Archimedean components of triangular norms

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Abstract — The Archimedean components of triangular norms (which turn the closed unit interval into an abelian, totally ordered semigroup with neutral element 1) are studied, in particular their extension to triangular norms, and some construction methods for Archimedean components are given. The triangular norms which are uniquely determined by their Archimedean components are characterized. Using ordinal sums and additive generators, new types of left-continuous triangular norms are constructed.

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1 Introduction

Equipped with a triangular norm as binary operation, the unit interval is an abelian, totally ordered semigroup having $1$ as neutral element [7, 16]. Triangular norms were introduced by B. Schweizer and A. Sklar [32] in the framework of probabilistic metric spaces (see [13, 34]), and they were originally used to carry over the triangle inequality from classical metric spaces to this more general setting, following some ideas outlined by K. Menger [24] in 1942.

Although no representation of the class of all t-norms is known so far, a well-known fact is that continuous t-norms are just ordinal sums [5] of continuous Archimedean t-norms [23, 34]. However, one issue (also linked to the structure of continuous t-norms) is not yet fully understood in the case of general (i.e., not necessarily continuous) t-norms: their Archimedean components. The aim of this paper is to contribute to this understanding.

Archimedean components of triangular norms are convex subsemigroups of the unit interval, containing an element $x$ if and only if each $n$-th power of $x$ is also an element thereof. The relationship between t-norms and their Archimedean components is studied, in particular, under which conditions a t-norm is uniquely determined by its Archimedean components. This naturally leads to the construction of t-norms from Archimedean components and, subsequently, to the construction of Archimedean components themselves.

We characterize the class of triangular norms which are uniquely determined by their Archimedean components (Theorem 4.4). In general, we give explicit formulae for the strongest t-norm having a given system of Archimedean components, and (under some additional assumptions) for the weakest t-norm which can be written as an ordinal sum, as well as for the weakest t-norm having among their Archimedean components the given ones. A general construction of t-norms from Archimedean components is given, and several construction methods for non-trivial Archimedean components are presented. In particular, the construction suggested in Proposition 5.4 always yields left-continuous Archimedean components generated by continuous, non-increasing additive generators which, in combination with Theorem 2.7, gives rise to a rich class of left-continuous t-norms.

The problems studied here add new insights into the structure of general t-norms and bring new construction methods for t-norms. This has an impact to several fields of applications where the need for a deeper understanding of (left-continuous) t-norms has significantly increased over the last years. In many-valued logics based on residuated lattices [12, 14, 19, 36] (compare also [4, 31]) the conjunction is evaluated by lower semicontinuous functions, which are just left-continuous t-norms if the set of truth values equals $[0, 1]$. Similar is the situation in the field of probabilistic metric spaces, in the general theory of non-additive measures and integrals [22, 30], in chaos theory and dynamical systems [35], and in the modelling of preference structures [9] and of cooperative games [2].

The paper is organized as follows. After presenting the most important facts about triangular norms and ordinal sums we introduce and discuss in Section 3 Archimedean classes and components. Section 4 deals with extensions of Archimedean components to triangular norms, paying special attention to the uniqueness of such extensions. Finally we present some construction methods for Archimedean components.
2 Triangular norms and ordinal sums

Triangular norms (t-norms for short) $T$ are binary operations on the closed unit interval $[0, 1]$ such that $([0, 1], T)$ is an abelian semigroup with neutral element 1 which is totally ordered, i.e., for all $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ we have $T(x_1, y_1) \leq T(x_2, y_2)$, where $\leq$ is the natural order on $[0, 1]$ (we shall reserve the symbol $\leq$ exclusively to this order relation).

Basic examples of t-norms are the minimum $T_M$, the product $T_P$, the Łukasiewicz t-norm $T_L$ given by $T_L(x, y) = \max(x+y-1, 0)$, and the drastic product $T_D$ with $T_D(1, x) = T_D(x, 1) = x$, and $T_D(x, y) = 0$ otherwise. Clearly, $T_M$ is the strongest and $T_D$ is the weakest t-norm, i.e., for each t-norm $T$ we have $T_D \leq T \leq T_M$. For more details on triangular norms we refer to [19].

Several specific notions, properties and constructions from semigroup theory were studied for t-norms, such as the Archimedean property and the cancellation law, nilpotent elements and zero divisors, ordinal sums, etc. [19, 34]

We only recall that a t-norm $T$ is said to be Archimedean if for each $(x, y) \in [0, 1]^2$ there is an $n \in \mathbb{N}$ with $x^{(n)} < y$, where $n$-th power $x^{(n)}$ of $x$ with respect to the operation $T$ is defined inductively by $x_T^{(1)} = x$ and $x_T^{(n+1)} = T(x_T^{(n)}, x)$.

Continuous t-norms turn $[0, 1]$ into a topological semigroup [3] (more precisely, into an $I$-semigroup [8, 27, 29]). Therefore, we have the following representations of continuous (Archimedean) t-norms [19, 23, 34].

Theorem 2.1 A function $T : [0, 1]^2 \rightarrow [0, 1]$ is a continuous Archimedean t-norm if and only if there is a continuous, strictly decreasing function $t : [0, 1] \rightarrow [0, \infty]$ with $t(1) = 0$ such that

$$T(x, y) = t^{(-1)}(t(x) + t(y)), \quad (2.1)$$

where the pseudo-inverse $t^{(-1)} : [0, \infty] \rightarrow [0, 1]$ of $t$ is given by

$$t^{(-1)}(u) = \sup\{x \in [0, 1] \mid t(x) > u\}.$$ 

In general, given a (not necessarily continuous) t-norm $T$, each strictly decreasing function $t : [0, 1] \rightarrow [0, \infty]$ which is right-continuous in 0, satisfies $t(1) = 0$ and, for all $(x, y) \in [0, 1]^2$, $t(x) + t(y) \in \text{Ran}(t) \cup \{t(0), \infty\}$ such that (2.1) holds, is called an additive generator of $T$, and it is uniquely determined by $T$ up to a positive multiplicative constant. Thus, Theorem 2.1 states that continuous Archimedean t-norms are characterized by having continuous additive generators. Note that each t-norm possessing an additive generator is necessarily Archimedean.

