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Triangular norms. Position paper II: general constructions and parameterized families

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Abstract

This second part (out of three) of a series of position papers on triangular norms (for Part I see Triangular norms. Position paper I: basic analytical and algebraic properties, Fuzzy Sets and Systems, in press) deals with general construction methods based on additive and multiplicative generators, and on ordinal sums. Also included are some constructions leading to non-continuous t-norms, and a presentation of some distinguished families of t-norms.

Keywords: Triangular norm; Additive generator; Ordinal sum; Parameterized families of triangular norms

1. Introduction

This is the second part (out of three) of a series of position papers on triangular norms. The monograph [40] provides a rather complete and self-contained overview about triangular norms and their applications.

Part I [42] considered some basic analytical properties of t-norms, such as continuity, and important classes such as Archimedean, strict and nilpotent t-norms. Also the dual operations, the triangular conorms, and De Morgan triples were mentioned. Finally, a short historical overview on the development of t-norms and their way into fuzzy sets and fuzzy logics was given.

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In this Part II we present general construction methods based on additive and multiplicative generators, and on ordinal sums. We also include some constructions leading to non-continuous t-norms, and a presentation of some distinguished families of t-norms.

To keep the paper readable, we have omitted all proofs (usually giving a source for the reader interested in them).

Finally, Part III will concentrate on continuous t-norms, in particular on their representation by additive and multiplicative generators and ordinal sums.

Recall that a triangular norm (briefly t-norm) is a binary operation T on the unit interval $[0,1]$ which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $T:[0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$:

(T1) $T(x,y) = T(y,x)$,
(T2) $T(x,T(y,z)) = T(T(x,y),z)$,
(T3) $T(x,y) \leqslant T(x,z)$ whenever $y \leqslant z$,
(T4) $T(x,1) = x$.

A function $F:[0,1]^2 \to [0,1]$ which satisfies, for all $x,y,z \in [0,1]$, properties (T1)–(T3) and

$$F(x,y) \leqslant \min(x,y) \tag{1}$$

is called a t-subnorm (as introduced in [30], see [42, Definition 2.3]).

2. Additive and multiplicative generators

It is straightforward that, given a t-norm T and a strictly increasing bijection $\varphi:[0,1]\to[0,1]$, the function $T_\varphi:[0,1]^2\to[0,1]$ given by

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x),\varphi(y))) \tag{2}$$

is again a t-norm.

In other words, the t-norms T and T_φ are isomorphic in the sense that for all $(x,y)\in[0,1]^2$

$$\varphi(T_\varphi(x,y)) = T(\varphi(x),\varphi(y)).$$

From the point of view of semigroup theory, for each t-norm T an increasing bijection $\varphi:[0,1]\to[0,1]$ is exactly an automorphism between the semigroups $([0,1],T)$ and $([0,1],T_\varphi)$. Note also that for all strictly increasing bijections $\varphi,\psi:[0,1]\to[0,1]$ and for each t-norm T we obtain

$$(T_\varphi)_\psi = T_{\varphi\circ\psi},$$

$$(T_\varphi)_{\varphi^{-1}} = (T_{\varphi^{-1}})_\varphi = T.$$

The only t-norms which are invariant with respect to construction (2) under arbitrary strictly increasing bijections are the two extremal t-norms $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$, i.e., if a t-norm T is only isomorphic to itself then either $T=T_{\mathbf{M}}$ or $T=T_{\mathbf{D}}$.

It is also trivial that construction (2) preserves the continuity, the Archimedean property, and the strictness, as well as the existence of idempotent and nilpotent elements, and the existence of zero divisors (see [40,42]) of the t-norm we started with.

This illustrates both the strength and the weakness of (2): it can be applied to any t-norm T, but the resulting t-norm T_φ has exactly the same algebraic properties.

Construction (2) uses the inverse of the function $\varphi : [0,1] \to [0,1]$ and, therefore, requires φ to be bijective.

If we want to construct t-norms as transformations of the additive semigroup $([0,\infty],+)$ and the multiplicative semigroup $([0,1],\cdot)$, respectively, monotone (but not necessarily bijective) functions are used, and a generalized inverse, the so-called pseudo-inverse [39,63] (see also [40, Section 3.1]) is needed.

Definition 2.1. Let $f : [a,b] \to [c,d]$ be a monotone function, where $[a,b]$ and $[c,d]$ are closed subintervals of the extended real line $[-\infty,\infty]$. The *pseudo-inverse* $f^{(-1)}(y) : [c,d] \to [a,b]$ of f is defined by

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a,b] \mid f(x) < y\} & \text{if } f(a) < f(b), \\ \sup\{x \in [a,b] \mid f(x) > y\} & \text{if } f(a) > f(b), \\ a & \text{if } f(a) = f(b). \end{cases} \tag{3}$$

Example 2.2. Consider the function $f : [-1,1] \to [c,d]$ with $[1.5,2.5] \subseteq [c,d]$ specified by $f(x) = (x+4)/2$. Then its pseudo-inverse $f^{(-1)} : [c,d] \to [-1,1]$ is given by

$$f^{(-1)}(x) = \max(\min(2x-4,1),-1).$$

Visualizations of the pseudo-inverse of non-continuous non-bijective monotone functions are given in Fig. 1. These pictures also indicate how to construct the graph of the pseudo-inverse $f^{(-1)}$ of a non-constant monotone function $f : [a,b] \to [c,d]$:

(1) Draw vertical line segments at discontinuities of f.
(2) Reflect the graph of f in the first median, i.e., in the graph of the identity function $\mathrm{id}_{[-\infty,\infty]}$.
(3) Remove any vertical line segments from the reflected graph except for their lowest points.

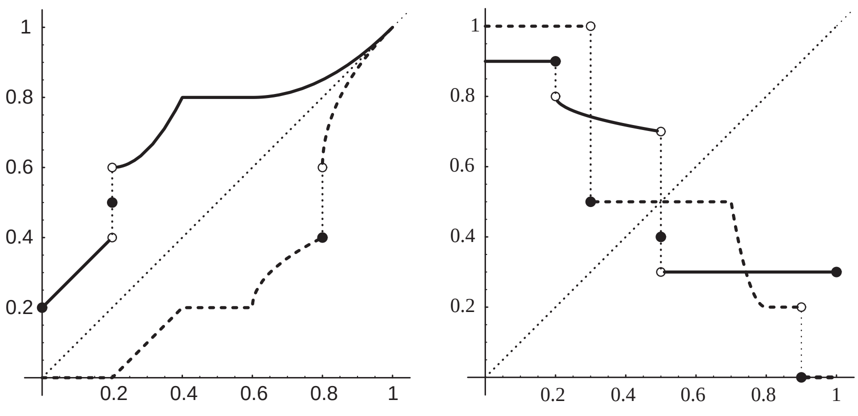

Fig. 1. Two monotone functions from $[0,1]$ to $[0,1]$ together with their pseudo-inverses (dashed graphs).

The basic idea of additive generators is already contained in a result of Abel [1] from 1826 who gave a sufficient condition for operations on the real line to be associative and showed how such functions may be constructed by means of a continuous, strictly monotone one-place function whose range is closed under addition. The following result [40, Theorem 3.23] is more general in the sense that the continuity of the one-place function is not needed, that the requirement of the closedness of the range under addition can be relaxed, and that the inverse function is replaced by the pseudo-inverse. On the other hand, it is slightly more special since we want to construct an operation on the unit interval with neutral element 1.

Theorem 2.3. *Let $f:[0,1]\to[0,\infty]$ be a strictly decreasing function with $f(1)=0$ such that f is right-continuous at 0 and*

$$f(x)+f(y)\in \mathrm{Ran}(f)\cup[f(0),\infty] \tag{4}$$

for all $(x,y)\in[0,1]^2$. The following function $T:[0,1]^2\to[0,1]$ is a t-norm:

$$T(x,y)=f^{(-1)}(f(x)+f(y)). \tag{5}$$

In Theorem 2.3, the pseudo-inverse $f^{(-1)}$ may be replaced by any monotone function $g:[0,\infty]\to[0,1]$ with $g|_{\mathrm{Ran}(f)}=f^{(-1)}|_{\mathrm{Ran}(f)}$. In some very abstract settings (see, e.g., [62]), such a function g (which may be non-monotone) is called a quasi-inverse of f.

It is obvious that a multiplication of f in Theorem 2.3 by a positive constant does not change the resulting t-norm T.

Definition 2.4. An *additive generator* $t:[0,1]\to[0,\infty]$ of a t-norm T is a strictly decreasing function which is right-continuous at 0 and satisfies $t(1)=0$, such that for all $(x,y)\in[0,1]^2$ we have

$$t(x)+t(y)\in \mathrm{Ran}(t)\cup[t(0),\infty], \tag{6}$$

$$T(x,y)=t^{(-1)}(t(x)+t(y)). \tag{7}$$

Starting with the function $t:[0,1]\to[0,\infty]$ given by $t(x)=1-x$ we get the Łukasiewicz t-norm $T_{\mathbf{L}}$, and $t(x)=-\ln x$ produces the product $T_{\mathbf{P}}$. The drastic product $T_{\mathbf{D}}$ (which is right-continuous but not continuous) is obtained putting

$$t(x)=\begin{cases}2-x & \text{if } x\in[0,1[,\\ 0 & \text{if } x=1.\end{cases}$$

In Fig. 2, an example of a rather complicated non-continuous t-norm together with its additive generator is given.

