Technical Report
FLLL–TR–0203

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Measure-based aggregation operators

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Abstract — In analogy to the representation of the standard probabilistic average as an expected value of a random variable, a geometric approach to aggregation is proposed. Several properties of such aggregation operators are investigated, and the relationship with distinguished classes of aggregation operators is discussed.

Key words — *Aggregation operator, fuzzy measure, Choquet integral, triangular norm*

Submitted for publication

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1 Introduction

Given a probability space $(X, \mathcal{A}, P)$, standard probabilistic averaging is done by means of the expected value of a random variable $\xi\colon X \longrightarrow [0,1]$ via

$$E(\xi) = \int_0^1 P(\{\xi \geq t\})\, dt, \tag{1}$$

Similarly, the mean value of an integrable function $f\colon [0,1] \longrightarrow [0,1]$ is given by

$$E(f) = \int_0^1 \lambda(\{f \geq t\})\, dt, \tag{2}$$

where λ is the Lebesgue measure on $\mathbb{R}$.

In both cases, the surface of the endograph of a non-increasing function $h_{m,f}\colon [0,1] \longrightarrow [0,1]$ given by $h_{m,f}(t) = m(\{f \geq t\})$ is computed, where m is a (σ-additive) measure and f represents the input to be aggregated.

We generalize (1) and (2), replacing the σ-additivity of the measure m by significantly weaker properties. This will lead to a rather general class of geometrical aggregation operators, and many well-known classes of aggregation operators are special cases thereof.

The paper is organized as follows. In the next section, based on two fuzzy measures m and μ, the (m,μ)-aggregation operator is introduced, and its properties are discussed in Section 3. Several types of integrals (such as Choquet or Sugeno integrals) are linked to σ-additive fuzzy measures determined by some copula, as will be shown in Section 4. In a similar way, maxitive fuzzy measures will be discussed in Section 5.

2 Geometric approach

Let X be a non-empty index set and $f\colon X \longrightarrow [0,1]$ the input system to be aggregated. Let $(X, \mathcal{A}, m)$ be a fuzzy measure space, i.e., $\mathcal{A}$ is a σ-algebra of subsets of X (in the case of a finite set X we usually take $\mathcal{A} = 2^X$), and $m\colon \mathcal{A} \longrightarrow [0,1]$ a fuzzy measure as introduced in [18], thus satisfying $m(\emptyset) = 0$, $m(X) = 1$ and $m(A) \leq m(B)$ whenever $A \subseteq B$.

If X is a finite set and $\mathcal{A} = 2^X$, a fuzzy measure $m\colon \mathcal{A} \longrightarrow [0,1]$ is called symmetric if it is invariant under bijective transformations of X, i.e., for each bijection $\varphi\colon X \longrightarrow X$ and for each $E \in \mathcal{A}$ we have $m(\varphi^{-1}(E)) = m(E)$. This implies

$$m(E) = h\left(\tfrac{\operatorname{card} E}{\operatorname{card} X}\right)$$

for some non-decreasing function $h\colon [0,1] \longrightarrow [0,1]$ with $h(0) = 0$ and $h(1) = 1$.

For each set $A \subseteq \mathbb{R}^n$, let $\mathcal{B}(A)$ be the σ-algebra of all Borel subsets of A. Denote by $\mathcal{L}(\mathcal{A})$ the set of all $\mathcal{A}$-measurable functions from X to $[0,1]$. If $f \in \mathcal{L}(\mathcal{A})$, define the function $h_{m,f}\colon [0,1] \longrightarrow [0,1]$ by

$$h_{m,f}(t) = m(\{f \geq t\}).$$

Obviously, $h_{m,f}$ is non-increasing (and, therefore, Borel measurable) and satisfies $h_{m,f}(0) = 1$. Consider the endograph of $h_{m,f}$, i.e., the set

$$D_{m,f} = \{(x, y) \in \,]0, 1[^2 \mid y < h_{m,f}(x)\}.$$

Note that the Borel measurability of $h_{m,f}$ implies that $D_{m,f}$ is a Borel subset of the open unit square $]0, 1[^2$. Note also that $D_{m,f}$ has a root structure, i.e., for each $(x_0, y_0) \in D_{m,f}$ we have $(x, y) \in D_{m,f}$ whenever $(x, y) \in \,]0, x_0[\, \times \,]0, y_0[$.

The surface of the set $D_{m,f}$ can now be evaluated by means of some fuzzy measure on $]0, 1[^2$.