Theorem 2.2 A function $T : [0, 1]^2 \rightarrow [0, 1]$ is a continuous t-norm if and only if there exist a family $(T_\alpha)_{\alpha \in A}$ of continuous Archimedean t-norms and a family $(a_\alpha, b_\alpha)_{\alpha \in A}$ of non-empty, pairwise disjoint open subintervals of $[0, 1]$ such that $T$ equals the ordinal sum $(a_\alpha, b_\alpha, T_\alpha)_{\alpha \in A}$, i.e.,

$$T(x, y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-b_\alpha}{b_\alpha-a_\alpha}\right) & \text{if } (x, y) \in [a_\alpha, b_\alpha]^2, \\ \min(x, y) & \text{otherwise}. \quad (2.2) \end{cases}$$

Note that Theorem 2.2 states that continuous t-norms are exactly ordinal sums of continuous Archimedean t-norms [5, 33]. Let us now recall some well-known facts about Archimedean t-norms.
Each Archimedean t-norm satisfies \( T(x, x) < x \) for all \( x \in ]0, 1[ \). The converse is not true, in general:

**Example 2.3** The t-norm \( T \) given by

\[
T(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in \left[0, \frac{1}{2}\right]^2, \\
2(x - \frac{1}{2})(y - \frac{1}{2}) + \frac{1}{2} & \text{if } (x, y) \in \left[\frac{1}{2}, 1\right]^2, \\
\min(x, y) & \text{otherwise,}
\end{cases}
\]

satisfies \( T(x, x) < x \) for all \( x \in ]0, 1[ \) without being Archimedean.

However, if a right-continuous t-norm \( T \) satisfies \( T(x, x) < x \) for all \( x \in ]0, 1[ \) then it is Archimedean.

The continuous Archimedean t-norms without zero divisors are called *strict*, and each of them is isomorphic to the product t-norm \( TP \). Each strict t-norm satisfies the cancelation law on \( ]0, 1[^2 \), and each of its additive generators \( t \) is unbounded, i.e., satisfies \( t(0) = \infty \).

The non-strict continuous Archimedean t-norms are called *nilpotent*, and each of them is isomorphic to the Łukasiewicz t-norm \( TL \). Each nilpotent t-norm \( T \) satisfies the cancellation law on its positive domain \( T^{-1}(]0, 1[) \), each of its additive generators \( t \) is bounded, i.e., satisfies \( t(0) < \infty \), and each \( x \in ]0, 1[ \) is both a zero divisor and a nilpotent element of \( T \).

As an immediate consequence of these facts, a t-norm \( T \) is continuous if and only if it is isomorphic to some ordinal sum of t-norms \( \langle a_{\alpha}, b_{\alpha}, T_{\alpha} \rangle_{\alpha \in A} \) where each summand \( T_{\alpha} \) equals either \( TP \) or \( TL \).

Although the structure of continuous t-norms as ordinal sums of continuous Archimedean t-norms has been known for several decades, a deeper study of general (i.e., not necessarily continuous) t-norms from the semigroup theory point of view only recently gained new momentum. Taking into account the unique infinite dyadic representation of numbers \( x \in ]0, 1[ \) via \( x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \), where \( (x_n)_{n \in \mathbb{N}} \) is a strictly increasing sequence of positive integers, the construction by means of isomorphisms was applied in the following examples [19]:

**Example 2.4** The functions \( T_1, T_2 : [0, 1]^2 \rightarrow [0, 1] \) defined by

\[
T_1(x, y) = \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{2^{x_n+y_n}} & \text{if } (x, y) \in ]0, 1[^2, \\
\min(x, y) & \text{otherwise,}
\end{cases}
\]

\[
T_2(x, y) = \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{2^{x_n+y_n-n}} & \text{if } (x, y) \in ]0, 1[^2, \\
0 & \text{otherwise,}
\end{cases}
\]

are strictly monotone t-norms (i.e., they satisfy the cancellation law on \( ]0, 1[^2 \)) which are non-continuous in each point \( (x, y) \in ]0, 1[^2 \) having at least one coordinate with a finite dyadic representation. We have \( T_1(x, x) < x \) and \( T_2(x, x) < x \), respectively, for all \( x \in ]0, 1[ \), and \( T_2 \) is left-continuous. However, only \( T_1 \) is Archimedean, while \( T_2 \) is not.

Also the exact relationship of general t-norms and ordinal sums of semigroups was clarified recently. Some special semigroups introduced as t-subnorms in [17] and as tosabs in [21] proved to be very useful in this context.
Definition 2.5  
(i) Let $I$ be a non-empty subinterval of $[0, 1]$. A totally ordered abelian semigroup $(I, \ast)$ where the semigroup operation $\ast$ is bounded from above by the minimum, i.e., satisfies $x \ast y \leq \min(x, y)$ for all $(x, y) \in I^2$, will be called a tosab.

(ii) A tosab $([0, 1], \ast)$ is called a t-subnorm.

In [21, Proposition 1.7] it was shown that the only way to obtain a t-norm as an ordinal sum of semigroup operations in the sense of A. H. Clifford [5] is to construct certain ordinal sums of tosabs.

Proposition 2.6 Let $([0, 1], \ast)$ be the ordinal sum of a family $((X_\alpha, \ast_\alpha))_{\alpha \in A}$ of semigroups, i.e., $\{X_\alpha \mid \alpha \in A\}$ is a partition of $[0, 1]$ and for all $(x, y) \in [0, 1]^2$ we have

$$x \ast y = \begin{cases} x \ast_\alpha y & \text{if } (x, y) \in X_\alpha^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Then the operation $\ast$ is a t-norm if and only if each $(X_\alpha, \ast_\alpha)$ is a tosab, if the order on $A$ is compatible with the usual order on $[0, 1]$, and if there is an $\alpha_0 \in A$ such that $1$ is the neutral element of $\ast_{\alpha_0}$.

In [17] ordinal sums of t-subnorms yielding t-norms were studied, and in [21, Theorem 3.1] it was shown that this construction is the most general way to obtain triangular norms as ordinal sums of semigroups in the sense of [5].

