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37th Linz Seminar on Fuzzy Set Theory

Abstracts

Enriched Category Theroy and Related Topics

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Abstracts

Isar Stubbe
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Editors

LINZ 2017

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ENRICHED CATEGORY THEORY AND RELATED TOPICS

ABSTRACTS

Isar Stubbe, Ulrich Höhle,
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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2017 will be the 37th seminar carrying on this tradition and is devoted to the theme "Enriched Category Theory and Related Topics". The goal of the seminar is to present and to discuss recent advances in enriched category theory and its various applications in pure and applied mathematics.

A considerable amount of interesting contributions were submitted for possible presentation at LINZ 2017 and subsequently reviewed by PC members. This volume contains the abstracts of the accepted contributions. These regular contributions are complemented by four invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

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Contents

Applications of fundamental categorical duality theorem to L-fuzzy sets and separated M-valued sets

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Abstract. Fundamental Categorical Duality Theorem states the duality between the full subcategory of an abstract category $\mathbf{C}$ with L-spatial objects and the full subcategory of the category of $\mathbf{C}$-$\mathcal{M}$-L-spaces with L-sober objects, and serves to prove new and existing dualities under the same framework. We apply the theorem to L-fuzzy sets and separated M-valued sets, and then obtain two new dualities.

1 Introduction

A dual equivalence-alias duality-between categories $\mathbf{C}$ and $\mathbf{D}$ is a well-known issue in category theory. In case $\mathbf{C}$ and $\mathbf{D}$ are concrete categories over the category of sets, it is shown in [14] that every schizophrenic object for $\mathbf{C}$ and $\mathbf{D}$ induces a dual adjunction between $\mathbf{C}$ and $\mathbf{D}$, and such adjunction restricts to a duality between two special subcategories of $\mathbf{C}$ and $\mathbf{D}$. There are many famous dualities (e.g., Stone duality [2, 12], Priestley duality [2] and localic duality [12]) between a category of ordered algebraic structures (e.g., Boolean algebras, distributive lattices and spatial frames) and a category of structured topological spaces (e.g., Stone spaces, Priestley spaces and sober topological spaces). The dualities induced by schizophrenic objects are able to describe these famous dualities [2, 14]. However, if one takes $\mathbf{C}$ to be an abstract category instead of a category of ordered algebraic structures, and if a suitable category $\mathbf{D}$-viewed as an abstraction of a category of structured topological spaces-is asked to have the property of a dual adjunction, or of a duality, with $\mathbf{C}$, then the approach in [14] does not give an answer to this question. An alternative approach, providing a satisfactory answer to the question, is proposed in [4]. This approach, for a given category $\mathbf{C}$ with set-indexed products and an essential $(\mathcal{E}, \mathcal{M})$-factorization structure, formulates the asked category $\mathbf{D}$ to be the category $\mathbf{C}$-$\mathcal{M}$-L-**Top** of $\mathbf{C}$-$\mathcal{M}$-L-spaces, and establishes a dual adjunction between $\mathbf{C}$ and $\mathbf{C}$-$\mathcal{M}$-L-**Top**, i.e. an adjoint situation

$$(\eta, \varepsilon) : L\Omega_{\mathcal{M}} \dashv LPt_{\mathcal{M}} : \mathbf{C}^{op} \to \mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top}.$$

The unit η and co-unit ε allow us to define largest subcategories $(\mathcal{E}, \mathcal{M})$-$L$-**Spat**-$\mathbf{C}$ of $\mathbf{C}$ and $\mathbf{C}$-$\mathcal{M}$-L-**SobTop** of $\mathbf{C}$-$\mathcal{M}$-L-**Top**, to which the functors $L\Omega_{\mathcal{M}}$ and $LPt_{\mathcal{M}}$ can be restricted in order to obtain a duality. The resulting duality between $(\mathcal{E}, \mathcal{M})$-$L$-**Spat**-$\mathbf{C}$ and $\mathbf{C}$-$\mathcal{M}$-L-**SobTop** is called Fundamental Categorical Duality Theorem (FCDT). In addition to the applications to Q-preordered sets, augmented posets and quasivarieties [4], FCDT also serves to prove famous dualities (e.g., Stone, Priestley, Heyting and localic dualities) [5, 6]. Our aim in this study is to apply FCDT to L-fuzzy sets and separated M-valued sets. The next section briefly explains these applications.

2 Applications to L-fuzzy sets and separated M-valued sets

2.1 Goguen's category of L-fuzzy sets

Let $(L, \leq)$ be a poset. Recall that Goguen [8, 9] defined the category Set(L) of L-fuzzy sets as the category with objects all pairs (X, α), where X is a set and $\alpha : X \to L$ is a map, and with morphisms $f : (X, \alpha) \to (Y, \beta)$ such that $f : X \to Y$ is a function satisfying $\alpha(x) \leq \beta(f(x))$.

Definition 1. *A triple (X, τ, μ) is called an L-primal measure space if X is a set, τ is a subset of the power set $\mathcal{P}(X)$ of X and $\mu : \tau \to L$ is a map.*

Various kinds of measure spaces (e.g., the classical measure spaces, probability spaces [13], plausibility spaces [7] and L-possibility spaces [3]) are special L-primal measure spaces.

Proposition 1. *L-primal measure spaces form a category **PMEAS**(L) whose morphisms $f : (X, \tau_X, \mu_X) \to (Y, \tau_Y, \mu_Y)$ are functions $f : X \to Y$ satisfying*

- *$f^{\leftarrow}(G) \in \tau_X$ for all $G \in \tau_Y$,*
- *$\mu_Y(G) \leq \mu_X(f^{\leftarrow}(G))$ for all $G \in \tau_Y$,*

where $f^{\leftarrow}(G)$ is the preimage of G under f.

We say that an L-primal measure space (X, τ, μ) is sober if the map $X \to \mathcal{P}(\tau)$, $x \mapsto \{G \in \tau \mid x \in G\}$, is a bijection.

If the poset $(L, \leq)$ has a greatest element, then we will establish, as an application of FCDT, a duality between Set(L) and the full subcategory of **PMEAS**(L) consisting of all sober L-primal measure spaces.

2.2 Höhle's category of separated M-valued sets

Let $M = (L, \leq, *)$ be a GL-monoid. A separated M-valued set is a pair (X, E) consisting of a set X and a separated M-valued equality E on X, i.e., a map $E : X \times X \to L$ provided with the following conditions [10]:

- $E(x, y) \leq E(x, x) \wedge E(y, y)$,
- $E(x, y) = E(y, x)$,
- $E(x, y) * (E(y, y) \to E(y, z)) \leq E(x, z)$,
- $E(x, x) \vee E(y, y) \leq E(x, y)$ implies $x = y$, $\forall x, y \in X$.

SM-**SET** denotes the category of separated M-valued sets [11] whose morphisms are $f : (X, E) \to (Y, F)$ such that $f : X \to Y$ is a function preserving extent of existence and equality, i.e.,

- $F(f(x), f(x)) \leq E(x, x)$,
- $E(x, y) \leq F(f(x), f(y))$.

A map $\mu : X \to L$ is called a strict and extensional L-fuzzy subset of a given separated M-valued set (X, E) [10] iff μ satisfies

- $\mu(x) \leq E(x, x)$,
- $\mu(x) * (E(x, x) \to E(x, y)) \leq \mu(y)$.

The set of all strict and extensional L-fuzzy subsets of (X, E) is denoted by $p(X, E)$.

Proposition 2. *For a set X and $c \in L$, denote the constant map $X \to L$ with value c by c_X. The set $R(X, L) = \left\{(c, \mu) \in L \times L^X \mid \mu \leq c_X\right\}$ carries a separated M-valued equality $E_{R(X,L)}$ defined by*

$$E_{R(X,L)}\left((c_1, \mu_1), (c_2, \mu_2)\right) = \bigwedge_{x \in X} \left[(c_1 * (\mu_1(x) \to \mu_2(x))) \wedge (c_2 * (\mu_2(x) \to \mu_1(x)))\right].$$

Proposition 3. *Every map $f : X \to Y$ determines a map $f_L^R : R(Y, L) \to R(X, L)$ by*

$$f_L^R(c, \mu) = (c, \mu \circ f).$$

Definition 2. *Let M-**SP** stand for a category with objects all pairs $(X, \mathcal{U})$ where X is a set and $\mathcal{U}$ is a subset of $R(X, L)$. Morphisms of the category are all $f : (X, \mathcal{U}_X) \to (Y, \mathcal{U}_Y)$ provided that $f : X \to Y$ is a function satisfying $f_L^R(c, \mu) \in \mathcal{U}_X$ for all $(c, \mu) \in \mathcal{U}_Y$.*

To each M-**SP**-object $(X, \mathcal{U})$, we associate a map $\rho_{(X,\mathcal{U})} : X \to p(\mathcal{U}, E_{\mathcal{U}})$, given by

$$\rho_{(X,\mathcal{U})}(x)(c, \mu) = \mu(x),$$

where $E_{\mathcal{U}}$ is the restriction of $E_{R(X,L)}$ to $\mathcal{U}$. Sobriety of such $(X, \mathcal{U})$ is defined to be the bijectivity of $\rho_{(X,\mathcal{U})}$. As the second application of FCDT, we will show a duality between **S**M-**SET** and the full subcategory of M-**SP** with sober objects.

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Enriched topology and asymmetry

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Mathematically modeling the question of how to satisfactorily compare, in a many-valued setting, both bitstrings and the predicates which they might satisfy—a surprisingly intricate question when the conjunction of predicates is non-commutative—involves notions of enriched categories and enriched functors. Particularly relevant is the notion of a set enriched by a po-groupoid, which turns out to be a many-valued preordered set, along with enriched functors extended as to be variable-basis. This positions us to model the above question by constructing topological systems enriched by many-valued preorders, systems whose associated extent spaces motivate the notion of topological spaces enriched by many-valued preorders and spaces which are non-commutative when the underlying lattice-theoretic base has a non-commutative tensor product. Of special interest are crisp and many-valued specialization preorders generated by many-valued topological spaces, orders which have the following consequences for many-valued spaces: they characterize the well-established L-T_0 separation axiom, define the L-$T_1(1)$ separation axiom—logically equivalent under appropriate lattice-theoretic conditions to the L-T_1 axiom of T. Kubiak, and define an apparently new L-$T_1(2)$ separation axiom. Along with the consequences of these ideas for many-valued spectra, these orders show that asymmetry has a home in many-valued topology comparable in at least some respects to its home in traditional topology.

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Non-commutativity and many-valuedness: the topological representation of the spectrum of C^*-algebras

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In the past there have been made various attempts to define the spectrum of a non-commutative C^*-algebra. But all these definitions have certain drawbacks — e.g. C.J. Mulvey's definition does not coincide with the standard definition of the spectrum in the commutative case (cf. [4]). The aim of our talk is to give an alternative definition of the spectrum which does not suffer under this deficit — i.e. coincides with the standard situation in the commutative setting. For this purpose we recall some properties of balanced and bisymmetric quantales, introduce a definition of the spectrum of a C^*-algebra working for the general case and develop subsequently its topological representation.

1 The category of balanced and bisymmetric quantales

A quantale $(\mathfrak{Q}, *)$ is *balanced* if the universal upper bound is idempotent. A quantale is *bisymmetric* if the following property holds for all $\alpha,\ \beta,\ \gamma \in \mathfrak{Q}$:

$$(\alpha * \beta) * (\gamma * \delta) = (\alpha * \gamma) * (\beta * \delta).$$

A quantale is *semi-unital* if the relations $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ hold for all $\alpha \in \mathfrak{Q}$. Every semi-unital quantale is balanced, and the semi-unitalization of every quantale exists. A quantale is *semi-integral* if the relation $\alpha * \top * \beta \leq \alpha * \beta$ holds for all $\alpha, \beta \in \mathfrak{Q}$.

Example 1. Every idempotent and left-sided (right-sided) quantale is semi-unital, bisymmetric and semi-integral.

Let $\mathfrak{Q}$ and $\mathfrak{P}$ be quantales. A strong homomorphism is a join preserving map $\mathfrak{Q} \xrightarrow{h} \mathfrak{P}$ satisfying the following conditions for all $\alpha, \beta \in \mathfrak{Q}$:

$$h(\alpha * \beta) = h(\alpha) * h(\beta) \quad \text{and} \quad h(\top) = \top.$$

Balanced and bisymmetric quantales with strong homomorphisms form a category denoted by $\mathsf{BSQuant}$.

Theorem 1. ([1]) $\mathsf{BSQuant}$ *is complete and cocomplete.*

In some important special cases the coproduct in $\mathsf{BSQuant}$ can be expressed by the tensor product of quantales.

Theorem 2. ([1]) *Let $(X, *)$ and $(Y, *)$ be bisymmetric quantales. If $(X, *)$ is strictly left-sided and $(Y, *)$ is strictly right-sided, then the tensor product $(X \otimes Y, \star)$ with the embeddings $X \xrightarrow{j_X} X \otimes Y$ and $Y \xrightarrow{j_Y} X \otimes Y$ as coprojections (i.e. $j_X(x) = x \otimes \top$ and $j_Y(y) = \top \otimes y$) is the coproduct of $(X, *)$ and $(Y, *)$ in* $\mathtt{BSQuant}$.

Remark 1. (Permanence properties of the tensor product) The tensor product preserves the structure of balanced quantales and semi-unital quantales. Moreover, in the case of balanced quantales the tensor product preserves also bisymmetry and semi-integrality.

Theorem 3. ([1]) *Let $(X, *)$ be a strictly left-sided quantale and $(Y, *)$ be a strictly right-sided quantale. Further, let $X \underset{\vartheta_Y}{\overset{\vartheta_X}{\rightleftarrows}} Y$ be a pair of join preserving anti-homomorphisms provided with the property $\vartheta_Y \circ \vartheta_X = 1_X$ and $\vartheta_X \circ \vartheta_Y = 1_Y$. Then there exists a unique isotone involution ι on the tensor product $(X \otimes Y, \star)$ such that $(X \otimes Y, \star, \iota)$ is an involutive quantale and the following diagram is commutative*:

$$\begin{tikzcd} X \otimes Y \arrow[r, "\iota"] & X \otimes Y \arrow[r, "\iota"] & X \otimes Y \\ X \arrow[u, "j_X"] \arrow[r, "\vartheta_X"] & Y \arrow[u, "j_Y"] \arrow[r, "\vartheta_Y"] & X \arrow[u, "j_X"'] \end{tikzcd} \tag{I}$$

In the setting of Theorem 2 and Theorem 3 we consider now a further balanced and bisymmetric quantale $(Z, *)$ and strong homomorphisms $Z \xrightarrow{q_X} X$ and $Z \xrightarrow{q_Y} Y$ satisfying the following properties

$$\vartheta_X \circ q_X = q_Y \quad \text{and} \quad \vartheta_Y \circ q_Y = q_X. \tag{1}$$

Then the pushout of q_X and q_Y in the sense of $\mathtt{BSQuant}$ is the coequalizer

$$Z \underset{j_Y \circ q_Y}{\overset{j_X \circ q_X}{\rightrightarrows}} X \otimes Y \xrightarrow{\pi} S.$$

Moreover, there exists a unique involution $'$ on $(S, *)$ such that $(S, *, ')$ is an involutive quantale and π is an involutive homomorphism.

2 Spectrum of a C^*-algebra

Let A be a C^*-algebra with unit. Then we apply Section 1 in the following setting:

- $\mathbb{L}(A)$ is the quantale of all closed left ideals of A.
- $\mathbb{R}(A)$ is the quantale of all closed right ideals of A.
- $\mathbb{I}(A)$ is the quantale of all closed two-sided ideals of A.

There exists a pair of anti-homomorphism $\mathbb{L}(A)X \underset{\vartheta_{\mathbb{R}(A)}}{\overset{\vartheta_{\mathbb{L}(A)}}{\rightleftarrows}} \mathbb{L}(A)$ given by the formation of adjoint ideals. Finally, let $\mathbb{I}(A) \xhookrightarrow{q_{\mathbb{L}(A)}} \mathbb{L}(A)$ and $\mathbb{I}(A) \xhookrightarrow{q_{\mathbb{R}(A)}} \mathbb{R}(A)$ be the respective embeddings. Since closed two-sided ideals are self-adjoint, the strong homomorphisms $q_{\mathbb{L}(A)}, q_{\mathbb{R}(A)}, \vartheta_{\mathbb{L}(A)}, \vartheta_{\mathbb{R}(A)}$ satisfy (1).

The *spectrum* $\mathbb{S}(A)$ is defined as the pushout of $\mathbb{I}(A) \xhookrightarrow{q_{\mathbb{L}(A)}} \mathbb{L}(A)$ and $\mathbb{I}(A) \xhookrightarrow{q_{\mathbb{R}(A)}} \mathbb{R}(A)$ in $\mathsf{BSQuant}$. Hence the following diagram is commutative:

$$\begin{tikzcd}
 & \mathbb{S}(A) & \\
\mathbb{L}(A) \arrow[ur, "\varphi_L"] \arrow[r, "j_{\mathbb{L}(A)}"] & \mathbb{L}(A) \otimes \mathbb{R}(A) \arrow[u, dotted, "\pi"'] & \mathbb{R}(A) \arrow[ul, "\varphi_R"'] \arrow[l, "j_{\mathbb{R}(A)}"'] \\
 & \mathbb{I}(A) \arrow[ul, hook, "q_L"] \arrow[ur, hook, "q_R"'] &
\end{tikzcd}$$

In particular $\mathbb{S}(A)$ is a semi-unital, semi-integral, bisymmetric and involutive quantale and π is a surjective and involutive homomorphism.

Since the commutative case is characterized by $\mathbb{I}(A) = \mathbb{L}(A) = \mathbb{R}(A)$, the previous definition of the spectrum of a C^*-algebra coincides with the standard definition in the commutative case.

3 Topological representation of C^*-algebras

We begin with the definition of prime elements for semi-unital quantales which goes back to W. Krull in the case of integral quantales 1928 (cf. [3]).

An element $p \in \mathfrak{Q}$ is called *prime* if and only if $p \neq \top$ and the following implication holds for all $\alpha, \beta \in \mathfrak{Q}$:

$$\alpha * \beta \leq p \quad \Longrightarrow \quad \alpha * \top \leq p \text{ or } \top * \beta \leq p.$$

Theorem 4. ([1]) (a) *Let $\mathfrak{Q}$ be a semi-integral quantale. Then every maximal left-sided (right-sided) element is prime.*

(b) *Let* $(X,*)$ *be a strictly left-sided quantale and* $(Y,*)$ *be a strictly right-sided quantale. An element* $p \in X \otimes Y$ *is a prime element if and only if there exists prime elements* x *in* $(X,*)$ *and* y *in* $(Y,*)$ *such that the following relation holds*:

$$p = (x \otimes \top) \vee (\top \otimes y).$$

Now let $C = \{\bot, a, \top\}$ be the chain with three elements. Then C_ℓ is the non-commutative, idempotent and left-sided 3-chain, and C_r is the non-commutative, idempotent and right-sided 3-chain. Obviously, the identity 1_C determines a pair of anti-homomorphisms between C_ℓ and C_r. Then we view the tensor product $C_\ell \otimes C_r$ as the *quantisation of* $\mathbf{2} = \{0,1\}$ which can be visualized by the following Hasse diagram:

$$\begin{tikzcd}[row sep=small, column sep=small] & \top \arrow[d, dash] & \\ & \alpha \arrow[dl, dash] \arrow[dr, dash] & \\ \lambda \arrow[dr, dash] & & \varrho \arrow[dl, dash] \\ & \beta \arrow[d, dash] & \\ & \bot & \end{tikzcd} \qquad \text{where} \quad \begin{cases} \alpha = (a \otimes \top) \vee (\top \otimes a), \\ \lambda = a \otimes \top, \\ \varrho = \top \otimes a, \\ \beta = a \otimes a. \end{cases}$$

Theorem 5. ([2]) *Let* $(X,*)$ *be a semi-unital quantale. Then every prime element* $p \in X$ *can be identified with a strong homomorphism* $X \xrightarrow{h} C_\ell \otimes C_r$ *and vice versa. In particular, this relationship is determined by the following property:*

$$p = \bigvee\{x \in X \mid h(x) \leq \alpha\}.$$

Topological representation of $\mathbb{S}(A)$

Let $\sigma(\mathbb{S}(A))$ be the set of all prime elements of $\mathbb{S}(A)$. Every element $f \in \mathbb{S}(A)$ induces a map $\sigma(\mathbb{S}(A)) \xrightarrow{\mathbb{A}_f} C_\ell \otimes C_r$ by:

$$\mathbb{A}_f(p) = h_p(f), \qquad p \in \mathbb{S}(A)$$

where h_p is the strong homomorphism corresponding to the prime element p. Then $\tau_A = \{\mathbb{A}_f \mid f \in \mathbb{S}(A)\}$ is a non-commutative six-valued topology on $\sigma(\mathbb{S}(A))$ which coincides with the Gelfand topology in the commutative case. In this sense τ_A can be regarded the *non-commutative Gelfand topology* of A and is obviously isomorphic to the spatial reflection of $\mathbb{S}(A)$.

If A is a *simple* C^*-algebra — i.e. $\mathbb{I}(A) = \{0, A\} \cong \mathbf{2}$, then τ_A is the six-valued product topology of the three-valued topology induced by the quantale $\mathbb{L}(A)$ of all closed left ideals with three-valued topology induced by the quantale $\mathbb{R}(A)$ of all closed right ideals. In this context, if L is a maximal left ideal and R is a maximal right ideal, then the pair (L,R) represents a typical prime element of the spectrum $\mathbb{S}(A) = \mathbb{L}(A) \otimes \mathbb{R}(A)$ of A.

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Applications of modules on unital quantales

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Let $\mathtt{Sup}$ be the monoidal closed category of complete lattices with join preserving maps. A unital quantale $(\mathfrak{Q}, *, e)$ is a monoid in $\mathtt{Sup}$. A right $\mathfrak{Q}$-module $(X, \boxdot)$ is a complete lattice X with a right action $X \otimes \mathfrak{Q} \xrightarrow{\boxdot} X$ over $(\mathfrak{Q}, *, e)$. Finally, let $[X, X]$ be the complete lattice of all join preserving self-maps of X and $([X, X], \circ, 1_X)$ be the corresponding unital quantale. It is well known that every right action $\boxdot$ on X over $\mathfrak{Q}$ can be identified with a unital quantale homomorphism $\mathfrak{Q} \xrightarrow{h} [X, X]$ such that the relation $x \boxdot \alpha = h(\alpha)(x)$ holds for all $x \in X$ and $\alpha \in \mathfrak{Q}$.

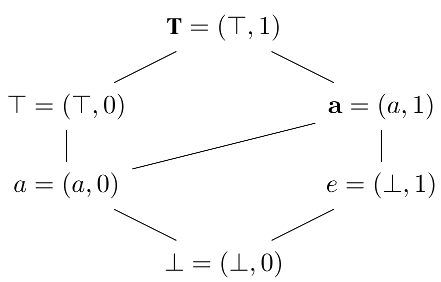
Theorem 1. *Let $(\mathfrak{Q}, *)$ be a (not necessary unital) quantale and $(\widehat{\mathfrak{Q}}, \star, e)$ be its unitalization. Then $\mathfrak{Q}$ is always a right-$\widehat{\mathfrak{Q}}$-module w.r.t. the following right action $\boxdot$ defined by*

$$(\alpha, 1)\boxdot\beta = (\alpha, 1)\star(\beta, 0) = (\alpha*\beta)\vee\beta, \quad (\alpha, 0)\boxdot\beta = (\alpha, 0)\star(\beta, 0) = \alpha*\beta, \;\; \alpha, \beta \in \mathfrak{Q}.$$

Example 1. Let $C_3 = \{\bot, a, \top\}$ be the three chain provided with the structure of a non-unital, non-commutative, idempotent, left-sided quantale. In particular, the corresponding multiplication is determined by:

$$a * \top = a, \quad \top * a = \top, \quad a * a = a.$$

Further, let $\widehat{C_3^\ell}$ be the unitalization of $(C_3, *)$. Then $\widehat{C_3^\ell} = C_3 \times \{0, 1\}$ can be visualized by the Hasse diagram and the corresponding multiplication table:



$\star$	$\bot$	a	$\top$	e	$\mathbf{a}$	$\mathbf{T}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\top$	a	a	$\top$
$\top$	$\bot$	a	$\top$	$\top$	$\top$	$\top$
e	$\bot$	a	$\top$	e	$\mathbf{a}$	$\mathbf{T}$
$\mathbf{a}$	$\bot$	a	$\top$	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{T}$
$\mathbf{T}$	$\bot$	$\top$	$\top$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$

If we now identify the right action $\boxdot$ with the unital quantale homomorphism $\widehat{C_3^\ell} \xrightarrow{h} [C_3, C_3]$, then we obtain $h(\mathbf{T}) = \varphi_1 = h(\top), \quad h(\mathbf{a}) = 1_{C_3} = h(e)$, $\varphi_3 := h(a)$ and $\varphi_4 := h(\bot)$ where

$$\varphi_1(\top) = \varphi_1(a) = \top,\; \varphi_1(\bot) = \bot, \quad \varphi_3(\top) = \varphi(a) = a,\; \varphi_3(\bot) = \bot, \quad \varphi_4 \equiv \bot.$$

Then we have four transition maps φ_4, φ_3, 1_{C_3} and φ_1 which constitute the chain C_4 of four element w.r.t. the pointwisely defined order — i.e.

$$\varphi_4 \leq \varphi_3 \leq 1_C \leq \varphi_1$$

and induce the structure of a non-integral, idempotent, non-commutative, unital quantale on C_4 where the multiplication is given by the composition of maps.

