ELSEVIER

Triangular norms. Position paper I: basic analytical and algebraic properties

Erich Peter Klement[a,*], Radko Mesiar[b], Endre Pap[c]

[a]*Department of Algebra, Stochastics and Knowledge-Based Mathematical Systems,
Johannes Kepler University, 4040 Linz, Austria*
[b]*Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering,
Slovak Technical University, 81 368 Bratislava, Slovakia*
[c]*Department of mathematics and Informatics, University of Novi Sad, 21000 Novi Sad, Serbia and Montenegro*

Abstract

We present the basic analytical and algebraic properties of triangular norms. We discuss continuity as well as the important classes of Archimedean, strict and nilpotent t-norms. Triangular conorms and De Morgan triples are also mentioned. Finally, a brief historical survey on triangular norms is given.
© 2003 Elsevier B.V. All rights reserved.

Keywords: Triangular norms

1. Introduction

Triangular norms (briefly t-norms) are an indispensable tool for the interpretation of the conjunction in fuzzy logics [27] and, subsequently, for the intersection of fuzzy sets [67]. They are, however, interesting mathematical objects for themselves.

Triangular norms, as we use them today, were first introduced in the context of probabilistic metric spaces [54,57,58], based on some ideas presented in [43] (see Section 7 for details). They also play an important role in decision making [21,26], in statistics [47] as well as in the theories of non-additive measures [39,50,61,64] and cooperative games [11]. Some parameterized families of t-norms (see, e.g. [22]) turn out to be solutions of well-known functional equations.

Algebraically speaking, t-norms are binary operations on the closed unit interval $[0,1]$ such that $([0,1], T, \leqslant)$ is an abelian, totally ordered semigroup with neutral element 1 [28].

* Corresponding author. Tel.: +43-732-2468-9151; fax: +43-732-2468-1351.
  E-mail addresses: ep.klement@jku.at (E.P. Klement), mesiar@math.sk (R. Mesiar), pap@im.ns.ac.yu, pape@eunet.yu (E. Pap).

For the closely related concept of uninorms (which turn [0,1] into an abelian, totally ordered semigroup with neutral element $e \in]0,1[$) see [38,66].

A recent monograph [38] provides a rather complete overview about triangular norms and their applications.

In a series of three papers we want to summarize in a condensed form the most important facts about t-norms. This Part I deals with the basic analytical properties, such as continuity, and with important classes such as Archimedean, strict and nilpotent t-norms. We also mention the dual operations, the triangular norms, and De Morgan triples. Finally we give a short historical overview on the development of t-norms and their way into fuzzy sets and fuzzy logics.

To keep the paper readable, we have omitted all proofs (usually giving a source for the reader interested in them) and rather included a number of (counter-)examples, in order to motivate and to illustrate the abstract notions used.

Part II will be devoted to general construction methods based mainly on pseudo-inverses, additive and multiplicative generators, and ordinal sums, adding also some constructions leading to non-continuous t-norms, and to a presentation of some distinguished families of t-norms.

Finally, Part III will concentrate on continuous t-norms, in particular, on their representation by additive and multiplicative generators and ordinal sums.

2. Triangular norms

The term triangular norm appeared for the first time (with slightly different axioms) in [43]. The following set of independent axioms for triangular norms goes back to Schweizer and Sklar [53–61].

Definition 2.1. *A triangular norm* (briefly t-*norm*) is a binary operation T on the unit interval [0, 1] which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $T : [0,1]^2 \rightarrow [0,1]$ such that for all $x, y, z \in [0,1]$:

(T1) $T(x,y) = T(y,x)$,
(T2) $T(x, T(y,z)) = T(T(x,y),z)$,
(T3) $T(x,y) \leqslant T(x,z)$ whenever $y \leqslant z$,
(T4) $T(x,1) = x$.

Since a t-norm is an algebraic operation on the unit interval [0,1], some authors (e.g., in [48]) prefer to use an infix notation like $x * y$ instead of the prefix notation $T(x,y)$. In fact, some of the axioms (T1)–(T4) then look more familiar: for all $x, y, z \in [0,1]$

(T1) $x * y = y * x$,
(T2) $x * (y * z) = (x * y) * z$,
(T3) $x * y \leqslant x * z$ whenever $y \leqslant z$,
(T4) $(x * 1) = x$.

Because of the importance of some functional aspects (e.g., continuity) and since we prefer to keep a unified notation throughout this paper, we shall consistently use the prefix notation for t-norms (and t-conorms).
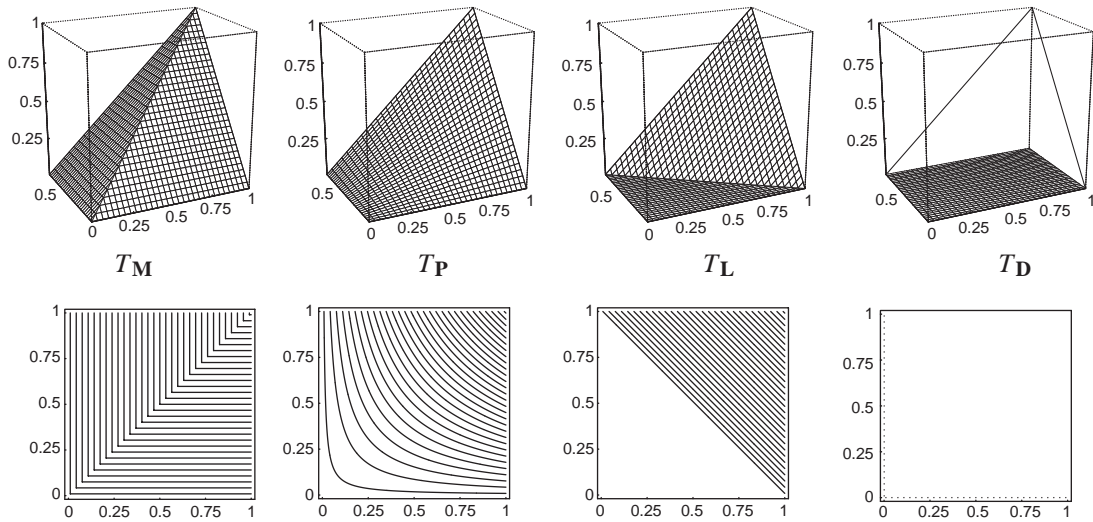
Fig. 1. 3D plots (top) and contour plots (bottom) of the four basic t-norms T_M, T_P, T_L, and T_D (observe that there are no contour lines for T_D).

Since t-norms are obviously extensions of the Boolean conjunction, they are usually used as interpretations of the conjunction Λ in [0, 1]-valued and fuzzy logics.

There exist uncountably many t-norms. In [38, Section 4] some parameterized families of t-norms are presented which are interesting from different points of view.

The following are the four basic t-norms, namely, the *minimum* T_M, the *product* T_P, the *Łukasiewicz* t-*norm* T_L, and the *drastic product* T_D (see Fig. 1 for 3D and contour plots), which are given by, respectively:

$$T_M(x, y) = \min(x, y), \tag{1}$$

$$T_P(x, y) = x \cdot y, \tag{2}$$

$$T_L(x, y) = \max(x + y - 1, 0), \tag{3}$$

$$T_D(x, y) = \begin{cases} 0 & \text{if}(x, y) \in [0, 1[^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{4}$$

These four basic t-norms are remarkable for several reasons. The drastic product T_D and the minimum T_M are the smallest and the largest t-norm respectively (with respect to the pointwise order). The minimum T_M is the only t-norm where each $x \in [0, 1]$ is an idempotent element (compare Definition 6.1), whereas the product T_P and the Łukasiewicz t-norm T_L are prototypical examples of two important subclasses of t-norms, namely, of the classes of strict and nilpotent t-norms, respectively.

It should be mentioned that the t-norms T_M, T_P, T_L, and T_D were denoted M, Π, W, and Z, respectively, in [57].

