{M}}):(\mathbb{L}_1,\mathbb{M}_1,u_1,l_1)\to(\mathbb{L}_2,\mathbb{M},u_2,l_2)$ as morphisms. ($id_{\mathbb{M}}:\mathbb{M}\to\mathbb{M}$ stands for an identity mapping.) In particular, in case when $\mathbb{M}$ is a two-point lattice we obtain the category $\mathbf{AS}^{\mathbf{2}}$.

Theorem 5. *Every source $F_i:(\mathbb{L}_1,\mathbb{M}_1)\to(\mathbb{L}_i,\mathbb{M}_i,u_i,l_i),\ i\in\mathrm{I}$ in **AS** has a unique initial lift $F_i:(\mathbb{L}_1,\mathbb{M}_1,u_1,l_1)\to(\mathbb{L}_i,\mathbb{M}_i,u_i,l_i),\ i\in\mathrm{I}$.*

Theorem 6. *Every sink $F_i:(\mathbb{L}_i,\mathbb{M}_i,u_i,l_i)\to(\mathbb{L}_1,\mathbb{M}_1),\ i\in\mathrm{I}$ in **AS** has a unique final lift $F_i:(\mathbb{L}_i,\mathbb{M}_i,u_i,l_i)\to(\mathbb{L}_1,\mathbb{M}_1,u_1,l_1),\ i\in\mathrm{I}$*

Corollary 1. *Category **AS** is topological over the category $\mathbf{IDL}^{op}\times\mathbf{CLAT}^{op}$ with respect to the forgetful functor $\mathfrak{F}:\mathbf{AS}\longrightarrow\mathbf{IDL}^{op}\times\mathbf{CLAT}^{op}$.*

We study also the categorical properties of the full subcategories of **AS** whose objects are tight, self-dual, and (weakly) semicontinuous from above approximate systems as well as some other classes of approximate systems. In particular, we show that

Theorem 7. *Category* D-**AS** *of self-dual approximate systems is topological over the category* $\mathbf{IDLC}^{op} \times \mathbf{CLAT}^{op}$ *with respect to the forgetful functor* $\mathfrak{F} : \text{D-}\mathbf{AS} \to \mathbf{IDLC}^{op} \times \mathbf{CLAT}^{op}$.

Some subcategories of **AS** determined by restricted classes of morphisms will be also in the scope of our interest. Finally we will discuss different concrete categories related to fuzzy (bi)topology and fuzzy rough sets regarded as subcategories of **AS**.

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Category Theory in Statistical Learning?

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David Corfield was asked recently by someone for his opinion on the possibility that category theory might prove useful in machine learning. First of all, he would not want to give the impression that there are signs of any imminent breakthrough. For other areas of computer science the task would be easier (nowadays!). Category theory features prominently in theoretical computer science as described in books such as [2].

And what about statistics? One direct help may be a probability theory. In a couple of web posts Corfield discussed a construction of probability theory in terms of a monad. He pointed out a natural inclination of the Bayesian to think about distributions over distributions fits this construction well.

Moreover, Graphical models, which include directed graphs, together with Bayesian networks, may sometimes form a symmetric monoidal category.

Another dimension to spaces of probability distributions is that they can be studied by differential geometry in a field known as information geometry. For an insightful treatment in the context of nonlinear models see [5], general treatment may be found in [1].

Beside the above mentioned issues, one practical application for empirical statistics, the *"categorization"* of inference function will be discussed. In [3] we have realized (by empirical research) a need of non-crisp monotonicity for Fisher information of experiments under heteroscedasticity. The classical Fisher information is based on the "classical" score function, used by the pioneers of modern statistics (Karl Pearson, Francis Y. Edgeworth and Sir Ronald A. Fisher) have been introduced as a local change of log-likelihood w.r.t. to a parameter of interest, more less in case to case studies. However, an alternative score can be defined ([4]) and proven to have some desirable properties ([6] and [7]) in classical statistical inference. In nonparametrics, a similar inference function, so called influence function is used. A practical discussion of this aspects in a context of the *"categorization"* of inference function will be given.