The aim of the following considerations is to show that the previous example has a real world application in the context of medical date given by qualifications in ICD and ICF.

1 An example patient case of functional decline accelerated by a conglomerate of disorders

John[4], 80 years old, needs support from his spouse, 77 years old. They have children, good family connections, and a supportive social network. John suffers from multiple diseases and uses multiple drugs. He was a smoker until he suffered from a cardiac infarct, and was thereafter also diagnosed with diabetes type 2. His basic and instrumental activities in his daily life are no longer what they used to be. He has become slower in walking, taking smaller and shuffling steps. His gait is still symmetrical, but he has some postural control problems. He has fallen in the garden, and been close too free falling at home. he is still driving their car, but only locally. His spouse is in a fairly good condition. She has become more clumsy, but is still fully functional in the household. Family members also help out. A year later, John's walking slows down, and car driving is more difficult. His spouse and family becomes more and more aware and concerned about his decline. He continues to have smaller fall or near-fall incidents. Fall and fall injury risk increases. There are now several possibilities to use scopes of assessment of John's functional condition. One is ICF[5]. An ICF profiling can now be done independently of the scope of multiple diseases, but can also be done as related to a selected main disease. The medical domain speaks about the distinction between comorbidity and multi-morbidity. Yet another year passes, and John's spouse now brings him to investigations. There are neurological findings in his basal ganglia which brings attention to the family of Parkinson diseases. Differential diagnosis in that group is not easy, in particular if it turns out not to be a typical Parkinson's disease (ICD[6] code G20),

[4] This patient case is not real cases, even if it is intended as a very realistic case. Any resemblance therefore with existing data in patient records is purely accidental.

[5] WHO's International Classification of Functioning, Disability and Health

[6] WHO's International Classification of Diseases, at this point in its ICD-10 version, with ICD-11 expected by 2018.

also since anti-Parkinsonian drugs had no effect. John eventually receives a diagnosis within the group related to other degenerative diseases of basal ganglia (G23). Some of the disorders in this group progresses faster, and leads to difficulties to manage John in his own home. He wanders at night, and sedatives make him dizzy during the day. Unluckily, he falls, because of a TIA (transient ischemic attack, G45.9), with a resulting injury, a fracture on his lower forearm (S52.5). Surgery is successful, but his ICF profiles obviously changes because of the fall, and progrediation given his neurological condition continues. The way his disorder (ICD) profile affects his functioning (ICF) profile can now intuitively be viewed as a disorder profile acting upon a functioning profile $\mathtt{a} : \mathtt{ICD} \times \mathtt{ICF} \to \mathtt{ICF}$. Clearly, this can be reflected in the ICF classifications in various ways and given different objectives. One such objective is explained in [4], where a specific ICF profile for Parkinson's disease is presented. Cain ICF related concern comes with involuntary movement functions (ICF code b765), muscle tone functions (b735) and emotional functions (b152). Similar cases could be described for seniors having dementia, a pulmonary condition, a heart condition, or other medical conditions as their main disorder in a conglomerate of diseases.

The specification if the module transition maps can also be done in several ways, depending on the objectives, so that it reflects and explains the particular situation of interest. Doing all this obviously invites to generalize the situation e.g. in direction of neuro-degenerative diseases, and as in connection more broadly with the issue of falls and fall injury prevention, also in a broader multi-professional care scope. However, we must underline that there is no canonic quantalization or C^*-algebraization of the ICD terminological structure, and similarly none for ICD. Terminologies like those for SNOMED are unstructured and simply relational, which in fact has invited IHTSDO[7] to adopt description logic (DL) as the ontology logic for its health ontology. However, it is not at all clear that this is useful, as pointed out in [5], where a many-valued and typed generalization of DL was proposed.

2 Medical semantics of *Example 1*

Qualification in ICD is bivalent, even if disorder may be more or less severe, but quantification in in ICF follows its generic scale of 5 items, with a sixth for 'unspecificed'. Similarly ICD's bivalence could be extended with a third 'unspecificed' or 'not (yet) known' value. In [1] we showed that there are many candidates for representing ICF's generic scale as a quantale. An interpretation of ICF's constructs in relation to diseases (ICD) suggests viewing ICF's generic scale as a quantale in form of $ICF_d = ICD^\ell \otimes ICD^r$ (see [1] for detail) reflecting the situation that a valuation of a multi-morbidity medical condition-condition interaction of ICD codes corresponds to the way valuation of functioning is done with respect to ICF codes. Similar relations and structure can be provided as involving several other classifications, like those involving drugs (ATC/CCC), lab data (LOINC) or surgical codes (NOMESCO). A larger and more systematic treatment of a wider range of aspects concerning relations between

[7] International Health Terminology Standards Development Organisation

nomenclatures, will be provided in a full version of this abstract. In this abstract and in this section we therefore focus only on qualification as related to disorder, and doing so for the three-valued structure for qualification of ICD codes.

The meaning of the elements of $C = \{\bot, a, \top\}$ now becomes interesting, and several interpretations are possible. For example, using a as the 'unspecified' or 'not (yet) known' value, we could then have the following:

$$
\begin{aligned}
\top &= \text{not sufficient evidence in support of diagnosis} \\
a &= \text{diagnosis suspected} \\
\bot &= \text{diagnosis confirmed and registered}
\end{aligned}
$$

Then, a pairing like $(G20, a)$ would mean that presence of Parkinson's disease is suspected, and invoking $(G20, \bot)$ in parallel with $(G23, a)$ means that the Parkinson's disease suspicion has been rejected in favour of a suspicion for other degenerative diseases of basal ganglia. Since deterioration in G23 is slower than in some of the G23 specific disorders, the transition from the combination of $(G23, \top)$, $(G49.5, a)$ and $(S52.5, \top)$ on the ICD side to a corresponding qualification of b152 on the ICF side, would show a different ICF qualification for b152 e.g. in the case of $(G20, a)$ and $(S52.5, \top)$, i.e., with Parkinson's disease only suspected, and TIA not registered as a cause of the fall.

Question. What is the medical semantic of the transition maps φ_1, φ_3, 1_{C_3}, $\varphi_4 \equiv \bot$?

This is just a first a preliminary example, and more detail and elaborations will be included in the extended paper related to this abstract.

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Tensor products and relation quantales

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Abstract. We report on our recent paper [3] on tensor products for closure spaces and posets and their quantales of relations.

1 Introduction

Tensor products have their place in algebra, (point-free) topology, order theory, category theory and other mathematical disciplines. In the realm of ordered sets, they are intimately related to the concept of Galois connections (see e.g. [1, 7]). The aim of this talk is to show how such tensor products give rise to certain quantales whose members are specific relations between complete lattices, partially ordered sets (posets) or closure spaces.

Before focussing on tensor products, let us recall briefly the fundamental notions in the theory of Galois connections. Given two posets A and B, let $Ant(A, B)$ denote the pointwise ordered set of all *antitone*, i.e. order-reversing maps from A to B, and $Gal(A, B)$ the subposet of all *Galois maps*, i.e. maps from A to B such that the preimage of any principal filter is a principal ideal [5, 7]. If A and B are complete lattices, $Ant(A, B)$ is a complete lattice, too, and $Gal(A, B)$ is the complete lattice of all $f\colon A \to B$ satisfying $f(\bigvee X) = \bigwedge f[X]$ for all $X \subseteq A$. Galois maps are closely tied to *Galois connections*; these are dual adjunctions between posets A and B, that is, pairs (f, g) of maps $f\colon A \to B$ and $g\colon B \to A$ such that

$$x \leq g(y) \Leftrightarrow y \leq f(x) \text{ for all } x \in A \text{ and } y \in B,$$

or equivalently, pairs of maps $f \in Ant(A, B)$ and $g \in Ant(B, A)$ with

$$x \leq g(f(x)) \text{ for all } x \in A \text{ and } y \leq f(g(y)) \text{ for all } y \in B.$$

Either partner in a Galois connection determines the other by the formula

$$g(y) = f^*(y) = \max\{x \in A : f(x) \geq y\},$$

and the Galois maps are nothing but the partners of Galois connections. Clearly, (f, g) is a Galois connection iff (g, f) is one, and consequently, $Gal(A, B) \simeq Gal(B, A)$. Both composites of the partners of a Galois connection are closure operations, and their ranges are dually isomorphic.

We introduce three kinds of tensor products for posets A and B as follows: $A \otimes_r B$ denotes the collection of all *right tensors*, i.e. down-sets T in $A \times B$ such that $xT = \{y \in B : (x,y) \in T\}$ is a principal ideal of B for each $x \in A$. The system $A\,{}_\ell\!\otimes B$ of *left tensors* is defined in the opposite manner, and the *tensor product* $A\,{}_\ell\!\otimes_r B$ consists of all (*two-sided*, i.e. left and right) *tensors*; if A and B are complete, it is denoted by $A \otimes B$; in that case, a down-set in $A \times B$ is a right tensor iff $\{x\} \times Y \subseteq T$ implies $(x, \bigvee Y) \in T$, a left tensor iff $X \times \{y\} \subseteq T$ implies $(\bigvee X, y) \in T$, and a tensor iff $X \times Y \subseteq T$ implies $(\bigvee X, \bigvee Y) \in T$ (see [7] for alternative characterizations).

A bijective connection between posets of antitone maps and tensor products of posets is provided by the assignments

$$f \mapsto T_f = \{(x,y) \in A \times B : f(x) \geq y\} \text{ and}$$
$$T \mapsto f_T \text{ with } f_T \colon A \to B, x \mapsto \max xT\,.$$

In fact, these maps are mutually inverse isomorphisms between $A \otimes_r B$ and $Ant(A,B)$, and they induce isomorphisms between $A\,{}_\ell\!\otimes_r B$ and $Gal(A,B)$.

If some poset B has a least element $0 = 0_B$, we may build the "truncated" poset $\breve{B} = B \smallsetminus \{0\}$. Now, given complete lattices A, B, C and $f \in Ant(A,B)$, $g \in Ant(B,C)$, define $g \odot f \colon A \to C$ by

$$g \odot f(x) = \bigvee \{z \in C : (x,z) \in E_{f,g}\},$$

where $E_{f,g}$ denotes the tensor generated by the set

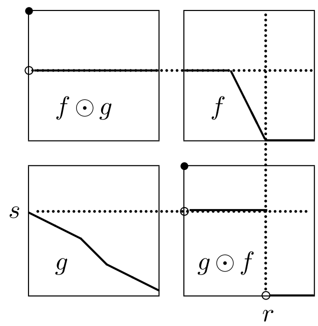
$$T_{f,g} = \{(x,z) \in A \times C : \exists\, y \in \breve{B} : f(x) \geq y \text{ and } g(y) \geq z\}.$$

Proposition 3.1 of [6] shows that the so-defined $g \odot f$ is in fact a Galois map from A to C. This gives a way of composing antitone maps and Galois maps or connections, so that the composed map is again antitone, which almost never would happen with the usual composition of maps. In certain cases, the alternate composition $\odot$ appears somewhat mysterious.

Example 1. If $\mathbb{I}$ denotes the real unit interval $[\,0,1\,]$ with the usual order, the composite $g \odot f$ of $f, g \in Ant(\mathbb{I}, \mathbb{I})$ is always a step function! Explicitly,

$$\begin{aligned} T_{f,g} &= \{(x,z) : \exists\, y > 0\ (f(x) \geq y,\ g(y) \geq z)\} \\ &= \{(x,z) : f(x) > 0,\ \exists\, y > 0\ (g(y) \geq z)\}, \\ E_{f,g} &= \{(a,c) : a \leq r = \bigvee \{x : f(x) > 0\},\ c \leq s = \bigvee \{g(y) : y > 0\}\} \cup \overline{\emptyset}, \end{aligned}$$

where $\overline{\emptyset} = (\{0\} \times \mathbb{I}) \cup (\mathbb{I} \times \{0\})$. Therefore, $g \odot f(0) = 1$, $g \odot f(a) = s$ if $0 < a \leq r$, and $g \odot f(a) = 0$ otherwise.



In particular, the new composition $\odot$ of any two involutions (that is, antitone bijections) of $\mathbb{I}$ yields the constant function 1.

In [6] it is also shown that $\odot$ makes $Gal(B, B)$ a quantale whenever B is a frame (locale). Hence, with any frame B, there is associated not only a quantale of (isotone, i.e. order preserving) residuated maps [2], but also a quantale of (antitone!) Galois maps. One of our main goals is to characterize those complete lattices B for which $Gal(B, B)$ or $Ant(B, B)$, respectively, together with the multiplication $\odot$ becomes a quantale. They are precisely the pseudocomplemented lattices. Surprisingly, that quantale has a unit element only in very special cases, namely, when the closure system is an atomic Boolean algebra (hence isomorphic to a powerset).

In most cases, it will be technically more comfortable to work with tensor products than with $Gal(A, B)$ or $Ant(A, B)$.

2 Tensor products of closure spaces and posets

We start by introducing more general kinds of tensor products for closure spaces. The approach via closure spaces unifies and facilitates the arguments considerably. It allows to extend the theory of tensor products for complete lattices in diverse directions, in particular, from complete lattices to arbitrary posets. The usual trick is here to replace joins with cuts, and then, in a more courageous step, cuts by arbitrary closed sets in closure spaces.

A *tensor* between closure spaces A and B is a subset T of $A \times B$ such that all "slices" xT and Ty are closed, and the tensor product $A \otimes B$ is the closure system of all such tensors. Any *augmented poset* $A_{\mathcal{X}} = (A, \mathcal{X})$ (where A is a poset and $\mathcal{X}$ a collection of subsets of A) may be interpreted as a closure space, by considering the closure system of all $\mathcal{X}$-ideals, i.e. down-sets I containing the cut closure ΔX whenever $X \in \mathcal{X}$ and $X \subseteq I$. Then, the tensor product $A_{\mathcal{X}} \otimes B_{\mathcal{Y}}$ of two augmented posets consist of all *$\mathcal{X}\mathcal{Y}$-ideals* or *$\mathcal{X}\mathcal{Y}$-tensors*, i.e. down-sets T in $A \times B$ such that for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, $X \times Y \subseteq T$ implies $\Delta X \times \Delta Y \subseteq T$ (resp. $(\bigvee X, \bigvee Y) \in T$ if the involved joins exist). If A and B are complete lattices then the tensor product $A_{\mathcal{X}} \otimes B = A_{\mathcal{X}} \otimes B_{\mathcal{P}B}$ is isomorphic to the complete lattice $Ant_{\mathcal{X}}(A, B)$ of maps $f\colon A \to B$ satisfying $f(\bigvee X) = \bigwedge f[X]$ for all $X \in \mathcal{X}$.

These tensor products have the expected universal property with respect to the appropriate bimorphisms. Under mild restrictions, they satisfy the expected (finite and infinite) associative and distributive laws, but the proofs are rather involved. The appropriate ingredient for quantale constructions is here distributivity at the bottom, a generalization of pseudocomplementedness.

3 Truncated tensor products

A considerable simplification is achieved by passing to *truncated tensor products* $A_{\mathcal{X}} \breve{\otimes} B_{\mathcal{Y}}$, cutting off the least tensor from all tensors. Their elements are down-sets in the direct product $\breve{A} \times \breve{B}$ (with $\breve{A} = A \setminus \overline{\emptyset}$ and $\breve{B} = B \setminus \overline{\emptyset}$) such that the conditions in the two coordinates hold for *nonempty* "rectangles". One advantage of that reduction is that the pure tensors $a \otimes b$ have no longer the rather complicated form $\overline{(a,b)} \cup (\overline{\emptyset} \times B) \cup (A \times \overline{\emptyset})$ but become simply point closures, resp. principal ideals. Another, and more important, advantage is that now the quantale constructions are much easier, since the tensor multiplication corresponding to $\odot$ is obtained by forming the (right) tensor closure of the usual relation product, and then the order isomorphism between $A_\ell \otimes_r B$ and $Gal(A,B)$, resp. between $A \otimes_r B$ and $Ant(A,B)$, also transports the multiplication.

The main result will be that for any complete lattice B, the truncated tensor product $B \breve{\otimes} B$, resp. the isomorphic tensor product $B \otimes B \simeq Gal(B,B)$, becomes a quantale iff B is pseudocomplemented, and a unital quantale iff B is an atomic boolean complete lattice.

4 Semicategories with tensor products as hom-sets

Our constructions also provide a *semicategory* (missing identity morphisms) of pseudocomplemented complete lattices together with the (truncated) tensors or antitone maps, respectively, as morphisms. In that semicategory, the atomic boolean complete lattices (isomorphic copies of power set lattices) form the greatest subcategory, and the latter is equivalent to the category of sets and relations as morphisms.

Similar (more general) results are obtained for augmented posets and for closure spaces instead of complete lattices. Crucial is here the observation that a closure system is pseudocomplemented iff all *polars* $x^\perp = \{y : \overline{x} \cap \overline{y} = \overline{\emptyset}\}$ are closed.

By the relevant distributive laws that follow from our results, our semicategories are even enriched in the monoidal category of complete lattices and supremum preserving functions. Hence, they are *semiquantaloids* (as considered by Stubbe [8]).

Notes. In the case of a frame, our quantale constructions via Galois connections or tensor products have important applications to the point-free treatment of uniform structures [4]. The diverse relation products discussed here fit, of course, into the general category-theoretical framework of relations and their multiplication (see [4, Section 2] for details).

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$\top$-$\mathcal{Q}$-filters and their applications

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Abstract. In this paper, we recover the concept of $\top$-filters proposed by Höhle and obtain a concept of $\top$-$\mathcal{Q}$-filters on a $\mathcal{Q}$-category, and a closed relation between $\top$-$\mathcal{Q}$-filters and $\mathcal{Q}$-filters is established. Further, we present two applications of $\top$-$\mathcal{Q}$-filters: (1) we demonstrate that a subclass of $\mathcal{Q}$-topologies, namely strong $\mathcal{Q}$-topologies, can be characterized by crisp systems of its $\top$-$\mathcal{Q}$-neighborhoods firstly; (2) if the underlying quantaloid is $\mathcal{B}(L)$, here L is a compelte Boolean algebra, there exists a kind of crisp systems of $\top$-$\mathcal{B}(L)$-neighborhoods, which is equivalent to strong $\mathcal{B}(L)$-topologies on a discrete $\mathcal{B}(L)$-categories.

1 Motivation and Preliminaries

The axioms of a crisp system of L-valued neighborhoods appears for the first time in Höhle's paper [2], which can characterize L-probabilistic topologies. Note that so called a crisp system of L-valued neighborhood is a $\top$-filter satisfying a additional condition. The closed relationship between $\top$-filters and many valued filters could be found in J. Gutiáerrez García [1] and Höhle [3]. here the later is useful to a nontrivial convergence theory in many valued topological spaces. Recently, in [4], Höhle studied the categorical foundations of topology based ordered monad in one hand and at the same time, the double presheaf monad on the category of $\mathcal{Q}$-categories is introduced. Further, In order to capture traditional axioms of topology, the submonad of $\mathcal{Q}$-filters was established and obtained a result that his neighborhood systems can be characterized by $\mathcal{Q}$-topologies.

Following Höhle's works of [2,4], we find it is possible to recover the concept of $\top$-filters on a $\mathcal{Q}$-category (Cf. Stubbe [8, 9]) in a kind of quantaloid [6] setting and obtain the the concept of $\top$-$\mathcal{Q}$-filters. Based on $\top$-$\mathcal{Q}$-filters, a crisp system of $\top$-$\mathcal{Q}$-neighborhoods could be established and using it, a subclass of $\mathcal{Q}$-topology, namely strong $\mathcal{Q}$-topology, can be characterized by its crisp systems of $\top$-$\mathcal{Q}$-neighborhoods.

At the end of the section, we agree on a quantaloid $\mathcal{Q}$ is integral [9] in the sense that each identity arrow in $\mathcal{Q}$ is the biggest endomorphism $\top_{X,X}$ of hom-set $\mathcal{Q}(X, X)$, and the smallest endomorphism of hom-set $\mathcal{Q}(X, X)$ will be denoted by $\perp_{X,X}$. For a $\mathcal{Q}$-category $\mathbb{A}$, $\mathbb{A}_X$ denote the set of all objects with the type $X \in \mathcal{Q}_0$.

2 $\top$-$\mathcal{Q}$-filters.

In this section, we present the axioms of $\top$-$\mathcal{Q}$-filter and a relationship between $\top$-$\mathcal{Q}$-filters and a kind of $\mathcal{Q}$-filters [4], is showed.

Definition 1. *Let $\mathbb{A}$ be a $\mathcal{Q}$-category, and $\mathbb{F}$ be a subset of $(\mathcal{P}\mathbb{A})_X$ for an $X \in \mathcal{Q}_0$. The pair $(X, \mathbb{F})$, briefly $\mathbb{F}$, is called a $\top$-$\mathcal{Q}$-filter on $\mathbb{A}$ if it satisfies the following axioms:*

(QK) *For each $\lambda \in (\mathcal{P}\mathbb{A})_X$, $\bigvee_{\mu\in\mathbb{F}} \mathcal{P}\mathbb{A}(\mu, \lambda) = \top_{X,X}$ implies $\lambda \in \mathbb{F}$.*
(QM) *For two μ,λ in $(\mathcal{P}\mathbb{A})_X$, both $\mu \in \mathbb{F}$ and $\lambda \in \mathbb{F}$ means $\mu \wedge \lambda \in \mathbb{F}$.*
(QN) *For all $\mu \in \mathbb{F}$, $(\perp_{X,X} \swarrow \bigvee_{x\in\mathbb{A}_X} \mu(x)) = \perp_{X,X}$.*

The set of all $\top$-$\mathcal{Q}$-filters on $\mathbb{A}$ is denoted by $\mathbb{F}_\top(\mathbb{A})$.

Example 1. Let $\mathbb{A}$ be a $\mathcal{Q}$-category and $x \in \mathbb{A}_0$. (1) a pair $(tx, [x])$, given by $[x] := \{\lambda \in (\mathcal{P}\mathbb{A})_{tx} \mid \lambda(x) = \top_{tx,tx}\}$, is a $\top$-$\mathcal{Q}$-filter in the sense of Definition 1.

(2) Let $\mathbb{T}$ be a strong $\mathcal{Q}$-topology (Cf. Zhang [11], Höhle [2], Yue and Fang [10]) on a $\mathcal{Q}$-category $\mathbb{A}$, i.e., $\mathbb{T}$ is a subcategory of $\mathcal{P}\mathbb{A}$ satisfying the following axioms: (QO0) $\perp_X \in \mathbb{T}_X$ for $X \in \mathcal{Q}_0$,(QO1) $\top_X \in \mathbb{T}_X$ for $X \in \mathcal{Q}_0$, (QO2) $\bigvee_{j\in J} \mu_j \in \mathbb{T}_X$ for any family $\{\mu_j| \ j \in J\} \subseteq \mathbb{T}_X$, (QO3) $\mu \wedge \lambda \in \mathbb{T}_X$ for any $\mu, \lambda \in \mathbb{T}_X$, (QOs) $\alpha \circ \mu \in \mathbb{T}_X$ for $\mu \in \mathbb{T}_0$ and $\alpha \in \mathcal{Q}(t\mu, X)$, and (QOss) $\alpha \searrow \mu \in \mathbb{T}_X$ for $\mu \in \mathbb{T}_0$ and $\alpha \in \mathcal{Q}(X, t\mu)$. For an $x \in \mathbb{A}_0$, a pair $(tx,\mathbb{U}^x_\mathbb{T})$ can be given by

$$\mathbb{U}^x_\mathbb{T} := \Big\{\mu \in (\mathcal{P}\mathbb{A})_{tx} \mid \bigvee_{\lambda\in\mathbb{T}_{tx}} \mathcal{P}\mathbb{A}(\lambda, \mu) \circ \lambda(x) = \top_{tx,tx}\Big\}.$$

Then the pair $(tx, \mathbb{U}^x_\mathbb{T})$ is a $\top$-$\mathcal{Q}$-filter on $\mathbb{A}$.