Sometimes we shall visualize t-norms (and functions $F : [0, 1]^2 \rightarrow [0, 1]$ in general) in different forms: as 3D plots, i.e., as surfaces in the unit cube, as contour plots showing the curves (or,

more generally, the sets) where the function in question has constant (equidistant) values, and, occasionally, as diagonal sections, i.e., as graphs of the function $x \to F(x,x)$.

The boundary condition (T4) and the monotonicity (T3) were given in their minimal form. Together with (T1) it follows that, for all $x \in [0,1]$, each t-norm T satisfies

$$T(0,x) = T(x,0) = 0, \tag{5}$$

$$T(1,x) = x. \tag{6}$$

Therefore, all t-norms coincide on the boundary of the unit square $[0,1]^2$.

The monotonicity of a t-norm T in its second component (T3) is, together with the commutativity (T1), equivalent to the (joint) monotonicity in both components, i.e., to

$$T(x_1, y_1) \leqslant T(x_2, y_2) \quad \text{whenever } x_1 \leqslant x_2 \text{ and } y_1 \leqslant y_2. \tag{7}$$

Since t-norms are just functions from the unit square into the unit interval, the comparison of t-norms is done in the usual way, i.e., pointwise.

Definition 2.2. If, for two t-norms T_1 and T_2, we have $T_1(x,y) \leqslant T_2(x,y)$ for all $(x,y) \in [0,1]^2$, then we say that T_1 is *weaker* than T_2 or, equivalently, that T_2 is *stronger* than T_1, and we write in this case $T_1 \leqslant T_2$.

We shall write $T_1 < T_2$ if $T_1 \leqslant T_2$ and $T_1 \neq T_2$, i.e., if $T_1 \leqslant T_2$ and if $T_1(x_0, y_0) < T_2(x_0, y_0)$ for some $(x_0, y_0) \in [0,1]^2$.

As an immediate consequence of (T1), (T3) and (T4), the drastic product $T_{\mathbf{D}}$ is the weakest, and the minimum $T_{\mathbf{M}}$ is the strongest t-norm, i.e., for each t-norm T we have

$$T_{\mathbf{D}} \leqslant T \leqslant T_{\mathbf{M}}. \tag{8}$$

Between the four basic t-norms we have these strict inequalities

$$T_{\mathbf{D}} < T_{\mathbf{L}} < T_{\mathbf{P}} < T_{\mathbf{M}}. \tag{9}$$

A slight modification of axiom (T4) leads to the following notion, introduced in [30,31].

Definition 2.3. A function $F : [0,1]^2 \to [0,1]$ which satisfies, for all $x,y,z \in [0,1]$, the properties (T1)–(T3) and

$$F(x,y) \leqslant \min(x,y) \tag{10}$$

is called a t-*subnorm*.

Clearly, each t-norm is a t-subnorm, but not vice versa: for example, the zero function is a t-subnorm but not a t-norm.

Each t-subnorm can be transformed into a t-norm by redefining (if necessary) its values on the upper right boundary of the unit square [38, Corollary 1.8].

Proposition 2.4. *If* $F : [0,1]^2 \rightarrow [0,1]$ *is a* t-*subnorm then the function* $T : [0,1]^2 \rightarrow [0,1]$ *defined by*

$$T(x,y) = \begin{cases} F(x,y) & if \ (x,y) \in [0,1[^2, \\ \min(x,y) & otherwise, \end{cases}$$

is a triangular norm.

An interesting question is whether a t-norm is determined uniquely by its values on the diagonal of the unit square. In general, this is not the case, but the two extremal t-norms T_D and T_M are completely determined by their diagonal sections, i.e., by their values on the diagonal of the unit square.

The associativity (T2) allows us to extend each t-norm T (which was introduced as a binary operation) in a unique way to an *n*ary operation for arbitrary $n \in \mathbb{N} \cup \{0\}$ by induction:

$$\mathop{\mathsf{T}}_{i=1}^{n} x_i = \begin{cases} 1 & \text{if } n = 0, \\ T\left(x_n, \mathop{\mathsf{T}}_{i=1}^{n-1} x_i\right) & \text{otherwise.} \end{cases} \tag{11}$$

We also shall use the notation

$$T(x_1, x_2, \ldots, x_n) = \mathop{\mathsf{T}}_{i=1}^{n} x_i.$$

If, in particular, $x_1 = x_2 = \cdots = x_n = x$, we shall briefly write

$$x_T^{(n)} = T(x, x, \ldots, x). \tag{12}$$

The *n*-ary extensions of the minimum T_M and the product T_P are obvious. For the Łukasiewicz t-norm T_L and the drastic product T_D we get

$$T_L(x_1, x_2, \ldots, x_n) = \max\left(\sum_{i=1}^{n} x_i - (n-1), 0\right),$$

$$T_D(x_1, x_2, \ldots, x_n) = \begin{cases} x_i & \text{if } x_j = 1 \text{ for all } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that each t-norm T is weaker than T_M implies that, for each sequence $(x_i)_{i \in \mathbb{N}}$ of elements of [0,1], the sequence

$$\left(\mathop{\mathsf{T}}_{i=1}^{n} x_i\right)_{n \in \mathbb{N}}$$

is non-increasing and bounded from below and, subsequently, convergent. We therefore can extend T to a (countably) infinitary operation putting

$$\mathop{\mathsf{T}}_{i=1}^{\infty} x_i = \lim_{n \to \infty} \mathop{\mathsf{T}}_{i=1}^{n} x_i. \tag{13}$$

However, similarly as for infinite series of numbers, then some desirable properties such as the generalized associativity may be violated (for more details see [44]).

3. Triangular conorms

In [55] triangular conorms were introduced as dual operations of t-norms. We give here an independent axiomatic definition.

Definition 3.1. A *triangular conorm* (t-*conorm* for short) is a binary operation S on the unit interval [0,1] which is commutative, associative, monotone and has 0 as neutral element, i.e., it is a function $S : [0,1]^2 \to [0,1]$ which satisfies, for all $x, y, z \in [0,1]$, (T1)–(T3) and

(S4) $S(x,0) = x$.

The following are the four basic t-conorms, namely, the *maximum* S_M, the *probabilistic sum* S_P, the *Łukasiewicz* t-*conorm* or (*bounded sum*) S_L, and the *drastic sum* S_D (see Fig. 2 for 3D and contour plots), which are given by, respectively:

$$S_M(x, y) = \max(x, y), \tag{14}$$

$$S_P(x, y) = x + y - x \cdot y, \tag{15}$$

$$S_L(x, y) = \min(x + y, 1), \tag{16}$$

$$S_D(x, y) = \begin{cases} 1 & \text{if } (x, y) \in]0,1]^2, \\ \max(x, y) & \text{otherwise.} \end{cases} \tag{17}$$

The t-conorms S_M, S_P, S_L, and S_D were denoted M^*, Π^*, W^* and Z^*, respectively, in [57].

The original definition of t-conorms given in [55] is completely equivalent to the axiomatic definition given above: a function $S : [0,1]^2 \to [0,1]$ is a t-conorm if and only if there exists a t-norm
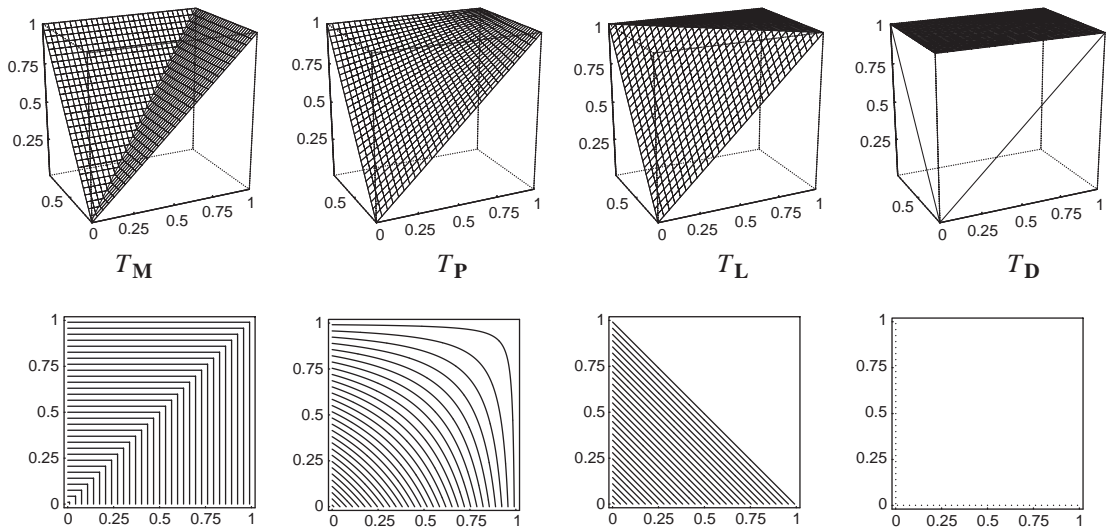

Fig. 2. 3D plots (top) and contour plots (bottom) of the four basic t-conorms S_M, S_P, S_L, and S_D.