If $t:[0,1]\to[0,\infty]$ is an additive generator of a t-norm T, then we clearly have for all $x_1,x_2,\dots,x_n\in[0,1]$

$$T(x_1,x_2,\dots,x_n)=t^{(-1)}\left(\sum_{i=1}^{n} t(x_i)\right),$$

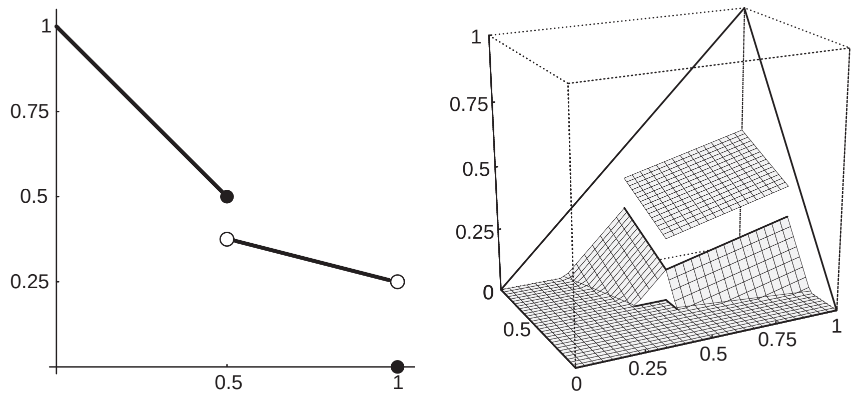

Fig. 2. A non-continuous t-norm together with its additive generator.

where, for $n > 2$, the expression $T(x_1, x_2, \ldots, x_n)$ is defined recursively by

$$T(x_1, x_2, \ldots, x_n) = T(T(x_1, x_2, \ldots, x_{n-1}), x_n).$$

An immediate consequence of Proposition 2.7 is that a t-norm with a non-trivial idempotent element (e.g., a non-trivial ordinal sum, see Definition 3.2) cannot have an additive generator. In particular, the minimum $T_{\mathbf{M}}$ has no additive generator, a fact which was mentioned first in [48]. In this context, the classical result in [5] (for an extension see [47]), where it was shown that there are no continuous real functions $f : \mathbb{R} \to \mathbb{R}$ and $g, h : [0,1] \to \mathbb{R}$ such that for all $(x, y) \in [0,1]^2$

$$\min(x, y) = f(g(x) + h(y))$$

is of interest.

There is a strong connection [40, Proposition 3.26] between the (left-)continuity of additive generators and the (left-)continuity of the t-norm constructed by (7).

Proposition 2.5. *Let T be a t-norm which has an additive generator $t : [0,1] \to [0, \infty]$. Then the following are equivalent*:

(i) *T is continuous*,
(ii) *T is left-continuous at the point* $(1,1)$,
(iii) *t is continuous*,
(iv) *t is left-continuous at* 1.

An obvious and useful consequence of this result is that a left-continuous t-norm which has an additive generator is automatically continuous.

The right-continuity of a t-norm having an additive generator is equivalent to the existence of a right-continuous additive generator [40, Proposition 3.27].

Proposition 2.6. *Let T be a t-norm which has an additive generator $t : [0,1] \to [0, \infty]$. Then T is right-continuous if and only if it has a right-continuous additive generator.*

The following proposition shows that triangular norms constructed by means of additive generators are always Archimedean [40, Proposition 3.29]. The converse, however, is not true: the t-norm introduced in [42, Example 6.14(vii)] is Archimedean and continuous at $(1,1)$ but not continuous whence, because of Proposition 2.5, it cannot have an additive generator.

Proposition 2.7. *If a t-norm T has an additive generator $t:[0,1]\to[0,\infty]$, then T is necessarily Archimedean. Moreover, we have*

(i) *the t-norm T is strictly monotone if and only if $t(0)=\infty$,*
(ii) *each element of $]0,1[$ is a nilpotent element of T if and only if $t(0)<\infty$,*
(iii) *if t is not continuous and satisfies $t(0)<\infty$ then there exists an $n_0\in\mathbb{N}$ such that for all $x\in[0,1[$ we have $x_T^{(n_0)}=0$.*

As an immediate consequence of Proposition 2.7(i) and (ii) we obtain, in the particular case of continuous additive generators of a (necessarily continuous Archimedean) t-norm (see Proposition 2.5), the following.

Corollary 2.8. *If $t:[0,1]\to[0,\infty]$ is a continuous additive generator of a continuous Archimedean t-norm T, then we have*

(i) *T is strict if and only if $t(0)=\infty$,*
(ii) *T is nilpotent if and only if $t(0)<\infty$.*

Starting from an arbitrary Archimedean t-norm T with additive generator it is easy to construct additive generators of t-norms which are isomorphic to T [40, Proposition 3.31].

Proposition 2.9. *Let T be an Archimedean t-norm with additive generator $t:[0,1]\to[0,\infty]$:*

(i) *If $\varphi:[0,1]\to[0,1]$ is a strictly increasing bijection then the function $t\circ\varphi:[0,1]\to[0,\infty]$ is an additive generator of the Archimedean t-norm T_φ given in (2).*
(ii) *If the additive generator t is continuous and if $\psi:[0,\infty]\to[0,\infty]$ is a strictly increasing bijection then the function $\psi\circ t:[0,1]\to[0,\infty]$ is an additive generator of a continuous Archimedean t-norm which is isomorphic to T.*

The result of Proposition 2.9(ii) makes it possible to construct, starting from a continuous Archimedean t-norm, interesting families of t-norms.

Example 2.10. Let T be a continuous Archimedean t-norm and $t:[0,1]\to[0,\infty]$ an additive generator of T.

(i) For each index $\lambda\in]0,\infty[$ the function $t^\lambda:[0,1]\to[0,\infty]$ defined by $t^\lambda(x)=(t(x))^\lambda$ is an additive generator of a continuous Archimedean t-norm which we shall denote by $T^{(\lambda)}$, and the family $(T^{(\lambda)})_{\lambda\in]0,\infty[}$ is strictly increasing with respect to the parameter λ. Adding the limit cases $T^{(0)} = T_{\mathbf{D}}$ and $T^{(\infty)} = T_{\mathbf{M}}$, we obtain some well-known families of t-norms in this way, depending on the t-norm T we start with. For instance, $((T_{\mathbf{L}})^{(\lambda)})_{\lambda\in[0,\infty]}$ is the family of Yager t-norms (see Example 5.4), $((T_{\mathbf{P}})^{(\lambda)})_{\lambda\in[0,\infty]}$ is the family of Aczél–Alsina t-norms (see [2,40, Section 4.8]), and $((T_0^{\mathbf{H}})^{(\lambda)})_{\lambda\in[0,\infty]}$, where $T_0^{\mathbf{H}}$ is the Hamacher t-norm with parameter 0 (see [19,25,26,40, Section 4.3]), is just the family of Dombi t-norms (see [14,40, Section 4.6]).

(ii) Let T^* be a strict t-norm with additive generator $t^* : [0,1] \to [0,\infty]$. Then, for each $\lambda \in]0,\infty[$, the function $t_{(t^*,\lambda)} : [0,1] \to [0,\infty]$ defined by

$$t_{(t^*,\lambda)}(x) = t((t^*)^{-1}(\lambda t^*(x)))$$

is an additive generator of a continuous Archimedean t-norm which we shall denote by $T_{(T^*,\lambda)}$. As a concrete example, for each $\lambda \in]0,\infty[$ the t-norm $T_{\mathbf{P}(T_0^{\mathbf{H}},\lambda)}$ equals the Hamacher t-norm $T_\lambda^{\mathbf{H}}$ (see [19,25,26,40, Section 4.3]).

(iii) If in (ii) we take $T^* = T_{\mathbf{P}}$ and, subsequently, $t^*(x) = -\ln x$ for all $x \in [0,1]$, then we obtain $t_{(t^*,\lambda)}(x) = t(x^\lambda)$, and the t-norm $T_{(T_{\mathbf{P}},\lambda)}$ will be denoted $T_{(\lambda)}$ for simplicity, and its additive generator by $t_{(\lambda)}$. As a concrete example, the subfamily $(T_\lambda^{\mathbf{SS}})_{\lambda\in]-\infty,\infty]}$ of the family of Schweizer–Sklar t-norms (see Example 5.1) is obtained as follows: for each $\lambda \in]0,\infty[$ we have $T_\lambda^{\mathbf{SS}} = (T_{\mathbf{L}})_{(\lambda)}$, and for $\lambda \in]-\infty,0[$ we have $T_\lambda^{\mathbf{SS}} = (T_0^{\mathbf{H}})_{(-\lambda)}$ (again $T_0^{\mathbf{H}}$ is the Hamacher t-norm with parameter 0).

There is a concept completely dual to additive generators, the so-called multiplicative generators of t-norms. Of course, the basis for this duality is that the exponential function and the logarithm are natural isomorphisms between the additive semigroup $([0,\infty],+)$ and the multiplicative semigroup $([0,1],\cdot)$.

If $t : [0,1] \to [0,\infty]$ is an additive generator of the t-norm T and if we define the strictly increasing function $\theta : [0,1] \to [0,1]$ by

$$\theta(x) = \mathrm{e}^{-t(x)},$$

then it is obvious that for all $(x,y) \in [0,1]^2$

$$T(x,y) = \theta^{(-1)}(\theta(x)\cdot\theta(y)).$$

The following result about multiplicative generators, (which can be obtained also via pseudo-inverse functions or in duality to additive generators), is a generalization of Theorem 5.2.1 in [62]).

Corollary 2.11. *Let $\theta : [0,1] \to [0,1]$ be a strictly increasing function which is right-continuous at 0 and satisfies $\theta(1) = 1$ and for all $(x,y) \in [0,1]^2$*

$$\theta(x)\cdot\theta(y) \in \mathrm{Ran}(\theta) \cup [0,\theta(0)].$$

Then the function $T : [0,1]^2 \to [0,1]$ given by

$$T(x,y) = \theta^{(-1)}(\theta(x)\cdot\theta(y))$$

is a triangular norm.