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Quantaloid-enriched categories for multi-valued logic and other purposes

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In this lecture I shall aim to give an overview of the basic concepts in the theory of quantaloid-enriched categories, giving as many examples as time permits. First I shall recall what quantales and quantaloids are, and how one computes extensions and liftings in them. Then I shall define categories, functors and distributors enriched in a quantaloid, saying something about the universal property of quantaloid-enrichment too. I shall explain how every functor between quantaloid-enriched categories determines a left adjoint distributor, and that this very fact is at the heart of quantaloid-enriched category theory. By way of illustration I shall show how to define adjunctions, presheaves, (co)limits, (co)completions, and so on. Further I shall say a word about the symmetrisation of quantaloid-enriched categories. And finally I shall indicate the link between quantaloid-enriched categories on the one hand, and modules on a quantaloid on the other. This lecture should provide (more than) the background that is needed for H. Heymans' lecture on sheaf theory via quantaloid-enrichment.

On the characterisation of regular left-continuous t-norms

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1 Introduction

Quantales are complete lattices endowed with an associative binary operation $\odot$ distributing from both sides over arbitrary joins [Ros]. A quantale is called strictly two-sided if there is a top element that is neutral w.r.t. $\odot$, and it is called commutative if $\odot$ is commutative. In the special case that the complete lattice is the real unit interval endowed with the natural order, a strictly two-sided, commutative quantale is an algebra well-known in fuzzy logic: a left-continous (l.-c.) t-norm algebra [KMP].

We consider this type of structure from a constructive point of view, being interested in its complete description. Our viewpoint is algebraic as we classify l.-c. t-norm algebras up to isomorphism only. However, we also make use of methods from analysis, in particular from the theory of functional algebras.

Let $([0,1];\leq,\odot,0,1)$ be a l.-c. t-norm algebra. We denote by $(\Lambda_\odot;\leq,\circ,\overline{0},\mathrm{id})$ the associated translation tomonoid. That is, $\Lambda_\odot$ consists of all (inner right) translations

$$\lambda_a^\odot\colon [0,1]\to[0,1]\colon x\mapsto x\odot a$$

by some $a\in[0,1]$; $\leq$ is the pointwise order; $\circ$ is the function composition; and $\overline{0}$ is the zero constant function, id the identical function. The isomorphism $a\mapsto\lambda_a^\odot$ of the semigroup $([0,1];\odot)$ and its translation semigroup $(\Lambda_\odot;\circ)$ [ClPr] extends to an isomorphism between $([0,1];\leq,\odot,0,1)$ and $(\Lambda_\odot;\leq,\circ,\overline{0},\mathrm{id})$. We have [Vet]:

Theorem 1. *Let $\odot$ be a l.-c. t-norm. Then $\Lambda_\odot$ is a set of functions from $[0,1]$ to $[0,1]$ with the following properties:*

(T1) *Every f is increasing.*

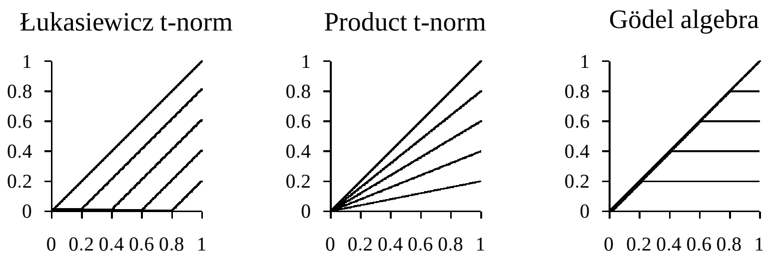
(T2) *Every f and g commute.*

(T3) *For every $t\in[0,1]$, there is exactly one f such that $f(1)=t$.*

(T4) *Every f is left-continuous.*

Conversely, let Λ be a set of functions from $[0,1]$ to $[0,1]$ fulfilling (T1)–(T4). *Then there is a unique l.-c. t-norm $\odot$ such that $\Lambda=\Lambda_\odot$.*

The following heuristic argument may illustrate how the present work was motivated. Consider the following depictions of the translation tomonoids of the three basic continuous t-norms:



Observe that, in each of these cases, if the picture was almost completely covered and we were able to inspect an arbitrarily narrow stripe below the identity line only, we would be able to reconstruct the whole functional algebra. In fact, the functions in a neighborhood of the identity either generate the whole algebra, or it can be concluded that all functions are idempotent and thus uniquely determined by the intersection of their graphs with the identity line.