T such that for all $(x, y) \in [0, 1]^2$ either one of the two equivalent equalities holds:

$$S(x, y) = 1 - T(1 - x, 1 - y), \tag{18}$$

$$T(x, y) = 1 - S(1 - x, 1 - y). \tag{19}$$

The t-conorm given by (18) is called the *dual* t-*conorm* of T and, analogously, the t-norm given by (19) is said to be the *dual* t-*norm* of S. Obviously, (T_M, S_M), (T_P, S_P), (T_L, S_L), and (T_D, S_D) are pairs of t-norms and t-conorms which are mutually dual to each other.

Considering the standard negation $N_s(x) = 1 - x$ (compare (20)) as complement of x in the unit interval, Eq. (18) explains the name t-conorm. We shall keep this original notion and avoid the term s-*norm* which sometimes is used synonymously in the literature.

The duality expressed in (18) allows us to translate many properties of t-norms into the corresponding properties of t-conorms, including the n ary and infinitary extensions of a t-conorm.

The duality changes the order: if, for some t-norms T_1 and T_2 we have $T_1 \leqslant T_2$, and if S_1 and S_2 are the dual t-conorms of T_1 and T_2, respectively, then we get $S_1 \geqslant S_2$.

If (T, S) is a pair of mutually dual t-norms and t-conorms, then dualities (18) and (19) can be generalized as follows (here I can be an arbitrary finite or countably infinite index set):

$$\mathop{S}_{i \in I} x_i = 1 - \mathop{T}_{i \in I} (1 - x_i),$$
$$\mathop{T}_{i \in I} x_i = 1 - \mathop{S}_{i \in I} (1 - x_i).$$

In fuzzy logics, t-conorms are usually used as an interpretation of the disjunction $\vee$.

4. Negations and De Morgan triples

Finally, let us have a brief look at negations.

Definition 4.1.
(i) A non-increasing function $N : [0, 1] \to [0, 1]$ is called a *negation* if

(N1) $N(0) = 1$ and $N(1) = 0$.

(ii) A negation $N : [0, 1] \to [0, 1]$ is called a *strict negation* if, additionally,

(N2) N is continuous.
(N3) N is strictly decreasing.

(iii) A strict negation $N : [0, 1] \to [0, 1]$ is called a *strong negation* if it is an involution, i.e., if

(N4) $N \circ N = \mathrm{id}_{[0,1]}$.

It is obvious that $N : [0, 1] \to [0, 1]$ is a strict negation if and only if it is a strictly decreasing bijection.

The most important and most widely used strong negation is the *standard negation* $N_s : [0, 1] \to [0, 1]$ given by

$$N_s(x) = 1 - x. \tag{20}$$

Note that $N : [0, 1] \to [0, 1]$ is a strong negation if and only if there is a monotone bijection $g : [0, 1] \to [0, 1]$ such that for all $x \in [0, 1]$

$$\varphi(x) = g^{-1}(N_s(g(x))), \tag{21}$$

i.e., each strong negation is a monotone transformation of the standard negation [62].

The negation $N : [0, 1] \to [0, 1]$ given by $N(x) = 1 - x^2$ is strict, but not strong.

An example of a negation which is not strict and, subsequently, not strong, is the *Gödel negation* $N_G[0, 1] \to [0, 1]$ given by

$$N_G(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in]0, 1]. \end{cases} \tag{22}$$

The standard negation N_s was used, e.g., in [53,54] when introducing t-conorms as duals of t-norms, or in [67] when modeling the complement of a fuzzy set.

Given a t-norm T and a strict negation N, one obtains a t-conorm $S : [0, 1]^2 \to [0, 1]$, which is N-dual to T in the sense of

$$S(x, y) = N^{-1}(T(N(x), N(y))). \tag{23}$$

Note, however, that if N is a non-strict negation, formula (23) cannot be applied.

If N is a strong negation, then, applying the construction in (23) to the t-conorm S, we get back the t-norm T we started with.

A triple (T, S, N), where T is a t-norm, S is a t-conorm and N is a negation is called a *De Morgan triple* if for all $(x, y) \in [0, 1]^2$ we have

$$T(x, y) = N(S(N(x), N(y))),$$
$$S(x, y) = N(T(N(x), N(y))).$$

This means that, given a t-norm $T, (T, S, N)$ is a De Morgan triple if and only if N is a strong negation and S is the N-dual of T.

Let $s : [0, 1] \to [0, 1]$ be a strictly increasing bijection. Then $S : [0, 1]^2 \to [0, 1]$ defined by

$$S(x, y) = s^{-1}(\min(s(x) + s(y), 1))$$

is a t-conorm (in fact, S is a nilpotent t-conorm with additive generator s [38, Definition 3.39]). Moreover, $N : [0, 1] \to [0, 1]$ given by

$$N(x) = \inf\{y \in [0, 1] \,|\, S(x, y) = 1\}$$

is a strong negation. If T is t-norm which is N-dual to S then we have

$$T(x, y) = s^{-1}(T_L(s(x), s(y))),$$
$$S(x, y) = s^{-1}(S_L(s(x), s(y))),$$
$$N(x) = s^{-1}(N_s(s(x))),$$

which means that the De Morgan triple (T, S, N) is isomorphic to the Łukasiewicz De Morgan triple $(T_\mathbf{L}, S_\mathbf{L}, N_s)$.

Even if (T, S, N) is a De Morgan triple, we do not necessarily have $T(x, N(x)) = 0$ and $S(x, N(x)) = 1$ for all $x \in [0, 1]$, i.e., the law of the excluded middle (which is one of the crucial features of the classical, two-valued Boolean logic) may be violated. For instance, if the t-norm T in the De Morgan triple (T, S, N_s) has no zero divisors, i.e., if $T(x, y) > 0$ whenever $x > 0$ and $y > 0$ (see Definition 6.1(iii)), then the law of the excluded middle never holds. On the other hand, in the De Morgan triple $(T_\mathbf{L}, S_\mathbf{L}, N_s)$ and, a fortiori, in each De Morgan triple (T, S, N_s) with $T \leqslant T_\mathbf{L}$, we have a many-valued analogue of the classical law of the excluded middle.

It is noteworthy that, given a De Morgan triple (T, S, N), the tuple $([0, 1], T, S, N, 0, 1)$ can never be a Boolean algebra: in order to satisfy distributivity we must have $T = T_\mathbf{M}$ and $S = S_\mathbf{M}$ (see Proposition 6.18), in which case it is impossible to have both $T(x, N(x)) = 0$ and $S(x, N(x)) = 1$ for all $x \in [0, 1]$.

5. Continuity

As can be seen from the drastic product $T_\mathbf{D}$ and its dual $S_\mathbf{D}$, t-norms and t-conorms (viewed as functions in two variables) need not be continuous (in fact, they need not, even be Borel measurable functions [38, Example 3.75]). Nevertheless, for a number of reasons continuous t-norms and t-conorms play an important role. Therefore, we shall discuss here continuity as well as left- and right-continuity.