Theorem 2.7 Let $T$ be a t-norm. Then $([0, 1], T)$ is an ordinal sum of semigroups if and only if $T$ is an ordinal sum of t-subnorms, i.e., there exist a family $(\ast_\alpha)_{\alpha \in A}$ of t-subnorms and a family $\{(a_\alpha, b_\alpha) \mid \alpha \in A\}$ of pairwise disjoint open subintervals of $[0, 1]$ such that, whenever $b_{\alpha_0} = 1$ for some $\alpha_0 \in A$, then $\ast_{\alpha_0}$ is a t-norm, and whenever $b_{\alpha_0} = a_{\beta_0}$ for some $\alpha_0, \beta_0 \in A$ then either $\ast_{\alpha_0}$ is a t-norm or $\ast_{\beta_0}$ has no zero divisors, and such that

$$T(x, y) = \begin{cases} a_\alpha + (b_\alpha - a_\alpha) \cdot \frac{x - a_\alpha}{b_\alpha - a_\alpha} \ast_\alpha \frac{y - a_\alpha}{b_\alpha - a_\alpha} & \text{if } (x, y) \in [a_\alpha, b_\alpha]^2, \\
\min(x, y) & \text{otherwise.} \end{cases}$$

3 Archimedean classes and components

When investigating the structure of t-norms, their Archimedean subsemigroups play an important role (compare [10]).

Definition 3.1 Let $T$ be a t-norm. Two elements $x, y \in [0, 1]$ are called $T$-Archimedean equivalent if there is an $n \in \mathbb{N}$ such that $x^{(n)}_T \leq y \leq x$ or $y^{(n)}_T \leq x \leq y$. For each $x \in [0, 1]$ the equivalence class $I_x$ containing $x$ is called a $T$-Archimedean class.

In [15] it was shown that each T-Archimedean class $I_x$ is a convex subset, i.e., a subinterval, of $[0, 1]$. Obviously, it is possible to generalize Definition 3.1 and to define, in complete analogy, the Archimedean classes of tosabs and, in particular, of t-subnorms.

Proposition 3.2 Let $T$ be a t-norm.
Lemma 3.3 A totally ordered abelian semigroup $(I, \ast)$ is a $T$-Archimedean component of some t-norm $T$ if and only if either $I = \{1\}$ or $I$ is a convex subset of $[0, 1]$ such that for all $x \in I$ we have $\lim_{n \to \infty} x^{(n)}_\ast = \inf I$.

Observe that in the case $I = [0, 1]$ or $I = [0, 1]$ the pair $(I, \ast)$ is a $T$-Archimedean component of some t-norm $T$ if and only if for each $x \in ]0, 1]$ we have $\lim_{n \to \infty} x^{(n)} = 0$, i.e., if and only if $T$ is Archimedean.

Whenever there is no need of stressing the particular t-norm they are linked with, from now on semigroups as described in Lemma 3.3 will be called simply Archimedean components.

Clearly, for each t-norm $T$ the set $\{I_x \mid x \in [0, 1]\}$ of all $T$-Archimedean components forms a partition of $[0, 1]$. Moreover, if for each non-empty subset $A \subseteq [0, 1]$ we put $I_A = \bigcup_{x \in A} I_x$, then $(I_A, T|_{I_A^2})$ is a totally ordered abelian semigroup whose semigroup operation is bounded from above by the minimum. That means, if the set $A$ is convex then $(I_A, T|_{I_A^2})$ is a tosab. For each $x \in [0, 1]$, $(I_x, T|_{I_x^2})$ is the maximal Archimedean subsemigroup containing $x$.

For an arbitrary t-norm $T$ the ordinal sum of its $T$-Archimedean components as introduced in Proposition 2.6 has the same Archimedean components as $T$.

Proposition 3.4 Let $T$ be a t-norm and $\{(I_x, T|_{I_x^2}) \mid x \in [0, 1]\}$ the set of the $T$-Archimedean components. Then the ordinal sum of the $T$-Archimedean components is the strongest t-norm which has the same Archimedean components as $T$.

Proof: Observe first that $\bigcup_{x \in [0, 1]} I_x = [0, 1]$, that the ordinal sum $\ast$ of the $T$-Archimedean components is a t-norm because of Proposition 2.6 and that $\ast$ and $T$ coincide on $\bigcup_{x \in [0, 1]} I_x^2$. 

Therefore, the ordinal sum $\ast$ has the same Archimedean components as $T$. Moreover, since on $[0,1]^2 \setminus \bigcup_{x \in [0,1]} I_x^2$ the ordinal sum $\ast$ coincides with the strongest t-norm $T_M$, $\ast$ is the strongest t-norm with this property. □

Example 3.5 For the nilpotent minimum $T^{nM}$ [19] given by

$$T^{nM}(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

the only non-trivial $T^{nM}$-Archimedean component is $(\left[0, \frac{1}{2}\right], \ast_1)$ with $x \ast_1 y = 0$ for all $(x, y) \in \left[0, \frac{1}{2}\right]^2$. The ordinal sum $\ast$ of the $T^{nM}$-Archimedean components is given by

$$x \ast y = \begin{cases} 0 & \text{if } (x, y) \in \left[0, \frac{1}{2}\right]^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Clearly, $\ast$ and $T^{nM}$ have the same Archimedean components. Moreover, for each set $A$ with $\left[0, \frac{1}{2}\right]^2 \subseteq A \subseteq [0,1]^2 \setminus \left[\frac{1}{2}, 1\right]^2$ satisfying, for each $(x, y) \in A$, both $(y, x) \in A$ and $[0, x] \times [0, y] \subseteq A$, the function $T_A : [0,1]^2 \to [0,1]$ defined by

$$T_A(x, y) = \begin{cases} 0 & \text{if } (x, y) \in A, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

is a t-norm [19] which has the same Archimedean components as $T^{nM}$.

Remark 3.6 (i) Because of Theorem 2.2, each continuous t-norm $T$ is uniquely determined by its non-trivial $T$-Archimedean components.

