# Aggregation Functions Constructions, Characterizations, and Functional Equations

Habilitationsschrift

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# Preface

Aggregation is an important tool in any discipline where the fusion of different pieces of information is of interest and as such relates to several fields of applied and pure mathematics, of operations research, computer science, and many other applied fields like economics and finance, pattern recognition and image processing, or data fusion. Also inside mathematics "aggregation" is used to denote different processes and models in various subfields like matrix algorithms, population dynamics, partial differential equations, risk theory, reasoning under uncertainty, social choice, group preference modelling, and multi-criteria decision making.

Aggregation functions focus on special subclasses of aggregation problems, namely those which can be formally expressed by a function taking arbitrary but finitely many arguments and mapping them to a single value being representative for the set of arguments or some of its aspects. Arguments and representative value are from the same domain, most often a bounded lattice or some numerical scale. Means are prototypical examples of representative values resulting from an aggregation process carried out by an aggregation function.

This habilitation thesis is a collection of refereed articles published (or accepted) in scientific journals and edited volumes. Its focus is set on problems of constructions (and existence) and of characterizations of aggregation functions with an emphasis on particular types of functional equations and inequalities. The aggregation functions dealt with operate on a bounded lattice and fulfill additional monotonicity and boundary conditions. Several of the aggregation functions investigated have their roots in probabilistic metric spaces, more specifically, in case that the bounded lattice is simply the unit interval, some of these functions have been introduced as triangular norms or copulas. As such the classes of aggregation functions discussed relate to algebra, many-valued logics, and probability theory as well as to applications fields like, e.g., multicriteria decision making and preference modelling.

The collection of articles is preceded by this introductory part outlining the overall structure of the thesis and giving all necessary definitions, notions, investigated problems and some of the results in a condensed and therefore reduced way. The first chapter contains an introduction to aggregation functions and their properties, moreover, a detailed outline of the contents of thesis, its structure, and full referential details of the included articles. The articles are combined into three parts. Part I focusses on construction and characterizations results for special semigroup operations as well as bivariate copulas and quasi-copulas. Part II deals with aggregation functions on the unit interval, especially with t-norms, and discusses the functional inequality of dominance. Part III contains two contributions relating to application problems in the context of preference modelling and decision making touching again construction and characterization problems and functional equations.

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Carrying out mathematical research in this way has been made possible by the support through several European activities, projects, and grants, in particular I could benefit from the CEEPUS Network SK-42-Fuzzy Control and Fuzzy Logic, the COST Action 274: TARSKI-Theory and Application of Relational Systems as Knowledge Systems, from the following bilateral actions the Action Austria-Czech Republic (Projects 2007/17 and 41p19), the Action Austria-Slovakia (Project 42s2), the WTZ "Acciones Integradas 2008-2009" (Project ES04/2008), and the WTZ Austria-Poland (Project PL 03/2008)—as well as from the Erwin Schrödinger-Research Fellowship J2636-N15 "The Property of Dominance—From Products of Probabilistic Metric Spaces to (Quasi-)Copulas and Applications" granted by the Austrian Science Fund (FWF) for a one-year postdoc stay at the Dipartimento di Matematica "Ennio De Giorgi" of the Università del Salento, in Lecce, Italy. The support is gratefully acknowledged. Further I would like to thank the Johannes Kepler University and the Linzer Hochschulfonds for the support of research visits abroad and for the support in hosting visitors at our department.

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# Chapter 1

# Introduction

## 1.1 Aggregation — an approximation

"Aggregation" is used in everyday life and in mathematics in very different contexts. According to the Oxford Advanced Learner's Dictionary [154] "aggregate" bears the following meanings, expressing that aggregation in general relates to a process during which a group of items (numbers, amounts, results) is merged into a total:

#### aggregate

- *noun*: 1 a total number or amount made up of smaller amounts that are collected together 2 (*technical*) sand or broken stone that is used to make concrete or for building roads, etc. in (the) 'aggregate' (*formal*) added together as a total or single amount on 'aggregate' (*sport*) when the scores of a number of games are added together: *They* won 4–2 on aggregate.
- *adj.*: (*economics* or *sports*) made up of several amounts that are added together to form a total number: aggregate demand/investment/turnover.
- verb: (formal or technical) to put together different items, amounts, etc. into a single group or total ► aggregation noun.

Database queries in Zentralblatt and MathSciNet for articles published more recently than 2007 and having the word "aggregation" listed in the title, yield more than hundred publications each and show that also inside mathematics aggregation is spread over different areas and is used to denote a variety of different processes and mathematical models. The topics of aggregation range from various fields, like matrix algorithms, population dynamics, and partial differential equations over risk theory to reasoning under uncertainty and, to a larger extent, to social choice, group preference modelling, and multi-criteria decision making. It is therefore necessary to clarify which kind of aggregation models will be discussed in the present thesis.

Mathematically speaking, its focus is set on aggregation processes which can be expressed by a function

$$\mathbf{A} \colon \bigcup_{n \in \mathbb{N}} D^n \to D \tag{1.1}$$

mapping arbitrary, but finitely many arguments from a set D to an object in D which is representative for the set of arguments itself or for one of its aspects. The actual set D as well as the function  $\mathbf{A}$  and their additional properties clearly depend on the application resp. the model currently being investigated. One of the most prominent fields of applications of such aggregation processes comprise, e.g., multi-criteria decision problems and group preference processes. Also in economics and statistics, various sorts of means are frequently used for determining indices and values representing a given set of data points or some of its aspects. Clearly, means and their generalizations are also aggregation functions in the meaning introduced above.

#### **Related** problems

The major research lines in aggregation can roughly be divided into the following categories (compare also [194]):

• Constructions (and existence): Problems of this category refer to questions of constructions and existence of aggregation functions allowing to model the theoretical demands and needs of a (practical) aggregation problem.

As one of the most famous results w.r.t. the existence of an appropriate aggregation function, we may quote Arrow's (im)possibility theorem [11]. The application setting Arrow is dealing with is group preference modelling and he investigates the following problem:

Provided that there are at least three alternatives which are ordered according to the preferences of at least two individuals, does there exist a *social welfare function* such that the social ordering of alternatives fulfills the following conditions?

- (1) Among all the alternatives there is a set S of three alternatives such that, for any set of individual orderings  $T_1, \ldots, T_n$  of the alternatives in S, there is an admissible set of individual orderings  $R_1, \ldots, R_n$  of all the alternatives such that, for each individual i,  $xR_iy$  if and only if  $xT_iy$  for  $x, y \in S$ . (This condition allows that the *a priori* knowledge about the occurrence of individual orderings is incomplete).
- (2) Let R<sub>1</sub>,..., R<sub>n</sub> and R'<sub>1</sub>,..., R'<sub>n</sub> be two sets of individual ordering relations, R and R' the corresponding social orderings, and P and P' the corresponding social preference relations. Suppose that for each *i* the two individual ordering relations are connected in the following ways: for x' and y' distinct from a given alternative x, x'R'<sub>i</sub>y' if and only if x'R<sub>i</sub>y'; for all y', xR<sub>i</sub>y' implies xR'<sub>i</sub>y'; for all y', xP<sub>i</sub>y' implies xP'<sub>i</sub>y'. Then, if xPy, xP'y. (The social ordering shall respond positively, at least not negatively, to alterations and enhancements of individual values; sometimes this property is referred to as monotonicity).
- (3) Let  $R_1, \ldots, R_n$  and  $R'_1, \ldots, R'_n$  be two sets of individual orderings and let C(S) and C'(S) be the corresponding social choice functions. If, for all individuals *i* and all *x* and *y* in a given environment *S*,  $xR_iy$  if and only if  $xR'_iy$ , then C(S) and C'(S) are the same (*independence of irrelevant alternatives*).
- (4) The social welfare function is not to be *imposed*. A social welfare function is said to be *imposed*, if, for some pair of distinct alternatives x and y, xRy for any set of individual orderings  $R_1, \ldots, R_n$ , where R were the social ordering corresponding to  $R_1, \ldots, R_n$ , i.e., some preferences are taboo and can not be influenced by the group members.
- (5) The social welfare function is not to be *dictatorial*. A social welfare function is said to be *dictatorial*, if there exists an individual *i* such that, for all *x* and *y*,  $xP_iy$  implies xPy regardless of the orderings  $R_1, \ldots, R_n$  of all individuals other than *i*, where *P* is the social preference relation corresponding to  $R_1, \ldots, R_n$ .

Note that by a *social welfare function* Arrow means (see Definition 4 in [11])

... a process or rule which, for each set of individual orderings  $R_1, \ldots, R_n$  for alternative social states (one ordering for each individual), states a corresponding social ordering of alternative social states, R.

Further note that orderings in the sense of Arrow are connected and transitive relations on the set of alternatives. Therefore, Arrow is considering an aggregation problem in the above sense, i.e., he is looking for some function  $\mathbf{A} \colon \bigcup_{n \in \mathbb{N}} D^n \to D$  such that

$$R = \mathbf{A}(R_1, \dots, R_n)$$

with D being the set of all possible orderings over the set of alternatives,  $R_i$  the individual orderings, and R the social order.

Arrow showed that, for three alternatives and at least two individuals, there is no social welfare function fulfilling all the demanded conditions at the same time. If there are at least three alternatives which the members of the society are free to order in arbitrary way, then every social welfare function satisfying Conditions 2 (monotonicity) and 3 (independence of irrelevant alternatives) and yielding a social ordering must be either imposed or dictatorial.

Problems of construction refer, beside the existence of an appropriate aggregation function, also to modifications, adoptions, extensions, and restrictions of existing aggregation functions to new ones, like, e.g., the introduction of weights into a given aggregation process, transformations, composed aggregation, or constructions like ordinal sums (see, e.g., [23]). Typical related questions would be whether the procedures applied yield again an aggregation function of a particular type fulfilling the theoretical and practical demands imposed.

• *Characterizations:* Characterization problems aim at a most comprehensive and exhaustive description of the aggregation function used. They also touch problems of finding equivalent, but possibly more expressive, descriptions of aggregation functions in order to allow an easy decision about the applicability of the aggregation function in different application settings and to allow for an additional understanding of the aggregation procedure.

As an example let us mention two classical characterization results for quasi-arithmetic means, i.e., functions  $M_f \colon \bigcup_{n \in \mathbb{N}} [a, b]^n \to [a, b], [a, b] \subseteq \mathbb{R}$ , such that

$$M_f(x_1, \dots, x_n) = f^{-1}\left(\frac{f(x_1) + \dots + f(x_n)}{n}\right)$$

with  $f: [a, b] \rightarrow [a, b]$  a continuous and strictly increasing function.

The first characterization has been provided in 1930 by Kolmogoroff [121] and at the same time by Nagumo [139] and reads as follows:

A continuous, strictly increasing function  $M \colon \bigcup_{n \in \mathbb{N}} [a, b]^n \to [a, b]$  is symmetric, idempotent, i.e., fulfills, for all  $x \in [a, b]$ ,

$$M(x, ..., x) = x$$

is decomposable, i.e., for all  $n \in \mathbb{N}$ , for all  $k \in \{1, \ldots, n\}$ , and all  $x_i \in [a, b], i \in \{1, \ldots, n\}$ ,

 $M(x_1,\ldots,x_k,x_{k+1},\ldots,x_n) = M(\underbrace{M(x_1,\ldots,x_k),\ldots,M(x_1,\ldots,x_k)}_{k \text{ times}},x_{k+1},\ldots,x_n),$ 

and fulfills M(x) = x for all  $x \in [a, b]$  if and only if there exists a continuous, strictly increasing function f on [a, b] such that  $M = M_f$ .

In [1], Aczél gave another characterization result for binary quasi-arithmetic means (compare also [2]):

A continuous, strictly increasing function  $M: [a,b]^2 \to [a,b]$  is symmetric, idempotent, and bisymmetric, i.e., for all  $x_{ij} \in [a,b]$ ,  $i, k \in \{1,2\}$ ,

$$M(M(x_{11}, x_{12}), M(x_{21}, x_{22})) = M(M(x_{11}, x_{21}), M(x_{12}, x_{22}))$$

if and only if there exists a continuous, strictly increasing function f on [a, b] such that  $M(x_1, x_2) = f^{-1}\left(\frac{f(x_1) + f(x_2)}{2}\right).$ 

These examples illustrate, that characterization results provide different viewpoints on the same aggregation function. Moreover, most often, functional equations and inequalities are instrumental in the description of relevant properties.

• Selection and optimization: The last main group of problems in aggregation processes relates to the selection of a particular aggregation function possibly from a class of aggregation functions determined by a parameter set, i.e., refers to choosing an appropriate class of functions, to optimizing a parameter set, or to fitting (parameters of) functions to a given set of input-output data pairs.

## 1.2 Focus of the thesis

The focus of the present thesis is set on problems of constructions (and existence), of characterizations of aggregation functions with an emphasis on particular types of functional equations. The aggregation procedures investigated restrict to those being expressible by aggregation functions, i.e., to the aggregation of arbitrary but finitely many arguments. Mathematically speaking, we restrict to aggregation processes which can be described by a function of type (1.1). Aggregation of infinitely many arguments and (finite as well as infinite) aggregation by (generalized) integrals are not in the focus of this thesis.

In all cases considered in this thesis, we will assume that  $(D, \land, \lor, 0, 1)$  is a bounded lattice. Since each bounded lattice is also a bounded poset and since most often the order aspect is of prior interest in our investigations we use the notation  $(D, \leq, 0, 1)$  only. The order aspect allows us to formulate monotonicity and boundary conditions for **A**. We briefly summarize a few basic notions and properties of aggregation functions acting on bounded lattice which will be of relevance in later investigations:

**Definition 1.1.** Consider a bounded lattice  $(L, \leq, 0, 1)$ . A function  $\mathbf{A} : \bigcup_{n \in \mathbb{N}} L^n \to L$  is called an *aggregation function on* L if the following conditions are fulfilled, for all  $n \in \mathbb{N}$  and for all  $x_i, y_i \in L$ ,  $i \in \{1, \ldots, n\}$ :

- (i)  $\mathbf{A}(x_1,\ldots,x_n) \leq \mathbf{A}(y_1,\ldots,y_n)$ , whenever  $x_i \leq y_i$ , for all  $i \in \{1,\ldots,n\}$ ,
- (ii)  $\mathbf{A}(x_1) = x_1$ ,
- (iii) A(0,...,0) = 0 and A(1,...,1) = 1.

If L = [0, 1], we refer to  $\mathbf{A} \colon \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$  simply as an aggregation function.

Note that every aggregation function **A** on a lattice L can be represented by a family  $(\mathbf{A}_{(n)})_{n \in \mathbb{N}}$ of *n*-ary aggregation functions on L, i.e., by functions  $\mathbf{A}_{(n)} \colon L^n \to L$  given by

$$\mathbf{A}_{(n)}(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_n)$$

where  $\mathbf{A}_{(1)} = \mathrm{id}_L$  and, for  $n \geq 2$ , each  $\mathbf{A}_{(n)}$  is non-decreasing in each argument and satisfies  $\mathbf{A}_{(n)}(0,\ldots,0) = 0$  and  $\mathbf{A}_{(n)}(1,\ldots,1) = 1$ . Usually, the aggregation function  $\mathbf{A}$  on L and the family  $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$  of *n*-ary aggregation functions on L are identified with each other.

Note that depending on the application setting and therefore for corresponding lattices, other notions for aggregation functions can be found in the literature, e.g., in operations research and reliability theory,  $L = \{0, ..., M\}$  and represents different states of a system (component), aggregation functions being referred to as *structure functions* (see, e.g., [16], and also [123]). In social choice theory,  $L = 2^X$  where  $A \in L$  represents the set of propositions that a group accepts, the corresponding aggregation process is more specifically called *judgment aggregation* (see, e.g., [51]). In many-valued logics L is interpreted as the lattice of truth values (e.g., [0, 1], some discrete chain, or the diamond lattice, see also Chapter 2). In decision theory, L = [0, 1] might serve as the range of monotone and normalized set functions modelling the importance or evaluation of subsets of the involved criteria by individuals or a group of individuals, the aggregation function involved being referred to as *consensus function* (compare, e.g., [55, 71, 124, 197]).

**Definition 1.2.** Consider a bounded lattice  $(L, \leq, 0, 1)$  and an aggregation function  $\mathbf{A} \colon \bigcup_{n \in \mathbb{N}} L^n \to L$  on L.

(i) A is called *symmetric* (or, depending on the application context, also *commutative*, *anony-mous*, or *neutral*) if, for all  $n \in \mathbb{N}$  and for all  $x_i \in L$ ,  $i \in \{1, \ldots, n\}$ ,

$$\mathbf{A}(x_1,\ldots,x_n) = \mathbf{A}(x_{\alpha(1)},\ldots,x_{\alpha(n)}) \tag{1.2}$$

for all permutations  $\alpha = (\alpha(1), \ldots, \alpha(n))$  of  $\{1, \ldots, n\}$ .

(ii) **A** is called *associative* if, for all  $n, m \in \mathbb{N}$  and all  $x_i, y_j \in L$  with  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ ,

$$\mathbf{A}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \mathbf{A}(\mathbf{A}(x_1,\ldots,x_n),\mathbf{A}(y_1,\ldots,y_m)).$$
(1.3)

(iii) **A** is called *bisymmetric* if, for all  $n, m \in \mathbb{N}$  and all  $x_{i,j} \in L$  with  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ ,

$$\mathbf{A}_{(m)} \left( \mathbf{A}_{(n)}(x_{1,1}, \dots, x_{1,n}), \dots, \mathbf{A}_{(n)}(x_{m,1}, \dots, x_{m,n}) \right) = \mathbf{A}_{(n)} \left( \mathbf{A}_{(m)}(x_{1,1}, \dots, x_{m,1}), \dots, \mathbf{A}_{(m)}(x_{1,n}, \dots, x_{m,n}) \right).$$
(1.4)

(iv) An element  $e \in L$  is called *neutral element* of **A** if, for all  $n \in \mathbb{N}$ , for all  $x_i \in L$ ,  $i \in \{1, ..., n\}$ , and each  $j \in \{2, ..., n-1\}$  it holds that

$$\mathbf{A}(x_1, \dots, x_{j-1}, e, x_{j+1}, \dots, x_n) = \mathbf{A}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$
(1.5)

as well as  $\mathbf{A}(e, x_2, \dots, x_n) = \mathbf{A}(x_2, \dots, x_n)$  and  $\mathbf{A}(x_1, \dots, x_{n-1}, e) = \mathbf{A}(x_1, \dots, x_{n-1})$ .

(v) An element  $a \in L$  is called *annihilator* of **A** if, for all  $n \in \mathbb{N}$ , for all  $x_i \in L$ ,  $i \in \{1, ..., n\}$ , and each  $j \in \{2, ..., n-1\}$ , it holds that

$$\mathbf{A}(x_1, \dots, x_{j-1}, a, x_{j+1}, \dots, x_n) = a$$
(1.6)

as well as 
$$A(a, x_2, ..., x_n) = A(x_1, ..., x_{n-1}, a) = a$$

(vi) An element  $d \in [0, 1]$  is called an *idempotent element* of **A**, if  $\mathbf{A}(d, \ldots, d) = d$  for all  $n \in \mathbb{N}$ . We will abbreviate the set of idempotent elements by  $\mathcal{I}(\mathbf{A}) = \{d \in L \mid \mathbf{A}(d, \ldots, d) = d\}$ . In case that  $\mathcal{I}(\mathbf{A}) = L$ , the aggregation function is called *idempotent*.

Because of (1.3), associative aggregation functions  $\mathbf{A}$  on L are completely characterized by their binary aggregation functions  $\mathbf{A}_{(2)}$  on L since all *n*-ary, n > 2, aggregation functions  $\mathbf{A}_{(n)}$  can be constructed by the recursive application of the binary aggregation function  $\mathbf{A}_{(2)}$ . Associative and symmetric aggregation functions on L are also bisymmetric. On the other hand, bisymmetric aggregation functions on L with some neutral element are associative and symmetric.

**Definition 1.3.** Consider two bounded lattices  $(L_1, \leq_1, 0_1, 1_1)$  and  $(L_2, \leq_2, 0_2, 1_2)$  and a order reversing or order preserving lattice isomorphism  $\varphi: L_2 \to L_1$ . Further let **A** be an aggregation function on  $L_1$ . Then the *isomorphic transformation*  $\mathbf{A}_{\varphi}$  is defined by

$$\mathbf{A}_{\varphi}(x_1,\ldots,x_n) = \varphi^{-1}(\mathbf{A}(\varphi(x_1),\ldots,\varphi(x_n)))$$

and is an aggregation function on  $L_2$ . If for two aggregation functions  $\mathbf{A}$ ,  $\mathbf{B}$ , on possibly different bounded lattices, there exists a lattice isomorphism  $\varphi$  such that  $\mathbf{A} = \mathbf{B}_{\varphi}$  or  $\mathbf{A}_{\varphi} = \mathbf{B}$ , then we refer to  $\mathbf{A}$  and  $\mathbf{B}$  as *isomorphic aggregation functions*. For binary (aggregation) functions on the unit interval we introduce the following additional properties:

**Definition 1.4.** Consider a binary (aggregation) function  $\mathbf{A}_{(2)}$ :  $[0,1]^2 \rightarrow [0,1]$ .

(i)  $\mathbf{A}_{(2)}$  is called 2-increasing (or, depending on the application context, also supermodular, superadditive, quasi-monotone, or fulfilling moderate growth) if, for all  $x_1, x_2, y_1, y_2 \in [0, 1]$  with  $x_1 \leq x_2$  and  $y_1 \leq y_2$ ,

$$\Delta_{x_1,x_2}^{y_1,y_2}(\mathbf{A}_{(2)}) = \mathbf{A}_{(2)}(x_1,y_1) - \mathbf{A}_{(2)}(x_1,y_2) - \mathbf{A}_{(2)}(x_2,y_1) + \mathbf{A}_{(2)}(x_2,y_2) \ge 0.$$
(1.7)

The expression  $\Delta_{x_1,x_2}^{y_1,y'_2}(\mathbf{A}_{(2)})$  is called the  $\mathbf{A}_{(2)}$ -volume of the rectangle  $[x_1,x_2] \times [y_1,y_2]$ .

(ii)  $\mathbf{A}_{(2)}$  is 1-*Lipschitz* if, for all  $x_1, y_1, x_2, y_2 \in [0, 1]$ ,

$$|\mathbf{A}_{(2)}(x_1, y_1) - \mathbf{A}_{(2)}(x_2, y_2)| \le |x_1 - x_2| + |y_1 - y_2|.$$
(1.8)

For more details and thorough expositions on different aspects of aggregation functions see the edited volumes [24, 93] and the monographs [12, 92, 194]. Several aggregation functions on (special) lattice structures have also been investigated in [A01, A02, A03] and, e.g., in [36, 42, 44, 47, 102, 171, 200].

## **1.3** Outline of the thesis

The focus of the present thesis is set on problems of constructions (and existence), of characterizations of aggregation functions with an emphasis on particular types of functional equations. The schematic structure of the thesis is the following:

Part I, entitled "Aggregation Functions: Constructions and Characterizations", is dedicated to constructions and characterizations results for two classes of functions. First for triangular norms and triangle functions which are both ordered Abelian semigroups acting on a bounded lattice whose top element serves also as the neutral element of the semigroup operation. Triangular norms are well-known concepts for modelling the evaluation of conjunctions in many-valued logics (see the monographs [8, 112, 113] for thorough expositions). Triangle functions are a necessary tool for an appropriate formulation of the triangle inequality in probabilistic metric spaces (see also [181]). Whereas triangular norms act on a bounded lattice of truth values, in classical cases most often the unit interval, the interpretation of the underlying domain of triangle functions is different. Triangle functions are defined on a subset of distribution functions, called distance distribution functions. W.r.t. the usual pointwise order of [0, 1]-valued functions, the set of distance distribution functions with its greatest and smallest element constitutes again a bounded lattice.

The second class of functions discussed in Part I relates to (binary) copulas and quasi-copulas. For both classes of functions the underlying lattice is the closed unit-interval equipped with the standard order. Copulas are functions which join multivariate distribution functions with their univariate marginal distribution functions (see also [108, 140]). In fact, according to *Sklar's theorem* [187], for each random vector  $(X_1, \ldots, X_n)$  there is a copula *C* (uniquely defined whenever all  $X_i$ ,  $i \in \{1, \ldots, n\}$ , are continuous) such that the joint distribution function  $F_{X_1,\ldots,X_n}$  of  $(X_1,\ldots,X_n)$  may be represented, for all  $x_i \in \mathbb{R}$ ,  $i \in \{1,\ldots,n\}$ , by

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = C_{X_1,\ldots,X_n}(F_{X_1}(x_1),\ldots,F_{X_n}(x_n))),$$

where, for all  $i \in \{1, ..., n\}$ ,  $F_{X_i}$  is the distribution function  $X_i$ . It is worth noting that the copula C completely captures the dependence structure of the random vector  $(X_1, ..., X_n)$ .

Quasi-copulas characterize operations on distribution functions induced by operations on random variables defined on the same probability space [9, 82]. It is clear, that copulas and quasicopulas are of interest in statistics and probability theory, however, more recently they also become more important in, e.g., finance [26, 130], hydrology [155], preference modelling [48, 49, 50] and also in many-valued logics [98].

For both classes of functions — triangular norms and triangle functions as well as bivariate copulas and quasi-copulas — construction and characterizations problems are investigated in this thesis.

The roots of all these functions can be traced back to the fields of probabilistic metric spaces, earlier called statistical metric spaces. It has been the investigation of products of such spaces which brought a special functional inequality, called *dominance*, to the fore. Several results on dominance have been achieved in the framework of probabilistic metric spaces (compare also, e.g., [6, 68, 190, 191], but also [180] and [8, 181] and the references therein), but several problems remained open and of interest for many years.

Part II, entitled "Aggregation Functions: Dominance — A Functional Inequality", focusses on the functional inequality of dominance. Especially, dominance between triangular norms and the question whether it constitutes a transitive and therefore also an order relation has been of interest for many years. The results presented in Part II show the contributions to the (negative) solution of this long open problem and provide several results for tools and techniques showing that dominance, although not transitive in general, is transitive on several (parameterized) subsets of triangular norms. The articles and results included discuss dominance between t-norms, copulas, quasi-copulas, and conjunctors. Note that in [171], we have turned back to the roots of dominance and discuss functional equations and inequalities, among which also dominance, between triangle functions and operations on distance distribution functions.

Part III — "Aggregation and Decision Modelling: Two Case Studies" — finally focusses on two application problems arising in the context of preference modelling and decision making touching again construction and characterization problems as well as special functional equation.

The first article contains representation and construction results for so-called *self-dual* and N-invariant aggregation functions unifying and extending two existing characterization results for self-dual aggregation functions. Self-dual aggregation functions are important in aggregating [0, 1]-valued relations which express individual intensities for a preference between two alternatives. In order to rule out incomparability, it is often required that the degree to which some alternative a is preferred to some alternative b should be in some sense complementary to the degree to which b is preferred to a. This naturally leads to the use of reciprocal preference relations  $R_i$ , i.e., relations for which  $R_i(a, b) + R_i(b, a) = 1$  for all alternatives a, b. Aggregating such preference relations  $R_i$  into a collective group preference relation R by preserving reciprocity demands self-dual aggregation functions.

The second article touches the problem of two-step aggregation procedures in multi-person multi-criteria decision problems. Several alternatives are evaluated by several criteria and by several experts. Aggregating partial results first w.r.t. the criteria and than by experts should lead to the same result as aggregating first w.r.t. the experts' judgements and than by combining partial results w.r.t. the evaluation criteria, i.e., the final result shall not depend on the order in which the single aggregation steps are performed. The aggregation functions involved have to commute in order to guarantee this demand. Commuting is expressed as a functional equation between the aggregation functions involved and denotes a special case of the generalized bisymmetry equation which is of relevance also in consistent aggregation result for such functions, in particular if one of the aggregation functions involved is symmetric, associative, and has a neutral element which is not a boundary element of the unit interval. Such functions, called uninorms, are also relevant in bipolar decision making in which the level of neutrality splits the evaluation scale into a positive and a negative part, such that the presented results are also interesting for bipolar decision making.

Finally, we would like to stress that the following chapters are intended to give a rough overview on the basic notions, the problems investigated, and the nature of the achieved results. Since it is not possible to touch all aspects and results in full detail, unless repeating the included contributions completely, the contents of these introductory parts do provide only a carefully chosen, but restricted selection of the results contained in the articles. In some cases the most important findings have been quoted, in other cases we have decided to restrict to special cases only illustrating the nature of the general results without discussing their complexity and generality to the full extent. Throughout the chapters we have tried to outline which approach has been applied. Nevertheless, we kindly invite the reader to still draw his/her attention to the attached original contributions containing all details, additional aspects and proofs.

## 1.4 List of included articles

Part I. Aggregation Functions: Constructions and Characterizations

Triangular norms and triangle functions — two special semigroups

- A01. S. Saminger. On ordinal sums of triangular norms on bounded lattices. *Fuzzy Sets and Systems*, vol. 157, no. 10, pp. 1403–1416, 2006.
- A02. S. Saminger-Platz, E.P. Klement, R. Mesiar. On extensions of triangular norms on bounded lattices. *Indagationes Mathematicae*, vol. 19, no. 1, pp. 135–150, 2008.
- A03. S. Saminger-Platz, C. Sempi. A primer on triangle functions I. Aequationes Mathematicae, vol. 76, pp. 201–240, 2008.

Copulas and quasi-copulas — aggregation functions reflecting dependence structures

- A04. J.J. Quesada-Molina, S. Saminger-Platz, C. Sempi. Quasi-copulas with a given subdiagonal section. *Nonlinear Analysis*, vol. 69, pp. 4654–4673, 2008.
- A05. F. Durante, S. Saminger-Platz, P. Sarkoci. On representations of 2-increasing binary aggregation functions. *Information Sciences*, vol. 178, pp. 4634–4541, 2008.
- **A06.** F. Durante, S. Saminger-Platz, P. Sarkoci. Rectangular patchwork for bivariate copulas and tail dependence. (accepted for publication in *Communications in Statistics Theory and Methods*).
- Part II. Aggregation Functions: Dominance A Functional Inequality

Dominance between ordinal sums — on the (non-)transitivity of dominance of t-norms

- A07. S. Saminger, B. De Baets, H. De Meyer. On the dominance relation between ordinal sums of conjunctors. *Kybernetika*, vol. 42, no. 2, pp. 337–350, 2006.
- A08. S. Saminger, P. Sarkoci, B. De Baets. The dominance relation on the class of continuous ordinal sum t-norms. In H.C.M. de Swart, E. Orłowska, M. Roubens, and G. Schmidt, editors, Theory and Applications of Relational Structures as Knowledge Instruments II, Springer, pp. 337–357, 2006.

Dominance between continuous Archimedean t-norms — easy-to-check conditions

- A09. S. Saminger-Platz, B. De Baets, H. De Meyer. A generalization of the Mulholland inequality for continuous Archimedean t-norms. *Journal of Mathematical Analysis and Applications*, vol. 345, pp. 607–614, 2008.
- A10. S. Saminger-Platz, B. De Baets, H. De Meyer. Differential inequality conditions for dominance between continuous Archimedean t-norms. (accepted for publication in *Mathematical Inequalities & Applications*).
- A11. S. Saminger-Platz. The dominance relation in some families of continuous Archimedean t-norms and copulas. (accepted for publication in *Fuzzy Sets and Systems*, doi:10.1016/j.fss.2008.12.009).

- Part III. Aggregation and Decision Modelling: Two Case Studies
  - A12. K. Maes, S. Saminger, B. De Baets. Representation and construction of self-dual aggregation operators. *European Journal of Operation Research*, vol. 177, pp. 472–487, 2007.
  - A13. S. Saminger-Platz, R. Mesiar, D. Dubois. Aggregation operators and commuting. IEEE Transactions on Fuzzy Systems, vol. 15, no. 6, pp. 1032–1045, 2007.

# Part I

Aggregation Functions:

**Constructions and Characterizations** 

## Chapter 2

# Triangular norms and triangle functions

Two special semigroups

## 2.1 Triangular norms on bounded lattices

#### Triangular norms on the unit interval

Triangular norms (briefly t-norms) on the unit interval were first introduced in the context of probabilistic metric spaces [178, 180, 182], based on some ideas by Menger [131] aiming at an extension of the triangle inequality for such spaces. Later on, they turned out to be indispensable tools for the interpretation of the conjunction in many-valued logics [10, 86, 87, 97, 103], in particular in fuzzy logics where the unit interval serves as set of truth values. Further, triangular norms on the unit interval play an important role in various further fields like decision making [72, 94], statistics [140], as well as the theories of non-additive measures [118, 147, 188, 198] and cooperative games [22]. The formal definition of t-norms on the unit interval reads as follows:

**Definition 2.1.** A binary operation  $T: [0,1]^2 \to [0,1]$  is called a *triangular norm* (briefly *t-norm*) if the following conditions are fulfilled, for all  $x, y, z \in [0,1]$ ,

(i) $T(x,z) \le T(y,z)$ whenever $x \le y$ ,	(monotonicity)
(ii) $T(x,y) = T(y,x),$	(commutativity)

(iii) T(x, T(y, z)) = T(T(x, y), z), (associativity)

(iv) 
$$T(x,1) = x$$
. (neutral element)

In other words, a t-norm T is a commutative, associative aggregation function with neutral element 1, or a t-norm T turns [0, 1] into an ordered Abelian semigroup with neutral element 1.

Triangular norms on the unit interval and their properties have been studied extensively. In the sequel, we restrict to those properties only which will be necessary for a complete understanding of the following parts. Thorough overviews on triangular norms on the unit interval (including proofs, further details and references) can be found in the monographs [8, 113], the edited volume [112] and the articles [115, 116, 117].

It is an immediate consequence that, due to the boundary and monotonicity conditions as well as the commutativity, any t-norm T fulfills, for all  $x \in [0, 1]$ ,

$$T(0,x) = T(x,0) = 0,$$
(2.1)

$$T(1,x) = x. (2.2)$$



Figure 2.1: 3D plots of the four basic t-norms  $T_{\mathbf{M}}$ ,  $T_{\mathbf{P}}$ ,  $T_{\mathbf{L}}$ , and  $T_{\mathbf{D}}$ .

Therefore, all t-norms coincide on the boundary of the unit square  $[0, 1]^2$ .

**Example 2.2.** The most prominent examples of t-norms on the unit interval are the *minimum*  $T_{\mathbf{M}}$ , the *product*  $T_{\mathbf{P}}$ , the *Lukasiewicz t-norm*  $T_{\mathbf{L}}$  and the *drastic product*  $T_{\mathbf{D}}$  (see also Figure 2.1). They are given by:

$$T_{\mathbf{M}}(x,y) = \min(x,y), \tag{2.3}$$

$$T_{\mathbf{P}}(x,y) = x \cdot y, \tag{2.4}$$

$$T_{\mathbf{L}}(x,y) = \max(x+y-1,0),$$
 (2.5)

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$
(2.6)

Obviously, the basic t-norms  $T_{\mathbf{M}}, T_{\mathbf{P}}$  and  $T_{\mathbf{L}}$  are continuous, whereas the drastic product  $T_{\mathbf{D}}$  is not. Note that for a t-norm T its continuity is equivalent to the continuity in each component (see also [113, 115]), for arbitrary  $x_0, y_0 \in [0, 1]$  both the vertical section  $T(x_0, ): [0, 1] \to [0, 1]$  and the horizontal section  $T(;y_0): [0, 1] \to [0, 1]$  are continuous functions in one variable.

The comparison of two t-norms is done pointwisely, i.e., if, for all  $x, y \in [0, 1]$ , it holds that  $T_1(x, y) \ge T_2(x, y)$ , then we say that  $T_1$  is *stronger* than  $T_2$  and denote it by  $T_1 \ge T_2$ . For each t-norm T it holds that  $T_{\mathbf{D}} \le T \le T_{\mathbf{M}}$ . Moreover, the four basic t-norms are ordered in the following way:  $T_{\mathbf{D}} < T_{\mathbf{L}} < T_{\mathbf{P}} < T_{\mathbf{M}}$ .

Ordinal sum t-norms are based on a construction for semigroups which goes back to A.H. Clifford [29] (see also [30, 100, 150]) based on ideas presented in [31, 111]. It has been successfully applied to t-norms in [74, 125, 179].

**Definition 2.3.** Consider an at most countable index set *I*. Let  $(]a_i, b_i[)_{i \in I}$  be a family of nonempty, pairwise disjoint open subintervals of [0, 1] and let  $(T_i)_{i \in I}$  be a family of t-norms. Then the function  $T: [0, 1]^2 \to [0, 1]$ , defined, for all  $x, y \in [0, 1]$ , by

$$T(x,y) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}), & \text{if } (x,y) \in [a_i, b_i]^2, \\ \min(x, y), & \text{otherwise.} \end{cases}$$
(2.7)

is called the *ordinal sum* and will be denoted by  $(\langle a_i, b_i, T_i \rangle)_{i \in I}$ .