Recall that a t-norm $T : [0, 1]^2 \to [0, 1]$ is continuous if for all convergent sequences $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ we have

$$T \left(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n \right) = \lim_{n \to \infty} T(x_n, y_n).$$

Obviously, the continuity of a t-conorm S is equivalent to the continuity of the dual t-norm T. Since the unit square $[0, 1]^2$ is a compact subset of the real plane $\mathbb{R}^2$, the continuity of a t-norm $T : [0, 1]^2 \to [0, 1]$ is equivalent to its uniform continuity.

Obviously, the basic t-norms $T_\mathbf{M}, T_\mathbf{P}$ and $T_\mathbf{L}$ as well as their dual t-conorms $S_\mathbf{M}, S_\mathbf{P}$ and $S_\mathbf{L}$ are continuous, and the drastic product $T_\mathbf{D}$ and the drastic sum $S_\mathbf{D}$ are not continuous.

In general, a real function of two variables, e.g, with domain $[0, 1]^2$, may be continuous in each variable without being continuous on $[0, 1]^2$. Because of their monotonicity, triangular norms (and conorms) are exceptions from this:

Proposition 5.1. *A t-norm $T : [0, 1]^2 \to [0, 1]$ is continuous if and only if it is continuous in each component, i.e., if for all $x_0, y_0 \in [0, 1]$ both the vertical section $T(x_0, \cdot) : [0, 1] \to [0, 1]$ and the horizontal section $T(\cdot, y_0) : [0, 1] \to [0, 1]$ are continuous functions in one variable.*

Obviously, because of the commutativity (T1), for a t-norm or a t-conorm its continuity is equivalent to its continuity in the first component.

For applications, e.g., in probabilistic metric spaces, many-valued logics or decomposable measures, quite often weaker forms of continuity are sufficient. Since we have a similar result as
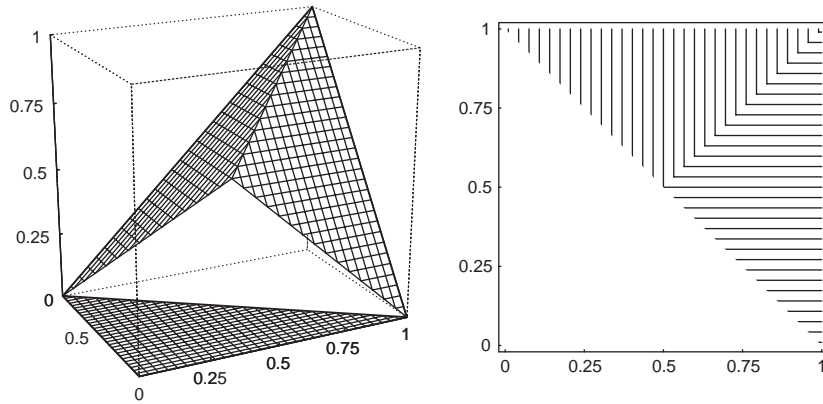
Fig. 3. 3D plot (left) and contour plot of the nilpotent minimum $T^{\mathbf{nM}}$ defined by (24).

Proposition 5.1 for left- and right-continuous t-norms, these definitions are given in one component only.

Definition 5.2. A t-norm $T : [0,1]^2 \to [0,1]$ is said to be *left-continuous* (*right-continuous*) if for each $y \in [0,1]$ and for all non-decreasing (non-increasing) sequences $(x_n)_{n \in \mathbb{N}}$ we have

$$\lim_{n \to \infty} T(x_n, y) = T\left(\lim_{n \to \infty} x_n, y\right).$$

Clearly, a t-norm is continuous if and only if it is both left- and right-continuous.

The *nilpotent minimum* $T^{\mathbf{nM}}$ (mentioned in [20,51,52], for a visualization see Fig. 3) defined by

$$T^{\mathbf{nM}}(x, y) = \begin{cases} 0 & \text{if } x + y \leqslant 1, \\ \min(x, y) & \text{otherwise} \end{cases} \tag{24}$$

is a t-norm which is left-continuous but not right-continuous. The drastic product $T_{\mathbf{D}}$, on the other hand, is right-continuous but not left-continuous. An example of a t-norm which is neither left- nor right-continuous can be found in Example 6.14(iv).

Clearly, a t-norm T is left-continuous if and only if its dual t-conorm given by (18) is right-continuous, and vice versa.

6. Algebraic properties

In the language of algebra, T is a t-norm if and only if $([0,1], T, \leqslant)$ is a fully ordered commutative semigroup with neutral element 1 and annihilator (zero element) 0. Therefore, it is natural to consider additional algebraic properties a t-norm may have.

Our first focus are idempotent and nilpotent elements, and zero divisors. Since for each $n \in \mathbb{N}$ we trivially have $0_T^{(n)} = 0$ and $1_T^{(n)} = 1$, only elements of $]0,1[$ will be considered as candidates for nilpotent elements and zero divisors in the following definition.

Definition 6.1. Let T be a t-norm.

(i) An element $a \in [0,1]$ is called an *idempotent element* of T if $T(a,a) = a$. The numbers 0 and 1 (which are idempotent elements for each t-norm T) are called *trivial* idempotent elements of T, each idempotent element in $]0,1[$ will be called a *non-trivial* idempotent element of T.

(ii) An element $a \in]0,1[$ is called a *nilpotent element* of T if there exists some $n \in \mathbb{N}$ such that $a_T^{(n)} = 0$.

(iii) An element $a \in]0,1[$ is called a *zero divisor* of T if there exists some $b \in]0,1[$ such that $T(a,b) = 0$.

The set of idempotent elements of the minimum $T_{\mathbf{M}}$ equals $[0, 1]$ (actually, $T_{\mathbf{M}}$ is the only t-norm with this property). For the Łukasiewicz t-norm $T_{\mathbf{L}}$ as well as for the drastic product $T_{\mathbf{D}}$, both the set of nilpotent elements and the set of zero divisors equal $]0,1[$. The minimum $T_{\mathbf{M}}$ and the product $T_{\mathbf{P}}$ have neither nilpotent elements nor zero divisors, and $T_{\mathbf{P}}, T_{\mathbf{L}}$, and $T_{\mathbf{D}}$ possess only trivial idempotent elements.

The set of idempotent elements of the nilpotent minimum $T^{\mathbf{nM}}$ defined in (24) equals $\{0\} \cup]0.5, 1]$, its set of nilpotent elements is $]0,0.5]$, and its set of zero divisors equals $]0, 1[$.

The idempotent elements of t-norms can be characterized in the following way, which involves the operation minimum [38, Proposition 2.3].

Proposition 6.2. (i) *An element $a \in [0,1]$ is an idempotent element of a t-norm T if and only if for all $x \in [a, 1]$ we have $T(a,x) = \min(a,x)$.*

(ii) *If T is a continuous t-norm, then $a \in [0,1]$ is an idempotent element of T if and only if for all $x \in [0,1]$ we have $T(a,x) = \min(a,x)$.*

Remark 6.3. For arbitrary t-norms some general observations concerning idempotent and nilpotent elements and zero divisors can be formulated.

(i) No element of $]0,1[$ can be both idempotent and nilpotent.

(ii) Each nilpotent element a of a t-norm T is also a zero divisor of T, but not conversely ($T^{\mathbf{nM}}$ is a counterexample).

(iii) If a t-norm T has a nilpotent element a then there is always an element $b \in]0,1[$ such that $b_T^{(2)} = 0$.

(iv) If $a \in]0,1[$ is a nilpotent element of a t-norm T then each number $b \in]0, a[$ is also a nilpotent element of T, i.e., the set of nilpotent elements of a t-norm T can either be the empty set (as for $T_{\mathbf{M}}$ or $T_{\mathbf{P}}$) or an interval of the form $]0, c[$ or $]0, c]$. The same is true for zero divisors.

Example 6.4. For the t-norm T [57, Example 5.3.13] given by

$$T(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 0.5]^2, \\ 2(x - 0.5)(y - 0.5) + 0.5 & \text{if } (x, y) \in]0.5, 1]^2, \\ \min(x, y) & \text{otherwise}, \end{cases} \tag{25}$$

its set of nilpotent elements and its set of zero divisors both equal $]0, 0.5]$, and for each element of the family $(T_c)_{c \in]0,1]}$ of t-norms defined by

$$T_c(x, y) = \begin{cases} \max(0, x + y - c) & \text{if } (x, y) \in [0, c]^2 \\ \min(x, y) & \text{otherwise,} \end{cases}$$

the set of nilpotent elements and the set of zero divisors of T_c equal $]0, c[$.