(ii) As a consequence, each continuous t-norm $T$ is uniquely determined when all the sequences $(x^n_T)_{n \in \mathbb{N}}$ are known (this was shown for strict t-norms in [18]).

(iii) Since the only non-trivial Archimedean classes of an Archimedean t-norm are $[0,1]$ or $[0,1]$, two continuous t-norms with the same Archimedean classes are isomorphic.

4 Extension of Archimedean components to t-norms

As already observed in Remark 3.6, a continuous t-norm $T$ is uniquely determined by its $T$-Archimedean components. Then, however, it is important to determine the Archimedean tosabs related to continuous t-norms. The proof of the following is obvious.

Lemma 4.1 A non-trivial Archimedean tosab $(I, \ast)$ is a $T$-Archimedean component of some continuous t-norm $T$ if and only if either $I = [a, b]$ or $I = \]a, b]$ for some $a, b \in [0,1]$, and if $\ast$ is continuous and satisfies $\lim_{y \to b} x \ast y = x$.

In general, starting with an arbitrary family of pairwise disjoint Archimedean tosabs $((I_\alpha, \ast_\alpha))_{\alpha \in A}$, by adding a family of trivial Archimedean tosabs $((I_\beta, \ast_\beta))_{\beta \in B}$ with $B = [0,1] \setminus \bigcup_{\alpha \in A} I_\alpha$ and $I_\beta = \{\beta\}$, we obtain a family of pairwise disjoint Archimedean tosabs...
is a semigroup with the desired properties, which implies:

Since $b$ is a t-norm, $b$ is a t-norm if and only if $1 \not\in \bigcup_{\alpha \in A} I_{\alpha}$. Obviously, $b$ is the strongest t-subnorm (respectively t-norm) having all the tosabs the same construction as in Proposition 3.4, i.e., defining the binary operation $\ast$ on $[0, 1]$ by

$$x \ast y = \begin{cases} x \ast y & \text{if } (x, y) \in I_{\alpha}^2, \\ \min(x, y) & \text{otherwise}, \end{cases}$$

it is clear that $\ast$ is always a t-subnorm, and that $\ast$ is a t-norm if and only if $1 \not\in I_{\alpha}^2$. Obviously, $\ast$ is the strongest t-subnorm (respectively t-norm) having all the tosabs we started with as $\ast$-Archimedean components. However, as we have seen in Example 3.5, $\ast$ is not necessarily the unique t-(sub)norm having all these Archimedean components.

The following result, whose proof is again straightforward, will be helpful for determining the uniqueness of triangular norms with given Archimedean components.

**Lemma 4.2** Let $T$ be a t-norm and $\{I_x \mid x \in [0, 1]\}$ the set of its $T$-Archimedean components. Then the following are equivalent:

(i) For each t-norm $\tilde{T}$ with $\tilde{T} \neq T$ there is an $x \in [0, 1]$ such that the $\tilde{T}$-Archimedean component $(I_x, T_{|I_x}^2)$ and the $T$-Archimedean component $(I_x, T_{|I_x}^2)$ are different.

(ii) For all $(x, y) \in [0, 1]^2$ with $x \leq y$ there is a unique totally ordered abelian semigroup $(I_{\{x,y\}}, \ast)$, where the operation $\ast$ is bounded from above by the minimum, such that both $(I_x, T_{|I_x}^2)$ and $(I_y, T_{|I_y}^2)$ are subsemigroups of $(I_{\{x,y\}}, \ast)$.

Observe that, for each t-norm $T$ and for each $x \in [0, 1]$ whose $T$-Archimedean class is a singleton (i.e., $I_x = \{x\}$), assertion (ii) in Lemma 4.2 holds because of the monotonicity and boundary conditions of triangular norms. As a consequence, it suffices to consider non-trivial Archimedean classes. We shall look at some important special cases.

**Lemma 4.3** Assume that $I_u = [a, b]$ or $I_u = [a, b]$ and let $(I_u, \ast_u)$ be a $T$-Archimedean component of some t-norm $T$ such that $\ast_u$ is continuous and satisfies $\lim_{y \to b} x \ast_u y > a$ for each $x \in [a, b]$ as well as the conditional cancellation law, i.e., that $y = z$ whenever $x \ast_u y = x \ast_u z > a$. Then there is a unique totally ordered abelian semigroup $(I_u \cup I_b, \ast)$ such that both $(I_u, \ast_u)$ and $(I_b, T_{|I_b}^2)$ are subsemigroups of $(I_u \cup I_b, \ast)$.

**Proof:** Since $I_b$ is an interval with $b \in I_b$, also $I_u \cup I_b$ is an interval. Suppose that $(I_u \cup I_b, \ast)$ is a semigroup with the desired properties, which implies $b \ast b = b$. Fix an arbitrary $x \in I_u$, put $v = x \ast b$ and assume $v < x$. Then necessarily $x > a$ and, because of $\lim_{y \to b} x \ast_u y > a$, also $v > a$, implying $z \ast b = v$ for all $z \in [v, x]$. Moreover, for all $y, z \in I_u$ we have $b \ast (v \ast y) = (b \ast v) \ast y = v \ast_u v$ and $b \ast (x \ast z) = (b \ast x) \ast z = v \ast_u z$. Because of the limit property $\lim_{y \to b} v \ast_u y > a$ there is a $y \in I_u$ with $v \ast_u y > a$, and because of the continuity and monotonicity of $\ast_u$ there is also some $z \in I_u$ such that $x \ast_u z = v \ast_u y$, and the conditional cancellation law implies $z \neq y$. Then, however, we obtain $v \ast_u z = b \ast (x \ast_u z) = b \ast (v \ast_u y) = v \ast_u y$, contradicting our assumption. Therefore, for each $x \in I_u$ we necessarily have $x \ast b = x$ and, subsequently, $x \ast y = x$ for all $y \in I_b$, i.e., the unique semigroup $(I_u \cup I_b, \ast)$ with the desired properties is just the ordinal sum of the semigroups $(I_u, \ast_u)$ and $(I_b, T_{|I_b}^2)$ in the sense of [5].