In quantaloid setting, there still exists a closed relation between $\top$-$\mathcal{Q}$-filters and $\mathcal{Q}$-filters, here $\mathcal{F} \in (\mathcal{P}^\dagger\mathcal{P}\mathbb{A})_0$ is said to be a $\mathcal{Q}$-filter [4] if it satisfies the following conditions: (QF0) $\mathcal{F}(\lambda) = \perp_{t\mathcal{F},X}$ for all $X \in \mathcal{Q}_0$, where $\lambda(x) =\perp_{tx,X}$ for all $x \in \mathbb{A}_X$, (QF1) $\mathcal{F}(\top_X) = \top_{t\mathcal{F},X}$ for all $X \in \mathcal{Q}_0$, where $\top_X(x) = \top_{tx,X}$ for all $x \in \mathbb{A}_0$, (QF2) $\mathcal{F}(\mu \wedge \lambda) = \mathcal{F}(\mu) \wedge \mathcal{F}(\lambda)$ for any $\mu, \lambda \in (\mathcal{P}\mathbb{A})_X$, and (QFs) $\alpha \circ \mathcal{F}(\mu) \leq \mathcal{F}(\alpha \circ \mu)$ for all $\alpha \in \mathcal{Q}(t\mu, -)$. The relationship between $\top$-$\mathcal{Q}$-filters and $\mathcal{Q}$-filters is presented in the theorem below, which precisely say that each one could be constructed by the other (For one object $\mathcal{Q}$, see U. Höhle [3], J. Gutiáerrez García [1]).

Theorem 1. *If $\mathcal{F}$ is a $\mathcal{Q}$-filter on a $\mathcal{Q}$-category $\mathbb{A}$, then the pair $(t\mathcal{F}, \mathbb{F}_\mathcal{F})$ given by is a $\top$-$\mathcal{Q}$-filter on $\mathbb{A}$. Conversely, if $\mathcal{Q}$ is a divisible Girard quantaloid [6, 9, 7], then for a $\top$-$\mathcal{Q}$-filter $(X, \mathbb{F})$, a $\mathcal{Q}$-filter $\mathcal{F}_\mathbb{F}$ with the type X on $\mathbb{A}$ can be constructed as follows:*

$$\forall \mu \in (\mathcal{P}\mathbb{A})_0, \quad \mathcal{F}_\mathbb{F}(\mu) := \bigvee_{\lambda\in\mathbb{F}} \mathcal{P}\mathbb{A}(\lambda, \mu).$$

3 $\top$-$\mathcal{Q}$-neighborhood systems determined by strong $\mathcal{Q}$-topologies.

In this section, by using $\top$-$\mathcal{Q}$-filters, we explore that there exists a kind of $\top$-$\mathcal{Q}$-neighborhood systems induced by a strong $\mathcal{Q}$-topology. Further, every strong $\mathcal{Q}$-topology on a $\mathcal{Q}$-category could be recovered by its $\top$-$\mathcal{Q}$-neighborhood systems.

Let $\mathbb{T}$ be a strong $\mathcal{Q}$-topology on a $\mathcal{Q}$-category $\mathbb{A}$. For an $x \in \mathbb{A}_0$, by Example 1 (2), a $\top$-$\mathcal{Q}$-filter $(tx, \mathbb{U}^x_\mathbb{T})$ could be obtained, which will be called a $\top$-$\mathcal{Q}$-neighborhood system at x. In this way, a map $\mathbb{U}_\mathbb{T} \colon \mathbb{A}_0 \to \mathbb{F}_\top(\mathbb{A})$ is given by $\mathbb{U}_\mathbb{T}(x) = (tx, \mathbb{U}^x_\mathbb{T})$, which will be called a $\top$-$\mathcal{Q}$-neighborhood system induced by $\mathbb{T}$. In order to recover a strong $\mathcal{Q}$-topology by its $\top$-$\mathcal{Q}$-neighborhood system, some lemmas below are needed.

Lemma 1. *If $\mu \in \mathbb{U}_{\mathbb{T}}$ is a $\top$-$\mathcal{Q}$-neighborhood at x wrt. a strong $\mathcal{Q}$-topology $\mathbb{T}$ on a $\mathcal{Q}$-category $\mathbb{A}$, then $\mu(x) = \top_{tx,tx}$. Conversely, for an $x \in \mathbb{A}_0$, if μ is in $\mathbb{T}_{tx}$ such that $\mu(x) = \top_{tx,tx}$, then μ is a $\top$-$\mathcal{Q}$-neighborhood at x.*

Lemma 2. *Let $\mathbb{T}$ be a strong $\mathcal{Q}$-topology on a $\mathcal{Q}$-category $\mathbb{A}$. For a $\mu \in (\mathcal{P}\mathbb{A})_0$ and an $x \in \mathbb{A}_0$, $\lambda \in \mathbb{U}^x_{\mathbb{T}}$ means $\mathcal{P}\mathbb{A}(\lambda, \mu) \leq \bigvee_{\sigma \in \mathbb{T}_{tx}} \mathcal{P}\mathbb{A}(\sigma, \mu) \circ \sigma(x)$.*

Lemma 3. *If $\mathbb{U}_{\mathbb{T}}$ is the $\top$-$\mathcal{Q}$-neighborhood system induced by a strong $\mathcal{Q}$-topology on a $\mathcal{Q}$-category $\mathbb{A}$, then $\mathbb{U}_{\mathbb{T}}$ satisfies the following condition:*

(TU) for each $\mu \in \mathbb{U}^x_{\mathbb{T}}$, there exists a $\lambda \in \mathbb{U}^x_{\mathbb{T}}$ such that $\lambda \leq \mu$ and there is $\lambda_y \in \mathbb{U}^y_{\mathbb{T}}$ satisfying $\lambda(y) \leq \mathcal{P}\mathbb{A}(\lambda_y, \mu)$ for any $y \in \mathbb{A}_0$.

Although the following lemma is trivial in classical topology, it presents the rights of the matter in strong $\mathcal{Q}$-topology.

Lemma 4. *Let $\mathbb{T}$ be a strong $\mathcal{Q}$-topology on a $\mathcal{Q}$-category $\mathbb{A}$ and $x \in \mathbb{A}_0$. For every $\sigma \in \mathbb{T}_0$, there is a $\sigma_x \in \mathbb{T}_{tx}$ such that $\sigma_x(x) = \top_{tx,tx}$, and for each $\mu \in (\mathcal{P}\mathbb{A})_0$,*

$$\mathcal{P}\mathbb{A}(\sigma, \mu) \circ \sigma(x) \leq \mathcal{P}\mathbb{A}(\sigma_x, \mu).$$

Now, by using Lemmas 1–4, we present the main result in the section, which say that a strong $\mathcal{Q}$-topology can be recovered by its $\top$-$\mathcal{Q}$-neighborhood system, indeed.

Theorem 2. *Let $\mathbb{A}$ be a $\mathcal{Q}$-category. Then for every strong $\mathcal{Q}$-topology $\mathbb{T}$ on $\mathbb{A}$, $\mu \in \mathbb{T}_0$ if and only if $\mu(y) \leq \bigvee_{\lambda \in \mathbb{U}^y_{\mathbb{T}}} \mathcal{P}\mathbb{A}(\lambda, \mu)$ for all $y \in \mathbb{A}_0$. Further, we have $\mathbb{T} = \mathbb{T}_{\mathbb{U}_{\mathbb{T}}}$.*

4 $\top$-$\mathcal{Q}$-neighborhood systems.

Let L be a complete Boolean algebra, which is a divisible Girard quantale in the sense of $\forall \alpha, \beta \in L,\ \alpha/(\beta\backslash\alpha) = (\beta \vee \alpha) = (\alpha/\beta)\backslash\alpha$. A divisible Girard quantaloid $\mathcal{B}(L)$ [5, 9, 4] can be constructed such that

- The objects of $\mathcal{B}(L)$ is equal to L;
- The hom-set $\mathcal{B}(L)(\alpha, \beta) := \{\gamma \in L \mid \gamma \leq \alpha \wedge \beta\}$ for all objects α, β.

Throughout this section, assume the underlying quantaloid is $\mathcal{B}(L)$, here L is a a complete Boolean algebra and the corresponding $\mathcal{B}(L)$-category $\mathbb{A}$ is discrete.

Definition 2. *Let $\mathbb{U} : \mathbb{A}_0 \to \mathbb{F}_{\top}(\mathbb{A})$ be a map such that $x \mapsto (tx, \mathbb{U}^x)$, denoted by $\mathbb{U} := \{(tx, \mathbb{U}^x)\}_{x \in \mathbb{A}_0}$ briefly. $\mathbb{U}$ is said to be a $\top$-$\mathcal{B}(L)$-neighborhood system if it fulfils the following axioms:*

(TV) *For every $x \in \mathbb{A}_0$, $\mu \in \mathbb{U}^x$ implies $\mu(x) = \top_{tx,tx}$.*

If, in addition, $\mathbb{U}$ fulfils

(TU) *For every $x \in \mathbb{A}_0$, $\mu \in \mathbb{U}^x$ implies there exists a $\lambda \in \mathbb{U}^x$ such that $\lambda \leq \mu$, and for every $y \in \mathbb{A}_0$, there is a $\lambda_y \in \mathbb{U}^y$ with the property of $\lambda(y) \leq \mathcal{P}\mathbb{A}(\lambda_y, \mu)$,* *then we say the $\top$-$\mathcal{B}(L)$-neighborhood system $\mathbb{U}$ is strongly $\mathcal{B}(L)$-topological.*

Let $\mathbb{U}$ be a $\top$-$\mathcal{B}(L)$-neighborhood system on a $\mathcal{B}(L)$-category $\mathbb{A}$. Define

Theorem 3. *Let $\mathbb{U}$ be a $\top$-$\mathcal{B}(L)$-neighborhood system on a $\mathcal{B}(L)$-category $\mathbb{A}$. Then $\mathbb{T}_{\mathbb{U}}$ is a strong $\mathcal{B}(L)$-topology on $\mathbb{A}$.*

In order to confirm if strongly $\mathcal{B}(L)$-topological $\top$-$\mathcal{B}(L)$-neighborhood systems is equivalent to strong $\mathcal{B}(L)$-topologies, the following lemma is needed.

Lemma 5. *If $\mathbb{U}$ is a $\top$-$\mathcal{B}(L)$-neighborhood system on a discrete $\mathcal{B}(L)$-category $\mathbb{A}$, then $\mathbb{U}^x_{\mathbb{T}_{\mathbb{U}}} \subseteq \mathbb{U}^x$ for any $x \in \mathbb{A}_0$.*

Lemma 6. *Let $\mathbb{U}$ be a strongly $\mathcal{Q}$-topological $\top$-$\mathcal{B}(L)$-neighborhood system on a discrete $\mathcal{B}(L)$-category $\mathbb{A}$. Then for any $x \in \mathbb{A}_0$ and $\mu \in \mathbb{U}^x$, there exists $\mathbb{T}_{\mathbb{U}}$-open λ_μ with the type of tx such that $\lambda_\mu \leq \mu$, here the λ_μ could be, concretely, constructed by*

$$\forall y \in \mathbb{A}_0, \quad \lambda_\mu(y) := \bigvee_{\lambda \in \mathbb{U}^y} \mathcal{P}\mathbb{A}(\lambda, \mu).$$

By Theorem 3, Lemmas 5, 6, we conclude that

Theorem 4. *Let $\mathbb{U}$ be a strongly $\mathcal{B}(L)$-topological $\mathcal{B}(L)$-neighborhood system on a discrete $\mathcal{B}(L)$-category $\mathbb{A}$. Then $\mathbb{U} = \mathbb{U}_{\mathbb{T}_{\mathbb{U}}}$ holds.*

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States of free product algebras and their integral representation

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1 Introduction

In his monograph [9], Hájek established theoretical basis for a wide family of fuzzy (thus, many-valued) logics which, since then, has been significantly developed and further generalized, giving rise to a discipline that has been named as Mathematical Fuzzy logic (MFL). Hájek's approach consists in fixing the real unit interval as standard domain to evaluate atomic formulas, while the evaluation of compound sentences only depends on the chosen operation which provides the semantics for the so called *strong conjunction* connective. His general approach to fuzzy logics is grounded on the observation that, if strong conjunction is interpreted by a continuous t-norm [10], then any other connective of a logic has a natural standard interpretation.

Among continuous t-norms, the so called Łukasiewicz, Gödel and product t-norms play a fundamental role. Indeed, Mostert-Shields' Theorem [10] shows that a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum. In other words, a t-norm is continuous if and only if it is an ordinal sum of Łukasiewicz, Gödel and product t-norms. These three operations determine three different algebraizable propositional logics (bringing the same names as their associated t-norms), whose equivalent algebraic semantics are the varieties of MV, Gödel and Product algebras respectively.

Within the setting of MFL, *states* were first introduced by Mundici [11] as maps averaging the truth-value in Łukasiewicz logic. In his work, states are functions mapping any MV-algebra $\mathbf{A}$ in the real unit interval $[0,1]$, satisfying a normalization condition and the additivity law. Such functions suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, besides corresponding to convex combinations of valuations in Łukasiewicz propositional logic. However, states and probability measures were previously studied in [5] (see also [6, 13]) on Łukasiewicz tribes (σ-complete MV-algebras of fuzzy sets) as well as on other t-norm based tribes with continuous operations. MV-algebraic states have been deeply studied in recent years, as they enjoy several important properties and characterizations (see [8] for a survey).

One of the most important results of MV-algebraic state theory is Kroupa-Panti theorem [12, §10], a representation theorem showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Moreover, the correspondence between states and regular Borel probability measures is one-to-one.

Many attempts of defining states in different structures have been made (see for instance [8, §8] for a short survey). In particular, in [2], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that results in corresponding to the integration of the truth value functions induced by Gödel formulas, with respect to Borel probability measures on the real unit cube $[0,1]^n$. Moreover, such states correspond to convex combinations of finitely many truth-value assignments.

The aim of this contribution is to introduce and study states for product logic, the remaining fundamental many-valued logic for which such a notion is still lacking. In particular, our axiomatization will result in characterizing Lebesgue integrals of the functions belonging to the free n-generated product algebra, i.e. the Lindenbaum algebra of product logic over n variables, with respect to Borel probability measures on $[0,1]^n$. In this sense, our states will correctly correspond to finitely additive probability measures in this context, and they will interestingly represent an axiomatization of the Lebesgue integral as an operator acting on product logic formulas. Moreover, and quite surprisingly since in the axiomatization of states the product t-norm operation is only indirectly involved via a condition concerning double negation, we prove that every state belongs to the convex closure of product logic valuations.

2 States of free product algebras and their integral representation

Product algebras are BL-algebras satisfying two further equations:

$$x \wedge \neg x = 0 \text{ and } \neg\neg x \to ((y \cdot x \to z \cdot x) \to (y \to z)) = 1.$$

They constitute a variety that is the equivalent algebraic semantics for Product logic. In what follows, $\mathscr{F}_{\mathbb{P}}(n)$ will denote the free product algebra over n generators. We invite the interested reader to consult [1] and [7] for more details.

The functional representation theorem for free product algebras (cf. [1, Theorem 3.2.5]), shows that, up to isomorphism, every element of $\mathscr{F}_{\mathbb{P}}(n)$ is a *Product logic function*, i.e. $[0,1]$-valued function defined on $[0,1]^n$ associated to a product logic formula built over n propositional variables. These functions are for Product logic the equivalent counterpart of McNaughton functions for Łukasiewicz logic.

Next we introduce the notion of state of $\mathscr{F}_{\mathbb{P}}(n)$.

Definition 1. *A* state of $\mathscr{F}_{\mathbb{P}}(n)$ *is a map* $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$ *satisfying the following conditions:*

S1. $s(1) = 1$ *and* $s(0) = 0$*,*
S2. $s(f \wedge g) + s(f \vee g) = s(f) + s(g)$*,*

S3. *If $f \leq g$, then $s(f) \leq s(g)$,*
S4. *If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$.*

By the previous definition, it follows that states of a free product algebra are lattice valuations (axioms S1–S3) as introduced by Birkhoff in [4]. However, if we compare Definition 1 with states of an MV-algebra, it is evident that, while for the case of MV-algebras the monoidal operation is directly involved in the axiomatization of states, the unique axiom that we impose and that, indirectly, involves the multiplicative connectives of product logic is S4.

Product logic functions in $\mathscr{F}_{\mathbb{P}}(n)$ are not continuous, unlike the case of free MV-algebras, and there are infinitely many, unlike the case for (finitely generated) free Gödel algebras. However, it is always possible to consider a *finite* partition of their domain, which depends on the Boolean skeleton of $\mathscr{F}_{\mathbb{P}}(n)$, upon which the restriction of each product function is continuous. By exploiting this fact, one can show the following integral representation theorem.

Theorem 1 (Integral representation). *For a $[0,1]$-valued map s on $\mathscr{F}_{\mathbb{P}}(n)$, the following are equivalent:*

(i) s is a state,
(ii) there is a unique Borel probability measure $\mu : \mathscr{B}([0,1]^n) \to [0,1]$ such that, for every $f \in \mathscr{F}_{\mathbb{P}}(n)$,

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

3 The state space and its extremal points

In the light of the previous Theorem 1, for n being a natural number, let us introduce the following notation: $\mathscr{S}(n)$ stands for the set of all states of $\mathscr{F}_{\mathbb{P}}(n)$, while $\mathscr{M}(n)$ denotes the set of all regular Borel probability measures on the Borel subsets of $[0,1]^n$. It is quite obvious that $\mathscr{S}(n)$ and $\mathscr{M}(n)$ are convex subsets of $[0,1]^{\mathscr{F}_{\mathbb{P}}(n)}$ and $[0,1]^{2^{[0,1]^n}}$ respectively, whence, by Krein-Milman Theorem they coincide with the convex hull of their extremal points. As for $\mathscr{M}(n)$ it is known that its extremal elements are Dirac measures, i.e., for each $x \in [0,1]^n$, those $\delta_x : 2^{[0,1]^n} \to [0,1]$ such that $\delta_x(B) = 1$ iff $x \in B$ and $\delta_x(C) = 0$ otherwise (see for instance [12, Cor. 10.6]).

Let $\delta : \mathscr{S}(n) \to \mathscr{M}(n)$ be the map that associates to every state its corresponding measure via Theorem 1. Thus, it is easy to prove that δ is bijective and affine. A direct consequence is that the extremal points of $\mathscr{S}(n)$, i.e., *extremal states* are mapped into extremal points of $\mathscr{M}(n)$, i.e., Dirac measures. Now, it is not hard to show that Dirac measures correspond univocally to the homomorphisms of $\mathscr{F}_{\mathbb{P}}(n)$ into $[0,1]$, that is to say, to the valuations of the logic, that hence are exactly the extremal states.

Theorem 2. *The following are equivalent for a state $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$*

1. *s is extremal;*
2. *$\delta(s)$ is a Dirac measure;*
3. *s is a product homomorphism.*

Thus, via Krein-Milman Theorem, we obtain the following:

Corollary 1. *For every $n \in \mathbb{N}$, the state space $\mathscr{S}(n)$ is the convex closure of the set of product homomorphisms from $\mathscr{F}_{\mathbb{P}}(n)$ into* $[0,1]$.

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Recognition, logic and codensity monads

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The notion of recognition by monoids in the theory of automata and regular languages has been shown to be a special case of Stone-Jonsson-Tarski duality for Boolean algebras with additional operations. This realisation makes it possible to consider generalisations of the profinite methods of the theory of automata and regular languages that apply to classes from complexity theory. As most of the complexity classes of interest have been shown to be model classes of various first- and higher-order logic fragments, we want to understand the semantic counterpart of adding a layer of quantifiers.

We concentrate on a family of quantifiers, including the classical existential quantifier as well as modular quantifiers, which are given by the monad determined by a finite commutative semiring and show that, on recognisers, the action of these quantifiers may be described using associated codensity monads.

In this talk we will give an introduction to the subjects concerned and outline recent results obtained in collaboration with Daniela Petrisan and Luca Reggio.

What are dual $\mathfrak{Q}$-preordered sets?

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Let $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ be a unital quantale. A $\mathfrak{Q}$-*preordered set* is a pair (X, p) where is X is a set and $X \times X \xrightarrow{p} \mathfrak{Q}$ is a $\mathfrak{Q}$-*preorder* on X — i.e. a map satisfying the following axioms:

(O1) $e \leq p(x, x)$ for each $x \in X$,
(O2) $p(x_1, x_2) * p(x_2, x_3) \leq p(x_1, x_3)$ for each $x_1, x_2, x_3 \in X$.

Every $\mathfrak{Q}$-preorder p on X induces a preorder $\leq_p$ by $x_1 \leq_p x_2$ iff e$\leq p(x_1, x_2)$. We call $\leq_p$ the *intrinsic preorder of* p. A $\mathfrak{Q}$-preorder p is *antisymmetric* if its underlying preorder $\leq_p$ is antisymmetric.

A $\mathfrak{Q}$-*homomorphism* between two $\mathfrak{Q}$-preordered sets (X, p) and (Y, q) is a map $X \xrightarrow{h} Y$ such that $p(x_1, x_2) \leq q(h(x_1), h(x_2))$ for all $x_1, x_2 \in X$. $\mathfrak{Q}$-preordered sets and $\mathfrak{Q}$-homomorphisms form a category denoted by $\mathtt{Preord}(\mathfrak{Q})$. $\mathfrak{Q}$-preordered sets (X, p) and (Y, q) are isomorphic in the sense of $\mathtt{Preord}(\mathfrak{Q})$ if there exists a bijective $X \xrightarrow{h} Y$ such that both h and h^{-1} are $\mathfrak{Q}$-homomorphisms. Finally, a $\mathfrak{Q}$-homomorphism $(X, p) \xrightarrow{h} (Y, q)$ is *left adjoint* to a $\mathfrak{Q}$-homomorphism $(Y, q) \xrightarrow{k} (X, p)$ if the relation $q(h(x), y) = p(x, k(y))$ holds for all $x \in X$ and $y \in Y$ (see also [3]).

Example 1. On $\mathfrak{Q}$ there exist two different $\mathfrak{Q}$-preorders $\pi_1^{\mathfrak{Q}}$ and $\pi_2^{\mathfrak{Q}}$ determined by the right- and left-implication respectively —i.e. if $\alpha, \beta \in \mathfrak{Q}$, then

$$\pi_1^{\mathfrak{Q}}(\alpha, \beta) = \alpha \searrow \beta = \bigvee\{\gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta\},$$

and

$$\pi_2^{\mathfrak{Q}}(\alpha, \beta) = \alpha \swarrow \beta = \bigvee\{\gamma \in \mathfrak{Q} \mid \gamma * \beta \leq \alpha\}.$$

Since the intrinsic preorder underlying $\pi_1^{\mathfrak{Q}}$ coincides with the order of $\mathfrak{Q}$ and the intrinsic preorder underlying $\pi_2^{\mathfrak{Q}}$ is the *dual order* of $\mathfrak{Q}$, we ask the following

QUESTION (1). How can we define the concept of a dual $\mathfrak{Q}$-preorder p^{op} of a given $\mathfrak{Q}$-preorder p on X in such a way that in the special situation given by the previous example $\big(\mathfrak{Q}, \big(\pi_1^{\mathfrak{Q}}\big)^{op}\big)$ is isomorphic to $(\mathfrak{Q}, \pi_2^{\mathfrak{Q}})$, and reciprocally, $\big(\mathfrak{Q}, \big(\pi_2^{\mathfrak{Q}}\big)^{op}\big)$ is isomorphic to $\big(\mathfrak{Q}, \pi_1^{\mathfrak{Q}}\big)$?

Further, let (X, p) be a $\mathfrak{Q}$-preordered set. A map $X \xrightarrow{f} \mathfrak{Q}$ is called a *contravariant $\mathfrak{Q}$-enriched presheaf on* (X, p) if f is left-extensional — i.e. if $p(x_2, x_1) * f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in X$.

Obviously, f is a contravariant $\mathfrak{Q}$-enriched presheaf iff $(X, p) \xrightarrow{f} \big(\mathfrak{Q}, \pi_2^{\mathfrak{Q}}\big)$ is a $\mathfrak{Q}$-homomorphism.

> QUESTION (2). How can we define the concept of a dual $\mathfrak{Q}$-preorder p^{op} of a given $\mathfrak{Q}$-preorder p on X in such a way that each contravariant $\mathfrak{Q}$-enriched presheaf f can be identified with a $\mathfrak{Q}$-homomorphism $(X, p^{op}) \xrightarrow{\widetilde{f}} (\mathfrak{Q}, \pi_1^{\mathfrak{Q}})$ and vice versa?