2 Regular l.-c. t-norms: the simple case

The exact facts have been examined in the paper [Vet], of which the present work is the continuation. As might be expected, the above observations do not apply for all l.-c. t-norm algebras. We restrict our attention to the following subclass.

Definition 1. *A l.-c. t-norm $\odot$ is called regular if the following conditions hold:*

(1) *There is an $n < \omega$ such that each $f \in \Lambda_\odot$ has at most n discontinuity points.*
(2) *For $t \in [0,1]$, put $e(t) = \inf\{s\colon s \odot t = t\}$. Then there are $0 = v_0 < v_1 < \ldots < v_k = 1$ such that for each $i = 0,\ldots,k-1$, the map $e|_{(v_i,v_{i+1})}$ is continuous and one of the following possibilities holds:*
   (a) *$e|_{(v_i,v_{i+1})}$ is constant r, and we have $r \odot t = t$ for all $t \in (v_i, v_{i+1})$;*
   (b) *$e|_{(v_i,v_{i+1})}$ is strictly monotonous.*

Even if this condition looks special, the class of t-norm algebras based on regular l.-c. t-norm is not neglible – in the sense that it generates the whole variety of MTL-algebras.

Regular l.-c. t-norm algebras can be decomposed in a specific way. Namely, let $(\Lambda_\odot; \leq, \circ, 0, id)$ be the translation tomonoid of the regular l.-c. t-norm $\odot$. Then we may determine a characteristic sequence of points $(v_0, \ldots, v_k)$ – cf. the definition of regularity –, called a *frame* for $\odot$. For each *basic* interval $(v_i, v_{i+1}]$, we consider the induced translation tomonoid:

$$\Lambda_{(v_i,v_{i+1}]} = \{f_{(v_i,v_{i+1}]}\colon f \in \Lambda_\odot\},$$

where $f_{(v_i,v_{i+1}]}\colon (v_i, v_{i+1}] \to (v_i, v_{i+1}]\colon a \mapsto f(a) \vee v_i$. We call $(\Lambda_{(v_i,v_{i+1}]}; \leq, \circ, 0, id)$ a *basic tomonoid* of $\odot$.

Theorem 2. *Any basic tomonoid associated to some l.-c. t-norm belongs to one out of six isomorphism classes.*

3 Regular l.-c. t-norms: the general case

Knowing the basic tomonoids associated to a l.-c. t-norm $\odot$ means to know how the translations by the elements of each basic interval act on this same interval. This knowledge alone may or may not determine the whole t-norm algebra. In the former case, a l.-c. t-norm is fully characterised by (1) the size k of a frame, (2) the type of each of the k basic tomonoids, and (3) the intervals parametrising the basic tomonoids.

The question how the translations by the elements of one basic interval act on the remaining intervals has not yet been addressed; this is the topic of the present work.

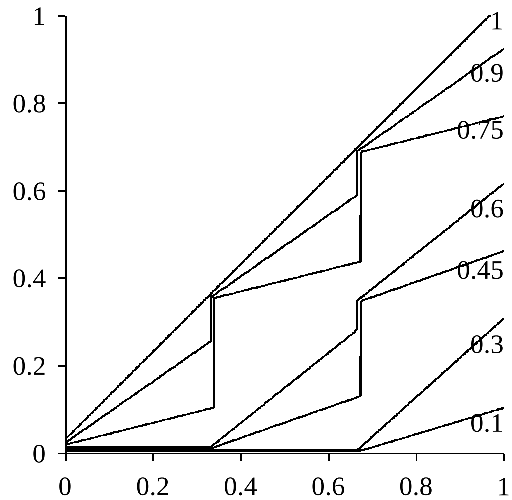
Let $(\Lambda_\odot; \leq, \circ, 0, id)$ be the translation tomonoid of the regular l.-c. t-norm $\odot$. Let $(v_0, \ldots, v_k)$ be a frame for $\odot$. For each pair of distinct intervals $(v_i, v_{i+1}]$ and $(v_j, v_{j+1}]$, put

$$H_{(v_i,v_{i+1}]}^{(v_j,v_{j+1}]} = \{ f_{(v_i,v_{i+1}]}^{(v_j,v_{j+1}]} : f \in \Lambda_\odot \},$$

where $f_{(v_i,v_{i+1}]}^{(v_j,v_{j+1}]} : (v_i, v_{i+1}] \to (v_j, v_{j+1}] : a \mapsto (f(a) \vee v_j) \wedge v_{j+1}$. We call $H_{(v_i,v_{i+1}]}^{(v_j,v_{j+1}]}$ a *lower tomonoid* of $\odot$.