Note that according to the fact that all  $[a_i, b_i] \subseteq [0, 1]$  and all  $a_i$  as well as  $b_i$  are ordered by the natural order on  $\mathbb{R}$  there exists a linearly ordered index set  $(J, \leq)$ ,  $J \neq \emptyset$  and a family of intervals  $[a_j, b_j]$  such that  $[0, 1] = \bigoplus_{j \in J} [a_j, b_j]$ , i.e.,  $[0, 1] = \bigcup_{j \in J} [a_j, b_j]$  and  $a_{j_1} \leq a_{j_2}$  whenever  $j_1 \leq j_2$  for all  $j_1, j_2 \in J$ . On each  $[a_j, b_j]$  associative operations  $*_j$  can be defined either as isomorphic, in fact affine, transformations of the corresponding t-norms or by the minimum such that the ordinal sum of t-norms is indeed an ordinal sum of semigroups in the original sense of Clifford [29]. Moreover, note that an ordinal sum of t-norms yields again a t-norm, which is the (largest) extension of t-norms acting on the subintervals  $[a_i, b_i]$  (see also Fig. 2.2).



Figure 2.2: Example of an ordinal sum t-norm  $T = (\langle a_1, b_1, T_1 \rangle, \langle a_2, b_2, T_2 \rangle, \langle a_3, b_3, T_3 \rangle).$ 

It is further remarkable that the concept of ordinal sums of semigroups did not only provide a method for constructing new triangular norms from given ones, but also led to a representation of continuous triangular norms as (trivial or non-trivial) ordinal sums of isomorphic images of the product and the Łukasiewicz t-norm [125, 137, 180] (see also Remark 4.6 later). Note that a full characterization of all, also non-continuous, t-norms is not yet known. For more results on triangular norms and ordinal sums see, e.g., [106, 112, 113, 114].

#### Triangular norms on bounded lattices

Many-valued logics are usually based on a bounded lattice  $(L, \leq, 0, 1)$  of truth values [86, 97, 126, 149, 189, 193], not necessarily being a chain (compare [13, 33, 58, 86]). In [83, 85] the unit interval was already replaced by a bounded lattice, stimulating some investigations in topology [84, 101, 104, 152] and logic [70, 102]. In all these cases, the conjunction is interpreted by some triangular norm on L. Since the structure of t-norms is known for some special cases only it was quite natural to study triangular norms from a more general viewpoint and on bounded lattices [37, 109, 200], including special cases such as discrete chains [129], product lattices [36, 107], or the lattice  $L^* = \{(x, y) \in [0, 1]^2 \mid x + y \leq 1\}$  [43, 45, 46].

Triangular norms on a bounded lattice  $(L, \leq, 0, 1)$  (see also [37]) are defined in analogy to triangular norms on the unit interval (compare Definition 2.1), fulfilling monotonicity, commutativity, associativity, and having neutral element 1. Therefore, a t-norm T on a bounded lattice  $(L, \leq, 0, 1)$  turns L into an ordered Abelian semigroup with neutral element 1.

Note that the structure of the lattice L heavily influences which and how many t-norms on L can be defined. However, for each, non-trivial, lattice L there exist at least two t-norms, i.e., the minimum  $T_{\mathbf{M}}^{L}(x,y) = x \wedge y$  and the drastic product

$$T_{\mathbf{D}}^{L}(x,y) = \begin{cases} x \wedge y, & \text{if } 1 \in \{x,y\}, \\ 0, & \text{otherwise,} \end{cases}$$

which are always the greatest and smallest possible t-norms on the lattice L. Observe that up to the trivial cases when  $|L| \leq 2$ , we always have  $T_{\mathbf{D}}^{L} \neq T_{\mathbf{M}}^{L}$ . In case that |L| = 2, there is a unique t-norm on L which is, in fact, the standard boolean conjunction. Finally, if |L| = 1, there is only one binary operation on L. Although in many of the cases mentioned before the lattices of truth values involved tend to be distributive, no additional assumptions on the lattice structure apart from its boundedness are imposed.

#### 2.1.1 Problem statements

Inspired by the investigations and results on ordinal sums of t-norms on the unit interval, the following problems have been formulated:

- Can strongest and weakest extensions of t-norms on bounded (complete) sublattices be found?
- Does an ordinal sum construction, similar to the one for ordinal sum t-norms on [0, 1], yield again a t-norm independently of the choice of the sublattices and the choice of t-norms on these sublattice?
- In case that not, for which lattices are such arbitrary choices possible?

#### 2.1.2 Main Results

The article [A01], entitled "On ordinal sums of triangular norms on bounded lattices", addresses all these questions for the case of a strongest extension T of t-norms  $T_i$  acting on subintervals  $([a_i, b_i])_{i \in I}$  of a bounded lattice  $(L, \leq, 0, 1)$  and discusses such extensions for the particular case of L being a product lattice and for the case of  $L = L^*$  as introduced above. In the article "On extensions of triangular norms on bounded lattices" [A02] these results are further extended to strongest and weakest extensions of a t-norm acting on a (complete) sublattice, not necessarily being a subinterval, of L to a t-norm acting on L. We summarize a few of the most relevant results, but refer for proofs, further details and results to the original contributions.

The approach for the strongest extension is inspired by the ideas of Clifford and by the concepts for ordinal sum t-norms on the unit interval. More precisely, consider a bounded lattice  $(L, \leq, 0, 1)$ , a bounded sublattice  $(S, \leq, a, b)$  of L, and a t-norm  $T^S: S^2 \to S$  on S. Then the extension  $T^L_{TS}: L^2 \to L$  of  $T^S$  to an operation on L is defined by

$$T_{T^S}^L(x,y) = \begin{cases} T^S(x,y), & \text{if } (x,y) \in S^2, \\ x \wedge y, & \text{otherwise.} \end{cases}$$
(2.8)

Obviously,  $T_{T^S}^L$  as defined by (2.8) is commutative and has neutral element 1. In case that it yields a t-norm it is the strongest possible extension of  $T^S$  on S to L. Since S is also a sublattice of  $([a,b], \leq, a, b)$  with  $[a,b] = \{x \in L \mid a \leq x \leq b\}$ , we have

$$T_{T^S}^L = T_{T_{T^S}^{[a,b]}}^L,$$

i.e., we may first extend  $T^S$  to [a, b] via (2.8) and repeat the same procedure to extend  $T_{T^S}^{[a,b]}$  to L. Because of

$$T_{T^S}^{[a,b]} = T_{T^S}^L|_{[a,b]^2}$$

a necessary condition for  $T_{T^S}^L$  to be a t-norm is that  $T_{T^S}^{[a,b]}$  is a t-norm. Therefore, without loss of generality we may restrict ourselves first to sublattices of L having the same bottom and top element as L.

**Proposition 2.4.** [A02, Proposition 2.1] Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $(S, \leq, 0, 1)$  a bounded sublattice of L. The following are equivalent:

- (i) For arbitrary t-norm  $T^S \colon S^2 \to S$  on S, the operation  $T^L_{TS}$  is a t-norm on L.
- (ii) For all  $(x, y) \in (S \setminus \{1\}) \times (L \setminus S)$  we have  $x \wedge y \in \{0, x\}$  and for all  $(x, y) \in (L \setminus S)^2$  it holds that  $x \wedge y \in S \Rightarrow x \wedge y = 0$ .

Note that condition (ii) equivalently expresses that for all  $x \in S \setminus \{1\}$  and for all  $y \in L \setminus S$  either  $x \wedge y = 0$  or  $x \leq y$  is fulfilled and for all  $x \in S \setminus \{0, 1\}$  and all  $y, z \in L \setminus S$ , such that  $x \leq y$  and  $x \leq z$ , also  $y \wedge z \in L \setminus S$ .

In [A01], the second extension step, i.e., the case of S being a subinterval [a, b] of the bounded lattice  $(L, \leq, 0, 1)$  has been treated extensively, revealing a series of necessary and sufficient conditions, in particular incomparability conditions, for the lattice in order to guarantee that  $T_{T^{[a,b]}}^{L}$  is indeed a t-norm extending arbitrary t-norm  $T^{[a,b]}$  acting on a fixed subinterval [a, b] to L: **Theorem 2.5.** [A01, Theorem 4.8] Consider some bounded lattice  $(L, \leq, 0, 1)$  and a subinterval [a, b] of L. Then the following are equivalent:

- (i) For arbitrary t-norm  $T^{[a,b]}$  on [a,b], the operation  $T^L_{T^{[a,b]}}$  defined by (2.8) is a t-norm on L.
- (ii) For all  $x \in L$  it holds that
  - (a) if x is incomparable to a, then it is incomparable to all  $u \in [a, b]$ ,
  - (b) if x is incomparable to b, then it is incomparable to all  $u \in [a, b]$ .

Note that condition (ii) can be further equivalently expressed by the fact that for all maximal chains  $C \subseteq L$  connecting 0 and 1, it holds that,

$$[a,b] \cap C \neq \emptyset \quad \Rightarrow \quad [a,b] \subseteq C,$$

or, compare also [A02, Proposition 3.1], that for all  $x \in L$ ,

$$\{x \in L \mid \exists y \in [a, b] : x \le y \text{ or } y \le x\} = [0, a] \cup [a, b] \cup [b, 1].$$

In [A01], the necessary and sufficient conditions are illustrated by several examples and the results are applied to product lattices and to  $L = L^*$  showing that the strongest extension as introduced above is, up to some trivial cases, not an appropriate way to create t-norms on product lattice resp. on  $L^*$ . Moreover, only special lattices allow for an arbitrary choice of the sublattice as well as the t-norm involved:

**Theorem 2.6.** [A01, Theorem 4.9] Consider a bounded lattice  $(L, \leq, 0, 1)$ . Then the following are equivalent:

- (i) For arbitrary interval [a, b] and arbitrary t-norm  $T^{[a,b]}$  on [a, b], the operation  $T^{L}_{T^{[a,b]}}$  defined by (2.8) is a t-norm on L.
- (ii) For all  $x, y \in L$  it holds that  $\{x \land y, x \lor y\} \subseteq \{0, 1, x, y\}$ .
- (iii) L is a horizontal sum of chains.

Based on the results obtained for S being a subinterval of L, the following statement can be made:

**Corollary 2.7.** [A02, Corollary 3.2] Let  $(L, \leq, 0, 1)$  be a bounded lattice,  $(S, \leq, a, b)$  a bounded sublattice of L and  $T^S: S^2 \to S$  a t-norm on S. Assume that for each  $(x, y) \in (S \setminus \{b\}) \times ([a, b] \setminus S)$  we have  $x \land y \in \{a, x\}$ , that for each  $(x, y) \in ([a, b] \setminus S)^2$  it follows that  $x \land y \in S$  implies  $x \land y = a$ , and that, in case  $]a, b[ \neq \emptyset$ , condition (ii) in Theorem 2.5 holds. Then  $T_{T^S}^L$  is a t-norm on L.

In both articles [A01, A02], also the cases of several sublattices resp. subintervals as well as further properties of the t-norms involved like, e.g., the intermediate value property and the relationship to residuated lattices are discussed and investigated.

However, by the previous results it becomes already obvious that the strongest extension of arbitrary t-norms on arbitrary sublattices leads to rather restrictive demands on the underlying lattices. Quite different is the situation when looking for the weakest possible extension of some  $T^S$  on some (complete) sublattice S.

**Definition 2.8.** [A02, Definition 6.1] Let  $(L, \leq, 0, 1)$  be a bounded lattice,  $(S, \leq, a, b)$  a complete and bounded sublattice, and  $T^S$  a t-norm on the corresponding sublattice S. Then define  $T^{S \cup \{0,1\}} : (S \cup \{0,1\})^2 \to (S \cup \{0,1\})$  by

$$T^{S \cup \{0,1\}}(x,y) = \begin{cases} x \land y, & \text{if } 1 \in \{x,y\}, \\ 0, & \text{if } 0 \in \{x,y\}, \\ T(x,y), & \text{otherwise.} \end{cases}$$
(2.9)

Further define  $W_{T^S}^L \colon L^2 \to L$  by

$$W_{T^{S}}^{L}(x,y) = \begin{cases} x \wedge y, & \text{if } 1 \in \{x,y\}, \\ T^{S \cup \{0,1\}}(x^{*},y^{*}), & \text{otherwise}, \end{cases}$$
(2.10)

with  $x^* = \sup\{z \mid z \le x, z \in S \cup \{0, 1\}\}$  for all  $x \in L$ .

**Proposition 2.9.** [A02, Propositions 6.3, 6.4] Let  $(L, \leq, 0, 1)$  be a bounded lattice and assume some complete, bounded sublattice  $(S, \leq, a, b)$ . Let  $T^S$  be a t-norm on the corresponding sublattice S. Then  $W_{T^S}^L: L^2 \to L$  defined by (2.10) is a t-norm on L and it is the smallest possible t-norm extension of  $T^S$  on L.

For t-norms on several sublattices the weakest extension is defined by in the following way:

**Definition 2.10.** [A02, Definition 6.5] Let  $(L, \leq, 0, 1)$  be a bounded lattice and I some index set. Further, let  $(S_i, \leq, a_i, b_i)_{i \in I}$  be a family of complete and bounded sublattices of L such that the family  $(]a_i, b_i]_{i \in I}$  consists of pairwise disjoint subintervals of L. Finally, let  $(T^{S_i})_{i \in I}$  be a family of t-norms on the corresponding sublattices  $S_i$ . Then define  $W_{T^{S_i}}^L: L^2 \to L$  by

$$W_{T^{S_i}}^L(x,y) = \begin{cases} x \wedge y, & \text{if } 1 \in \{x,y\}, \\ T^{S_i \cup \{0,1\}}(x_i^*, y_i^*), & \text{otherwise,} \end{cases}$$
(2.11)

with  $x_i^* = \sup\{z \mid z \le x, z \in S_i \cup \{0, 1\}\}$  and define  $W: L^2 \to L$  by

$$W(x,y) = \sup_{i \in I} W_{T^{S_i}}^L(x,y).$$
(2.12)

Note that, by definition, W is a symmetric and monotone operation on L which has neutral element 1. However, further restrictions on the family of sublattices have to be applied in order to guarantee that W is indeed an extension of arbitrary t-norms  $T^{S_i}$  on the sublattices  $S_i$ .

**Proposition 2.11.** [A02, Proposition 6.6] Let  $(L, \leq, 0, 1)$  be a bounded lattice and I some index set. Further, let  $(S_i, \leq, a_i, b_i)_{i \in I}$  be a family of complete and bounded sublattices of L such that the family  $(]a_i, b_i[)_{i \in I}$  consists of pairwise disjoint subintervals of L. Further assume that for all  $i, j \in I$  with  $i \neq j$  it holds that

- (i) if  $x \in S_j$  then  $x_i^* \notin S_i \setminus \{a_i, b_i\}$ , i.e.,  $x_i^* \in \{0, a_i, b_i\}$ ,
- (ii) if  $x \in S_j \setminus \{b_j\}$  and  $x_i^* = a_i$ , then  $(a_j)_i^* \ge a_i$ , and
- (iii) if  $x \in S_j \setminus \{b_j\}$  and  $x_i^* = b_i$ , then  $(a_j)_i^* = b_i$ .

Then for all t-norms  $T^{S_i}$  on  $S_i$  and for all t-norms  $T^{S_j}$  on  $S_j$  with  $i \neq j$  it holds that  $W^L_{T^{S_i}}(x,y) \leq T^{S_j}(x,y)$  for all  $(x,y) \in S^2_j$  and  $W^L_{T^{S_i}}(x,y) \leq T^{S_i}(x,y)$  for all  $(x,y) \in S^2_i$ , i.e.,

$$W_{T^{S_i}}^L|_{S_i}^2 \le T^{S_j}$$
 and  $W_{T^{S_j}}^L|_{S_i}^2 \le T^{S_i}$ .

Moreover, W given by (2.12) is a monotone and symmetric extension of each  $T^{S_i}$ , i.e.,  $W|_{S_i^2} = T^{S_i}$  for all  $i \in I$ , which has neutral element 1.

It is also shown in [A02] that, although associativity of W can not be proven in general, for several important cases, like Cartesian products and ordinal sums of bounded sublattices, or Lbeing a chain, also the associativity of W holds, i.e., that the construction indeed yields again a t-norm on L.

## 2.2 Triangle functions

From a historical point of view, triangle functions were introduced by Šerstnev in [183, 196] in his definitive formulation of the triangle inequality in probabilistic metric spaces (see, e.g., [176] for a historical introduction to these spaces). Triangle functions constitute an important class of binary operations on a subspace of distribution functions, namely distance distribution functions which form the basic objects in the discussion of probabilistic metric spaces (see [180] and [181] and the references therein for an extensive discussion of such spaces). We briefly recall the definition of distance distribution functions and of triangle functions as introduced by Šerstnev:

**Definition 2.12.** A function  $F \colon \mathbb{R} \to [0,1]$ , with  $\mathbb{R}$  denoting the extended real line, is called a *distance distribution function* if it is non-decreasing, left-continuous on  $\mathbb{R}$ , and fulfills  $F(\infty) = 1$ , and F(0) = 0. The set of all distance distribution functions will be denoted by  $\Delta^+$ .

The elements of  $\Delta^+$  are partially ordered by the usual pointwise order

 $F \leq G$  if and only if  $F(x) \leq G(x)$  for all  $x \in \overline{\mathbb{R}}$ .

Moreover,  $(\Delta^+, \leq, \varepsilon_{\infty}, \varepsilon_0)$  is a bounded lattice with bottom and top element given, for all  $x \in \overline{\mathbb{R}}$ , by

$$\varepsilon_{\infty}(x) = \begin{cases} 1, & \text{if } x = \infty, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \varepsilon_{0}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 2.13.** A triangle function is a binary commutative and associative operation on  $\Delta^+$  which is non-decreasing in each argument and has neutral element  $\varepsilon_0$ .

In fact, triangle functions are triangular norms on the special bounded lattice  $(\Delta^+, \leq, \varepsilon_{\infty}, \varepsilon_0)$ .

### 2.2.1 Problem statement, results and additional remarks

Similar as in the case of t-norms on the unit interval a full characterization of all triangle functions is not yet known. Already in 1983, in [180] several open problems have been formulated focusing on a clarification of the structure of triangle functions in general resp. for special subclasses. We briefly quote a few of them:

- **Problem 7.9.1:** [...] In particular determine all continuous triangle functions and, if possible, find a representation corresponding to the one given in Theorems  $5.3.8^1$  and  $5.4.1.^2$ .
- **Problem 7.9.5:** [...] Suppose that T is a continuous t-norm. To what extent does the structure of T determine the structure of  $\tau_{T,L}^3$ ? In particular, if T is an ordinal sum, is  $\tau_{T,L}$  an ordinal sum?
- **Problem 7.9.6:** Find conditions on T on L that are both necessary and sufficient (rather than merely sufficient) for  $\tau_{T,L}$  to be a triangle function. [...]

Partial answers to these problems can be found in the literature (see also Chapter 7 in the notes of [181] and the references therein). However, several of the proofs of these results are not always easily accessible. Moreover, additional and new results clarifying further properties of triangle functions could be achieved in collaboration with Carlo Sempi.

The article [A03], entitled "A primer on triangle functions I", contains all these results. Since, as its title already indicates, it has also been the intention to offer not only new results, but also a handy reference for an (updated and extended) introduction to triangle functions we refrain from quoting single results from this forty pages contribution but refer directly to the

<sup>&</sup>lt;sup>1</sup>Representation of continuous t-norms on some real-valued interval [a, e] as minimum, continuous Archimedean t-norm, or ordinal sums thereof.

<sup>&</sup>lt;sup>2</sup>Representation of continuous Archimedean t-norms on some real-valued interval [a, e] by means of generators. <sup>3</sup>Denoting a special class of triangle functions based on a t-norm T and an operation L.

original article. Note that the first part of the primer mainly focusses on constructions of triangle functions, important subclasses and their properties. Functional equations and inequalities are extensively treated in the second part entitled "A primer on triangle functions II", submitted to *Aequationes Mathematicae* in Spring 2008.

However, a few additional remarks w.r.t. the concept of strongest and weakest extensions of t-norms on bounded lattices, as discussed in the previous section, can and should be made:

The set  $E^+$  of step functions, i.e., of distance distribution functions  $\varepsilon_a \colon \mathbb{R} \to [0,1], a \in [0,\infty]$ , for  $a < \infty$ , being defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{if } x \le a, \\ 1, & \text{otherwise,} \end{cases}$$

forms an important (complete) sublattice  $(E^+, \leq, \varepsilon_{\infty}, \varepsilon_0)$  of the lattice  $(\Delta^+, \leq, \varepsilon_{\infty}, \varepsilon_0)$ , i.e., it allows to embed the real line into probabilistic metric spaces.

Since  $E^+$  is a complete sublattice and, due to Proposition 2.9, the weakest extension  $W_{\tau_E}^{\Delta^+}$ of some triangle function  $\tau_E$  acting on  $E^+$  as defined by (2.10) yields indeed a triangle function on  $\Delta^+$ . However, for the strongest extension as defined by (2.8), the incomparability conditions of Proposition 2.4 are not fulfilled, i.e., the strongest extension yields not a triangle function  $\Delta^+$  for arbitrary triangle functions on  $E^+$ . Similar arguments hold for the strongest and weakest extension of triangle functions acting on some subinterval  $[\varepsilon_a, \varepsilon_b] \subset \Delta^+$ ,  $a, b \in ]0, \infty[$ , to a triangle function on  $\Delta^+$ .

# Chapter 3

# Bivariate copulas and quasi-copulas

## Aggregation functions reflecting dependence structures

Copulas were first introduced by Sklar in 1959 in [187]. Bivariate copulas are functions that join bivariate distribution functions with their univariate marginal distribution functions. Moreover, the copula of a random pair (X, Y) completely captures the dependence structure of (X, Y) due to *Sklar's theorem* [187] which states that for each random vector (X, Y) there is a copula  $C_{X,Y}$ (uniquely defined whenever X and Y are continuous), such that the joint distribution function  $F_{X,Y}$  of (X, Y) may be represented by

$$F_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y))$$

with  $F_X$  and  $F_Y$  the marginal distribution functions of X and Y, respectively.

In addition, every copula is the restriction of a bivariate distribution function to the unit square whose marginals are uniform on [0, 1] (see [108, 140] for a thorough introduction to copulas). We introduce (bivariate) copulas as special classes of binary aggregation functions:

**Definition 3.1.** A binary aggregation function  $C: [0,1]^2 \rightarrow [0,1]$  is called a *copula* if it is 2-increasing and has neutral element 1.

Note that every copula C has also annihilator 0, i.e., C(x, 0) = C(0, x) = 0 for every  $x \in [0, 1]$ . Moreover, they are 1-Lipschitz and for every copula C it holds that  $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$ . Therefore, in the fields of copulas,  $T_{\mathbf{L}}$  and  $T_{\mathbf{M}}$  are also referred to as the *Fréchet-Hoeffding bounds* of copulas, most often denoted by W resp. M. Note that also  $T_{\mathbf{P}}$  is a copula, in the framework of copulas usually denoted by  $\Pi$  and referred to as the *independence copula*. We follow the notation W,  $\Pi$ , M throughout this section. Note also that an associative copula is also a (continuous) t-norm on [0, 1] and that 1-Lipschitz t-norms are copulas (see, e.g., [113]).

Quasi-copulas were introduced by Alsina et al. in [9] and characterized by Genest et al. in [82]; they characterize binary operations on distribution functions induced by operations on random variables defined on the same probability space.

**Definition 3.2.** A binary aggregation function  $Q: [0,1]^2 \to [0,1]$  is called a *quasi-copula*, if it is 1-Lipschitz, has neutral element 1 and annihilator 0.

Every copula is a quasi-copula, however not every quasi-copula is also 2-increasing, i.e., a copula. Such quasi-copulas are usually referred to as *proper* quasi-copulas. Note that also for every quasi-copula Q it holds that  $W \leq Q \leq M$ .

Copulas and quasi-copulas are of considerable interest in several fields of applications, for instance, in finance [26, 130], hydrology [155], but also preference modelling [48, 49, 50] and, due to the close relationship between copulas and t-norms, also in many-valued logics [98]. Therefore, having at one's disposal a large number of examples of (quasi-)copulas is of great practical and theoretical interest.

During the last few years several researchers have focussed their attention on new methods for constructing families of bivariate copulas and quasi-copulas with desirable properties and a stochastic interpretation. Some of these approaches have been devoted to copulas and quasi-copulas with given values along specified sections or subsets of the unit square, like, e.g., diagonals [60, 61, 69, 75, 141], horizontal, vertical, or affine sections connecting opposite sides of the unit square [62, 119, 120], or grid structures [34, 185]. Additionally, best-possible bounds for the functions thus constructed [76, 119, 142, 143] have been investigated.

## 3.1 Problem statements and results

#### Quasi-copulas with a given sub-diagonal section

In the spirit mentioned above, in [A04] "Quasi-copulas with a given sub-diagonal section" we studied (quasi-)copulas with a given sub-diagonal section, i.e., with given values along an affine section with slope one connecting perpendicular sides of the unit square. More precisely the following problems have been tackled:

- Given a sub-diagonal  $\delta_{x_0}$ , i.e., an function being admissible for serving as a sub-diagonal section of a copula (for more details see [**A04**, Section 2]), does there exist a copula or a quasi-copula  $Q_{\delta_{x_0}}$  whose sub-diagonal section  $\delta_{x_0}^Q$  coincides with  $\delta_{x_0}$ , i.e., for which  $\delta_{x_0}^Q = \delta_{x_0}$ ?
- Given a sub-diagonal  $\delta_{x_0}$ , if  $\mathcal{Q}_{\delta_{x_0}}$  denotes the set of all quasi-copulas whose sub-diagonal sections coincide with  $\delta_{x_0}$ , what are the best-possible bounds for  $\mathcal{Q}_{\delta_{x_0}}$ ?

The first question has been answered by a series of constructions for (quasi-)copulas with a given sub-diagonal section, relevant aspects being:

- W-ordinal sums (see Section 3 in [A04]) allowing to determine (quasi-)copulas with a given sub-diagonal section from (quasi-)copulas with a given diagonal section,
- patchwork resp. splicing techniques for obtaining new quasi-copulas from two given (quasi-)copulas, all coinciding in the corresponding subdiagonal section (see Section 4 in [A04]), and
- symmetrization techniques for obtaining symmetric quasi-copulas with a given sub-diagonal section (see Section 5 in [A04]).

Some of these constructions allow to obtain also new copulas from given ones [A04, Section 3], in other cases sufficient (and necessary) conditions for yielding a copula could be provided [A04, Theorem 5, Corollary 7, Corollary 8]. In Section 7 in [A04] the construction of a symmetric copula with a given sub-diagonal section is proven for particular cases of sub-diagonals. Although all these constructions allow to find (quasi-)copulas Q with a given subdiagonal section  $\delta_{x_0}^Q$ , only the lower bound of the set of all quasi-copulas  $Q_{\delta_{x_0}}$  coinciding in their sub-diagonal section can be obtained by these methods. Note that the lower bound is not only a quasi-copula but also a copula. In [A04, Section 8], also the existence and structure of the upper bound, being a quasi-copula, is proven. For the readers' convenience we summarize the result on upper and lower bounds of  $Q_{\delta_{x_0}}$ :



Figure 3.1: Sub-domains of  $[0, 1]^2$ .

**Theorem 3.3.** [A04, Theorems 13,14] Consider  $x_0 \in [0, 1[$  and a sub-diagonal  $\delta_{x_0}$ . We distinguish the following sub-domains of the unit square (see also Fig. 3.1):

$$T_U(x_0) = \{(u, v) \in [0, 1]^2 \mid u - x_0 \le v\},\$$
  
$$T_L(x_0) = \{(u, v) \in [0, 1]^2 \mid u - x_0 \ge v\},\$$
  
$$S_2(x_0) = [x_0, 1] \times [0, 1 - x_0].$$

Then the copula  $B_{\delta_{x_0}}: [0,1]^2 \to [0,1]$  and the quasi-copula  $G_{\delta_{x_0}}: [0,1]^2 \to [0,1]$  defined by

$$\begin{split} B_{\delta_{x_0}}(u,v) &= \begin{cases} m_{x_0}(u,v) - h_{x_0}(u,v), & \text{if } (u,v) \in S_2(x_0), \\ \max(u+v-1,0), & \text{otherwise}, \end{cases} \\ G_{\delta_{x_0}}(u,v) &= \begin{cases} \min(u,v,M'_{x_0}(u,v) - q_{x_0}(u,v)), & \text{if } (u,v) \in T_U(x_0), \\ \min(m'_{x_0}(u,v),M'_{x_0}(u,v) - q_{x_0}(u,v)), & \text{otherwise}, \end{cases} \end{split}$$

where

$$m_{x_0}(u, v) = \max(\min(u - x_0, v), 0), \qquad M_{x_0}(u, v) = \min(\max(u - x_0, v), 1 - x_0), m'_{x_0}(u, v) = \min(u - x_0, v), \qquad M'_{x_0}(u, v) = \max(u - x_0, v),$$

and

$$h_{x_0}(u,v) = \min(t - \delta_{x_0}(t) \mid t \in [m_{x_0}(u,v), M_{x_0}(u,v)]),$$
  
$$q_{x_0}(u,v) = \max(t - \delta_{x_0}(t) \mid t \in [m_{x_0}(u,v), M_{x_0}(u,v)]).$$

are the smallest resp. greatest (quasi-)copula whose sub-diagonal section at  $x_0$  coincides with  $\delta_{x_0}$ , i.e.,  $B_{\delta_{x_0}} \leq Q \leq G_{\delta_{x_0}}$  for all  $Q \in \mathcal{Q}_{\delta_{x_0}}$ .

#### 2-increasing (aggregation) functions

Several of the already mentioned constructions with given values along sections or subsets relate or at least touch the problem of determining binary 2-increasing functions on a rectangular subset of the unit square with prescribed marginal behavior. In case of grid construction methods, as introduced by De Baets and De Meyer [34], the rectangular substructure of the domain and the fixed values are obvious. In case of copulas with a given horizontal and/or vertical section [62, 120] the underlying domain is split into two resp. four rectangles, the values of the copulas along all its margins of the subset being determined either by the boundary conditions of a copula or the corresponding sections. Similarly the gluing method of Siburg and Stoimenov for binary copulas leads to a distinction of two subrectangles of the unit square [185]. But also in case of affine sections connecting perpendicular sides of the unit square, like it is the case for (quasi-)copulas with a given sub-diagonal section (see also Fig. 3.1), symmetrization techniques lead to subrectangles resp. subsquares of the unit square where the behavior of the copula is determined by its marginal behavior w.r.t. that subdomain.

Although binary 2-increasing functions with given margins acting on some rectangular subdomain of the unit square appear in all these fields, a full investigation and characterization of such functions had not been undertaken before. First results had been achieved in [63] where 2-increasing aggregation functions, acting on  $[0, 1]^2$ , had been investigated leading to some constructions and special properties.

However, a full characterization of all binary 2-increasing functions with given upper as well as upper and lower margins has been achieved later independently by Saminger-Platz, at that time on a sabbatical leave at Lecce university, and by Durante together with Sarkoci in Linz. The results are published in the co-authored article [A05], entitled "On representations of 2-increasing binary aggregation functions", and we briefly recall the most important theorems contained therein providing characterizations of 2-increasing aggregation functions with given upper resp. upper and lower margins (for further constructions, properties and bounds as well as proofs and examples see additionally [A05]).

Note that the 2-increasingness property has been introduced for binary (aggregation) functions only, such that in this section we restrict to binary aggregation functions only, denoting them simply by A. Its corresponding margins are, for all  $a, b \in [0, 1]$ , given by the functions  $h_A^a$ ,  $v_A^a: [0, 1] \to [0, 1]$ , defined, for all  $x, y \in [0, 1]$ , by

$$h^a_A(x) = A(x,a)$$
 and  $v^b_A(y) = A(b,y).$ 

**Theorem 3.4.** [A05, Theorem 10] Consider a binary, 2-increasing aggregation function A with upper margins  $h_A^1$  and  $v_A^1$ , then there exists a copula C such that  $A(x, y) = C(h_A^1(x), v_A^1(y))$  for all  $x, y \in [0, 1]$ .

**Theorem 3.5.** [A05, Theorem 17] Consider a binary 2-increasing aggregation function A with margins  $h_A^0$ ,  $h_A^1$ ,  $v_A^0$ ,  $v_A^1$  such that  $\lambda_A = V_A([0,1]^2) > 0$ . Then there exists a copula C such that

$$A(x,y) = \lambda_A C(\varphi_1(x), \varphi_2(y)) + h^0_A(x) + v^0_A(y)$$
(3.1)

with

$$\begin{split} \varphi_1 \colon [0,1] \to [0,1], \quad \varphi_1(x) &= \frac{1}{\lambda_A} \left( h_A^1(x) - h_A^0(x) - h_A^1(0) \right), \\ \varphi_2 \colon [0,1] \to [0,1], \quad \varphi_2(y) &= \frac{1}{\lambda_A} \left( v_A^1(y) - v_A^0(y) - v_A^1(0) \right). \end{split}$$

Note that for a binary 2-increasing aggregation function A,  $\lambda_A = V_A([0,1]^2) = 0$  is equivalent to the fact that A is a modular function, i.e.,  $A(x,y) = h_0^A(x) + v_0^A(y) = h_1^A(x) + v_1^A(y) - 1$  for all  $x, y \in [0,1]$  (compare also [63, Propositions 2.3, 3.6]).

Although the previous theorems might look very basic at first sight, they are of considerable value. In particular Theorem 3.5 allows to obtain a full characterization of continuous, binary, 2-increasing and non-decreasing functions with given margins and acting on some rectangular subdomain of the unit square as shown in the article "**Rectangular patchwork for bivariate copulas and tail dependence**" [A06].

**Theorem 3.6.** [A06, Theorem 2.1] Consider a binary 2-increasing function  $F: [a_1, a_2] \times [b_1, b_2] \rightarrow [c_1, c_2]$ , continuous and non-decreasing in each argument, with margins  $h_F^{b_1}$ ,  $h_F^{b_2}$ ,  $v_F^{a_1}$ , and  $v_F^{a_2}$  and  $\operatorname{Ran}_F = [c_1, c_2]$ . Put  $\lambda_F = V_F([a_1, a_2] \times [b_1, b_2])$ . If  $\lambda_F = 0$ , then

$$F(x,y) = h_F^{b_1}(x) + v_F^{a_1}(y) - h_F^{b_1}(a_1).$$

If  $\lambda_F > 0$ , then there exists a unique copula C such that

$$F(x,y) = \lambda_F C\left(\frac{\varphi_1^F(x)}{\lambda_F}, \frac{\varphi_2^F(y)}{\lambda_F}\right) + h_F^{b_1}(x) + v_F^{a_1}(y) - h_F^{b_1}(a_1),$$
(3.2)

with

$$\begin{aligned} \varphi_1^F(x) &= V_F([a_1, x] \times [b_1, b_2]) = h_F^{b_2}(x) - h_F^{b_2}(a_1) - h_F^{b_1}(x) + h_F^{b_1}(a_1), \\ \varphi_2^F(y) &= V_F([a_1, a_2] \times [b_1, y]) = v_F^{a_2}(y) - v_F^{a_2}(b_1) - v_F^{a_1}(y) + v_F^{a_1}(b_1). \end{aligned}$$

Based on this characterization it is possible to obtain a full characterization of rectangular patchworks of copulas and to allow a different viewpoint on many of the constructions of copulas mentioned above with given grid structure resp. given horizontal and/or vertical sections. By a rectangular patchwork we denote the following construction:

**Definition 3.7.** Consider a copula C, a family  $(R_i)_{i \in I}$  of subrectangles of  $[0, 1]^2$  such that  $R_i \cap R_j \subseteq \partial R_i \cap \partial R_j$  whenever  $i \neq j$ , i.e.,  $R_i$  and  $R_j$  have common points just on their boundaries. Moreover, for every  $i \in I$ , let us consider a continuous mapping  $F_i \colon R_i \to [0, 1]$ , which is non-decreasing in each argument, such that  $C = F_i$  on  $\partial R_i$ . We call the function  $F \colon [0, 1]^2 \to [0, 1]$  defined, for all  $x, y \in [0, 1]$ , by

$$F(x,y) = \begin{cases} F_i(x,y), & \text{if } (x,y) \in R_i, \\ C(x,y), & \text{otherwise,} \end{cases}$$
(3.3)

the patchwork of  $(F_i)_{i \in I}$  into the copula C.

Moreover, the function F is a copula if and only if, for all  $i \in I$ ,  $F_i$  is 2-increasing on  $R_i$  [34]. Note that one of the oldest rectangular patchwork construction for copulas are *ordinal sums* of copulas (see [140]), a construction already encountered in Section 2.1 on triangular norms.

Ordinal sums of copulas are defined in an analogous way to ordinal sums of t-norms. In terms of rectangular patchwork they are obtained by considering C to be equal to M, every  $R_i$  to be a square of the type  $[a_i, b_i]^2$ , where  $0 \le a_i < b_i \le a_{i+1} < b_{i+1} \le 1$  and the functions  $F_i$  being all affine isomorphic transformations of copulas.