Definition 2.1 Consider two fuzzy measure spaces $(X, \mathcal{A}, m)$ and $(]0, 1[^2, \mathcal{B}(]0, 1[^2), \mu)$. The functional $M_{m,\mu}\colon \mathcal{L}(\mathcal{A}) \longrightarrow [0, 1]$ given by

$$M_{m,\mu}(f) = \mu(D_{m,f}) \tag{3}$$

will be called (m, μ)*-aggregation operator*.

It is easy to see that $M_{m,\mu}$ is an aggregation operator in the sense of [10, 11]: obviously we have $M_{m,\mu}(0) = 0$ and $M_{m,\mu}(1) = 1$, and, if $f \le g$, then clearly $D_{m,f} \subseteq D_{m,g}$ and, because of the monotonicity of μ, $M_{m,\mu}(f) \le M_{m,\mu}(g)$.

Obviously, (1) is a (P, λ^2)-aggregation operator and (2) is a (λ, λ^2)-aggregation operator (generally, λ^n will denote the Lebesgue measure on the n-dimensional space $\mathbb{R}^n$).

Observe that, in the case $X = \{1, 2\}$, each $f \in \mathcal{L}(2^X)$ can be identified with a two-dimensional vector $(x, y) \in [0, 1]^2$ via $x = f(1)$ and $y = f(2)$, and each fuzzy measure $m\colon 2^X \longrightarrow [0, 1]$ is uniquely determined by the two values $a = m(\{1\})$ and $b = m(\{2\})$. Then the endograph $D_{m,f}$ consists of at most two rectangles. More precisely, for arbitrary $(a, b) \in [0, 1]^2$ (representing $m\colon 2^X \longrightarrow [0, 1]$) and arbitrary $(x, y) \in [0, 1]^2$ with $x \le y$, (i.e., either (x, y) or (y, x) represents $f \in \mathcal{L}(2^X)$), the endographs $D_{(a,b),(x,y)}$ and $D_{(a,b),(y,x)}$ are the union of (at most) two rectangles, namely,

$$D_{(a,b),(x,y)} = (]0, x] \times]0, 1[\cup]x, y] \times]0, b[) \cap]0, 1[^2 ,$$
$$D_{(a,b),(y,x)} = (]0, x] \times]0, 1[\cup]x, y] \times]0, a[) \cap]0, 1[^2 .$$

Clearly, if $a = b$ then for each fuzzy measure $\mu\colon \mathcal{B}(]0, 1[^2) \longrightarrow [0, 1]$ the corresponding (m, μ)-aggregation operator $M_{m,\mu}$ is symmetric. If, say $a < b$, then we necessarily have the strict inclusion $D_{(a,b),(y,x)} \subset D_{(a,b),(x,y)}$ for each $(x, y) \in [0, 1]^2$ with $x < y$. Due to the monotonicity of the fuzzy measure μ, the corresponding (m, μ)-aggregation operator $M_{m,\mu}$ then fulfills $M_{m,\mu}(y, x) \le M_{m,\mu}(x, y)$.

In general, for each binary (m, μ)-aggregation operator $M_{m,\mu}\colon [0, 1]^2 \longrightarrow [0, 1]$, the expression

$$(x - y)\big(M_{m,\mu}(x, y) - M_{m,\mu}(y, x)\big) \tag{4}$$

never changes its sign, i.e., it is either always non-negative, or it is always non-positive. Conversely, each binary aggregation operator $A\colon [0, 1]^2 \longrightarrow [0, 1]$ not changing the sign of the expression (4) can be expressed in the form $A = M_{m,\mu}$ for suitable fuzzy measures m and μ.

Indeed, if (4) is always non-negative it suffices to put $a = \frac{1}{3}$ and $b = \frac{2}{3}$ and to define the fuzzy measure $\mu_A \colon \mathcal{B}(]0,1[^2) \longrightarrow [0,1]$ by

$$\mu_A(E) = \sup\{A(f) \mid f \in 2^X, D_{m,f} \subseteq E\}.$$

Note that sufficient conditions for the expressability of the binary aggregation operator as (m,μ)-aggregation operator are, for example, the symmetry or the idempotency.

Similar observations can be done for arbitrary $n \in \mathbb{N}$, i.e., for finite sets $X = \{1,2,\ldots,n\}$. Again, the class of (m,μ)-aggregation operators contains all symmetric as well as all idempotent aggregation operators. Obviously, our approach to aggregation can also be applied to arbitrary abstract measurable space $(X,\mathcal{A})$.