It turns out that the lower tomonoids are largely determined by the basic tomonoids: for each i, j, the algebra $H_{(v_i,v_{i+1}]}^{(v_j,v_{j+1}]}$ is determined as follows. There is a totally ordered set of functions H, uniquely determined by $\Lambda_{(v_i,v_{i+1}]}$ and $\Lambda_{(v_j,v_{j+1}]}$, such that $H_{(v_i,v_{i+1}]}^{(v_j,v_{j+1}]}$ is an interval of H. As a consequence, for the description of a general regular t-norm, we need in addition to (1), (2), and (3) above to specify (4) the relevant intervals of the lower tomonoids, and (5) the intervals parametrising the lower tomonoids.

The three-part Hájek t-norm



As an example, we consider a t-norm that was proposed in a modified form by P. Hájek [Haj]:

$$a \odot b = \begin{cases} a(3b-2) & \text{if } a \leq \frac{1}{3} \text{ and } b > \frac{2}{3}, \\ 3ab - 2a - b + 1 & \text{if } \frac{1}{3} < a \leq \frac{2}{3} \text{ and } b > \frac{2}{3}, \\ 3ab - 2a - 2b + 2 & \text{if } a, b > \frac{2}{3}, \\ 0 & \text{if } a \leq \frac{1}{3} \text{ and } b \leq \frac{2}{3}, \\ 3ab - a - b + \frac{1}{3} & \text{if } \frac{1}{3} < a, b \leq \frac{2}{3} \end{cases}$$

for $a, b \in [0,1]$. We have the following characteristic data. (1) Size of frame: 3. (2) Type of basic tomonoids: product; product; product. (3) Parametrising intervals: $[\frac{2}{3}, 1)$; $[\frac{2}{3}, 1)$; $[\frac{2}{3}, 1)$. (4) Intervals of the lower algebras: full; full; full. (5) Intervals parametrising the lower algebras: $[\frac{1}{3}, \frac{2}{3})$; $[\frac{1}{3}, \frac{2}{3})$; $[0, \frac{1}{3})$.

4 Conclusion

We have shown that any left-continous t-norm fulfilling the condition of regularity allows a particular type of decomposition into finitely many constituents. Namely, the real unit interval may be divided into finitely many subintervals and the tomonoids of translations by the elements of one interval restricted to another interval may be indicated by means of six isomorphism classes. In short, we may associate to a regular l.-c. t-norm its characteristic data, describing how the t-norm is composed from a finite set of specific constituents.

Conversely, it is not difficult to check if given data to construct a l.-c. t-norm is actually the characteristic data of a l.-c. t-norm. An easy criterion to decide this question is, however, not known to us.

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Type-2 operations on finite chains

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1 Introduction

The algebra of truth values for fuzzy sets of type-2 consists of all mappings from the unit interval into itself, with operations certain convolutions of these mappings with respect to pointwise max and min. This algebra has been studied extensively as indicated in the references below. The basic theory depends on the fact that $[0,1]$ is a complete chain, so lends itself to various generalizations and consideration of special cases. This paper develops the theory where each copy of the unit interval is replaced by a finite chain. Most of the theory goes through, but there are several special circumstances of interest.

2 The Algebra $\mathbf{M}(\mathbf{m}^{\mathbf{n}})$

For a positive integer n, let $\mathbf{n}$ be the set $\{1,2,\ldots,n\}$. This set has its usual linear order which we denote by $\leq$, max and min operations denoted $\vee$ and $\wedge$, negation given by $\neg k = n - k + 1$, and the obvious constants 1 and n. With these operations, $\mathbf{n}$ becomes a De Morgan algebra, in fact a Kleene algebra since it also satisfies $a \wedge \neg a \leq b \vee \neg b$.