We finish this introduction with special case of *commutative* unital quantales. In this setting $\mathfrak{Q}$ plays the role of a *symmetric* monoidal closed category and the concept of a dual $\mathfrak{Q}$-enriched category is already given in Kelly's book 1982 ([3]) which suggests to introduce the dual $\mathfrak{Q}$-preorder $X \times X \xrightarrow{p^{op}} \mathfrak{Q}$ as follows:

$$p^{op}(x_1, x_2) = p(x_2, x_1), \qquad x_1, x_2 \in X. \tag{D}$$

Now it is easily seen that both questions have a positive answer. Hence the previous problems only arise when the underlying quantale is *non-commutative*.

1 Involutive and unital quantales and right $\mathfrak{Q}$-modules

We do not give an ad hoc definition of a dual $\mathfrak{Q}$-preorder here, but motivate the approach by Stubbe's Theorem that the category of right $\mathfrak{Q}$-modules is isomorphic to the category of join-complete $\mathfrak{Q}$-valued lattices ([4]). For this purpose we provide the set $\mathbb{P}(X, p)$ of all contravariant $\mathfrak{Q}$-enriched presheaves on (X, p) with the following $\mathfrak{Q}$-preorder:

$$d(f, g) = \bigwedge_{x \in X} \big(f(x) \searrow g(x)\big), \qquad f, g \in \mathbb{P}(X, p).$$

The $\mathfrak{Q}$-enriched Yoneda embedding $(X, p) \xrightarrow{\eta_{(X,p)}} \big(\mathbb{P}(X, p), d\big)$ is given by:

$$\eta_{(X,p)}(x) = \widetilde{x}, \qquad \widetilde{x}(z) = p(z, x), \qquad x, z \in X,$$

and the relation $f(x) = d(\widetilde{x}, f)$ holds for all $x \in X$ and $f \in \mathbb{P}(X, p)$.

A $\mathfrak{Q}$-preordered set (X, p) is said to be *join-complete* if there exists a $\mathfrak{Q}$-homomorphism $\mathbb{P}(X, p) \xrightarrow{\xi} X$ such that for all $x \in X$ and $f \in \mathbb{P}(X, p)$ the relation

$$p(\xi(f), x) = d(f, \widetilde{x})$$

holds — i.e. ξ is left adjoint to $\eta_{(X,p)}$ and is therefore uniquely determined by $\eta_{(X,p)}$ up to an equivalence. We denote ξ by $\mathsf{sup}_{(X,p)}$. A *join-complete $\mathfrak{Q}$-valued lattice* is a join-complete $\mathfrak{Q}$-valued preordered set (X, p) such that p is antisymmetric.

On the other hand, a right $\mathfrak{Q}$-module is a complete lattice X provided with a right action $\boxdot$ over $\mathfrak{Q}$ in the sense of the category Sup of complete lattices and join preserving maps — i.e. $X \times \mathfrak{Q} \xrightarrow{\boxdot} X$ is a map which is join preserving in each variable separately and satisfies the following axioms for each $x \in X$ and $\alpha, \beta \in \mathfrak{Q}$:

$$x \boxdot e = x \quad \text{and} \quad (x \boxdot \alpha) \boxdot \beta = x \boxdot (\alpha * \beta) \quad \text{where } e \text{ is the unit of } \mathfrak{Q}.$$

Let $(X, \boxdot)$ be a right $\mathfrak{Q}$-module. Then there exists a $\mathfrak{Q}$-valued preorder $p_\boxdot$ on X determined by:

$$p_\boxdot(x_1, x_2) = \bigvee\{\alpha \in \mathfrak{Q} \mid x_1 \boxdot \alpha \le x_2\}, \qquad x_1, x_2 \in X.$$

The intrinsic preorder $\le_{p_\boxdot}$ coincides with the partial order in X, and the map $\mathbb{P}(X, p_\boxdot) \xrightarrow{\mathsf{sup}_{(X,p_\boxdot)}} X$ defined by $\mathsf{sup}_{(X,p_\boxdot)}(f) = \bigvee_{x\in X} x \boxdot f(x)$ is left adjoint to the Yoneda embedding $\eta_{(X,p_\boxdot)}$. Hence $(X, p_\boxdot)$ is a join-complete $\mathfrak{Q}$-valued lattice.

On the other hand, given a join-complete $\mathfrak{Q}$-valued lattice (X, p), then $(X, \le_p)$ is a complete lattice and there exists a unique right action $X \times \mathfrak{Q} \xrightarrow{\boxdot} X$ in the sense of Sup satisfying the following properties for each $x_1, x_2 \in X$ and $f \in \mathfrak{Q}$:

$$p(x_1, x_2) = \bigvee\{\alpha \in \mathfrak{Q} \mid x_1 \boxdot \alpha \le_p x_2\}, \quad \mathsf{sup}_{(X,p)}(f) = \bigvee_{x\in X} x \boxdot f(x).$$

Finally, we assume an order preserving involution $'$ on $\mathfrak{Q}$ such $(\alpha * \beta)' = \beta' * \alpha'$ holds for each $\alpha, \beta \in \mathfrak{Q}$ — i.e. $(\mathfrak{Q}, *, e, ')$ is an involutive and unital quantale. Then the category of right $\mathfrak{Q}$-modules is self-dual (cf. [1]) and the *dual* right $\mathfrak{Q}$-module of $(X, \boxdot)$ is given by the following right action $\boxdot^{op}$ on X^{op}:

$$x \boxdot^{op} \alpha = \bigvee\{z \in X \mid z \boxdot \alpha' \le x\}, \qquad x \in X,\ \alpha \in \mathfrak{Q}.$$

If $p_\boxdot$ is the $\mathfrak{Q}$-preorder associated with $(X, \boxdot)$, then the $\mathfrak{Q}$-preorder $p_{\boxdot^{op}}$ associated with $(X^{op}, \boxdot^{op})$ is given by:

$$\begin{aligned} p_{\boxdot^{op}}(x_1, x_2) &= \bigvee\{\alpha \in \mathfrak{Q} \mid x_1 \boxdot^{op} \alpha \le^{op} x_2\} = \bigvee\{\alpha \in \mathfrak{Q} \mid x_1 \boxdot^{op} \alpha \ge x_2\} \\ &= \bigvee\{\alpha \in \mathfrak{Q} \mid x_2 \boxdot \alpha' \le x_1\} = p_\boxdot(x_2, x_1)'. \end{aligned}$$

The previous formula motivates to define the *dual $\mathfrak{Q}$-preorder* p^{op} for any $\mathfrak{Q}$-preordered set (X, p) in the case of an involutive and unital quantale $(\mathfrak{Q}, *, e, ')$ by

$$p^{op}(x_1, x_2) = p(x_2, x_1)', \qquad x_1, x_2 \in X. \tag{D'}$$

(Note that in any unital and commutative quantale the identity map is an order preserving involution and hence the construction in (D') extends that in (D).)

Now we return to the questions (1) and (2) and make the following observations. Because of $\big(\pi_1^{\mathfrak{Q}}\big)^{op}(\alpha, \beta) = (\beta \searrow \alpha)' = \alpha' \swarrow \beta' = \pi_2^{\mathfrak{Q}}(\alpha', \beta')$ the $\mathfrak{Q}$-preorder $\pi_2^{\mathfrak{Q}}$ is not the dual $\mathfrak{Q}$-preorder of $\pi_1^{\mathfrak{Q}}$, but isomorphic to $\big(\pi_1^{\mathfrak{Q}}\big)^{op}$ and the corresponding $\mathfrak{Q}$-isomorphism is given by the involution $'$. With regard to question (2) it is easily seen that each contravariant $\mathfrak{Q}$-enriched presheaf f can be identified with the $\mathfrak{Q}$-homomorphism $(X, p^{op}) \xrightarrow{f'} (\mathfrak{Q}, \pi_1)$. Hence question (2) has a positive answer provided we enrich the underlying quantale by an involution $'$.

2 Non-involutive quantales and transposed $\mathfrak{Q}^\tau$-preorders

A quantale $(\mathfrak{Q}, *)$ is called *non-involutive* if and only if there does not exist an order preserving involution on $\mathfrak{Q}$ which is also an anti-homomorphism w.r.t. $*$. Hence every non-involutive quantale is non-commutative. Simple examples of non-involutive and unital quantales are already given by the chain with four elements.

In the non-involutive setting the map p^{op} constructed in Section 1 is no longer a $\mathfrak{Q}$-preorder. In order to overcome this obstacle we are changing the underlying quantale and moving from the quantale $(\mathfrak{Q}, *)$ to its *transposed quantale* $\mathfrak{Q}^\tau = (\mathfrak{Q}, *^\tau)$ — i.e. the underlying complete lattice is the same, while its multiplication $*^\tau$ is given by $\alpha *^\tau \beta = \beta * \alpha$ for each $\alpha, \beta \in \mathfrak{Q}$. Instead of p^{op} we now construct the *transposed* $\mathfrak{Q}^\tau$*-preorder* $X \times X \xrightarrow{p^\tau} \mathfrak{Q}^\tau$ of p and put down the following definition (cf. [2]):

$$p^\tau(x_1, x_2) = p(x_2, x_1), \qquad x_1, x_2 \in X. \tag{T}$$

It is important to notice that p^τ is not a $\mathfrak{Q}$-preorder, but a $\mathfrak{Q}^\tau$-preorder. Consequently this does not solve the questions (1) and (2), but we can live with the definition (T) provided we accept to deal with both types of many-valued preorders ($\mathfrak{Q}$-preorders and $\mathfrak{Q}^\tau$-preorders) simultaneously.

If we now replace the quantale $\mathfrak{Q}$ by its transposed quantale $\mathfrak{Q}^\tau$ in the previous *Example*, then we observe that $\pi_1^{\mathfrak{Q}^\tau}$ is the transposed $\mathfrak{Q}^\tau$-preorder of $\pi_2^{\mathfrak{Q}}$ and $\pi_2^{\mathfrak{Q}^\tau}$ is the transposed $\mathfrak{Q}^\tau$-preorder of $\pi_1^{\mathfrak{Q}}$. Further, a map $X \xrightarrow{f} \mathfrak{Q}$ is a contravariant $\mathfrak{Q}$-enriched presheaf on (X, p) if and only if f is a $\mathfrak{Q}^\tau$-homomorphism $(X, p^\tau) \xrightarrow{f} (\mathfrak{Q}^\tau, \pi_1^{\mathfrak{Q}^\tau})$.

Finally, if $(X, \boxdot)$ is a right $\mathfrak{Q}$-module, then there exists a right action $\boxdot^\tau$ on X^{op} in the sense of the transposed quantale $\mathfrak{Q}^\tau$ given by:

$$x \boxdot^\tau \alpha = \bigvee\{z \in X \mid z \boxdot \alpha \le x\}, \qquad \alpha \in \mathfrak{Q},\ x \in X.$$

Because of $z \le x \boxdot^\tau \alpha \iff x \boxdot^\tau \alpha \le^{op} z \iff z \boxdot \alpha \le x$ the relation $\boxdot = (\boxdot^\tau)^\tau$ follows. Hence $(X^{op}, \boxdot^\tau)$ is called the *transposed right* $\mathfrak{Q}^\tau$*-module* of $(X, \boxdot)$.

If $p_\boxdot$ is the associated $\mathfrak{Q}$-preorder of $(X, \boxdot)$, then the associated $\mathfrak{Q}^\tau$-preorder of $(X^{op}, \boxdot^\tau)$ is given by:

$$\begin{aligned} p_{\boxdot^\tau}(x_1, x_2) &= \bigvee\{\alpha \in \mathfrak{Q} \mid x_1 \boxdot^\tau \alpha \le^{op} x_2\} = \bigvee\{\alpha \in \mathfrak{Q} \mid x_1 \boxdot^\tau \alpha \ge x_2\} \\ &= \bigvee\{\alpha \in \mathfrak{Q} \mid x_2 \boxdot \alpha \le x_1\} = p_\boxdot(x_2, x_1) = (p_\boxdot)^\tau(x_1, x_2). \end{aligned}$$

Hence $p_{\boxdot^\tau}$ coincides with the transposed $\mathfrak{Q}^\tau$-preorder of $p_\boxdot$.

We summarize the previous results as follows. Because of the construction of transposed right $\mathfrak{Q}^\tau$-modules, the concept of *transposed* $\mathfrak{Q}^\tau$*-preorders* play a remarkable role in the setting of non-involutive and unital quantales. Obviously, this concept is a substitute for the missing concept of *dual* $\mathfrak{Q}$-preorders.

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Tensors products of complete lattices in many valued topology

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1 Introduction

A few ways have been proposed in the literature to define tensor products for the category of all complete lattices with their join preserving maps (e.g. [1, 9, 12]). One way is to view the tensor product $M \otimes L$ of complete lattices M and L as the family of join reversing maps between M and L. This means in short that the complete lattice $M \otimes L$ (ordered pointwise) is the range of the universal bimorphism from $M \times L$ into $M \otimes L$. We will present this in more detail based on [2, 3].

The principal aim of this talk is to describe two examples of occurrence of tensor products of complete lattices in many-valued topology which were announced or developed in [3–5]. They are concerned with lower semicontinuous lattice-valued maps and with the Hutton's unit interval [8].

2 Lower semicontinuity

Lattice-valued lower semicontinuous maps play an important role in many-valued topology. They are tool to make link between ordinary topology and many-valued topology via omega-functors (as e.g. in [7, 10]).

More specifically, let X be a topological space (with topology $\mathcal{O}(X)$) and let L be an arbitrary complete lattice. A map $f\colon X \to L$ is called lower semicontinuous if

$$f(x) = \bigvee \{ \bigwedge f(U) : U \text{ is an open nbhd of } x \}$$

for each x in X (cf. [7]). The family $\mathrm{Lsc}(X, L)$ of all lower semicontinuous maps from X to L is closed under arbitrary pointwise joins, and if L is, say, meet-continuous, then $\mathrm{Lsc}(X, L)$ is an L-topology on the set X.

For the case in which L is a continuous lattice, we have the following remarkable result:

The complete lattice $\mathrm{Lsc}(X, L)$ *is the tensor product* $\mathcal{O}(X) \otimes L$.

For X a discrete space, $\mathrm{Lsc}(X, L) = L^X$ becomes the tensor product of $\mathcal{P}(X)$ and L with no restriction on L (this result goes back to [9]).

3 The Hutton's interval

Various modifications have been made to the antitone variant of Hutton's L-interval $I(L)$ where $I = [0, 1]$ (cf. [5]). With L a complete chain, Lowen [11] defined $I(L)$ to consist of all antitone and left-continuous maps from I into L. Zhang and Liu [13] went from antitonicity to isotonicity and considered the set $M(L)$ of all join preserving maps between two completely distributive lattices M and L.

Let M be completely distributive and let L be complete. We shall keep at antitonicity by defining $f\colon M \to L$ to be left-continuous if

$$f(t) = \bigwedge\{f(s) : s \lhd t\}$$

for all $t \in M$, where $\lhd$ is the totally below relation (Raney). Since f is left-continuous iff it is join reversing, the tensor product $M \otimes L$ becomes the right generalization of $I(L)$, and $\lhd$ successfully plays the role of the strictly less-than relation in I. For a large class of lattices M (we mean $\lhd$-separability in the sense of [6]) many results on continuous $I(L)$-valued maps will continue to hold in the setting of $(M \otimes L)$-valued maps.

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Enriched perspectives on duality theory

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The principal aim of this talk is to present several perspectives on classical duality theory (for sober spaces, Stone spaces, spectral and Priestley spaces, Esakia spaces, stably compact spaces, ...) from the point of view quantale-enriched category theory. For all of these perspectives, the starting point is the well-known equivalence

$$\mathsf{Pos}^{\mathrm{op}} \simeq \mathsf{TAL} \tag{1}$$

between the dual of the category of Pos of partially ordered sets and monotone maps and the category TAL of totally algebraic lattices and $\sup$- and $\inf$-preserving maps. Here a complete lattice is totally algebraic if it is completely distributive and every element is the supremum of the subset of all totally compact elements totally below it.

To begin, we consider categories enriched in a quantale V (or even a quantaloid) instead of partially ordered sets. Our principal motivation here is Lawvere's groundbreaking paper [17] presenting generalised metric spaces as enriched categories. One amazing insight of [17] is a characterisation of the notion of Cauchy completeness for metric spaces using adjoint distributors, which allows us to speak of Cauchy complete V-categories in general. Furthermore, based on the presentation of *ordered sets via adjunction* [27], the notions of "completely distributive" and "totally algebraic" are brought into the context of quantaloid-enriched categories in [24], and from that one obtains the equivalence

$$\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathsf{V}\text{-}\mathsf{TAL}$$

between the dual category of $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{cc}}$ of Cauchy complete V-categories and V-functors and the category $\mathsf{V}\text{-}\mathsf{TAL}$ of totally algebraic V-categories and $\sup$- and $\inf$-preserving V-functors.

Next we move to the framework of a topological theory $\mathcal{T}$ introduced in [8], consisting of a monad $\mathbb{T}$, a quantale V and a $\mathbb{T}$-algebra structure on V compatible with the monad and the quantale structure. A $\mathcal{T}$-category is defined as a pair (X, a) consisting of a set X and a map

$$a : TX \times X \to \mathsf{V},$$

subject to two axioms which resemble the axioms of a category. We write $\mathcal{T}\text{-}\mathsf{Cat}$ for the category of $\mathcal{T}$-categories and (appropriately defined) $\mathcal{T}$-functors. We note that the notion of $\mathcal{T}$-category embodies ordered, topological, metric and approach structures (for the latter, see [18]); and an extensive presentation of these structures and be found in [13]. Taking now $\mathcal{T}\text{-}\mathsf{Cat}$ instead of Pos in (1), what category should be considered now instead of TAL?

To give one possible answer, we explain the path taken in [14] and define the notion of $\mathcal{T}$-colimit as a particular colimit in a V-category. A complete and cocomplete

V-category in which limits distribute over $\mathcal{T}$-colimits is to be thought of as the generalisation of a (co-)frame to this categorical level. We construct a pair of functors

$$\begin{tikzcd} \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[r, bend right, "\Omega"'] & \mathcal{T}\text{-}\mathsf{Frm} \arrow[l, bend right, "\mathrm{pt}"'] \end{tikzcd}$$

between the dual category of $\mathcal{T}$-Cat and the category $\mathcal{T}$-Frm of $\mathcal{T}$-frames and homomorphisms. Moreover, there is a natural transformation $\mathrm{Id} \to \mathrm{pt} \cdot \Omega$; and the $\mathcal{T}$-categories for which the comparison $X \to \mathrm{pt}(\Omega(X))$ is surjective are precisely those which are *Cauchy complete*. Looking at our examples, we see that

- for V-categories, $\mathcal{T}$-frame means completely distributive V-category;
- for topological spaces, a $\mathcal{T}$-frame is a frame in the usual sense, and a topological space is Cauchy complete if and only if it is sober;
- in the case of approach spaces, our construction relates to the duality between the categories of sober approach spaces and non-expansive maps and the category of spatial approach frames and homomorphisms studied in [2, 25, 26].

A rather different approach was taken in [10] where, inspired by [27, 24], the notion of *completely distributive* $\mathcal{T}$-category is introduced. To explain this idea, we develop enriched category theory for $\mathcal{T}$-categories and introduce distributors, the contravariant presheaf monad, weighted colimit and the Yoneda lemma for $\mathcal{T}$-categories (see [4, 9]). We then construct a dual adjunction between $\mathcal{T}$-Cat and the category of completely distributive $\mathcal{T}$-categories and appropriate $\mathcal{T}$-functors. On the $\mathcal{T}$-categorical side, the fixed objects of this adjunction turn out to be again the Cauchy complete objects.

The second part of this talk is inspired by the fact that the equivalence (1) can be generalised to the category of partially ordered sets and monotone *relations* (see [20], for instance); or, turning this perspective upside down, the duality result for categories of *functions* is a restriction of the one for *relations*. Another example of this type is Halmos's duality theorem [7] which affirms that the category of Stone spaces and Boolean relations (relations which are continuous in an appropriate sense) is dually equivalent to the category of Boolean algebras with "hemimorphisms", that is, maps preserving finite suprema but not necessarily finite infima. Similar results for spectral spaces and Priestley spaces are obtained in [3, 19]. We also observe that the category of Stone spaces and continuous relations can be viewed as the Kleisli category of the Vietoris monad on the category of Stone spaces and continuous maps, a fact which allows us to use the theory of monads in this context.

Furthermore, our leading example (1) as well as the classical Stone-dualities for Boolean algebras and distributive lattices (see [21–23]) are obtained using the two-element space or the two-element lattice. Due to this fact, we can only expect dualities for categories somehow cogenerated by 2 with an appropriate structure. Indeed, a Stone space can be defined as a compact Hausdorff space X where the cone $(f : X \to 2)_f$ is point-separating and initial; and a similar fact holds for Priestley spaces. In order to obtain duality results involving all compact Hausdorff spaces, we need to work with a cogenerator of $\mathsf{CompHaus}$ rather than the 2-element discrete space. Of course, this is exactly what is done in the classical Gelfand duality theorem (see [5]) or in several papers on lattices of the continuous functions (see [15, 16] and [1]) where functions into the unit disc or the unit interval are considered. Keeping in mind that ordered sets (and hence in particular lattices and Boolean algebras) can be identified as categories enriched in the two-element quantale 2, our thesis is *that the passage from the two-element space to the compact Hausdorff space* $[0, \infty]$ *should be matched by a move from ordered structures to metric structures on the other side*. We will point out how some results about lattices of real-valued continuous functions secretly talk about (ultra)metric spaces. However, for technical reasons, in this talk we will consider structures enriched in a quantale based on $[0, 1]$ rather than in $[0, \infty]$.

To obtain enriched versions of Halmos's duality, we restrict ourselves to topological theories based on the ultrafilter monad and quantales with underlying lattice the unit interval $[0, 1]$. Following the path of [12], we

- introduce compact Hausdorff $[0, 1]$-categories which should be seen as an enriched version of Nachbin's partially ordered compact Hausdorff spaces;
- show that the category of compact Hausdorff $[0, 1]$-categories and homomorphisms is equivalent to a certain subcategory of $\mathcal{T}\text{-}\mathsf{Cat}$, similarly to the identification or partially ordered compact Hausdorff spaces with stably compact spaces (see [6]);
- introduce the covariant presheaf monad $\mathbb{V}$ on $\mathcal{T}\text{-}\mathsf{Cat}$ which turns out to be the lower Vietoris monad in the case of topological spaces (see [11]);
- show that this monad restricts to the category of compact Hausdorff $[0, 1]$-categories and homomorphisms;
- obtain, for certain quantale structures on $[0, 1]$, a full embedding

$$\begin{aligned}&(\text{compact Hausdorff } [0,1]\text{-categories})_{\mathbb{V}}\\ &\quad \longrightarrow (\text{“finitely cocomplete” } [0,1]\text{-categories})^{\mathrm{op}};\end{aligned}$$

- discuss how to describe the image of the functor above and its restriction to the category of compact Hausdorff $[0, 1]$-categories and homomorphisms.

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Topology from enrichment: the curious case of partial metrics

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Abstract. For any small quantaloid $\mathcal{Q}$, there is a new quantaloid $\mathcal{D}(\mathcal{Q})$ of diagonals in $\mathcal{Q}$. If $\mathcal{Q}$ is divisible then so is $\mathcal{D}(\mathcal{Q})$ (and vice versa), and then it is particularly interesting to compare categories enriched in $\mathcal{Q}$ with categories enriched in $\mathcal{D}(\mathcal{Q})$. Taking Lawvere's quantale of extended positive real numbers as base quantale, $\mathcal{Q}$-categories are generalised metric spaces, and $\mathcal{D}(\mathcal{Q})$-categories are generalised partial metric spaces, i.e. metric spaces in which self-distance need not be zero and with a suitably modified triangular inequality. We show how every small quantaloid-enriched category has a canonical closure operator on its set of objects: this makes for a functor from quantaloid-enriched categories to closure spaces. Under mild necessary-and-sufficient conditions on the base quantaloid, this functor lands in the category of topological spaces; and an involutive quantaloid is Cauchy-bilateral (a property discovered earlier in the context of distributive laws) if and only if the closure on any enriched category is identical to the closure on its symmetrisation. As this now applies to metric spaces and partial metric spaces alike, we demonstrate how these general categorical constructions produce the "correct" definitions of convergence and Cauchyness of sequences in generalised partial metric spaces. Finally we describe the Cauchy-completion, the Hausdorff contruction and exponentiability of a partial metric space, again by application of general quantaloid-enriched category theory.

1 Introduction

Following Fréchet [5], a metric space (X, d) is a set X together with a real-valued function d on $X \times X$ such that the following axioms hold:

[M0] $d(x, y) \geq 0$,
[M1] $d(x, y) + d(y, z) \geq d(x, z)$,
[M2] $d(x, x) = 0$,
[M3] if $d(x, y) = 0 = d(y, x)$ then $x = y$,
[M4] $d(x, y) = d(y, x)$,
[M5] $d(x, y) \neq +\infty$.