An example of an aggregation operator $A \colon [0,1]^2 \longrightarrow [0,1]$, which changes the sign of the expression (4) and which, subsequently, is not an (m,μ)-aggregation operator, is given by

$$A(x,y) = \min\left(\sqrt{1-(1-x)^2}, \tfrac{1+y}{2}\right).$$

Then $A(0.4,1) > A(1,0.4)$ while $A(0.1,0.2) < A(0.2,0.1)$, i.e., we have different signs in the expression (4).

3 Properties of (m,μ)-aggregation operators

The following properties of (m,μ)-aggregation operators are consequences of the properties of the fuzzy measures m and μ.

Proposition 3.1 *Consider two fuzzy measure spaces $(X,\mathcal{A},m)$ and $(]0,1[^2, \mathcal{B}(]0,1[^2),\mu)$ and the corresponding (m,μ)-aggregation operator $M_{m,\mu} \colon \mathcal{L}(\mathcal{A}) \longrightarrow [0,1]$. Then we have:*

(i) $M_{m,\mu}$ is idempotent for each fuzzy measure m on $(X,\mathcal{A})$ if and only if for all $x \in]0,1[$ we have $\mu(]0,x] \times]0,1[) = x$.

(ii) The fuzzy measure m on $(X,\mathcal{A})$ can be reproduced from $M_{m,\mu}$ via $m(E) = M_{m,\mu}(\mathbf{1}_E)$ if and only if $\mu(]0,1[\times]0,x[) = x$ for all $x \in \mathrm{Ran}(m)$.

(iii) If X is a finite set and if the fuzzy measure m is symmetric, then $M_{m,\mu}$ is symmetric for each fuzzy measure μ on $(]0,1[^2, \mathcal{B}(]0,1[^2))$.

(iv) If the fuzzy measures m and μ are continuous then the (m,μ)-aggregation operator $M_{m,\mu}$ is also continuous.

Proof: Suppose first that the (m,μ)-aggregation operator $M_{m,\mu}$ is idempotent, i.e., $M_{m,\mu}(c) = c$ for all constant functions $c \colon X \longrightarrow [0,1]$ with $c \in [0,1]$. Then, however, for each $c \in]0,1[$ we get $D_{m,c} =]0,c] \times]0,1[$, i.e.,

$$c = M_{m,\mu}(c) = \mu(]0,c] \times]0,1[).$$

The converse in (i) is obvious.

Concerning (ii), it suffices to observe that for all $E \in \mathcal{A}$

$$D_{m,\mathbf{1}_E} =]0,1[\times]0, m(E)[\,.$$

If the fuzzy measure m is symmetric, then for each bijection $\varphi\colon X \longrightarrow X$ we obtain the equality of the endographs $D_{m,f} = D_{m,f\circ\varphi}$ for all $f \in 2^X$. Then necessarily

$$M_{m,\mu}(f) = \mu(D_{m,f}) = \mu(D_{m,f\circ\varphi}) = M_{m,\mu}(f\circ\varphi),$$

i.e., $M_{m,\mu}$ is a symmetric aggregation operator.

Finally, (iv) follows from the continuity of the endographs which is a consequence of the continuity of m. □

It should be noted that the symmetry of fuzzy measures (and of aggregation operators) can be formulated also for every bounded Borel subset X of $\mathbb{R}^n$ with $n \in \mathbb{N}$ and $\mathcal{A} = \mathcal{B}(X)$. Then the symmetry is taken as invariance with respect to bijective transformations on X preserving the Lebesgue measure. This again means that a symmetric fuzzy measure $m\colon \mathcal{A} \longrightarrow [0,1]$ is a distorted Lebesgue measure on $\mathcal{A}$, i.e.,

$$m(E) = h\big(\tfrac{\lambda^n(E)}{\lambda^n(X)}\big)$$

for some non-decreasing function $h\colon [0,1] \longrightarrow [0,1]$ with $h(0) = 0$ and $h(1) = 1$.

The following results follow immediately from Definition 2.1.

Corollary 3.2 *Consider two fuzzy measure spaces* $(X, \mathcal{A}, m)$ *and* $(]0,1[^2, \mathcal{B}(]0,1[^2), \mu)$ *and the corresponding* (m, μ)*-aggregation operator* $M_{m,\mu}\colon \mathcal{L}(\mathcal{A}) \longrightarrow [0,1]$*. Then we have:*

(i) If m is the weakest fuzzy measure, i.e., if $m(A) = 0$ for all $A \neq]0,1[^2$, then $M_{m,\mu}$ is the weakest aggregation operator, i.e., we have $M_{m,\mu}(f) = 0$ for all $f \neq 1$ and for all fuzzy measures μ.