We denote by $\mathbf{m}^{\mathbf{n}}$ the set $\{f : \mathbf{n} \to \mathbf{m}\}$ of all mappings from the set $\mathbf{n}$ into the set $\mathbf{m}$. The algebra $\mathbf{M}(\mathbf{m}^{\mathbf{n}})$ consists of the set $\mathbf{m}^{\mathbf{n}}$ with operations given in the following definition.

Definition 1. *On* $\mathbf{m}^{\mathbf{n}}$, *let*

$$(f \sqcup g)(i) = \bigvee_{j \vee k = i} (f(j) \wedge g(k))$$

$$(f \sqcap g)(i) = \bigvee_{j \wedge k = i} (f(j) \wedge g(k))$$

$$\neg f(i) = \bigvee_{j = \neg i} f(j) = f(\neg i)$$

$$\overline{1}(i) = \begin{cases} m & if\ i = m \\ 1 & if\ i \neq m \end{cases} \quad and\ \overline{0}(i) = \begin{cases} m & if\ i = 1 \\ 1 & if\ i \neq 1 \end{cases}$$

Thus we have the algebra

$$\mathbf{M}(\mathbf{m}^{\mathbf{n}}) = (\mathbf{m}^{\mathbf{n}}, \sqcup, \sqcap, \neg, \overline{0}, \overline{1})$$

There are two other operations on the functions in $\mathbf{m^n}$, namely pointwise max and min. We also denote these by $\vee$ and $\wedge$, respectively. Just as in the case $\mathbf{M}([0,1]^{[\mathbf{0},\mathbf{1}]})$, these operations help in determining the properties of the algebra $\mathbf{M}(\mathbf{m^n})$ via the auxiliary operations $f^L(i) = \vee_{j\leq i} f(j)$ and $f^R(i) = \vee_{j\geq i} f(j)$.

The operations $\sqcup$ and $\sqcap$ in $\mathbf{M}(\mathbf{m^n})$ can be expressed in terms of the pointwise max and min of functions in two different ways, as follows.

Theorem 1. *The following hold for all $f, g \in \mathbf{M}(\mathbf{m^n})$.*

$$f \sqcup g = \left(f \wedge g^L\right) \vee \left(f^L \wedge g\right) = (f \vee g) \wedge \left(f^L \wedge g^L\right)$$

$$f \sqcap g = \left(f \wedge g^R\right) \vee \left(f^R \wedge g\right) = (f \vee g) \wedge \left(f^R \wedge g^R\right)$$

Using these auxiliary operations, it is fairly routine to verify the following properties of the algebra $\mathbf{M}(\mathbf{m^n})$. The details of the proofs are almost exactly the same as for the algebra $\mathbf{M}([0,1]^{[0,1]})$, which are given for example in [9].

Corollary 1. *Let f, g, $h \in \mathbf{M}(\mathbf{m^n})$. Some basic equations follow.*

1. $f \sqcup f = f$; $f \sqcap f = f$
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
4. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
5. $\overline{1} \sqcap f = f$; $\overline{0} \sqcup f = f$
6. $\neg\neg f = f$
7. $\neg(f \sqcup g) = \neg f \sqcap \neg g$; $\neg(f \sqcap g) = \neg f \sqcup \neg g$

The elements of $\mathbf{M}(\mathbf{m^n})$ may be deonoted by n-tuples $(a_1, a_2, \ldots, a_n)$ of elements of $\mathbf{m}$. Note that with this notation, in $\mathbf{m^n}$ the element $\overline{1}$ is $(1,1,\ldots,1,m)$ and $\overline{0}$ is $(m,1,1,\ldots,1)$. Further note that the algebra has an absorbing element $(1,1,\ldots,1)$. Finally, $\neg(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$.

3 The Main Results

Each of $\sqcup$ and $\sqcap$, being idempotent, commutative and associative, gives rise to a partial order. These partial orders are defined by $f \leq_\sqcup g$ if $f \sqcup g = g$ and $f \leq_\sqcap g$ if $f \sqcap g = f$.

Theorem 2. *The partial orders $\leq_\sqcup$ and $\leq_\sqcap$ are lattice orders.*

The equations listed above do not form an equational basis for $\mathbf{M}(\mathbf{m^n})$. We do not know an equational basis for $\mathbf{M}(\mathbf{m^n})$ nor even if a finite one exists. However, similar to the case of $\mathbf{M}([\mathbf{0},\mathbf{1}]^{[0,1]})$ [4], we do get the following.