The categorical content of this definition, as first observed by Lawvere [16], is that the extended real interval $[0, \infty]$ underlies a quantale $([0, \infty], \bigwedge, +, 0)$, so that a "generalised metric space" (i.e. a structure as above, minus the axioms M3-M4-M5) is exactly a category enriched in that quantale.

More recently, see e.g. [17], the notion of a partial metric space (X,p) has been proposed to mean a set X together with a real-valued function p on $X \times X$ satisfying the following axioms:

[P0] $p(x,y) \geq 0$,
[P1] $p(x,y) + p(y,z) - p(y,y) \geq p(x,z)$,
[P2] $p(x,y) \geq p(x,x)$,
[P3] if $p(x,y) = p(x,x) = p(y,y) = p(y,x)$ then $x = y$,
[P4] $p(x,y) = p(y,x)$,
[P5] $p(x,y) \neq +\infty$.

The categorical content of *this* definition was discovered in two steps: first, Höhle and Kubiak [13] showed that there is a particular quantaloid of positive real numbers, such that categories enriched in that quantaloid correspond to ("generalised") partial metric spaces; and second, we realised in [20] that Höhle and Kubiak's quantaloid of real numbers is actually a universal construction on Lawvere's quantale of real numbers: namely, the quantaloid $\mathcal{D}[0,\infty]$ of diagonals in $[0,\infty]$.

It was shown in [12] that to any category enriched in a symmetric quantale one can associate a closure operator on its collection of objects. For a metric space (X,d), viewed as an $[0,\infty]$-enriched category, that "categorical closure" on X coincides precisely with the metric (topological) closure defined by d. And Lawvere [16] famously reformulated the Cauchy completeness of a metric space in terms of adjoint distributors. It is however not that complicated to extend the construction of the "categorical closure" to general quantaloid-enriched categories, thus making it applicable to partial metric spaces viewed as $\mathcal{D}[0,\infty]$-enriched categories. And then it is only natural to see if and how Lawvere's arguments for metric spaces go through in the case of partical metrics. This is what we set out to do in this paper—whence its title.

2 Topology from quantaloid-enrichment

Let $\mathcal{Q}$ be a small quantaloid. A functor $F\colon \mathbb{C} \to \mathbb{D}$ between $\mathcal{Q}$-categories is fully faithful when $\mathbb{C}(y,x) = \mathbb{D}(Fy,Fx)$ for every $x \in \mathbb{C}_0$ and $y \in \mathbb{D}$; equivalently, this says that the unit of the adjunction of distributors $F_* \dashv F^*$ (graph and cograph of F) is an equality (instead of a mere inequality). The complementary notion to fully faithfulness will be of importance to us in this section:

Definition 1. *A functor $F\colon \mathbb{C} \to \mathbb{D}$ between $\mathcal{Q}$-categories is* **fully dense** *if the counit of the adjunction of distributors $F_* \dashv F^*$ is an equality (instead of a mere inequality).*

Whenever $\mathbb{C}$ is a $\mathcal{Q}$-category, any two subsets $S \subseteq T \subseteq \mathbb{C}_0$ determine an inclusion of full subcategories $\mathbb{S} \hookrightarrow \mathbb{T} \hookrightarrow \mathbb{C}$. Slightly abusing terminology we shall say that S is fully dense in T whenever the canonical inclusion $\mathbb{S} \hookrightarrow \mathbb{T}$ is fully dense.

Definition 2. *Let $\mathbb{C}$ be a $\mathcal{Q}$-category. The* **categorical closure** *of a subset $S \subseteq \mathbb{C}_0$ is the largest subset $\overline{S} \subseteq \mathbb{C}_0$ in which S is fully dense; that is to say,*

$$\overline{S} = \bigcup\{T \subseteq \mathbb{C}_0 \mid S \textit{ is fully dense in } T\}.$$

Proposition 1. *For every $\mathcal{Q}$-category $\mathbb{C}$, $(\mathbb{C}_0, \overline{(\cdot)})$ is a closure space, and for every functor $F\colon \mathbb{C} \to \mathbb{D}$, $F\colon (\mathbb{C}_0, \overline{(\cdot)}) \to (\mathbb{D}_0, \overline{(\cdot)})$ is a continuous function. This makes for a functor* $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Clos}$.

Suppose now that $\mathcal{Q}$ is an involutive quantaloid (and, as usual, write $f \mapsto f^\circ$ for the involution). When $\mathbb{C}$ is a $\mathcal{Q}$-category and $S \subseteq \mathbb{C}_0$ determines the full subcategory $\mathbb{S} \hookrightarrow \mathbb{C}$, then that same set S also determines a full subcategory $\mathbb{S}_\mathsf{s} \hookrightarrow \mathbb{C}_\mathsf{s}$ of the symmetrisation $\mathbb{C}_\mathsf{s}$ of $\mathbb{C}$. Thus we may compute *two* closures of S: for notational convenience, let us write $\overline{S}$ for its closure in $\mathbb{C}$, and $\widehat{S}$ for its closure in $\mathbb{C}_\mathsf{s}$. It is straightforward that $\widehat{S} \subseteq \overline{S}$, and this inclusion can be strict—but we have that:

Proposition 2. *For an involutive quantaloid $\mathcal{Q}$, the following conditions are equivalent:*

1. *for every $\mathcal{Q}$-category $\mathbb{C}$ and every subset $S \subseteq \mathbb{C}_0$, the closure of S in $\mathbb{C}$ coincides with the closure of S in $\mathbb{C}_\mathsf{s}$,*
2. *$\mathcal{Q}$ is strongly Cauchy bilateral: for every family $(f_i\colon X \to Y_i, g_i\colon Y_i \to X)_{i\in I}$ of morphisms in $\mathcal{Q}$, $1_X \leq \bigvee_i g_i \circ f_i$ implies $1_X \leq \bigvee (g_i \wedge f_i^\circ) \circ (g_i^\circ \wedge f_i)$.*

The final issue we wish to address here in full generality, concerns the topologicity of the closure associated with a $\mathcal{Q}$-category $\mathbb{C}$.

Proposition 3. *For any quantaloid $\mathcal{Q}$, if every identity arrow is finitely join-irreducible*[3] *then the closure associated to any $\mathcal{Q}$-category $\mathbb{C}$ is topological. For any integral quantaloid $\mathcal{Q}$ the converse holds too.*

3 Topology from partial metrics

For Lawvere's quantale $R = ([0,\infty]^{\mathsf{op}}, +, 0)$, and adopting common notations, an R-category $\mathbb{X}$ consists of a set $X = \mathbb{X}_0$ together with a function $d = \mathbb{X}(-,-)\colon X \times X \to [0,\infty]$ such that $d(x,y) + d(y,z) \geq d(x,z)$ and $0 = d(x,x)$. Such an (X,d) is a *generalised metric space* [16]; adding symmetry ($d(x,y) = d(y,x)$), separatedness (if $d(x,y) = 0 = d(y,x)$ then $x = y$) and finiteness ($d(x,y) < \infty$) makes it a metric space in the sense of Fréchet [5]. Upon identifying two R-categories $\mathbb{X}$ and $\mathbb{Y}$ with two generalised metric spaces (X,d_X) and (Y,d_Y), it is straightforward to verify that an R-functor $F\colon \mathbb{X} \to \mathbb{Y}$ can be identified with a 1-Lipschitz function $f\colon X \to Y$, i.e. $d_X(x',x) \geq d_Y(fx', fx)$. We shall write GMet for the category $\mathsf{Cat}(R)$.

On the other hand, a $\mathcal{D}(R)$-category $\mathbb{X}$ is precisely a set $X := \mathbb{X}_0$ together with a function $p := \mathbb{X} : X \times X \to [0,\infty]$ satisfying

$$p(y,x) \geq p(x,x) \vee p(y,y) \text{ and } p(z,y) - p(y,y) + p(y,x) \geq p(z,x).$$

We call such a structure (X,p) a *generalised partial metric space*—indeed, upon imposing *finiteness*, *symmetry* and *separatedness*, we recover exactly the partial metric spaces of [17], whose definition we recalled in the Introduction. A partial functor

[3] We mean here that, for any object X of $\mathcal{Q}$, if $1_X \leq f_1 \vee ... \vee f_n$ ($n \in \mathbb{N}$) then $1_X \leq f_i$ for some $i \in \{1, ..., n\}$. In other words, $1_X \neq 0_X$ and for any $1_X \leq f \vee g$ we have $1_X \leq f$ or $1_X \leq g$.

$f\colon (X,p) \to (Y,q)$ between such spaces is a non-expansive map $f\colon X \to Y : x \mapsto fx$ satisfying furthermore $p(x,x) = q(fx,fx)$; these objects and morphisms thus form the (locally ordered) category PMet.

Taking advantage of the categorical properties of the quantale R, and using appropriate base changes involving $\mathcal{D}(R)$, leads to:

Proposition 4. *The categorical closure on a generalised partial metric space (X,p) is topological, and is identical to the closure on the associated symmetric generalised partial metric space (X,p_{s}) (where $p_{\mathsf{s}}(y,x) = p(y,x) \vee p(x,y)$). Furthermore, the categorical topology on a generalised partial metric space (X,p) is identical to the topology on the generalised metric space (X,p_0) (where $p_0(y,x) := p(y,x) - p(x,x)$). Therefore we can conclude that the categorical topology on a generalised partial metric space is always metrisable by means of a symmetric generalised metric.*

One could consider this a disappointment: there are not more "partially metrisable topologies" then there are metrisable ones. Still, one must realise that it is not always trivial to interpret topological and/or metric phenomena in a given partial metric (X,p) by passing to some metric (X,d) which just happens to define the same topology.

Theorem 1. *In a generalised partial metric space (X,p), equipped with its categorical topology, we have a convergent sequence $(x_n)_n \to x$ if and only if all three limits*

$$\lim_{n\to\infty} p(x,x_n),\ \lim_{n\to\infty} p(x_n,x_n) \textit{ and } \lim_{n\to\infty} p(x_n,x)$$

(exist and) are equal to $p(x,x)$.

We now turn to the study of Cauchy sequences in, and completion of, partial metric spaces (for the categorical topology). With the benefit of hindsight we define:

Definition 3. *A sequence $(x_n)_n$ in a generalised partial metric space (X,p) is* **Cauchy** *if $(p(x_n,x_m))_{(n,m)}$ is a Cauchy net in $[0,\infty]$.*

Theorem 2. *A generalised partial metric space (X,p) is sequentially Cauchy complete (meaning that every Cauchy sequence in (X,p) converges) if and only if (X,p) is categorically Cauchy complete (meaning that every Cauchy distributor on (X,p) qua $\mathcal{D}(R)$-enriched category is representable).*

The above results, stated for *partial* metric spaces, of course apply to metric spaces too; and note that they produce exactly the "usual" results. In the next example we see how also the computation of the Cauchy completion of a partial metric space generalises the usual case:

Example 1. Let (X,p) be a generalised partial metric, and view it as a $\mathcal{D}(R)$-category $\mathbb{X}$. The categorical theory of generalised partial metric spaces tells us that its Cauchy completion $\mathbb{X}_{\mathsf{cc}} = (X,p)_{\mathsf{cc}}$ has as elements the equivalence classes of Cauchy sequences in (X,p) (equivalently, the Cauchy distibutors on $\mathbb{X}$), and the partial distance between two equivalence classes of Cauchy sequences $[(x_n)_n]$ and $[(y_n)_n]$ is

$$p([(x_n)_n],[(y_n)_n]) \stackrel{(1)}{=} \bigwedge_{z\in X} \lim_{n\to\infty} p(x_n,z) - p(z,z) + \lim_{n\to\infty} p(z,y_n) \stackrel{(2)}{=} \lim_{n\to\infty} p(x_n,y_n).$$

In [19] we developed a general theory of 'Hausdorff distance' for quantaloid-enriched categories; applied to the quantaoid $\mathcal{D}(R)$ this produces the following results for partial metrics.

Example 2. The **Hausdorff space** $\mathcal{H}(X,p) = (\mathcal{H}X, p_{\mathcal{H}})$ of a generalised partial metric space (X,p) is the new generalised partial metric space with elements

$$\mathcal{H}X = \{S \subseteq X \mid \forall x, x' \in S \colon p(x,x) = p(x',x')\}$$

(i.e. the **typed subsets** of X) and partial distance

$$p_{\mathcal{H}}(T,S) = \bigvee_{t \in T} \bigwedge_{s \in S} p(t,s). \tag{1}$$

The inclusion $(X,p) \to \mathcal{H}(X,p) \colon x \mapsto \{x\}$ is the unit for the so-called **Hausdorff doctrine** $\mathcal{H} \colon \mathsf{GPMet} \to \mathsf{GPMet}$, and as such enjoys a universal property: it is the universal conical cocompletion (see [19, Section 5]). (Note: the naive extension of the formula in (1) to *arbitrary* subsets of (X,p) fails to produce a partial metric!)

We gave a general characterisation of exponentiable quantaloid-enriched categories and functors in [4]; this specialises to the case of partial metric spaces as follows.

Example 3. A generalised partial metric space (X,p) is **exponentiable** in the (cartesian) category GPMet if and only if

$$\begin{array}{l} \text{for all } x_0, x_2 \in X,\ u,v,w \in [0,\infty[\text{ and } \varepsilon > 0 \\ \text{such that } p(x_0,x_2) \leq u - v + w,\ p(x_0,x_0) \vee v \leq u, \text{ and } p(x_2,x_2) \vee v \leq w \\ \text{there exists } x_1 \in X \text{ such that} \\ \quad p(x_1,x_1) = v,\ p(x_0,x_1) \leq u + \varepsilon, \text{ and } p(x_1,x_2) \leq w + \varepsilon. \end{array} \tag{2}$$

This immediately implies that an exponentiable partial metric space is either empty, or has all distances equal to ∞, or has for every $r \in [0,\infty[$ at least one element with self-distance r. In particular a generalised metric space (X,d) exponentiable in GPMet if and only if it is empty (even though a non-empty (X,d) may still be exponentiable in GMet!). Furthermore, with the same proof as in [10, Theorem 5.3 and Corollary 5.4], we obtain that every injective partial metric space is exponentiable; moreover, the full subcategory of GPMet defined by all injective partial metric spaces is Cartesian closed.

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Fixed points of adjoint $\mathcal{Q}$-functors and their applications

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Abstract. This talk aims to establish general representation theorems for fixed points of adjoint functors between categories enriched in a quantaloid, which set up a common framework for representation theorems of (i) concept lattices in formal concept analysis (FCA) and rough set theory (RST), and (ii) fixed points of Galois correspondences between concrete categories.

1 Fixed points of Galois connections

We start the introduction from the classical case. A Galois connection $s \dashv t$ between posets C, D consists of monotone maps $s\colon C \longrightarrow D$, $t\colon D \longrightarrow C$ such that $s(x) \leq y \iff x \leq t(y)$ for all $x \in C$, $y \in D$. By a fixed point of $s \dashv t$ is meant an element $x \in C$ with $x = ts(x)$ or, equivalently, an element $y \in D$ with $y = st(y)$, since

$$\mathsf{Fix}(ts) := \{x \in C \mid x = ts(x)\} \quad \text{and} \quad \mathsf{Fix}(st) := \{y \in D \mid y = st(y)\}$$

are isomorphic posets with the inherited order from C and D, respectively.

Theorem 1. *Let $s \dashv t\colon D \longrightarrow C$ be a Galois connection between posets. A poset X is isomorphic to $\mathsf{Fix}(ts)$ if, and only if, there exist surjective maps $l\colon C \longrightarrow X$ and $r\colon D \longrightarrow X$ such that*

$$\forall c \in C, \forall d \in D:\ s(c) \leq d \textit{ in } D \iff l(c) \leq r(d) \textit{ in } X.$$

It is well known that if C, D are complete lattices, then so is $\mathsf{Fix}(ts) \cong \mathsf{Fix}(st)$. The above representation theorem can be strengthened to the following one in terms of $\bigvee$-dense and $\bigwedge$-dense maps providing the completeness of C, D:

Theorem 2. *Let $s \dashv t\colon D \longrightarrow C$ be a Galois connection between complete lattices. A complete lattice X is isomorphic to $\mathsf{Fix}(ts)$ if, and only if, there exist $\bigvee$-dense maps $f\colon A \longrightarrow X$, $k\colon A \longrightarrow C$ and $\bigwedge$-dense maps $g\colon B \longrightarrow X$, $h\colon B \longrightarrow D$ such that*

$$\forall a \in A, \forall b \in B:\ sk(a) \leq h(b) \textit{ in } D \iff f(a) \leq g(b) \textit{ in } X.$$

$$\begin{tikzcd}
C \arrow[rr, shift left, "s"] \arrow[rr, phantom, "\perp"] \arrow[dr, "l"'] & & D \arrow[ll, shift left, "t"] \arrow[dl, "r"] \\
& X &
\end{tikzcd}$$

Theorem 1

$$\begin{tikzcd}
& C \arrow[r, shift left, "s"] \arrow[r, phantom, "\perp"] & D \arrow[l, shift left, "t"] & \\
A \arrow[ur, "k"] \arrow[drr, "f"'] & & & B \arrow[ul, "h"'] \arrow[dl, "g"] \\
& & X &
\end{tikzcd}$$

Theorem 2

2 Fixed points of adjoint $\mathcal{Q}$-functors

More generally, given a (possibly large) quantaloid $\mathcal{Q}$, a pair of $\mathcal{Q}$-functors $F\colon \mathbb{A}{\longrightarrow}\mathbb{B}$, $G\colon \mathbb{B}{\longrightarrow}\mathbb{A}$ between (possibly large) $\mathcal{Q}$-categories forms an adjunction $F \dashv G\colon \mathbb{B}{\longrightarrow}\mathbb{A}$ in $\mathcal{Q}$-$\mathbf{CAT}$ if $\mathbb{B}(F-,-) = \mathbb{A}(-,G-)$. For a $\mathcal{Q}$-functor $F\colon \mathbb{A} \longrightarrow \mathbb{A}$, we denote by

$$\mathsf{Fix}(F) := \{x \in \mathbb{A}_0 \mid Fx \cong x\}$$

for the $\mathcal{Q}$-subcategory of $\mathbb{A}$ consisting of fixed points of F.

Theorem 3. *Let $S \dashv T\colon \mathbb{D} \longrightarrow \mathbb{C}$ be a pair of adjoint $\mathcal{Q}$-functors. A $\mathcal{Q}$-category $\mathbb{X}$ is equivalent to $\mathsf{Fix}(TS)$ if, and only if, there exist essentially surjective $\mathcal{Q}$-functors $L\colon \mathbb{C} \longrightarrow \mathbb{X}$ and $R\colon \mathbb{D} \longrightarrow \mathbb{X}$ with $\mathbb{D}(S-,-) = \mathbb{X}(L-,R-)$.*

A $\mathcal{Q}$-category $\mathbb{A}$ is *total* if the Yoneda embedding $\mathsf{Y}_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathsf{P}\mathbb{A}$ admits a left adjoint. The $\mathcal{Q}$-version of Theorem 2 is stated as:

Theorem 4. *Let $S \dashv T\colon \mathbb{D} \longrightarrow \mathbb{C}$ be a pair of adjoint $\mathcal{Q}$-functors between total $\mathcal{Q}$-categories. A total $\mathcal{Q}$-category $\mathbb{X}$ is equivalent to $\mathsf{Fix}(TS)$ if, and only if, there exist dense $\mathcal{Q}$-functors $F\colon \mathbb{A} \longrightarrow \mathbb{X}$, $K\colon \mathbb{A} \longrightarrow \mathbb{C}$ and codense $\mathcal{Q}$-functors $G\colon \mathbb{B} \longrightarrow \mathbb{X}$, $H\colon \mathbb{B} \longrightarrow \mathbb{D}$ with $\mathbb{D}(SK-,H-) = \mathbb{X}(F-,G-)$.*

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, shift left, "S"] \arrow[rr, phantom, "\perp"] \arrow[dr, "L"'] & & \mathbb{D} \arrow[ll, shift left, "T"] \arrow[dl, "R"] \\
& \mathbb{X} &
\end{tikzcd}$$

Theorem 3

$$\begin{tikzcd}
& \mathbb{C} \arrow[rr, shift left, "S"] \arrow[rr, phantom, "\perp"] & & \mathbb{D} \arrow[ll, shift left, "T"] & \\
\mathbb{A} \arrow[ur, "K"] \arrow[drr, "F"'] & & & & \mathbb{B} \arrow[ul, "H"'] \arrow[dll, "G"] \\
& & \mathbb{X} & &
\end{tikzcd}$$

Theorem 4

3 Applications

3.1 Concept lattices in FCA and RST

In this subsection we assume $\mathcal{Q}$ to be a small quantaloid and consider small $\mathcal{Q}$-categories. A small $\mathcal{Q}$-category is usually called *complete* if it is total.

Each $\mathcal{Q}$-distributor $\varphi\colon \mathbb{A} \rightarrow\!\!\!\!\!\circ\, \mathbb{B}$ induces an *Isbell adjunction* $\varphi_{\uparrow} \dashv \varphi^{\downarrow}\colon \mathsf{P}^{\dagger}\mathbb{B}{\longrightarrow}\mathsf{P}\mathbb{A}$ and a *Kan adjunction* $\varphi^{*} \dashv \varphi_{*}\colon \mathsf{P}\mathbb{A}{\longrightarrow}\mathsf{P}\mathbb{B}$, which respectively present the $\mathcal{Q}$-categorical version of the central operators in formal concept analysis (FCA) and rough set theory (RST). Their fixed points, $\mathsf{M}\varphi$ and $\mathsf{K}\varphi$, generalize concept lattices in FCA and RST, respectively.

The following representation theorems in FCA and RST are consequences of Theorems 3 and 4:

Theorem 5. [3] *For any $\mathcal{Q}$-distributor $\varphi\colon \mathbb{A} \relbar\!\!\!\circ\!\!\!\rightarrow \mathbb{B}$, a separated complete $\mathcal{Q}$-category $\mathbb{X}$ is isomorphic to $\mathsf{M}\varphi$ if, and only if, there exist a dense $\mathcal{Q}$-functor $F\colon \mathbb{A} \longrightarrow \mathbb{X}$ and a codense $\mathcal{Q}$-functor $G\colon \mathbb{B} \longrightarrow \mathbb{X}$ with $\varphi = \mathbb{X}(F-, G-)$.*

In particular, when $\mathcal{Q} = \mathbf{2}$, a complete lattice X is isomorphic to $\mathsf{M}\varphi$ if, and only if, there exist a $\bigvee$-dense map $f\colon A \longrightarrow X$ and a $\bigwedge$-dense map $g\colon B \longrightarrow X$ such that

$$\forall a \in A,\ \forall b \in B:\ a\varphi b \iff f(a) \leq g(b) \text{ in } X.$$

Theorem 6. *For any $\mathcal{Q}$-distributor $\varphi\colon \mathbb{A} \relbar\!\!\!\circ\!\!\!\rightarrow \mathbb{B}$, a separated complete $\mathcal{Q}$-category $\mathbb{X}$ is isomorphic to $\mathsf{K}\varphi$ if, and only if, there exist a dense $\mathcal{Q}$-functor $F\colon \mathbb{B} \longrightarrow \mathbb{X}$ and a codense $\mathcal{Q}$-functor $G\colon \overline{\mathbb{A}} \longrightarrow \mathbb{X}$ with $\varphi^{\triangleright} = \mathbb{X}(F-, G-)$, where $\varphi^{\triangleright}$ is the relative pseudo-complement of φ.*

3.2 Fixed points of Galois correspondences

Given a ("base") category $\mathcal{B}$ with small hom-sets, a category $\mathcal{E}$ that comes equipped with a faithful functor $|\text{-}|\colon \mathcal{E} \longrightarrow \mathcal{B}$ is called *concrete* over $\mathcal{B}$. The 2-equivalence [1, 2]

$$\mathbf{CAT} \Downarrow_c \mathcal{B} \simeq \mathcal{Q}_{\mathcal{B}}\text{-}\mathbf{CAT}$$

between concrete categories over $\mathcal{B}$ and categories enriched in the free quantaloid $\mathcal{Q}_{\mathcal{B}}$ generated by $\mathcal{B}$ allow us to exploit representation theorems for fixed points of Galois correspondences through Theorems 3 and 4:

Theorem 7. *Let $S \dashv T\colon \mathcal{E} \longrightarrow \mathcal{D}$ be a Galois correspondence between concrete categories over $\mathcal{B}$. A category $\mathcal{X}$ over $\mathcal{B}$ is concretely equivalent to $\mathsf{Fix}(TS)$ if, and only if, there exist essentially surjective concrete functors $L\colon \mathcal{D} \longrightarrow \mathcal{X}$ and $R\colon \mathcal{E} \longrightarrow \mathcal{X}$ with $\mathcal{E}(SX, Y) = \mathcal{X}(LX, RY)$ for all $X \in \operatorname{ob}\mathcal{D}$, $Y \in \operatorname{ob}\mathcal{E}$.*

Theorem 8. *Let $S \dashv T\colon \mathcal{E} \longrightarrow \mathcal{D}$ be a Galois correspondence between topological categories over $\mathcal{B}$. A topological category $\mathcal{X}$ over $\mathcal{B}$ is concretely equivalent to $\mathsf{Fix}(TS)$ if, and only if, there exist finally dense concrete functors $F\colon \mathcal{D}' \longrightarrow \mathcal{X}$, $K\colon \mathcal{D}' \longrightarrow \mathcal{D}$ and initially dense concrete functors $G\colon \mathcal{E}' \longrightarrow \mathcal{X}$, $H\colon \mathcal{E}' \longrightarrow \mathcal{E}$ with $\mathcal{E}(SKX, HY) = \mathcal{X}(FX, GY)$ for all $X \in \operatorname{ob}\mathcal{D}'$, $Y \in \operatorname{ob}\mathcal{E}'$.*

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Regularity vs. constructive complete (co)distributivity

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Abstract. A relation φ between sets is regular if, and only if, $\mathcal{K}\varphi$ is completely distributive, where $\mathcal{K}\varphi$ is the complete lattice consisting of fixed points of the Kan adjunction induced by φ. For a small quantaloid $\mathcal{Q}$, we investigate the $\mathcal{Q}$-enriched version of this classical result, and prove that (i) the dual of $\mathcal{K}\varphi$ is completely distributive $\Longrightarrow$ (ii) φ is regular $\Longrightarrow$ (iii) $\mathcal{K}\varphi$ is completely distributive for any $\mathcal{Q}$-distributor φ. Although the converse implications do not hold in general, in the case that $\mathcal{Q}$ is a commutative integral quantale, we show that these three statements are equivalent for any φ if, and only if, $\mathcal{Q}$ is a Girard quantale.