Definition 2.12. A *multiplicative generator* of a t-norm T is a strictly increasing function $\theta:[0,1]\to[0,1]$ which is right-continuous at 0 and satisfies $\theta(1)=1$, such that for all $(x,y)\in[0,1]^2$ we have

$$\theta(x)\cdot\theta(y)\in \mathrm{Ran}(\theta)\cup[0,\theta(0)],$$

$$T(x,y)=\theta^{(-1)}(\theta(x)\cdot\theta(y)).$$

Because of the duality between t-norms and t-conorms, that T is a t-norm if and only if the function $S:[0,1]^2\to[0,1]$ given by

$$S(x,y)=1-T(1-x,1-y) \tag{8}$$

is a t-conorm, additive and multiplicative generators of t-conorms can also be considered. Without presenting technical details and proofs, this can be summarized as follows.

Let T be a t-norm, S the dual t-conorm, $t:[0,1]\to[0,\infty]$ an additive generator of T and $\theta:[0,1]\to[0,1]$ a multiplicative generator of T. If we define the functions $s:[0,1]\to[0,\infty]$ and $\xi:[0,1]\to[0,1]$ by

$$s(x)=t(1-x),$$
$$\xi(x)=\theta(1-x),$$

then it is obvious that for all $(x,y)\in[0,1]^2$ we get

$$S(x,y)=s^{(-1)}(s(x)+s(y)),$$
$$S(x,y)=\xi^{(-1)}(\xi(x)\cdot\xi(y)).$$

In complete analogy to Theorem 2.3 and Corollary 2.11 we get the following result for triangular conorms.

Corollary 2.13. (i) *Let $s:[0,1]\to[0,\infty]$ be a strictly increasing function with $s(0)=0$ such that s is left-continuous at 1 and*

$$s(x)+s(y)\in\mathrm{Ran}(s)\cup[s(1),\infty]$$

for all $(x,y)\in[0,1]^2$. Then the function $S:[0,1]^2\to[0,1]$ given by

$$S(x,y)=s^{(-1)}(s(x)+s(y))$$

is a triangular conorm.

(ii) *Let $\xi:[0,1]\to[0,\infty]$ be a strictly decreasing function which is left-continuous at 1 and satisfies $\xi(0)=1$ such that for all $(x,y)\in[0,1]^2$*

$$\xi(x)\cdot\xi(y)\in\mathrm{Ran}(\xi)\cup[0,\xi(1)].$$

Then the function $S:[0,1]^2\to[0,1]$ given by

$$S(x,y)=\xi^{(-1)}(\xi(x)\cdot\xi(y))$$

is a triangular conorm.

It is therefore quite natural to define additive and multiplicative generators of t-conorms as follows.

Definition 2.14. (i) An *additive generator* of a t-conorm S is a strictly increasing function $s:[0,1] \to [0,\infty]$ which is left-continuous at 1 and satisfies $s(0)=0$, such that for all $(x,y)\in[0,1]^2$ we have

$$s(x)+s(y) \in \mathrm{Ran}(s) \cup [s(1),\infty],$$

$$S(x,y) = s^{(-1)}(s(x)+s(y)).$$

(ii) A *multiplicative generator* of a t-conorm S is a strictly decreasing function $\xi:[0,1]\to[0,1]$ which is left-continuous at 1 and satisfies $\xi(0)=1$, such that for all $(x,y)\in[0,1]^2$ we have

$$\xi(x)\cdot\xi(y) \in \mathrm{Ran}(\xi) \cup [0,\xi(1)],$$

$$S(x,y) = \xi^{(-1)}(\xi(x)\cdot\xi(y)).$$

The exact relationship between the classes of additive and multiplicative generators of t-norms and t-conorms, respectively, is exhibited by the commutative diagram in Fig. 3, where the operators N, E and L assign to each function $f:[0,1]\to[0,\infty]$ the functions $\mathsf{N}f$, $\mathsf{E}f$, $\mathsf{L}f:[0,1]\to[0,\infty]$ given by

$$\mathsf{N}f(x) = f(1-x), \tag{9}$$

$$\mathsf{E}f(x) = \mathrm{e}^{-f(x)}, \tag{10}$$

$$\mathsf{L}f(x) = -\ln(f(x)). \tag{11}$$

Note that the same function can be an additive generator for a t-norm and a multiplicative generator for a t-conorm, and vice versa. Put, e.g., $f(x)=1-x$ and $g(x)=x$ for $x\in[0,1]$. Then f is an additive generator of $T_{\mathbf{L}}$ and a multiplicative generator of $S_{\mathbf{P}}$, while g is an additive generator of $S_{\mathbf{L}}$ and a multiplicative generator of $T_{\mathbf{P}}$.

The concept of additive and multiplicative generators of t-norms can be further generalized. We only mention that requirement (4), i.e., the closedness of the range, can be relaxed and one still obtains a t-norm [68]. On the other hand, the strict monotonicity and the boundary condition of the generator can also be relaxed, in which case one obtains a t-subnorm, in general [51,52].

Theorem 2.15. *Let* $f:[0,1]\to[0,\infty]$ *be a non-increasing function such that*

$$f(x)+f(y) \in \mathrm{Ran}(f) \cup [f(0),\infty]$$

for all $(x,y)\in[0,1]^2$. *The following function* $F:[0,1]^2\to[0,1]$ *is a t-subnorm*:

$$F(x,y) = f^{(-1)}(f(x)+f(y)). \tag{12}$$

Observe that for an arbitrary continuous non-increasing function $f:[0,1]\to[0,\infty]$ the operation F given by (12) is a left-continuous t-subnorm [51].

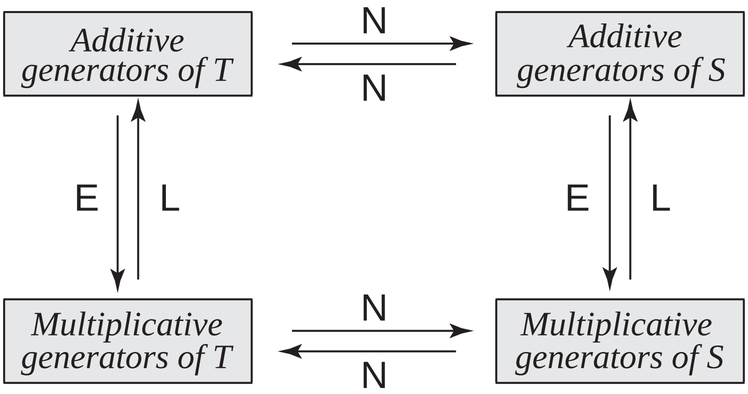

Fig. 3. The relationship between additive and multiplicative generators of a t-norm T and its dual t-conorm S: a commutative diagram.

3. Ordinal sums

The construction of a new semigroup from a family of given semigroups using ordinal sums goes back to Clifford [11] (see also [12,27,59]), and it is based on ideas presented in [13,37]. It has been successfully applied to t-norms in [20,48,61] (for a proof of the following result see [40, Theorem 3.43]; a visualization is given in Fig. 4).

Theorem 3.1. *Let* $(T_\alpha)_{\alpha\in A}$ *be a family of t-norms and* $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ *be a family of non-empty, pairwise disjoint open subintervals of* $[0,1]$. *Then the following function* $T:[0,1]^2\to[0,1]$ *is a t-norm*:

$$T(x,y)=\begin{cases} a_\alpha+(e_\alpha-a_\alpha)\cdot T_\alpha\left(\dfrac{x-a_\alpha}{e_\alpha-a_\alpha},\dfrac{y-a_\alpha}{e_\alpha-a_\alpha}\right) & \textit{if } (x,y)\in[a_\alpha,e_\alpha]^2,\\ \min(x,y) & \textit{otherwise.}\end{cases} \tag{13}$$

This allows us to adapt the general concept of ordinal sums of abstract semigroups to the case of t-norms as follows.

Definition 3.2. Let $(T_\alpha)_{\alpha\in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm T defined by (13) is called the *ordinal sum* of the *summands* $\langle a_\alpha, e_\alpha, T_\alpha\rangle$, $\alpha\in A$, and we shall write

$$T=(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}.$$

In the same spirit it is possible to introduce the ordinal sum of other binary operations on the unit interval $[0,1]$. Examples for this are the ordinal sum of t-conorms (which is again a t-conorm, see Corollary 3.7), of copulas (introduced in [64], for a recent survey see [55]), always yielding a copula, and of t-subnorms (which always leads to a t-subnorm, sometimes even to a t-norm as