Although the set of nilpotent elements is in general a subset of the set of zero divisors, for each t-norm the existence of zero divisors is equivalent to the existence of nilpotent elements, i.e., a t-norm has zero divisors if and only if it has nilpotent elements [38, Proposition 2.5].

For right-continuous t-norms (in fact, the right-continuity of T on the diagonal of the unit square is sufficient) it is possible to obtain each idempotent element as the limit of the powers of a suitable $x \in [0, 1]$ [38, Proposition 2.6].

Proposition 6.5. *Let T be a t-norm which is right-continuous on the diagonal $\{(x, x) \mid x \in [0, 1]\}$ of the unit square $[0, 1]^2$, and let $a \in [0, 1]$. The following are equivalent*:

(i) *a is an idempotent element of T.*
(ii) *There exists an $x \in [0, 1]$ such that $a = \lim_{n \to \infty} x_T^{(n)}$.*

It is well-known that, for continuous t-norms, its set of idempotent elements is a closed subset of the unit interval [0,1]. As a consequence of [38, Corollary 2.8], this is also true for t-norms which are right-continuous in some specific points of the diagonal of the unit square and, consequently, for t-norms which are right-continuous:

Corollary 6.6. *Let T be a t-norm such that for each $a \in [0, 1[$*

$$T(a, a) = a \quad \text{whenever} \quad \lim_{x \searrow a} T(x, x) = a.$$

Then the set of idempotent elements of T is a closed subset of [0,1].

The t-norm T given in (25) shows that the converse implication does not necessarily hold in Corollary 6.6 (just consider the case $a = 0.5$).

Some t-norms have additional algebraic properties. The first group of such properties centers around the notions of strict monotonicity and the Archimedean property, which play an important role in many algebraic concepts, e.g., in semigroups.

Definition 6.7. For an arbitrary t-norm T we consider the following properties:

(i) The t-norm T is said to be *strictly monotone* if

(SM) $\quad T(x, y) < T(x, z) \quad$ whenever $x > 0 \quad$ and $\quad y < z$.

(ii) The t-norm T satisfies the *cancellation law* if

(CL) $\quad T(x, y) = T(x, z) \quad$ implies $x = 0 \quad$ or $\quad y = z$.

(iii) The t-norm T satisfies the *conditional cancellation law* if

   (CCL) $T(x, y) = T(x, z) > 0$ implies $y = z$.

(iv) The t-norm T is called *Archimedean* if

   (AP)   for each $(x, y) \in \,]0, 1[^2$ there is an $n \in \mathbb{N}$ with $x_T^{(n)} < y$.

(v) The t-norm T has the *limit property* if

   (LP)   for all $x \in \,]0, 1[$: $\lim\limits_{n \to \infty} x_T^{(n)} = 0$.

Example 6.8.   (i) The minimum $T_\mathbf{M}$ has none of these properties, and the product $T_\mathbf{P}$ satisfies all of them. The Łukasiewicz t-norm $T_\mathbf{L}$ and the drastic product $T_\mathbf{D}$ are Archimedean and satisfy the conditional cancellation law (CCL) and the limit property (LP), but none of the other properties.

(ii) If a t-norm T satisfies the cancellation law (CL) then it obviously fulfills the conditional cancellation law (CCL), but not conversely (see, e.g., $T_\mathbf{L}$).

(iii) The algebraic properties introduced in Definition 6.7 are independent of the continuity: the continuous t-norm $T_\mathbf{M}$ shows that continuity implies none of these properties. Conversely, $T_\mathbf{D}$ and the non-continuous t-norm T given by

$$T(x, y) = \begin{cases} \dfrac{xy}{2} & \text{if } (x, y) \in [0, 1[^2, \\ \min(x, y) & \text{otherwise,} \end{cases} \tag{26}$$

which is strictly monotone and satisfies the cancellation law (CL), are examples demonstrating that none of the algebraic properties implies the continuity of the t-norm under consideration.

The strict monotonicity (SM) of a t-norm is related to the other properties as follows [38, Proposition 2.11]:

Proposition 6.9. *Let T be a* t-*norm. Then we have*:

 (i) *T is strictly monotone if and only if it satisfies the cancellation law (CL).*
 (ii) *If T is strictly monotone then it has only trivial idempotent elements.*
(iii) *If T is strictly monotone then it has no zero divisors.*

The Archimedean property (AP) of a t-norm can be characterized in the following way [38, Theorem 2.12].

Proposition 6.10. *For a* t-*norm T the following are equivalent*:

 (i) *T is Archimedean.*
 (ii) *T satisfies the limit property (LP).*

(iii) *T has only trivial idempotent elements and, whenever*

$$\lim_{x \searrow x_0} T(x,x) = x_0$$

   for some $x_0 \in]0,1[$, there exists a $y_0 \in]x_0,1[$ such that $T(y_0, y_0) = x_0$.

Combining the continuity with some algebraic properties, we obtain two extremely important classes of t-norms.

Definition 6.11.

 (i) A t-norm T is called *strict* if it is continuous and strictly monotone.
 (ii) A t-norm T is called *nilpotent* if it is continuous and if each $a \in]0,1[$ is a nilpotent element of T.

Example 6.12. (i) The product $T_\mathbf{P}$ is a strict t-norm, and the Łukasiewicz t-norm $T_\mathbf{L}$ is a nilpotent t-norm. In fact [38, Propositions 5.9, 5.10] each strict t-norm is isomorphic to $T_\mathbf{P}$ and each nilpotent t-norm is isomorphic to $T_\mathbf{L}$.
   (ii) Because of Proposition 6.9(i), a t-norm T is strict if and only if it is continuous and satisfies the cancellation law (CL).
   (iii) Each strict and each nilpotent t-norm fulfills the conditional cancellation law (CCL).

The following result gives a number of sufficient conditions for a t-norm to be Archimedean [38, Proposition 2.15].

Proposition 6.13. *For an arbitrary* t-*norm T we have*:

 (i) *If T is right-continuous and has only trivial idempotent elements then it is Archimedean.*
 (ii) *If T is right-continuous and satisfies the conditional cancellation law (CCL) then it is Archimedean.*
 (iii) *If $\lim_{x \searrow x_0} T(x,x) < x_0$ for each $x_0 \in]0,1[$ then T is Archimedean.*
 (iv) *If T is strict then it is Archimedean.*
 (v) *If each $x \in]0,1[$ is a nilpotent element of T then T is Archimedean.*

In [40] it was shown that each left-continuous Archimedean t-norm is necessarily continuous.
   All the implications between the algebraic properties of t-norms considered so far are summarized and visualized in Fig. 4. The following are counterexamples showing that there are no other logical relations between these algebraic properties.

Example 6.14. (i) The Łukasiewicz t-norm $T_\mathbf{L}$ shows that an Archimedean t-norm need not be strictly monotone, and that the limit property (LP) does not imply the cancellation law (CL). The product $T_\mathbf{P}$ is an example of a continuous Archimedean t-norm without nilpotent elements. The drastic product $T_\mathbf{D}$ is an example of a non-continuous Archimedean t-norm for which each $a \in]0,1[$ is a nilpotent element.
   (ii) The t-norm given in (26) shows that a strictly monotone t-norm need not be continuous and, subsequently, not necessarily strict.
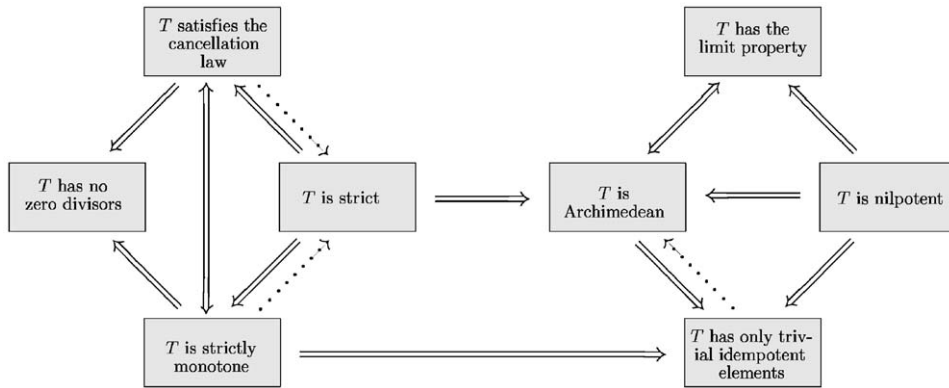
Fig. 4. The logical relationship between various algebraic properties of t-norms: a double arrow indicates an implication, a dotted arrow means that the corresponding implication holds for *continuous* t-norms.