Summarizing these results, we obtain the following sufficient condition for the uniqueness of t-norms with given Archimedean components.
Theorem 4.4 Let $T$ be a t-norm and suppose that each of its non-trivial components satisfies the hypotheses of Lemma 4.3. Then there is no other t-norm $\tilde{T}$ having the same Archimedean components as $T$.

Theorem 4.4 allows the relationship between continuous t-norms and their Archimedean components to be strengthened (not supposing explicitly the continuity of $T$).

Corollary 4.5 Let $T$ be a t-norm, suppose that each of its non-trivial components satisfies the hypotheses of Lemma 4.3 and, additionally, $\lim_{x \to b_x} T(y, z) = y$ whenever $x \in [0, 1]$, $y \in I_x$ and $b_x = \sup I_x$. Then $T$ is a continuous t-norm, and it is uniquely determined by its Archimedean components.

Observe that the t-norms considered in Examples 2.3, 2.4 and 3.5 do not satisfy the hypotheses of Lemma 4.3.

Example 4.6 Assume that $T$ is a t-norm whose $T$-Archimedean components are $\left(\left[0, \frac{1}{2}\right], \ast_1\right)$ with $x \ast_1 y = x \cdot y$, $\left(\left[\frac{1}{2}, 1\right], \ast_2\right)$ with $x \ast_2 y = \frac{1}{2}$, and the trivial component $\left(\{1\}, \ast\right)$. Then we get

$$T(x, y) = \begin{cases} x \cdot y & \text{if } (x, y) \in \left[0, \frac{1}{2}\right]^2, \\ \frac{1}{2} & \text{if } (x, y) \in \left[\frac{1}{2}, 1\right]^2, \\ \min(x, y) & \text{otherwise}, \end{cases}$$

i.e., $T$ necessarily is the ordinal sum of its Archimedean components.

As already observed, each family $\{(I_\alpha, \ast_\alpha)\}_{\alpha \in A}$ of pairwise disjoint Archimedean tosabs satisfying $I_{\alpha_0} = \{1\}$ for some $\alpha_0 \in A$ whenever $1 \in \bigcup_{\alpha \in A} I_\alpha$ can be extended to a t-norm with no additional non-trivial Archimedean components by means of the ordinal sum construction in the sense of [5] (after filling, if necessary, the gaps in the given system of Archimedean tosabs by singleton components), and the resulting t-norm $T$ is the strongest t-norm coinciding, for each $\alpha \in A$, with $\ast_\alpha$ on $I_\alpha^2$. This fact is due to the simple observation that, for each subinterval $I$ of $[0, 1]$, $(I, \ast)$ with $x \ast y = \min(x, y)$ is the strongest tosab acting on $T$ and that it is an Archimedean component only if $\ast$ is a singleton. Looking for weakest tosabs acting on a subinterval $I$ of $[0, 1]$, such a tosab only exists if $c = \inf I \in I$, in which case the semigroup operation $\ast : I^2 \to I$ is given by $x \ast y = c$.

Corollary 4.7 Let $\{(I_\alpha, \ast_\alpha)\}_{\alpha \in A}$ be a family of pairwise disjoint Archimedean tosabs satisfying $I_{\alpha_0} = \{1\}$ for some $\alpha_0 \in A$ whenever $1 \in \bigcup_{\alpha \in A} I_\alpha$, and assume that we can write $[0, 1] \setminus \bigcup_{\alpha \in A} I_\alpha = \bigcup_{\beta \in B} I_\beta$ with $c_\beta = \inf J_\beta \in J_\beta$. Then the function $T_w : [0, 1]^2 \to [0, 1]$ given by

$$T_w(x, y) = \begin{cases} x \ast_\alpha y & \text{if } (x, y) \in I_{\alpha}^2, \\ c_\beta & \text{if } (x, y) \in J_{\beta}^2, \\ \min(x, y) & \text{otherwise}, \end{cases} \quad (4.1)$$

is a t-norm. It is the weakest t-norm which coincides, for each $\alpha \in A$, with $\ast_\alpha$ on $I_{\alpha}^2$ and which can be written as an ordinal sum of tosabs.
If the hypotheses of Corollary 4.7 are satisfied then the t-norm \( T_w \) given by (4.1) will be called a \( w \)-ordinal sum. Let \((T_\alpha)_{\alpha \in A}\) be a family of t-norms and \((a_\alpha, b_\alpha)_{\alpha \in A}\) be a family of non-empty, pairwise disjoint open subintervals of \([0, 1]\).

The most important feature of the ordinal sum \( T = \langle (a_\alpha, b_\alpha, T_\alpha)_{\alpha \in A} \rangle \) of t-norms [19] given by (2.2) is that \( T \) coincides with the appropriate linear transformations of the t-norms \( T_\alpha \) on the set \( \bigcup_{\alpha \in A} [a_\alpha, b_\alpha]^2 \). By filling the gaps with the minimum t-norm \( T_M \) (which is the strongest of all t-norms), we obviously obtain the strongest possible t-norm which coincides with the linear transformations of the t-norms \( T_\alpha \) on the set \( \bigcup_{\alpha \in A} [a_\alpha, b_\alpha]^2 \).

If the topological closure of the set \( \bigcup_{\alpha \in A} [a_\alpha, b_\alpha] \) is a proper subset of \([0, 1]\), it is always possible to construct other t-norms which are ordinal sums of semigroups in the sense of [5] and which coincide with the linear transformations of the t-norms \( T_\alpha \) on the set \( \bigcup_{\alpha \in A} [a_\alpha, b_\alpha]^2 \).

Recall that the set \([0, 1] \setminus \bigcup_{\alpha \in A} [a_\alpha, b_\alpha] \) can always be written as the union of pairwise disjoint intervals \( \bigcup_{\beta \in Y_\beta} [a_\beta, b_\beta] \), where each \( Y_\beta \) is a component with respect to connectedness containing a smallest element, i.e., each \( Y_\beta \) is of the form \([c_\beta, d_\beta]\) or \([c_\beta, d_\beta]\).