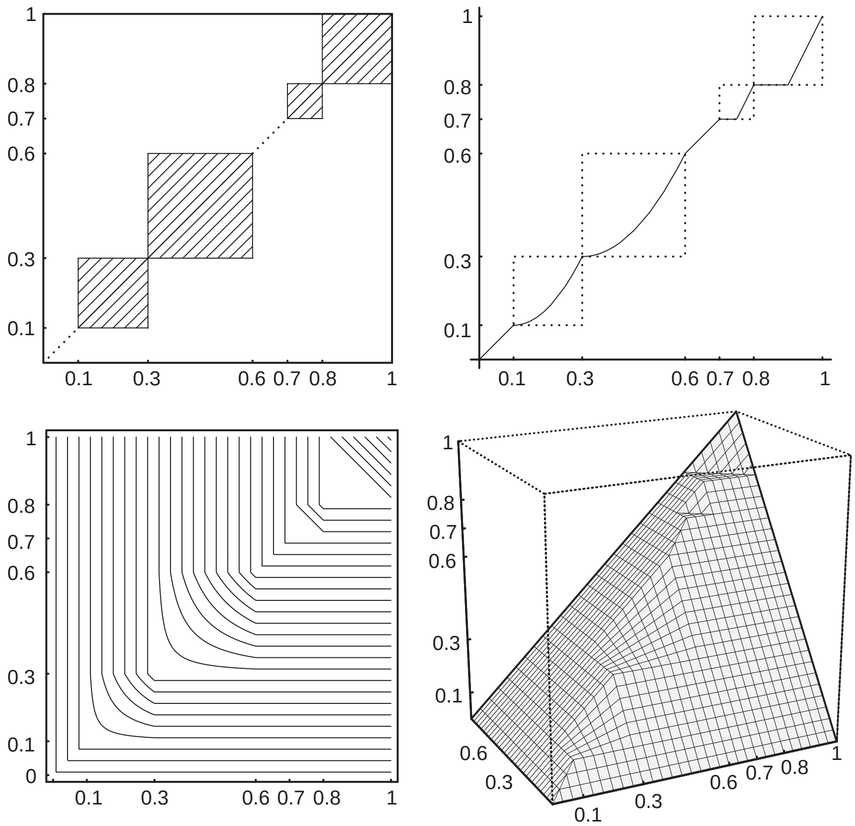

Fig. 4. Construction of ordinal sums: the "important" parts of the domain of $(\langle 0.1, 0.3, T_{\mathbf{P}}\rangle, \langle 0.3, 0.6, T_{\mathbf{P}}\rangle, \langle 0.7, 0.8, T_{\mathbf{L}}\rangle, \langle 0.8, 1, T_{\mathbf{L}}\rangle)$ (top left), top right its diagonal section, bottom left its contour plot, and bottom right its 3D plot.

in Example 4.5(ii), compare also [30,33] and Theorem 3.8). In [41], it was shown that the most general way to obtain a t-norm as an ordinal sum of semigroups in the spirit of [11] is to build ordinal sums of suitable t-subnorms.

Clearly, each t-norm T can be viewed as a trivial ordinal sum with one summand $\langle 0, 1, T\rangle$ only, i.e., we have $T = (\langle 0, 1, T\rangle)$.

Also, the minimum $T_{\mathbf{M}}$ is a neutral element of the ordinal sum construction in the following sense: if $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$ is an ordinal sum of t-norms and if $T_{\alpha_0} = T_{\mathbf{M}}$ for some $\alpha_0 \in A$, then the summand $\langle a_{\alpha_0}, e_{\alpha_0}, T_{\alpha_0}\rangle$ can be omitted, i.e.,

$$(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A} = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A\setminus\{\alpha_0\}}.$$

In particular, an empty ordinal sum of t-norms, i.e., an ordinal sum of t-norms with index set $\emptyset$, yields the minimum $T_{\mathbf{M}}$:

$$T_{\mathbf{M}} = (\emptyset) = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in\emptyset}.$$

An ordinal sum of t-norms may have infinitely many summands. For instance, the ordinal sum $T=(\langle 1/2^n, 1/2^{n-1}, T_{\mathbf{P}}\rangle)_{n\in\mathbb{N}}$ is given by

$$T(x,y)=\begin{cases}\dfrac{1}{2^n}+2^n\left(x-\dfrac{1}{2^n}\right)\left(y-\dfrac{1}{2^n}\right) & \text{if } (x,y)\in\left[\dfrac{1}{2^n},\dfrac{1}{2^{n-1}}\right]^2,\\ \min(x,y) & \text{otherwise.}\end{cases}$$

However, if $T=(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$ is an ordinal sum of t-norms, then each of the intervals $]a_\alpha, e_\alpha[$ is non-empty and, therefore, contains some rational number. Consequently, the cardinality of the index set A cannot exceed the cardinality of the set of rational numbers (in $[0,1]$), i.e., A must be finite or countably infinite.

It is possible to construct parameterized families of t-norms using ordinal sums. Two examples are the family $(T_\lambda^{\mathbf{DP}})_{\lambda\in]0,1]}$ of *Dubois–Prade t-norms* (first introduced in [17]) and the family $(T_\lambda^{\mathbf{MT}})_{\lambda\in]0,1]}$ of *Mayor–Torrens t-norms* (see [50]) defined by, respectively,

$$T_\lambda^{\mathbf{DP}}=(\langle 0,\lambda,T_{\mathbf{P}}\rangle), \tag{14}$$

$$T_\lambda^{\mathbf{MT}}=(\langle 0,\lambda,T_{\mathbf{L}}\rangle). \tag{15}$$

By construction, the set of idempotent elements of an ordinal sum $T=(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$ of t-norms contains the set

$$M=[0,1]\Big\backslash\bigcup_{\alpha\in A}]a_\alpha,e_\alpha[\tag{16}$$

as a subset, and for each idempotent element a of T with $a\in M$ and for all $x\in[0,1]$ we have $T(a,x)=\min(a,x)$.

Moreover, the set M given in (16) equals the set of idempotent elements of T if and only if each T_α has only trivial idempotent elements.

If $T=(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$ is a non-trivial ordinal sum of t-norms, i.e., if $A\neq\emptyset$ and if no $]a_\alpha, e_\alpha[$ equals $]0,1[$, then T necessarily has non-trivial idempotent elements and, as a consequence, cannot be Archimedean.

An ordinal sum $T=(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$ of t-norms has zero divisors (nilpotent elements) if and only if there is an $\alpha_0\in A$ such that $a_{\alpha_0}=0$ and T_{α_0} has zero divisors (nilpotent elements).

There is a very close relationship between the existence of non-trivial idempotent elements and ordinal sums [40, Proposition 3.48].

Proposition 3.3. *Let T be a t-norm and $a_0\in]0,1[$ such that $T(a_0,x)=\min(a_0,x)$ for all $x\in[0,1]$. Then a_0 is a non-trivial idempotent element of T if and only if there are t-norms T_1 and T_2 such that $T=(\langle 0,a_0,T_1\rangle,\langle a_0,1,T_2\rangle)$.*

The continuity of an ordinal sum of t-norms is equivalent to the continuity of all of its summands, i.e., an ordinal sum $T=(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$ of t-norms with $A\neq\emptyset$ is continuous if and only if T_α is continuous for each $\alpha\in A$ [40, Proposition 3.49].

It is easy to see that the representation of a t-norm as an ordinal sum of t-norms is not unique, in general. For instance, for each subinterval $[a,e]$ of $[0,1]$ we have

$$T_{\mathbf{M}}=(\emptyset)=(\langle 0,1,T_{\mathbf{M}}\rangle)=(\langle a,e,T_{\mathbf{M}}\rangle).$$

This gives rise to a natural question: which t-norms T have a unique ordinal sum representation (in which case it necessarily must be the trivial representation $T=(\langle 0,1,T\rangle))$?

Definition 3.4. A t-norm T which only has a trivial ordinal sum representation (i.e., there is no ordinal sum representation of T different from $T=(\langle 0,1,T\rangle))$ is called *ordinally irreducible*.

The proof of the two subsequent results can be found in [40, Propositions 3.53 and 3.54].

Proposition 3.5. *For each t-norm T the following are equivalent*:

(i) *T is not ordinally irreducible*,
(ii) *there is an $x_0 \in]0,1[$ such that $T(x_0,y)=\min(x_0,y)$ for all $y\in[0,1]$*,
(iii) *there exists a non-trivial idempotent element x_0 of T such that the vertical section $T(x_0,\cdot): [0,1]\to[0,1]$ is continuous.*

Proposition 3.6. *For each t-norm $T\neq T_{\mathbf{M}}$ the following are equivalent*:

(i) *T can be uniquely represented as an ordinal sum of ordinally irreducible t-norms*,
(ii) *the set*

$$M_T=\{x\in[0,1]\mid T(x,y)=\min(x,y) \textit{ for all } y\in[0,1]\}$$

is a closed subset of $[0,1]$.

Each Archimedean t-norm has only trivial idempotent elements and is, therefore, ordinally irreducible.

The nilpotent minimum $T^{\mathbf{nM}}$ which was introduced in [18] (cf. also [57,58], for a visualization see [42, Fig. 3]) and which is given by

$$T^{\mathbf{nM}}(x,y)=\begin{cases}0 & \text{if } x+y\leqslant 1,\\ \min(x,y) & \text{otherwise}\end{cases}$$

is an example of an ordinally irreducible t-norm with non-trivial idempotent elements.

If T is a t-norm and if a,b are numbers with $a<b$ and $a,b\in M_T$ then $T_{ab}:[0,1]^2\to[0,1]$ defined by

$$T_{ab}(x,y)=\frac{T(a+(b-a)x,a+(b-a)y)}{b-a}$$

is a t-norm which has an ordinal sum representation where one of the summands equals $\langle a,b,T_{ab}\rangle$.

For the t-norm T defined by

$$T(x,y)=\begin{cases}0 & \text{if } (x,y)\in[0,\frac{1}{2}]^2,\\[4pt] \dfrac{n+4}{2n+4} & \text{if } (x,y)\in\left]\dfrac{n+4}{2n+4},\dfrac{n+3}{2n+2}\right[^2,\\[4pt] \min(x,y) & \text{otherwise,}\end{cases}$$

the set $M_T = \{0\} \cup \{(n+3)/(2n+2) \mid n \in \mathbb{N}\}$ is not closed, showing that T has no ordinal sum representation where the t-norms in the summands are ordinally irreducible.

In the case of t-conorms, a construction dual to the one in Theorem 3.1 can be applied. The roles of the neutral element (which is 0 in this case) and the annihilator (which is 1) are interchanged in this case, and we have to replace the operation min by max.

Corollary 3.7. *Let $(S_\alpha)_{\alpha\in A}$ be a family of t-conorms and $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the function $S : [0,1]^2 \to [0,1]$ defined by*

$$S(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot S_\alpha\left(\dfrac{x-a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y-a_\alpha}{e_\alpha - a_\alpha}\right) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \max(x,y) & \text{otherwise} \end{cases}$$

is a t-conorm which is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, S_\alpha\rangle$, $\alpha\in A$, and we shall write $S = (\langle a_\alpha, e_\alpha, S_\alpha\rangle)_{\alpha\in A}$.