(iii) The non-continuous t-norm given in (25) shows that a t-norm with only trivial idempotent elements is not necessarily strictly monotone or Archimedean.

(iv) A t-norm may satisfy both the strict, monotonicity (SM) and the Archimedean property (AP) without being continuous and, subsequently, without being strict. One example for this is the t-norm introduced in (26), another t-norm with these features is the following [10]: recall that each $(x, y) \in]0, 1]^2$ is in a one-to-one correspondence with a pair $((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}})$ of strictly increasing sequences of natural numbers given by the unique infinite dyadic representations

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{x_n}} \quad \text{and} \quad y = \sum_{n=1}^{\infty} \frac{1}{2^{y_n}}$$

of the numbers x and y, respectively. Using this notion, then the function $T : [0, 1]^2 \to [0, 1]$ given by

$$T(x, y) = \begin{cases} \sum_{n=1}^{\infty} \dfrac{1}{2^{x_n + y_n}} & \text{if } (x, y) \in]0, 1[^2, \\ \min(x, y) & \text{otherwise} \end{cases}$$

is a t-norm which is strictly monotone, Archimedean, and left-continuous on $]0, 1[^2$. However, T is discontinuous in each point $(x, y) \in]0, 1]^2$ where at least one coordinate is a dyadic rational number (i.e., of the form $m/2^n$ for some $m, n \in \mathbb{N}$ with $m \leqslant 2^n$; observe that the set of discontinuity points of T is dense in $[0, 1]^2$). Consequently, T is not strict.

(v) A modification of the t-norm in (iv) yields a t-norm which is strictly monotone but neither Archimedean nor continuous (compare [67]): keeping the notation of (iv), the function $T : [0, 1]^2 \to [0, 1]$, which is defined by

$$T(x, y) = \begin{cases} \sum_{n=1}^{\infty} \dfrac{1}{2^{x_n + y_n - n}} & \text{if } (x, y) \in]0, 1]^2, \\ 0 & \text{otherwise,} \end{cases}$$

is a t-norm which is strictly monotone, left-continuous on $[0,1]^2$, but discontinuous in each point $(x,y) \in \,]0,1[^2$ where at least one coordinate is a dyadic rational number. However, T is not Archimedean.

(vi) The function $T : [0,1]^2 \rightarrow [0,1]$ defined by

$$T(x,y) = \begin{cases} xy & \text{if } (x,y) \in [0,0.5]^2, \\ 2(x-0.5)(y-0.5)+0.5 & \text{if } (x,y) \in \,]0.5,1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is a t-norm which has only trivial idempotent elements, no zero divisors, is not Archimedean and not strictly monotone.

(vii) Recall that each $x \in \,]0,1]$ has a unique infinite dyadic representation $x = \sum_{n=1}^{\infty} 1/2^{x_n}$, where $(x_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers, and consider the function $f : [0,1] \rightarrow [0,1]$ defined by

$$f(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{x_n}} & \text{if } x = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{x_n}}, \\ 0 & \text{if } x = 0. \end{cases}$$

Then the function $T : [0,1]^2 \rightarrow [0,1]$ (introduced in [59], compare [38, Example 3.21]) given by

$$T(x,y) = \begin{cases} f(f^{(-1)}(x) \cdot f^{(-1)}(y)) & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

where $f^{(-1)} : [0,1] \rightarrow [0,1]$ is the pseudoinverse of f (observe that $f^{(-1)}$ is also known as Cantor function) given by

$$f^{(-1)}(x) = \sup\{z \in [0,1] \mid f(z) < x\},$$

is an Archimedean t-norm which is continuous in the point $(1,1)$, but which has no zero divisors and which is not strictly monotone. A more complicated example of this type is the Krause t-norm [38, Appendix B.1], which is also a non-continuous t-norm with a continuous diagonal, thus providing a counterexample to an open problem stated in [57].

It turns out that among the continuous Archimedean t-norms there are only two classes: the nilpotent and the strict t-norms. The existence of nilpotent elements (or zero divisors) provides a simple check for that [38, Theorem 2.18], see also (Fig. 5).

Theorem 6.15. *Let T be a continuous Archimedean* t*-norm. Then the following are equivalent*:

(i) *T is nilpotent*.
(ii) *There exists some nilpotent element of T*.
(iii) *There exists some zero divisor of T*.
(iv) *T is not strict*.

Remark 6.16. (i) A consequence of Proposition 6.10 is that a t-norm T is Archimedean if and only if it fulfills the limit property (LP). Note that, e.g., for topological semigroups, the Archimedean property is usually defined by means of the limit property (LP) (see [12,45]).
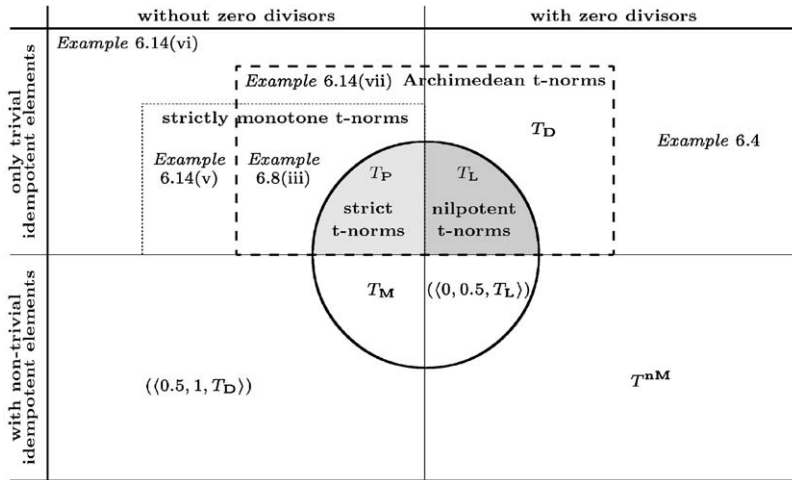
Fig. 5. Different classes of t-norms, each of them with a typical representative: within the central circle one finds the continuous t-norms, and the classes of strict and nilpotent t-norms are marked in gray (for the definition of the ordinal sums $(\langle 0, 0.5, T_L \rangle)$ and $(\langle 0.5, 1, T_D \rangle)$ see [38, Definition 3.44]).

(ii) An immediate consequence of Theorem 6.15 and Example 6.12(iii) is that a continuous t-norm is Archimedean if and only if it satisfies the conditional cancellation law (CCL).

(iii) From Theorem 6.15 it follows that a continuous t-norm T is strict if and only if for each $x \in]0, 1[$ the sequence $(x_T^{(n)})_{n \in \mathbb{N}}$ is strictly decreasing and converges to 0. Again, this is the usual way to define the strictness of topological semigroups.

The strict monotonicity of t-conorms as well as strict, Archimedean and nilpotent t-conorms can be introduced using dualities (18) and (19). Without presenting all the technical details, we only mention that it suffices to interchange the words t-norm and t-conorm and the roles of 0 and 1, respectively, and sometimes to reverse the inequalities involved, in order to obtain the proper definitions and results for t-conorms. For instance, a t-conorm S is strictly monotone if

$$(\text{SM}^*) \qquad S(x, y) < S(x, z) \quad \text{whenever} \quad x < 1 \quad \text{and} \quad y < z.$$

The Archimedean property is an example where it is necessary to reverse the inequality, so a t-conorm S is Archimedean if

$$(\text{AP}^*) \qquad \text{for each } (x, y) \in]0, 1[^2 \quad \text{there is an } n \in \mathbb{N} \text{ such that } x_S^{(n)} > y.$$

Of course, a t-conorm fulfills any of these properties if and only if the dual t-norm fulfills it.