The rectangular patchwork in combination with the result obtained in Theorem 3.6 allows to construct new copulas:

**Theorem 3.8.** [A06, Theorem 2.2] Consider a family of copulas  $(C_i)_{i \in I}$  and a family of rectangles  $(R_i = [a_1^i, a_2^i] \times [b_1^i, b_2^i])_{i \in I}$  of  $[0, 1]^2$  such that  $R_i \cap R_j \subseteq \partial R_i \cap \partial R_j$ , for every  $i \neq j$ .

Consider further a copula C and put  $\lambda_i = V_C(R_i)$ . Then the function  $\widetilde{C} : [0,1]^2 \to [0,1]$  defined, for every  $x, y \in [0,1]$ , by

$$\widetilde{C}(x,y) = \begin{cases} \lambda_i C_i \left( \frac{V_C(\left[a_1^i, x\right] \times \left[b_1^i, b_2^i\right]\right)}{\lambda_i}, \frac{V_C(\left[a_1^i, a_2^i\right] \times \left[b_1^i, y\right])}{\lambda_i} \right) \\ +h_C^{b_1^i}(x) + v_C^{a_1^i}(y) - h_C^{b_1^i}(a_1^i), \quad if (x, y) \in R_i \text{ with } \lambda_i \neq 0, \\ C(x, y), & otherwise, \end{cases}$$

is a copula. We denote such a copula  $\widetilde{C}$  by  $(\langle R_i, C_i \rangle)_{i \in I}^C$ .

Note that by the construction, the rectangles  $R_i$  are not bound to a particular shape or position within the unit square, like, e.g., squares along the main diagonal as it is the case for ordinal sum copulas. Moreover, copulas  $C_i$  allow to distribute the mass  $\lambda_i$ , assigned to the rectangle  $R_i$  by C, differently on  $R_i$ . Choosing  $C_i = W$  resp.  $C_i = M$  for all  $i \in I$  leads therefore to the smallest resp. largest copula coinciding on the margins of all  $R_i$  and being equal to C on  $[0,1]^2 \setminus \bigcup_{i \in I} R_i$ . Clearly, every copula C can be represented as a rectangular patchwork  $(\langle [0,1]^2, C \rangle)^C$ , however, in general this representation is not unique. Moreover, for M,  $\Pi$ , and W it holds that  $(\langle R_i, C \rangle)_{i \in I}^C = C$  for all families of rectangles  $(R_i)_{i \in I}$  and  $C \in \{M, \Pi, W\}$ .

In [A06], but also [64] it is illustrated how several constructions like, e.g., W-ordinal sums, copulas with given horizontal and/or vertical sections, and binary glued copulas can be interpreted as rectangular patchwork copulas. Moreover, several additional aspects like, e.g., copulas with different tail dependencies or absolutely continuous copulas with given diagonal section, are discussed and illustrated by examples in [A06].

# Part II

**Aggregation Functions:** 

Dominance — A Functional Inequality

# Chapter 4

# Preliminaries

## 4.1 Motivation

In 1942, Karl Menger introduced the concept of *probabilistic* (originally denoted as *statistical*) *metric spaces* [131], in which the distance between two objects p and q is characterized by a probability distribution function, more precisely by a distance distribution function,  $F_{pq}$  rather than by a real number. For any positive number x, the value  $F_{pq}(x)$  is interpreted as the probability that the distance between p and q is less than x. The metric in such spaces has been defined in analogy to the axioms of (pseudo-)metric spaces, and the most disputable axiom has been the probabilistic analogue of the triangle inequality (see also, e.g., [177, 180, 181] for more details on the historic developments). Its actual variant goes back to Šerstnev [182, 183] and reads as follows, for all objects p, q, r,

$$F_{pr} \ge \tau(F_{pq}, F_{qr})$$

with  $\tau$  being a triangle function (see Definition 2.13). During the investigation of topological aspects of probabilistic metric spaces, products and quotients of such spaces have been touched (e.g., [6, 68, 190, 191]) such that the property of dominance came to the fore. In [190], Tardiff introduced the notion of a  $\sigma$ -product of two probabilistic metric spaces with  $\sigma$  some triangle function. He further showed that the  $\sigma$ -product of two probabilistic metric spaces under the triangle function  $\tau$  is again a probabilistic metric space under  $\tau$  if and only if  $\sigma$  dominates  $\tau$  ( $\sigma \gg \tau$ ), i.e., if, for all distance distribution functions  $F_1, F_2, G_1, G_2 \in \Delta^+$ , the following inequality is fulfilled

$$\sigma(\tau(F_1, G_1), \tau(F_2, G_2)) \ge \tau(\sigma(F_1, F_2), \sigma(G_1, G_2)).$$

Therefore, dominance constitutes a binary relation on the class of all triangle functions [190], and this notion was soon generalized to operations on an arbitrary partially ordered set [180] and therefore also for t-norms.

Dominance is further instrumental for the preservation of a variety of properties most often expressed by some inequality, during (dis-)aggregation processes and when construction fuzzy (equivalence and order) relations on product spaces (for particular examples and details see also [17, 18, 19, 35, 50, 157, 163]). The dominance property was therefore introduced also in the framework of aggregation functions where it enjoyed further development.

In addition, dominance has an additional interpretation in aggregation processes, in particular in two-step evaluation procedures of given data matrices as will be illustrated immediately. It is worth mentioning that, besides these application points of view, the property of dominance turned out to be an interesting mathematical notion per se, since it constitutes a binary relation on a set of operations defined by a functional inequality of the operations involved. In the next sections we will illustrate various aspects of the discussion on dominance — in the framework of aggregation functions as well as in particular for t-norms.

## 4.2 Definitions, properties, and problems

The definition of dominance between binary operations on a partially ordered set as introduced by Schweizer and Sklar [180] reads as follows:

**Definition 4.1.** Consider a partially ordered set  $(P, \leq)$  and two associative binary operations f, g on P with common neutral element e. Then f dominates g  $(f \gg g)$ , if, for all  $x, y, u, v \in P$ ,

$$f(g(x,y),g(u,v)) \ge g(f(x,u),f(y,v)).$$

Note that f and g need not be monotone and as such need not be (binary) aggregation functions. However, Schweizer and Sklar indicated already in [180] that due the common neutral element, dominance implies an ordering between the operations involved (choose y = u = e). That the converse is in general not true as has been shown by Tardiff in [192].

#### Dominance between aggregation functions

Since associativity, contrary to monotonicity, is not a defining property of aggregation functions, let us look at the definition of dominance in the framework of aggregation functions (compare also [163]:

**Definition 4.2.** Consider two aggregation functions  $\mathbf{A}, \mathbf{B}$  on a bounded lattice  $(L, \leq, 0, 1)$ . Then  $\mathbf{A}$  dominates  $\mathbf{B}$   $(\mathbf{A} \gg \mathbf{B})$ , if, for all  $n, m \in \mathbb{N}$  and for all  $x_{ij} \in L$  with  $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$ , it holds that

$$\mathbf{A}(\mathbf{B}(x_{11},\ldots,x_{m1}),\ldots,\mathbf{B}(x_{1n},\ldots,x_{mn})) \ge \mathbf{B}(\mathbf{A}(x_{11},\ldots,x_{1n}),\ldots,\mathbf{A}(x_{m1},\ldots,x_{mn})).$$
(4.1)

In Fig. 4.1, dominance between two aggregation functions  $\mathbf{A}$  and  $\mathbf{B}$  is illustrated. It shows that dominance can be interpreted in a nice way in two-step aggregation procedures: Arguments  $(x_{ij})$ , given as a data matrix, shall be evaluated in two steps. The evaluation along the rows shall be carried out by  $\mathbf{A}$  whereas the evaluation along the columns is done by  $\mathbf{B}$ . In the first aggregation step partial results, either  $a_i$ 's or  $b_i$ 's, are computed which are then mapped to final values  $b = \mathbf{A}(b_1, \ldots, b_n)$  resp.  $a = \mathbf{B}(a_1, \ldots, a_m)$ . If  $\mathbf{A}$  dominates  $\mathbf{B}$ , then  $b \ge a$ , i.e., independently of the actual arguments  $x_{ij}$ . An evaluation first by columns and then by rows will always lead to a greater result than aggregating first rows and then columns. The aggregation procedure depends on the "agenda" in which the aggregation steps are carried out, however, dominance among the aggregation functions involved at least guarantees that the results will always be ordered in a given way. Clearly, it is also of interest to determine for which aggregation functions  $\mathbf{A}$  and  $\mathbf{B}$  the final results are the same for arbitrary arguments. Such operations are called to *commute* with each other and are discussed in Chapter 8 with an emphasis on (bipolar) decision making in a multi-criteria multi-person decision problem.

Figure 4.1: Dominance in two-step aggregation procedures.
#### **Related** problems

As mentioned already earlier, dominance constitutes a binary relation on the set of all aggregation functions on a bounded lattice. As such the following questions arise naturally:

- Which additional properties does dominance have as a binary relation on the set of all aggregation functions?
- Which properties does it have for special subclasses of aggregation functions? In particular when does it constitute a reflexive, antisymmetric, transitive, i.e., an order relation?
- For a given aggregation function **A**, how does the set of dominated resp. dominating aggregation functions look like?

Several properties of dominance, in particular constructions of dominating functions, in case of aggregation functions on [0, 1] have been investigated in [163] and in case of aggregation functions on  $\Delta^+$  in [171]. A characterization of the set of dominating aggregation functions for the four basic t-norms is provided in [163].

Since later on we will focus on dominance between commutative and associative aggregation functions with a common neutral element, we briefly summarize relevant results (compare also [163, 180]):

**Proposition 4.3.** Consider two aggregation functions A, B on a bounded lattice  $(L, \leq, 0, 1)$ .

• If **A** resp. **B** are associative, then **A** dominates **B** if and only if  $\mathbf{A}_{(2)}$  dominates **B** resp. **A** dominates  $\mathbf{B}_{(2)}$ . If both **A** and **B** are associative, then **A** dominates **B** if and only if  $\mathbf{A}_{(2)}$  dominates  $\mathbf{B}_{(2)}$ , i.e., if, for all  $x, y, u, v \in L$ ,

$$\mathbf{A}(\mathbf{B}(x,y),\mathbf{B}(u,v)) \ge \mathbf{B}(\mathbf{A}(x,u),\mathbf{A}(y,v)).$$

- Assume that  $\mathbf{A}$  resp.  $\mathbf{B}$  possess neutral elements  $e_{\mathbf{A}}$  resp.  $e_{\mathbf{B}}$ . Then  $\mathbf{A} \gg \mathbf{B}$  implies  $e_{\mathbf{A}} \ge e_{\mathbf{B}}$ . If  $e_{\mathbf{A}} = e_{\mathbf{B}}$ , then dominance implies ordering, i.e.,  $\mathbf{A} \gg \mathbf{B}$  implies  $\mathbf{A} \ge \mathbf{B}$ .
- A dominates itself if and only if it is bisymmetric.
- Consider a bounded lattice  $(K, \leq_K, 0_K, 1_K)$ . Then the following are equivalent:
  - (i) A dominates B.
  - (ii)  $\mathbf{A}_{\varphi}$  dominates  $\mathbf{B}_{\varphi}$  for all order preserving isomorphisms  $\varphi \colon L \to K$ .
  - (iii)  $\mathbf{A}_{\varphi}$  is dominated by  $\mathbf{B}_{\varphi}$  for all order reversing isomorphisms  $\varphi \colon L \to K$ .

### 4.3 On the transitivity of dominance

Summarizing the basic properties, dominance constitutes a reflexive relation on any set of associative and symmetric aggregation functions. Moreover, it is also antisymmetric in case there is common neutral element. The question whether dominance is also transitive, and therefore an order relation, has been of interest for many years:

• Already in 1983, in [180, Problem 12.11.3] Schweizer and Sklar pose the following open problem for binary associative operations on a partially ordered set  $(P, \leq)$  with common neutral element  $e \in P$ :

Is the relation "dominates" always transitive? If not, under which conditions is it transitive?

 In 2003, in [7, Problem 17] Alsina, Frank and Schweizer formulate the question in a more explicit way for t-norms: Is the dominance relation transitive, hence a partial order, on the set of all t-norms? If not, for what subsets is this the case?

A counterexample for the first problem, has been shown by Sherwood and has been published in 2006 in [8]:

**Example 4.4.** Consider a linearly ordered set  $P = \{0, 1, 2\}, 0 < 1 < 2$ , and the binary operations F, G, H on P defined by the following tables

$\mathbf{F}$	0	1	2	G	0	1	2	Η	0	1	2
0	0	1	2	0	0	1	2	 0	0	1	2
1	1	0	2	1	1	1	2	1	1	2	2
2	2	2	2	2	2	2	2	2	2	2	2

Then F, G, H are commutative, associative operations with common neutral element 0 and fulfill

$$H \gg G$$
 and  $G \gg F$  but  $H \not\gg F$ .

Note that F is not increasing in each argument, and therefore not an aggregation function. A counterexample for dominance between aggregation functions (on the unit interval) has been found by Saminger and published in 2005 in [157]:

**Example 4.5.** Consider aggregation functions on the unit interval, in particular, the weakest aggregation function  $\mathbf{A}_w$ , given by,

$$\mathbf{A}_w(x_1,\ldots,x_n) = \begin{cases} 1, & \text{if } x_1 = \ldots = x_n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

the minimum min and the arithmetic mean  $\mathbf{M}$ . Then

 $\mathbf{A}_w \gg \min$  and  $\min \gg \mathbf{M}$  but  $\mathbf{A}_w \not\gg \mathbf{M}$ .

#### Dominance between triangular norms

Recall that, because of the properties of t-norms, dominance is already a reflexive and antisymmetric relation on the set of t-norms. Moreover, it holds that for all t-norms T

$$T_{\mathbf{M}} \gg T$$
 and  $T \gg T_{\mathbf{D}}$ .

The question on the transitivity of dominance in particular for t-norms remained unanswered for quite some time. Several results on dominance in special families of t-norms had been achieved, by applying different proof techniques, and had been published until 2005 (see, e.g., [A07] and [113, 173, 184]). Fig. 4.2 provides a condensed and brief overview of these results. As is clear from the corresponding Hasse-diagrams these partial results supported the conjecture that dominance would indeed be transitive, either due to its rare occurrence within the family considered or due to its abundant occurrence, in accordance with the parameter of the family. Finally, in 2006, the conjecture was disproved by Sarkoci [175]: dominance is not transitive on the class of (continuous) t-norms. The counterexample was found among ordinal sum t-norms and was based on properties of dominance proven in the more general framework of aggregation functions with neutral element 1 ([A07]).

Although the long open problem is now answered to the negative, the question for which subsets of t-norms dominance still constitutes an order relation remains open. Of particular interest are continuous t-norms, for which a complete characterization in terms of ordinal sums and continuous Archimedean t-norms is available (see also Section 2.1). We briefly summarize the most important and relevant facts and notions on continuous t-norms:

Family of t-norms	$T_{\lambda} \gg T_{\mu}$ if and only if	Hasse- diagram	Reference
Schweizer-Sklar $(T_{\lambda}^{\mathbf{SS}})_{\lambda \in [-\infty,\infty]}$	$\lambda \leq \mu$	ļ	Sherwood, 1984 [184]
Aczél-Alsina $(T_{\lambda}^{\mathbf{AA}})_{\lambda \in [0,\infty]}$ Dombi $(T_{\lambda}^{\mathbf{D}})_{\lambda \in [0,\infty]}$ Yager $(T_{\lambda}^{\mathbf{Y}})_{\lambda \in [0,\infty]}$	$\lambda \geq \mu$	•	Klement et al., 2000 [113]
Frank $(T_{\lambda}^{\mathbf{F}})_{\lambda \in [0,\infty]}$ Hamacher $(T_{\lambda}^{\mathbf{H}})_{\lambda \in [0,\infty]}$	$\lambda = 0,  \lambda = \mu,  \mu = \infty$		Sarkoci, 2005 [173]
Mayor-Torrens $(T_{\lambda}^{\mathbf{MT}})_{\lambda \in [0,1]}$ Dubois-Prade $(T_{\lambda}^{\mathbf{DP}})_{\lambda \in [0,1]}$	$\lambda=0, \lambda=\mu$		Saminger et al., 2005 [159]

Figure 4.2: Dominance in selected families of t-norms.

**Remark 4.6** (Representation of continuous t-norms (see also [113, 125, 137, 180])). A t-norm T is continuous if and only if it is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exist an index set I, a family  $(]a_i, b_i[)_{i \in I}$  of non-empty pairwise disjoint open subintervals of [0, 1], a family  $(T_i)_{i \in I}$  of continuous Archimedean t-norms such that  $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$  (see Definition 2.3).

- Therefore, for arbitrary continuous t-norm T exactly one of the following cases holds:
- (i) T is the minimum  $T_{\mathbf{M}}$ , i.e.,  $I = \emptyset$ ,
- (ii) T is a continuous Archimedean t-norm (|I| = 1), i.e., there exists a continuous strictly decreasing function  $t: [0, 1] \rightarrow [0, \infty]$  fulfilling t(1) = 0 such that

$$T(x,y) = t^{(-1)}(t(x) + t(y))$$

for all  $x, y \in [0, 1]$ . The function t is referred to as the *additive generator* of the t-norm T and the function  $t^{(-1)}: [0, \infty] \to [0, 1]$ , defined, for all  $x \in [0, \infty]$ , by  $t^{(-1)}(x) = t^{-1}(\min(t(0), x))$  denotes the *pseudo-inverse* of t. In case that  $t(0) = \infty$ , then T is *strict*, i.e., there exists an order isomorphism  $\varphi: [0, 1] \to [0, 1]$  such that  $T_{\varphi} = T_{\mathbf{P}}$ . For  $t(0) < \infty$ , T is *nilpotent*, i.e., there exists an order isomorphism  $\varphi: [0, 1] \to [0, 1] \to [0, 1]$  such that  $T_{\varphi} = T_{\mathbf{L}}$ .

(iii) T is a non-trivial ordinal sum with strict or nilpotent summand t-norms, i.e.,  $I \neq \emptyset$  and no  $]a_i, b_i[$  equals ]0, 1[.

Since  $T_{\mathbf{M}}$  dominates all t-norms, the problem of dominance between continuous t-norms can be reduced to the following cases: dominance between ordinal sum t-norms, between continuous Archimedean t-norms as well as between ordinal sum t-norms and continuous Archimedean t-norms. For all cases the following problems are relevant:

- Find, if possible, necessary and/or sufficient conditions for dominance between t-norms of the corresponding classes.
- If possible, formulate the conditions in such a way that they can be checked easily, i.e., based on the ordinal sum structure or on the basis of the additive generators involved.

In the following two chapters we provide answers to these questions, giving an overview on the most important findings of the articles [A07–A11].

# Chapter 5

# Dominance between ordinal sums

On the (non-)transitivity of dominance between t-norms

#### 5.1 Problem statement

As outlined in the previous chapter, dominance between ordinal sum operations, in particular ordinal sum t-norms, is of interest for clarifying the structure of dominance between continuous t-norms. In [A07], entitled "On the dominance relationship between ordinal sums of conjunctors", the dominance relation between ordinal sums of conjunctors has been investigated. Note that *conjunctors* are aggregation functions with neutral element 1 and are also referred to as *semicopulas* [66]. For any conjunctor C it holds that  $T_{\mathbf{D}} \leq C \leq T_{\mathbf{M}}$ . Clearly, t-norms are associative and commutative conjunctors, quasi-copulas are 1-Lipschitz conjunctors, and copulas are 2-increasing conjunctors. Therefore, the results on dominance between ordinal sums of conjunctors are valid also for ordinal sums of t-norms and (quasi-)copulas.

Throughout this chapter we consider  $C_1 = (\langle a_i, b_i, C_{1,i} \rangle)_{i \in I}$  and  $C_2 = (\langle a_j, b_j, C_{2,j} \rangle)_{j \in J}$  to be two ordinal sum conjunctors defined in complete analogy to ordinal sums of t-norms, i.e., I and Jbeing at most countable index sets,  $(]a_i, b_i]_{i \in I}$  resp.  $(]a_j, b_j]_{j \in J}$  families of pairwise disjoint open subsets of [0, 1], and  $(C_{1,i})_{i \in I}$  resp.  $(C_{2,j})_{j \in J}$  families of conjunctors, such that, for all  $x, y \in [0, 1]$ ,

$$C_{1}(x,y) = \begin{cases} a_{i} + (b_{i} - a_{i})C_{1,i}(\frac{x - a_{i}}{b_{i} - a_{i}}, \frac{y - a_{i}}{b_{i} - a_{i}}), & \text{if } (x,y) \in [a_{i}, b_{i}]^{2}, \\ \min(x,y), & \text{otherwise,} \end{cases}$$
(5.1)

and for  $C_2$  accordingly.

Since  $T_{\mathbf{M}}$  dominates all conjunctors we may additionally assume that  $I \neq \emptyset \neq J$ . Note that I and J might, in general, be different (see also Fig. 5.1). In case that  $I \neq J$ , we additionally assume that the summand operations  $C_{1,i}$  and  $C_{2,j}$  are all ordinally irreducible, i.e., have themselves no other ordinal sum representation than  $(\langle 0, 1, C_{1,i} \rangle)$  resp.  $(\langle 0, 1, C_{2,j} \rangle)$ . Therefore, the representations of  $C_1$  and  $C_2$  based on their summand carriers  $]a_i, b_i[$  resp.  $]a_j, b_j[$  and summand operations  $C_{1,i}$  resp.  $C_{2,j}$  are the finest possible ordinal sum representations of  $C_1$  resp.  $C_2$ . For continuous ordinal sum t-norms this means that all summand operations involved are continuous Archimedean t-norms. Note that the present representation covers dominance between non-trivial continuous ordinal sum t-norms as well as dominance between an ordinal sum t-norm and a continuous Archimedean t-norm, in the latter case it holds that either |I| = 1 or |J| = 1 and the corresponding summand t-norm being continuous Archimedean.

Based on these notions we can pose the following problem statement which has been investigated in [A07]:

For two ordinal sum conjunctors  $C_1 = (\langle a_i, b_i, C_{1,i} \rangle)_{i \in I}$  and  $C_2 = (\langle a_j, b_j, C_{2,j} \rangle)_{j \in J}$ , provide necessary and sufficient conditions such that  $C_1$  dominates  $C_2$ .



Figure 5.1: Examples of two ordinal sum conjunctors  $C_1$  and  $C_2$  differing in their summand carriers.

## 5.2 Main results

If  $C_1$  and  $C_2$  coincide in their summand carriers, i.e., I = J, but differ only in their summand operations, which in this case need not be ordinally irreducible, then the following holds:

**Proposition 5.1.** [A07, Proposition 4] Consider two ordinal sum conjunctors  $C_1 = (\langle a_i, b_i, C_{1,i} \rangle)_{i \in I}$ and  $C_2 = (\langle a_i, b_i, C_{2,i} \rangle)_{i \in I}$ . Then  $C_1$  dominates  $C_2$  if and only if, for all  $i \in I$ ,  $C_{1,i}$  dominates  $C_{2,i}$ , i.e., all summand operations must be in the corresponding dominance relationship.

We now consider the case that  $I \neq J$  and assume w.l.o.g. that  $C_1$  and  $C_2$  are represented by ordinally irreducible summand operations only. Since all conjunctors are bounded from above by  $T_{\mathbf{M}}$  and, because of the common neutral element, dominance implies ordering, it holds immediately that if  $C_1$  dominates  $C_2$ , then  $C_1(x, y) = T_{\mathbf{M}}(x, y)$  whenever  $C_2(x, y) = T_{\mathbf{M}}(x, y)$  [A07, Proposition 5]. Geometrically speaking, if an ordinal sum conjunctor  $C_1$  dominates an ordinal sum conjunctor  $C_2$ , then it must necessarily consist of more regions where it acts as  $T_{\mathbf{M}}$  than does  $C_2$ . Two such cases are displayed in Fig. 5.1 (a) and (b).

**Corollary 5.2.** [A07, Corollary 1] Consider two ordinal sum conjunctors  $C_1 = (\langle a_{1,i}, b_{1,i}, C_{1,i} \rangle)_{i \in I}$ and  $C_2 = (\langle a_{2,j}, b_{2,j}, C_{2,j} \rangle)_{j \in J}$  with ordinally irreducible summand operations only. If  $C_1$  dominates  $C_2$  then for all  $i \in I$  there exists some  $j \in J$  such that  $[a_{1,i}, b_{1,i}] \subseteq [a_{2,j}, b_{2,j}]$ .

Note that each  $[a_{2,j}, b_{2,j}]$  can contain several or even none of the summand carriers  $[a_{1,i}, b_{1,i}]$  (see also Fig. 5.1 (a) and (b)). Hence, for each  $j \in J$ , we can consider the subset  $I_J$  of I

$$I_j = \{i \in I \mid [a_{1,i}, b_{1,i}] \subseteq [a_{2,j}, b_{2,j}]\}.$$
(5.2)

Based on these notions and due to Proposition 5.1, dominance between two ordinal sum conjunctors can be characterized in the following way:

**Proposition 5.3.** [A07, Proposition 6] Consider two ordinal sum conjunctors  $C_1$ ,  $C_2$ , i.e.,  $C_1 = (\langle a_{1,i}, b_{1,i}, C_{1,i} \rangle)_{i \in I}$  and  $C_2 = (\langle a_{2,j}, b_{2,j}, C_{2,j} \rangle)_{j \in J}$ , with ordinally irreducible summand operations only. Then  $C_1$  dominates  $C_2$  if and only if

- (i)  $I = \bigcup_{j \in J} I_j$ ,
- (ii)  $C_1^j \gg C_{2,j}$  for all  $j \in J$  with  $C_1^j = (\langle \varphi_j(a_{1,i}), \varphi_j(b_{1,i}), C_{1,i} \rangle)_{i \in I_j}$  and  $\varphi_j \colon [a_{2,j}, b_{2,j}] \to [0,1], \varphi_j(x) = \frac{x a_{2,j}}{b_{2,j} a_{2,j}}.$

As a consequence of Proposition 5.3, the study of dominance between ordinal sum conjunctors can be reduced to the study of dominance of an ordinal sum conjunctor over a single ordinally irreducible conjunctor.



Figure 5.2: Three ordinal sum t-norms based on  $T_{\mathbf{L}}$  only violating, for  $\lambda \in [0, 0.5]$ , the transitivity of the dominance relation (compare also [175] and [A08]).

#### **Idempotent elements**

Idempotent elements allow to formulate a necessary and easy-to-check condition for dominance between conjunctors. Recall that we denote the set of idempotent elements of a conjunctor C by  $\mathcal{I}(C)$ , i.e.,  $\mathcal{I}(C) = \{x \in [0,1] \mid C(x,x) = x\}$ .

**Proposition 5.4.** [A07, Proposition 7] If a conjunctor  $C_1$  dominates a conjunctor  $C_2$ , then the following hold:

- (i)  $\mathcal{I}(C_2) \subseteq \mathcal{I}(C_1)$ ,
- (ii)  $\mathcal{I}(C_1)$  is closed under  $C_2$ .

As shown by Sarkoci (see [172, 174] and [A08]), in case of ordinal sum t-norms where all summand operations are exclusively equal to  $T_{\mathbf{L}}$ , the necessary condition turns into a characterization allowing to determine counterexamples to the conjecture of the transitivity of dominance on the set of continuous ordinal sum t-norms.

**Example 5.5.** [175, Section 3] Consider the three t-norms  $T_1 = (\langle 0, 0.5, T_L \rangle)$ ,  $T_2 = (\langle 0, 0.5, T_L \rangle)$ ,  $\langle 0.5, 1, T_L \rangle$ ) and  $T_3 = T_L$  (see also Fig. 5.2). Then  $T_1$  can be expressed as the ordinal sum t-norm  $(\langle 0, 0.5, T_L \rangle, \langle 0.5, 1, T_M \rangle)$  and therefore, based on Proposition 5.1,  $T_1$  dominates  $T_2$  since  $T_L \gg T_L$  and  $T_M \gg T_L$ .  $T_2$  dominates  $T_3$  since its set of idempotent element, namely  $\mathcal{I}(T_2) = \{0, 0.5, 1\}$ , is closed under  $T_3$ . However,  $T_1$  does not dominate  $T_3 = T_L$ . To see this choose x = u = 0.75 and y = v = 0.5, then

 $T_1(T_3(x,y), T_3(u,v)) = T_1(0.25, 0.25) = 0,$  $T_3(T_1(x,u), T_1(y,v)) = T_3(0.75, 0.5) = 0.25.$ 

Therefore,  $T_1 \gg T_2$  and  $T_2 \gg T_3$ , but  $T_1 \not\gg T_3$ .

### 5.3 Additional remarks

The results presented in the previous section focus on ordinal sums of conjunctors. In [A07] additional results on dominance among conjunctors, not necessarily being ordinal sums, are presented. Moreover, first proofs for dominance in the family of Mayor-Torrens and Dubois-Prade t-norms are formulated. The contribution [A08], entitled "The dominance relation on the class of continuous t-norms from an ordinal sum point of view", provides, among others, an overview on the ordinal sum results in terms of t-norms, different proofs for dominance in the family of Mayor-Torrens and Dubois-Prade t-norms, additional results on families of ordinal sum t-norms based on  $T_{\mathbf{L}}$  resp.  $T_{\mathbf{P}}$  only as well as a counterexamples to the transitivity of dominance for ordinal sums involving either only  $T_{\mathbf{L}}$  or  $T_{\mathbf{P}}$ . Moreover, a geometrical interpretation of the dominance of some t-norm T over  $T_{\mathbf{L}}$  resp.  $T_{\mathbf{P}}$  is offered.

# Chapter 6

# Dominance between continuous Archimedean t-norms

Easy-to-check conditions

## 6.1 Problem statement

Due to the representation of continuous Archimedean t-norms by continuous additive generators (see also Remark 4.6 (ii)), dominance between continuous Archimedean t-norms can be expressed in terms of their additive generators, invoking another functional inequality, the (generalized) Mulholland inequality. Already in 1984, Tardiff [192] showed the relationship between the Mulholland inequality and dominance among two strict t-norms. The Mulholland inequality, introduced by Mulholland in 1950 [138], is a generalization of the Minkowski inequality and has been studied, mainly, independently from the context of dominance in the framework of functional equations [105, 138, 191, 192]: Mulholland, already in [138], proved some necessary and a sufficient condition for a continuous, strictly increasing function  $h: [0, \infty] \to [0, \infty]$  with h(0) = 0 to fulfill the inequality. Tardiff, in [192], showed a different sufficient condition, and in 2002, Jarczyck and Matkowski clarified the relationship between the two sufficient conditions, showing that Tardiff's condition implies that of Mulholland [105] (see also [A09, Section 4] for an overview on the corresponding relevant conditions). It is remarkable that, although the relationship between the Mulholland inequality and dominance between two strict t-norms has been known since years, these properties have hardly ever been used for proving or disproving dominance between strict t-norms, one exception being the case when the Mulholland inequality turns into the Minkowski inequality leading to the dominance relationship in the families of Aczél-Alsina, Dombi, and Yager t-norms [113]. Moreover, it is clear that the Mulholland inequality and its various corresponding sufficient and necessary conditions are applicable for the case of strict t-norms only. Therefore, the following problems have been investigated:

- Provide an equivalent description of dominance between two continuous Archimedean t-norms in terms of their additive generators (*generalized Mulholland inequality*).
- Formulate, if possible, sufficient and necessary conditions for some function *h* to fulfill the generalized Mulholland inequality.
- In this case, find in addition equivalent easy-to-check-conditions of the sufficient and necessary conditions in terms of the additive generators involved to apply the achieved results for proving or disproving dominance between continuous Archimedean t-norms.

In the contribution "A generalization of the Mulholland inequality for continuous Archimedean t-norms" [A09] the first two questions have been investigated and solved. Note that the sufficient and necessary conditions for a function h to fulfill the generalized Mulholland inequality have been proven independently of the context of dominance and are therefore valid for a larger class of functions than needed for the solution to the dominance problem.

The third problem of easy-to-check conditions and their applications has been studied and solved in the article "Differential inequality conditions for dominance between continuous **Archimedean t-norms**" [A10] in which also the strength of the conditions has been demonstrated by several results. New results on dominance in families of t-norms and copulas as well as a comprehensive overview on the conditions are provided in the contribution "The dominance relation in some families of continuous Archimedean t-norms and copulas" [A11]. We briefly quote the most relevant findings in the following sections, however refer for proofs, more details and additional results to the articles mentioned before.

## 6.2 The generalized Mulholland inequality

Let us first introduce the equivalent formulation of dominance between two continuous Archimedean t-norms by means of their additive generators:

**Theorem 6.1.** [A09, Theorem 1] Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . Then  $T_1$  dominates  $T_2$  if and only if the function  $h: [0, \infty] \to [0, \infty]$  defined by  $h = t_1 \circ t_2^{(-1)}$  fulfills, for all  $a, b, c, d \in [0, t_2(0)]$ ,

$$h^{(-1)}(h(a) + h(c)) + h^{(-1)}(h(b) + h(d)) \ge h^{(-1)}(h(a+b) + h(c+d)).$$
(6.1)

with  $h^{(-1)}$ :  $[0,\infty] \to [0,\infty]$  the pseudo-inverse of the non-decreasing function h, given by  $h^{(-1)} = t_2 \circ t_1^{(-1)}$ .

Note that, for two additive generators  $t_1, t_2$ , the function h fulfills h(0) = 0 and is constant on  $[t_2(0), \infty]$ , i.e.,  $h(x) = t_1(0)$  for all  $x \in [t_2(0), \infty]$ . In case that  $T_2$  is strict, then necessarily  $T_1$  has to be strict (otherwise it leads to a contradiction to the dominance relationship resp. the induced order) and therefore  $t_2(0) = \infty$  and h being a strictly increasing bijection with  $h^{(-1)}$  being the standard inverse  $h^{-1}$  of h.

In case some function  $h: [0, \infty] \to [0, \infty]$  fulfills (6.1), for all  $a, b, c, d \in [0, \infty]$ , we say that it fulfills the *generalized Mulholland inequality*. For the investigation of some function h to fulfill the generalized Mulholland inequality properties like convexity, the geometric convexity, and the logarithmic convexity of a function showed up to be most relevant.

**Definition 6.2.** A function  $h: [0, \infty[ \to [0, \infty[$  is called *geometric convex* (*geo-convex* for short) on ]0, t[, with  $t \in ]0, \infty[$ , if, for all  $x, y \in ]0, t[$ ,

$$h(\sqrt{xy}) \le \sqrt{h(x)h(y)} \,.$$

It is called *logarithmic convex* (*log-convex* for short) on ]0,t[ if the function  $\log \circ h: [0,\infty[ \rightarrow [-\infty,\infty[$  is convex on ]0,t[.

For a continuous function h such that  $h(]0, \infty[) \subseteq ]0, \infty[$ , its geo-convexity on ]0, t[ is equivalent to the convexity of the function  $\log \circ h \circ \exp$  on  $]-\infty, \log(t)[$ . Clearly, if h(0) = 0, then the geoconvexity holds also on [0, t[. Further, if h is strictly increasing, then its log-convexity on ]0, t[implies its geo-convexity on ]0, t[.

Based on these notions we can formulate the sufficient and necessary conditions for a function h to fulfill the generalized Mulholland inequality:

**Theorem 6.3.** [A09, Theorem 6] Consider a function  $h: [0, \infty] \to [0, \infty]$  and some fixed value  $t \in ]0, \infty[$  such that

- (i) h is continuous and strictly increasing on [0, t] as well as convex on ]0, t[,
- (ii) h(0) = 0 and  $h(x) \ge h(t)$  whenever  $x \ge t$ ,
- (iii) h is geo-convex on ]0, t[.

Define the functions  $g: [0, \infty] \to [0, \infty]$  and  $H: [0, \infty]^2 \to [0, \infty]$  by

$$g(x) = \begin{cases} h^{-1}(x), & \text{if } x \in [0, h(t)], \\ t, & \text{otherwise,} \end{cases}$$
(6.2)

$$H(x,y) = g(h(x) + h(y)).$$
(6.3)

Then the following inequality holds, for all  $a, b, c, d \in [0, \infty]$ ,

$$H(a+b, c+d) \le H(a, c) + H(b, d).$$
 (6.4)

**Proposition 6.4.** [A09, Proposition 9] Consider a function  $h: [0, \infty] \to [0, \infty]$  and some fixed value  $t \in ]0, \infty[$  such that

- (i) h is continuous and strictly increasing on [0, t] as well as convex on ]0, t[,
- (ii) h(0) = 0 and  $h(x) \ge h(t)$  whenever  $x \ge t$ ,
- (iii) h is differentiable on ]0,t[ and h' is geo-convex on ]0,t[.