All results in this section given for t-norms remain valid for t-conorms with the obvious changes where necessary. In particular, if $(\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$ is an ordinal sum of t-norms then the dual t-conorm given by (8) is an ordinal sum of t-conorms, i.e., $(\langle 1-e_\alpha, 1-a_\alpha, S_\alpha\rangle)_{\alpha\in A}$, where each t-conorm S_α is the dual of the t-norm T_α. Note, however, that the t-norm T_α and the t-conorm S_α, in general, act on different intervals.

Observe that the most general construction of t-norms by means of ordinal sums of semigroups in the sense of Clifford [11] is the subsequent modification of Theorem 3.1 (see [41], compare also [33]).

Theorem 3.8. *Let $(F_\alpha)_{\alpha\in A}$ be a family of t-subnorms and $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Further, if $e_{\alpha_0} = 1$ for some $\alpha_0 \in A$ then assume that F_{α_0} is a t-norm, and if $e_{\alpha_0} = a_{\beta_0}$ for some $\alpha_0, \beta_0 \in A$ then assume either that F_{α_0} is a t-norm or that F_{β_0} has no zero divisors. Then the following function $T : [0,1]^2 \to [0,1]$ is a t-norm:*

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot F_\alpha\left(\dfrac{x-a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y-a_\alpha}{e_\alpha - a_\alpha}\right) & \text{if } (x,y) \in]a_\alpha, e_\alpha]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Recently, some other methods for constructing new t-norms related to ordinal sums were introduced. We only mention the matrix composition [15] (which even allows t-norms to be constructed on certain abstract lattices).

4. Other constructions

The following result shows that certain binary operations acting on subintervals of the half-open unit interval $[0,1[$ always can be extended to a t-norm [40, Proposition 3.60].

Proposition 4.1. *Let A be a subinterval of the half-open unit interval $[0,1[$ and let $*: A^2 \to A$ be an operation on A which satisfies for all $x,y,z \in A$ properties* (T1)–(T3) *and* (1). *Then the function $T:[0,1]^2 \to [0,1]$ defined by*

$$T(x,y) = \begin{cases} x * y & \text{if } (x,y) \in A^2, \\ \min(x,y) & \text{otherwise} \end{cases} \tag{17}$$

is a t-norm.

An immediate consequence of Proposition 4.1 is that each t-subnorm F can be transformed into a t-norm by redefining (if necessary) its values on the upper right boundary of the unit square.

Corollary 4.2. *If F is a t-subnorm then the function $T:[0,1]^2 \to [0,1]$ given by*

$$T(x,y) = \begin{cases} F(x,y) & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is a triangular norm.

Another consequence of Proposition 4.1 is that, given a t-norm T and a non-decreasing function $f:[0,1] \to [0,1]$ which in some sense is compatible with T, the pseudo-inverse allows us to construct a new t-norm (see [40, Theorem 3.6]). We will not state this result in its most general form and, therefore, restrict ourselves to continuous functions f, in which case f is compatible with each t-norm T.

Proposition 4.3. *Let $f:[0,1] \to [0,1]$ be a continuous, non-decreasing function and let T be a t-norm. Then the following function $T_{[f]}:[0,1]^2 \to [0,1]$ is a t-norm:*

$$T_{[f]}(x,y) = \begin{cases} f^{(-1)}(T(f(x), f(y))) & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{18}$$

It is easy to see that construction (18) leads to a t-subnorm (as introduced in [30], see [42, Definition 2.3]).

Corollary 4.4. *Let $f:[0,1] \to [0,1]$ be a continuous, non-decreasing function and let T be a t-norm. Then the function $F:[0,1]^2 \to [0,1]$ given by*

$$F(x,y) = f^{(-1)}(T(f(x), f(y)))$$

is a t-subnorm.

If $f,g:[0,1] \to [0,1]$ are two continuous, non-decreasing functions and if T is a t-norm, then we have

$$(T_{[f]})_{[g]} = T_{[f \circ g]}.$$

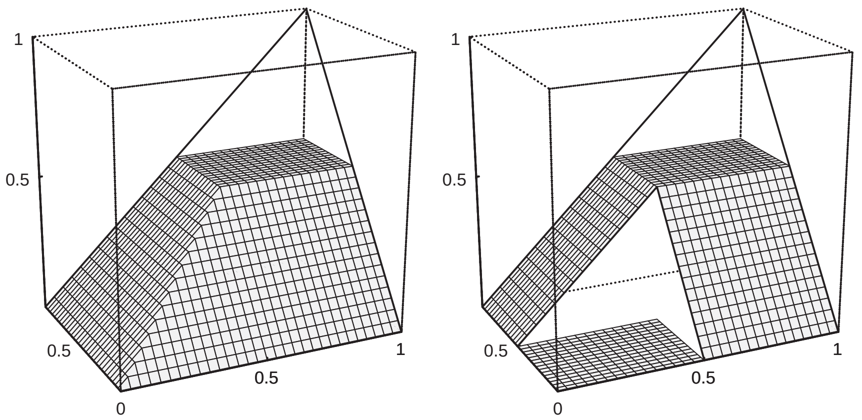

Fig. 5. The t-norms $(T_{\mathbf{M}})_{[f]}$ (left) and $(T_{\mathbf{D}})_{[f]}$ (right) induced by the function $f:[0,1]\to[0,1]$ given by $f(x)=\min(2x,1)$.

This means in particular that the original T can be reconstructed if $f\circ g=\mathrm{id}_{[0,1]}$, e.g., if $\mathrm{Ran}(f)=[0,1]$ and $g=f^{(-1)}$, in which case we get

$$T=(T_{[f]})_{[f^{(-1)}]}.$$

The minimum $T_{\mathbf{M}}$ and the drastic product $T_{\mathbf{D}}$ are no longer invariant under the construction of Proposition 4.3 (see Fig. 5), and if $f:[0,1]\to[0,1]$ is a constant function then, for an arbitrary t-norm T, formula (18) always yields the drastic product $T_{\mathbf{D}}$.

The general form of Proposition 4.3 (cf. [40, Theorem 3.6]) can be applied in many special cases which cover a wide range of constructions of t-norms mentioned in the literature.

In general, construction (18) preserves neither the continuity (see $T_{\mathbf{M}[f]}$ in Fig. 5) nor any of the algebraic properties the original t-norm may have: for instance, $T_{\mathbf{D}}$ is Archimedean but $T_{\mathbf{D}[f]}$ in Fig. 5 is not (see also Fig. 6).

It also may be that a non-continuous t-norm T gives rise to a continuous t-norm $T_{[f]}$ (see Fig. 7).

Many more examples of t-norms constructed by means of (18) can be found in [39,40, Section 3.1].

Example 4.5. (i) Let $[a,b]$ be a closed subinterval of $[0,1[$, let $f:[a,b]\to[0,\infty]$ be a continuous, non-increasing function, and define the binary operation $*$ on $[a,b]$ by

$$x*y=f^{(-1)}(f(x)+f(y))$$

Clearly, $*$ satisfies all the requirements in Proposition 4.1 (cf. Theorem 2.3) and, consequently, the function $T:[0,1]^2\to[0,1]$ given by (17) is a left-continuous t-norm.

(ii) If in (i) we put $a=0$ and $f(x)=0$ for all $x\in[0,b]$, then the t-norm T introduced by (17) is given by

$$T(x,y)=\begin{cases}0 & \text{if } (x,y)\in[0,b]^2,\\ \min(x,y) & \text{otherwise.}\end{cases}$$

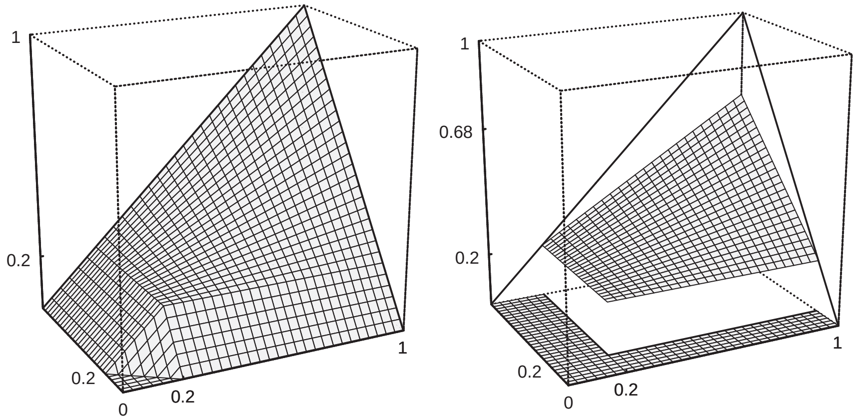

Fig. 6. Starting with the ordinal sum $T=(\langle 0,0.2,T_{\mathbf{L}}\rangle,\langle 0.2,1,T_{\mathbf{P}}\rangle)$ (left, which has nilpotent elements) and the function $f:[0,1]\to[0,1]$ given by $f(x)=\frac{1}{5}+\frac{4}{5}x$ we obtain the strict t-norm $T_{[f]}=T_{\mathbf{P}}$; on the other hand, the function $g:[0,1]\to[0,1]$ given by $g(x)=\frac{3}{4}\max(x-\frac{1}{5},0)$ induces the non-strict t-norm $(T_{\mathbf{P}})_{[g]}$ (right).