Finally let us have a brief look at the distributivity of t-norms and t-conorms.

Definition 6.17. Let T be a t-norm and S be a t-conorm. Then we say that T *is distributive over S* if for all $x, y, z \in [0, 1]$

$$T(x, S(y, z)) = S(T(x, y), T(x, z)),$$

and that *S is distributive over T* if for all $x, y, z \in [0, 1]$

$$S(x, T(y, z)) = T(S(x, y), S(x, z)).$$

If T is distributive over S and S is distributive over T, then (T, S) is called a *distributive pair* (of t-norms and t-conorms).

In the context of distributivity the minimum $T_\mathbf{M}$ and the maximum $S_\mathbf{M}$ play a distinguished role (compare also [8]).

Proposition 6.18. *Let T be a* t-*norm and S a* t-*conorm. Then we have*:

  (i) *S is distributive over T if and only if $T = T_\mathbf{M}$.*
 (ii) *T is distributive over S if and only if $S = S_\mathbf{M}$.*
(iii) *(T, S) is a distributive pair if and only if $T = T_\mathbf{M}$ and $S = S_\mathbf{M}$.*

7. Historical remarks

The history of triangular norms started with Menger's paper "Statistical metrics" [43]. The main idea was to study metric spaces where probability distributions rather than numbers are used to model the distance between the elements of the space in question. Triangular norms naturally came into the picture in the course of the generalization of the classical triangle inequality to this more general setting. The original set of axioms for t-norms was somewhat weaker, including among others also triangular conorms.

Consequently, the first field where t-norms played a major role was the theory of probabilistic metric spaces (as statistical metric spaces were called after 1964). Schweizer and Sklar [53–61] provided the axioms of t-norms, as they are used today, and a redefinition of statistical metric spaces given in [58] led to a rapid development of the field. Many results concerning t-norms were obtained in the course of this development, most of which are summarized in the monograph [57] of Schweizer and Sklar.

Mathematically speaking, the theory of (continuous) t-norms has two rather independent roots, namely, the field of (specific) functional equations and the theory of (special topological) semigroups.

Concerning functional equations, t-norms are closely related to the equation of associativity (which is still unsolved in its most general form). The earliest source in this context seems to be Abel [1], further results in this direction were obtained in [9,13,2,29]. Especially Aczél's monograph [3,4]. had (and still has) a big impact on the development of t-norms. The main result based on this background was the full characterization of continuous Archimedean t-norms by means of additive generators in [41] (for the case of strict t-norms see [55]).

Another direction of research was the identification of several parameterized families of t-norms as solutions of some (more or less) natural functional equations. The perhaps most famous result in this context has been proven in [22], showing that the family of Frank t-norms and t-conorms (together with ordinal sums thereof) are the only solutions of the so-called Frank functional equation.

The study of a class of compact, irreducibly connected topological semigroups was initiated in [19], including a characterization of such semigroups, where the boundary points (at the same time annihilator and neutral element, respectively) are the only idempotent elements and where no nilpotent elements exist. In the language of t-norms, this provided a full representation of strict t-norms. In [45] all such semigroups, where the boundary points play the role of annihilator and neutral element, were characterized (see also [49]). Again in the language of t-norms, this provided a representation of all continuous t-norms [41].

Several construction methods from the theory of semigroups, such as (isomorphic) transformations (which are closely related to generators mentioned above) and ordinal sums (based on the work of Clifford [14], and foreshadowed in [34,15]), have been successfully applied to construct whole families of t-norms from a few given prototypical examples [56]. Summarizing, starting with only three t-norms, namely, the minimum T_M, the product T_P and the Łukasiewicz t-norm T_L, it is possible to construct all continuous t-norms by means of isomorphic transformations and ordinal sums [41].

Non-continuous t-norms, such as the drastic product T_D, have been considered from the very beginning [54]. In [41] even an additive generator for this t-norm was given. However, a general classification of non-continuous t-norms is still not known.

In his seminal paper "Fuzzy sets", Zadeh [67] introduced the theory of fuzzy sets as a generalization of the classical Cantorian set theory whose logical basis is the two-valued Boolean logic (compare also Klaua [32,33]). It was suggested in [67] to use the minimum T_M, the maximum S_M, and the standard negation N_s to model the intersection, union, and complement of fuzzy sets, respectively. However, also the product T_P, the probabilistic sum S_P and the Łukasiewicz t-conorm S_L (the latter in a restricted form) were already mentioned as possible candidates for intersection and union of fuzzy sets, respectively, in this very first paper.

The use of general t-norms and t-conorms for modeling the intersection and the union of fuzzy sets seems to have at least two independent roots. On the one hand, there was a series of seminars devoted to this topic, held in the seventies by Trillas at the Departament de Matemàtiques i Estadística de l'Escola Técnica Superior d'Arquitectura of the Universitat Politecnica de Barcelona. On the other hand, there were suggestions by Höhle during the First International Symposium on Policy Analysis and Information Systems (Durham, NC, 1979) and the First International Seminar on Fuzzy Set Theory (Linz, Austria, 1979). The canonical reason for this was that the axioms of commutativity, associativity, monotonicity as well as the boundary conditions were (and still are) generally considered as reasonable, even indispensable properties of meaningful extensions of the Cantorian intersection and union (a notable exception from this are the compensatory operators which may be non-associative, compare Zimmermann and Zysno [68], Dombi [16], Luhandjula [42], Türksen [63], Alsina et al. [5], Yager and Filev [65], and Klement et al. [37]).

Very early traces of (some slight variations of) t-norms and t-conorms in the context of integration of fuzzy sets with respect to non-additive measures can be found in the Ph.D. Thesis of M. Sugeno [61], first concepts for a unified theory of fuzzy sets (based on T_M and S_M) were presented in [46] and S. Gottwald [23–26]. The first papers using general t-norms and t-conorms for operations on fuzzy sets were Anthony and Sherwood [7], Alsina et al. [6], Dubois [17], and Klement [35,36] (see also Dubois and Prade [18]). A full characterization of strong negations as models of the complement of fuzzy sets can be found in [62].

Acknowledgements

References

[1] N.H. Abel, Untersuchungen der Funktionen zweier unabhängigen veränderlichen Größen x und y wie $f(x, y)$, welche die Eigenschaft haben, dass $f(z, f(x, y))$ cine symmetrische Funktion von x, y und z ist, J. Reine Angew. Math. 1 (1826) 11–15.

[2] J. Aczél, Sur les opérations definies pour des nombres réels, Bull. Soc. Math. France 76 (1949) 59–64.

[3] J. Aczél, Vorlesungen über Funktionalgleichungen und ihre Anwendungen, Birkhäuser, Basel, 1961.

[4] J. Aczél, Lectures on Functional Equations and their Applications, Academic Press, New York, 1966.

[5] C. Alsina, G. Mayor, M.S. Tomás, J. Torrens, A characterization of class of aggregation functions, Fuzzy Sets and Systems 53 (1993) 33–38.

[6] C. Alsina, E. Trillas, L. Valverde, On non-distributive logical connectives for fuzzy sets theory, BUSEFAL 3 (1980) 18–29.

[7] J.M. Antony, H. Sherwood, Fuzzy groups redefined, J. Math. Anal. Appl. 69 (1979) 124–130.

[8] R. Bellman, M. Giertz, On the analytic formalism of the theory of fuzzy sets, Inform. Sci. 5 (1973) 149–156.

[9] L.E.J. Brouwer, Die Theorie der endlichen kontinuierlichen Gruppen unabhängig von den Axiomen von Lie, Math. Ann. 67 (1909) 246–267.