Define the function  $g: [0,\infty] \to [0,\infty]$  by (6.2) and the function  $H: [0,\infty]^2 \to [0,\infty]$  by (6.3). Then the following inequality holds, for all  $a, b, c, d \in [0,\infty]$ ,

$$H(a+b, c+d) \le H(a, c) + H(b, d).$$

**Proposition 6.5.** [A09, Proposition 10] Consider a function  $h: [0, \infty] \to [0, \infty]$  and some fixed value  $t \in [0, \infty]$  such that

- (i) h is continuous and strictly increasing on [0, t],
- (ii) h(0) = 0 and and  $h(x) \ge h(t)$  whenever  $x \ge t$ .

Define the function  $g: [0, \infty] \to [0, \infty]$  by (6.2) and the function  $H: [0, \infty]^2 \to [0, \infty]$  by (6.3). If H fulfills (6.4), for all  $a, b, c, d \in [0, \infty]$ , then h is convex on ]0, t[.

For two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ , the corresponding function  $h: [0, \infty] \to [0, \infty]$ , defined by  $h = t_1 \circ t_2^{(-1)}$  is continuous and strictly increasing on  $[0, t_2(0)]$ . It fulfills h(0) = 0 as well as  $h(x) = h(t_2(0)) = t_1(0)$  for all  $x \ge t_2(0)$ . Moreover, it holds that  $H(x, y) = h^{(-1)}(h(x) + h(y))$ , in accordance with Theorem 6.1. Therefore and by taking into account that the log-convexity of h implies its geo-convexity, the above results can be restated for dominance between continuous Archimedean t-norms (see also [A09, Propositions 11–13] and [A10, Propositions 3–5]):

**Proposition 6.6.** Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$  and define the function  $h: [0, \infty] \to [0, \infty]$  by  $h = t_1 \circ t_2^{(-1)}$ . Then the following hold:

- (i) If h is convex on  $]0, t_2(0)[$  and log- or geo-convex on  $]0, t_2(0)[$ , then  $T_1$  dominates  $T_2$ .
- (ii) If h is differentiable and convex on ]0, t<sub>2</sub>(0)[ and if h' is log- or geo-convex on ]0, t<sub>2</sub>(0)[, then T<sub>1</sub> dominates T<sub>2</sub>.
- (iii) If  $T_1$  dominates  $T_2$ , then h is convex on  $]0, t_2(0)[$ .

The relationships between the sufficient conditions for dominance are summarized in Fig. 6.1.

Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . If the function

$$h: [0,\infty] \to [0,\infty], h = t_1 \circ t_2^{(-1)}$$

is convex on  $]0, t_2(0)[$  and  $\ldots$ 

$$\begin{array}{ccc} h' \text{ exists and} & h' \text{ exists and} \\ h' \text{ is log-convex on } ]0, t_2(0)[ \implies h' \text{ is geo-convex on } ]0, t_2(0)[ \\ & \downarrow \\ h \text{ is log-convex on } ]0, t_2(0)[ \implies h \text{ is geo-convex on } ]0, t_2(0)[ \\ & \downarrow \\ & h \text{ fulfills } (6.1) \\ & \text{ for all } a, b, c, d \in [0, t_2(0)] \\ & \uparrow \\ & T_1 \gg T_2 \end{array}$$

Figure 6.1: Sufficient conditions for dominance between two continuous Archimedean t-norms  $T_1$  and  $T_2$ .

### 6.3 Easy-to-check conditions

Although the, sufficient as well as necessary, conditions can be visualized easily, concrete proofs might become cumbersome, in particular for two members of a parametric family, because h is a composed function of an additive generator and the pseudo-inverse of another additive generator. Therefore, we aim at equivalent conditions expressed directly by the additive generators involved which could be achieved in case the additive generators have derivatives of sufficiently high order. The corresponding conditions, therefore, offer local descriptions of the corresponding properties of h. Again we summarize the most important results and refer for proofs and further details to [A10, Section 4]:

In the sequel,  $T_1$  and  $T_2$  denote two continuous Archimedean t-norms with continuous additive generators  $t_1$  and  $t_2$ . Then the function  $h: [0, \infty] \to [0, \infty], h = t_1 \circ t_2^{(-1)}$  is continuous and strictly increasing on  $]0, t_2(0)[, h(0) = 0$  and  $h(]0, t_2(0)[) \subseteq ]0, t_1(0)[$ . Moreover, we assume that  $t_1$  and  $t_2$ are sufficiently often (i.e., once, twice or three times) differentiable.

**Proposition 6.7.** [A10, Proposition 6] The function h is convex on  $]0, t_2(0)[$ , i.e.,  $h''(x) \ge 0$ , for all  $x \in ]0, t_2(0)[$ , if and only if, for all  $u \in ]0, 1[$ ,

$$t_1'(u)t_2''(u) - t_1''(u)t_2'(u) \ge 0.$$

**Proposition 6.8.** [A10, Proposition 7] The function h is log-convex on  $]0, t_2(0)[$ , i.e.,

$$h(x)h''(x) - {h'}^2(x) \ge 0.$$

for all  $x \in [0, t_2(0)[$ , if and only if, for all  $u \in [0, 1[$ ,

$$t_1'^{2}(u)t_2'(u) + t_1(u)(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)) \ge 0.$$

**Proposition 6.9.** [A10, Proposition 8] The function h is geo-convex on  $]0, t_2(0)[$ , i.e.,

$$h(x)h'(x) + x\left(h(x)h''(x) - {h'}^{2}(x)\right) \ge 0,$$

for all  $x \in [0, t_2(0)[$ , if and only if, for all  $u \in [0, 1[$ ,

$$\frac{t_1'^2(u) - t_1(u)t_1''(u)}{t_1(u)t_1'(u)} \ge \frac{t_2'^2(u) - t_2(u)t_2''(u)}{t_2(u)t_2'(u)}$$

**Corollary 6.10.** [A10, Corollary 10] The function h' is log-convex on  $]0, t_2(0)[$ , i.e., for all  $x \in ]0, t_2(0)[$ ,

$$h'(x)h'''(x) - h''^{2}(x) \ge 0,$$

if and only if, for all  $u \in [0, 1[$ ,

$$t_{1}'^{2}(u)\left(2t_{2}''^{2}(u)-t_{2}'(u)t_{2}'''(u)\right) \geq t_{2}'^{2}(u)\left(t_{1}''^{2}(u)-t_{1}'(u)t_{1}'''(u)\right)+t_{1}'(u)t_{1}''(u)t_{2}'(u)t_{2}''(u).$$

**Corollary 6.11.** [A10, Corollary 11] The function h' is geo-convex on  $]0, t_2(0)[$ , i.e., for all  $x \in ]0, t_2(0)[$ ,

$$h'(x)h''(x) + x\left(h'(x)h'''(x) - h''^{2}(x)\right) \ge 0,$$

if and only if, for all  $u \in [0, 1[$ ,

$$t_{2}(u)\Big(t_{1}'(u)t_{2}'(u)\big(t_{1}'''(u)t_{2}'(u)-t_{2}'''(u)t_{1}'(u)\big) \\ -\big(t_{1}''(u)t_{2}'(u)-t_{2}''(u)t_{1}'(u)\big)\big(2t_{1}'(u)t_{2}''(u)+t_{1}''(u)t_{2}'(u)\big)\Big) \\ > t_{1}'(u)t_{2}'^{2}(u)\big(t_{1}'(u)t_{2}''(u)-t_{1}''(u)t_{2}'(u)\big).$$

#### 6.4 Further results

Since these new conditions allow to investigate dominance among strict or nilpotent as well as between strict and nilpotent t-norms, they have been applied for proving resp. disproving dominance in various families of t-norms resp. Archimedean copulas but also for dominance between members of different families of t-norms:

In Section 5.1 of [A10] another proof for dominance between members of the family of Schweizer-Sklar t-norms is provided. In Section 6 of [A10] and in [161] the dominance relation between members of the family of Dombi t-norms and Yager t-norms resp. Aczél-Alsina t-norms have been investigated.

**Proposition 6.12.** Consider the families of Dombi t-norms  $(T_{\lambda}^{\mathbf{D}})_{\lambda \in [0,\infty]}$ , of Yager t-norms  $(T_{\mu}^{\mathbf{Y}})_{\mu \in [0,\infty]}$ , and of Aczél-Alsina t-norms  $(T_{\mu}^{\mathbf{AA}})_{\mu \in [0,\infty]}$ . For all  $\lambda, \mu \in [0,\infty]$ , it holds that  $T_{\lambda}^{\mathbf{D}}$  dominates  $T_{\mu}^{\mathbf{Y}}$  if and only if  $\lambda \geq \mu$ . For all  $\lambda, \mu \in [0,\infty]$  with  $\mu \leq 1 \leq \lambda$ , it follows that  $T_{\lambda}^{\mathbf{D}}$  dominates  $T_{\mu}^{\mathbf{AA}}$ .

In Section 5.2 of [A10] the dominance relationship between members of the family of Sugeno-Weber t-norms has been studied.

**Proposition 6.13.** Consider the family of Sugeno-Weber t-norms  $(T_{\lambda}^{\mathbf{SW}})_{\lambda \in [0,\infty]}$ . For all  $\lambda, \mu \in [0,\infty]$  such that either  $\lambda \leq \min(1,\mu)$ , or  $1 < \lambda \leq \mu \leq t^*$ , with  $t^* = 6.00914$  denoting the second root of  $\log^2(t) + \log(t) - t + 1 = 0$ , it holds that  $T_{\lambda}^{\mathbf{SW}} \gg T_{\mu}^{\mathbf{SW}}$ . On the other hand, if  $T_{\lambda}^{\mathbf{SW}} \gg T_{\mu}^{\mathbf{SW}}$ , then  $\lambda \leq \mu$ .

Moreover, note that dominance is not a linear order on the class of Sugeno-Weber t-norms, since, e.g., neither  $T_{51}^{\mathbf{SW}}$  dominates  $T_{101}^{\mathbf{SW}}$  nor  $T_{101}^{\mathbf{SW}}$  dominates  $T_{51}^{\mathbf{SW}}$ .

In [A11, Section 5] the dominance relation in five additional families of t-norms resp. Archimedean copulas is laid bare, i.e., proving that dominance constitutes an order relation on all these families. All families are taken from the book on associative functions by Alsina et al. [8] and the corresponding notations refer to the ones used in this book. The results are summarized in Fig. 6.4 providing the corresponding Hasse diagram and an indication by which sufficient condition the result has been achieved. Note that all these families of t-norms contain subfamilies of (Archimedean) copulas (see also [140]).

Family of t-norms	$T_{\lambda} \gg T_{\mu}$ if and only if	Hasse- diagram	Condition
$(T^{8}_{\lambda})_{\lambda \in [0,\infty]}$	$\lambda \leq \mu$	Ţ	log-convexity of $h'$
$(T_{\lambda}^{10})_{\lambda \in [0,\infty]}$	$\lambda \leq \mu$	•	geo-convexity of $h$
$(I_{\lambda}^{-})_{\lambda \in [0,\infty]}$ $(T^{23})$	$\lambda \leq \mu$ $\lambda \leq \mu$	Ļ	geo-convexity of $h$
$(I_{\lambda})_{\lambda \in [0,\infty]}$	$\lambda \ge \mu$	*	geo-convexity of n
$(T^{9}_{\lambda})_{\lambda\in[0,\infty]}$	$\lambda=\infty,\lambda=\mu,\mu=0$		

Figure 6.2: Results on dominance in five additional families of t-norms [A11].

# 6.5 Concluding remarks

Summarizing the results in this part we can say that dominance is not a transitive and therefore not an order relation on the class of all (continuous) t-norms. However, there are several families of (continuous Archimedean) t-norms and (Archimedean) copulas for which dominance is an order relation, either due to the rare or abundant occurrence of a dominance relationship between its members. For more details see the articles included in this thesis.

Apart from the framework of t-norms and copulas, several properties of dominance for aggregation functions on [0, 1], in particular constructions of dominating functions and characterizations of the set of aggregation functions dominating one of the four basic t-norms, have been investigated in [163]. In [171] dominance is, among other functional (in)equalities, like, e.g., convexity or Cauchy's equation, discussed for aggregation functions on  $\Delta^+$  resp. for triangle functions.

# Part III

# Aggregation and Decision Modelling: Two Case Studies

# **Decision Modelling**

The last part focusses on two case studies of aggregation functions in the field of decision making. It has been already mentioned in Chapter 1 that decision making and preference modelling form an important field of application for aggregation procedures, leading to problems of existence, construction and characterization of aggregation functions, involving functional equations and inequalities.

In this spirit we emphasize, in the following two chapters, the application setting and the appearance of the related, mathematical, problems for the aggregation functions involved. We will rather focus on the discussion of the results than on an exhaustive presentation of all the results of the articles "**Representation and construction of self-dual aggregation operators**" [A12] and "Aggregation operators and commuting" [A13]. The articles touch problems of aggregation in decision making — the first one mainly addresses representation and construction problems whereas the second one focusses on a functional equation arising in (bipolar) decision making. Before turning to concrete details, let us illustrate the general application setting:

A basic constituent in any preference and decision problem is a set of alternatives A, which we assume to be finite and stable, i.e., to remain the same during the whole investigation. Note that depending on the application setting A is also referred to as the set of actions or the choice set (see also [148]).

The alternatives are evaluated w.r.t. some *criteria* or *attributes*. Note that the notion of "criterion" or "attribute" should rather be thought of in an abstract way. Since the evaluation of alternatives by different members of a jury leading to a final group decision can be identified with a decision problem of a single decision maker, namely the jury, who evaluates the alternatives w.r.t. several criteria, e.g., each member of the jury being responsible for or providing different viewpoints. Therefore, evaluations by means of different criteria or by means of different members of a jury are often identified with each other in the literature, i.e., in group preference modelling the alternatives are evaluated by experts forming a jury, in multicriteria decision problems the alternatives are evaluated by a set of criteria in its original sense of meaning. In the sequel, we assume that the set of "criteria" is finite, moreover, that they are all of the same type, i.e., providing (valued) preferences or numerical values from the same scale.

Note that the determination of the set of criteria is a non-trivial task. If possible, it should, e.g., represent all important aspects of the application problem and as such the set of criteria should be complete. Redundancies should be avoided and the set of criteria shall be as minimal as possible in order to keep the complexity as low as possible (see also [110, 153]). Moreover, aspects of independence, of ambiguity, or of different importance and interaction of criteria have also to be mentioned and taken into account when modelling the preference or decision problem at hand.

According to Vincke ([195], see also [148]), a multicriteria decision problem is a situation in which, having defined a set of alternatives A and a consistent family of criteria G on A, one wishes

- to determine a subset of actions considered to be best w.r.t. G (choice, selection problem),
- to divide A into subsets according to some norm (sorting, classification problem),
- to rank the actions of A from best to worst (ranking problem),
- or a mixture thereof.

In all these settings, aggregation of evaluations w.r.t. single criteria for obtaining a final decision is a major task, one of the interesting and challenging aspects being the modelling of the consistency of the family of criteria. Consistency in case of preference modelling might relate to the modelling of a rational behaviour of each expert (and most often also of the whole commission), leading to additional demands on the (valued) preference structure expressing the expert's opinion over the different alternatives as well as to the aggregation procedure applied. Such a situation will be in the focus of the following chapter dealing with reciprocal relations and their aggregation.

On the other hand, problems arise if the set of criteria corresponds to a multi-step evaluation procedures. Irrelevancy of the order by which evaluations are carried out and its consequence for the aggregation functions involved are discussed in the last chapter.

Further, it has to be mentioned that modelling dependencies or compensation effects between different criteria asks for aggregation procedures involving capacities (fuzzy measures) resp. bipolar capacities (see, e.g., [57, 88, 89, 90, 91, 95, 96]). Clearly, preservation of additional properties of these capacities resp. bi-capacities during the aggregation process applied leads to additional demands for the aggregation function applied. We briefly touch this aspects in the last chapter. Note that additional and more detailed results on decomposable bi-capacities and their aggregation can be found, e.g., in [156, 162].

# Chapter 7

# Self-dual aggregation functions

## Aggregating reciprocal relations

In many decision problems, the question arises how to determine a collective decision, preference or opinion, based on several individual decisions, preferences or opinions. One possible strategy is simply to carry out an aggregation process based on the experts' decisions, preferences or opinions which is usually done by some aggregation function.

In preference modelling, [0, 1]-valued relations R are used to express the individual intensity of preference (compare also [20, 72] and the references therein). Consider a finite set of alternatives  $A = \{a_1, \ldots, a_m\}$  and n experts. The opinion of expert k is represented by a relation  $R_k \colon A^2 \to [0, 1]$ , such that  $R_k(a_i, a_j)$  expresses the degree to which expert k prefers alternative  $a_i$  to alternative  $a_j$ . In order to avoid inconsistent preferences it is often required that the degree to which  $x_i$  is preferred to  $x_j$  is in some sense complementary to the degree to which  $x_j$  is preferred to  $x_i$ . The latter can be obtained by using reciprocal preference relations  $R_k$ , i.e.,  $R_k(x_i, x_j) + R_k(x_j, x_i) = 1$  (see, e.g., [27, 78, 79, 81]). In this case two alternatives  $x_i$  and  $x_j$  are indifferent to expert k if  $R_k(x_i, x_j) = R_k(x_j, x_i) = \frac{1}{2}$ .

Note that reciprocal relations are known under various names such as *ipsodual* relations or *probabilistic* relations [52]. Moreover, they appear in various fields such as, e.g., game theory [41, 40, 52, 67], voting theory [78, 144], psychological studies on preference [52] and the comparison of random variables [38, 39].

The determination of a collective preference relation R from the individual preferences  $R_k$  is carried out by an aggregation function  $\mathbf{A}$ , i.e., for all alternatives  $a_i, a_j \in A$ ,

$$R(a_i, a_k) = \mathbf{A}(R_1(a_i, a_k), \dots, R_n(a_i, a_k)).$$

Note that when aggregating reciprocal relations  $R_k \colon A^2 \to [0, 1]$ , the commutativity of the aggregation function involved expresses the equal treatment of all judgements during the aggregation process (no weights are assigned to the different experts). Moreover, a neutral element gives an expert the opportunity to abstain from the decision process without influencing it, whereas an annihilator allows to model veto situations.

As introduced in Section 1.2 aggregation functions **A** can be transformed by means of monotone bijections  $\varphi \colon [0,1] \to [0,1]$ , i.e., the function  $\mathbf{A}_{\varphi} \colon \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ , is defined, for all  $n \in \mathbb{N}$  and all  $x_i \in [0,1], i \in \{1,\ldots,n\}$ , by

$$\mathbf{A}_{\varphi}(x_1,\ldots,x_n) = \varphi^{-1} \big( \mathbf{A}(\varphi(x_1),\ldots,\varphi(x_n)) \big)$$

and is an aggregation function on [0, 1].

If for some fixed  $\varphi \colon [0,1] \to [0,1]$  it holds that  $\mathbf{A} = \mathbf{A}_{\varphi}$  we say that  $\mathbf{A}$  is *invariant* w.r.t.  $\varphi$  (see, e.g., [135, 145]). A particularly important transformation is induced by  $\mathcal{N} \colon [0,1] \to [0,1]$ ,

 $\mathcal{N}(x) = 1 - x$ , also referred to as the *standard negation* in the framework of many-valued logics. The transformation  $A^d = A_N$  is called the *dual* aggregation function of **A**. Aggregation functions invariant with respect to  $\mathcal{N}$  are referred to as *self-dual* aggregation functions (see also [23, 186]).

It was soon recognized that the collective preference relation R as introduced above is itself reciprocal if and only if **A** is a self-dual aggregation function [78, 80, 81], i.e., for all  $x_i \in [0, 1]$ ,  $i \in \{1, ..., n\}$ 

$$\mathbf{A}(x_1, \dots, x_n) = \mathbf{A}_{\mathcal{N}}(x_1, \dots, x_n) = 1 - \mathbf{A}(1 - x_1, \dots, 1 - x_n)$$
(7.1)

Examples of self-dual aggregation functions are the arithmetic mean and all weighted means, as well as quasi-arithmetic means  $\mathbf{M}_f$  for which the strictly monotone, continuous function  $f: [0,1] \rightarrow [-\infty,\infty]$  fulfills f(1-x) = 1 - f(x). Recall that quasi-arithmetic means  $\mathbf{M}_f$  are, for all  $n \in \mathbb{N}$  and for all  $x_i \in [0,1], i \in \{1,\ldots,n\}$ , defined by  $\mathbf{M}_f(x_1,\ldots,x_n) = f^{-1}(\frac{1}{n}(f(x_1) + \ldots, f(x_n)))$ .

Note that any self-dual and commutative binary aggregation function **A** necessarily satisfies  $\mathbf{A}(x, 1-x) = \frac{1}{2}$  for all  $x \in [0, 1]$  such that no t-norm is self-dual.

Various names for self-dual aggregation functions comprise, e.g., *neutral* [79] or *reciprocal* aggregation functions [78, 80, 81]. Continuous, commutative and self-dual aggregation functions have been referred to as *symmetric sums* [56, 72, 186]. Dombi [53] has investigated strictly increasing, associative symmetric sums on ]0,1[. Idempotent symmetric sums have been discussed by Dubois [54].

### 7.1 Problem statement and results

Therefore, self-dual aggregation functions have been of interest for many years and have been studied from different perspectives. It is remarkable that two different characterizations of self-dual aggregation functions have been provided — one by Calvo et al. in [23] inspired by the results on symmetric sums by Silvert [186] and another one by García-Lapreseta et al. in [80] based on the arithmetic mean. We briefly quote these two results:

**Proposition 7.1.** [23, 80] An aggregation function  $\mathbf{A}$  is self-dual if and only if one of the following properties holds:

• there exists an aggregation function **B** such that

$$\mathbf{A}(x_1,\ldots,x_n) = \frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(x_1,\ldots,x_n) + 1 - \mathbf{B}_{\mathcal{N}}(x_1,\ldots,x_n)}$$

with convention  $\frac{0}{0+0} = \frac{1}{2}$ ,

• there exists an aggregation function **B** such that

$$\mathbf{A}(x_1,\ldots,x_n) = \frac{\mathbf{B}(x_1,\ldots,x_n) + \mathbf{B}_{\mathcal{N}}(x_1,\ldots,x_n)}{2}.$$

What has been striking is that both expressions are of the form

$$\mathbf{A}(x_1,\ldots,x_n) = \mathcal{C}(\mathbf{B}(x_1,\ldots,x_n),\mathbf{B}_{\mathcal{N}}(x_1,\ldots,x_n))$$
(7.2)

for some binary operation C characterizing aggregation functions being invariant w.r.t. the standard negation N. One of the aims of the article [A12] "Representation and construction of self-dual aggregation operators" has therefore been to extend these result for other invariant aggregation functions, moreover to determine other binary operations C being admissible for enabling a full characterization of these functions. More precisely, the following problems have been addressed in [A12]:

• Is it possible to find a characterization similar to (7.2) for aggregation functions **A** which are invariant w.r.t. an *involutive* negation, i.e., w.r.t. an order-reversing bijection  $N: [0,1] \rightarrow [0,1]$  additionally fulfilling N(N(x)) = x?

• In case, characterize the binary operations C which enable a full characterization of all N-invariant aggregation functions, i.e., for which it holds that an n-ary operation  $\mathbf{A}$  is N-invariant if and only if there exists an aggregation function  $\mathbf{B}$  such that  $\mathbf{A} = C_{\mathbf{B},N}$  with  $C_{\mathbf{B},N}$  being defined, for all  $x_1, \ldots, x_n \in [0, 1]$ , by

$$\mathcal{C}_{\mathbf{B},N}(x_1,\ldots,x_n) = \mathcal{C}(\mathbf{B}(x_1,\ldots,x_n),\mathbf{B}_N(x_1,\ldots,x_n)).$$
(7.3)

Both questions could be answered to the positive, the main result being the following:

**Theorem 7.2.** [A12, Theorem 1] Consider an involutive negation N. A binary operation C enables a full characterization of all N-invariant aggregation functions if and only if the following conditions hold:

- (i) C is a binary aggregation function,
- (ii) C(x, y) = N(C(N(y), N(x))), for all  $x, y \in [0, 1]$ ,
- (iii) there exists a non-decreasing function  $f: [0,1] \to [0,1]$  such that f(0) = 0, f(1) = 1 and  $\mathcal{C}(f(x), N(f(N(x))) = x$  for all  $x \in [0,1]$ .<sup>1</sup>

Note that  $\mathbf{A} = \mathcal{C}_{\mathbf{A},N}$  if and only if  $\mathcal{C}$  is idempotent, i.e.,  $\mathcal{C}(x,x) = x$  for all  $x \in [0,1]$ . In that case, it is sufficient to choose  $f = \mathrm{id}_{[0,1]}$ . Moreover, there exists no binary operation  $\mathcal{C}$  which enables a characterization of N-invariant aggregation functions for *all* involutive negations N, i.e., the admissibility of  $\mathcal{C}$  depends on the particular choice of the negation N.

For the standard negation  $\mathcal{N}$  and therefore for the characterization of self-dual aggregation functions, the above theorem can be rephrased in the following way:

Theorem 7.3. Consider a binary aggregation function C fulfilling

- (i)  $\mathbf{C}(x, y) + \mathbf{C}(1 y, 1 x) = 1$ , for all  $x, y \in [0, 1]$ ,
- (ii) there exists a non-decreasing function  $f: [0,1] \rightarrow [0,1]$  such that f(0) = 0, f(1) = 1 and  $\mathbf{C}(f(x), 1 f(1-x)) = x$  for all  $x \in [0,1]$ .

Then an aggregation function **A** is self-dual if and only if there exists an aggregation function **B** such that  $\mathbf{A} = \mathbf{C}_{\mathbf{B},\mathcal{N}}$ , i.e., for all  $x_1, \ldots, x_n \in [0, 1]$ ,

$$\mathbf{A}(x_1, \dots, x_n) = \mathbf{C}(\mathbf{B}(x_1, \dots, x_n), 1 - \mathbf{B}(1 - x_1, \dots, 1 - x_n)).$$

Note that for  $\mathbf{C}(x, y) = \frac{x}{x+1-y}$ , with the convention  $\frac{0}{0} = \frac{1}{2}$ , the previous theorem is just the characterization of self-dual aggregation functions as shown in [23, 186], for  $\mathbf{C}(x, y) = \frac{x+y}{2}$  it turns out to be the characterization shown in [80]. Another example for an admissible operation  $\mathbf{C}$  would be  $\mathbf{C}(x, y) = \text{med}(x, y, 0.5)$ .

### 7.2 Further results

Once C is fixed for a particular involutive negation N and in accordance with Theorem 7.2,  $C_{\mathbf{B},N}$  as defined by (7.3) is an N-invariant aggregation function for arbitrary aggregation function  $\mathbf{B}$ . Therefore, having at one's disposal admissible operations C allows to construct N-invariant aggregation functions  $\mathbf{A}$  from arbitrary aggregation functions  $\mathbf{B}$ , in particular allows to construct

<sup>&</sup>lt;sup>1</sup>Note that Theorem 2 in **[A12]** seems to give an equivalent formulation of Theorem 1, namely of condition (iii). Unfortunately, it contains a small mathematical inaccuracy which, however, does not influence or weaken all following results contained in the article. The inaccuracy refers to condition (3) given as "C reaches every element of  $[0, a_N]$ " which should be replaced by "The graph of C contains an non-decreasing (w.r.t. three space coordinates) curve whose z-coordinate reaches every number in  $[0, a_N]$ ". This minor mistake had been indicated to the editors of the journal already in Oct, 2006, however, the final version of the article could not be updated anymore.

self-dual aggregation functions  $\mathbf{A}$  relevant for the aggregation of reciprocal relations in group preference modelling.

Additionally we can state that to every N-invariant aggregation function **A** there corresponds at least one aggregation function **B** such that  $\mathbf{A} = C_{\mathbf{B},N}$ . Moreover note, for some N-invariant **A**, there might even exist operations B, i.e., not necessarily aggregation functions, such that  $\mathbf{A} = C_{B,N}$ . However, the admissibility of **B** resp. B are intrinsically bound to C as well as to N. Therefore, in [**A12**, Section 4], minimal conditions for operations B ensuring that  $\mathbf{A} = C_{B,N}$  is a self-dual aggregation function have been studied for the two special cases of  $C(x, y) = \frac{x}{x+1-y}$  and  $C(x, y) = \frac{x+y}{2}$ .

Since for some given C and an involutive negation N, several aggregation functions **B** resp. operations B might generate the same N-invariant aggregation function **A**, in [**A12**, Section 5] the equivalence classes of operations determining the same N-invariant aggregation function are studied and illustrated again for the cases of  $C(x, y) = \frac{x}{x+1-y}$  and  $C(x, y) = \frac{x+y}{2}$  with arbitrary involutive negation N.

# Chapter 8

# Commuting aggregation functions Agenda independent two-step aggregation procedures

In various applications where information fusion or multicriteria evaluation is needed, an aggregation process is carried out as a two-step procedure whereby several local aggregation steps are performed in parallel and then the results are merged into a global result. It may happen that in practice the two steps can be exchanged because there is no reason to perform either of the steps first. For instance, in a multi-person multi-criteria decision problem, each alternative is evaluated by a matrix of ratings where, e.g., the rows represent evaluations by persons and the columns represent evaluations by criteria (see also Section 4.2). One may, for each row, merge the ratings according to each column with some aggregation function  $\mathbf{A}$  and form as such the global rating of each person, and then merge the experts' opinions using another aggregation function  $\mathbf{B}$ . On the other hand, one may decide first to combine the ratings in each column using the aggregation function  $\mathbf{B}$ , thus forming the global ratings according to each criterion, and then merge these social evaluations across the criteria by the aggregation function  $\mathbf{A}$ . The problem is that it is not guaranteed that the results of the two procedures will be the same, while one would expect them to be so in any sensible approach. When the two procedures yield the same results the aggregation functions  $\mathbf{A}$  and  $\mathbf{B}$  are said to *commute*.

**Definition 8.1.** Consider two aggregation functions **A** and **B**. Then we say that  $\mathbf{A}_{(n)}$  commutes with  $\mathbf{B}_{(m)}$ , if for all  $x_{ij} \in [0, 1]$ ,  $i \in \{1, \ldots, m\}$ ,  $j \in \{1, \ldots, n\}$ , it holds that

$$\mathbf{A}_{(n)} \left( \mathbf{B}_{(m)}(x_{11}, \dots, x_{m1}), \dots, \mathbf{B}_{(m)}(x_{1n}, \dots, x_{mn}) \right) = \mathbf{B}_{(m)} \left( \mathbf{A}_{(n)}(x_{11}, \dots, x_{1n}), \dots, \mathbf{A}_{(n)}(x_{m1}, \dots, x_{mn}) \right).$$
(8.1)

If  $\mathbf{A}_{(n)}$  commutes with  $\mathbf{B}_{(m)}$ , for all  $n, m \in \mathbb{N}$ , then **A** commutes with **B**.

Clearly, the property of commuting is related to dominance, i.e., two aggregation functions commute if and only if they dominate each other. Further, any aggregation function commuting with itself is bisymmetric and vice versa. For two associative aggregation functions, commuting between their binary operations is necessary and sufficient for their commuting in general.

Moreover, commuting as expressed by (8.1) is a special case of the so-called *generalized bisymmetry* equation as introduced and discussed in [4, 5] and plays a key role in consistent aggregation (see, e.g., [3]). Commuting aggregation functions are further relevant, e.g., in utility theory [55], but also in extension theorems for functional equations, e.g., [146]. Moreover, the commuting property is instrumental in the preservation of some property during an aggregation process, in particular of some form of additivity resp. decomposability when aggregating set functions, in particular capacities, also known as fuzzy measures (compare also [25, 28, 147]), or bipolar capacities resp. bi-capacities (see [88, 89, 90, 91, 95]). Bi-capacities extend the concept of capacities, acting

most often on the unit interval, to capacities acting on pairs of disjoint sets taking values on some bipolar scale, usually [-1, 1]:

**Definition 8.2.** Consider some finite universe X and denote by  $\mathcal{Q}(X) = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \cap B = \emptyset\}$  the set of all disjoint pairs of subsets of X.

- A set function  $m: \mathcal{P}(X) \to [0,1]$  is called a *capacity* (or *fuzzy measure*) on X if it fulfills  $m(\emptyset) = 0, m(X) = 1$ , and  $m(A) \le m(B)$  whenever  $A \subseteq B$  (monotonicity).
- A function  $v: \mathcal{Q}(X) \to \mathbb{R}$  is a *bi-capacity* if  $v(\emptyset, \emptyset) = 0$ , and  $A \subseteq B$  implies that  $v(A, C) \leq v(B, C)$  and  $v(C, A) \geq v(C, B)$  for all  $C \in \mathcal{P}(X \setminus B)$ . Furthermore, v is normalized if  $v(X, \emptyset) = 1$  and  $v(\emptyset, X) = -1$ .

Depending on the application setting, several additional properties for capacities, like, e.g., *S*-decomposability [198], *k*-additivity [136, 133, 134], *k*-maxitivity [132], and for bi-capacities, like, e.g., *S*-decomposability [88], *U*-decomposability [162], *C*-decomposability [77] *k*-additivity [90, 162, 77], additivity and of CPT type [88, 89] have been introduced, on the one hand expressing a special structure of the (bi-)capacity but also reducing the complexity necessary for the definition of the (bi-)capacity itself.

Properties like the S-decomposability or U-decomposability of (bi)-capacities can be expressed by some functional equation involving a t-conorm S resp. a uninorm U. Preserving these properties during aggregation processes naturally relates to the property of commuting between the aggregation function imposed and the operation used for expressing the corresponding decomposition property (compare also, e.g., [55, 156]).

Note that *t*-conorms and uninorms belong, as t-norms, to the class of commutative and associative aggregation functions with neutral element. For t-conorms it holds that the neutral element e equals 0, for t-norms it holds that e = 1 and for uninorms we have  $e \in [0, 1[$  [199]. Indeed the neutral element of the uninorm separates the unit interval into a bipolar evaluation scale with a positive [0, e[ and a negative part ]e, 1] (compare also [57]).

### 8.1 Problem statement and results

Commuting aggregation functions are therefore relevant in two-step aggregation procedures and in the preservation of properties during aggregation processes. For special cases of aggregation functions like, e.g., for weighted arithmetic means, results on commuting exist for many years in connection with the problem of consensus functions for probabilities [124], more recently also for t-norms and conorms in connection with generalized utility theory [55]. However, a more general treatment has still been missing. Therefore, the following problems have been investigated in the article "Aggregation operators and commuting" [A13]:

- Reveal as much as possible about the structure of commuting aggregation functions.
- Investigate in particular aggregation functions which are commuting with t-norms, t-conorms resp. uninorms.

#### On the structure of commuting aggregation functions

First of all, it could be shown that there do not exist different aggregation functions which commute with each other and possess both some neutral element:

**Proposition 8.3.** [A13, Proposition 22] Consider two aggregation functions A and B with neutral elements  $e_a$  resp.  $e_b$ . If A commutes with B then  $e_a = e_b$  and therefore also A = B.

Note that this expresses in particular that t-norms, t-conorms, and uninorms never commute with each other, up to the trivial case that, due to their commutativity and associativity, they commute with themselves. Therefore, any aggregation function commuting with any of these functions necessarily may not have a neutral element. For the application setting of a two-step evaluation procedure carried out by two different aggregation functions this means that the experts can express their neutrality w.r.t. one of the steps applied only.

Moreover, it could be shown that commuting between two aggregation functions heavily relates to unary functions being distributive over one of the two aggregation functions involved. On the one hand, such functions can be constructed from commuting aggregation functions, on the other hand they allow to construct commuting operations. Therefore, functions being distributive over an aggregation function had been studied:

For an *n*-ary aggregation function  $\mathbf{A}_{(n)}$  the set of all non-decreasing functions  $f: [0, 1] \to [0, 1]$ distributing with  $\mathbf{A}_{(n)}$  is denoted by  $\mathcal{F}_{\mathbf{A}_{(n)}}$ . Note that, since  $\mathbf{A}_{(1)}$  is the identity mapping,  $\mathcal{F}_{\mathbf{A}_{(1)}}$ is the set of all non-decreasing functions  $f: [0, 1] \to [0, 1]$  which is abbreviated simply by  $\mathcal{F}$ . Therefore,

$$\mathcal{F}_{\mathbf{A}_{(n)}} = \{ f \in \mathcal{F} \mid f(\mathbf{A}_{(n)}(x_1, \dots, x_n)) = \mathbf{A}_{(n)}(f(x_1), \dots, f(x_n)) \}.$$

Evidently,  $\mathcal{F}_{\mathbf{A}} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_{\mathbf{A}_{(n)}}$  denotes the set of all functions  $f \in \mathcal{F}$  that are distributive over the aggregation function  $\mathbf{A}$ . Note that  $\mathcal{F}_{\mathbf{A}_{(n)}}$ ,  $n \in \mathbb{N}$ , as well as  $\mathcal{F}_{\mathbf{A}}$  contain at least the following functions

$$\begin{aligned} \mathbf{0} \colon [0,1] \to [0,1], & x \mapsto 0, \\ \mathbf{1} \colon [0,1] \to [0,1], & x \mapsto 1, \\ \mathrm{id} \colon [0,1] \to [0,1], & x \mapsto x \end{aligned}$$

and are therefore not empty for arbitrary aggregation function A.