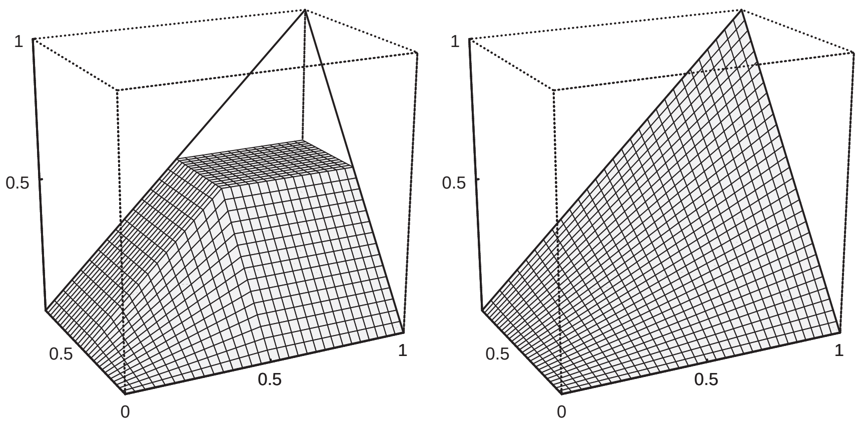

Fig. 7. Using the function $f:[0,1]\to[0,1]$ given by $f(x)=x/2$, the non-continuous ordinal sum $(\langle 0,0.5,T_{\mathbf{P}}\rangle,\langle 0.5,1,T_{\mathbf{D}}\rangle)$ (left, see Definition 3.2) induces the continuous t-norm $T_{[f]}=T_{\mathbf{P}}$.

Observe that T is not an ordinal sum of ordinally irreducible t-norms (cf. Definition 3.4), but rather an ordinal sum of t-subnorms (see Theorem 3.8).

If A is a subinterval of the half-open unit interval $[0,1[$ and if the binary operation $*$ on A satisfies the requirements in Proposition 4.1, then the t-norm T given by (17) is continuous if and only if $*$ is continuous and if for all $x\in A$ we have $\sup\{x*y\mid y\in A\}=x$.

Proposition 4.6. *Let A be a subset of* $]0,1[^2$ *with the following properties*:

(i) *A is symmetric, i.e.,* $(x,y)\in A$ *implies* $(y,x)\in A$.
(ii) *For all* $(x,y)\in A$ *we have* $]0,x]\times]0,y]\subseteq A$.

Then the following function $T_A : [0,1]^2 \to [0,1]$ *is a t-norm*:

$$T_A(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Remark 4.7. Let $A \subseteq\,]0,1[^2$ be a set which satisfies the conditions in Proposition 4.6:

(i) A t-norm T is of the form T_A if and only if we have $T(x,y) \in \{0, \min(x,y)\}$ for all $(x,y) \in [0,1]^2$.
(ii) The t-norm T_A is left-continuous if and only if the set

$$(\{0\} \times [0,1]) \cup ([0,1] \times \{0\}) \cup A$$

is a closed subset of $[0,1]^2$ (compare also Example 4.5(ii)), and it is continuous only in the trivial case $A = \emptyset$ (in which case we obtain $T_A = T_{\mathbf{M}}$).
(iii) If $A \neq \emptyset$, then T_A always has nilpotent elements and zero divisors.
(iv) The only Archimedean case is $A =]0,1[^2$ (which means $T_A = T_{\mathbf{D}}$), in all the other cases T_A has non-trivial idempotent elements.
(v) If $f : [0,1] \to [0,1]$ is a non-decreasing function, then the set

$$A_f = \{(x,y) \in]0,1[^2 \mid f(x) + f(y) \leqslant 1\}$$

satisfies all the properties in Proposition 4.6, and T_{A_f} is a t-norm (this is also true if we replace the constant 1 by any other real number).

Example 4.8. (i) In the special case $f = \mathrm{id}_{[0,1]}$ we obtain the nilpotent minimum $T^{\mathbf{nM}}$.

(ii) If $f : [0,1] \to [0,1]$ is a strictly increasing bijection and if T is the nilpotent t-norm defined by

$$T(x,y) = f^{-1}(T_{\mathbf{L}}(f(x), f(y))),$$

i.e., if f is the unique isomorphism between T and the Łukasiewicz t-norm $T_{\mathbf{L}}$, then T_{A_f} is the strongest t-norm which vanishes exactly at the same points of the unit square $[0,1]^2$ as T.

The following construction even leads to an Archimedean t-norm.

Proposition 4.9. *Let A be a subset of* $]0,1[^2$ *which satisfies the conditions in Proposition* 4.6, *and assume that* $a \in]0,1[$ *fulfills* $\{a\} \times]0,1[\subseteq A$. *Then the following function* $T_{A,a} : [0,1]^2 \to [0,1]$ *is an Archimedean t-norm*:

$$T_{A,a}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A, \\ a & \text{if } (x,y) \in]0,1[^2 \setminus A, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

As an immediate consequence of Proposition 4.9 we obtain the following sufficient condition.

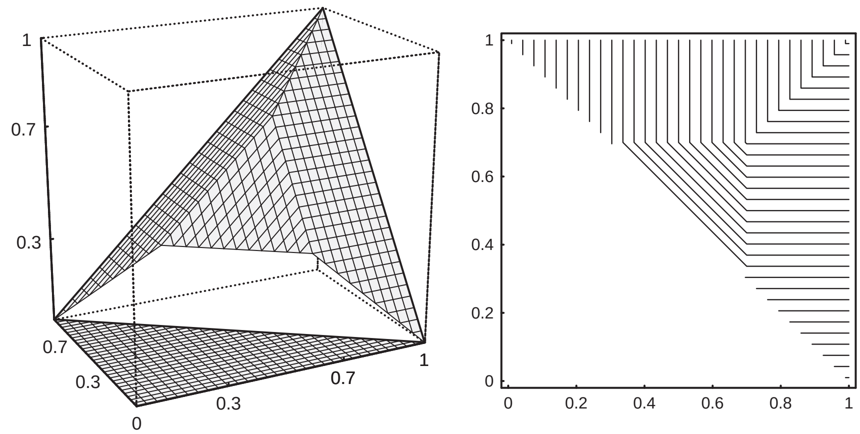

Fig. 8. Three-dimensional and contour plot of the Jenei t-norm $T^{\mathbf{J}}_{0.3}$ (see Example 4.11).

Corollary 4.10. *If* $f:[0,1]\to[0,1]$ *is a non-decreasing function and if* $a \in]0, f^{(-1)}(1-f(1))[$ *then* $T_{A_f,a}$ *is an Archimedean t-norm.*

Several methods for constructing and characterizing left-continuous t-norms T satisfying $T(x,y)=0$ if and only if $x+y\leqslant 1$ have been proposed recently in [28–32,34,36]. Such t-norms T are interesting in the context of fuzzy logics (compare [22,23]), since in this case the negation associated with T equals the standard negation N given by $N(x)=1-x$. An overview of all these methods can be found in [35]. The following family of non-continuous t-norms is an example of such t-norms [29]. Moreover, these are (up to isomorphism) the only left-continuous t-norms T such that the associated residual implications $\to_T$ [22,23,36] satisfy

$$x \to_T y = N(y) \to_T N(x).$$

Example 4.11. The family $(T^{\mathbf{J}}_{\lambda})_{\lambda\in[0,0.5]}$ of *Jenei t-norms* is given by (see Fig. 8)

$$T^{\mathbf{J}}_{\lambda}\lambda(x,y) = \begin{cases} 0 & \text{if } x+y \leqslant 1, \\ \lambda + x + y - 1 & \text{if } x+y > 1 \text{ and } (x,y)\in[\lambda, 1-\lambda]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Several peculiar non-continuous t-norms have been constructed, mostly in order to show (by providing counterexamples) that

(i) the continuity of the diagonal does not imply the continuity of the whole t-norm (e.g. the Krause t-norm, see [40, Appendix B.1]),
(ii) a strictly monotone t-norm need not be Archimedean nor continuous [7,24,65],
(iii) there are non-Archimedean t-norms constructed by means of strictly decreasing non-continuous functions from $[0,1]$ to $[0,\infty]$ [67],

(iv) there are t-norms which are (as functions from $[0,1]^2$ to $[0,1]$) not Borel measurable [38].

5. Families of t-norms and t-conorms

We now give a quick overview of some of the most important parameterized families of t-norms and t-conorms (Table 1). In the literature, several other parameterized families are mentioned, e.g., the families of Aczél–Alsina t-norms [2], Dombi t-norms [14], Dubois–Prade t-norms (14), Jenei t-norms (see Example 4.11), and Mayor–Torrens t-norms (15). Extensive surveys of families of t-norms and t-conorms can be found in [40,49,53] (Figs. 9 and 10).

5.1. Schweizer–Sklar t-norms

Already in [60] an interesting family of t-norms was presented, and in [61] the index set was extended to the whole real line (our notation follows the monograph [62], i.e., our index λ corresponds to $-p$ in the original papers). This family of t-norms is remarkable in the sense that it contains all four basic t-norms.

Example 5.1. (i) The family $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$ of *Schweizer–Sklar t-norms* is given by

$$T_\lambda^{\mathbf{SS}}(x,y)=\begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda=-\infty,\\ T_{\mathbf{P}}(x,y) & \text{if } \lambda=0,\\ T_{\mathbf{D}}(x,y) & \text{if } \lambda=\infty,\\ (\max((x^\lambda+y^\lambda-1),0))^{1/\lambda} & \text{if } \lambda\in]-\infty,0[\cup]0,\infty[. \end{cases}$$

(ii) Additive generators $t_\lambda^{\mathbf{SS}}:[0,1]\to[0,\infty]$ of the continuous Archimedean members of the family of Schweizer–Sklar t-norms $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$ are given by

$$t_\lambda^{\mathbf{SS}}(x)=\begin{cases} -\ln x & \text{if } \lambda=0,\\ \dfrac{1-x^\lambda}{\lambda} & \text{if } \lambda\in]-\infty,0[\cup]0,\infty[. \end{cases}$$

In Example 2.10(iii) a construction for the additive generators of the continuous Archimedean Schweizer–Sklar t-norms was given.