(ii) If m is the strongest fuzzy measure, i.e., if $m(A) = 1$ for all $A \neq \emptyset$, then $M_{m,\mu}$ is the strongest aggregation operator, i.e., we have $M_{m,\mu}(f) = 1$ for all $f \neq 0$ and for all fuzzy measures μ.

(iii) For each increasing function $h\colon [0,1] \longrightarrow [0,1]$ with $h(0) = 0$ and $h(1) = 1$, also $h \circ \mu$ is a fuzzy measure and $M_{m,h\circ\mu} = h \circ M_{m,\mu}$.

(iv) If $A\colon [0,1]^n \longrightarrow [0,1]$ is an aggregation operator and if $\mu_1, \mu_2, \ldots, \mu_n$ are fuzzy measures such that $\mu = A(\mu_1, \mu_2, \ldots, \mu_n)$, then

$$M_{m,\mu} = A(M_{m,\mu_1}, M_{m,\mu_2}, \ldots, M_{m,\mu_n}).$$

Finally, we prove a result concerning the relationship between endographs $D_{m,f}$ and a pseudo-additivity. Recall that a binary operation $\oplus\colon [0,1]^2 \longrightarrow [0,1]$ is called a pseudo-addition if it is a continuous t-conorm [9], i.e., an associative symmetric binary aggregation operator which is continuous and has 0 as neutral element. Further, recall that two measurable functions $f, g\colon X \longrightarrow [0,1]$ are called comonotone [1, 3, 14] if they are measurable with respect to the same chain $\mathcal{C}$ in

$\mathcal{A}$. Equivalently, comonotonicity of the functions f and g can be characterized as follows: for each finite $E \in \mathcal{A}$ we have $\min\{f(x) \mid x \in E\} = f(z_0)$ if and only if $\min\{g(x) \mid x \in E\} = g(z_0)$.

If $(X, \mathcal{T})$ is a compact topological space and if $\mathcal{A}$ is the Borel σ-algebra generated by $\mathcal{T}$, then the comonotonicity can also be formulated in a more general setting for continuous functions $f, g\colon X \longrightarrow [0,1]$.

Proposition 3.3 *Consider a fuzzy measure $(X, \mathcal{A}, m)$ where either X is a finite set and $\mathcal{A} = 2^X$ or $(X, \mathcal{T})$ is a compact topological space and $\mathcal{A}$ is the Borel σ-algebra generated by $\mathcal{T}$. For all comonotone (continuous) functions $f, g\colon X \longrightarrow [0,1]$ and for each pseudo-addition $\oplus\colon [0,1]^2 \longrightarrow [0,1]$ we obtain*

$$D_{m,f\oplus g} = D_{m,f} \oplus D_{m,g}, \tag{5}$$

where

$$D_{m,f} \oplus D_{m,g} = D_{m,f} \cup D_{m,g} \cup D^{\oplus}$$

and

$$D^{\oplus} = \{(u \oplus v, y) \mid (u,y) \in D_{m,f}, (v,y) \in D_{m,g}\}.$$

Proof: Obviously $f \oplus g \geq f$ and $f \oplus g \geq g$, i.e., $D_{m,f} \cup D_{m,g} \subseteq D_{m,f\oplus g}$. If $(u,y) \in D_{m,f}$ and $(v,y) \in D_{m,g}$, i.e., $0 < y < m(\{f \geq u\})$ and $0 < y < m(\{g \geq v\})$ then clearly

$$\{f \geq u\} \cap \{g \geq v\} \subseteq \{f \oplus g \geq u \oplus v\}.$$

Because of the comonotonicity of f and g, all three sets $\{f \geq u\}, \{g \geq v\}$ and $\{f \oplus g \geq u \oplus v\}$ are contained in the same chain $\mathcal{C}$ in $\mathcal{A}$. Then the monotonicity of m implies

$$m(\{f \oplus g \geq u \oplus v\}) \geq \min(m(\{f \geq u\}), m(\{g \geq v\})),$$

and, therefore, $y < m(\{f \oplus g \geq u \oplus v\})$, proving $(u \oplus v, y) \in D_{m,f\oplus g}$. As a consequence we have

$$D_{m,f} \oplus D_{m,g} \subseteq D_{m,f\oplus g}.$$

To show the converse inclusion, observe first that for each $t \in \,]0,1[$ the set $E = \{f \oplus g \geq t\}$ is closed. If $E \neq \emptyset$ let $z_0 \in E$ be the point at which f, g and thus also $f \oplus g$ attain their respective minimum on E. For each point $(t,y) \in D_{m,f\oplus g}$ we have