In [A13, Section III] several properties of  $\mathcal{F}_{\mathbf{A}}$  and  $\mathcal{F}_{\mathbf{A}_{(n)}}$  have been revealed and examples for important aggregation functions like, e.g., different kinds of means, provided. We briefly quote just a few of the results:

Proposition 8.4. [A13, Propositions 7–9, 11]

- The set  $\mathcal{F}_{\mathbf{A}}$  is maximal in case of lattice polynomials only.
- If  $\mathbf{A}_{(n)}$  commutes with some  $\mathbf{B}_{(m)}$ , then  $f_{d,i,\mathbf{A}_{(n)}}: [0,1] \to [0,1]$  defined, for all  $x \in [0,1]$ , by

$$i\text{-th position} \\ f_{d,i,\mathbf{A}_{(n)}}(x) = \mathbf{B}_{(m)}(d,\ldots,d,\overset{\downarrow}{x},d,\ldots,d)$$

with  $i \in \{1, \ldots, m\}$  and d some idempotent element of  $\mathbf{A}_{(n)}$ , fulfills,  $f_{d,i,\mathbf{A}_{(n)}} \in \mathcal{F}_{\mathbf{A}_{(n)}}$ .

- For a bisymmetric aggregation function  $\mathbf{A}$  it holds that  $\mathcal{F}_{\mathbf{A}_{(n)}}$  is closed under  $\mathbf{A}_{(n)}$ , i.e., for all  $f_i \in \mathcal{F}_{\mathbf{A}_{(n)}}$ ,  $i \in \{1, \ldots, n\}$ , also the function  $g: [0, 1] \to [0, 1]$  defined, for all  $x \in [0, 1]$ , by  $g(x) = \mathbf{A}_{(n)}(f_1(x), \ldots, f_n(x))$  fulfills  $g \in \mathcal{F}_{\mathbf{A}_{(n)}}$ .
- In case **A** is associative, it holds that  $f \in \mathcal{F}_{\mathbf{A}}$  if and only if  $\mathcal{F}_{\mathbf{A}_{(2)}}$ .

#### Operations commuting with bisymmetric aggregation functions

Based on the investigation of distributive functions the structure of operations commuting with some bisymmetric aggregation function could be revealed. The main two results read as follows:

**Proposition 8.5.** [A13, Proposition 18] Let  $\mathbf{A}$  be a bisymmetric aggregation function. Then, for all  $n \in \mathbb{N}$  and all  $f_i \in \mathcal{F}_{\mathbf{A}}$ , the n-ary operation  $B: [0,1]^n \to [0,1]$  defined by

$$B(x_1,\ldots,x_n) = \mathbf{A}(f_1(x_1),\ldots,f_n(x_n))$$

commutes with A.

Note that B need not be an aggregation function in general. If for n = 1,  $f_1 = id \in \mathcal{F}_A$  and for n > 1, the functions  $f_i \in \mathcal{F}_A$ ,  $i \in \{1, ..., n\}$ , are chosen such that

$$B(0,...,0) = \mathbf{A}(f_1(0),...,f_n(0)) = 0,$$
  

$$B(1,...,1) = \mathbf{A}(f_1(1),...,f_n(1)) = 1$$

then B is also an n-ary aggregation function.

In case that  $\mathbf{A}$  is bisymmetric and possesses also a neutral element, i.e., it is also associative and commutative and therefore either a t-norm, a t-conorm, or a uninorm, the above construction turns into a characterization:

**Proposition 8.6.** [A13, Proposition 21] Let  $\mathbf{A}$  be a bisymmetric aggregation function with neutral element e. Then an n-ary operation  $B: [0,1]^n \to [0,1]$  commutes with  $\mathbf{A}$  if and only if there exist  $f_i \in \mathcal{F}_{\mathbf{A}}, i \in \{1,\ldots,n\}$ , such that

$$B(x_1,\ldots,x_n) = \mathbf{A}(f_1(x_1),\ldots,f_n(x_n)).$$

As mentioned earlier in Section 2.1, a full characterization of t-norms is still not available. The same applies to t-conorms and even the more for uninorms. Since the characterization of the set of unary functions distributing with any of these operations heavily depends on the structure of the underlying operation, only special cases of operations have been treated further in [A13], namely continuous t-norms, continuous t-conorms, and particular classes of uninorms. Note that, due to the results quoted above on associative aggregation functions and their set of distributive functions, it suffices to investigate functions f which distribute over a binary operation \* with \* denoting either a t-norm, a t-conorm, or a uninorm, such that  $f \in \mathcal{F}_*$  is equivalent to the fact that f fulfills a *Cauchy* like equation, i.e., for all  $x, y \in [0, 1]$ ,

$$f(x * y) = f(x) * f(y).$$
 (8.2)

For continuous t-conorms (8.2) has been solved by Benvenuti et al. in [15] and as such by duality also for continuous t-norms (note that if T is a t-norm, then S, defined, for all  $x, y \in [0, 1]$ , by S(x, y) = 1 - T(1 - x, 1 - y) is a t-conorm and vice versa). In [A13, Section V] the results on functions distributing over a continuous t-conorm resp. continuous t-norm are summarized and illustrated for basic t-(co)norms.

Section VI in [A13] extensively discusses the case of uninorms. First the general structure of functions f commuting with some uninorm U is discussed showing that the set of idempotent elements of U (notice that  $\{0, e, 1\} \subseteq \mathcal{I}(U)$ ) and the range of some  $f \in \mathcal{F}_U$  play a special role, in particular whether  $e \in \operatorname{Ran}_f$  or not. After these general considerations two important classes of uninorms resp. functions being distributive over uninorms of these classes are investigated. Several examples illustrate the general structure of functions f distributing with such uninorms.

Note that a uninorm U can be interpreted as a combination of some t-norm T and some t-conorm S, i.e.,

$$U_{(n)}(x_1, \dots, x_n) = U_{(2)}(T(\min(x_1, e), \dots, \min(x_n, e)), S(\max(x_1, e), \dots, \max(x_n, e)))$$

with T some t-norm acting on [0, e] and S some t-conorm acting on [e, 1]. Such created uninorms cover a large class of aggregation functions since on the remainder of their domains they can be chosen such that the monotonicity and associativity condition are not violated, but otherwise arbitrarily. However, due to its properties, any uninorm U fulfills

$$\min(x, y) \le U(x, y) \le \max(x, y)$$

whenever  $\min(x, y) \leq e$  and  $e \leq \max(x, y)$  for all  $x, y \in [0, 1]$ , giving rise to the particular classes  $U_{T,S,\min}, U_{T,S,\max}$  of uninorms, the first special subclasses of uninorms discussed in [A13].

Note further that there exists no uninorm which is continuous on the whole domain [73]. Therefore, the second important subclass of uninorms investigated in [A13] are uninorms generated by some additive generator which are continuous on the whole domain up to the points (0, 1) and (1, 0). A uninorm  $U: [0, 1]^2 \rightarrow [0, 1]$  is called a *generated* uninorm with *additive generator* h where  $h: [0, 1] \rightarrow [-\infty, \infty]$  is a monotone bijection such that

$$U(x,y) = h^{-1}(h(x) + h(y)),$$

with convention  $+\infty + (-\infty) = -\infty$ . Note that the neutral element *e* of such a generated uninorm is given by  $h^{-1}(0) = e$ . For both special classes of uninorms, distributive functions  $f \in \mathcal{F}_U$  are investigated and illustrated by several examples as well examples of commuting operations provided in Section VI of [A13].

Chapter 8. Two-step aggregation processes

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## Articles

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- A02. S. Saminger-Platz, E.P. Klement, R. Mesiar. On extensions of triangular norms on bounded lattices. *Indagationes Mathematicae*, vol. 19, no. 1, pp. 135–150, 2008.
- A03. S. Saminger-Platz, C. Sempi. A primer on triangle functions I. Aequationes Mathematicae, vol. 76, pp. 201–240, 2008.

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# On ordinal sums of triangular norms on bounded lattices $\stackrel{\leftrightarrow}{\sim}$

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#### Abstract

Ordinal sums have been introduced in many different contexts, e.g., for posets, semigroups, t-norms, copulas, aggregation operators, or quite recently for hoops. In this contribution, we focus on ordinal sums of t-norms acting on some bounded lattice which is not necessarily a chain or an ordinal sum of posets. Necessary and sufficient conditions are provided for an ordinal sum operation yielding again a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. By such also the structure of the underlying bounded lattice is investigated. Further, it is shown that up to trivial cases there are no ordinal sum t-norms on product lattices in general. © 2006 Elsevier B.V. All rights reserved.

Keywords: Triangular norm; Ordinal sum; Horizontal sum; Bounded lattice

#### 1. Introduction

Triangular norms were originally studied in the framework of probabilistic metric spaces [39–42] aiming at an extension of the triangle inequality and following some ideas of Menger [33]. Later on, they turned out to be interpretations of the conjunction in many-valued logics [1,16–18,22], in particular in fuzzy logics, where the unit interval serves as set of truth values.

Since triangular norms are special compact semigroups, the concept of ordinal sums in the sense of Clifford [5] provided a method to construct new triangular norms from given ones, but also led to the remarkable representation of continuous triangular norms as ordinal sums of isomorphic images of the product and the Łukasiewicz t-norm [31,34]. For more results on triangular norms and ordinal sums see, e.g., [24,28–30].

In [13,15] the unit interval was replaced by some more general structure, i.e., a bounded lattice, stimulating some investigations in topology [14,20,23,37] and logic [12,21]. Therefore, it was quite natural to study triangular norms on bounded lattices [8,10,26,43], including special cases such as discrete chains [32] or the lattice  $L^* = \{(x, y) \in [0, 1]^2 | x + y \leq 1\}$  [11] for which also ordinal sum operations have been provided. It is worth mentioning that ordinal sum operations have also been introduced in the frameworks of, e.g., copulas [35], aggregation operators [9] or general algebraic structures such as hoops [3,4]. As such several types of ordinal sums are known in the literature and since many of these profit from or are based on a close relationship between the ordinal sum operation and the structure of

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the underlying lattice, it seemed quite natural to investigate this relationship in more detail, especially for t-norms on bounded lattices by also taking into account the concept of ordinal sums of posets (see, e.g., [2]).

Therefore, we will focus our considerations on ordinal sum t-norms on bounded lattices which are defined through their restrictions to subintervals of the lattice and the lattice-infimum as such reflecting a kind of plug-in strategy for building new operations. On the other hand we investigate which types of lattices are appropriate candidates for allowing an arbitrary selection of the subintervals as well as an arbitrary selection of the corresponding t-norms but guaranteeing that the ordinal sum operation yields again a t-norm on the lattice. Therefore, we will concentrate in this contribution on ordinal sum t-norms built from summand t-norms only, although other summand operations could also be taken into account (compare also, e.g., [24,30]).

Note that due to the results of Clifford [5], see also [6,19,27], we know that an ordinal sum of semigroups (as introduced in [5]) whose carriers are (bounded) lattices is again a semigroup with a carrier equal to the ordinal sum (in the sense of Birkhoff [2]) of the summand lattices. However, conversely, a straightforward application of Clifford's ordinal sum theorem to subsets of some fixed lattice L requires L to be an ordinal sum of its sublattices. But, and as we will show later, there exist ordinal sum t-norms on bounded lattices which are not an ordinal sum of some of their sublattices, i.e., ordinal sum t-norms on bounded lattices need not be ordinal sums in the sense of Clifford.

We investigate the problem by the following steps. The next section is dedicated to an overview on relevant concepts of ordinal sums, namely, ordinal sums in the sense of Birkhoff, in the sense of Clifford, as well as ordinal sums of t-norms on the unit interval. Section 3 briefly deals with t-norms on bounded lattices and introduces ordinal sum t-norms. In Section 4 we investigate such t-norms with one summand only first on some fixed subinterval then on an arbitrary subinterval revealing necessary and sufficient conditions concerning the underlying lattice. We close this section with a discussion about ordinal sum t-norms on product lattices. Subsequently, we generalize the results to ordinal sum t-norms with arbitrarily many summands in Section 5 and close this contribution by a short summary and further perspectives.

#### 2. On some types of ordinal sums

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#### 2.1. Ordinal sums in the sense of Birkhoff

In [2], Birkhoff provides a definition for building the ordinal sum  $X \oplus Y$  of two disjoint posets X, Y. Due to the associativity of this construction we immediately extend this concept to families of pairwise disjoint posets for some linearly ordered index set  $(I, \preccurlyeq_I)$ ,  $I \neq \emptyset$ . Note that ordinal sums of disjoint posets in the sense of Birkhoff are also referred to as linear sums of posets [7].

**Definition 2.1.** Consider a linearly ordered index set  $(I, \preccurlyeq_I)$ ,  $I \neq \emptyset$  and a family of pairwise disjoint posets  $(X_i, \leqslant_i)_{i \in I}$ . The *ordinal sum*  $\bigoplus_{i \in I} X_i$  is defined as the set  $\bigcup_{i \in I} X_i$  equipped with the following order  $\leqslant$ 

$$x \leqslant y : \Leftrightarrow (\exists i \in I : x, y \in X_i \land x \leqslant_i y) \lor (\exists i, j \in I : x \in X_i \land y \in X_j \land i \prec_I j).$$
(1)

If necessary, we will refer to such ordinal sums as ordinal sums in the sense of Birkhoff explicitly.

Since the order relation for elements from different summand carriers is inherited from the linearly ordered index set, ordinal sums formally minimize the number of incomparable elements of a poset with carrier  $\bigcup_{i \in I} X_i$  which extends the posets  $(X_i, \leq_i)$ . As such ordinal sums of posets are in general not symmetric, i.e.,  $X \oplus Y \neq Y \oplus X$ .

Note that based on the linear order of the index set  $(I, \preccurlyeq_I)$  and the order  $\leqslant$  on  $\bigcup_{i \in I} X_i$  defined by (1) the condition of pairwise disjointness can be even relaxed.

**Lemma 2.2.** Consider some linearly ordered index set  $(I, \preccurlyeq_I)$  and a family of posets  $(X_i, \leqslant_i)_{i \in I}$ . If for all  $i, j \in I$  with  $X_i \cap X_j = A \neq \emptyset$  it holds that  $(A, \leqslant_i) = (A, \leqslant_j)$ ,  $X_i = (X_i \setminus A) \oplus A$ ,  $X_j = A \oplus (X_j \setminus A)$ , and for each  $k \in I$  either  $k \preccurlyeq_I i \prec_I j$  or  $i \prec_I j \preccurlyeq_I k$ , then there exists some index set K and a family of pairwise disjoint posets  $(X_k, \leqslant_k)_{k \in K}$  such that

- for each  $k \in K$  there exists some  $i \in I$  with  $X_k \subseteq X_i$  and
- the ordinal sum  $\bigoplus_{k \in K} X_k$  is isomorphic to the poset  $(\bigcup_{i \in I} X_i, \leq)$  where  $\leq$  is defined by (1).

Note that the strategy just described focusses on the union of the carriers and an order compatible with the order of the underlying posets. Another possible way for introducing ordinal sums of non pairwise disjoint sets is to replace each  $X_i$  by  $(i, X_i)$  and thus creating isomorphic but pairwise disjoint sets for which the ordinal sum construction in the sense of Birkhoff can be applied. We will follow our approach since it does not lead to multiple copies of elements common in several posets. Especially, for two intervals which overlap in at most one point we introduce the following concept of ordinal sums of intervals.

In case that  $(X, \leq x)$  has a smallest and a largest element,  $a_X$  resp.  $b_X$ , we denote X by  $[a_X, b_X] = \{x \in X \mid a_X \leq x \leq x \leq x > b_X\}$  and use notions of other types of intervals accordingly.

**Definition 2.3.** Consider a linearly ordered index set  $(I, \preccurlyeq)$ ,  $I \neq \emptyset$  and a family of intervals  $([a_i, b_i])_{i \in I}$  such that for all  $i, j \in I$  with  $i \prec j$  either  $[a_i, b_i]$  and  $[a_j, b_j]$  are disjoint or  $b_i = a_j$ . The ordinal sum  $\bigoplus_{i \in I} [a_i, b_i]$  is the set  $\bigcup_{i \in I} [a_i, b_i]$  equipped with the order  $\leqslant$  defined by

$$x \leq y : \Leftrightarrow (\exists i \in I : x, y \in [a_i, b_i] \land x \leq y) \lor (\exists i, j \in I : x \leq b_i \land a_i \leq y \land i \prec j).$$

If necessary, we refer to this kind of ordinal sum as ordinal sum of intervals.

So far, we have discussed ordinal sums of posets focussing on the preservation or the construction of an order relation on the union of sets. We now turn to ordinal sums of semigroups and therefore shifting the focus from orders to operations.

#### 2.2. Ordinal sums in the sense of Clifford

In [5], see also [6,19,27], ordinal sums have been introduced in the context of abstract semigroups aiming at a construction of a new semigroup from a given family of semigroups. The basic idea is to extend an ordinally ordered system of non-overlapping semi-groups into a single semigroup whose carrier equals the union of the original carriers.

**Definition 2.4** (*Clifford* [5]). Let  $(I, \preccurlyeq)$ ,  $I \neq \emptyset$  be a linearly ordered index set,  $(X_i)_{i \in I}$  a family of pairwise disjoint sets, and  $(G_i)_{i \in I}$  with  $G_i = (X_i, *_i)$  a family of semigroups. Put  $X = \bigcup_{i \in I} X_i$  and define the binary operation \* on X by

 $x * y = \begin{cases} x *_i y & \text{if } (x, y) \in X_i \times X_i, \\ x & \text{if } (x, y) \in X_i \times X_j \text{ and } i \prec j, \\ y & \text{if } (x, y) \in X_i \times X_j \text{ and } i \succ j. \end{cases}$ 

Then we say that (X, \*) is the *ordinal sum* of all  $(X_i, *_i)_{i \in I}$ . If necessary, we refer to this type of ordinal sum as *ordinal sum in the sense of Clifford*.

**Proposition 2.5** (*Clifford* [5]). With all the assumptions of the previous definition the ordinal sum (X, \*) is also a semigroup, i.e., \* is an associative operation on X.

Note that ordinality in the sense of Clifford refers to the linear order of the index set I involved. The elements of some  $X_i$ ,  $i \in I$ , need not fulfill any special order relation. On the other hand, taking into account that equality is an order relation on any set we immediately can state the following corollary.

**Corollary 2.6.** Any ordinal sum in the sense of Clifford can be expressed as an associative operation on an ordinal sum of a family of sets in the sense of Birkhoff.

The semigroup operation \* on X is linked to the order structure of the index set and therefore also to the order of the deduced ordinal sum in the sense of Birkhoff.

Similar as in the case of ordinal sums of Birkhoff the condition of disjointness can be and has been relaxed in the case of ordinal sums in the sense of Clifford.

**Proposition 2.7** (*Clifford* [5]). Let  $(I, \preccurlyeq)$ ,  $I \neq \emptyset$  be a linearly ordered set,  $(X_i)_{i \in I}$  a family of sets, and  $(G_i)_{i \in I}$  with  $G_i = (X_i, *_i)$  a family of semigroups.

Assume that for all  $i, j \in I$  with  $i \prec j$  the sets  $X_i$  and  $X_j$  are either disjoint or that  $X_i \cap X_j = \{x_{ij}\}$ , where  $x_{ij}$  is both the unit element of  $G_i$  and the annihilator of  $G_j$ , and where for each  $k \in I$  with  $i \prec k \prec j$  we have  $X_k = \{x_{ij}\}$ . Put  $X = \bigcup_{i \in I} X_i$  and define the binary operation \* on X by

$$x * y = \begin{cases} x *_i y & if(x, y) \in X_i \times X_i, \\ x & if(x, y) \in X_i \times X_j \text{ and } i \prec j, \\ y & if(x, y) \in X_i \times X_j \text{ and } i \succ j. \end{cases}$$

Then (X, \*) is a semigroup.

#### 2.3. Ordinal sums of t-norms on the unit interval

We now concentrate on a special class of semigroups with carrier [0, 1], namely t-norms which are commutative semigroups, non-decreasing in each coordinate and having 1 as neutral element.

**Definition 2.8.** Let  $(]a_i, b_i[]_{i \in I}$  be a family of pairwise disjoint open subintervals of [0, 1] and let  $(T_i)_{i \in I}$  be a family of t-norms. Then the *ordinal sum*  $T = (\langle a_i, b_i, T_i \rangle)_{i \in I} : [0, 1]^2 \to [0, 1]$  is given by

$$T(x, y) = \begin{cases} a_i + (b_i - a_i)T_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right) & \text{if } (x, y) \in [a_i, b_i]^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

If necessary, we refer to this type of ordinal sums as ordinal sums of t-norms.

Note that some differences to previous definitions can be stressed:

- The carrier of the ordinal sum is the unit interval [0, 1] which in general need not equal the union of all  $[a_i, b_i], i \in I$  of the family of intervals involved.
- The members of the family of t-norms  $(T_i)_{i \in I}$  are not acting on the subdomains  $[a_i, b_i]$ , but on [0, 1].

According to the fact that all  $[a_i, b_i] \subseteq [0, 1]$  and all  $a_i$  as well as  $b_i$  are ordered by the natural order on  $\mathbb{R}$  an index set  $J, J \neq \emptyset$  and a family of intervals  $[a_j, b_j]$  can be found such that  $[0, 1] = \bigoplus_{j \in J} [a_j, b_j]$  (see also [29]). On each  $[a_j, b_j]$  associative operations  $*_j$  can be defined either as isomorphic, in fact linear, transformations of the corresponding t-norms or by the minimum such that in this case the ordinal sum of t-norms is just an ordinal sum in the sense of Clifford. Note that such an ordinal sum of t-norms in the sense above where each summand t-norm is isomorphic either to the product or the Łukasiewicz t-norm [31,34].

#### 3. Triangular norms on bounded lattices

Extensions of triangular norms on lattices in the framework of fuzzy sets and fuzzy logic (see also [13,15,18]) always require the top resp. bottom element of *L* to play the role of a neutral element resp. of an annihilator. Therefore, we concentrate on bounded lattices *L* only with top and bottom elements, denoted by 1 and 0, respectively, in the sequel.

**Definition 3.1.** Consider some bounded lattice  $(L, \land, \lor, 1, 0)$  and denote by  $\leq_L$  the corresponding lattice order. A binary operation  $T: L^2 \to L$  is called a *triangular norm* (*t-norm*) on L if the following conditions are fulfilled for all  $x, y, z \in L$ :

(i) $T(x, y) = T(y, x)$ ,	(commutativity)
(ii) $T(x, z) \leq T(y, z)$ whenever $x \leq Ly$ ,	(monotonicity)
(iii) T(x, 1) = x,	(neutral element)
(iv) $T(x, T(y, z)) = T(T(x, y), z).$	(associativity)

Note that the structure of the lattice *L* heavily influences which and how many t-norms on *L* can be defined. However, for each lattice *L* there exist at least two t-norms, i.e., the minimum  $T_{\mathbf{M}}^{L}(x, y) = x \wedge y$  and the drastic product

$$T_{\mathbf{D}}^{L}(x, y) = \begin{cases} x \land y & \text{if } 1 \in \{x, y\} \\ 0 & \text{otherwise,} \end{cases}$$

which are always the greatest and smallest possible t-norms on that lattice *L*. Observe that up to the trivial cases when  $|L| \leq 2$ , we always have  $T_{\mathbf{D}}^{L} \neq T_{\mathbf{M}}^{L}$ . In case that |L| = 2, there is a unique t-norm on *L* which is, in fact, the standard boolean conjunction. Finally, if |L| = 1, there is only one binary operation on *L*.

Classical t-norms are then just t-norms on the lattice  $([0, 1], \land, \lor, 1, 0)$ . The intervals  $[a_i, b_i]$ ,  $i \in I$ , used as carriers for the ordinal sum of t-norms are subintervals with bottom element  $a_i$  and top element  $b_i$ . The isomorphic transformations of t-norms are t-norms on these subintervals in the sense of the definition above.

We turn to ordinal sums of t-norms on an arbitrary lattice L. Since each of its summands is described by a sublattice  $L_i$  of L and a t-norm  $T_i$  acting on that sublattice, the sublattice has to possess a largest element acting as the neutral element of the t-norm involved and a smallest element acting as its annihilator. Moreover, monotonicity of the final operation should be provided such that appropriate candidates for such sublattices are intervals  $[a, b] = \{x \in L \mid a \leq Lx \leq Lb\}$  with  $a \leq Lb$  (see also [38]). Therefore, we define ordinal sums of t-norms on lattices in the following way.

**Definition 3.2.** Consider some lattice  $(L, \land, \lor, 0, 1)$  and some linearly ordered index set *I*. Further, let  $(]a_i, b_i[)_{i \in I}$  be a family of pairwise disjoint subintervals of *L* and  $(T^{[a_i,b_i]})_{i \in I}$  a family of t-norms on the corresponding intervals  $([a_i, b_i])_{i \in I}$ . Then the *ordinal sum*  $T = (\langle a_i, b_i, T^{[a_i,b_i]} \rangle)_{i \in I} \colon L^2 \to L$  is given by

$$T(x, y) = \begin{cases} T^{[a_i, b_i]}(x, y) & \text{if } x, y \in [a_i, b_i], \\ x \wedge y & \text{otherwise.} \end{cases}$$
(2)

In the sequel we concentrate on conditions such that an operation  $T: L^2 \to L$  defined by Eq. (2) is a t-norm on the lattice L.

#### 4. Ordinal sums with one summand only

#### 4.1. Ordinal sums with one summand only on some fixed subinterval

Consider some bounded lattice  $(L, \land, \lor, 1, 0)$  and fix some subinterval  $[a, b] = \{x \in L \mid a \leq Lx \leq Lb\}$ , with  $\leq_L$  the lattice order. Further assume a t-norm  $T^{[a,b]}$  on [a, b]. Then  $T: L^2 \to L$  defined by

$$T(x, y) = \begin{cases} T^{[a,b]}(x, y) & \text{if } x, y \in [a, b], \\ x \land y & \text{otherwise.} \end{cases}$$
(3)

is an *ordinal sum* ( $\langle a, b, T^{[a,b]} \rangle$ ) of  $T^{[a,b]}$  on L with one summand only. By this the following questions arise quite naturally:

- For which lattices L is T defined by Eq. (3) a t-norm for arbitrary t-norms  $T^{[a,b]}$  on [a,b]?
- If T defined by Eq. (3) is a t-norm for arbitrary t-norms  $T^{[a,b]}$  on [a, b], what do we know about the structure of the underlying lattice L?
- Do we end up with the same answers to the previous two questions?

#### 4.1.1. From lattices to t-norms

The following lemma provides a partial answer to the first question.

**Lemma 4.1.** Consider some lattice  $(L, \land, \lor, 1, 0)$  and a subinterval [a, b] of L. If L is an ordinal sum of intervals such that  $L = [0, a] \oplus [a, b] \oplus [b, 1]$ , then  $T: L^2 \to L$  defined by Eq. (3) is a t-norm for arbitrary t-norm  $T^{[a,b]}$  on [a, b].



Fig. 1. T-norms on some special lattice (Example 4.2).

**Proof.** Note that due to the structure of the lattice as ordinal sum of intervals, it holds that for all  $x \notin [a, b]$  either  $x \leq a$  or  $b \leq x$  is fulfilled. The latter conditions are equivalent to  $x \wedge a = x$  or  $x \wedge b = b$ , respectively.

If  $[a, b] = \{a, b\}$ , the t-norm  $T^{[a,b]}$  on [a, b] is necessarily  $T_{\mathbf{M}}^{[a,b]}$ , implying  $T = T_{\mathbf{M}}^{L}$ . Therefore, we assume that  $[a, b] \neq \emptyset$ , i.e., there exists some  $u \in L$  such that  $a <_{L} u <_{L} b$ .

Commutativity holds due to the commutativity of  $\wedge$  and  $T^{[a,b]}$ . It is also straightforward to show that 1 is the neutral element of *T*. The preservation of associativity is a direct consequence of Clifford's theorem on ordinal sums of semigroups (see also Proposition 2.7).

Finally, monotonicity is preserved due to the monotonicity of  $T^{[a,b]}$  on [a, b] and of  $\wedge$  on the lattice L in all cases but two, i.e., if  $z \in [a, b]$  and either  $x \in [a, b]$  or  $y \in [a, b]$  with  $x \leq Ly$ . First assume that  $x \in [a, b]$  and  $y \notin [a, b]$ . Since  $y \geq Lx$  it follows that y > Lb, then

$$T(x,z) = T^{[a,b]}(x,z) \leqslant_L T^{[a_i,b_i]}_{\mathbf{M}}(x,z) \leqslant_L z = y \land z = T(y,z).$$

Secondly, if  $x <_L a$  and  $y, z \in [a, b]$  then

$$T(x,z) = x \wedge z = x \leq La \leq LT^{[a,b]}(y,z) = T(y,z).$$

Therefore, T defined by Eq. (3) is a t-norm independently of the choice of  $T^{[a,b]}$  on [a,b].

The previous proposition only provides a sufficient and not a necessary condition for T being a t-norm, as the following example shows.

**Example 4.2.** Consider the bounded lattice  $(L, \land, \lor, 0, 1)$  with  $L = \{0, x, a, u, 1\}$  as shown in Fig. 1, further the subinterval  $[a, b] = [a, 1] = \{a, u, 1\}$  and the operation T as defined by Fig. 1. It can be easily checked that T is defined by Eq. (3) and is, moreover, a t-norm although L is not an ordinal sum of intervals.

#### 4.1.2. From t-norms to the structure of the lattice

Before turning to necessary and sufficient conditions for *T* defined by Eq. (3) being a t-norm we prove some basic lemmata revealing insight into the structure of the underlying lattice. Note that throughout this section we consider some bounded lattice  $(L, \land, \lor, 0, 1)$  and a fixed subinterval [a, b] of *L*.

**Lemma 4.3.** Assume that T defined by Eq. (3) is a t-norm for arbitrary  $T^{[a,b]}$  on [a, b] and choose  $x \in L$  arbitrarily. If there exists some  $u \in [a, b]$  such that  $x \leq Lu$  then  $x \in [0, a]$  or  $x \in [a, b]$ , i.e., x is comparable to a.

**Proof.** Assume that *T* defined by Eq. (3) is a t-norm for arbitrary  $T^{[a,b]}$  on [a,b] and let  $x \in L$  and  $u \in [a,b]$  such that  $x \leq Lu$ , i.e.,  $x \wedge u = x$ . If u = a the proposition is trivially fulfilled, therefore we demand that  $u \in [a, b]$ .

Assume that x is incomparable to a. By such x is not contained in [a, b], i.e.,  $x \notin [a, b]$ . Due to the associativity of T and the fact that T is also a t-norm for  $T^{[a,b]} = T_{\mathbf{D}}^{[a,b]}$  the following equality must hold

$$a \wedge x = T(a, x) = T(T^{[a,b]}(u, u), x) = T(T(u, u), x)$$
  
= T(u, T(u, x)) = T(u, u \land x) = T(u, x) = x,

being equivalent to  $x \leq La$  and contradicting the assumption of incomparability of x to a.  $\Box$ 

Note that the previous lemma is equivalent to the following corollary focusing on the incomparability to *a*.

**Corollary 4.4.** Assume that T defined by Eq. (3) is a t-norm for arbitrary  $T^{[a,b]}$  on [a, b] and let  $x \in L$  arbitrarily. If x is incomparable to a, then x is also incomparable to all  $u \in [a, b]$ .

We now turn to the incomparability with the top element *b* of the subinterval.

**Lemma 4.5.** Assume that T defined by Eq. (3) is a t-norm for arbitrary  $T^{[a,b]}$  on [a, b]. If some  $x \in L$  is incomparable to b then x is incomparable to all  $u \in [a, b]$ .

**Proof.** Consider some lattice  $(L, \land, \lor, 0, 1)$  and a subinterval [a, b]. Assume that *T* defined by Eq. (3) is a t-norm for arbitrary  $T^{[a,b]}$  on [a, b]. Let  $x \in L$  be incomparable to *b* and therefore  $x \notin [a, b]$ .

Suppose that there exists some  $u \in [a, b]$  to which x can be compared. If  $x \leq Lu$ , also  $x \leq Lu \leq Lb$  contradicting the incomparability to b. Therefore, let  $x \geq Lu$  from which we can conclude that

 $b \geq L b \wedge x \geq L b \wedge a = a,$ 

i.e.,  $b \wedge x \in [a, b]$ , even  $b \wedge x \in [a, b]$  since  $b \wedge x = b$  would lead to a contradiction to the incomparability of *b* and *x*. Since *T* is a t-norm also for  $T^{[a,b]} = T^{[a,b]}_{\mathbf{D}}$  and is therefore associative the following equation is fulfilled

$$u = T^{[a,b]}(b, u) = T(b, x \wedge u) = T(b, T(x, u))$$
  
=  $T(T(b, x), u) = T(b \wedge x, u) = T^{[a,b]}(b \wedge x, u) = a$ 

contradicting that  $u \in [a, b]$  and showing that x has to be incomparable to all  $u \in [a, b]$  in case that it is incomparable to b.  $\Box$ 

Note that the previous lemma also implies that if x is incomparable to b and there exists some  $u \in [a, b]$  to which x can be compared, it follows necessarily that u = a.

Lemma 4.1 shows that if the underlying lattice can be described as an ordinal sum of intervals, any ordinal sum with summands according to these intervals is again a t-norm. Corollary 4.4 and Lemma 4.5 already indicate that from the fact that *T* defined by Eq. (3) is a t-norm for some fixed subinterval [a, b] of *L* we can draw some conclusions about the structure of the underlying lattice *L*. The following corollary states under which conditions the lattice *L* even fulfills  $L = [0, a] \oplus [a, b] \oplus [b, 1]$ . It is an immediate consequence of the previous lemmata resp. a consequence of Theorem 4.8 proven later.

**Corollary 4.6.** Consider some lattice  $(L, \land, \lor, 0, 1)$  and a subinterval [a, b]. Assume that T defined by Eq. (3) is a *t*-norm for arbitrary  $T^{[a,b]}$  on [a,b]. If for all  $x \in L$  there exist some  $u \in ]a, b[$  such that x can be compared with u, then  $L = [0, a] \oplus [a, b] \oplus [b, 1]$ .

**Remark 4.7.** Example 4.2 shows that there exist ordinal sum t-norms on lattices *L* which are not ordinal sums of intervals, i.e.,  $L \neq [0, a] \oplus [a, b] \oplus [b, 1]$ . Note that those  $x \in L \setminus [0, a] \oplus [a, b] \oplus [b, 1]$  are incomparable to all  $u \in ]a, b[$ . They are further at most comparable to either *a* or *b*, but not to both at the same time (see also Fig. 2).



Fig. 2. Lattice cases.

#### 4.1.3. Necessary and sufficient conditions

The following theorem provides necessary and sufficient conditions for an ordinal sum being a t-norm and providing insight into the structure of the underlying lattice.

**Theorem 4.8.** Consider some bounded lattice  $(L, \land, \lor, 0, 1)$  and a subinterval [a, b] of L. Then the following are equivalent:

(i) The ordinal sum  $T: L^2 \to L$  defined by Eq. (3) is a t-norm for arbitrary  $T^{[a,b]}$  on [a,b].

(ii) For all  $x \in L$  it holds that

- (a) if x is incomparable to a, then it is incomparable to all  $u \in [a, b[,$
- (b) if x is incomparable to b, then it is incomparable to all  $u \in [a, b]$ .

**Proof.** Consider a bounded lattice  $(L, \land, \lor, 0, 1)$  and a subinterval [a, b] of *L*. The necessity follows directly from Lemma 4.5 and Corollary 4.4.

For proving the sufficiency, the claim is trivially fulfilled if  $[a, b] = \{a, b\}$ , since then  $T = T_{\mathbf{M}}^{L}$ . Therefore, we assume that  $[a, b] \neq \emptyset$ , i.e., there exists some  $u \in L$  such that  $a <_{L} u <_{L} b$ .

Commutativity holds due to the commutativity of  $\wedge$  and  $T^{[a,b]}$ . It is also straightforward to show that 1 is the neutral element of *T*.

*Monotonicity*: Monotonicity is preserved due to the monotonicity of  $T^{[a,b]}$  on [a, b] and the monotonicity of  $\wedge$  on the lattice L in all cases but one, namely if  $x \notin [a, b]$  and  $y \in [a, b]$  with  $x \leq Ly$ . We have to show that  $T(x, z) \leq T(y, z)$  is fulfilled for all  $z \in [a, b]$  and x, y as described just before.

If x < a and  $y, z \in [a, b]$  it holds that

$$T(x,z) = x \wedge z \leq a \wedge z = a \leq T^{[a,b]}(y,z) = T(y,z).$$

In case that x is incomparable to a and therefore to all  $v \in [a, b]$  it follows necessarily that y = b. Moreover, it holds for all  $z \in [a, b]$  that

$$T(x, z) = x \wedge z \leq Lz = T^{[a,b]}(b, z) = T(y, z).$$

Further if x < a or x incomparable to  $a, y \in [a, b]$ , but  $z \notin [a, b]$   $T(x, z) \leq T(y, z)$  due to the monotonicity of  $\wedge$ .