**Corollary 4.8** Let \( T = \langle (a_\alpha, b_\alpha, T_\alpha)_{\alpha \in A} \rangle \) be an ordinal sum of t-norms. Then the \( w \)-ordinal sum \( T_w : [0, 1]^2 \rightarrow [0, 1] \) of the summands \((a_\alpha, b_\alpha, T_\alpha)\) given by

\[
T_w(x, y) = \begin{cases} 
    a_\alpha + (b_\alpha - a_\alpha) \cdot T_\alpha \left( \frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha} \right) & \text{if } (x, y) \in [a_\alpha, b_\alpha]^2, \\
    c_\beta & \text{if } (x, y) \in Y_\beta^2, \\
    \min(x, y) & \text{otherwise},
\end{cases}
\]

is the weakest t-norm which is an ordinal sum of semigroups in the sense of [5] and which contains all the original summands of \( T \).

**Proof:** This follows directly from Proposition 2.6 taking into account that the weakest semigroup operation \( *_\beta \) on \( Y_\beta \) is given by \( x *_\beta y = c_\beta \). \( \square \)

Obviously, all non-trivial \( T \)-Archimedean components are also \( T_w \)-Archimedean components. However, additional non-trivial \( T_w \)-Archimedean components occur if at least one interval \( Y_\beta \) is non-trivial.

It is not difficult to see that we have \( T_w = T \) in Corollary 4.8 if and only if \([0, 1] \) equals the topological closure of \( \bigcup_{\alpha \in A} [a_\alpha, b_\alpha] \).

In general, the \( w \)-ordinal sum \( T_w \) cannot be written as an ordinal sum of t-norms containing all the original summands: for example, starting with the ordinal sum of t-norms \( T = \langle (\frac{n+1}{2n+4}, \frac{n+3}{2n+2}, T_M) \rangle \) \( n \in \mathbb{N} \) then \( T_w \) coincides with the t-norm \( * \) given in Example 3.5, which is not an ordinal sum of t-norms (only an ordinal sum of t-subnorms) containing all the original summands. However, for a finite index set \( A = \{1, 2, \ldots, n\} \) the \( w \)-ordinal sum \( T_w \) turns out to be an ordinal sum of t-norms containing the original summands. Put \( B = \{0, 1\} \cup \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\} = \{c_0, c_1, \ldots, c_m\} \) with \( 0 = c_0 < c_1 < \cdots < c_m = 1 \) and, for \( i \in \{1, 2, \ldots, m\} \),

\[
\tilde{T}_i = \begin{cases} 
    T_j & \text{if } a_j = c_{i-1}, \\
    T_D & \text{otherwise},
\end{cases}
\]

(4.2)

where \( T_D \) is the drastic product. Then \( T_w \) can be written as an ordinal sum of t-norms as follows:

\[
T_w = \langle (c_{i-1}, c_i, \tilde{T}_i) \rangle_{i \in \{1, 2, \ldots, m\}}.
\]

(4.3)
Note that in (4.2) the t-norm $T_D$ can be replaced by an arbitrary t-norm and formula (4.3) still will yield a t-norm which coincides with $T$ on the square $[a_i, b_i]^2$ for each $i \in \{1, 2, \ldots, n\}$. If, for example, we replace $T_D$ by the Łukasiewicz t-norm $T_L$ and if $T_1, T_2, \ldots, T_n$ are all copulas [20, 28], then formula (4.3) gives the weakest copula containing all the summands $(a_i, b_i, T_i)$.

Observe that the w-ordinal sum $T_w$ given in Corollary 4.8 is the weakest t-norm coinciding with $T$ on the square $[a_\alpha, b_\alpha]^2$ for each $\alpha \in \mathcal{A}$ if and only if the topological closure of $\bigcup_{\alpha \in \mathcal{A}} [a_\alpha, b_\alpha]$ is a closed interval of the form $[0, c]$ for some $c \in ]0, 1]$. It was shown in [25] that the weakest t-norm $T$ coinciding with $T$ on the square $[a_\alpha, b_\alpha]^2$ for each $\alpha \in \mathcal{A}$ (which in general is not an ordinal sum of semigroups in the sense of [5] containing all the original summands) is given by

$$
\tilde{T}(x, y) = \begin{cases} 
\sup \left\{ a_\alpha + (b_\alpha - a_\alpha) \cdot T_\alpha \left( \frac{\min(b_\alpha, x) - a_\alpha}{b_\alpha - a_\alpha}, \frac{\min(b_\alpha, y) - a_\alpha}{b_\alpha - a_\alpha} \right) \mid \alpha \in \mathcal{A}, a_\alpha \leq \min(x, y) \right\} & \text{if } \max(x, y) < 1, \\
\min(x, y) & \text{otherwise}. 
\end{cases}
$$

Note that, using the notation of Corollary 4.8, the t-norm $\tilde{T}$ can be rewritten as follows:

$$
\tilde{T}(x, y) = \begin{cases} 
a_\alpha + (b_\alpha - a_\alpha) \cdot T_\alpha \left( \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha} \right) & \text{if } (x, y) \in [a_\alpha, b_\alpha]^2, \\
c_\beta & \text{if } \min(x, y) \in Y_\beta, \max(x, y) < 1, \\
\min(x, y) & \text{otherwise}. 
\end{cases}
$$

## 5 Construction of Archimedean components

As we have seen in Section 4, Archimedean components (tosabs) are, on one hand, essential tools when constructing triangular norms. Indeed, the knowledge of all $T$-Archimedean components of a t-norm $T$ induces the full information about the trajectories $(x_T^{(n)})_{n \in \mathbb{N}}$ for each $x \in ]0, 1]$. To conclude from the trajectories $(x_T^{(n)})_{n \in \mathbb{N}}$ to the $T$-Archimedean components is possible in special cases only, e.g., if the t-norm $T$ is continuous. It is for this reason that the trajectories of a continuous t-norm $T$ determine this t-norm uniquely [18] (compare Remark 3.6).

On the other hand, in order to construct new t-norms (e.g., as ordinal sums of semigroups) it is important to know a rich variety of possible Archimedean components.