$$0 < y < m(\{f \oplus g \geq t\}),$$

i.e., $E = \{f \oplus g \geq t\} \neq \emptyset$. Put $u = f(z_0)$ and $v = g(z_0)$. Then $u \oplus v = t$ and thus $E \subset \{f \geq u\}$ and $E \subset \{g \geq v\}$, i.e., $0 < y < m(\{f \geq u\})$ and $0 < y < m(\{g \geq v\})$, proving $(t,y) \in D_{m,f} \oplus D_{m,g}$ whenever $u > 0$ and $v > 0$. If $u = 0$, i.e., $v = t$, then $(t,y) \in D_{m,g}$. Similarly, if $v = 0$ then $u = t$ and thus $(t,y) \in D_{m,f}$. Hence

$$D_{m,f\oplus g} \subseteq D_{m,f} \oplus D_{m,g},$$

i.e., equality (5) holds. □

If, as a special case, $\oplus$ equals the maximum $S_{\mathbf{M}}$ (which is the only idempotent pseudo-addition), then we simply have

$$\max(D_{m,f}, D_{m,g}) = D_{m,f} \cup D_{m,g},$$

i.e., for comonotone (continuous) functions f and g we obtain

$$D_{m,\max(f,g)} = D_{m,f} \cup D_{m,g}.$$

On the other hand, if $\oplus$ equals the bounded sum $S_{\mathbf{L}}$, i.e., $x \oplus y = \min(x+y,1)$, then for two comonotone (continuous) functions f and g satisfying $f+g \leq 1$ we have

$$D_{m,f+g} = S_{\mathbf{L}}\left(D_{m,f}, D_{m,g}\right),$$

and for each $y \in]0,1[$ we have

$$\begin{aligned} &\lambda(\{x \in]0,1[\mid (x,y) \in D_{m,f+g}\}) \\ &\qquad = \lambda(\{u \in]0,1[\mid (u,y) \in D_{m,f}\}) \\ &\qquad\quad + \lambda(\{v \in]0,1[\mid (v,y) \in D_{m,g}\}). \end{aligned}$$

4 σ-additivity of μ

As we shall see later, especially important is the case when we are constructing an (m,μ)-aggregation operator $M_{m,\mu}$ by means of some σ-additive measure $\mu\colon \mathcal{B}(]0,1[^2) \longrightarrow [0,1]$, in which case μ is a probability measure on the product space $(]0,1[^2, \mathcal{B}(]0,1[^2))$. As a consequence of Sklar's Theorem [17] (see also [13, Theorem 2.4.1]), in this case there exists a copula $C\colon [0,1]^2 \longrightarrow [0,1]$ such that for all $(x,y) \in]0,1[^2$

$$\mu(]0,x[\times]0,y[) = C(P_1(]0,x[), P_2(]0,y[)),$$

where P_1 and P_2 are the respective marginal probabilities of μ.

As an immediate consequence of Proposition 3.3, we have the following result concerning comonotone additivity.

Corollary 4.1 *Let $\mu\colon \mathcal{B}(]0,1[^2) \longrightarrow [0,1]$ be a probability measure with marginals $P_1 = \lambda$ (the Lebesgue measure) and $P_2 = P$ linked by the product copula $T_{\mathbf{P}}$, i.e., $\mu(]0,x[\times]0,y[) = x \cdot P(]0,y[)$ for all $(x,y) \in]0,1[^2$. Let $(X,\mathcal{A},m)$ be a fuzzy measure space such that the underlying topological space $(X,\mathcal{T})$ is compact. Then the (m,μ)-aggregation operator $M_{m,\mu}$ is comonotone additive, i.e., for all measurable (continuous) comonotone functions $f,g\colon X \longrightarrow [0,1]$ with $f+g \leq 1$ we have*

$$M_{m,\mu}(f+g) = M_{m,\mu}(f) + M_{m,\mu}(g).$$

Keeping all the notations and hypotheses of Corollary 4.1 and introducing a new fuzzy measure $P \circ m\colon \mathcal{A} \longrightarrow [0,1]$ by $P \circ m(E) = P\left(]0, m(E)[\right)$, it is not difficult to check that the (m,μ)-aggregation operator $M_{m,\mu}$ discussed in Corollary 4.1 equals the Choquet integral [2, 3, 14] with respect to $P \circ m$, i.e.,