Associativity: Associativity holds trivially if all arguments are either from [a, b] or from  $L \setminus [a, b]$ . For the remaining possibilities we distinguish the following two cases:

*Case* 1: Suppose that exactly one argument involved is from [a, b]. We assume  $x \in [a, b]$  and  $y, z \notin [a, b]$ . If  $y \land z \in [a, b[$ , necessarily  $y \land z = a$ . Then we can conclude that

$$T(x, T(y, z)) = T(x, y \land z) = T(x, a) = a$$
$$= x \land a = x \land y \land z = T(T(x, y), z).$$

All other cases can be shown analogously or are fulfilled due to the associativity of  $\wedge$ .

*Case 2*: Suppose that exactly two elements involved are from the subinterval [a, b]. We choose  $x, y \in [a, b]$  and  $z \notin [a, b]$ . It can be easily verified that in case that either x or y are equal to one of the boundaries of [a, b] associativity of T is fulfilled due to the associativity and monotonicity of  $\wedge$ . Therefore, we assume that  $x, y \in [a, b]$ .

In case that  $x \leq Lz$  or  $y \leq Lz$ , z is comparable to all [a, b] and fulfills necessarily  $z \geq Lb$ . Moreover,

$$T(T(x, y), z) = T(x, y) \land z = T(x, y) = T(x, y \land z) = T(x, T(y, z)).$$

Similarly, if  $x \ge Lz$  or  $y \ge Lz$ , z is comparable to all [a, b] and  $z \le La$  such that

$$T(T(x, y), z) = T(x, y) \land z = z = x \land z = T(x, y \land z) = T(x, T(y, z)).$$

In case that z is incomparable to x and y it holds that  $u \wedge z = a \wedge z$  for all  $u \in [a, b[$ .

$$T(T(x, y), z) = T(x, y) \land z = a \land z$$
  
=  $x \land a \land z = T(x, y \land z) = T(x, T(y, z)).$ 

The remaining cases can be shown analogously such that associativity is proven.  $\Box$ 

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#### 4.2. Ordinal sums with one summand on arbitrary subintervals

The previous results deal with t-norms on some fixed subinterval. How does L look like if T defined by Eq. (3) is a t-norm not only for arbitrary t-norms  $T^{[a,b]}$  but also for arbitrary subintervals [a, b]?

Recall that a bounded poset  $(X_i \leq 0, 1)$  is called a *horizontal sum* of the bounded posets  $((X_i \leq 0, 1))_{i \in I}$  if  $X = \bigcup_{i \in I} X_i$  with  $X_i \cap X_j = \{0, 1\}$  whenever  $i \neq j$ , and  $x \leq y$  if and only if there is an  $i \in I$  such that  $\{x, y\} \subseteq X_i$  and  $x \leq i y$  (compare, e.g., horizontal sums of effect algebras [36]).

**Theorem 4.9.** Let  $(L, \land, \lor, 0, 1)$  be a bounded lattice. Then the following are equivalent:

- (i) for any  $[a, b] \subseteq L$  and any t-norm  $T^{[a,b]}$  the ordinal sum operation T on L given by (3) is a t-norm on L,
- (ii) for all  $x, y \in L$ :  $\{x \land y, x \lor y\} \subseteq \{0, 1, x, y\},\$
- (iii) L is a horizontal sum of chains.

**Proof.** The equivalence of (ii) and (iii) is obvious.

(i) $\Rightarrow$ (ii): Assume that *T* defined by Eq. (3) is a t-norm for an arbitrary subinterval [a, b] and an arbitrary t-norm  $T^{[a,b]}$  on [a, b]. Consider some  $x, y \in L$  being incomparable and further fulfilling  $x \lor y = z \neq 1$ . Applying Corollary 4.4 to [x, 1] we get a contradiction since y is incomparable to x but comparable to  $z \in [x, 1]$  due to  $y \leq Lx \lor y = z$ . Similarly, Lemma 4.5 leads to a contradiction if  $x \land y = z \neq 0$  for some incomparable  $x, y, \in L$ . In case that  $x, y \in L$  are comparable we clearly get  $\{x \land y, x \lor y\} = \{x, y\}$ .

(ii) $\Rightarrow$ (i): Up to the trivial case where a = 0 and b = 1, i.e., [a, b] = L, we get in all other cases that [a, b] is a proper subchain of *L*. If some  $x \in L$  is incomparable to *a* then necessarily  $a \neq 0$  and *x* belongs to another subchain  $L_i$  of *L* such that  $L_i \cap [a, b] = \emptyset$ . Therefore, *x* is also incomparable to all  $u \in [a, b]$ . Analogously, we can argue about some *x* being incomparable to *b* such that condition (ii) of Theorem 4.8 is fulfilled for each subinterval [a, b] of *L* and thus showing that *T* is a t-norm on *L*.  $\Box$ 

The theorem shows under which condition a kind of plug-in strategy for building t-norms can be applied. Provided that the lattice is describable as a disjoint union of chains with some common bottom and top element we can choose any subinterval of the lattice and plug-in arbitrary t-norm on that subinterval such that the final operation is still a t-norm on the original lattice.

**Example 4.10.** Consider the lattice  $L = \{0, a, b, c, 1\}$  as displayed in Fig. 3 then the following cases can be distinguished:

- For any interval [u, v] which consists of at most of two elements  $T^{[u,v]}$  is clearly  $T_{\mathbf{M}}^{[u,v]}$  and as such  $T = T_{\mathbf{M}}^{L}$  trivially a t-norm.
- For any  $[u, v] \supseteq \{a, b\}$  and for any t-norm  $T^{[u,v]}$  on [u, v], T is a t-norm on L.
- The only subintervals leading to contradictory results are [a, 1] and [b, 1]. We illustrate this for the subinterval [a, 1]. There are only two t-norms on [a, 1]. For  $T^{[a,1]} = T_{\mathbf{M}}^{[a,1]}$  also  $T = T_{\mathbf{M}}^{[0,1]}$ . For  $T^{[a,1]} = T_{\mathbf{D}}^{[a,1]}$  we get

$$T(T(c, c), b) = T(a, b) = 0 \neq b = T(c, b) = T(c, T(c, b)),$$

and therefore associativity is violated.



Fig. 3. Lattice discussed in Example 4.10.

Т	(0,0)	(0,1)	(1,0)	(1,1)	
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	
(0,1)	(0,0)	\$	(0,0)	(0,1)	
(1,0)	(0,0)	(0,0)	$\diamond$	(1,0)	
(1,1)	(0,0)	(0,1)	(1,0)	(1,1)	
with $\diamond \in \{(0, 0), (0, 1)\}$					

Table 1 Triangular norms on  $\{0, 1\} \times \{0, 1\}$ 

Example 4.11. Consider the poset

 $L = \{(-1, -1), (1, 1), (-x, 1 - x), (x, x - 1) \mid x \in ]0, 1[\}$ 

equipped with the product order on  $\mathbb{R}^2$ . It is a lattice with top element (1, 1) and bottom element (-1, -1). Moreover, it is a suitable candidate for defining ordinal sum t-norms ( $\langle a, b, T^{[a,b]} \rangle$ ) for arbitrary subintervals [a, b] and arbitrary t-norms  $T^{[a,b]}$  on [a, b], since removing the top and the bottom element leads to two disjoint chains of incomparable elements.

#### 4.3. Ordinal sums on special bounded lattices

Motivated by the last example we may ask whether there are ordinal sum t-norms on product lattices with the order on the product lattice being defined coordinate-wisely. Taking into account Theorem 4.9 we know that subintervals of the product lattice and t-norms on it can be chosen arbitrarily if and only if the product lattice without top and bottom element is a disjoint union of chains. In fact, these are just product lattices which are either isomorphic to a chain, i.e., all factor lattices are singletons up to one which forms a chain, or they are isomorphic to the product lattice  $\{0, 1\} \times \{0, 1\}$ , i.e., all factors are singletons up to two containing exactly two elements.

In the first case any t-norm on the quite trivial product lattice acts as a corresponding t-norm on the chain. In the second case, there exists no subinterval of the product lattice containing more than two elements besides the product lattice itself. Therefore there exist only four different t-norms on  $\{0, 1\} \times \{0, 1\}$  which are displayed in Table 1. Note that the values of T((0, 1), (0, 1)) and T((1, 0), (1, 0)) can be chosen independently of each other. Due to the incomparability of the involved elements monotonicity is not violated.

We have seen by the previous remark that ordinal sum t-norms consisting of a summand with an arbitrary subinterval as well as an arbitrary t-norm on the subinterval are possible just for very trivial product lattices. Therefore, is it possible to build an ordinal sum t-norm on a more complex product lattice taking into account not arbitrary but some special subinterval?

**Proposition 4.12.** Consider some lattices  $(L_i, \wedge_i, \vee_i, 0_i, 1_i)$ ,  $i \in \{1, ..., n\}$ ,  $n \in \mathbb{N}$ ,  $n \ge 2$  with  $|L_i| > 1$  for all i and the corresponding product lattice  $L = \prod_{i=1}^{n} L_i$ . The order on the product lattice is defined coordinate-wisely, i.e.,  $(a_1, ..., a_n) \le L(b_1, ..., b_n)$  if and only if  $a_i \le L_i b_i$  for all  $i \in \{1, ..., n\}$ . An interval  $[a, b] = \prod_{i=1}^{n} [a_i, b_i]$  and an arbitrary t-norm  $T^{[a,b]}$  on that interval lead to an ordinal sum t-norm T on L as defined by Eq. (3) if and only if either [a, b] = L or  $[a, b] = \emptyset$ .

**Proof.** Assume lattices  $(L_i, \wedge_i, \vee_i, 0_i, 1_i)$  as given above and the corresponding product lattice *L* as well as some interval  $[a, b] = \prod_{i=1}^{n} [a_i, b_i]$ . If [a, b] = L or  $]a, b[ = \emptyset$ , the ordinal sum  $T: L^2 \to L$  defined by Eq. (3) is clearly a t-norm for arbitrary  $T^{[a,b]}$  on [a, b]. Note once again that in the latter case the only t-norm on [a, b] is  $T_{\mathbf{M}}^{[a,b]}$  and as such  $T = T_{\mathbf{M}}^{L}$ .

In order to show the necessity assume that [a, b] fulfills  $[a, b] \neq L$  and  $]a, b[ \neq \emptyset$  and that  $T: L^2 \rightarrow L$  defined by Eq. (3) is a t-norm for arbitrary  $T^{[a,b]}$ . Due to the structure of [a, b] there necessarily exists some  $k \in \{1, ..., n\}$ such that  $[a_k, b_k] \neq L_k$  and some  $u = (u_1, ..., u_n) \in ]a, b[$  and that there exist further some  $l, m \in \{1, ..., n\}$  not necessarily different from each other or from k such that the following conditions are fulfilled:

- $0_k <_{L_k} a_k$  or  $b_k <_{L_k} 1_k$  due to the fact that  $[a_k, b_k] \neq L_k$ ,
- $a_l <_{L_l} u_l$  and  $u_m <_{L_m} b_m$  because of u being an interior element of [a, b].

 $b = (b_1, \dots, b_k, \dots, b_l, \dots, b_m, \dots, b_n),$  $u = (u_1, \dots, u_k, \dots, u_l, \dots, u_m, \dots, u_n),$  $a = (a_1, \dots, a_k, \dots, a_l, \dots, a_m, \dots, a_n),$  $x = (x_1, \dots, x_k, \dots, x_l, \dots, x_m, \dots, x_n),$ 

We distinguish different cases for k, l and m and show that for all possibilities we find two elements of L contradicting the incomparability conditions of Theorem 4.8.

 $k \neq l$ : Choose  $x \in L$  such that  $x_k = 0_k$ ,  $x_l = u_l$  and  $x_i = a_i$  for all  $i \in \{1, ..., n\} \setminus \{k, l\}$ . Since  $0_k <_{L_k} a_k \leq_{L_k} u_k$ , as well as  $x_l = u_l >_{L_l} a_l$  and  $u_i \geq_{L_i} a_i = x_i$ , x is incomparable to a but fulfills  $x \leq_L u$  with  $u \in [a, b]$  leading to a contradiction to Theorem 4.8.

 $k = l \neq m$ : We select two elements x and y from L in the following way:

 $x = (a_1, \dots, a_{k-1}, 0_k, a_{k+1}, \dots, a_{m-1}, b_m, a_{m+1}, \dots, a_n),$  $y = (a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_{m-1}, b_m, a_{m+1}, \dots, a_n)$ 

then  $y \in [a, b]$  since  $a_k <_{L_k} b_k$  and  $a_m \leq_{L_m} u_m <_{L_m} b_m = y_m$ . Furthermore,  $x \leq_L y$  although x is incomparable to a because of  $a_m <_{L_m} x_m$  and  $x_k = 0_k <_{L_k} a_k$  and as such contradicting again Theorem 4.8.

k = l = m: Note that in this case  $0_k <_{L_k} a_k <_{L_k} u_k <_{L_k} b_k \leq_{L_k} 1_k$  form a sub-chain of  $L_k$ . Since the number *n* of lattices involved in the product lattice is at least two,  $n \ge 2$ , we know that there exists some further  $j \in \{1, ..., n\}$  such that  $[a_j, b_j] \subseteq L_j$  with  $|L_j| > 1$ .

Assume that  $a_j \neq b_j$ , i.e.,  $a_j <_{L_j} b_j$ . We choose  $x, y \in L$  by

$$x = (a_1, \dots, a_{k-1}, 0_k, a_{k+1}, \dots, a_{j-1}, b_j, a_{j+1}, \dots, a_n),$$
  

$$y = (a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_{j-1}, b_j, a_{j+1}, \dots, a_n),$$

then again  $y \in [a, b], x \leq y$  but x being incomparable to a leading to a contradiction to Theorem 4.8.

If  $a_i = b_i$  then either  $0_i <_{L_i} a_i$  or  $b_i <_{L_i} 1_j$ . In the first case, we build  $x, y \in L$  by

 $x = (a_1, \dots, a_{k-1}, u_k, a_{k+1}, \dots, a_{j-1}, 0_j, a_{j+1}, \dots, a_n),$  $y = (a_1, \dots, a_{k-1}, u_k, a_{k+1}, \dots, a_{j-1}, u_j, a_{j+1}, \dots, a_n),$ 

then  $y \in [a, b]$ ,  $x \leq Ly$  but x being incomparable to a. In the second case, choose  $x, y \in L$  by

$$x = (a_1, \dots, a_{k-1}, u_k, a_{k+1}, \dots, a_{j-1}, 1_j, a_{j+1}, \dots, a_n),$$
  

$$y = (a_1, \dots, a_{k-1}, u_k, a_{k+1}, \dots, a_{j-1}, b_j, a_{j+1}, \dots, a_n),$$

then  $y \in [a, b]$ ,  $x \leq y$  but x being incomparable to b such that both cases contradict Theorem 4.8.

In case that  $b_k <_{L_k} 1_k$ , elements  $x, y \in L$  contradicting the incomparability conditions of Theorem 4.8 can be constructed in an analogous way.  $\Box$ 

Note that the previous proposition just provides insight why the method of building ordinal sum t-norms is, up to some trivial cases, not appropriate for creating t-norms on product lattices. However, there exist several other ways for defining t-norms on such lattices, e.g., [8,25].

**Remark 4.13.** Similar arguments hold for the case of t-norms on the lattice  $(L^*, \land, \lor, 0_{L^*}, 1_{L^*})$  with  $L^* = \{(x_1, x_2) | (x_1, x_2) \in [0, 1]^2 \land x_1 + x_2 \leq 1\}$ , the join and the meet operation defined as follows:

$$(x_1, x_2) \land (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2))$$

$$(x_1, x_2) \lor (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2))$$

for arbitrary  $(x_1, x_2)$ ,  $(y_1, y_2) \in L^*$ , and bottom and top element given by  $0_{L^*} = (0, 1)$  resp.  $1_{L^*} = (1, 0)$ .

As one can see immediately the removal of the bottom and top element from the lattice does not lead to a disjoint union of chains expressing that no ordinal sum t-norm in the sense of Definition 3.2 can be built on  $L^*$ . Moreover, if some subinterval is fixed, elements contradicting the incomparability condition of Theorem 4.8 can in general be found

similar as before in the case of product lattices. Surely there exist other strategies for defining t-norms on  $L^*$  being inspired by a kind of plug-in strategy, i.e., by using transformations and projections of t-norms on  $L^*$  as been carried out in, e.g., [11].

#### 5. Ordinal sums with more summands

Theorems 4.8 and 4.9 deal with ordinal sums with one summand either based on some arbitrary or fixed subinterval. We now extend these results to ordinal sums with more summands (see Definition 3.2) and end up again with a close relationship between the structure of the underlying lattice and the operation defined on that lattice.

**Proposition 5.1.** Consider some bounded lattice  $(L, \land, \lor, 0, 1)$  and some linearly ordered index set  $(I, \preceq)$ ,  $I \neq \emptyset$ . Then the following are equivalent:

(i) The ordinal sum T as defined by Eq. (2) is a t-norm for arbitrary families of pairwise disjoint subintervals  $(]a_i, b_i]_{i \in I}$  and for arbitrary t-norms  $T^{[a_i, b_i]}$  on the corresponding  $[a_i, b_i]$ .

(ii) *L* is a horizontal sum of chains.

**Proof.** Consider some bounded lattice  $(L, \land, \lor, 0, 1)$  and some linearly ordered index set  $(I, \preceq)$ ,  $I \neq \emptyset$ . If *T* defined by Eq. (2) is a t-norm for arbitrary summands, then for sure also for only one arbitrary summand leading to *L* being a horizontal sum of chains due to Theorem 4.9.

On the other hand, suppose that L is a horizontal sum of chains. We want to show that  $T: L^2 \to L$  defined by

$$T(x, y) = \begin{cases} T^{[a_i, b_i]} & \text{if } x, y \in [a_i, b_i], \\ x \land y & \text{otherwise,} \end{cases}$$

is a t-norm for an arbitrary family  $([a_i, b_i])_{i \in I}$  and arbitrary t-norms  $T^{[a_i, b_i]}$  on  $[a_i, b_i]$ . Note that the family of subintervals is such that all members are pairwise disjoint regarding their interiors, i.e., up to boundary elements. Commutativity of T is preserved because of the commutativity of  $\wedge$  in L and  $T^{[a_i, b_i]}$  on  $[a_i, b_i]$ . Furthermore, the neutral element of T is obviously 1, leaving associativity and monotonicity to be proven.

*Monotonicity*: We have to show that whenever  $x \leq Ly$  holds then  $T(x, z) \leq LT(y, z)$  for all  $z \in L$ . Consider some  $x, y \in L$  and assume that  $x \leq Ly$ . If x = 0 or y = 1 the inequality is trivially fulfilled for arbitrary  $z \in L$ .

Therefore, suppose that  $x, y \in L \setminus \{0, 1\}$ . Since  $L \setminus \{0, 1\}$  is a disjoint union of chains and  $x \leq Ly$  by assumption, x and y have to belong to one of theses chains, i.e.,  $x, y \in C^*$  for some chain  $C^* \subseteq L \setminus \{0, 1\}$ .

For all  $z \in L \setminus C^*$  the above inequality is trivially fulfilled since T(x, z) = 0 = T(y, z) due to the incomparability of x and z, resp. y and z.

If  $z \in C^* \cup \{0, 1\}$  monotonicity can be proven analogously to Lemma 4.1 by describing  $C^* \cup \{0, 1\}$  as an ordinal sum of intervals in the following way: Choose  $a = \bigwedge \{a_i \mid i \in I, a_i \in C^*\}$  and  $b = \bigvee \{b_i \mid i \in I, b_i \in C^*\}$ . Then  $C^* \cup \{0, 1\} = [0, a] \oplus [a, b] \oplus [b, 1]$  is an ordinal sum of intervals and *T* acts as an ordinal sum t-norm with one summand on that domain.

Associativity: We have to prove that T(x, T(y, z)) = T(T(x, y), z) is fulfilled for arbitrary  $x, y, z \in L$ . If one of the arguments is equal to 0 or 1 the equality is trivially fulfilled, such that we assume  $x, y, z \in L \setminus \{0, 1\}$  arbitrarily. In case that all three arguments belong to the same chain  $C^*$  associativity can be shown again analogously to Lemma 4.1 by describing  $C^*$  as an ordinal sum of intervals as mentioned before. In case that at least one of the arguments is from some other sub-chain associativity is trivially fulfilled since T(x, T(y, z)) = 0 = T(T(x, y), z).

In case that L is a chain, the previous propositions show immediately that the subintervals and the t-norms on them can be chosen arbitrarily up to the fact that the subintervals have to have pairwise disjoint interiors. In any case T defined by Eq. (2) is surely a t-norm on L. The previous proposition emphasizes again the special role of the so-called horizontal sum in case that the family of subintervals can be chosen arbitrarily.

In case that the family of subintervals is fixed we get the following necessary and sufficient conditions, being in fact a direct generalization of Theorem 4.8.

**Proposition 5.2.** Consider some bounded lattice  $(L, \land, \lor, 0, 1)$ , some linearly ordered index set  $(I, \leq)$ ,  $I \neq \emptyset$  and a family of pairwise disjoint subintervals  $(]a_i, b_i[]_{i \in I}$  of L. Then the following are equivalent:

(i) The ordinal sum  $T: L^2 \to L$  defined by Eq. (2) is a t-norm for arbitrary  $T^{[a_i,b_i]}$  on  $[a_i, b_i]$ .

(ii) For all  $x \in L$  and for all  $i \in I$  it holds that

- (a) if x is incomparable to  $a_i$ , then it is incomparable to all  $u \in [a_i, b_i[,$
- (b) if x is incomparable to  $b_i$ , then it is incomparable to all  $u \in [a_i, b_i]$ .

**Proof.** Fix some bounded lattice  $(L, \land, \lor, 0, 1)$  and a family of pairwise disjoint subintervals  $(]a_i, b_i[)_{i \in I}$  of L w.r.t. some linearly ordered index set I. The necessity of the proposition is a direct result of Theorem 4.8 since by assumption that T is a t-norm for arbitrary  $T^{[a_i,b_i]}$  on  $[a_i, b_i]$  we can choose  $T^{[a_{i_0},b_{i_0}]}$  arbitrarily for some  $i_0 \in I$  and for all other  $i \neq i_0$  we define  $T^{[a_i,b_i]} = T^{[a,b]}_{\mathbf{M}}$  fulfilling the necessary conditions of Theorem 4.8.

For proving the sufficiency assume that *T* is defined by Eq. (2) and all  $x \in L$  fulfill the incomparability conditions for all  $i \in I$ . Commutativity again holds due to the commutativity of  $\wedge$  and all  $T^{[a_i,b_i]}$ ,  $i \in I$  as well as 1 is clearly the neutral element of *T*. Monotonicity is fulfilled in most cases due to the monotonicity of  $\wedge$ . The remaining cases can be proven analogously to the proof of Theorem 4.8. Similarly, regarding associativity, note that T(T(x, y), z) = $T(x, T(y, z)) = x \wedge y \wedge z$  is fulfilled up to the case where at least two elements involved are from the same subinterval  $[a_i, b_i]$ , which in turn can be proven analogously to the proof of Theorem 4.8.  $\Box$ 

**Remark 5.3.** Remember that in case of ordinal sum t-norms on the unit interval with summands based on some fixed family of subintervals  $([a_i, b_i])_{i \in I}$ , the unit interval could be described as an ordinal sum of some family  $([a_j, b_j])_{j \in J}$  being a covering of  $([a_i, b_i])_{i \in I}$ .

In the case of ordinal sum t-norms on some lattice L w.r.t. some fixed family of subintervals  $([a_i, b_i])_{i \in I}$  there exists a family of chains  $(C_j)_{j \in J}$ , pairwise disjoint up to their boundaries, not necessarily forming a covering of  $([a_i, b_i])_{i \in I}$ , such that all  $[a_i, b_i]$  and L itself can be constructed by applying either the ordinal sum or the horizontal sum construction principle to these chains resp. such built intervals consecutively. We would like to illustrate this by the following examples. As such we denote by  $C_1 \dot{\cup}_h C_2$  the horizontal sum built as the disjoint union of  $C_1$  and  $C_2$  with identifying their bottom and top elements.

- The lattice *L* discussed in Example 4.2, see also Fig. 1 is describable by  $C_1 = \{0, a, u, 1\}$  and  $C_2 = \{0, x, 1\}$  through  $L = C_1 \dot{\cup}_h C_2$ .
- Lattice L discussed in Example 4.10, see also Fig. 3 is equal to

 $(\{0, a, c\} \dot{\cup}_h \{0, b, c\}) \oplus \{c, 1\}.$ 

Note that for any t-norm T[0, c] on [0, c], T defined by (3) is a t-norm on L.

• Similarly  $\{0, 1\}^2$  is equivalent to

{(0, 0), (0, 1), (1, 1)} $\dot{\cup}_h$ {(0, 0), (1, 0), (1, 1)}.

Note that building new ordinal or horizontal sum of the previous lattices and keeping the corresponding t-norms on the sublattices leads again to a t-norm on the such constructed lattice.

#### 6. Conclusion

We have investigated the ordinal sum construction principle for building t-norms on bounded lattices. We have shown that the structure of the underlying lattice has quite an influence on how such ordinal sum t-norms can be determined. It turned out that lattices built as ordinal and horizontal sums of chains are the most important and appropriate ones. Further that there exist no t-norms on product lattices constructed through the proposed ordinal sum construction principle except for some trivial cases regarding the t-norms or the lattices involved. Note that we have focused on subintervals of the bounded lattice as the carriers of the summand t-norms. Further investigations of sublattices, not necessarily being a subinterval, as summand carriers are discussed in [38].

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#### On extensions of triangular norms on bounded lattices

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#### ABSTRACT

Smallest and largest possible extensions of triangular norms on bounded lattices are discussed. As such ordinal and horizontal sum like constructions for t-norms on bounded lattices are investigated. Necessary and sufficient conditions for the lattice guaranteeing that the extension is again a t-norm are revealed.

1. INTRODUCTION

Many-valued logics are usually based on a bounded lattice  $(L, \leq, 0, 1)$  of truth values [17,18,25,31,36,37], not necessarily being a chain (a first attempt in this direction is described in [17, Section 15.2], compare [4,12] and also the *paraconsistent logic* in [8]). In such a case, the conjunction is interpreted by some triangular norm on L. The structure of t-norms (fulfilling the intermediate value property) is known in some special cases only (closed real intervals, especially the unit interval, finite chains), see [3,21]. In this paper we are interested in the problem of extending a t-norm acting on a (complete) sublattice of L to a t-norm acting on L, discussing the largest and smallest possible extensions. Although in many of the before mentioned cases the lattices involved tend to be distributive we will not make any additional assumptions on the lattice structure except for its boundedness.

Let  $(L, \leq, 0, 1)$  be a bounded lattice. An operation  $T: L^2 \to L$  which turns L into an ordered abelian semigroup with neutral element 1 will be called a *triangular* norm or, briefly, a *t-norm* on L [10]. In fact, (T, L) is a commutative integral

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*l*-monoid [20] (compare also Examples 1.1–1.4 of commutative semigroups in [16]) if and only if  $T: L^2 \to L$  is a triangular norm on *L* additionally fulfilling  $T(x, y \lor z) = T(x, y) \lor T(x, z)$  for all  $x, y, z \in L$ .

Note that the structure of the lattice L heavily influences which and how many t-norms on L can be defined. However, on each bounded lattice L with |L| > 2 there are at least two t-norms, the minimum  $\wedge$  and the drastic product  $T_{\mathbf{D}}^{L}$  defined by

$$T_{\mathbf{D}}^{L}(x, y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ 0 & \text{otherwise,} \end{cases}$$

which are also the greatest and smallest t-norms on the lattice *L* (if |L| = 2 then  $\wedge$  and  $T_{\mathbf{D}}^{L}$  coincide with the standard boolean conjunction).

Now consider a bounded sublattice  $(S, \leq, a, b)$  of *L* and a t-norm  $T^S: S^2 \to S$  on *S*. We are investigating the strongest and weakest possible extension of  $T^S$  leading to a t-norm *T* on the lattice *L*.

Inspired by ideas of Clifford [7] (in the context of ordinal sums of abstract semigroups) and [14,24,29,34,35] (ordinal sums of t-norms on the unit interval), define the binary operation  $T_{TS}^{L}: L^2 \to L$  by

(1.1) 
$$T_{TS}^{L}(x, y) = \begin{cases} T^{S}(x, y) & \text{if } (x, y) \in S^{2}, \\ x \wedge y & \text{otherwise.} \end{cases}$$

More recently, similar constructions (towers of irreducible hoops [1,6]) have been applied to characterize BL-chains [18]. Evidently,  $T_{TS}^{L}$  is an extension of  $T^{S}$ . Moreover, if  $T_{TS}^{L}$  is a t-norm then it clearly is the strongest t-norm extending  $T^{S}$ .

In the following sections we shall investigate under which conditions, starting from an arbitrary t-norm  $T^S$  on some sublattice S, the extension  $T_{TS}^L$  always is a t-norm on L. We will show that the arbitrariness of the choice of  $T^S$  on S, for  $T_{TS}^L$  to be always a t-norm on L, leads to some restrictions on the structure of the sublattice S. As a consequence also to restrictions on the structure of L, in case that not only any choice of  $T^S$  but also any choice of S shall be admissible. Based on these results we further discuss the strongest extension of families of arbitrary t-norms on some corresponding families of arbitrary sublattices and a few further properties of triangular norms. Finally, we turn to the determination of the smallest possible extension  $W_{TS}^L$  of a t-norm  $T^S$  on a bounded and complete sublattice S.

#### 2. S AND L WITH COMMON BOTTOM AND TOP ELEMENTS

Fix a bounded lattice  $(L, \leq, 0, 1)$  and consider a bounded sublattice  $(S, \leq, a, b)$  of *L* and a t-norm  $T^S: S^2 \to S$  on *S*. Obviously,  $T_{T^S}^L$  as defined by (1.1) is commutative and has neutral element 1. Since *S* is also a sublattice of  $([a, b], \leq, a, b)$  with  $[a, b] = \{x \in L \mid a \leq x \leq b\}$ , we have

$$T_{TS}^L = T_{T_{TS}^{[a,b]}}^L$$

i.e., we may first extend  $T^S$  to [a, b] via (1.1) and repeat the same procedure to extend  $T_{T^S}^{[a,b]}$  to L. Because of

$$T_{TS}^{[a,b]} = T_{TS}^{L}|_{[a,b]^2},$$

a necessary condition for  $T_{TS}^{L}$  to be a t-norm is that  $T_{TS}^{[a,b]}$  is a t-norm. Therefore, without loss of generality we may restrict ourselves first to sublattices of L having the same bottom and top element as L.

**Proposition 2.1.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $(S, \leq, 0, 1)$  a sublattice of *L*. The following are equivalent:

- (i) For all  $(x, y) \in (S \setminus \{1\}) \times (L \setminus S)$  we have  $x \wedge y \in \{0, x\}$  and for all  $(x, y) \in (L \setminus S)^2$  it holds that  $x \wedge y \in S \Rightarrow x \wedge y = 0$ .
- (ii) For each t-norm  $T^S: S^2 \to S$  on S, the operation  $T^L_{TS}$  is a t-norm on L.

**Proof.** To show necessity assume that condition (i) is fulfilled. It is immediate to see that  $T_{TS}^L$  defined by (1.1) is commutative and has neutral element 1. Since for each t-norm *T* we additionally have  $T(x, y) \leq x \wedge y$ , for the monotonicity of  $T_{TS}^L$  it suffices to check if  $T_{TS}^L(x, y) \leq T_{TS}^L(x^*, y)$  for  $x \leq x^*$  in case  $x \notin S$ ,  $x^* \in S$ , and  $y \in S \setminus \{1\}$ .

If  $x^* \neq 1$ , then, because of condition (i),  $x^* \wedge x = x \in \{0, x^*\} \subset S$ , contradicting the assumption  $x \notin S$ . Therefore,  $x^* = 1$ , and we can conclude  $T_{TS}^L(x, y) = x \wedge y \in \{0, y\}, T_{TS}^L(x^*, y) = T^S(x^*, y) = y$ , and, obviously,  $T_{TS}^L(x, y) \leq y = T_{TS}^L(x^*, y)$ .

{0, y},  $T_{T^S}^L(x^*, y) = T^S(x^*, y) = y$ , and, obviously,  $T_{T^S}^L(x, y) \leq y = T_{T^S}^L(x^*, y)$ . For proving the associativity, i.e., T(x, T(y, z)) = T(T(x, y), z), it is obvious that it holds whenever either all  $x, y, z \in S$  or all  $x, y, z \in L \setminus S$  as well as if  $0 \in \{x, y, z\}$ or  $1 \in \{x, y, z\}$ . Therefore, let us first assume that  $x, y \notin S$  and  $z \in S \setminus \{0, 1\}$ . Then,  $x \wedge z \in \{0, z\}, y \wedge z \in \{0, z\}$ , and if  $x \wedge y \in S$  then  $x \wedge y = 0$  such that in all cases it follows

$$T(x, T(y, z)) = x \wedge y \wedge z = T(T(x, y), z).$$

Similar arguments can be applied in case  $x, z \notin S$  and  $y \in S \setminus \{0, 1\}$  resp.  $y, z \notin S$  and  $x \in S \setminus \{0, 1\}$ .

In case that only one element involved is element of the sublattice, let us first assume that  $x \notin S$  and  $y, z \in S \setminus \{0, 1\}$ , then  $x \land y \in \{0, y\}$ ,  $x \land z \in \{0, z\}$ ,  $y \land z \in S \setminus \{1\}$ , and  $x \land T(y, z) \in \{0, T(y, z)\}$ . Then the following can be argued: If  $x \land y \land z = 0$  then associativity is trivially fulfilled. Otherwise, if  $x \land y \land z = y \land z > 0$ , such that T(x, y) = y and therefore T(T(x, y), z) = T(y, z) and  $T(x, T(y, z)) = x \land T(y, z) = T(y, z)$  since  $T(y, z) \leq y = x \land y \leq x$ . Analogous arguments can be applied for proving the case  $z \notin S$  and  $x, y \in S \setminus \{0, 1\}$ . Finally, it remains to show associativity for  $y \notin S$  and  $x, z \in S \setminus \{0, 1\}$ . If  $x \land y \land z = 0$ , then again it is trivially fulfilled. Otherwise, necessarily  $x \land y = x$  and  $y \land z = z$ , such that

$$T(x, T(y, z)) = T(x, y \land z) = T(x, z) = T(x \land y, z) = T(T(x, y), z).$$

Conversely, assume that  $T_{TS}^{L}$  is t-norm for each t-norm  $T^{S}$  on S and fix some  $(x, y) \in (S \setminus \{1\}) \times (L \setminus S)$  such that  $x \land y \notin \{0, x\}$ .

If  $x \land y \in S$  consider the t-norm  $T^S$  on S given by

$$T^{S}(u, v) = \begin{cases} 0 & \text{if } (u, v) \in ([0, x] \cap S)^{2} \setminus \{(x, x)\}, \\ u \wedge v & \text{otherwise,} \end{cases}$$

and we obtain  $T_{TS}^{L}(T_{TS}^{L}(x, x), y) = x \land y \neq 0 = T_{TS}^{L}(x, T_{TS}^{L}(x, y)).$ If  $x \land y \notin S$  then

$$T_{T_{\mathbf{D}}S}^{L}(T_{T_{\mathbf{D}}S}^{L}(x,x), y) = 0 \neq x \land y = T_{T_{\mathbf{D}}S}^{L}(x, T_{T_{\mathbf{D}}S}^{L}(x, y)).$$

Moreover, fix some  $(x, y) \in (L \setminus S)^2$  such that  $x \wedge y = z \in S \setminus \{0\}$ . Then

$$T_{T_{\mathbf{D}}S}^{L}(T_{T_{\mathbf{D}}S}^{L}(x, y), z) = T_{T_{\mathbf{D}}S}^{L}(z, z)$$
  
= 0 < z =  $T_{T_{\mathbf{D}}S}^{L}(x, z) = T_{T_{\mathbf{D}}S}^{L}(x, T_{T_{\mathbf{D}}S}^{L}(y, z)).$ 

Since in all cases the associativity is violated, this proves that (ii) implies (i).  $\Box$ 

Note that condition (i) equivalently expresses that for all  $x \in S \setminus \{1\}$  and for all  $y \in L \setminus S$  either  $x \wedge y = 0$  or  $x \leq y$  is fulfilled and for all  $x \in S \setminus \{0, 1\}$  and all  $y, z \in L \setminus S$ , such that  $x \leq y$  and  $x \leq z$ , also  $y \wedge z \in L \setminus S$ .