A complete lattice A is constructively completely distributive (ccd for short) if $\sup\colon \mathcal{P}A \longrightarrow A$, the monotone map sending each down set of A (here $\mathcal{P}A$ denotes the set of down sets of A, ordered by inclusion) to its supremum, admits a left adjoint in $\mathbf{Ord}$. It is well known that (ccd) and complete distributivity (cd for short), are equivalent if one assumes the axiom of choice [10]. Moreover, as one may describe a (ccd) lattice precisely by the existence of a string of adjunctions

$$T \dashv \sup \dashv \mathsf{y} : A \longrightarrow \mathcal{P}A$$

in $\mathbf{Ord}$, where y is the ($\mathbf{2}$-enriched) Yoneda embedding that sends each $x \in A$ to the principal down set $\downarrow x$, the notion of (ccd) can be extended to any (locally small) category; see [2–4] for discussions of such categories (called *totally distributive* categories there).

The closed relationship between regular relations (i.e., regular arrows in the category $\mathbf{Rel}$ of sets and relations) and (ccd) lattices was first discovered by Zareckiĭ in the case $B = A$ [13] (see also [12] for related discussions), and was extended to arbitrary relations by Xu and Liu [11]. Explicitly, each relation $\phi : A \to\!\!\!\!\!\!\!\;\;\;\; B$ between sets induces a *Kan adjunction* [7]

$$\phi^* \dashv \phi_* : \mathbf{2}^A \longrightarrow \mathbf{2}^B$$

between the powersets of A and B, with

$$\phi^* V = \{x \in A \mid \exists y \in V :\ x\phi y\},$$
$$\phi_* U = \{y \in B \mid \forall x \in A :\ x\phi y \Longrightarrow\ x \in U\}$$

for $V \subseteq B, U \subseteq A$, whose fixed points constitute a complete lattice

$$\mathcal{K}\varphi := \mathbf{Fix}(\phi_*\phi^*) = \{V \subseteq B \mid \phi_*\phi^* V = V\}.$$

A relation $\phi : A \nrightarrow B$ between sets is regular if, and only if, $\mathcal{K}\varphi$ is (ccd).

Since *distributors* [1] (also known as *profunctors* or *bimodules*) generalize relations as functors generalize maps, it is natural to consider the relationship between regularity of distributors and complete distributivity in the framework of category theory, with distributors in lieu of relations. The aim of this paper is to investigate this problem in a special case, i.e., for distributors between categories enriched in a small quantaloid $\mathcal{Q}$ [6, 8].

For a small quantaloid $\mathcal{Q}$, a $\mathcal{Q}$-distributor $\phi : \mathbb{A} \nrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories may be thought of as a multi-typed and multi-valued relation that respects $\mathcal{Q}$-categorical structures in its domain and codomain, and *regular* $\mathcal{Q}$-distributors are precisely regular arrows in the category $\mathcal{Q}$-$\mathbf{Dist}$ of $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors. Each $\phi : \mathbb{A} \nrightarrow \mathbb{B}$ induces a Kan adjunction [7]

$$\phi^* \dashv \phi_* : \mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B},$$

with $\phi^*(\lambda) = \lambda \circ \phi$ and $\phi_*(\mu) = \mu \swarrow \phi$ for all $\mu \in \mathcal{P}\mathbb{A}, \lambda \in \mathcal{P}\mathbb{B}$. The fixed points constitute a complete $\mathcal{Q}$-category

$$\mathcal{K}\varphi = \{\lambda \in \mathcal{P}\mathbb{B} | \phi_*\phi^*(\lambda) = \lambda\}.$$

Furthermore, A $\mathcal{Q}$-category $\mathbb{A}$ is *(ccd)* if one has a string of adjoint $\mathcal{Q}$-functors

$$T \dashv \sup \dashv \mathrm{y} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A},$$

where y is the ($\mathcal{Q}$-enriched) Yoneda embedding. Dually, $\mathbb{A}$ is ${}^{op}ccd$ if $\mathbb{A}^{op}$ is a (ccd) $\mathcal{Q}^{op}$-category.

It is shown that $\mathcal{K}\varphi$ is (ccd) if ϕ is a regular $\mathcal{Q}$-distributor as observed by Stubbe [9], but unfortunately, the converse statement is not true. In fact, the regularity of ϕ necessarily follows if $\mathcal{K}\varphi$ is ^{op}ccd! Hence, the chain of logic is essentially as follows:

$$\mathcal{K}\varphi \text{ is } {}^{op}\text{ccd} \implies \phi \text{ is regular} \implies \mathcal{K}\varphi \text{ is (ccd)}.$$

Moreover, when $\mathcal{Q}$ is a *Girard* quantaloid [5], it does hold that

$$\mathcal{K}\varphi \text{ is } {}^{op}\text{ccd} \iff \phi \text{ is regular} \iff \mathcal{K}\varphi \text{ is (ccd)}. \tag{1}$$

In particular, since **2** is a Girard quantale (i.e., a one-object Girard quantaloid), it recovers that ϕ is a regular relation $\iff$ $\mathcal{K}\varphi$ is completely distributive.

Finally, we wish to find the minimal requirement for $\mathcal{Q}$ to establish the equivalences (1). when $\mathcal{Q}$ is assumed to be a commutative integral quantale, we show that the equivalences (1) hold for any ϕ enriched in such $\mathcal{Q}$ if, and only if, $\mathcal{Q}$ is a Girard quantale.

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Continuous metric spaces and injective approach spaces

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In 1972, Dana Scott proved a fundamental result in domain theory: continuous lattices are precisely the specialization orders of injective T_0 topological spaces. Given a partially ordered set $(X, \leq)$, let $\Sigma(X, \leq)$ be the T_0 topological space obtained by endowing X with the Scott topology of $(X, \leq)$. Then Σ is functorial from the category of partially ordered sets and Scott continuous maps to the category of topological spaces. For each T_0 topological space $(X, \mathcal{O})$, X together with the specialization order of $(X, \mathcal{O})$ becomes a partially ordered set, denoted by $\Omega(X, \mathcal{O})$. The result of Scott says that if $(X, \leq)$ is a continuous lattice, then $\Sigma(X, \leq)$ is an injective T_0 topological space and $(X, \leq) = \Omega\Sigma(X, \leq)$; conversely, if $(X, \mathcal{O})$ is an injective T_0 topological space, then $\Omega(X, \mathcal{O})$ is a continuous lattice and $(X, \mathcal{O}) = \Sigma\Omega(X, \mathcal{O})$. Thus, the specialization-order functor Ω establishes an isomorphism between the category of injective T_0 topological spaces and that of continuous lattices and Scott continuous maps, with Σ being the inverse. This result reveals a deep connection between order-theoretic properties and topological properties.

Following Lawvere, metric spaces (not necessarily symmetric and finitary) are $[0, \infty]$-enriched ordered sets. In 1989, R. Lowen created approach spaces. As observed by D. Hofmann, approach spaces are $[0, \infty]$-enriched topological spaces. So, it is natural to ask whether there is a $[0, \infty]$-enriched version of the result of Scott, i.e., whether the category of "$[0, \infty]$-enriched continuous lattices" is isomorphic to the category of injective T_0 approach spaces?

In order to answer this question, we have to make clear what is a "$[0, \infty]$-enriched continuous lattice" and what is an injective T_0 approach space first. Since the notion of injective T_0 approach spaces is indisputable (as in any concrete categories), it remains to postulate "$[0, \infty]$-enriched continuous lattices". Our idea is to treat flat weights of metric spaces as $[0, \infty]$-version of ideals in ordered sets, then define continuous metric spaces in a "standard way". This definition of continuous metric spaces is well in accordance with that of Yoneda completeness of metric spaces — a $[0, \infty]$-version of directed completeness. Continuous metric spaces can be viewed as $[0, \infty]$-enriched domains, or, metric domains. "$[0, \infty]$-enriched continuous lattices" are then defined to be the complete and continuous separated metric spaces.

Next, the notion of Scott distance of metric spaces is introduced, making a metric space (X, d) into an approach space, denoted by $\Sigma(X, d)$. Scott distance of a metric space is a $[0, \infty]$-version of Scott topology on ordered sets.

It is proved that every injective T_0 approach space is a complete and continuous separated metric space endowed with the Scott distance. Precisely, if (X, δ) is an injective T_0 approach space, then $\Omega(X, \delta)$, the specialization metric space of (X, δ), is a complete and continuous separated metric space and $(X,\delta) = \Sigma\Omega(X,\delta)$. But, there is a complete and continuous separated metric space (X,d) such that $\Sigma(X,d)$ is not an injective approach space. So, the isomorphism between the categories continuous lattices and injective T_0 topological spaces is not valid in the $[0,\infty]$-valued setting.

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On the tensor product of Grothendieck k-linear categories

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Abstract. Let k be a commutative ring. We define a tensor product of k-linear Grothendieck sites, and a resulting tensor product of k-linear Grothendieck categories based upon their representations as categories of k-linear sheaves. As an example of our construction, we describe the tensor product of non-commutative projective schemes in terms of $\mathbb{Z}$-algebras, and show that for projective schemes our tensor product corresponds to the usual product scheme. In addition, we show that our tensor product is a special case of the tensor product of locally presentable k-linear categories. We also prove that the tensor product of locally coherent Grothendieck categories is again locally coherent if and only if the Deligne tensor product of their abelian categories of finitely presented objects exists.

1 Definition of the tensor product

Consider k a fixed commutative ring. Our work sits in the context of $\mathsf{Mod}(k)$-enriched categories or, as they are usually called, k-linear categories. All the concepts mentioned below are $\mathsf{Mod}(k)$-enriched, so, for the shake of brevity, it may not appear specified every time.

A Grothendieck category is a cocomplete abelian category with a generator and exact filtered colimits. Grothendieck categories play an important role in non-commutative algebraic geometry, where they are used as models for non-commutative spaces ([3], [4], [31]). Our initial motivation comes from algebraic geometry, where one of the most basic operations to be performed with schemes X and Y is taking their product scheme $X \times Y$. For affine schemes $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, this corresponds to taking the tensor product $A \otimes B$ of the underlying rings. Our aim is to define a tensor product $\mathcal{C} \boxtimes \mathcal{D}$ for arbitrary Grothendieck categories $\mathcal{C}$ and $\mathcal{D}$, such that for the particular case of rings A and B we have

$$\mathsf{Mod}(A) \boxtimes \mathsf{Mod}(B) = \mathsf{Mod}(A \otimes B). \tag{1}$$

As seen in [23], based on the original Gabriël-Popescu theorem [29], one can always realise any k-linear Grothendieck category $\mathcal{C}$ as a category of k-linear sheaves on a small k-linear category $\mathfrak{a}$ with respect to a certain topology $\mathcal{T}_{\mathfrak{a}}$ on $\mathfrak{a}$, i.e. $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$, or in simpler words, we can realise $\mathcal{C}$ as a "k-linear Grothendieck topos". In addition, using this language of (k-linear) topologies on k-linear categories $\mathfrak{a}$, a characterization of the k-linear functors $\mathfrak{a} \longrightarrow \mathcal{C}$ which induce an equivalence $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \subseteq \mathsf{Mod}(\mathfrak{a})$ can be found in [23].

Our approach to the definition of a tensor product of Grothendieck categories consists of the following steps:

(i) First, we define a suitable tensor product of k-linear sites $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ en $(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ to be $(\mathfrak{a}\otimes\mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$ for a certain tensor product topology $\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}$ on the standard tensor product of k-linear categories $\mathfrak{a} \otimes \mathfrak{b}$.

(ii) Next, we show that the definition

$$\mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \boxtimes \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) = \mathsf{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) \tag{2}$$

is a good definition for Grothendieck categories, as it is independent of the particular sites chosen in the sheaf category representations (up to equivalence of categories).

2 Geometrical example

We apply our tensor product to $\mathbb{Z}$-algebras and schemes. In [11], $\mathbb{Z}$-algebras $\mathfrak{a}$ are endowed with a certain *tails topology* $\mathcal{T}_{\mathrm{tails}}$ and the category $\mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathrm{tails}})$ is proposed as a replacement for the category of quasicoherent modules, which exists in complete generality. When applied to projective schemes X and Y, by looking at the $\mathbb{Z}$-algebras associated to defining graded algebras which are generated in degree 1, we obtain the following formula:

$$\mathsf{Qch}(X) \boxtimes \mathsf{Qch}(Y) = \mathsf{Qch}(X \times Y). \tag{3}$$

Formula (3) is expected to hold in greater generality, at least for schemes and suitable stacks.

3 Relation with other categorical tensor products

We also discuss the relation of our tensor product with other tensor products of categories in the literature. In [7], [9], [10], a tensor product of locally presentable categories is studied, based upon [2]. It is well known that Grothendieck categories are locally presentable. For locally α-presentable Grothendieck categories, we use canonical sheaf representations in terms of the sites of α-presentable objects in order to calculate our tensor product, and we show that it coincides with the tensor product as locally presentable categories. In particular, the tensor product is again locally α-presentable. As a special case, we observe that locally finitely presentable Grothendieck categories are preserved under tensor product. In contrast, the stronger property of local coherence, which imposes the category of finitely presented objects to be abelian, is *not* preserved under tensor product, as is already seen for rings. Hence, one can view our tensor product of Grothendieck categories as a solution, within the framework of abelian categories, to the non-existence, in general, of the Deligne tensor product of small abelian categories (see [21]).

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A representation theorem for Q-sup-algebras

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Abstract. Based on the notion of Q-sup-lattices (a fuzzy counterpart of complete join-semilattices valuated in a commutative quantale), we present the concept of Q-sup-algebras – Q-sup-lattices endowed with a collection of finitary operations compatible with the fuzzy joins. Similarly to the crisp case investigated in [10], we present basic characteristics of their subalgebras and quotients, and following Solovyov, we show a representation theorem for Q-sup-algebras.

1 Introduction

The topic of sets with fuzzy order relations valuated in complete lattices with additional structure has been quite active in the recent decade, and a number of papers have been published.

Based on a quantale-valued order relation and subset membership, counterparts to common order-theoretic notions can be defined, like monotone mappings, adjunctions, joins and meets, complete lattices, or join-preserving mappings, and one can consider a category formed from the latter two concepts. An attempt for systematic study of such categories of fuzzy complete lattices with quantale valuation ("Q-sup-lattices") with fuzzy join-preserving mappings has been made by the second author in his recent paper [5].

With some theory of Q-sup-lattices available, new concepts of algebraic structures in this category can easily be built. In this paper, we shall deal with general algebras with finitary operations, building on existing results obtained for algebras based on crisp sup-lattices ('sup algebras' as in [10]). We shall see that our fuzzy structures behave in strong analogy to their crisp counterparts.

We also highlight an important fact: that concepts based on a fuzzy order relation (in the sense of the quantale valuation as studied in this work) should not be treated as generalizations of their crisp variants – they are rather standard crisp concepts of order theory, satisfying certain additional properties. This fact also reduces the work needed to carry out proofs. Thus, even with the additional properties imposed, the theory of fuzzy-ordered structures develops consistently with its crisp counterpart.

The connection between fuzzy and crisp order concepts has also been justified by I. Stubbe in a general categorial setting of modules over quantaloids [9], and in the recent work of S. A. Solovyov in the quantale-fuzzy setting [8] where categories of quantale-valued sup-lattices are proved to be isomorphic to well-investigated categories of quantale modules. This isomorphism allows to directly transfer some of the fundamental constructions and properties known for quantale modules, to our framework.

With this paper we hope to contribute to the theory of quantales and quantale-like structures. It considers the notion of Q-sup-algebra and shows a representation theorem for such structures generalizing the well-known representation theorems for quantales and sup-algebras. In addition, we present some important properties of the category of Q-sup-algebras.

2 Q-sup-lattices

Let a unital commutative quantale Q be fixed from now on, with the multiplicative unit denoted 1 (which is not required to be the greatest element).

Definition 1. *Let X be a set. A mapping $e\colon X \times X \to Q$ is called a Q-*order *if for any $x, y, z \in X$ the following are satisfied:*

1. *$e(x, x) \geq 1$ (reflexivity),*
2. *$e(x, y) \cdot e(y, z) \leq e(x, z)$ (transitivity),*
3. *if $e(x, y) \geq 1$ and $e(y, x) \geq 1$, then $x = y$ (antisymmetry).*

*The pair (X, e) is then called a Q-*ordered set.

Given Q-ordered sets (X, e_X), (Y, e_Y), a mapping $f\colon X \to Y$ is called *Q-monotone* if $e_X(x, y) \leq e_Y(f(x), f(y))$ for any $x, y \in X$. A mapping $f\colon X \to X$ is *Q-inflating* (Q-deflating) if $1 \leq e_X(x, f(x))$ ($1 \leq e_X(f(x), x)$). An idempotent Q-monotone, Q-inflating mapping is called a *Q-order nucleus* (a *Q-order conucleus* if Q-deflating).

A *Q-subset* of a set X is an element of the set Q^X. Let M be a Q-subset of a Q-ordered set (X, e). An element s of X is called a *Q-join* of M, denoted $\bigsqcup M$ if:

1. $M(x) \leq e(x, s)$ for all $x \in X$, and
2. for all $y \in X$, $\bigwedge_{x \in X}(M(x) \to e(x, y)) \leq e(s, y)$.

If $\bigsqcup M$ exists for any $M \in Q^X$, we call (X, e) *Q-join complete*, or a *Q-sup-lattice*.

Let X and Y be sets, and $f\colon X \to Y$ be a mapping. *Zadeh's forward power set operator* for f maps Q-subsets of X to Q-subsets of Y by

$$f_Q^{\rightarrow}(M)(y) = \bigvee_{x \in f^{-1}(y)} M(x).$$

Let (X, e_X) and (Y, e_Y) be Q-ordered sets. We say that a mapping $f\colon X \to Y$ is *Q-join-preserving* if for any Q-subset M of X such that $\bigsqcup M$ exists, $\bigsqcup_Y f_Q^{\rightarrow}(M)$ exists and

$$f\left(\bigsqcup_X M\right) = \bigsqcup_Y f_Q^{\rightarrow}(M).$$

It is well know that the category Q-Sup of Q-sup-lattices and Q-join-preserving mappings is isomorphic to the category Q-Mod of right Q-modules (see [8, 9]) - a result which also holds when Q is replaced by an arbitrary quantaloid [9].

From any Q-order on a set X, an induced partial order relation can be defined by $x \leq y \iff e(x, y) \geq 1$. This induced partial order leads to an important characteristics of many of the Q-fuzzy concepts (posets, joins and meets, monotone maps etc.): rather than a generalization of ordinary notions of order theory, they should be regarded as specific instances of them, satisfying additional properties.

3 Q-sup-algebras

In what follows we need the notion of a Q-sup-algebra from [6] (see for motivation [4, 10]).

Definition 2. *A* Q-sup-algebra of type Ω *(shortly, a* Q-sup-algebra*) is a triple* $\mathcal{A} = (A, \bigsqcup, \Omega)$ *where* $(A, \bigsqcup)$ *is a* Q*-sup-lattice,* (A, Ω) *is an algebra of type* Ω*, and each operation* ω *is* Q*-join-preserving in any component, that is,*

$$\omega\left(a_1, \ldots, a_{j-1}, \bigsqcup M, a_{j+1}, \ldots, a_n\right) = \bigsqcup \omega(a_1, \ldots, a_{j-1}, -, a_{j+1}, \ldots, a_n)^{\rightarrow}_{Q}(M)$$

for any $n \in \mathbb{N}$*,* $\omega \in \Omega_n$ *(the subset of operations of arity* n*),* $j \in \{1, \ldots, n\}$*,* $a_1, \ldots, a_n \in A$*, and* $M \in Q^A$*.*

The induced partial order relation $\leq_e$ makes every Q-sup-algebra into a sup-algebra in the sense of [4, 10]. Like with e.g. quantales, or sup-algebras in general, quotients and subalgebras of Q-sup-algebras can be characterized by means of Q-order nuclei and conuclei acting on the carrier Q-sup-lattice that are also *subhomomorphisms* of the induced sup-algebras, i.e. mappings $f\colon A \to A$ satisfying $\omega(f(a_1), \ldots, f(a_n)) \leq_e f(\omega_A(a_1, \ldots, a_n))$ for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A$, and $\omega \leq_e f(\omega)$ for any $\omega \in \Omega_0$.

Proposition 1. [6]

1. *Let* $(A, \bigsqcup_A, \Omega)$ *and* $(B, \bigsqcup_B, \Omega)$ *be* Q*-sup-algebras, and let* $f\colon A \to B$ *be a surjective homomorphism. Then there exists a nucleus* j *on* A *such that* $B \cong A_j$*.*
2. *Let* $(A, \bigsqcup_A, \Omega)$ *and* $(B, \bigsqcup_B, \Omega)$ *be* Q*-sup-algebras, and let* $f\colon A \to B$ *be an injective homomorphism. Then there exists a conucleus* g *on*B *such that* $A \cong B_g$*.*

For any Q-sup-algebra A it can be shown that the set Q^A of all its Q-subsets is also a Q-sup-algebra. An analogy of the representation theorem for sup-algebras [10] and quantale algebras [7] can then be presented:

Theorem 1. *If* $(A, \bigsqcup_A, \Omega)$ *is a* Q*-sup-algebra, then there is a nucleus* j *on* Q^A *such that* $A \cong Q^A_j$*.*

Similarly as in [7] we also have:

Theorem 2. *The category of* Q*-sup-algebras of type* Ω *is a monadic construct.*

Corollary 1. *The category of* Q*-sup-algebras of type* Ω *is complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations. Moreover, monomorphisms are precisely those morphisms that are injective functions.*

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Sheaves on coverable groupoids

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In [1] it has been seen that for well behaved open groupoids (*coverable groupoids*), for instance Lie groupoids, there is a strong form of embedding of the quantale of a groupoid into the quantale of an étale groupoid that covers it (*quantale pairs*). In this talk I will show that an appropriate notion of action for such pairs yields an equivalence of categories G-*Loc* $\cong (Q, \mathcal{O})$-*Loc* where $\mathcal{O} = \mathcal{O}(G)$ is the quantale of a coverable open groupoid G. Moreover, by applying the latter equivalence I will provide an extension of the theory of quantale sheaves in such a way that coverable groupoid sheaves can be identified with quantale sheaves as in the étale case [2].