[10] M. Budinčević, M.S. Kurilić, A family of strict and discontinuous triangular norms, Fuzzy Sets and Systems 95 (1998) 381–384.

[11] D. Butnariu, E.P. Klement, Triangular Norm-Based Measures and Games with Fuzzy Coalitions, Kluwer Academic Publishers, Dordrecht, 1993.

[12] J.H. Carruth, J.A. Hildebrant, R.J. Koch, The Theory of Topological Semigroups, Lecture Notes in Mathematics, Marcel Dekker, New York, 1983.

[13] É. Cartan, La théorie des groupes finis et continus et l'Analysis Situs, in: Mém. Sci. Math., vol. 42, Gauthier-Villars, Paris, 1930.

[14] A.H. Clifford, Naturally totally ordered commutative semigroups, Amer. J. Math. 76 (1954) 631–646.

[15] A.C. Climescu, Sur l'équation fonctionelle de l'associativité, Bull. École Polytechn. Iassy 1 (1946) 1–16.

[16] J. Dombi, Basic concepts for a theory of evaluation: the aggregative operator, European J. Oper. Res. 10 (1982) 282–293.

[17] D. Dubois, Triangular norms for fuzzy sets, in: E.P. Klement (Ed.), Proc. 2nd Internat. Seminar on Fuzzy Set Theory, Linz, 1980, pp. 39–68.

[18] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.

[19] W. M. Faucett, Compact semigroups irreducibly connected between two idempotents, Proc. Amer. Math. Soc. 6 (1955) 741–747.

[20] J.C. Fodor, Contrapositive symmetry of fuzzy implications, Fuzzy Sets and Systems 69 (1995) 141–156.

[21] J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Dordrecht, 1994.

[22] M.J. Frank, On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$, Aequationes Math. 19 (1979) 194–226.

[23] S. Gottwald, Untersuchungen zur mehrwertigen Mengenlehre. I, Math. Nachr. 72 (1976) 297–303.

[24] S. Gottwald, Untersuchungen zur mehrwertigen Mengenlehre. II, Math. Nachr. 74 (1976) 329–336.

[25] S. Gottwald, Untersuchungen zur mehrwertigen Mengenlehre. III, Math. Nachr. 79 (1977) 207–217.

[26] M. Grabisch, H.T. Nguyen, E.A. Walker, Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference, Kluwer Academic Publishers, Dordrecht, 1995.

[27] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.

[28] K.H. Hofmann, J.D. Lawson, Linearly ordered semigroups: historic origins and A.H. Clifford's influence, in: K.H. Hofmann, M.W. Mislove (Eds.), Semigroup Theory and its Applications, London Mathematics Society Lecture Notes, vol. 231, Cambridge University Press, Cambridge, 1996, pp. 15–39.

[29] M. Hosszú, Some functional equations related with the associativity law, Publ. Math. Debrecen 3 (1954) 205–214.

[30] S. Jenei, Structure of left-continuous triangular norms with strong induced negations, (I) Rotation construction, J. Appl. Non-Classical Logics 10 (2000) 83–92.

[31] S. Jenei, Structure of Girard monoids on [0,1], in: S.E. Rodabaugh, E.P. Klement (Eds.), Topological and Algebraic Structures in Fuzzy Sets. A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Dordrecht, 2003. pp. 277–308.

[32] D. Klaua, Über einen Ansatz zur mehrwertigen Mengenlehre, Monatsb. Deutsch. Akad. Wiss. Berlin 7 (1965) 859–867.

[33] D. Klaua, Einbettung der klassischen Mengenlehre in die mehrwertige, Monatsb. Deutsch. Akad. Wiss. Berlin 9 (1967) 258–272.

[34] F. Klein-Barmen, Über gewisse Halbverbände und kommutative Semigruppen II, Math. Z 48 (1942–1943) 715–734.

[35] E.P. Klement, Some remarks on t-norms, fuzzy σ-algebras and fuzzy measures, in: E.P. Klement (Ed.), Proceedings Second International Seminar on Fuzzy Set Theory, Linz, 1980, pp. 125–142.

[36] E.P. Klement, Operations on fuzzy sets and fuzzy numbers related to triangular norms, in: Proc. 11th Internat. Symp. on Multiple-Valued Logic, Norman, IEEE Press, New York, 1981, pp. 218–225.

[37] E.P. Klement, R. Mesiar, E. Pap, On the relationship of associative compensatory operators to triangular norms and conorms, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 4 (1996) 129–144.

[38] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.

[39] E.P. Klement, S. Weber, Generalized measures, Fuzzy Sets and Systems 40 (1991) 375–394.

[40] A. Kolesárová, A note on Archimedean triangular norms, BUSEFAL 80 (1999) 57–60.

[41] C.M. Ling, Representation of associative functions, Publ. Math. Debrecen 12 (1965) 189–212.

[42] M.K. Luhandjula, Compensatory operators in fuzzy linear programming with multiple objectives, Fuzzy Sets and Systems 8 (1982) 245–252.

[43] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA 8 (1942) 535–537.

[44] R. Mesiar, H. Thiele, On T-quantifiers and S-quantifiers, in: V. Novák, I. Perfilieva (Eds.), Discovering the World with Fuzzy Logic, Physica-Verlag, Heidelberg, 2000, pp. 310–326.

[45] P.S. Mostert, A.L. Shields, On the structure of semi-groups on a compact manifold with boundary, Ann. of Math. Ser. II 65 (1957) 117–143.

[46] C.V. Negoita, D.A. Ralescu, Applications of Fuzzy Sets to Systems Analysis, Wiley, New York, 1975.

[47] R.B. Nelsen, An Introduction to Copulas, in: Lecture Notes in Statistics, vol. 139, Springer, New York, 1999.

[48] H.T. Nguyen, E. Walker, A First Course in Fuzzy Logic, CRC Press, Boca Raton, FL, 1997.

[49] A.B. Paalman-de Miranda, Topological Semigroups, Matematisch Centrum, Amsterdam, 1964.

[50] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht, 1995.

[51] P. Perny, Modélisation, agrégation et exploitation des préférences floues dans une problématique de rangement, Ph.D. Thesis, Université Paris-Dauphine, Paris, 1992.

[52] P. Perny, B. Roy, The use of fuzzy outranking relations in preference modelling, Fuzzy Sets and Systems 49 (1992) 33–53.

[53] B. Schweizer, A. Sklar, Espaces métriques aléatoires, C. R. Acad. Sci. Paris Sér. A 247 (1958) 2092–2094.

[54] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 313–334.

[55] B. Schweizer, A. Sklar, Associative functions and statistical triangle inequalities, Publ. Math. Debrecen 8 (1961) 169–186.

[56] B. Schweizer, A. Sklar, Associative functions and abstract semigroups, Publ. Math. Debrecen 10 (1963) 69–81.

[57] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland, New York, 1983.

[58] A.N. Šerstnev, Random normed spaces: problems of completeness, Kazan. Gos. Univ. Učen. Zap. 122 (1962) 3–20.

[59] D. Smutná, A note on non-continuous t-norms, BUSEFAL 76 (1998) 19–24.

[60] D. Smutná, On a peculiar t-norm, BUSEFAL 75 (1998) 60–67.

[61] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Thesis, Tokyo Institute of Technology, 1974.

[62] E. Trillas, Sobre funciones de negación en la teoría de conjuntas difusos, Stochastica 3 (1979) 47–60.

[63] I.B. Türksen, Inter-valued fuzzy sets and 'compensatory AND', Fuzzy Sets and Systems 51 (1992) 295–307.

[64] S. Weber, $\perp$-decomposable measures and integrals for Archimedean t-conorms $\perp$, J. Math. Anal. Appl. 101 (1984) 114–138.

[65] R.R. Yager, D.P. Filev, Essentials of Fuzzy Modelling and Control, Wiley, New York, 1994.

[66] R.R. Yager, A. Rybalov, Uninorm aggregation operators, Fuzzy Sets and Systems 80 (1996) 111–120.

[67] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

[68] H.-J. Zimmermann, P. Zysno, Latent connectives in human decision making, Fuzzy Sets and Systems 4 (1980) 37–51.