#### 3. EXTENSION OF T-NORMS ON AN ARBITRARY INTERVAL

First consider a fixed subinterval [a, b] of a bounded lattice  $(L, \leq, 0, 1)$  and an arbitrary t-norm  $T^{[a,b]}$  on [a,b]. We want to check under which conditions on the interval [a, b] (and on the lattice L) the operation  $T_{T^{[a,b]}}^L$  constructed by (1.1) will be a t-norm on L (see also Theorem 4.8 in [33]). Recall that the open interval [a, b] is defined by  $[a, b] \setminus \{a, b\}$ . Moreover, if  $[a, b] = \emptyset$ , then  $T^{[a,b]} = \wedge$  and also  $T_{T^{[a,b]}}^{L} = \wedge$  clearly being a t-norm on L, so without loss of generality we can restrict in the sequel to subintervals [a, b] with  $]a, b[\neq \emptyset$  only.

**Proposition 3.1.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and [a, b] a subinterval of *L* with  $]a, b[ \neq \emptyset$ . The following are equivalent:

- (i)  $\{x \in L \mid \exists y \in ]a, b[: x \leq y \text{ or } x \geq y\} = [0, a] \cup [a, b] \cup [b, 1].$ (ii) For each t-norm  $T^{[a,b]}: [a,b]^2 \to [a,b]$  on [a,b], the operation  $T^L_{T^{[a,b]}}$  is a t-norm on L.

**Proof.** Note that condition (i) expresses that whenever some lattice element x, not necessarily from [a, b], is comparable to an interior element of the subinterval, then it must be comparable to both boundaries of the subinterval, i.e., to a as well as to b.

Now assume that condition (i) is fulfilled. For the monotonicity of  $T_{T[a,b]}^{L}$  it suffices to check the case  $x \notin [a, b], \{y, z\} \subseteq [a, b]$  and  $x \leq y$ . If x < a then

$$T_{T[a,b]}^{L}(x,z) = x \wedge z \leq a \wedge z = a \leq T^{[a,b]}(y,z) = T_{T[a,b]}^{L}(y,z).$$

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If x and a are incomparable, then  $x \notin [0, a] \cup [a, b] \cup [b, 1]$ , i.e., x is incomparable to all  $u \in [a, b]$ , such that y = b and

$$T^{L}_{T^{[a,b]}}(x,z) = x \wedge z \leq z = T^{[a,b]}(y,z) = T^{L}_{T^{[a,b]}}(y,z)$$

Similarly, the associativity of  $T_{T[a,b]}^{L}$  can be checked case by case. We illustrate the case of  $x \in L$  being incomparable to *a* and  $y, z \in [a, b]$ . We prove the associativity for this case by a series of properties:

Since x is incomparable to a, it is incomparable to all  $u \in [a, b]$  and therefore it follows that, necessarily  $x \land v \notin [a, b]$  for all  $v \in [a, b]$ .

Further, for all  $v \in [a, b]$  it holds that  $x \wedge v = x \wedge a$ : If v = a, this is obviously true, therefore assume that  $v \in ]a, b[$ . In order to guarantee that  $x \wedge v \notin [a, b]$  it follows from  $x \wedge v \leqslant v < b$  that necessarily  $x \wedge v \leqslant a$  and further  $x \wedge v \leqslant a \wedge x \leqslant v \wedge x$  due to the monotonicity and the idempotency of  $\wedge$ .

Based on these properties we can now conclude for the associativity of some  $x \in L$  being incomparable to *a* and some  $y, z \in [a, b]$  with  $y \land z \in [a, b]$ :

$$T(x, T(y, z)) = x \wedge T(y, z) = x \wedge a = x \wedge a \wedge z$$
$$= T(x \wedge a, z) = T(x \wedge y, z) = T(T(x, y), z).$$

If y = z = b, then  $T(x, T(y, z)) = T(x, b) = x \land b = T(x \land b, b) = T(T(x, y), z)$ which concludes the case. The remaining cases for showing the associativity of *T* can be checked analogously, thus showing that (i) implies (ii). Clearly,  $T_{T[a,b]}^{L}$  is commutative and has 1 as a neutral element.

Conversely, let  $x \in L$  be incomparable to *b* and comparable to some  $u \in ]a, b[$ , i.e.,  $x \ge u$ , which implies  $b \land x \in ]a, b[$ . Then

$$u = T_{\mathbf{D}}^{[a,b]}(b,u) = T_{T_{\mathbf{D}}^{[a,b]}}^{L}(b,x \wedge u)$$
  
=  $T_{T_{\mathbf{D}}^{[a,b]}}^{L}(b, T_{T_{\mathbf{D}}^{[a,b]}}^{L}(x,u)) = T_{T_{\mathbf{D}}^{[a,b]}}^{L}(T_{T_{\mathbf{D}}^{[a,b]}}^{L}(b,x),u)$   
=  $T_{T_{\mathbf{D}}^{[a,b]}}^{L}(b \wedge x, u) = T_{\mathbf{D}}^{[a,b]}(b \wedge x, u) = a$ 

contradicting  $u \in ]a, b[$  and showing that the incomparability of x to b implies the incomparability to all elements of ]a, b[. In complete analogy we can show that the incomparability of x to a implies the incomparability to all elements of ]a, b[ by proving a contradiction to T(T(u, u), x) = T(u, T(u, x)) in case that x is comparable to some  $u \in ]a, b[$ , i.e., in case  $x \leq u$  and choosing  $T^{[a,b]} = T_{\mathbf{D}}^{[a,b]}$ , thus completing the proof that (ii) implies (i).  $\Box$ 

**Corollary 3.2.** Let  $(L, \leq, 0, 1)$  be a bounded lattice,  $(S, \leq, a, b)$  a bounded sublattice of L and  $T^S: S^2 \to S$  a t-norm on S. Assume that for each  $(x, y) \in (S \setminus \{b\}) \times ([a, b] \setminus S)$  we have  $x \land y \in \{a, x\}$ , that for each  $(x, y) \in ([a, b] \setminus S)^2$  it follows that  $x \land y \in S$  implies  $x \land y = a$ , and that, in case  $[a, b] \neq \emptyset$ , condition (i) in Proposition 3.1 holds. Then  $T_{TS}^L$  is a t-norm on L.

Note that the conditions in Proposition 3.1 heavily depend on the interval [a, b] and on the lattice *L*. Now we look for conditions on *L* only guaranteeing that for each subinterval each t-norm can be extended to a t-norm on *L*.

Recall that a bounded poset  $(X, \leq, 0, 1)$  is called a *horizontal sum* of the bounded posets  $((X_i, \leq_i, 0, 1))_{i \in I}$  if  $X = \bigcup_{i \in I} X_i$  with  $X_i \cap X_j = \{0, 1\}$  whenever  $i \neq j$ , and  $x \leq y$  if and only if there is an  $i \in I$  such that  $\{x, y\} \subseteq X_i$  and  $x \leq_i y$  (compare, e.g., horizontal sums of effect algebras [32]). A non-trivial example of a bounded lattice which is a horizontal sum of chains is given by

$$L = \{(-1, -1), (1, 1), (-x, 1 - x), (x, x - 1) \mid x \in ]0, 1[\}$$

equipped with the product order on  $\mathbb{R}^2$ .

**Proposition 3.3** ([33]). Let  $(L, \leq, 0, 1)$  be a bounded lattice. The following are equivalent:

- (i) *L* is a horizontal sum of chains.
- (ii) For all  $x, y \in L$ :  $\{x \land y, x \lor y\} \subseteq \{0, x, y, 1\}$ .
- (iii) For each subinterval [a,b] of L and each t-norm  $T^{[a,b]}:[a,b]^2 \to [a,b]$ on [a,b], the operation  $T^L_{T^{[a,b]}}$  is a t-norm on L.

4. T-NORMS ON HORIZONTAL SUMS OF CHAINS

Until now we have considered one subinterval of the bounded lattice  $(L, \leq, 0, 1)$  only. However, Proposition 3.3 can be generalized to a system of pairwise disjoint intervals.

**Definition 4.1.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and *I* some index set. Further, let  $(]a_i, b_i[]_{i \in I}$  be a family of pairwise disjoint subintervals of *L* and  $(T^{[a_i, b_i]})_{i \in I}$ a family of t-norms on the corresponding intervals  $[a_i, b_i]$ . Then the  $\land$ -extension  $T: L^2 \rightarrow L$ , denoted  $T = (\langle [a_i, b_i], T_i \rangle)_{i \in I}$ , is given by

(4.1) 
$$T(x, y) = \begin{cases} T^{[a_i, b_i]}(x, y) & \text{if } (x, y) \in [a_i, b_i]^2, \\ x \land y & \text{otherwise.} \end{cases}$$

**Corollary 4.2** ([33]). Let  $(L, \leq, 0, 1)$  be a bounded lattice. The following are equivalent:

- (i) L is a horizontal sum of chains.
- (ii) For each family of pairwise disjoint subintervals (]a<sub>i</sub>, b<sub>i</sub>[)<sub>i∈I</sub> of L and for each family of t-norms (T<sup>[a<sub>i</sub>,b<sub>i</sub>]</sup>)<sub>i∈I</sub> on the corresponding intervals [a<sub>i</sub>, b<sub>i</sub>] the ∧-extension (⟨[a<sub>i</sub>, b<sub>i</sub>], T<sub>i</sub>⟩)<sub>i∈I</sub> defined by (4.1) is a t-norm on L.

As an immediate consequence of Corollary 4.2 we obtain the ordinal sum construction [14,24,29,34,35] for t-norms on the unit interval (see also [22] for

a full investigation of the relationship with the concept of Clifford [7]) and on any chain.

Moreover, applying consecutively Proposition 2.1 and Corollary 4.2 we obtain the following general result:

**Proposition 4.3.** Let  $(L, \leq, 0, 1)$  be a bounded lattice which is a horizontal sum of some family  $(L_k)_{k \in K}$  of chains, and let  $(S_i, \leq, a_i, b_i)_{i \in I}$  be a family of bounded sublattices such that the sets  $S_{i,k}^*$  defined by  $S_{i,k}^* = ]a_i, b_i[ \cap L_k$  are pairwise disjoint.

If for each  $i \in I$  and for each  $(x, y) \in (S_i \setminus \{b_i\}) \times ([a_i, b_i] \setminus S_i)$  we have  $x \land y \in \{a_i, x\}$ , then for each family  $(T^{S_i})_{i \in I}$  of t-norms on the corresponding sublattices  $S_i$  the function  $T^L : L^2 \to L$  given by

$$T(x, y) = \begin{cases} T^{S_i}(x, y) & if(x, y) \in {S_i}^2\\ x \wedge y & otherwise, \end{cases}$$

is a t-norm on L.

**Proof.** If for all  $i \in I$  and all  $k \in K$ ,  $S_{i,k}^* = \emptyset$ , it follows that  $S_i = \{0, 1\}$  for all  $i \in I$  and therefore  $T = \wedge$ . Otherwise, assume that for some  $i \in I$  and for some  $k \in K$ ,  $S_{i,k}^* \neq \emptyset$ . It remains to show that for all  $(x, y) \in ([a_i, b_i] \setminus S_i)^2$  it follows that  $x \wedge y \in S_i$  implies  $x \wedge y = a_i$ .

In case  $a_i \neq 0$  or  $b_i \neq 1$ , for any  $(x, y) \in (S_{i,k}^* \setminus S_i)^2$  it follows that  $x \land y \in \{x, y\}$ such that  $x \land y \notin S_i$ . In case  $a_i = 0$  and  $b_i = 1$  it might be that  $x \in S_{i,k}^* \setminus S_i \subset [a_i, b_i] \setminus S_i$  and  $y \in S_{i,l}^* \setminus S_i \subset [a_i, b_i] \setminus S_i$  with  $k \neq l$ , however, then  $x \land y = 0 = a_i \in S_i$  follows immediately. Because of Proposition 2.1 we can further conclude that  $T|_{([a_i, b_i] \cap L_k)^2}$  is a t-norm on the bounded sublattice (subinterval)  $([a_i, b_i] \cap L_k = L_k$  and otherwise  $[a_i, b_i] \cap L_k = [a_i, b_i]$ .  $\Box$ 

The special structure of horizontal sums allows us to represent each t-norm as the  $\wedge$ -extension of its restrictions to the summands:

**Proposition 4.4.** Let  $(L, \leq, 0, 1)$  be a bounded lattice which is a horizontal sum of some family  $(L_k)_{k \in K}$  of bounded lattices. Then a binary operation  $T : L^2 \to L$  is a *t*-norm on *L* if and only if  $T = (\langle L_k, T |_{L_k^2} \rangle)_{k \in K}$ .

Proposition 4.4 allows us to give a representation of certain types of t-norms on horizontal sums of chains, thus generalizing the representation theorem [19,24,29, 30] of continuous t-norms on the unit interval and of t-norms on finite chains [26] fulfilling the intermediate value property by means of additive generators. Recall that a t-norm  $T: L^2 \rightarrow L$  on L fulfills the *intermediate value property* if it satisfies that for all  $x, y, z \in L$  with  $x \leq y$  and for each  $u \in [T(x, z), T(y, z)]$ , there is a  $v \in [x, y]$  such that T(v, z) = u.

**Corollary 4.5.** Let  $(L, \leq, 0, 1)$  be a bounded lattice which is a horizontal sum of bounded chains  $(C_k)_{k \in K}$ , where each chain  $C_k$  is either finite or isomorphic to

a non-trivial compact subinterval of the real line. If  $T: L^2 \to L$  is a t-norm on L fulfilling the intermediate value property, then there exist a family  $(S_i, \leq, a_i, b_i)_{i \in I}$  of subchains of L satisfying the hypothesis of Proposition 4.3 and a family of continuous, strictly decreasing real-valued functions  $(t_i: S_i \to [0, \infty])_{i \in I}$  satisfying  $t_i(b_i) = 0$  such that for each  $(x, y) \in L^2$ 

$$T(x, y) = \begin{cases} t_i^{-1}(\min(t_i(x) + t_i(y), t_i(a_i))) & \text{if } (x, y) \in S_i^2 \\ x \land y & \text{otherwise.} \end{cases}$$

**Proof.** From Proposition 4.4 we know  $T = (\langle C_k, T |_{C_k^2} \rangle)_{k \in K}$ . If  $C_k$  is finite then the t-norm  $T|_{C_k^2}$  fulfills the intermediate value property, and the existence of the subchains  $S_i$  of  $C_k$  and the functions  $t_i : S_i \to [0, \infty]$  with the desired properties follows from [26,27]. If  $C_k$  is isomorphic to a non-trivial compact subinterval of the real line, then  $(C_k, T|_{C_k^2})$  is isomorphic to an *I*-semigroup [13], and the result follows from [29] (compare also [21,30]).  $\Box$ 

Due to the well-known structure of t-norms on the real unit interval and on finite chains fulfilling the intermediate value property (in the latter case such t-norms are uniquely determined by their non-trivial idempotent elements), we are able to construct all such t-norms on bounded lattices which are horizontal sums of non-trivial compact subintervals of the real line and finite chains [21,26,29].

As an immediate consequence, the number of t-norms on a finite lattice L which is a horizontal sum of chains which fulfill the intermediate value property is given by  $2^{|L|-2}$  (compare the result of [26] for divisible t-norms on finite chains). Observe that the minimum  $\land$  always satisfies the hypothesis of Corollary 4.5 (the index set Ibeing empty in this case) whereas, e.g., for  $L = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$  the drastic product  $T_{\mathbf{D}}^{L}$ does not fulfill the intermediate value property.

Further note that in Corollary 4.5 the hypothesis that the infinite chains involved there be isomorphic to non-trivial compact subintervals of the real line cannot be relaxed, in general. Take the chain  $(L, \leq)$  with  $L = ]0, 1[^2 \cup \{(0, 0), (1, 1)\}$ and  $\leq$  being the lexicographic order. Then the function  $T: L^2 \rightarrow L$  given by  $T((x_1, y_1), (x_2, y_2)) = (x_1y_1, x_2y_2)$  is a t-norm which fulfills the intermediate value property and is not representable as a  $\wedge$ -extension of some t-norm possessing an additive generator since the semigroup  $(L \setminus \{(0, 0)\}, T|_{(L \setminus \{(0, 0)\})^2})$  is Archimedean and cancellative, but has anomalous pairs (e.g., (0.5, 0.6) and (0.5, 0.5)), compare [2,28] and see [15] for the corresponding notions and related results. Moreover, take the chain  $(L, \leq)$  with  $L = \{-1\} \cup [0, 1]$  and  $\leq$  the standard order on the real line. Then the function  $T: L^2 \rightarrow L$  defined by

$$T(x, y) = \begin{cases} x + y - 1 & \text{if } x + y \ge 1, \\ -1 & \text{otherwise,} \end{cases}$$

is an Archimedean t-norm fulfilling the intermediate value property but with no additive generator.

#### 5. FURTHER PROPERTIES

A lattice  $(L, 0, 1, \leq)$  equipped with some t-norm  $T: L^2 \to L$  is called *divisible* [20] if for all  $x, y \in L$  with  $y \leq x$  there exists some  $z \in L$  such that y = T(x, z)(compare also the natural ordering of groupoids in [15]). Note that the divisibility of a t-norm T is, in general, a weaker property than its intermediate value property as the following example shows.

**Example 5.1.** Consider the bounded lattice  $(L, \leq, 0, 1)$  with  $L = \{0, 1, a, b, c, d, e\}$  as displayed and define  $T : L^2 \to L$  by

Т	0	а	b	С	d	е	1	1
0	0	0	0	0	0	0	0	
а	0	0	0	0	0	0	а	$e \sim c$
b	0	0	0	0	0	а	b	
С	0	0	0	а	а	b	с	
d	0	0	0	а	0	а	d	$\forall a$
е	0	0	а	b	а	d	е	
1	0	а	b	с	d	е	1	0

Then *T* is a t-norm on *L* which is divisible but does not fulfill the intermediate value property (it suffices to choose for x = b, y = e and z = c).

However, for chains the intermediate value property and divisibility coincide. Moreover, the intermediate value property of a t-norm T on a lattice L which is a horizontal sum is equivalent to the intermediate value property of T restricted to the summands of L. Thus the requirement of the intermediate value property for Tin Corollary 4.5 can be relaxed to divisibility.

Further note that in many-valued logics, the algebraic background is mostly a residuated lattice  $(L, 0, 1, \leq, *, \rightarrow)$ , where  $*: L^2 \rightarrow L$  is a t-norm on *L*. The t-norm \* modelling the conjunction operator and the operator  $\rightarrow : L^2 \rightarrow L$  modelling the implication operator form adjoint operators linked to each other by the adjunction relation

 $x * y \leq z$  if and only if  $x \leq y \rightarrow z$ 

for all  $x, y, z \in L$ . Note that a such residuated lattice is divisible if and only if

$$(5.1) \qquad x * (x \to y) = x \land y$$

for all  $x, y \in L$  [20]. Observe that (5.1) is preserved by ordinal sums. However, this is not more true for horizontal sums of chains. To see this, consider any finite bounded lattice  $(L, 0, 1, \leq)$ . Choosing  $* = \wedge$ , then  $(L, 0, 1, \leq, \wedge, \rightarrow)$  is residuated if and only if it is distributive, i.e., it does not contain as a sublattice a non-trivial 5-point horizontal sum [5,23]. Thus the only non-trivial horizontal sum of chains which yields a residuated lattice  $(L, 0, 1, \leq, \wedge, \rightarrow)$  is the four-point diamond lattice

which is also the product of two chains with two elements. Note that for this lattice all t-norms *T* fulfill the intermediate value property and therefore divisibility, but (5.1) is only fulfilled for  $* = \wedge$ .

#### 6. WEAKEST POSSIBLE EXTENSION

It was mentioned already in the beginning that the  $\wedge$ -extension of some t-norm  $T^S$  on some bounded sublattice *S* as given by (1.1) is the strongest possible extension of  $T^S$ . We have shown that guaranteeing that the  $\wedge$ -extension is a t-norm independent of the choice of the t-norm  $T^S$  (and the sublattice *S*) demands rather restrictive conditions on the underlying lattice. Quite different is the situation when looking for the weakest possible extension of  $T^S$  on some single sublattice *S*.

**Definition 6.1.** Let  $(L, \leq, 0, 1)$  be a bounded lattice,  $(S, \leq, a, b)$  a complete and bounded sublattice, and  $T^S$  a t-norm on the corresponding sublattice *S*. Then define  $T^{S \cup \{0,1\}} : (S \cup \{0,1\})^2 \to (S \cup \{0,1\})$  by

(6.1) 
$$T^{S \cup \{0,1\}}(x, y) := \begin{cases} x \land y & \text{if } 1 \in \{x, y\}, \\ 0 & \text{if } 0 \in \{x, y\}, \\ T(x, y) & \text{if } (x, y) \in S^2 \end{cases}$$

Further define  $W_{TS}^L: L^2 \to L$  by

(6.2) 
$$W_{T^S}^L(x, y) := \begin{cases} x \land y & \text{if } 1 \in \{x, y\}, \\ T^{S \cup \{0, 1\}}(x^*, y^*) & \text{otherwise,} \end{cases}$$

with  $x^* = \sup\{z \mid z \le x, z \in S \cup \{0, 1\}\}.$ 

**Lemma 6.2.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and assume some complete, bounded sublattice  $(S, \leq, a, b)$ . Let  $T^S$  be a t-norm on the corresponding sublattice S. Then  $T^{S \cup \{0,1\}} : (S \cup \{0,1\})^2 \to (S \cup \{0,1\})$  defined by (6.1) is a t-norm on  $S \cup \{0,1\}$ . Moreover, it is the unique t-norm extension of  $T^S$  from S to  $S \cup \{0,1\}$ .

**Proof.** In case  $\{0, 1\} \subseteq S$ , then  $T^{S \cup \{0,1\}} = T^S$ . Moreover, clearly this "extension" is unique. For all other cases, it is immediate that  $T^{S \cup \{0,1\}} = T^{S \cup \{0,1\}}_{T^S}$ , i.e.,  $T^{S \cup \{0,1\}}$  coincides with the strongest possible extension provided by means of (1.1) such that indeed  $T^{S \cup \{0,1\}}$  is a t-norm. For any extension T' of  $T^S$  to the sublattice  $S \cup \{0,1\}$  which is also a t-norm it holds that  $T'(x, y) = T^S(x, y)$  for any  $(x, y) \in S^2$ . Moreover,  $T'(x, 0) = T'(0, x) = 0 = T^{S \cup \{0,1\}}(x, 0) = T^{S \cup \{0,1\}}(0, x)$  for any  $x \in S \cup \{0,1\}$ , and  $T'(x, 1) = T'(1, x) = x = T^{S \cup \{0,1\}}(x, 1) = T^{S \cup \{0,1\}}(1, x)$  for any  $x \in S \cup \{0,1\}$ , showing that  $T^{S \cup \{0,1\}}$  is the unique and as such the weakest and strongest possible t-norm extension of  $T^S$  on  $S \cup \{0,1\}$ .

**Proposition 6.3.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and assume some complete, bounded sublattice  $(S, \leq, a, b)$ . Let  $T^S$  be a t-norm on the corresponding sublattice S. Then  $W_{TS}^L: L^2 \to L$  defined by (6.2) is a t-norm on L.

**Proof.** First note that in case some x is smaller or incomparable to all elements of S, then  $x^* = 0$ . If x is greater than some element in S, then  $x^* \in S$  since S is a complete sublattice. Moreover, if  $x \in S$ , then  $x^* = x \in S$ . Since in any case  $x^* \leq x$  it is guaranteed that  $W_{T^S}^L$  is well defined.

Moreover, for any  $x, y \in L \setminus \{0, 1\}$  it holds that  $W_{TS}^L(x, y) \in S \cup \{0, 1\}$  and therefore  $W_{TS}^L(x, y)^* = W_{TS}^L(x, y)$ . It is immediate to see that  $W_{TS}^L$  has neutral element 1 and that it is symmetric.

Let us next focus on its monotonicity. Therefore, assume some  $x, x', y \in L$  such that  $x \leq x'$  and let us show  $T(x, y) \leq T(x', y)$ . Since  $x \leq x'$ , also  $x^* \leq (x')^*$ . Whenever  $1 \in \{x, y\}$ , monotonicity is trivially fulfilled. Therefore, assume that x' = 1 but  $x \neq 1$  and  $y \neq 1$ , then

$$W_{TS}^{L}(x, y) = T^{S \cup \{0, 1\}} (x^*, y^*) \leq x^* \wedge y^* \leq y = W_{TS}^{L}(1, y).$$

And, finally, for all other  $x, x', y \in L$  it holds that

$$W_{T^{S}}^{L}(x, y) = T^{S \cup \{0, 1\}} (x^{*}, y^{*}) \leq T^{S \cup \{0, 1\}} ((x')^{*}, y^{*}) = W_{T^{S}}^{L} (x', y).$$

It remains to prove associativity, i.e.,  $W_{TS}^L(x, W_{TS}^L(y, z)) = W_{TS}^L(W_{TS}^L(x, y), z)$ for all  $x, y, z \in L$ . Whenever all  $x, y, z \in S$ ,  $1 \in \{x, y, z\}$ , or  $0 \in \{x^*, y^*, z^*\}$ , this holds immediately. However, for all remaining cases, we have  $W_{TS}^L(x, y) = T^{S \cup \{0,1\}}(x^*, y^*) = T^{S \cup \{0,1\}}(x^*, y^*)^*$ , such that

$$W_{TS}^{L}(W_{TS}^{L}(x, y), z) = T^{S \cup \{0,1\}}(T^{S \cup \{0,1\}}(x^{*}, y^{*}), z^{*})$$
  
=  $T^{S \cup \{0,1\}}(x^{*}, T^{S \cup \{0,1\}}(y^{*}, z^{*})) = W_{TS}^{L}(x, W_{TS}^{L}(y, z))$ 

proving associativity and thus that  $W_{TS}^L$  is indeed a t-norm on L.

**Proposition 6.4.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and assume some complete, bounded sublattice  $(S, \leq, a, b)$ . Let  $T^S$  be a t-norm on the corresponding sublattice S. Then  $W_{TS}^L: L^2 \to L$  defined by (6.2) is the smallest possible t-norm extension of  $T^S$  on L.

**Proof.** Assume that T' is a t-norm extension of  $T^S$  on L. For all  $(x, y) \in (S \cup \{0, 1\})^2$ ,  $T'(x, y) = W_{T^S}^L(x, y)$ . Next, consider that either  $x \notin S \cup \{0, 1\}$  or  $y \notin S \cup \{0, 1\}$ , then  $x^* \leq x$  and  $y^* \leq y$ , and further

$$T'(x, y) \ge T'(x^*, y^*) = T^{S \cup \{0,1\}}(x^*, y^*) = W_{TS}^L(x, y)$$

such that  $W_{T^S}^L$  is indeed the smallest possible t-norm extension of  $T^S$  on L.  $\Box$ 

So far we have considered one complete sublattice *S* of the bounded lattice  $(L, \leq, 0, 1)$  only. Next, we aim at a generalization in case of families of complete sublattices and corresponding t-norms.

**Definition 6.5.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and *I* some index set. Further, let  $(S_i, \leq, a_i, b_i)_{i \in I}$  be a family of complete and bounded sublattices of L such that the family  $(]a_i, b_i[])_{i \in I}$  consists of pairwise disjoint subintervals of L. Finally, let  $(T^{S_i})_{i \in I}$  be a family of t-norms on the corresponding sublattices  $S_i$ . Then define  $W^L_{TS_i}: L^2 \to L$  by

(6.3) 
$$W_{T^{S_i}}^L(x, y) := \begin{cases} x \land y & \text{if } 1 \in \{x, y\}, \\ T^{S_i \cup \{0, 1\}}(x_i^*, y_i^*) & \text{otherwise,} \end{cases}$$

with  $x_i^* = \sup\{z \mid z \leq x, z \in S_i \cup \{0, 1\}\}$  and define  $W: L^2 \to L$  by

(6.4) 
$$W(x, y) := \sup_{i \in I} W_{T^{S_i}}^L(x, y).$$

Note that, by definition, W is a symmetric and monotone operation on L which has neutral element 1. However, further restrictions on the family of sublattices have to be applied in order to guarantee that W is indeed an extension of arbitrary t-norms  $T^{S_i}$  on the sublattices  $S_i$ .

**Proposition 6.6.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and I some index set. Further, let  $(S_i, \leq, a_i, b_i)_{i \in I}$  be a family of complete sublattices of L such that the family  $(]a_i, b_i[)_{i \in I}$  consists of pairwise disjoint subintervals of L. Further assume that for all  $i, j \in I$  with  $i \neq j$  it holds that

- (i) *if*  $x \in S_i$  *then*  $x_i^* \notin S_i \setminus \{a_i, b_i\}$ , *i.e.*,  $x_i^* \in \{0, a_i, b_i\}$ , (ii) if  $x \in S_j \setminus \{b_j\}$  and  $x_i^* = a_i$ , then  $(a_j)_i^* \ge a_i$ , and (iii) if  $x \in S_j \setminus \{b_j\}$  and  $x_i^* = b_i$ , then  $(a_j)_i^* = b_i$ .

Then for all t-norms  $T^{S_i}$  on  $S_i$  and for all t-norms  $T^{S_j}$  on  $S_j$  with  $i \neq j$  it holds that  $W^L_{T^{S_i}}(x, y) \leq T^{S_j}(x, y)$  for all  $(x, y) \in S^2_j$  and  $W^L_{T^{S_j}}(x, y) \leq T^{S_i}(x, y)$  for all  $(x, y) \in S_i^2$ , *i.e.*,

$$W_{TS_i}^L|_{S_i}^2 \leq T^{S_j}$$
 and  $W_{TS_i}^L|_{S_i}^2 \leq T^{S_i}$ .

Moreover, W given by (6.4) is a monotone and symmetric extension of each  $T^{S_i}$ , *i.e.*,  $W|_{S^2} = T^{S_i}$  for all  $i \in I$ , which has neutral element 1.

**Proof.** Without loss of generality fix some t-norms  $T^{S_i}$ ,  $T^{S_j}$  on  $S_i$  resp.  $S_j$  with  $i, j \in I, i \neq j$ , and let  $(x, y) \in S_j^2$ . Then  $x_i^*, y_i^* \in \{0, a_i, b_i\}$ . If  $x_i^* = 0$  or  $y_i^* = 0$ , it follows immediately that  $W_{T^{S_i}}^L(x, y) = 0 \leq T^{S_j}(x, y)$ . If  $x_i^* = b_i$  or  $y_i^* = b_i$ , then  $(a_i)_i^* = b_i$  such that

$$W_{TS_i}^L(x, y) \leq b_i \leq a_j \leq T^{S_j}(x, y).$$

Finally, for  $x_i^* = y_i^* = a_i$ , necessarily  $(a_j)_i^* \ge a_i$ , such that we can conclude

$$W_{\tau S_i}^L(x, y) = a_i \leqslant (a_j)_i^* \leqslant a_j \leqslant T^{S_j}(x, y).$$

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Therefore, for all  $j \in I$  and for all  $(x, y) \in S_i^2$ ,

$$\sup_{i \in I, i \neq j} W_T^L(x, y) \leqslant T^{S_j}(x, y)$$

and moreover, since  $W_{T^{S_j}}^L|_{S_j^2} = T^{S_j}$ ,  $W(x, y) = \sup_{i \in I} W_{T^{S_i}}^L(x, y) = T^{S_j}(x, y)$ showing that W is indeed an extension of  $T^{S_j}$ .  $\Box$ 

Further note that the supremum of arbitrary t-norms on a lattice L need not be a t-norm in general, compare also [11]. However, for particular and important classes of lattices the operation W as defined by (6.4) is associative, i.e., is a t-norm.

**Example 6.7.** Let  $(L_i, \leq, 0_i, 1_i)$ ,  $i \in \{1, ..., n\}$ ,  $n \in \mathbb{N}$ , be arbitrary complete and bounded lattices and consider their product lattice  $L = \prod_{i=1}^{n} L_i$ . Then for each  $i \in \{1, ..., n\}$ ,  $S_i = \{(0_1, ..., x_i, ..., 0_n) \mid x_i \in L_i\}$  is a complete and bounded sublattice of *L*. Moreover, for each t-norm  $T_i$  on  $L_i$ , the function  $T^{S_i} : S_i^2 \to S_i$  defined by

$$T^{S_i}(\mathbf{x}, \mathbf{y}) = (0_1, \dots, T_i(x_i, y_i), \dots, 0_n)$$

denotes a t-norm on  $S_i$ . Therefore,  $W: L^2 \to L$  as defined by (6.4) can be computed as

$$W(\mathbf{x}, \mathbf{y}) = \sup_{i=1,...,n} W_{TS_i}^L(\mathbf{x}, \mathbf{y}) = (T_1(x_1, y_1), \dots, T_n(x_n, y_n))$$

and is a t-norm on L for arbitrary t-norms  $T_i$  on  $L_i$ .

**Example 6.8.** Let  $(L, \leq, 0, 1)$  be a bounded lattice. Further, let  $(]a_i, b_i[]_{i \in I}$  be a family of pairwise disjoint, non-empty subintervals of L with  $(I, \leq)$  a linearly ordered index set such that

- $(\{[a_i, b_i[ | i \in I\}) \cup \{\{1\}\}\}$  forms a partition of *L* and
- whenever  $i \prec j$  then  $x \leq y$  for all  $x \in [a_i, b_i]$  and for all  $y \in [a_j, b_j]$ ,

i.e., *L* is a so-called ordinal sum of partially ordered sets  $([a_i, b_i[, \leq), i \in I], and (\{1\}, \leq), see e.g. [9]. Let$ *J*be a finite subset of*I* $, i.e., <math>J = \{i_1, \ldots, i_n\} \subseteq I$  for some  $n \in \mathbb{N}$ , such that  $i_1 < i_2 < \cdots < i_n$  and as a consequence  $a_{i_1} < a_{i_2} < \cdots < a_{i_n}$ . Additionally define  $a_{i_{n+1}} := 1$ .

Finally, let  $(T^{[a_{i_j}, b_{i_j}]})_{i_j \in J}$  be a family of t-norms on the corresponding intervals  $[a_{i_j}, b_{i_j}]$ . Then,  $([a_{i_j}, b_{i_j}], \leq)_{i_j \in J}$  forms a family of complete and bounded sublattices of *L* for which the requirements of Proposition 6.6 hold such that *W* defined by (6.4) can be computed as

$$W(x, y) = \begin{cases} x \land y & \text{if } 1 \in \{x, y\}, \\ T^{[a_{i_j}, b_{i_j}]}(x, y) & \text{if } (x, y) \in [a_{i_j}, b_{i_j}]^2, \\ x \land y \land b_{i_j} & \text{if } a_{i_j} \le x \land y < a_{i_{j+1}} \text{ and } b_{i_j} \le x \lor y < 1, \\ 0 & \text{if } x < a_{i_1} \text{ or } y < a_{i_1}. \end{cases}$$

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Moreover, W is associative, i.e., a t-norm on L.

**Remark 6.9.** Note that for J = I, it holds that  $a_{i_1} = 0$  and, for all  $i_j \in J$ ,  $b_{i_j} = a_{i_{j+1}}$  and therefore  $x \land y \land b_{i_j} = x \land y$  whenever  $x \land y \in [a_{i_j}, b_{i_j}]$  and  $b_{i_j} \leq x \lor y$ . As a consequence, for J = I, the weakest extension W and the strongest extension T as defined by (4.1) of t-norms ( $T^{[a_i,b_i]})_{i \in I}$  on corresponding intervals  $[a_i, b_i]$  coincide.

In case  $J \subsetneq I$ , always  $W \neq T$  such that the present example provides another way of obtaining t-norms on chains, in particular on [0, 1], which extend t-norms  $(T^{[a_i,b_i]})_{i\in I}$  on corresponding intervals  $[a_i, b_i]$ . Note further that for L = [0, 1], the weakest extension W is right-continuous whenever all  $(T^{[a_i,b_i]})_{i\in I}$  are rightcontinuous.

In case of chains the previous result can even be strengthened.

**Proposition 6.10.** Let  $(L, \leq, 0, 1)$  be a chain. Further, let  $(]a_i, b_i[]_{i \in I}$  be a family of pairwise disjoint, non-empty subintervals of L and  $(T^{[a_i,b_i]})_{i \in I}$  a family of *t*-norms on the corresponding subintervals with  $(I, \leq)$  a linearly ordered index set. Then W defined by (6.4) is associative, i.e., a *t*-norm on L.