**Lemma 5.1** Let $(G, \leq, *)$ be an abelian, totally ordered Archimedean semigroup. Then for all $(u, v) \in G^2$ we have $u * v \leq u$ and $u * v \leq v$, i.e., the operation $*$ is bounded from above by the minimum operator induced by the order $\leq$.

**Proof:** Suppose first that $u < u * u$ for some $u \in G$. Since $*$ is Archimedean, there is an $n \in \mathbb{N}$ such that $(u * u)^{(n)}_u = u^{(2n)}_u \leq u < u * u$. On the other hand, since $(G, \leq, *)$ is totally ordered, we get $u < u * u \leq u^{(3)}_u$ and, by induction, $u < u^{(2n)}_u$ for all $n \in \mathbb{N}$, which is a contradiction. Therefore, we necessarily have $u * u \leq u$ for all $u \in G$. Now suppose that $u < u * v$ for some $(u, v) \in G$ with $u < v$. Then $u < u * v \leq u * v^{(2)}_u = (u * v)^{(2)}_u$ and, again by induction, for each $n \in \mathbb{N}$ we get $u < u * v \leq u * v^{(n)}_u$. On the other hand, the Archimedean property of $*$ implies
the existence of some \( n \in \mathbb{N} \) with \( v^{(n)}_x \leq u < v \) and, since \((G, \preceq, \cdot)\) is totally ordered, also \( u \cdot v^{(n)}_x \preceq u \cdot u \). Because of \( u \preceq u \preceq u \) and, subsequently, \( u \cdot v^{(n)}_x \preceq u \), this is a contradiction. \( \Box \)

Because of Lemma 5.1 it is clear that, given an abelian, totally ordered Archimedean semigroup \((G, \preceq, \cdot)\) and an isomorphic transformation (i.e., an order preserving bijection) \( \varphi : I \to G \), where \( I \) is some subinterval of \([0, 1]\), then also \((I, \preceq, \varphi)\) with \( x \cdot y = \varphi^{-1}(\varphi(x) \cdot \varphi(y)) \) is a tosab and hence a \( T \)-Archimedean component of some \( t \)-norm \( T \). A semigroup \((G, \preceq, \cdot)\) which satisfies the hypotheses of Lemma 5.1 and which is isomorphic to some tosab will be called a fitting Archimedean semigroup.

**Example 5.2**

(i) The semigroup \([1, \infty],[\preceq, +]\) with \( x \preceq y \) if and only if \( x \geq y \) is a fitting Archimedean semigroup: for each open subinterval \([a, b]\) of \([0, 1]\) and for each order preserving bijection \( \varphi : [a, b] \to [1, \infty] \) the pair \([a, b], +_\varphi\) is an Archimedean component.

(ii) Coming back to the \( t \)-norms \( T_1 \) and \( T_2 \) in Example 2.4, equip the set \( G \) of strictly increasing sequences of positive integers with the inverse lexicographic order \( \preceq \), i.e., \((x_n)_{n \in \mathbb{N}} \preceq (y_n)_{n \in \mathbb{N}}\) if and only if there is \( n_0 \in \mathbb{N} \) such that \( x_{n_0} > y_{n_0} \) and \( x_n = y_n \) for all \( n < n_0 \).

Defining the operation \(*\) on \( G \) by \((x_n)_{n \in \mathbb{N}} \ast (y_n)_{n \in \mathbb{N}} = (x_n + y_n)_{n \in \mathbb{N}}\)

and the order preserving bijection \( \varphi : [0, 1], \leq \to (G, \preceq) \) by \( \varphi^{-1}((x_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \)

then \((G, \ast)\) is a fitting Archimedean semigroup and \([0, 1], T_1\) is a \( T_1 \)-Archimedean component.

Now, put \( H = G \cup \{(n)_{n \in \mathbb{N}}\} \), introduce the order \( \sqsubseteq \) on \( H \) by keeping the original order \( \preceq \) on \( G \) and by supposing that the sequence \((n)_{n \in \mathbb{N}}\) be the greatest element of \( H \), and define the binary operation \( \circ \) on \( H \) by \((x_n)_{n \in \mathbb{N}} \circ (y_n)_{n \in \mathbb{N}} = (x_n + y_n - n + 1)_{n \in \mathbb{N}}\).

Then, for a fixed \( m \in \mathbb{N} \), \( \psi : \left(1 - \frac{1}{2^{-m-1}}, 1 - \frac{1}{2^{-m}}\right], \leq \to (H, \sqsubseteq) \) given by \( \psi^{-1}((x_n)_{n \in \mathbb{N}}) = 1 - \frac{1}{2^{m-1}} + \sum_{n=1}^{\infty} \frac{1}{2^{m+n}} \)

is an order preserving bijection, and \((1 - \frac{1}{2^{-m-1}}, 1 - \frac{1}{2^{-m}}], \circ_\psi\) is a \( T_2 \)-Archimedean component.

Isomorphic transformations of fitting Archimedean semigroups can be further generalized. We present here three ways for doing so. The first of them has a straightforward proof, the second one uses pseudo-inverses of monotone functions [19], and the third one starts from a finite semigroup.

**Proposition 5.3** Let \((I, \ast)\) be an Archimedean component, let \( x \in I \) and put \( J_x = [0, x] \cap I \) and \( K_x = [0, x] \cap I \). Then \((J_x, \ast_{|J_x})\) and \((K_x, \ast_{|K_x})\) are Archimedean components.
Proposition 5.4 \textit{Let I be a subinterval of } [0, 1] \textit{and let } f : I \rightarrow [0, \infty] \textit{be a continuous non-increasing function which is unbounded if } \inf I \notin I. \textit{Let the pseudo-inverse } f^{(-1)} : [0, \infty] \rightarrow I \textit{be given by } f^{(-1)}(u) = \sup \{x \in I \mid f(x) > u\} \textit{(using the convention } \sup \emptyset = \inf I)\textit{, and define the operation } \ast \textit{ on } I \textit{by}
\[ x \ast y = f^{(-1)}(f(x) + f(y)). \quad (5.1) \]
\textit{Then } (I, \ast) \textit{ is an Archimedean component, and the semigroup operation } \ast \textit{ is left-continuous.}