Theorem 3. *For $m \geq 2$, the algebras $\mathbf{M}(\mathbf{m^n})$ and $\mathbf{M}(\mathbf{2^n})$ generate the same variety and thus satisfy the same equations.*

Theorem 4. *Let $\mathbf{n} \geq \mathbf{5}$. Then $\mathbf{M}(\mathbf{2^n})$ and $\mathbf{M}(\mathbf{2^5})$ generate the same variety and thus satisfy the same equations.*

One main objective of this paper was to show that the automorphism group of the retract $(\mathbf{m^n}, \sqcup, \sqcap)$ of $\mathbf{M}(\mathbf{m^n})$ is trivial, that is, has only one element. To effect this, the irreducible elements of $(\mathbf{m^n}, \sqcup, \sqcap)$ were determined.

Theorem 5. *Let $m, n \geq 2$. The irreducible elements of $(\mathbf{m^n}, \sqcup, \sqcap)$ are these.*

1. *The absorbing element $(1, 1, \ldots, 1)$.*
2. *The n-tuple with m_i in the i-th place and 1 elsewhere.*
3. *The element $m_1 \vee m_n$.*
4. *If $n = 2$, n-tuples that contain m, and the absorbing element.*

Using the theorem above and long sequence of lemmas, we get the following.

Theorem 6. *The automorphism group of $(\mathbf{m^n}, \sqcup, \sqcap)$ has only one element.*

4 Comments

One principal result of this paper is that the partial order given by the operation $\sqcup$ is a lattice, and analogously for $\sqcap$. For the operation $\sqcup$, the sup of two elements f and g is $f \sqcup g$, but the inf of the two elements is the sup of the set of all elements below both f and g. The elements f and g are n-tuples of elements of **m,** and the inf is given by some formula in the elements in these two n-tuples.

Problem 1. Find a formula for the inf of two elements in the lattice determined by $\sqcup$. And similarly, do the same for the lattice determined by $\sqcap$.

Problem 2. In the case of the algebra $([0,1]^{[0,1]}, \sqcup)$, determine whether or not the partial order determined by $\sqcup$ is a lattice.

In the case of $\mathbf{2^3}$, the lattices determined by $\sqcup$ and by $\sqcap$ are both distributive, but this is not true for all $\mathbf{m^n}$.

Problem 3. For which $\mathbf{m^n}$ are the lattices determined by $\sqcup$ and $\sqcap$ distributive? We conjecture none for m and $n \geq 3$.

The proof that $Aut(\mathbf{m^n}, \sqcup, \sqcap)$ consists of only the identity automorphism was effected by a long sequence of lemmas, etc. Hopefully, there is a much shorter and less computational proof.

Problem 4. Find a proof that $Aut(\mathbf{m^n}, \sqcup, \sqcap)$ is trivial that is more conceptual, less computational, and shorter.

In showing that the automorphism group of $(\mathbf{m^n}, \sqcup, \sqcap)$ consists of only the identity automorphism, we used in the proof that an automorphism preserved both $\sqcup$ and $\sqcap$. But small examples show that the automorphism group of $(\mathbf{m^n}, \sqcup)$ is just the identity automorphism, and we suspect that this is true in general, but have no proof.

Problem 5. Find the automorphism group of $(\mathbf{m^n}, \sqcup)$**.** (Since $(\mathbf{m^n}, \sqcup)$ and $(\mathbf{m^n}, \sqcap)$ are isomorphic, their automorphism groups will be isomorphic.)

Finally, there are many ways to specialize and to generalize the truth-value algebra $([0,1]^{[0,1]}, \sqcup, \sqcap, \neg, \bar{0}, \bar{1})$ of type-2 fuzzy sets. We have just taken a finite chain for each interval $[0,1]$. For example, one could take any two complete lattices instead, or substitute one finite chain for one of the intervals $[0,1]$, and so on. Such investigations may be of interest.