Note that the subfamily $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,1]}$ of Schweizer–Sklar t-norms is also a family of copulas, in which context (e.g., in [55]) it is referred to as the family of *Clayton copulas* [10].

5.2. Hamacher t-norms

In [25,26] an axiomatic approach for the logical connectives conjunction and disjunction, which can be expressed by rational functions, in many-valued logics with $[0,1]$ as set of truth values was presented. The original axioms for the conjunction $T:[0,1]^2\to[0,1]$ include continuity, associativity,

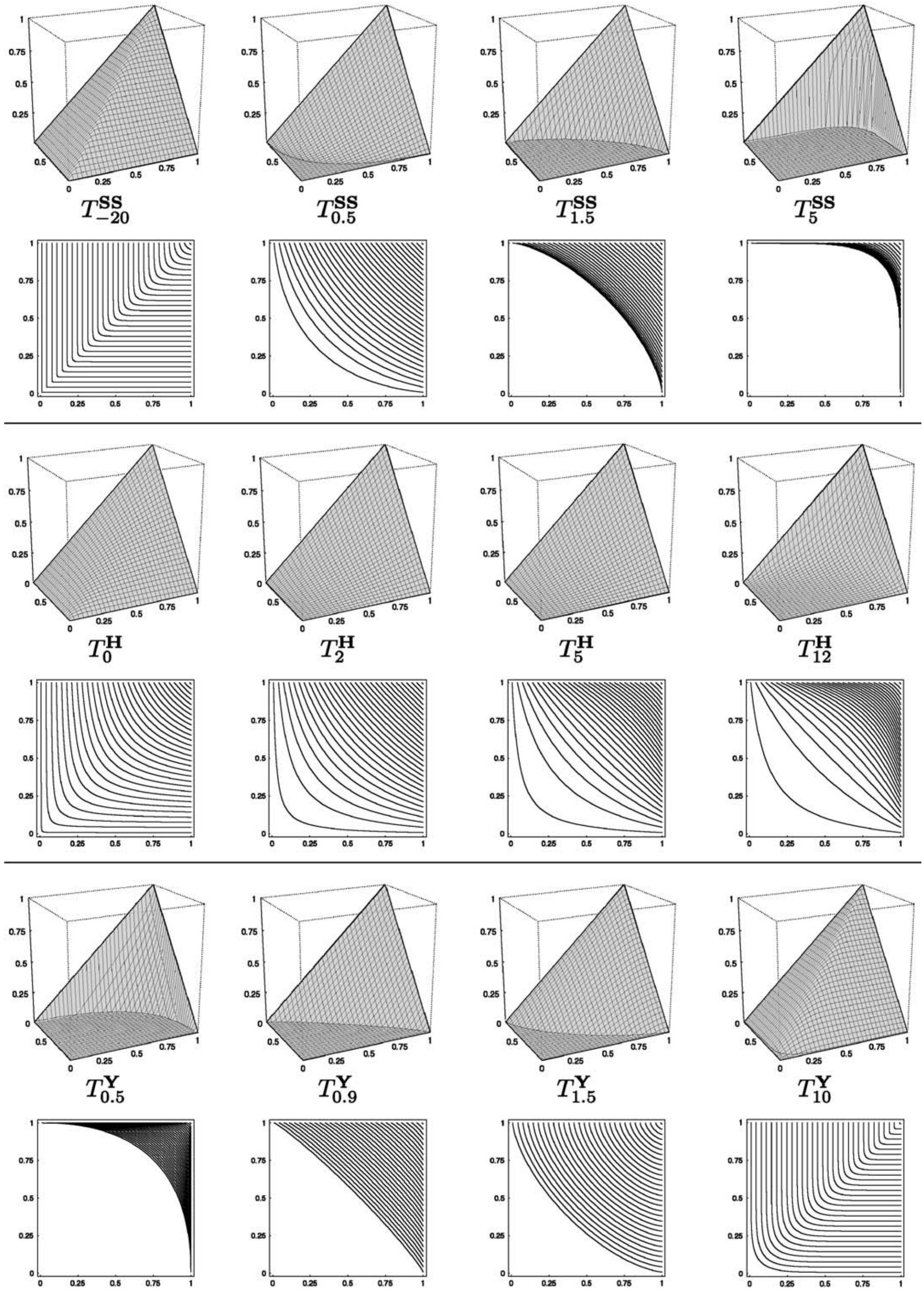

Fig. 9. Several Schweizer–Sklar (top), Hamacher (center) and Yager (bottom) t-norms.

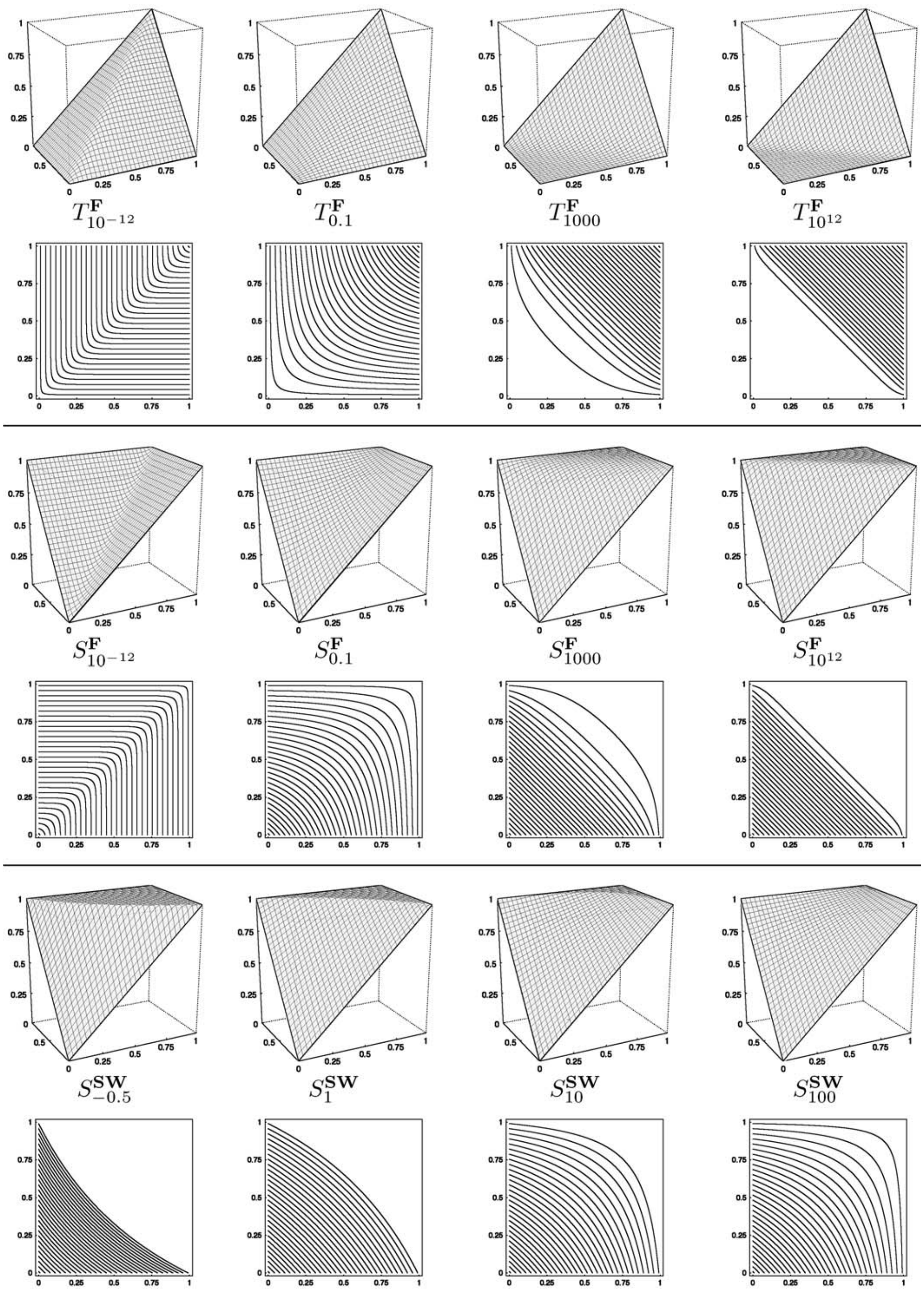

Fig. 10. Several Frank t-norms (top), Frank t-conorms (center) and Sugeno–Weber t-conorms (bottom).

strict monotonicity on $]0,1]^2$ in each component, and $T(1,1)=1$. The main result of [25,26] can be reformulated in this way: a continuous t-norm T is a rational function (i.e., a quotient of two polynomials) if and only if T belongs to the family $(T_\lambda^{\mathbf{H}})_{\lambda\in[0,\infty[}$ (in our presentation we also include the limit case $\lambda=\infty$).

Example 5.2. (i) The family $(T_\lambda^{\mathbf{H}})_{\lambda\in[0,\infty]}$ of *Hamacher t-norms* is given by

$$T_\lambda^{\mathbf{H}}(x,y)=\begin{cases} T_{\mathbf{D}}(x,y) & \text{if } \lambda=\infty,\\ 0 & \text{if } \lambda=x=y=0,\\ \dfrac{xy}{\lambda+(1-\lambda)(x+y-xy)} & \text{otherwise.}\end{cases}$$

(ii) Additive generators $t_\lambda^{\mathbf{H}}:[0,1]\to[0,\infty]$ of the strict members of the family of Hamacher t-norms are given by

$$t_\lambda^{\mathbf{H}}(x)=\begin{cases} \dfrac{1-x}{x} & \text{if } \lambda=0,\\ \ln\left(\dfrac{\lambda+(1-\lambda)x}{x}\right) & \text{if } \lambda\in]0,\infty[.\end{cases}$$

It is clear that $T_1^{\mathbf{H}}=T_{\mathbf{P}}$ and $T_0^{\mathbf{H}}=T_{-1}^{\mathbf{SS}}$ (the latter is sometimes called the *Hamacher product*).