\textbf{Proof:} Observe first that \ast is well-defined. Also, \( f^{(-1)} \) is right-continuous, and the continuity of \( f \) implies the strict monotonicity of \( f^{(-1)} \) on the range of \( f \) and also \( f \circ f^{(-1)}(x) = x \) for all \( x \) in the range of \( f \). Since the range of \( f \) is an interval, the operation \ast is associative and we have \( x_{s}^{(n)} = f^{(-1)}(n \cdot f(x)) \), implying \( \lim_{n \to \infty} x_{s}^{(n)} = \inf I \) for each \( x \in I \). The rest of the proof is straightforward. \qed

In combination with Theorem 2.7, the method described in Proposition 5.4 allows large classes of left-continuous t-norms to be constructed as follows (recall that exactly the left-continuous t-norms possess a residuum which is often used as interpretation of the implication in many-valued and fuzzy logics [12, 14]): given an arbitrary family \( \{[a_{\alpha}, a_{\beta}]\}_{\alpha \in A} \) of pairwise disjoint open subintervals of \( [0, 1] \) and a family \( \{f_{\alpha} : [a_{\alpha}, b_{\alpha}] \rightarrow [0, \infty]\}_{\alpha \in A} \) of continuous non-decreasing functions satisfying \( f_{\alpha}(1) = 0 \) whenever \( b_{\alpha} = 1 \) and such that, if \( b_{\alpha} = a_{\beta} \) for some \( \alpha, \beta \in A \), \( f_{\beta}(a_{\beta}) \) is finite only if \( f_{\alpha}(b_{\alpha}) = 0 \), then the function \( T : [0, 1]^{2} \rightarrow [0, 1] \) given by
\[ T(x, y) = \begin{cases} f_{\alpha}^{(-1)}(f_{\alpha}(x) + f_{\alpha}(y)) & \text{if } (x, y) \in [a_{\alpha}, b_{\alpha}]^{2}, \\ \min(x, y) & \text{otherwise}, \end{cases} \]
is a left-continuous t-norm.

Proposition 5.5 \textit{Consider a finite set } G \textit{ with cardinality } n \textit{ and an abelian, totally ordered Archimedean semigroup } (G, \circ) \textit{(observe that } \circ \textit{ is necessarily nilpotent). For each subinterval } [a, b] \textit{ of } [0, 1] \textit{ and for all } n \text{-element subsets } A = \{a_{0}, a_{1}, \ldots, a_{n-1}\} \textit{ with } a_{0} < a_{1} < \cdots < a_{n-1} < b \textit{ let } (A, \circ) \textit{ be the abelian, totally ordered Archimedean semigroup which is isomorphic to } (G, \circ). \textit{Putting } x \ast y = x_{A} \circ y_{A}, \textit{ where } x_{A} = \max \{u \in A \mid u \leq x\}, \textit{ then } ([a, b], \ast) \textit{ is an Archimedean component, and the semigroup operation } \ast \textit{ is right-continuous.}

\textbf{Proof:} The proof follows from the simple observation that \( (x \ast y)_{A} = x_{A} \circ y_{A} \) and from the right continuity of the function \( \tau : [a, b] \rightarrow A \text{ defined by } \tau(x) = x_{A}. \qed \)

Remark 5.6 \textit{(i)} If an Archimedean component \((I, \ast)\) is generated by some additive generator \( f : I \rightarrow ]0, \infty[ \) via (5.1) and if the Archimedean components \((J_{x}, \ast|_{J_{x}})\) and \((K_{x}, \ast|_{K_{x}})\) are obtained as in Proposition 5.3, then these semigroups are generated by the additive generators \( f|_{J_{x}} \) and \( f|_{K_{x}} \), respectively.

\textit{(ii)} As a consequence of [26] we have: if in an Archimedean component \((I, \ast)\) the operation \ast is generated by some additive generator \( f : I \rightarrow ]0, \infty[ \) via (5.1), then \ast is continuous if and only if \( f \) is strictly monotone on the set \( \{x \in I \mid \text{there is some } y \in I \text{ such that } x \leq y \ast y\} \).
Example 5.7 Let \( f : [0,1] \rightarrow [0, \infty] \) be given by \( f(x) = \max(10 - 18x, 1) \). Because of Remark 5.6 the operation \( * \) on \([0,1]\) given by (5.1) is continuous. According to Proposition 5.4, \(([0,1], *)\) is a \( T \)-Archimedean component of the uniquely determined t-norm \( T \) given by

\[
T(x,y) = \begin{cases} 
\min(x,y) & \text{if } \max(x,y) = 1, \\
\max\left(\min(x, \frac{1}{2}) + \min(y, \frac{1}{2}) - \frac{5}{9}, 0\right) & \text{otherwise.}
\end{cases}
\]

Putting \( x_0 = \frac{1}{2} \) we obtain \( J_{x_0} = [0, \frac{1}{2}] \) and \( f|J_{x_0}(x) = 10 - 18x \). Obviously, \( f|J_{x_0} \) generates \( *|J_{x_0} \) (which coincides with \( T|J_{x_0} \)).

6 Concluding remarks

Continuous t-norms are just ordinal sums of continuous Archimedean t-norms, the latter being generated by continuous additive generators (see Theorems 2.2 and 2.1). Left-continuous t-norms and the corresponding residual implications (which are linked by the Galois connection [11]) play a crucial role in many-valued logics, but no characterization of left-continuous t-norms is known so far. Concerning the relationship between left-continuous and continuous t-norms, note that left-continuous t-norms which are either Archimedean or which are generated by some additive generator are necessarily continuous [19]. Proposition 5.4, applied to continuous Archimedean components (i.e., to special threads [6, 16], for example to Archimedean components which are generated by continuous additive generators), gives rise to new left-continuous t-norms. Such left-continuous t-norms can be characterized in very special cases only, e.g., if their Archimedean components satisfy the cancellation law in which case they are generated because of [1]. However, if the Archimedean components of such left-continuous t-norms cannot be extended to \( I \)-semigroups [8, 27, 29] or if they do not satisfy the cancellation law, the characterization of those t-norms is still an open problem.

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