$$M_{m,\mu}(f) = (C)\int_X f\, dP \circ m.$$

Turning our attention to the case when $\mu\colon \mathcal{B}(]0,1[^2) \longrightarrow [0,1]$ is a probability measure with marginal probabilities $P_1 = P_2 = \lambda$, then Sklar's Theorem [17, 13] implies that μ is uniquely determined by the copula C linking P_1, P_2 and μ, i.e., for all $(x,y) \in]0,1[^2$

$$\mu(]0,x[\times]0,y[) = C(x,y).$$

To avoid any misunderstanding, such a fuzzy measure will be denoted by μ_C. However, then $\mu_C(]0,1[\times]0,x[) = x$ for all $x \in [0,1]$, and due to Proposition 3.1(ii), for each fuzzy measure space $(X,\mathcal{A},m)$, the (m,μ)-aggregation operator M_{m,μ_C} reproduces the underlying fuzzy measure m. Similarly, $\mu_C(]0,x]\times]0,1[) = x$ for every $x \in]0,1[$. However, then Proposition 3.1(i) implies the idempotency of the operator M_{m,μ_C}, independently of the fuzzy measure m. We therefore have proved the following result:

Proposition 4.2 *Let $C\colon [0,1]^2 \longrightarrow [0,1]$ be a copula and denote by μ_C the unique probability measure on $(]0,1[^2, \mathcal{B}(]0,1[^2)$ with $\mu_C(]0,x[\times]0,y[) = C(x,y)$ for all $(x,y) \in]0,1[^2$. Then, for each fuzzy measure space $(X,\mathcal{A},m)$, the (m,μ_C)-aggregation operator M_{m,μ_C} is an idempotent aggregation operator and we have $M_{m,\mu_C}(\mathbf{1}_A) = m(A)$ for all $A \in \mathcal{A}$.*

Note that such a copula-based approach to aggregation was originally proposed in [6] for the Frank family of t-norms (see, e.g., [4, 9]). Depending on the choice of the copula C, we obtain some well-known types of integrals.

Example 4.3 Keeping the notations of Proposition 4.2, we obtain the following special cases:

(i) If C equals the standard product $T_{\mathbf{P}}$, i.e., $\mu_{T_{\mathbf{P}}}$ is the Lebesgue measure on $\mathcal{B}(]0,1[^2)$, then $M_{m,\mu_{T_{\mathbf{P}}}}$ is just the Choquet integral with respect to m (see [2, 14]).

If, in addition, m is a σ-additive measure on $(X,\mathcal{A})$, then $M_{m,\mu_{T_{\mathbf{P}}}}$ coincides with the classical Lebesgue integral with respect to m, and for $X = \{1,2,\ldots,n\}$ we obtain a weighted mean.

If $X = \{1,2,\ldots,n\}$ and if m is a symmetric fuzzy measure on $(X,2^X)$ then $M_{m,\mu_{T_{\mathbf{P}}}}$ is an OWA operator [21].

(ii) If C equals the minimum $T_{\mathbf{M}}$ then

$$\mu_{T_{\mathbf{M}}}(A) = \lambda(\{x \in]0,1[\mid (x,x) \in A\}),$$

and $M_{m,\mu_{T_{\mathbf{M}}}}$ equals the Sugeno integral (see [18] and also [14]).

If $X = \{1,2,\ldots,n\}$ and if m is a symmetric fuzzy measure on $(X,2^X)$ then $M_{m,\mu_{T_{\mathbf{M}}}}$ is an WOWM (weighted ordered weighted maximum) operator [15].

(iii) If C equals the Łukasiewicz t-norm $T_{\mathbf{L}}$ then

$$\mu_{T_{\mathbf{L}}}(A) = \lambda(\{x \in]0,1[\mid (x,1-x) \in A\}),$$

and $M_{m,\mu_{T_{\mathbf{L}}}}$ is the so-called opposite Sugeno integral [6].

Note that each Choquet integral-based aggregation operator can be represented as an $M_{m,\mu_{T_{\mathbf{P}}}}$ operator, including all OWA operators. Similarly, each Sugeno integral-based aggregation operator, including all WOWM operators, is of the form $M_{m,\mu_{T_{\mathbf{M}}}}$.