The subfamily $(T_\lambda^{\mathbf{H}})_{\lambda\in[0,2]}$ of Hamacher t-norms is also a family of copulas [55], mentioned first in [3], in which context it is usually referred to as the family of *Ali–Mikhail–Haq copulas*.

5.3. Frank t-norms

The investigations of the associativity of duals of copulas in the framework of distribution functions have led to the following problem: characterize all continuous (or, equivalently, non-decreasing) associative functions $F:[0,1]^2\to[0,1]$, which satisfy for each $x\in[0,1]$ the boundary conditions $F(0,x)=F(x,0)$ and $F(x,1)=F(1,x)=x$, such that the function $G:[0,1]^2\to[0,1]$ given by

$$G(x,y)=x+y-F(x,y)$$

is also associative. In [20] it was shown that F has to be an ordinal sum of members of the following family of t-norms.

Example 5.3. (i) The family $(T_\lambda^{\mathbf{F}})_{\lambda\in[0,\infty]}$ of *Frank t-norms* (which were called *fundamental t-norms* in [8]) is given by

$$T_\lambda^{\mathbf{F}}(x,y)=\begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda=0,\\ T_{\mathbf{P}}(x,y) & \text{if } \lambda=1,\\ T_{\mathbf{L}}(x,y) & \text{if } \lambda=\infty,\\ \log_\lambda\left(1+\dfrac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right) & \text{otherwise.}\end{cases}$$

(ii) The family $(S_\lambda^{\mathbf{F}})_{\lambda\in[0,\infty]}$ of *Frank t-conorms* is given by

$$S_\lambda^{\mathbf{F}}(x,y)=\begin{cases} S_{\mathbf{M}}(x,y) & \text{if } \lambda=0,\\ S_{\mathbf{P}}(x,y) & \text{if } \lambda=1,\\ S_{\mathbf{L}}(x,y) & \text{if } \lambda=\infty,\\ 1-\log_\lambda\left(1+\dfrac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right) & \text{otherwise.}\end{cases}$$

(iii) Additive generators $t_\lambda^{\mathbf{F}}, s_\lambda^{\mathbf{F}}:[0,1]\to[0,\infty]$ of the continuous Archimedean members of the families of Frank t-norms and t-conorms are given by, respectively,

$$t_\lambda^{\mathbf{F}}(x)=\begin{cases} -\ln x & \text{if } \lambda=1,\\ 1-x & \text{if } \lambda=\infty,\\ \ln\left(\dfrac{\lambda-1}{\lambda^x-1}\right) & \text{if } \lambda\in]0,1[\,\cup\,]1,\infty[,\end{cases}$$

$$s_\lambda^{\mathbf{F}}(x)=\begin{cases} -\ln(1-x) & \text{if } \lambda=1,\\ x & \text{if } \lambda=\infty,\\ \ln\left(\dfrac{\lambda-1}{\lambda^{1-x}-1}\right) & \text{if } \lambda\in]0,1[\cup]1,\infty[.\end{cases}$$

All Frank t-norms are also copulas (see [55]) and have interesting statistical properties in the context of bivariate distributions [21,54].

The family of Frank t-norms is strictly decreasing, and the family of Frank t-conorms is strictly increasing (see [40, Proposition 6.8], a first proof of this result was given in [8, Proposition 1.12]).

The result of [20] (cf. [40, Theorem 5.14]) can be reformulated in the sense that a pair (T,S), where T is a continuous t-norm and S is a t-conorm, fulfills the Frank functional equation

$$T(x,y)+S(x,y)=x+y \tag{19}$$

for all $(x,y)\in[0,1]^2$ if and only if T is an ordinal sum of Frank t-norms and S is an ordinal sum of the corresponding dual Frank t-conorms, i.e., if

$$T=(\langle a_\alpha,e_\alpha,T_{\lambda_\alpha}^{\mathbf{F}}\rangle)_{\alpha\in A},$$

$$S=(\langle a_\alpha,e_\alpha,S_{\lambda_\alpha}^{\mathbf{F}}\rangle)_{\alpha\in A}.$$

Note, however, that the t-norm T and the t-conorm S are not necessarily dual to each other since the dual t-conorm of T is given by

$$(\langle 1-e_\alpha,1-a_\alpha,S_{\lambda_\alpha}^{\mathbf{F}}\rangle)_{\alpha\in A},$$

which coincides with S if and only if for each $\alpha\in A$ there is a $\beta\in A$ such that $\lambda_\alpha=\lambda_\beta$ and $a_\alpha+e_\beta=a_\beta+e_\alpha=1$.

The family of Frank t-norms plays a key role in the context of fuzzy logics [9,23,43]. Also for T-measures based on Frank t-norms it is possible to prove nice integral representations (see [8, Theorems 5.8, 6.2, 7.1] as well as a Liapounoff Theorem (see [6,8, Theorem 13.3]).

5.4. Yager t-norms

One of the most popular families for modeling the intersection of fuzzy sets is the following family of t-norms (which was first introduced in [71] for the special case $\lambda \geqslant 1$ only). The idea was to use the parameter λ as a reciprocal measure for the strength of the logical AND. In this context, $\lambda = 1$ expresses the most demanding (i.e., the smallest) AND, and $\lambda = \infty$ the least demanding (i.e., the largest) AND.

Example 5.4. (i) The family $(T_\lambda^{\mathbf{Y}})_{\lambda \in [0,\infty]}$ of *Yager t-norms* is given by

$$T_\lambda^{\mathbf{Y}}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y) & \text{if } \lambda = 0, \\ T_{\mathbf{M}}(x,y) & \text{if } \lambda = \infty, \\ \max(1 - ((1-x)^\lambda + (1-y)^\lambda)^{1/\lambda}, 0) & \text{otherwise.} \end{cases}$$

(ii) Additive generators $t_\lambda^{\mathbf{Y}} : [0,1] \to [0,\infty]$ of the nilpotent members $(T_\lambda^{\mathbf{Y}})_{\lambda \in]0,\infty[}$ of the family of Yager t-norms are given by

$$t_\lambda^{\mathbf{Y}}(x) = (1-x)^\lambda.$$

In Example 2.10(i) a construction for the additive generators of the nilpotent Yager t-norms was given.

It is trivial to see that $T_1^{\mathbf{Y}} = T_{\mathbf{L}}$. The subfamily $(T_\lambda^{\mathbf{Y}})_{\lambda \in [1,\infty]}$ of Yager t-norms is also a family of copulas [55].

The family of Yager t-norms is used in several applications of fuzzy set theory, e.g., in the context of fuzzy numbers (see [16]). In particular, for the addition of fuzzy numbers based on Yager t-norms it has been shown in [46] that the sum of piecewise linear fuzzy numbers again is a piecewise linear fuzzy number. The Yager t-norms appear also in the investigation of t-norms whose graphs are (partly) ruled surfaces [4].

5.5. Sugeno–Weber t-conorms

In [69], the use of the families of some special t-norms and t-conorms was suggested in order to model the intersection and union of fuzzy sets, respectively. These t-conorms are widely used in the context of decomposable measures [44,45,56,70], and they already appeared as possible generalized additions in the context of λ-fuzzy measures in [66].

Example 5.5. (i) The family $(S_\lambda^{\mathbf{SW}})_{\lambda \in [-1,\infty]}$ of *Sugeno–Weber t-conorms* is given by

$$S_\lambda^{\mathbf{SW}}(x,y) = \begin{cases} S_{\mathbf{P}}(x,y) & \text{if } \lambda = -1, \\ S_{\mathbf{D}}(x,y) & \text{if } \lambda = \infty, \\ \min(x + y + \lambda xy, 1) & \text{otherwise.} \end{cases}$$

(ii) Additive generators $s_\lambda^{\mathbf{SW}}:[0,1]\to[0,\infty]$ of the continuous Archimedean members of the family of Sugeno–Weber t-conorms are given by

$$s_\lambda^{\mathbf{SW}}(x)=\begin{cases} x & \text{if } \lambda=0,\\ -\ln(1-x) & \text{if } \lambda=-1,\\ \dfrac{\ln(1+\lambda x)}{\ln(1+\lambda)} & \text{if } \lambda\in]-1,\quad 0[\cup]0,\infty[. \end{cases}$$

Table 1
Some properties of the families of t-norms and t-conorms

Family	Continuous	Archimedean	Strict	Nilpotent
Schweizer–Sklar t-norms $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$	$\lambda\in[-\infty,\infty[$	$\lambda\in]-\infty,\infty]$	$\lambda\in]-\infty,0]$	$\lambda\in]0,\infty[$
Hamacher t-norms $(T_\lambda^{\mathbf{H}})_{\lambda\in[0,\infty]}$	$\lambda\in[0,\infty[$	All	$\lambda\in[0,\infty[$	None
Frank t-norms $(T_\lambda^{\mathbf{F}})_{\lambda\in[0,\infty]}$	All	$\lambda\in]0,\infty]$	$\lambda\in]0,\infty[$	$\lambda=\infty$
Yager t-norms $(T_\lambda^{\mathbf{Y}})_{\lambda\in[0,\infty]}$	$\lambda\in]0,\infty]$	$\lambda\in[0,\infty[$	None	$\lambda\in]0,\infty[$
Sugeno–Weber t-conorms $(S_\lambda^{\mathbf{SW}})_{\lambda\in[-1,\infty]}$	$\lambda\in[-1,\infty[$	All	$\lambda=-1$	$\lambda\in]-1,\infty[$

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