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* Joint work with Pedro Resende

Groupoid quantales

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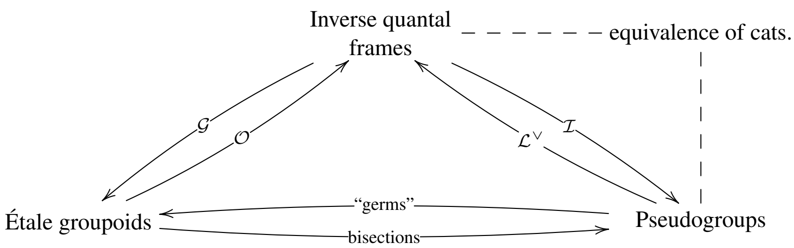
Quantales generalize locales and relate fruitfully to other mathematical subjects such as C*-algebras, groupoids, inverse semigroups, toposes, and logic. The purpose of this talk is to give an overview of what is currently known about their relation to groupoids and inverse semigroups, and to hint at some applications. This has two parts. The first concerns (localic) étale groupoids, which correspond, on one hand, to the class of quantales known as inverse quantal frames, and, on the other, to pseudogroups [9]:



The only correspondence which is an actual equivalence of categories is that between inverse quantal frames and pseudogroups, but in fact these three kinds of structures form bicategories and the arrows in the above diagram describe biequivalences [10]. The bicategorical structure is, for groupoids, based on biactions, and, for quantales, on bimodules. Based on the biequivalence between étale groupoids and inverse quantal frames one sees that the usual notion of homomorphism of inverse quantal frames corresponds to the so-called algebraic morphisms of groupoids, as in [1]. Groupoid functors do not in general yield homomorphisms of quantales (unless they are covering functors, see [5]), but rather lead to a bimodule based construction that corresponds to Hilsum–Skandalis maps of groupoids [3, 7]. In order to understand this, one needs the correspondence between groupoid sheaves and quantale sheaves, as in [12], and a quantale module description of principal bundles. The latter, along with a formulation in terms of left adjoint one-cells in the bicategories, is joint work with J.P. Quijano [11].

Despite the importance of étale groupoids, there are situations where more general groupoids, and more general quantales, are needed. For instance, in topos theory the general representation of Grothendieck toposes [4, 6] requires étale-complete groupoids — or Grothendieck quantales [2], which however are not directly related to étale-complete groupoids in the same way that étale groupoids and inverse quantal frames are. Étale-complete groupoids are still not general enough for some purposes, either, since in particular simply connected Lie groups are not étale-complete, but in order to cater for a whole realm of applications in differential geometry and physics one needs general Lie groupoids. The second part of my talk will describe the beginnings of a relation between quantales and groupoids of a general kind that encompasses Lie groupoids. Such groupoids, called coverable groupoids because they are equipped with certain coverings by étale groupoids, correspond to pairs of quantales $(Q, \mathcal{O})$ where Q is an inverse quantal frame and $\mathcal{O}$ is a suitable subquantale and ideal in Q [8]. The development of the functorial correspondence between coverable groupoids and such quantale pairs requires looking at actions, sheaves and principal bundles for such structures, and is the subject of joint work with J.P. Quijano [13].

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Statistical metrics for statistical learning

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Karl Menger [3] and Henri Poincaré introduced concept of probability in geometry by analyzing the issue of non-transitivity. Poincaré expressed this law as "the raw result of experience", i.e. physical equality is not a transitive relation. More formally, in 1942 Karl Menger generalised the concept of metric space to that of statistical metric space by generalising the notion of distance from that of a non negative real number to that of a distribution function. In Mengers notation, $F(x;p,q)$ is the probability that the distance of p and q is less than x. [4] discussed the topology of nervous nets, introduced the concept of heterarchy and non-transitivity. This no-transitivity is very useful in statistical learning, we can mention non-transitivity of correlation coefficient (see [1]).

In this talk I will address non-transitivity in statistics and causal dependence. According to [2], causal dependence between actual events is sufficient for causation, but not necessary: it is possible to have causation without causal dependence. Causation is transitive, however, causal dependence is not. Several important applications of non-transitivity to statistical learning will be given, some of them in relation to genetics [5].

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L^M-valued equalities, L^M-rough approximation operators and LM-fuzzy ditopologies

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Abstract. We introduce a certain many-valued generalization of the concept of an L-valued equality called an L^M-valued equality. Properties of L^M-valued equalities are studied and a construction of an L^M-valued equality from a pseudometric is presented. L^M-valued equalities are applied to introduce upper and lower L^M-rough approximation operators, which are essentially many-valued generalizations of Z. Pawlak's rough approximation operators [8] and of their fuzzy counterparts. We study properties of these operators and their mutual interrelations. In its turn, L^M-rough approximation operators are used to induce topological-type structures, called here LM-fuzzy ditopologies.

1 Prerequisites: The context of the work

In this work two objects, L and M, play the fundamental role. By L we denote an integral commutative cl-monoid (iccl-monoid for short) [6, 7] that is a tuple $(L, \wedge_L, \vee_L, *)$, where $(L, \leq_L, \wedge_L, \vee_L,)$ is a complete lattice, and $(L, *, 1_L)$ is a commutative monoid in which $*$ distributs over arbitrary joins, that is $\alpha * \left(\bigvee_{i\in I} \beta_i\right) = \bigvee_{i\in I}(\alpha * \beta_i)$ for all $\alpha \in L$ and for all $\{\beta_i \mid i \in I\} \subseteq L$.[1] We assume that $0_L \neq 1_L$ where 0_L and 1_L are respectively the bottom and the top elements of L. By M we denote a complete frame, that is an infinitely distributive lattice $(M, \wedge_M, \vee_M)$. As different from the lattice L, we do not exclude here the trivial case, that is M can be a one-element lattice $\bullet$. Although in the larger part of this work M can be an arbitrary frame, when applying our results for constructing LM-fuzzy ditopologies in Section 5, we additionally assume that M is completely distributive.

2 L^M-valued equalities and L^M-valued sets

Applying the standard definition of a fuzzy set to our situation we say that an L^M-fuzzy subset A of a set X is just a mapping $A : X \to L^M$. However, the special form of the range set L^M allows to interpret A either as a mapping assigning to each $x \in X$ the mapping $A(x) = \varphi_x : M \to L$, or as an L-fuzzy subset $\tilde{A} \in L^{X\times M}$ of $X \times M$. This interpretation of an L^M-fuzzy set A allows to represent it as the family $\{A^\alpha : \alpha \in M\}$

[1] Note that an iccl-monoid can be characterized also as an integral commutative quantale in the sense of K.I. Rosenthal.

of L-fuzzy subsets $A^\alpha \in L^X$ of X ordered by the elements of M, where the L-fuzzy sets A^α are defined by $A^\alpha(x) = A(x, \alpha)$.

Adjusting the defintion of an L-valued equuality [6] to our situation we come to the following:

Definition 1. *Given a set X, an L^M-valued equality on it is a mapping $E : X \times X \to L^M$ such that*

($1EL^M$) $E(x,x)(\alpha) = 1_L$ *for every* $x \in X$ *and every* $\alpha \in M$;
($2EL^M$) $E(x,y)(\alpha) = E(y,x)(\alpha)$ *for all* $x, y \in X$ *and every* $\alpha \in M$;
($3EL^M$) $E(x,y)(\alpha) * E(y,z)(\alpha) \le E(x,z)(\alpha)$ *for all* $x, y, z \in X$, $\alpha \in M$.
($4EL^M$) $\alpha < \beta \implies E(x,y)(\alpha) \ge E(x,y)(\beta)$ *for all* $x, y \in X$, $\alpha, \beta \in M$.

Definition 2. *An L^M-valued equality E will be called upper semi-continuous if*

($5EL^M$) $E(x,y)\left(\bigvee_{i \in I} \alpha_i\right) = \bigwedge_{i \in I} E(x,y)(\alpha_i)$ *for all* $x, y \in X$, $\{\alpha_i \mid i \in I\} \subseteq M$.

An L^M-valued equality E will be called lower semi-continuous if

($6EL^M$) $E(x,y)\left(\bigwedge_{i \in I} \alpha_i\right) = \bigvee_{i \in I} E(x,y)(\alpha_i)$ *for all* $x, y \in X$, $\{\alpha_i \mid i \in I\} \subseteq M$.

An L^M-valued equality E will be called global if it satisfies property ($7EL^M$):

($7EL^M$) $E(x,y)(0_M) = 1_L \forall x, y \in X$;
$E(x,y)(1_M) = 1_L$ *if* $x = y$ *and* $E(x,y)(1_M) = 0_L$ *otherwise* .

Proposition 1. *A mapping $E : X \times X \times M \to L$ is an L^M-valued equality on a set X if and only if for every $\alpha \in M$ the restriction E^α of E to $X \times X \times \{\alpha\}$ is an L-valued equality on X [6] and $\alpha \le \beta \Longrightarrow E^\alpha \ge E^\beta$. Thus an L^M- valued equality on a set X can represented as a non-increasing family of L-valued equalities on this set ordered by the elements of the lattice M.*

Definition 3. *An L^M-fuzzy set B is called extensional, if $B(x,\alpha) * E(x,x')(\alpha) \le B(x',\alpha)$ for every $x, x' \in X$ and for every $\alpha \in M$. By an L^M-etensioanal hull of an L-fuzzy set $A \in L^X$ we call the smallest extensional L^M-fuzzy set $B \in (L^M)^X$ which is larger or equal to A, that is $A(x) \le B(x,\alpha)$ for all $x \in X$ and for all $\alpha \in M$.*

It is easy to see, that an L^M-fuzzy set B is extensional if and only if for each $\alpha \in M$ the L-fuzzy set B^α is extensional. Specifically, an L^M-fuzzy set B is the extensional hull of the L^M-fuzzy set A if and only if for each $\alpha \in M$ B^α is the extensional hull of A^α.

3 L^M-rough approximation operators on an L^M-valued set

Let $E : X \times X \to L^M$ be an L^M-valued equality on a set X. Given an L-fuzzy set $A \in L^X$ we define L^M-fuzzy sets $u_E(A) \in (L^M)^X$ and $l_E(A) \in (L^M)^X$ as follows:

$$u_E(A)(x)(\alpha) \;= \bigvee\nolimits_{x'} \left(E(x,x')(\alpha) * A(x')\right),\; l_E(A)(x)(\alpha)$$
$$= \bigwedge\nolimits_{x'} \left(E(x,x')(\alpha) \mapsto A(x')\right).$$

In such a way we obtain operators $u_E : L^X \to= (L^M)^X$ and $l_E : L^X \to (L^M)^X$ that in an obvious way can be interpreted also as operators $u_E : L^X \to L^{M\times X}, l_E : L^X \to L^{M\times X}$. We call operators $u_E : L^X \to (L^M)^X$ and $l_E : L^X \to (L^M)^X$ by *an upper and lower L^M-fuzzy rough approximation operator* induced by the L^M-valued equality E respectively.

Such operators can be represented as families of L-fuzzy rough approximation operators $\{u_E^\alpha : L^X \to L^X : \alpha \in M\}$ defined by

$$u^\alpha(A)(x) = u(A)(x)\; \forall A \in L^X,\; \forall x \in X, l^\alpha(A)(x) = l(A)(x)\; \forall A \in L^X,\; \forall x \in X$$

which are ordered by elements of the lattice M in such a way that $\alpha \le \beta \Longrightarrow u_E^\alpha(A) \ge u_E^\beta(A)\; \forall A \in L^X$, and $\alpha \le \beta \Longrightarrow l_E^\alpha(A) \le l_E^\beta(A)\; \forall A \in L^X$,

In order to allow subsequent application of the L^M-rough approximation operators we define the *reduced composition* $u_E \odot u_E : L^X \to (L^M)^X$ for operator u_E by setting

$$(u_E \odot u_E)(A)(x)(\alpha) = u_E(u_E(A)(x)(\alpha))(x)(\alpha)\; \forall A \in L^X,\; \forall x \in X.$$

In an analogoues way reduced compositions $l_E \odot l_E : L^X \to (L^M)^X$, $u_E \odot l_E : L^X \to (L^M)^X$ and $l_E \odot u_E : L^X \to (L^M)^X$ are defined.

Proposition 2. *Let (X, E) be an L^M-valued set. Then the induced upper and lower L^M-fuzzy rough approximation operators $u_E : L^X \to (L^M)^X$, $l_E : L^X \to (L^M)^X$ have the following properties:*

- (1u) $u_E(0_X)(x, 0_M) = 0_L$ *for all* $x \in X$;
- (2u) $u_E(A)(x, \alpha) \ge A(x)$ *for every* $x \in X, \alpha \in M$.
- (3u) $u_E(\bigvee_i A_i) = \bigvee_i u_E(A_i)\; \forall \{A_i \mid i \in I\} \subseteq L^X$
- (4u) $(u_E \odot u_E)(A) = u_E(A)\; \forall A \in L^X$;
- (5u) $\alpha \le \beta \Rightarrow u_E(A)(x, \alpha) \ge u_E(A)(x, \beta)\; \forall x \in X$;
- (6u) *If E is upper semicontinuous, then* $u_E(A)(x, \bigwedge_i \alpha_i) = \bigvee_i u_E(A)(x, \alpha_i)$ *for every set* $\{\alpha_i \mid i \in I\} \subseteq M$;
- (7u) *If E is global, then* $u_E(A)(x, 0_M) = \bigvee_{x' \in X} A(x')$ *and* $u_E(A)(x, 1_M) = A(x)$.
- (1l) $l(1_X)(x, \alpha) = 1_L\; \forall \alpha \in M,\; \forall x \in X$;
- (2l) $A(x) \ge l_E(A)(x, \alpha)\; \forall A \in L^X,\; \forall \alpha \in M$;
- (3l) $l_E(\bigwedge_i A_i) = \bigwedge_i l_E(A_i)\; \forall \{A_i \mid i \in I\} \subseteq L^X$
- (4l) $(l_E \odot l_E)(A) = l_E^\alpha(A)\; \forall A \in L^X \forall \alpha \in M$;
- (5l) *If E is non-increasing, then* $\alpha \le \beta \Longrightarrow l_E(A)(x, \alpha \le l_E(A)(x, \beta)$;

(6l) *If E is upper semicontinuous, then $l_E(A)(x, \bigvee_i \alpha_i) = \bigwedge_i l_E(A)(x, \alpha_i)$;*
(7l) *If E is global, then $l_E(A)(x, 0_M) = \bigwedge_{x'} A(x')$ and $l_E(A)(x, 1_M) = A(x)$.*

Theorem 1. *For every L-fuzzy set A in an L^M-valued set (X, E) the upper L^M-rough approximation operator assignes to A its extensional hull $u_E(A)$ and the lower L^M-rough approximation operator l_E assigns to A its extensional kernel $l(A_E)$*

Proposition 3. *Given an L^M-valued set (X, E) it holds $u_E \odot l_E = l_E$ and $l_E \odot u_E = u_E$.*

4 LM-fuzzy ditopology induced by an L^M-valued equality

A. Skowron [9] and A. Wiweger [10] were, probably, the first ones who noticed deep relations between Pawlak's rough approximation operators [8] and topological interior and closure operators. Later the relations between fuzzy rough approximation operators [4] and (Chang-Goguen) L-topological spaces were studied by different authors, see e.g. [5]. However, in our opinion it is more correct to use in this research the term "L-ditopology" [1] instead of "L-topology" since the families of fuzzy open and fuzzy sets induced by fuzzy rough approximation operators are generally independent. In this section we use L^M-fuzzy rough approximation operations to induce LM-fuzzy ditopologies [2], that is pair of mutually independent mappings $\mathcal{T} : L^X \to (L^M)^X$ and $\mathcal{K} : L^X \to (L^M)^X$ satisfying axioms of LM-fuzzy topology and LM-fuzzy co-topology respectively.

The properties $(1l) - (4l)$ of l_E related to l_E^α can be reformulated as follows:

(1l^α) $l^\alpha(1_X) = 1_L$;
(2l^α) $A \geq l_E^\alpha(A)\ \forall A \in L^X$;
(3l^α) $l_E^\alpha(\bigwedge_i A_i) = \bigwedge_i l_E(A_i)\ \forall \{A_i \mid i \in I\} \subseteq L^X$
(4l^α) $l_E^\alpha(l_E^\alpha(A)) = l_E^\alpha(A)\ \forall A \in L^X$;

However, this means that $l_E^\alpha : L^X \times L^X$ can be interpreted as an L-fuzzy interior operator on the set X. Hence by setting $T_\alpha = \{A \in L^X : l_E^\alpha(A) = A\}$, we obtain the L-fuzzy topology corresponding to this Alexandroff L-fuzzy interior operator (see e.g. [5]): $1_X \in T_\alpha$; $\{A_i : i \in I\} \subseteq T_\alpha \Longrightarrow \bigwedge_i A_i \in T_\alpha$; $\{A_i : i \in I\} \subseteq T_\alpha \Longrightarrow \bigvee_i A_i \in T_\alpha$ Taking such L-fuzzy topologies for all $\alpha \in M$ we obtain an non-increasing family $\{T_\alpha : \alpha \in M\}$. Besides, since $l_E^\alpha \leq l_E^\beta$ whenever $\alpha \leq \beta$, we conclude that that is the family $\{T_\alpha : \alpha \in M\}$ is non-increasing.

Theorem 2. *If M is completely distributive, then $\mathcal{T}(A) = \bigvee\{\alpha \in M : A \in T_\alpha.\}$ is anLM-fuzzy topology on the L^M-valued set (X, E), that is $\mathcal{T} : L^X \to M$ satisfies the following axioms:*

1. *$\mathcal{T}(1_X) = 1_M$;*
2. *$\mathcal{T}(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ for every family $\{A_i : i \in I\} \subseteq L^X$;*
3. *$\mathcal{T}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ for every family $\{A_i : i \in I\} \subseteq L^X$;*

In a similar way, using the upper L^M-rough approximation operator u_E, and referring to Proposition 3.1, we construct a family $\{K_\alpha : \alpha \in M\}$ where $K_\alpha = \{A \in L^X : u_E^\alpha(A) = A\}$. It is easy to notice that K_α is an Alexandroff L-co-topology [3]. This means that for each α the family K_α has the following properties :
$1_X \in K_\alpha$; $\{A_i : i \in I\} \subseteq K_\alpha \Longrightarrow \bigvee_i A_i \in K_\alpha$; $\{A_i : i \in I\} \subseteq K_\alpha \Longrightarrow \bigwedge_i A_i \in K_\alpha$

Besides, since $u_E^\alpha \geq u_E^\beta$ whenever $\alpha \leq \beta$, we conclude that that the family $\{K_\alpha : \alpha \in M\}$ is non-increasing. We use this family of L-fuzzy co-topologies to define an (Alexandroff) L-fuzzy co-topology $\mathcal{K}$ on the set X, by setting $\mathcal{K}(A) = \bigvee\{\alpha \in M : A \in K_\alpha.\}$

Theorem 3. *If M is completely distributive, then $\mathcal{K}$ is an LM-fuzzy co-topology on the L^M-valued set (X, E). This means that the mapping $\mathcal{K} : L^X \to M$ satisfies the following axioms:*

1. $\mathcal{K}(1_X) = 1_M$*;*
2. $\mathcal{K}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{K}(A_i)$ *for every family* $\{A_i : i \in I\} \subseteq L^X$*;*
3. $\mathcal{K}(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ *for every family* $\{A_i : i \in I\} \subseteq L^X$*;*

5 Construction of an L^M-valued equality from a pseudo-metric

Let $L = M = [0, 1]$ be the unit intervals and let $* : L \times L \to L$ be a continuous t-norm. Further, let $\rho : X \times X \to [0, 1]$ be a pseudo-metric on a set X. We define a mapping $E_\rho : X \times X \times [0, 1] \to [0, 1]$ by setting

$$E_\rho(x, y)(\alpha) = \begin{cases} \frac{1-\alpha}{1-\alpha+\alpha\rho(x,y)} & \text{if } \alpha \neq 1 \text{ or } \rho(x, y) \neq 0 \\ 1 & \text{if } \alpha = 1 \text{ and } \rho(x, y) = 0. \end{cases}$$

Proposition 4. *For every pseudo-metric $\rho : X \times X \to [0, 1]$ the mapping $E_\rho : X \times X \times [0, 1] \to [0, 1]$ is continuous for all $x, y \in [0, 1]$ and satisfies conditions* ($1EL^M$), ($2EL^M$), ($4EL^M$), ($5EL^M$), ($6EL^M$), ($7EL^M$) and ($8EL^M$). *It satisfies condition (3EL^M) in cases of the product t-norm $* = \cdot$ and of the Łukasiewicz t-norm $* = *_L$. If ρ is an ultra pseudo-metric, then mapping $E_\rho : X \times X \times [0, 1] \to [0, 1]$ satisfies condition (3EL^M) in case of the minimum t-norm $* = \wedge$.*

Corollary 1. *In case $* = \cdot$ and $* = *_L$ the mapping $E_\rho : X \times X \to [0, 1] \to [0, 1]$ is a global continuous L^M-equality for any pseudo-metric $\rho : X \times X \to [0, 1]$. If ρ is an ultra pseudo-metric, then E_ρ is a global continuous L^M-valued equality in case $* = \wedge$.*

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Type-n arrow categories

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Allegories and Dedekind categories, in particular, [1, 5–8] provide an adequate categorical and algebraic framework for reasoning about relations. An obvious example for each of these categories is the category **REL** of sets and binary relation. A binary relation can be represented by its characteristic function, i.e., by a function that returns true if the pair is in the relation and false if not. The category **REL** is is not the only example of an allegory, of course. Given a complete Heyting algebra L, the category L-**REL** of sets and so called L-relations is also an example. L-relations differ from regular relations by assigning to each pair a degree of membership from L instead of true or false. Certain aspects of L-relations cannot be expressed in allegories or Dedekind categories. For example, consider the special case that an L-fuzzy relation R returns the smallest or the greatest element of L for each pair. Such a relation corresponds in an obvious way to a regular binary relation, and therefore they are called crisp. Even though several abstract notions of crispness in Dedekind categories have been proposed [2–4], it was shown that this property cannot be expressed in the language of allegories or Dedekind categories [9, 10]. Therefore, Goguen and arrow categories [9, 11] were introduced adding two additional operations to the theory of Dedekind categories covering the notion of crispness.

A higher-order or type-2 L-relation uses membership values from the L^L, i.e., it uses endofunctions on L as the degree of membership for each pair. Since L^L forms a complete Heyting algebra if L does, the category L^L-**REL** also forms an arrow category. To distinguish between L-**REL** and L^L-**REL** one normally speaks about type-1 L-relations and type-2 L-relations, respectively. By iterating the process we can define type-n L-relations for arbitrary n. In this paper we are interested in the relationship between type-1 and type-2 L-relations and the iteration process leading to type-n L-relations.

In [12, 13] the extension was on object was used to show that the category of type-2 L-relations can be constructed as a Kleisli category of type-1 L relations. The extension $A^\sharp$ of a set A is the set $A \times L$ of pairs of A elements and a lattice value. The corresponding isomorphism, i.e., the bijection between $A^\sharp$ and $A \times 1^\sharp$ with unit 1, was shown in the abstract setting. Furthermore, it was shown that the induced product functor together with appropriate natural transformations forms a monad so that the category of type-2 L-relations was obtained as the Kleisli category for this monad. These result were used in [14] to apply the abstract theory of arrow categories in the development of fuzzy controllers.

In this paper we want to show two major results. First, we want to remove the adhoc notion of an extension of an object completely from the construction of higher-order arrow categories. In [12, 13] it was already shown that $A^\sharp$ is isomorphic to $A \times 1^\sharp$. In this paper we will show that $1^\sharp$ is isomorphic to $\mathcal{P}_L(1)$ where $\mathcal{P}_L(A)$ is the L-power of A. The L-power of A is the abstract version of the L-fuzzy powerset. Please note that this construction is different from relational powers or the constructions given in power allegories. As an immediate consequence we obtain the following result in the abstract setting of arrow categories: If an arrow category has relational products and L-powers, then the arrow category of type-2 L relations can be defined as the Kleisli category induced by the monad above. Our second result shows that this process can be iterated, i.e., it verifies that the arrow category of type-2 L relations again has relational products and L-powers.

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Characterizations of topological quantaloid-enriched convergence spaces

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Abstract. In this talk, according to Höhle's $\mathcal{Q}$-enriched filter monad, we introduce strong $\mathcal{Q}$-filer monad and study (topological) pretopological $\mathcal{Q}$-ordered convergence spaces. Kowalsky's diagonal condition and Fischer's diagonal condition are given to characterize topological $\mathcal{Q}$-ordered convergence spaces. $\mathcal{Q}$-ordered convergence spaces are also studied from the viewpoint of lax algebra on $\mathcal{Q}$-**Dist**.

1 Introduction

Since quantaloid-enriched categories developed by Stubbe [10, 11] can give a reasonable explanation of the theories on fuzzy sets, it draw much attention in the field of fuzzy mathematics. Stubbe in [12] also gave an overview of this theory for the readership of fuzzy logicians and fuzzy set theorists. Based on quantaloid-enriched categories, Höhle in [5], Pu and Zhang in [6] and Shen in [9] studied different aspects of quantaloid-enriched topology, preordered set and quantaloid-enriched closure space, etc.