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Convergence and compactness in fuzzy metric spaces

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Abstract. As Höhle observed in [9] the fact that the topology associated to a probabilistic metric space is metrizable means that,from this topological point of view, probabilistic metric spaces are always equivalent to ordinary metric spaces,and the problem of topologization of probabilistic metric spaces is not satisfactorily solved. He proposed many-valued topologies as suitable tools for this purpose. Hence in [15], we endowed George and Veeramani's fuzzy metric (which has close relation to probabilistic metric) with many-valued structures-fuzzifying topology and fuzzifying uniformity. The aim of this paper is to go on studying the properties of George and Veeramani's fuzzy metric. We will give the concept of convergence degree and generalize the convergence and compactness theories in metric spaces to Veeramani's fuzzy metric spaces.

1 Introduction

Metric space plays an important role in the research and applications of topology. Convergence theory is an another important part in metric spaces and is the key tool in studying completeness. Probabilistic metric space, a generalization of the ordinary metric space, was first studied by Menger [12] and further developed by Schweizer and Sklar [14]. Inspired by the notion of probabilistic metric spaces, Kramosil and Michalek [10] in 1975 introduced the notion of fuzzy metric, a fuzzy set in the Cartesian product $X \times X \times \mathfrak{R}$ satisfying certain conditions (see Definition 2.12 for a similar form). George and Veeramani [1–3] slightly modified the definition of Kramosil and Michalek's fuzzy metric space and associated each fuzzy metric space to a Hausdorff topology.

Till now many topological structures and related theories have been defined and studied on the probabilistic metric space and George and Veeramani's fuzzy metric space. For example, Höhle [7, 8] studied the associated topologies and the fuzzy uniformities in the probabilistic metric space, J. Gutiérrez García and M.A. de Prada Vicente [6] studied the Hutton [0,1]-quasi-uniformities generated by the George and Veeramani's fuzzy metric. Gregori,etc,in [4, 5] studied the convergence and completeness in George and Veeramani's fuzzy metric spaces. Recall that the value $M(x,y,t)$ in the definition of George and Veeramani's fuzzy metric can be thought as the degree of the nearness between x and y with respect to t. Hence in this paper, we want to give the degree convergence theory of sequence in fuzzy metric spaces, and generalize the corresponding theory of convergence and compactness in classical metric spaces to fuzzy metric spaces.

2 Convergence in fuzzy metric spaces

Since the value $M(x,y,t)$ can be thought as the degree of the nearness between x and y with respect to t, in this section, we will give the definitions of degree convergence and study the relationship between them.

Definition 1. *Let (X,M) be a fuzzy metric space, $x \in X$ and $\{x_n\}$ be sequence. The degree to which $\{x_n\}$ converges to x is defined by*

$$Con(\{x_n\},x) = \bigwedge_{\varepsilon>0} \bigvee_{N\in\mathcal{N}} \bigwedge_{n>N} M(x_n,x,\varepsilon).$$

The degree to which $\{x_n\}$ accumulates to x is defined by

$$Ad(\{x_n\},x) = \bigwedge_{\varepsilon>0} \bigwedge_{N\in\mathcal{N}} \bigvee_{n>N} M(x_n,x,\varepsilon).$$

The degree to which $\{x_n\}$ is a Cauchy sequence is defined by

$$Cauchy(\{x_n\}) = \bigwedge_{\varepsilon>0} \bigwedge_{N\in\mathcal{N}} \bigvee_{n,m>N} M(x_n,x_m,\varepsilon).$$

Lemma 1. *Let (X,M) be a fuzzy metric space, $x \in X$ and $\{x_n\}$ be sequence. Then we have the following results:*

(1) $Con(\{x\},x) = Ad(\{x\},x) = Cauchy(\{x\}) = 1$, where $\{x\}$ is the constant sequence of x;

(2) $Con(\{x_n\},x) \leq Ad(\{x_n\},x)$;

(3) $Con(\{x_n\},x) \leq Cauchy(\{x_n\})$ for all $x \in X$;

(4) $Ad(\{x_n\},x) = \bigvee_{\{x_{n_k}\}} Con(\{x_{n_k}\},x)$;

(5) $Ad(\{x_n\},x) \leq \bigvee_{\{x_{n_k}\}} Con(\{x_{n_k}\},x)$.

(6) $Ad(\{x_n\},x) \wedge Cauchy(\{x_n\}) \leq Con(\{x_n\},x)$.