Recall that the dual operator $A^d\colon \mathcal{L}(\mathcal{A}) \longrightarrow [0,1]$ of an aggregation operator $A\colon \mathcal{L}(\mathcal{A}) \longrightarrow [0,1]$ is given by the formula

$$A^d(f) = 1 - A(1-f).$$

Similarly, for each fuzzy measure m on $(X, \mathcal{A})$ its dual $m^d \colon \mathcal{A} \longrightarrow [0,1]$ is defined by

$$m^d(E) = 1 - m(X \setminus E).$$

For an arbitrary copula C and the aggregation operator M_{m,μ_C} considered in Proposition 4.2 we obtain

$$\begin{aligned} M^d_{m,\mu_C}(f) &= 1 - M_{m,\mu_C}(1-f) \\ &= 1 - \mu_C(\{(u,v) \mid v < m(\{1-f \geq u\})\}) \\ &= \mu_C(\{(x,y) \mid 1-y \leq m^d(\{f > 1-x\})\}). \end{aligned}$$

Since the probability μ_C is non-atomic, the latter expression can be transformed as follows,

$$\begin{aligned} &\mu_C(\{(x,y) \mid 1-y \leq m^d(\{f > 1-x\})\}) \\ &\qquad = \mu_{\hat{C}}(\{(x,y) \mid y < m^d(\{f \geq x\})\}) \\ &\qquad = M_{m^d,\mu_{\hat{C}}}(f), \end{aligned}$$

where the survival copula $\hat{C} \colon [0,1]^2 \longrightarrow [0,1]$ linked to the copula C (see [13]) is given by

$$\hat{C}(x,y) = x + y - 1 + C(1-x, 1-y).$$

Therefore, we have just proved the following result (compare also [9, 8]):

Proposition 4.4 *Keeping the notations and hypotheses of Proposition 4.2, we have*

$$M^d_{m,\mu_C} = M_{m^d,\mu_{\hat{C}}}. \tag{6}$$

Observe that if a copula C coincides with its survival copula $\hat{C}$, then a special form of (6) holds, namely, $M^d_{m,\mu_C} = M_{m^d,\mu_C}$. All copulas with the property $C = \hat{C}$ were characterized in [7]. In particular, an associative copula C coincides with its survival copula $\hat{C}$ if and only if C is either a member of the family of Frank t-norms $(T^{\mathbf{F}}_\lambda)_{\lambda \in [0,\infty]}$ (see [4]) or if C is a symmetric ordinal sum of Frank t-norms [9, 7]. Because of $T^{\mathbf{F}}_0 = T_{\mathbf{M}}$, $T^{\mathbf{F}}_1 = T_{\mathbf{P}}$, and $T^{\mathbf{F}}_\infty = T_{\mathbf{L}}$, for all Sugeno, Choquet and opposite Sugeno integrals we have

$$\Big(\int_X f\, dm\Big)^d = \int_X f\, dm^d.$$

Now we show how a transformation of an aggregation operator is related to our geometric approach to aggregation. The proof of the following result is a matter of simple checking only—it is therefore omitted.

Proposition 4.5 *Consider two fuzzy measure spaces $(X, \mathcal{A}, m)$ and $(]0,1[^2, \mathcal{B}(]0,1[^2), \mu)$, the corresponding (m,μ)-aggregation operator $M_{m,\mu}$, and an increasing bijection $\varphi \colon [0,1] \longrightarrow [0,1]$. Then the function $(M_{m,\mu})_\varphi \colon \mathcal{L}(\mathcal{A}) \longrightarrow [0,1]$ defined by*

$$(M_{m,\mu})_\varphi(f) = \varphi^{-1}(M_{m,\mu}(\varphi \circ f))$$

is an (m_φ, μ_φ)-aggregation operator with respect to the two fuzzy measures $m_\varphi = \varphi^{-1} \circ m$ and $\mu_\varphi = \varphi^{-1} \circ \mu \circ \varphi$, the latter in the following sense:

$$\mu_\varphi(E) = \varphi^{-1}(\mu(\{(\varphi(x), \varphi(y)) \mid (x,y) \in E\})).$$

Combining the results of Proposition 4.5 and Example 4.3, we see that the class of weighted quasi-arithmetic means with respect to increasing bijections $\varphi\colon [0,1] \longrightarrow [0,1]$ (i.e., all cancellative weighted quasi-arithmetic means) is related to the class of S-measures m on $(X,\mathcal{A})$ and λ_φ on $(]0,1[^2, \mathcal{B}(]0,1[^2))$ with $\lambda_\varphi(]0,x[\times]0,y[) = T(x,y)$ for all $(x,y) \in [0,1]^2$, where S is the nilpotent t-conorm with additive generator φ and T the strict t-norm with multiplicative generator φ [9]. Observe also that then M_{m,λ_φ} is exactly the (S,T)-integral with respect to m, as introduced in [8].