In [5], Höhle studied the categorical foundations of topology based on ordered monad and used appropriate submonads of the double presheaf monad to introduce quantaloid-enriched topology. When Höhle studied $\mathcal{Q}$-enriched filter monad as a submonad of the double presheaf monad on $\mathcal{Q}$-**Cat**, he enriched the underlying quantaloid $\mathcal{Q}$ by an extra local binary operation–premultiplication satisfying certain conditions. As a special case, we use the meet in $\mathcal{Q}$ to replace the premultiplication, and slightly modify the definition of $\mathcal{Q}$-enriched filter to give strong $\mathcal{Q}$-filer monad. From this strong $\mathcal{Q}$-filter monad, there is a natural way to obtain strong quantaloid-enriched topology—-the Zhang's strong topology under the quantaloid setting. In this talk, we introduce quantaloid-enriched convergence spaces(we call $\mathcal{Q}$-ordered convergence space), and give some characterizations of topological $\mathcal{Q}$-ordered convergence structures by Kowalsky's diagonal condition and Fischer's diagonal condition. Following the idea of Clementino, Hofmann and Tholen in [1, 2], we will show that $\mathcal{Q}$-ordered convergence spaces can also be studied by lax algebra on $\mathcal{Q}$-**Dist**.

2 Strong $\mathcal{Q}$-filer monad

Definition 1. *Let $\mathbb{A}$ be a $\mathcal{Q}$-category and $\mathcal{F} \in \mathcal{P}^{\dagger}(\mathcal{P}\mathbb{A})_0$. $\mathcal{F}$ is called a strongly stratified $\mathcal{Q}$-filter if it satisfies the following conditions:*

(QF0) $\mathcal{F}(\bot_X) = \bot_{t(\mathcal{F}),X}$ *for all* $X \in \mathcal{Q}_0$*, where* $\bot_X(x) = \bot_{t(x),X}$ *for all* $x \in \mathbb{A}_0$*;*

(QF1) $\mathcal{F}(\top_X) = \top_{t(\mathcal{F}),X}$ *for all* $X \in \mathcal{Q}_0$*, where* $\top_X(x) = \top_{t(x),X}$ *for all* $x \in \mathbb{A}_0$*;*

(QF2) $\mathcal{F}(\mu \wedge \lambda) = \mathcal{F}(\mu) \wedge \mathcal{F}(\lambda)$ *for any* $\mu, \lambda \in \mathcal{P}\mathbb{A}_0$ *with the same type;*

(QFS) $f \circ \mathcal{F}(\mu) \leq \mathcal{F}(f \circ \mu)$ *for all* $f \in \mathcal{Q}(t(\mu), -)$*;*

(QFSS) $\mathcal{F}(f \searrow \mu) = f \searrow \mathcal{F}(\mu)$ *for* $\mu \in \mathcal{P}\mathbb{A}_0$ *and* $f \in \mathcal{Q}(-, t(\mu))$.

The set of all strongly stratified $\mathcal{Q}$-filters on $\mathbb{A}$ forms a underlying set of a subcategory of $\mathcal{P}^{\dagger}(\mathcal{P}\mathbb{A})$, denoted by $\mathbb{F}_{\mathcal{Q}S}(\mathbb{A})$.

Example 1. (1) Let $x \in \mathbb{A}_0$. Define $[x] : \mathcal{P}\mathbb{A}_0 \to \mathcal{Q}$ by $[x](\mu) = \mu(x)$. Then $[x]$ is a strongly stratified $\mathcal{Q}$-filter on $\mathbb{A}$ (the type of $[x]$ is $t(x)$).

(2) Let $F : \mathbb{A} \to \mathbb{B}$ be a $\mathcal{Q}$-functor and $\mathcal{F}$ be a strongly stratified $\mathcal{Q}$-filter on $\mathbb{A}$. Define $F^{\Rightarrow}(\mathcal{F})(\mu) = \mathcal{F}(\mu \circ F_{\sharp})$ for $\mu \in \mathcal{P}\mathbb{B}_0$. Then $F^{\Rightarrow}(\mathcal{F})$ is a strongly stratified $\mathcal{Q}$-filter on $\mathbb{B}$.

In order to give the $\mathcal{Q}$-neighborhood system, we fix the strong Q-filter monad $\mathbf{T} = (\mathbb{F}_{\mathcal{Q}S}, \eta, \iota)$ in the sense of [5] as follows: The functor $\mathbb{F}_{\mathcal{Q}S} : \mathcal{Q}\text{-}\mathbf{Cat} \to \mathcal{Q}\text{-}\mathbf{Cat}$ is given by sending $\mathbb{A}$ to $\mathbb{F}_{\mathcal{Q}S}(\mathbb{A})$ and F to $F^{\Rightarrow}$; the two natural transformations $\eta : 1_{\mathcal{Q}\text{-}\mathbf{Cat}} \to \mathbb{F}_{\mathcal{Q}S}$ and $\iota : \mathbb{F}_{\mathcal{Q}S} \circ \mathbb{F}_{\mathcal{Q}S} \to \mathbb{F}_{\mathcal{Q}S}$ are given by $\eta_{\mathbb{A}}(x) = [x]$ and $\iota_{\mathbb{A}}(\Im)(\mu) = \Im(e_{\mu})$, respectively. For $R : \mathbb{A} \to \mathbb{F}_{\mathcal{Q}S}(\mathbb{B})$ and $S : \mathbb{B} \to \mathbb{F}_{\mathcal{Q}S}(\mathbb{C})$, the Kleisli composition function $\bullet$ with respect to the strong Q-filter monad T as follows is

$$S \bullet R = \iota_{\mathbb{C}} \circ \mathbb{F}_{\mathcal{Q}S}(S) \circ R = \iota_{\mathbb{C}} \circ S^{\Rightarrow} \circ R.$$

From [5], a $\mathcal{Q}$-functor $\mathcal{N} : \mathbb{A} \to \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})$ is called a $\mathcal{Q}$-neighborhood system of $\mathbb{A}$ if $\mathcal{N}$ fulfills the conditon (QN1) $\mathcal{N} \leq \eta_{\mathbb{A}}$. A $\mathcal{Q}$-neighborhood system $\mathcal{N}$ is called topological it further satisfies the following (QN2):

(QT) $\mathcal{N} \bullet \mathcal{N} = \mathcal{N}$, i.e., $\iota_{\mathbb{A}}(N^{\Rightarrow}(N^x)) = N^x$ for all $x \in \mathbb{A}_0$.

Let $\mathcal{Q}$-**Nei** denote the category of $\mathcal{Q}$-neighborhood spaces and $\mathcal{Q}$-**TNei** denote the category of topological $\mathcal{Q}$-neighborhood spaces, where $\mathcal{Q}$-functor $F : (\mathbb{A}, \mathcal{N}) \to (\mathbb{B}, \mathcal{R})$ is called continuous if $\mathcal{R} \circ F \leq F^{\Rightarrow} \circ \mathcal{N}$ holds.

There are natural ways to show the relationship between (topological) $\mathcal{Q}$-neighborhood space, (topological) $\mathcal{Q}$-interior operator and $\mathcal{Q}$-topology. For example from topological $\mathcal{Q}$-neighborhood space, we can obtain the strong topology in the sense of Zhang under the quantaloid setting.

Lemma 1. *Let* $(\mathbb{A}, \mathcal{N})$ *be a (topological)* $\mathcal{Q}$*-neighborhood space and define a full subcategory* $\mathbb{T}$ *of* $\mathcal{P}\mathbb{A}$ *by* $\mathbb{T}_0 = \{\mu \in \mathcal{P}\mathbb{A}_0 | \ \forall x \in \mathbb{A}_0,\ N^x(\mu) = \mu(x)\}$*. Then* $\mathbb{T}$ *satisfies the following axioms:*

(QT1) $\forall \Phi \in \mathcal{P}\mathbb{T}_0$, $\sup_{\mathcal{P}\mathbb{A}} i^{\rightarrow}(\Phi) \in \mathbb{T}_0$*;*

(QT2) *For each* $\Psi \in \mathcal{P}^{\dagger}\mathbb{C}_0$ *with* $\mathbb{C}_0$ *is finite subset of* $\mathbb{T}_0$*, then* $\inf_{\mathcal{P}\mathbb{A}} i^{\nrightarrow}(\Psi) \in \mathbb{T}_0$.

3 $\mathcal{Q}$-ordered convergence space

Definition 2. *Let* $\mathbb{A}$ *be a* $\mathcal{Q}$*-category. A* $\mathcal{Q}$*-functor* $\lim : \mathbb{F}_{\mathcal{Q}S}(\mathbb{A}) \to \mathcal{P}^{\dagger}\mathbb{A}$ *is called a* $\mathcal{Q}$*-ordered convergence structure on* $\mathbb{A}$ *if it satisfies:*

(QC1) $\lim[x](x) \geq 1_{t(x)}$ *for all* $x \in \mathbb{A}_0$.

A $\mathcal{Q}$-functor $F : (\mathbb{A}, \lim_{\mathbb{A}}) \to (\mathbb{B}, \lim_{\mathbb{B}})$ between $\mathcal{Q}$-ordered convergence spaces is called continuous if $F^{\rightharpoondown} \circ \lim_{\mathbb{A}} \leq \lim_{\mathbb{B}} \circ F^{\Rightarrow}$. The category of $\mathcal{Q}$-ordered convergence spaces is denoted by $\mathcal{Q}$-**Conv**. $\mathcal{Q}$-**Conv** is topological on $\mathcal{Q}$-**Cat**.

Define functor $\Gamma : \mathcal{Q}\text{-}\mathbf{Conv} \to \mathcal{Q}\text{-}\mathbf{Nei}$ by $\Gamma((\mathbb{A}, \lim)) = (\mathbb{A}, \mathcal{N}_{\lim})$, where $\mathcal{N}_{\lim}(x) = \sup_{\mathbb{F}_{\mathcal{Q}S}(\mathbb{A})} \lim(-)(x)$. Define $\Omega : \mathcal{Q}\text{-}\mathbf{Nei} \to \mathcal{Q}\text{-}\mathbf{Conv}$ by $\Omega((\mathbb{A}, \mathcal{N})) = (\mathbb{A}, \lim_{\mathcal{N}})$, where $\lim_{\mathcal{N}} \mathcal{F}(x) = \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})(\mathcal{F}, N^x)$.

Lemma 2. *Γ is the left adjoint functor of Ω.*

Definition 3. *A $\mathcal{Q}$-ordered convergence structure* $\lim$ *on $\mathbb{A}$ is called pretopological if there exists a $\mathcal{Q}$-neighborhood system $\mathcal{N}$ such that* $\lim = \lim_{\mathcal{N}}$. *A $\mathcal{Q}$-ordered convergence structure* $\lim$ *on $\mathbb{A}$ is called topological if there exists a topological $\mathcal{Q}$-neighborhood system $\mathcal{N}$ such that* $\lim = \lim_{\mathcal{N}}$.

$(\mathbb{A}, \lim)$ is pretopological if and only if it satisfies the (QP) condition as follows:

$$\text{(QP)}\quad \forall \mathcal{F} \in \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})_0, \forall x \in \mathbb{A}_0, \quad \lim \mathcal{F}(x) = \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})(\mathcal{F}, N^x_{\lim}).$$

A pretopological $\mathcal{Q}$-ordered convergence structure $\lim$ on $\mathbb{A}$ is topological if and only if $\mathcal{N}_{\lim}$ is topological $\mathcal{Q}$-neighborhood system, i.e., it satisfies the (QT) condition as follows:

$$\text{(QT)} \qquad \mathcal{N}_{\lim} \leq \mathcal{N}_{\lim} \bullet \mathcal{N}_{\lim}.$$

It is routine to check that $\mathcal{Q}$-**PConv**–the category of pretopological $\mathcal{Q}$-ordered convergence spaces—is isomorphic to $\mathcal{Q}$-**Nei** and $\mathcal{Q}$-**TPConv**—the category of topological pretopological $\mathcal{Q}$-ordered convergence spaces—is isomorphic to $\mathcal{Q}$-**TNei**; $\mathcal{Q}$-**PConv** is a bireflective subcategory of $\mathcal{Q}$-**Conv** and $\mathcal{Q}$-**TPConv** is a bireflective subcategory of $\mathcal{Q}$-**PConv**.

Lemma 3. *If a $\mathcal{Q}$-ordered convergence structure* $\lim : \mathbb{F}_{\mathcal{Q}S}(\mathbb{A}) \to \mathcal{P}^{\dagger}\mathbb{A}$ *has a right adjoint, then* $\lim$ *must be pretopological.*

In fact, we have the following results:

Theorem 1. *Let* $\lim$ *be a $\mathcal{Q}$-ordered convergence structure on $\mathbb{A}$. Then the following statements are equivalent.*

(QP) $\forall \mathcal{F} \in \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})_0, \forall x \in \mathbb{A}_0,\ \lim \mathcal{F}(x) = \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})(\mathcal{F}, N^x_{\lim})$;

(QP*) *For each non-empty $\Theta \in \mathcal{P}(\mathbb{F}_{\mathcal{Q}S}(\mathbb{A}))_0$,* $\lim(\sup_{\mathbb{F}_{\mathcal{Q}S}(\mathbb{A})} \Theta) = \sup_{\mathcal{P}^{\dagger}\mathbb{A}} \lim^{\rightarrow} \Theta$*;*

(QP$^{\diamond}$) $\forall x \in \mathbb{A}_0$, $\lim N^x_{\lim}(x) \geq 1_{t(x)}$.

Theorem 1 gives us some characterizations of pretopological $\mathcal{Q}$-ordered convergence structure. Especially, the condition (QP*) says $\lim$ is pretopological if and only if $\lim$ is sup-preserving for non-empty presheaf of $\mathbb{F}_{\mathcal{Q}S}(\mathbb{A})$. Hence Lemma 3 is valid. In the following, we give two characterizations of topological $\mathcal{Q}$-ordered convergence spaces by two famous diagonal condition—Kowalsky's diagonal condition and Fischer's diagonal condition.

First, we give an extension of Kowalsky's diagonal condition to $\mathcal{Q}$-ordered convergence space in the following way:

(K-QT) For each $\mathcal{Q}$-functor $\Upsilon : \mathbb{A} \to \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})$ with $\lim \Upsilon(y)(y) \geq 1_{t(y)}$ for all $y \in \mathbb{A}_0$, it holds $\lim \leq \lim \circ \iota_{\mathbb{A}} \circ \Upsilon^{\Rightarrow}$.

Theorem 2. *If* $\lim$ *is a pretopological* $\mathcal{Q}$*-ordered convergence space, then (QT) and (K-QT) are equivalent.*

Then, the Fischer's diagonal condition can be given to $\mathcal{Q}$-ordered convergence space as follows:

(F-QT): For all $\mathcal{Q}$-functors $\Lambda : \mathbb{J} \to \mathbb{A}$ and $\Upsilon : \mathbb{J} \to \mathbb{F}_{\mathcal{Q}S}(\mathbb{A})$ with $\lim \Upsilon(j)(\Lambda(j)) \geq 1_{t(j)}$ for all $j \in \mathbb{J}_0$, it holds $\mathbb{F}_{\mathcal{Q}S}(\Lambda^{\Rightarrow}(\mathcal{F}), N^x_{\lim}) \leq \lim K\Upsilon\mathcal{F}(x)$ for all $\mathcal{F} \in \mathbb{F}_{\mathcal{Q}S}(\mathbb{J})_0$.

Theorem 3. *If* $\lim$ *is a* $\mathcal{Q}$*-ordered convergence space, then* $\lim$ *is topological if and only if* $\lim$ *satisfies (F-QT).*

4 $\mathcal{Q}$-ordered convergence space as lax algebra on $\mathcal{Q}$-Dist

Following the idea of Clementino, Hofmann and Tholen in [1, 2], the strong $\mathcal{Q}$-filter monad $\mathbf{T} = (\mathbb{F}_{\mathcal{Q}S}, \eta, \iota)$ can be lax extended to $\mathcal{Q}$-**Dist**. From [10], there is a one-to-one correspondence between $\mathcal{Q}\text{-}\mathbf{Cat}(\mathbb{F}_{\mathcal{Q}S}(\mathbb{A}), \mathcal{P}^{\dagger}\mathbb{A})$ and $\mathcal{Q}\text{-}\mathbf{Dist}(\mathbb{F}_{\mathcal{Q}S}(\mathbb{A}), \mathbb{A})$. Hence, for $\mathcal{Q}$-ordered convergence space $(\mathbb{A}, \lim)$ and reflexive lax algebra $(\mathbb{A}, \alpha)$, $\alpha_{\lim}$ defined by $\alpha_{\lim}(\mathcal{F}, x) = \lim \mathcal{F}(x)$ is a reflexive lax algebra, and $\lim_{\alpha}$ defined by $\lim_{\alpha} \mathcal{F}(x) = \alpha(\mathcal{F}, x)$ is a $\mathcal{Q}$-ordered convergence structure.

Theorem 4. $\mathcal{Q}$-$Alg(\mathbf{T}, \eta)$*—the category of reflexive lax algebras on* $\mathbf{T}$*— is isomorphic to* $\mathcal{Q}$-**Conv**.

Theorem 5. *Let* $(\mathbb{A}, \lim)$ *be* $\mathcal{Q}$*-ordered convergence space. Then* $(\mathbb{A}, \lim)$ *is a pretopological* $\mathcal{Q}$*-ordered convergence space if and only if* $(\mathcal{N}_{\lim})^{\sharp} \leq \alpha_{\lim}$.

Theorem 6. *Let* $(\mathbb{A}, \lim)$ *be pretopological* $\mathcal{Q}$*-ordered convergence space. Then* $(\mathbb{A}, \lim)$ *is topological if and only if* $\alpha_{\lim} \circ \mathcal{N}^{\Rightarrow}_{\lim} \circ \iota^{\sharp}_{\mathbb{A}} \leq \alpha_{\lim}$.

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The role of fuzzy logic in fuzzy order and fuzzy topology

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Since the introduction of the language of categories, functors, and natural transformations by Eilenberg and Mac Lane in 1945, category theory has penetrated deeply in our understanding of mathematics. Enriched categories generalize the idea of a locally small category by replacing hom-sets with objects from a fixed monoidal closed category.

A monoidal closed category consists of

- a category $\mathcal{V}$;
- a functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, called the tensor, such that for each object A both $(-) \otimes A : \mathcal{V} \longrightarrow \mathcal{V}$ and $A \otimes (-) : \mathcal{V} \longrightarrow \mathcal{V}$ have a right adjoint;
- an object k in $\mathcal{V}$;
- for every triple A, B, C of objects, an "associativity" isomorphism $\alpha_{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$;
- for every object A, an isomorphism $l_A : k \otimes A \longrightarrow A$ and an isomorphism $r_A : A \otimes k \longrightarrow A$.

These data must satisfy certain coherence axioms.

Following Lawvere, we interpret the object k as *true*, the tensor product $\otimes$ as the logic connective *conjunction*, and the right adjoints of $- \otimes A : \mathcal{V} \longrightarrow \mathcal{V}$ and $A \otimes - : \mathcal{V} \longrightarrow \mathcal{V}$ as *implications*. Then, no matter whatever the objects may be, a monoidal closed category looks like a *table of truth-values* for a (many-valued) logic. In his pioneering paper [18], Lawvere has displayed "many general results about metric spaces (which are categories enriched over the quantale $([0, \infty]^{\mathrm{op}}, +)$) as consequences of a *generalized pure logic* whose *truth-values* are taken in an arbitrary monoidal closed category."

While a quantale is a "fixed" table of truth-values, a quantaloid can be thought of as a"dynamic" table of truth-values for a fuzzy logic. So, fuzzy set theory is inevitably related to enriched categories in a close way. In particular, for a quantaloid $\mathcal{Q}$, a category enriched over $\mathcal{Q}$ can be thought of as a *fuzzy set* (or, a partially defined set) endowed with an order relation valued in $\mathcal{Q}$.

Given a quantale (or, more general, a quantaloid) $\mathcal{Q}$, theories of orders and topologies valued in $\mathcal{Q}$ have been established in the literature. In this talk, we are concerned with how the "logic features" of $\mathcal{Q}$ will affect the behaviors of $\mathcal{Q}$-orders and $\mathcal{Q}$-topologies. This interaction is what we mean by "the role of fuzzy logic" in the title. A number of examples related to fuzzy orders and fuzzy topologies in this regard will be surveyed.

Let $\mathcal{Q} = (Q, \&)$ be a commutative quantale. A $\mathcal{Q}$-topology on a set X is a subset $\tau \subseteq Q^X$ such that

(O1) for all $p \in Q$, the constant fuzzy set p_X belongs to τ;
(O2) $\lambda \wedge \mu \in \tau$ for all $\lambda, \mu \in \tau$;
(O3) $\bigvee_{j\in J} \lambda_j \in \tau$ for each subfamily $\{\lambda_j\}_{j\in J}$ of τ.

Let $\mathcal{Q} = (Q, \&)$ be a commutative quantale and τ be a $\mathcal{Q}$-topology on X. We say that

(1) τ is stratified if $p\&\lambda \in \tau$ for all $p \in Q$ and $\lambda \in \tau$.
(2) τ is co-stratified if $p \to \lambda \in \tau$ for all $p \in Q$ and $\lambda \in \tau$.
(3) τ is strong if it is both stratified and co-stratified.
(4) τ is Alexandroff if $\bigwedge_{j\in J} \lambda_j \in \tau$ for all $\{\lambda_j : j \in J\} \subseteq \tau$.

Example 1. [21, 35, 14] If the underlying lattice of $\mathcal{Q}$ is meet continuous, then the category of topological spaces can be embedded in that of $\mathcal{Q}$-topological spaces as a both bireflective and coreflective full subcategory if and only if $\mathcal{Q}$ is continuous.

It is trivial that a finite topological space is always Alexandroff, but, this is not always true in the many-valued setting.

Example 2. [4] Let $\mathcal{Q} = ([0,1], \&)$ with $\&$ being a continuous t-norm. Then every finite strong $\mathcal{Q}$-topological space is Alexandroff if and only if $\&$ is an ordinal sum of the Łukasiewicz t-norm and the set of idempotent elements of $\&$ is a well-ordered subset of $[0,1]$ with respect to the usual order.

A neighborhood of x is a fuzzy set $\lambda \in Q^X$ such that $\lambda^\circ(x) = 1$. The neighborhoods of x form a prefilter $\mathcal{N}_x$ on X, called the neighborhood prefilter of x. A CNS space is a $\mathcal{Q}$-topological space X such that

$$\lambda^\circ(x) = \bigvee_{\nu\in\mathcal{N}_x} S(\nu, \lambda) = \bigvee_{\nu\in\mathcal{N}_x} \bigwedge_{x\in X} (\nu(x) \to \lambda(x))$$

for all $\lambda \in Q^X$ and $x \in X$.

Example 3. [16, 24] Let $\mathcal{Q} = (Q, \&)$ be a continuous, commutative and integral quantale. Then the category of CNS spaces is simultaneously reflective and coreflective in the category of stratified $\mathcal{Q}$-topological spaces if and only if if for each $p \in Q$, the map

$$Q \longrightarrow Q, \quad x \mapsto \bigvee_{q\in\Uparrow p} (q \to x)$$

is Scott continuous.

Example 4. [13, 15] Let $\mathcal{Q} = (Q, \&)$ be a commutative and integral quantale. The following are equivalent:

(1) The bottom element in Q is a dualizing element of $\mathcal{Q}$, hence $\mathcal{Q}$ is a Girard quantale.

(2) Every complete $\mathcal{Q}$-lattice can be written as the concept lattice of some fuzzy context based on rough set theory.
(3) The opposite of a completely distributive $\mathcal{Q}$-lattice is completely distributive.
(4) The opposite of $(Q, \to)$ is completely distributive.

Example 5. [17] Let $\mathcal{Q} = ([0,1], \&)$ with $\&$ being a left continuous t-norm on $[0,1]$. Then the category of Yoneda complete $\mathcal{Q}$-ordered sets is monoidal closed if and only if $\&$ is continuous.

Example 6. [17] Let $\mathcal{Q} = ([0,1], \&)$ with $\&$ being a continuous t-norm. The following are equivalent:

(1) $\& = \min$.
(2) The category of $\mathcal{Q}$-ordered sets is Cartesian closed.
(3) The category of Yoneda complete $\mathcal{Q}$-ordered sets is Cartesian closed.

Given a quantale $\mathcal{Q}$, let $\mathsf{D}(\mathcal{Q})$ be the quantaloid of diagonals in $\mathcal{Q}$. Then $\mathsf{D}(\mathcal{Q})$-categories can be thought of ordered fuzzy sets (valued in $\mathcal{Q}$). The last example is concerned with the behavior of "directed complete" ordered fuzzy sets.

Example 7. [19] Let $\mathcal{Q}$ be the interval $[0,1]$ coupled with a continuous t-norm $\&$. The following are equivalent:

(1) $\&$ is either isomorphic to the Łukasiewicz t-norm or to the product t-norm.
(2) Each Yoneda complete $\mathsf{D}(\mathcal{Q})$-category with an isolated element is flat complete.
(3) Each bicomplete $\mathsf{D}(\mathcal{Q})$-category with an isolated element is Cauchy complete.

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