Example 1. Let d be an ordinary metric on X and M^d be the induced fuzzy metric. In the following, we know that the convergence in (X,M^d) is coincident with that in (X,d).

$$Con(\{x_n\},x) = \bigwedge_{\varepsilon>0} \bigvee_{N\in\mathcal{N}} \bigwedge_{n>N} \frac{\varepsilon}{\varepsilon + d(x_n,x)} = \bigwedge_{\varepsilon>0} \frac{\varepsilon}{\varepsilon + \bigwedge_{N\in\mathcal{N}} \bigvee_{n>N} d(x_n,x)} = \begin{cases} 1, & x_n \to x, \\ 0, & \text{others}, \end{cases}$$

$$\begin{aligned} Cauchy(\{x_n\},x) &= \bigwedge_{\varepsilon>0} \bigvee_{N\in\mathcal{N}} \bigwedge_{n,m>N} \frac{\varepsilon}{\varepsilon + d(x_n,x_m)} = \bigwedge_{\varepsilon>0} \frac{\varepsilon}{\varepsilon + \bigwedge_{N\in\mathcal{N}} \bigvee_{n,m>N} d(x_n,x_m)} \\ &= \begin{cases} 1, & \{x_n\} \text{is Cauchy}, \\ 0, & \text{others}, \end{cases} \end{aligned}$$

$$Ad(\{x_n\},x) = \bigwedge_{\varepsilon>0} \bigwedge_{N\in\mathcal{N}} \bigvee_{n>N} \frac{\varepsilon}{\varepsilon + d(x_n,x)} = \bigwedge_{\varepsilon>0} \frac{\varepsilon}{\varepsilon + \bigvee_{N\in\mathcal{N}} \bigwedge_{n>N} d(x_n,x)} = \begin{cases} 1, & x_n \infty x, \\ 0, & \text{others}, \end{cases}$$

Example 2. Let $X = \{x,y\}$ and $d : X \times X \times (0,+\infty) \to [0,1]$ be defined by

$$M(a,b,t) = \begin{cases} 1, & a = b = x, \\ 1, & a = b = y, \\ 1, & a \neq b, t > \frac{1}{2}, \\ \frac{1}{2} + t, & a \neq b, t \leq \frac{1}{2}, \end{cases}$$

Then d is a fuzzy metric on X and $Con(\{x\},y) = Ad(\{x\},y) = \frac{1}{2}$. If we take $\{x_n\} = \{x,y,x,y,x...\}$, then $Con(\{x_n\},x) = 0$ and $Ad(\{x_n\},x) = 1$.

3 Compactness in fuzzy metric spaces

In this section, we want to generalized the compactness in metric spaces to fuzzy setting according to the above convergence theory.

Definition 2. *Let (X,M) be a fuzzy metric space. The degree to which (X,M) is compact is defined by*

$$Comp(M) = \bigwedge_{\{x_n\}} \bigvee_{x \in X} Ad(\{x_n\},x).$$

The degree to which (X,M) is sequently compact is defined by

$$Scomp(M) = \bigwedge_{\{x_n\}} \bigvee_{\{x_{n_k}\}} \bigvee_{x \in X} Con(\{x_{n_k}\},x).$$

Definition 3. *Let (X,M) be a fuzzy metric space and $F \in 2^X$. The degree to which F is an ε-net of (X,M) is defined by*

$$\varepsilon - net(F) = \bigwedge_{x \in X} \bigvee_{y \in F} M(x,y,\varepsilon).$$

The degree to which (X,M) is totally bounded is defined by

$$Totallb(M) = \bigwedge_{\varepsilon > 0} \bigvee_{F \in 2^{(X)}} \varepsilon - net(F).$$

The degree to which (X,M) is complete is defined by

$$Complete(M) = \bigwedge_{\{x_n\}} (Cauchy(\{x_n\}) \to \bigvee_{x \in X} Con(\{x_n\},x))$$

Theorem 1. *Let (X,M) be a fuzzy metric space. Then $Comp(M) = Scomp(M)$.*

Theorem 2. *Let (X,M) be a fuzzy metric space. Then*

$$Totallb(M) = \bigwedge_{\{x_n\}} \bigvee_{\{x_{n_k}\}} Cauchy(\{x_{n_k}\}).$$

Theorem 3. *Let (X,M) be a fuzzy metric space. Then $Comp(M) = Complete(M) \wedge Totallb(M)$.*

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