Similarly, Choquet-like integrals as studied in [12] can be represented as M_{m,λ_φ}, where m can be an arbitrary fuzzy measure on $(X,\mathcal{A})$ in this case. If, in addition, m is a symmetric fuzzy measure, then M_{m,λ_φ} is an ordered weighted quasi-arithmetic mean. However, if we consider the strongest copula $T_\mathbf{M}$ and put $\mu = (\mu_{T_\mathbf{M}})_\varphi$ (see Propositions 4.2 and 4.5), then for each increasing bijection φ the resulting operator $M_{m,\mu}$ equals the Sugeno integral with respect to m (compare Proposition 4.2(ii)), independently of φ.

Finally, if $\mu = P_1 \otimes P_2$, i.e., the product copula $T_\mathbf{P}$ and the marginal probabilities P_1, and P_2 have determined the product measure μ, then we obtain

$$M_{m,\mu}(f) = (C)\int_X F_1 \circ f \; d(F_2 \circ m).$$

This means that $M_{m,\mu}$ is the Choquet integral of the distorted input $F_1 \circ f$ with respect to the distorted fuzzy measure $F_2 \circ m$, where F_1 and F_2 are the distribution functions of the probabilities P_1 and P_2, respectively, i.e., for all $x \in \,]0,1[$ and for $i = 1,2$ we have $F_i(x) = P_i(]0,x[)$.

5 Maxitivity of μ

Recall that a fuzzy measure $\mu\colon \mathcal{B}(]0,1[^2) \longrightarrow [0,1]$ is called maxitive [16, 19, 14] (or F-additive in [18]) if for all $E, F \in \mathcal{B}(]0,1[^2)$ we have

$$\mu(E \cup F) = \max(\mu(E), \mu(F)).$$

As an immediate consequence of Proposition 3.3, we have the following result for maxitive fuzzy measures:

Corollary 5.1 *Consider two fuzzy measure spaces $(X,\mathcal{A},m)$ (where the underlying topological space $(X,\mathcal{T})$ is compact) and $(]0,1[^2, \mathcal{B}(]0,1[^2)$, and two comonotone measurable (continuous) functions $f,g\colon X \longrightarrow [0,1]$. If the fuzzy measure μ is maxitive then we have*

$$M_{m,\mu}(\max(f,g)) = \max(M_{m,\mu}(f), M_{m,\mu}(g)),$$

i.e., the (m,μ)-aggregation operator $M_{m,\mu}$ is comonotone maxitive.

An interesting class of aggregation operators is obtained when considering a possibility measure π_φ on $(]0,1[^2, \mathcal{B}(]0,1[^2))$ defined by

$$\pi_\varphi(E) = \sup\{\varphi(x,y) \mid (x,y) \in E\}, \tag{7}$$

where $\varphi\colon \,]0,1[^2 \longrightarrow \,]0,1[$ is a function fulfilling $\sup\{\varphi(x,y) \mid (x,y) \in \,]0,1[^2\} = 1$. Note that each possibility measure π_φ is obviously maxitive. The proof of the following result is again a matter of checking the requirements of Definition 2.1 only.

Proposition 5.2 *Let* $\varphi\colon\ ]0,1[^2 \longrightarrow]0,1[$ *be a function which is non-decreasing and left-continuous in its second component, let* m *be a fuzzy measure on* $(X,\mathcal{A})$ *and let* π_φ *the possibility measure on* $(]0,1[^2, \mathcal{B}(]0,1[^2))$ *defined by (7). Then we have*

$$M_{m,\pi_\varphi}(f) = \sup\{\varphi(x, m(\{f \geq x\})) \mid x \in]0,1[\}.$$

Example 5.3 Keeping the notations of Proposition 5.2 we get:

(i) $M_{m,\pi_{T_\mathbf{M}}}$ is the Sugeno integral with respect to m [18].

(ii) $M_{m,\pi_{T_\mathbf{P}}}$ is the maxitive integral with respect to m [14, 16].

(iii) If T is a strict t-norm, then M_{m,π_T} is a generalization of the Sugeno integral with respect to m, as introduced in [20].

6 Concluding remarks

We have introduced a geometric interpretation of aggregation by means of fuzzy measures. Our constructive method allows us to introduce several aggregation operators either for finitely many inputs, or on abstract spaces. Moreover, many well-known aggregation operators, among them weighted means, OWA operators, quasi-arithmetic means, are special cases in our approach, which can be viewed as a unified approach to all these classes of aggregation operators.

Acknowledgment

This work was supported by two European actions (CEEPUS network SK-42 and COST action 274) as well as by grants VEGA 1/8331/01 and MNTS (Yugoslavia).

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