**Proof.** From Proposition 6.6 we can conclude that *W* is a monotone and symmetric extension of each  $T^{[a_i,b_i]}$  which has neutral element 1. Next choose arbitrary  $x, y, z \in L$ . In case  $1 \in \{x, y, z\}$  the associativity of *W* holds trivially, therefore assume that  $1 \notin \{x, y, z\}$ . In case  $x \wedge y \wedge z \in [a_i, b_i]$  for some  $i \in I$  we can conclude that

$$W(W(x, y), z) = T^{[a_i, b_i]}(T^{[a_i, b_i]}(x \land b_i, y \land b_i), z \land b_i)$$
  
=  $T^{[a_i, b_i]}(x \land b_i, T^{[a_i, b_i]}(y \land b_i, z \land b_i))$   
=  $W(x, W(y, z)).$ 

If  $m = x \land y \land z \in L \setminus \bigcup_{i \in I} [a_i, b_i]$ , then for all  $i \in I$  such that  $b_i < m$  it holds that

$$W_{T^{[a_i,b_i]}}^L(W_{T^{[a_i,b_i]}}^L(x,y),z) = W_{T^{[a_i,b_i]}}^L(x,W_{T^{[a_i,b_i]}}^L(y,z)) = b_i.$$

If  $a_i > m$ , then  $W_{T^{[a_i,b_i]}}^L(W_{T^{[a_i,b_i]}}^L(x, y), z) = W_{T^{[a_i,b_i]}}^L(x, W_{T^{[a_i,b_i]}}^L(y, z)) = 0$ . As a consequence  $W(W(x, y), z) = W(x, W(y, z)) = \sup\{b_i | b_i < m\}$ .  $\Box$ 

Proposition 6.10 can further be extended to horizontal sums of chains.

**Corollary 6.11.** Let  $(L, \leq, 0, 1)$  be a bounded lattice which is a horizontal sum of some family  $(L_k)_{k \in K}$  of chains. Further, let  $(]a_i, b_i[]_{i \in I}$  be a family of pairwise disjoint, non-empty subintervals of L and  $(T^{[a_i,b_i]})_{i \in I}$  a family of t-norms on the corresponding intervals  $[a_i, b_i]$ . Then, for any  $k \in K$ ,  $W|_{L_k^2}$  is a t-norm on  $L_k$  and of the form as described in Proposition 6.10. In case  $x \in L_k \setminus \{1\}$  and  $y \in L_l \setminus \{1\}$ with  $k \neq l$ , W(x, y) = 0, and in case  $1 \in \{x, y\}$ ,  $W(x, y) = x \land y$ , such that W is also a t-norm on L.
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Survey paper

# A primer on triangle functions I

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**Summary.** This primer aims at providing an overview of existing concepts and facts about triangle functions as they have been presented in [41]. Moreover, it contains new results on triangle functions and proofs for results not easily available. In this first part we present the most important classes of triangle functions, based on the recent notions of semicopula, quasicopula, as well as the more traditional ones of t-norm, copula and (generalized) convolution. We close this part by listing some basic results needed for the applications (inequalities, aspects of stability and invariance of subspaces) and outlining a few open questions.

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# 1. Introduction

Triangle functions constitute an important class of binary operations on a subspace of distribution functions. From a historical point of view, they were introduced by Šerstnev in [47, 48] in his definitive formulation of the triangle inequality in probabilistic metric spaces (see for a historical introduction to these spaces [35]).

The aim of the present primer is to collect and select known results about these functions, to provide proofs for results not easily available and to add new results; in this way we hope to help the reader to find her/his way in the existing literature, and to provide a handy reference when needed. More than twenty years have elapsed since the publication of the book *Probabilistic Metric Spaces* by Schweizer and Sklar [41], in which a whole chapter, the seventh, was devoted to triangle functions; the notes added by the authors in the second edition of this fundamental reference only point to corrections and to works published in the meantime. The present one seems to be a good time for writing a work like this one, since, on the one hand, the revived interest in the theory of probabilistic metric and normed spaces has necessarily brought triangle functions to the fore; on the other hand, the paper [36] by Schweizer seems to be the harbinger of new applications in a vast field such as that of fuzzy sets and related areas. In order to keep the length within reasonable bounds we split this primer into two parts of which the present one mainly deals with the known classes of triangle functions. In the second part we intend to discuss duality, conjugate transforms, functional equations and inequalities for triangle functions, in particular dominance.

The first part of this primer is devoted to a thorough introduction of basic notions and principles; triangle functions in general and particular classes of these constitute the middle and main part of this primer. Finally, a few selected topics related to triangle functions will be discussed.

#### 2. Preliminaries and notation

In this section we collect some of the preliminaries that are needed in the sequel and fix the notation.

**Definition 2.1.** A distribution function (briefly a d.f.) F is a function from the extended reals  $\overline{\mathbb{R}}$  into [0,1] such that

- (a) it is increasing<sup>1</sup>;
- (b) it is left-continuous on  $\mathbb{R}$ ;
- (c)  $F(-\infty) = 0$  and  $F(\infty) = 1$ .

The set of all d.f.'s will be denoted by  $\Delta$ . The subset of  $\Delta$  consisting of proper d.f.'s, namely of those elements F such that  $\ell^+F(-\infty) = F(-\infty) = 0$  and  $\ell^-F(+\infty) = F(+\infty) = 1$  will be denoted by  $\mathcal{D}$ . Here, for a function  $\varphi : \mathbb{R} \to \mathbb{R}$ , we have used the notations

$$\ell^- \varphi(t) := \lim_{\substack{s \to t \\ s < t}} \varphi(s) \quad \text{and} \quad \ell^+ \varphi(t) := \lim_{\substack{s \to t \\ s > t}} \varphi(s).$$

A distance distribution function (briefly, d.d.f.) is a d.f. F such that F(0) = 0. The set of all d.d.f.'s will be denoted by  $\Delta^+$ , while  $\mathcal{D}^+ := \mathcal{D} \cap \Delta^+$  will denote the set of proper d.d.f.'s.

For a d.f., or a d.d.f., F the set of its points of discontinuity will be denoted by D(F). It is well-known that D(F) is, at most, countable,  $\operatorname{card}(D(F)) \leq \aleph_0$ . A d.f. or a d.d.f. F may be decomposed as the sum of a continuous function  $F_c$  and of a discrete function  $F_d$ , i.e., for all  $x \in \overline{\mathbb{R}}$ ,

$$F(x) = F_c(x) + F_d(x).$$

Among d.f.'s the following will be met again and again:

$$\varepsilon_a(x) := \begin{cases} 0, & x \le a, \\ 1, & x > a, \end{cases}$$

 $<sup>\</sup>overline{1}$  The reader should be warned that by the term *increasing* we mean increasing in the weak sense, often termed *non-decreasing*.

where a is any number in  $[-\infty, +\infty[$ , while for  $a = +\infty$ ,

$$\varepsilon_{\infty}(t) := \begin{cases} 0, & t < +\infty, \\ 1, & t = +\infty. \end{cases}$$

The d.f.  $\varepsilon_a$  belongs to  $\Delta^+$  if, and only if, *a* is positive,  $a \ge 0$ . In the sequel, we refer to functions of type  $\varepsilon_a$  as step functions.

The following family of d.f.'s in  $\Delta^+$  will also be needed; for  $s \in [0, 1[,$ 

$$V_s(x) := \begin{cases} 0, & x \le 0, \\ s, & x \in ]0, +\infty[, \\ 1, & x = +\infty. \end{cases}$$
(2.1)

The elements of  $\Delta$  are partially ordered by the usual pointwise order

 $F \leq G$  if, and only if,  $F(x) \leq G(x)$  for all  $x \in \overline{\mathbb{R}}$ .

In this order  $\varepsilon_{-\infty}$  is the maximal element, while  $\varepsilon_{\infty}$  is the minimal element. The restriction to  $\Delta^+$  of the order just introduced on  $\Delta$  has now  $\varepsilon_0$  as the

maximal element, while the minimal element is still  $\varepsilon_{\infty}$ .

Moreover, if F is in  $\Delta$  and A is a subset of  $\overline{\mathbb{R}}$ , then

$$\sup\{F(x) \mid x \in A\} = F(\sup\{x \mid x \in A\})$$

because of the left-continuity of F.

The supremum of any set of d.f.'s is again a d.f., whereas the infimum need not be, since left-continuity might not be preserved.

For a d.f.  $F \in \Delta$  its (left-continuous) quasi-inverse  $F^{\wedge}$  is defined by

$$F^{\wedge}(x) := \sup\{t \mid F(t) < x\}.$$

The sets  $\Delta$ ,  $\Delta^+$ ,  $\mathcal{D}$  and  $\mathcal{D}^+$  can all be made into metric spaces by the introduction of several topologically equivalent distances. As is traditional, we shall select the *Sibley metric* (see [49]), which is called the *modified Lévy metric* in [41].

**Definition 2.2.** If F and G be d.f.'s, i.e.,  $F, G \in \Delta$ , and h is in [0, 1], denote by (F, G; h) the condition

$$\forall x \in \left] -\frac{1}{h}, \frac{1}{h} \right[ \quad F(x-h) - h \le G(x) \le F(x-h) + h.$$

The Sibley distance is the function  $d_S: \Delta \times \Delta \to [0,1]$  defined by

$$d_S(F,G) = \inf\{h \mid both \ (F,G;h) \text{ and } (G,F;h) \text{ hold}\}.$$
(2.2)

That (2.2) defines a *bona fide* metric on  $\Delta$  was proved in [49] (see also [34]). We keep denoting by  $d_S$  the restriction of the Sibley metric to  $\Delta^+$ , since no possible

confusion may arise. It was proved in [49] that  $d_S$  metrizes the topology of *weak* convergence for d.f.'s in  $\Delta$ : a sequence  $(F_n)_{n \in \mathbb{N}}$  of d.f.'s or of d.d.f.'s, converges weakly to a d.f. (respectively to a d.d.f.) F if

$$\lim_{n \to +\infty} F_n(x) = F(x)$$

at every point  $x \in \mathbb{R}$  at which F is continuous. Notice that this definition requires that also the limit function F is a d.f. or, respectively a d.d.f..

Although it is not the "right" metric on  $\Delta$ , we shall mention the *Lévy metric*  $d_L$ , which is defined on  $\mathcal{D}$  by

$$d_L(F,G) := \inf\{h > 0 : both \ (F,G;h) \text{ and } (G,F;h) \text{ hold}\}$$
(2.3)

where, for F and G in  $\Delta$ , (F, G; h) denotes the condition

$$\forall x \in \mathbb{R} \quad F(x-h) - h \le G(x) \le F(x+h) + h.$$

See, for instance, [26] for a proof of the fact that  $d_L$  is indeed a metric on  $\Delta$ and that it metrizes the weak convergence of d.f.'s belonging to  $\mathcal{D}$ . The spaces  $(\mathcal{D}, d_L)$  and  $(\mathcal{D}^+, d_L)$  are not compact, while so are both  $(\Delta, d_S)$  and  $(\Delta^+, d_S)$ .

Moreover, it follows immediately form their respective definitions that, for any two d.f.'s F and G in  $\Delta$ ,

$$d_S(F,G) \le d_L(F,G).$$

# 3. Operations on the range or on the domain of (distance) distribution functions

We adopt from [41, Section 7.1] the definitions of the sets of operations  $\mathfrak{T}$  and  $\mathfrak{L}$ . The first set of operations,  $\mathfrak{T}$ , deals with operations on the unit interval, which is the range of (distance) distribution functions, whereas the second class deals with operations on the extended positive real line,  $\mathbb{R}_+ := [0, +\infty]$ , which is the domain of every d.d.f.. However, in order to take into consideration also recent advances, we modify the notation of [41]. In the language that has come into use after the publication of the book [41], the elements of  $\mathfrak{T}$  are called *semicopulæ* (see [8, 9]) or *conjunctors* (see [7, 13, 33]).

**Definition 3.1.** A *semicopula* is a function  $S : [0,1]^2 \to [0,1]$  that satisfies the following two conditions:

- (S1) S is increasing in each place, viz., for every  $s \in [0, 1]$ , the functions  $t \mapsto S(t, s)$  and  $t \mapsto S(s, t)$  are increasing;
- (S2) for every  $t \in [0, 1]$ , S(t, 1) = S(1, t) = t.

We shall denote by S (rather than  $\mathfrak{T}$ ) the set of all semicopulas. Note that a semicopula need be neither commutative nor associative. Moreover, because of isotony and the fact that 1 serves as neutral element, any semicopula  $S \in S$  has also 0 as its unique null element.

Vice versa, the set of all binary operations  $S^*$  on [0,1] that are increasing in each place and which have 0 as the neutral element will be denoted by  $S^*$ . We will refer to its elements as *co-semicopulas*.

Further we distinguish the following subclasses of semicopulas:

**Definition 3.2.** A quasi-copula is a function  $Q : [0, 1]^2 \rightarrow [0, 1]$  that satisfies the following conditions:

- (Q1) Q is increasing in each place;
- (Q2) Q satisfies the Lipschitz condition, i.e., for all  $a, b, c, d \in [0, 1]$ ,

$$|Q(a,b) - Q(c,d)| \le |a - c| + |b - d|;$$
(3.1)

(Q3) for every  $t \in [0, 1]$ , Q(t, 1) = Q(1, t) = t.

Quasi-copulas were introduced in [2] and characterized in [12]; the set of quasicopulas will be denoted by Q.

Since it will be needed later, we further provide the definition of a *copula*, which was introduced by Sklar ([50], see also [31, 51]).

**Definition 3.3.** A *copula* is a function  $C : [0,1]^2 \to [0,1]$  that satisfies the following conditions:

- (C1) for every  $t \in [0, 1]$ , C(0, t) = C(t, 0) = 0 and C(1, t) = C(t, 1) = t;
- (C2) C is 2-increasing, i.e., for all s, s', t and t' in [0, 1], with  $s \leq s'$  and  $t \leq t'$ ,

$$C(s',t') - C(s',t) - C(s,t') + C(s,t) \ge 0.$$
(3.2)

It follows from the definition that every copula C is increasing in each place and that it satisfies the Lipschitz condition (3.1). As a consequence, every copula C is a quasi-copula, but there are quasi-copulas that are not copulas; these will be called *proper*. If C denotes the set of all copulas, we can write

 $\mathcal{C}\subset\mathcal{Q}\subset\mathcal{S}.$ 

In the sequel, the notion of t-norm will often be needed. This concept was introduced in a slightly different form by Menger in [27] and in its definitive form in [37, 38]. To this class of operations are devoted the important monographs [1, 17], see also [18, 19, 20, 16].

**Definition 3.4.** A triangular norm or, briefly, a *t*-norm is a function  $T : [0, 1]^2 \rightarrow [0, 1]$  that satisfies the following conditions:

- (T1) T is commutative, i.e., T(s,t) = T(t,s) for all s and t in [0,1];
- (T2) T is associative, i.e., T(T(s,t), u) = T(s, T(t, u)) for all s, t and u in [0, 1];
- (T3) T is increasing, i.e.,  $T(s,t) \leq T(s',t)$  for all  $t \in [0,1]$  whenever  $s \leq s'$ ;
- (T4) T satisfies the boundary condition T(1,t) = t for every  $t \in [0,1]$ .

Notice that by virtue of its commutativity, any t-norm T is increasing in each place. Moreover, since a t-norm is obviously a semicopula, any t-norm T has 0 as its unique null element. The set of all t-norms will be denoted by  $\mathcal{T}$ .

The most important t-norms are the minimum M, the product  $\Pi$ , the Lukasiewicz t-norm W and the drastic product D given by

$$\begin{split} M(x,y) &:= \min\{x,y\} = x \land y, \\ \Pi(x,y) &:= x y, \\ W(x,y) &:= \max\{0, x + y - 1\}, \\ D(x,y) &:= \begin{cases} \min\{x,y\}, & \max\{x,y\} = 1\\ 0, & \text{otherwise.} \end{cases} \end{split}$$

In the literature these *t*-norms are often called  $T_{\mathbf{M}}$ ,  $T_{\mathbf{P}}$ ,  $T_{\mathbf{L}}$  and  $T_{\mathbf{D}}$ , respectively.

Of these t-norms, M,  $\Pi$  and W are also copulas, while D is not. It has been shown by Moynihan ([29, Theorem 3.1], see also [41, Theorem 6.3.2], that a tnorm is a copula if, and only if, it satisfies the Lipschitz condition. Vice versa, any associative copula is a (continuous) t-norm (see also [17, Corollary 9.9]).

**Proposition 3.1.** If a quasi-copula Q is a t-norm, then it is an associative symmetric<sup>2</sup> copula.

*Proof.* A quasi-copula Q is a *t*-norm if, and only if, it is commutative and associative. Then, if Q fulfils these two properties, i.e., if it is a *t*-norm, it satisfies also, by definition, the Lipschitz condition (3.1). But following Moynihan's result, it is therefore a (symmetric) copula.

Corollary 3.2. No proper quasi-copula is a t-norm.

Further, from a function  $f : [0,1]^2 \to [0,1]$ , in particular, for a *t*-norm, for a quasi-copula, or for a copula, two other functions, also from  $[0,1]^2$  into [0,1], are defined, namely

$$f^*(s,t) := 1 - f(1 - s, 1 - t), \tag{3.3}$$

and

$$\overline{f}(s,t) := s + t - f(s,t). \tag{3.4}$$

If f is a t-norm T, then  $T^*$  is called the t-conorm<sup>3</sup> of T, while if it is a copula C, then  $\overline{C}$  is called the *dual copula* of C [31]. A dual copula is not a copula because it is not 2-increasing. We shall denote the corresponding sets of co-functions, resp.

 $<sup>^2\,</sup>$  The two terms "symmetric" and "commutative" are here equivalent; we shall use the word "symmetric" only when speaking of copulas or quasi-copulas

<sup>&</sup>lt;sup>3</sup> We should like to point out that in many books and publications related to *t*-norms, corresponding *t*-conorms are denoted by *S*. However, since in our case *S* serves as a symbol for a semicopula, we shall denote *t*-conorms by  $T^*$ .

duals, accordingly; for instance, the set of all co-semicopulas by  $S^*$ , the set of all dual semicopulas  $\overline{S}$ .

The following simple result is sometimes useful.

**Lemma 3.3.** For a function  $f : [0,1]^2 \to [0,1]$  the following statements are equivalent:

- (a) f satisfies the Lipschitz condition (3.1);
- (b)  $\overline{f}$  is increasing, in the sense that  $\overline{f}(s',t') \ge \overline{f}(s,t)$  for all s, t,s', t' in [0,1] with  $s \le s'$  and  $t \le t'$ .

We now turn to the other set of operations, which we denote by  $\mathfrak{L}$  as in [41].

**Definition 3.5.** The class  $\mathfrak{L}$  is the set of all binary operations on  $\overline{\mathbb{R}}_+$  such that (L1) L is onto, i.e.,  $\operatorname{Ran}_L = \overline{\mathbb{R}}_+$ ;

- (L2) L is increasing in each place;
- (L3) L is continuous on  $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$ , except possibly at the points  $(0, \infty)$  and  $(\infty, 0)$ .

Additional properties may be required of elements of  $\mathfrak{L}$ :

(LS) L is strictly increasing in the following sense

for all 
$$u_1, u_2, v_1, v_2 \in \mathbb{R}_+$$
 with  $u_1 < u_2, v_1 < v_2$ :  $L(u_1, v_1) < L(u_2, v_2)$ . (3.5)

(L0) L has 0 as its neutral element.

(LB) For every  $x \in [0, \infty)$  the set  $A_x = \{(u, v) \mid L(u, v) = x\}$  is bounded.

**Example 3.1.** Examples of operations in  $\mathfrak{L}$  are the minimum, the maximum, the sum, and

$$K_{\alpha}(x,y) := (x^{\alpha} + y^{\alpha})^{1/\alpha} \qquad (\alpha > 0).$$

Notice also that the minimum satisfies condition (LS) but neither condition (L0) nor condition (LB), the maximum, the sum, and  $K_{\alpha}$  for  $\alpha \geq 1$  satisfy (LS), (L0) and (LB).

## 4. Triangle functions

Since the introduction of *Probabilistic Metric Spaces* (=PM spaces) first, and, then of *Probabilistic Normed Spaces* (=PN spaces) (see [3, 4, 21, 22, 24, 41, 45, 46]) binary operations on  $\Delta^+$  have been the object of great interest, especially *triangle* functions, namely continuous (i.e., topological) semigroups on  $\Delta^+$ .

**Definition 4.1.** A triangle function is a binary operation on  $\Delta^+$  that is commutative, associative, and increasing in each place and has  $\varepsilon_0$  as identity. Explicitly, a triangle function  $\tau$  satisfies the following conditions, for all F, G and H in  $\Delta^+$ :

(TF1)  $\tau(F,G) = \tau(G,F);$ 

(TF2)  $\tau(\tau(F,G),H) = \tau(F,\tau(G,H));$ (TF3) if  $F \leq G$ , then  $\tau(F,H) \leq \tau(G,H);$ (TF4)  $\tau(\varepsilon_0,F) = \tau(F,\varepsilon_0) = F.$ 

Moreover, a triangle function is *continuous* if it is continuous in the metric space  $(\Delta^+, d_S)$ .

For every triangle function  $\tau$  and for every  $F \in \Delta^+$ , one has

$$\varepsilon_{\infty} \le \tau(\varepsilon_{\infty}, F) \le \tau(\varepsilon_{\infty}, \varepsilon_0) = \varepsilon_{\infty},$$
(4.1)

so that  $\varepsilon_{\infty}$  is the null element of  $\tau$ .

The order on  $\Delta^+$  induces an order on the set of triangle functions

 $\tau_1 \leq \tau_2 \quad \Longleftrightarrow \quad \forall F, G \in \Delta^+ \ \forall x \in \mathbb{R}_+ \quad \tau_1(F,G)(x) \leq \tau_2(F,G)(x).$ 

Many examples of triangle functions will be encountered in the next sections.

Since the result of applying a triangle function to a pair (F, G) of d.d.f.'s is again a d.d.f., the simplest way of constructing a triangle function is to compute the value at x directly from the values F and G taken at the same point x; such operations will be investigated in Sections 5 and 6.

A second possibility is to associate a pair of numbers (u, v) with the given argument x, i.e.,  $(u, v) \sim x$ , and to evaluate the d.d.f.'s involved at u resp. v. Formally, assume that two d.d.f.'s F and G are given; then consider a structure of the following form:

$$\Theta_{A,\sim,\Omega}(x) = \Omega_{(u,v)\sim x} \{ A(F(u), G(v)) \},$$
(4.2)

where A represents a rule on how to combine F(u) and G(v), ~ the relationship between (u, v) and x. Of course, several (u, v) might be related to x; therefore,  $\Theta_{A,\sim,\Omega}(x)$  will have to be determined from the set  $\{A(F(u), G(v)) \mid (u, v) \sim x\}$ , which we denote by means of the operation  $\Omega$ . Although it is simple to pose the following question, the answer to it will prove to be a real challenge:

For which classes of A,  $\sim$  and  $\Omega$  is  $\Theta_{A,\sim,\Omega}$  an operation resp. a triangle function on  $\Delta^+$ ?

In Sections 7 through 9 we shall deal with some operations of this type and provide a partial answer to the above question.

And finally, we shall turn to a third class of operations involving integrals and measures in Section 10.

### 5. Pointwise induced triangle functions

As mentioned above, operations on  $\Delta^+$  can be induced pointwise by operations on [0, 1] (see Definition 7.1.3 in [41]).

**Definition 5.1.** Let f be a mapping from  $[0,1]^2$  to [0,1]; for every pair of d.d.f.'s F and G and for every  $x \in \overline{\mathbb{R}}_+$ , the function  $\Pi_f : \Delta^+ \times \Delta^+ \to [0,1]^{\overline{\mathbb{R}}_+}$  is defined, by

$$\Pi_f(F,G)(x) = f(F(x), G(x)).$$
(5.1)

If  $\Pi_f$  maps  $\Delta^+ \times \Delta^+$  into  $\Delta^+$ , i.e.,  $\Pi_f(\Delta^+ \times \Delta^+) \subset \Delta^+$ , then  $\Pi_f$  will be referred to as the operation pointwise induced by f on  $\Delta^+$ . If it is also a triangle function, we shall speak of the triangle function pointwise induced by f.

It is of particular interest to investigate for which operations f,  $\Pi_f$  yields a binary operation on  $\Delta^+$ , or, even, a triangle function. It was already mentioned in [41] that a left-continuous *t*-norm is an appropriate choice for getting a triangle function; however, this result can be strengthened. But, before proceeding, the definition of left-continuity will be needed.

**Definition 5.2** ([41, Definition 7.1.6]). A binary operation f on [0, 1] that is increasing in each place is said to be *left-continuous* if

$$f(x, y) = \sup\{f(u, v) \mid 0 < u < x, 0 < v < y\}$$

for all x and y in [0, 1].

The following basic result for increasing functions will play an important role. Its proof is only a slight modification of that in [15] or [17, Proposition 1.22].

**Lemma 5.1.** A binary operation f on [0,1] that is increasing in each place is left-continuous if, and only if, it is left-continuous in each place.

Now we are ready to turn to the characterization of the triangle functions  $\Pi_T$ .

**Theorem 5.2.** Let T be a function from  $[0,1]^2$  into [0,1]. Then  $\Pi_T$  defined by (5.1) is a triangle function if, and only if, T is a left-continuous t-norm.

We shall prove this result by means of a series of lemmata.

**Lemma 5.3.** Let T be a function from  $[0,1]^2$  into [0,1]. If T is a left-continuous t-norm, then  $\Pi_T$  is a triangle function.

*Proof.* The left-continuity of T guarantees that  $\Pi_T$  is a binary operation on  $\Delta^+$ . Further, the associativity and commutativity of T imply that also  $\Pi_T$  is associative and commutative. It remains to verify that  $\varepsilon_0$  is the neutral element of  $\Pi_T$ . If x > 0, then

$$\Pi_T(F,\varepsilon_0)(x) = T(F(x),\varepsilon_0(x)) = T(F(x),1) = F(x),$$
  
while, for  $x = 0$ ,  $\Pi_T(F,\varepsilon_0)(0) = \Pi_T(F(0),\varepsilon_0(0)) = T(0,0) = 0 = F(0).$ 

**Lemma 5.4.** Let T be a function from  $[0,1]^2$  into [0,1]. If  $\Pi_T$  is a triangle function, then T is a t-norm.

*Proof.* Assume that  $\Pi_T$  is a triangle function. Then one has to show that the function  $T: [0,1]^2 \to [0,1]$  that induces  $\Pi_T$  through (5.1) is indeed commutative, associative, increasing in each component and has 1 as its neutral element.

Choose s and t arbitrarily in ]0,1]. Then, for every real a > 0,

$$T(s,t) = T(V_s(a), V_t(a)) = \Pi_T(V_s, V_t)(a) = \Pi_T(V_t, V_s)(a) = T(t,s),$$

which proves the commutativity of T for  $s, t \in [0, 1]$ . When s = 0 and t is in [0, 1], then

$$T(0,t) = T(\varepsilon_{\infty}(a), V_t(a)) = \Pi_T(\varepsilon_{\infty}, V_t)(a) = 0$$
  
=  $\Pi_T(V_t, \varepsilon_{\infty})(a) = T(V_t(a), \varepsilon_{\infty}(a)) = T(t, 0).$ 

This also proves that T(0,t) = T(t,0) = 0 for every  $t \in [0,1]$ . Further, one can prove, in a completely analogous manner, that T is associative and increasing as  $\Pi_T$  is also associative and increasing.

Finally, put s = 1 and notice that  $s = 1 = \varepsilon_0(a)$  so that

$$T(1,t) = T(\varepsilon_0(a), V_t(a)) = \Pi_T(\varepsilon_0, V_t)(a) = V_t(a) = t,$$

which shows that 1 is the neutral element of T. Thus T is indeed a t-norm whenever  $\Pi_T$  is a triangle function.

**Lemma 5.5.** Let T be a t-norm. If  $\Pi_T$  is a triangle function, then T is leftcontinuous.

*Proof.* We shall prove this result by contradiction. Assume, if possible, that  $\Pi_T$  is a triangle function and that T is not left-continuous. Because of Lemma 5.1, it is possible to study left-continuity for only one place, say the first one. Then there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  of elements of [0, 1] and some  $y_0 \in [0, 1]$  such that

$$\sup_{n \in \mathbb{N}} \{T(x_n, y_0)\} < T\left(\sup_{n \in \mathbb{N}} x_n, y_0\right).$$
(5.2)

Without loss of generality, it can be assumed that  $x_n \neq 0$  for all  $n \in \mathbb{N}$ . Indeed, if  $x_n = 0$  for all  $n \in \mathbb{N}$ , then equality would hold in (5.2). Therefore, there exists at least some natural n such that  $x_n \neq 0$ ; thus, by eliminating from  $(x_n)_{n \in \mathbb{N}}$  all zero elements (and by repeating elements if there are only finitely many non-zero elements left), another sequence can be constructed which again fulfils inequality (5.2). Therefore, we may assume that for  $(x_n)_{n \in \mathbb{N}}$ ,  $x_n \neq 0$  for all  $n \in \mathbb{N}$ .

Now consider the two d.d.f.'s U and  $V_{y_0}$ , where U is the d.f. of a random variable uniformly distributed on (0, 1).

$$U(x) = \begin{cases} 0, & x \le 0, \\ x, & x \in [0, 1], \\ 1, & x > 1. \end{cases}$$

Since, by assumption,  $\Pi_T$  is a triangle function,  $\Pi_T(U, V_{y_0})$  is a d.d.f., say G, i.e.,

$$G = \Pi_T \left( U, V_{y_0} \right).$$

However, G cannot belong to  $\Delta^+$ . In fact, consider the sequence  $(x_n)_{n \in \mathbb{N}}$  introduced above and notice that the left-continuity of both U and  $V_{y_0}$  yields

$$\sup_{n \in \mathbb{N}} G(x_n) = \sup_{n \in \mathbb{N}} T(U(x_n), V_{y_0}(x_n)) = \sup_{n \in \mathbb{N}} T(x_n, y_0)$$
  
$$< T\left(\sup_{n \in \mathbb{N}} x_n, y_0\right) = T\left(U\left(\sup_{n \in \mathbb{N}} x_n\right), V_{y_0}\left(\sup_{n \in \mathbb{N}} x_n\right)\right) = G\left(\sup_{n \in \mathbb{N}} x_n\right)$$

so that G is not left-continuous on  $]0, \infty[$ ; therefore  $\Pi_T$  is not a triangle function. This contradicts the assumption and concludes the proof.  $\square$ 

**Corollary 5.6.** If T is a left-continuous t-norm, then the triangle function  $\Pi_T$ is sup-preserving on  $\Delta^+$ , in the sense that, if  $(F_n)_{n\in\mathbb{N}}$  is a sequence of d.d.f.'s such that  $\sup_{n\in\mathbb{N}} F_n = F$ , then, for every  $G \in \Delta^+$ ,

$$\sup_{n\in\mathbb{N}}\Pi_T(F_n,G) = \Pi_T\left(\sup_{n\in\mathbb{N}}F_n,G\right)$$
(5.3)

*Proof.* For every x > 0, one has, because of the left-continuity of T,

$$\sup_{n \in \mathbb{N}} \Pi_T(F_n, G)(x) = \sup_{n \in \mathbb{N}} T\left(F_n(x), G(x)\right) = T\left(\sup_{n \in \mathbb{N}} F_n(x), G(x)\right)$$
$$= T(F(x), G(x)) = \Pi_T(F, G)(x) = \Pi_T\left(\sup_{n \in \mathbb{N}} F_n, G\right)(x),$$
ch proves (5.3).

which proves (5.3).

A triangle function that can be expressed as a function of the values of F and G at x, but cannot be reconducted to the type introduced in Definition 5.1 was introduced by Ying in [52]. Let a be in  $\overline{\mathbb{R}}_+$  and T be a left-continuous t-norm, then, see [52, Theorem 1], a triangle function  $\tau_{a,T}$  is defined via

$$\tau_{a,T}(F,G)(x) := \min \left\{ T\left(F(x), G(a \lor x)\right), T\left(F(a \lor x), G(x)\right) \right\},$$
(5.4)

which may also be written in the form

$$\tau_{a,T}(F,G)(x) = \max \left\{ T\left(F(x), G(x)\right), \min \{T\left(F(a), G(x)\right), T\left(F(x), G(a)\right) \} \right\}.$$

## 6. Pointwise induced operations and aggregation operators

Let us turn back to Definition 5.1 and ask which properties are required of a function f in order that  $\Pi_f$  is a binary operation, but not necessarily a triangle function, on  $\Delta^+$ . Are any functions  $f: [0,1]^2 \to [0,1]$  appropriate candidates?

**Theorem 6.1.** Let A be a mapping from  $[0,1]^2$  into [0,1]. Then  $\Pi_A$  defined by (5.1) is a binary operation on  $\Delta^+$  if, and only if, A is a left-continuous binary aggregation operator.

Again we provide the proof in two steps, which we state as lemmata. Recall that a binary aggregation operator  $A: [0,1]^2 \to [0,1]$  is defined by the following properties:

- (A1) A(0,0) = 0 and A(1,1) = 1,
- (A2)  $A(u_1, v_1) \leq A(u_2, v_2)$  for all  $u_1, u_2, v_1$  and  $v_2$  in [0, 1] with  $u_1 \leq u_2$  and  $v_1 \leq v_2$ .

**Lemma 6.2.** Let A be a mapping from  $[0,1]^2$  into [0,1]. If A is a left-continuous binary aggregation operator, then  $\Pi_A$  defined by (5.1) is a binary operation on  $\Delta^+$ .

*Proof.* We have to show that, for arbitrary F and G in  $\Delta^+$ ,  $\Pi_A(F,G)$  belongs to  $\Delta^+$ . Let F and G be arbitrary d.d.f.'s. By Definition 5.1, it is clear that  $\Pi_A(F,G)$  is indeed a function from  $\mathbb{R}$  into [0,1]. It is also increasing, since for all x and y in  $\overline{\mathbb{R}}$  with  $x \leq y$ ,

$$\Pi_A(F,G)(x) = A(F(x), G(x)) \le A(F(y), G(y)) = \Pi_A(F,G)(y).$$

Moreover,

$$\Pi_A(F,G)(0) = A(F(0),G(0)) = A(0,0) = 0,$$
  
$$\Pi_A(F,G)(+\infty) = A(F(+\infty),G(+\infty)) = A(1,1) = 1$$

For every sequence  $(x_n)_{n\in\mathbb{N}}$  of elements of  $\mathbb{R}_+$ , the left-continuity of A, F and G together with Lemma 5.1 implies

$$\sup_{n \in \mathbb{N}} \Pi_A(F, G)(x_n) = \sup_{n \in \mathbb{N}} A(F(x_n), G(x_n)) = A\left(\sup_{n \in \mathbb{N}} F(x_n), \sup_{n \in \mathbb{N}} G(x_n)\right)$$
$$= A\left(F\left(\sup_{n \in \mathbb{N}} x_n\right), G\left(\sup_{n \in \mathbb{N}} x_n\right)\right) = \Pi_A(F, G)\left(\sup_{n \in \mathbb{N}} x_n\right),$$
ich establishes the left-continuity of  $\Pi_A(F, G)$ .

which establishes the left-continuity of  $\Pi_A(F, G)$ .

**Lemma 6.3.** Let A be a mapping from  $[0,1]^2$  into [0,1]. If  $\Pi_A$  defined by (5.1) is a binary operation on  $\Delta^+$ , then A is a left-continuous binary aggregation operator.

*Proof.* Let  $\Pi_A$  be a binary operation on  $\Delta^+$ . Then necessarily,

 $\Pi_A(F,G)(0) = 0$  and  $\Pi_A(F,G)(+\infty) = 1$ 

for all F and G in  $\Delta^+$ , so that one also has A(0,0) = 0 and A(1,1) = 1.

In order to show that A is increasing in each place, let  $u_1, u_2, v_1$  and  $v_2$  be in [0, 1] and satisfy the conditions  $u_1 \leq u_2$  and  $v_1 \leq v_2$ . Then choose a and b in

 $[0, +\infty)$  with a < b. Define two d.d.f.'s F and G by

$$F(x) = \begin{cases} 0, & x \le 0, \\ u_1, & x \in ]0, a], \\ u_2, & x \in ]a, b], \\ 1, & x > b, \end{cases}$$

if  $u_1 < u_2$ , and by  $F = V_{u_1}$  when  $u_1 = u_2$ . Similarly

$$G(x) = \begin{cases} 0, & x \le 0, \\ v_1, & x \in ]0, a], \\ v_2, & x \in ]a, b], \\ 1, & x > b, \end{cases}$$

if  $v_1 < v_2$ , and by  $G = V_{v_1}$  when  $v_1 = v_2$ . Therefore,

$$A(u_1, v_1) = A(F(a), G(a)) \le A(F(b), G(b)) = A(u_2, v_2).$$

In the same manner as in the proof of Lemma 5.5 it can be shown that A is left-continuous in each place.

As a consequence of the previous results, the only admissible operations on [0, 1]guaranteeing that  $\Pi_A$  is indeed a binary operation on  $\Delta^+$  are left-continuous binary aggregation operators. Analogous arguments can be applied when extending this result to *n*-ary operations on  $\Delta^+$  induced by some *n*-ary aggregation operator Aon [0, 1] that is left-continuous in each place.

Further we remark that in [2] the authors studied pointwise induced operations on  $\Delta^+$ ; however, this study was restricted to operations induced by corresponding operations on the underlying random variables. It turned out that the induced operations on  $\Delta^+$  correspond only in very few cases to operations on random variables on the *same* probability space (namely when the induced operation is related to some quasi-copula). We quote the main result from [2].

**Theorem 6.4.** Suppose that  $\Phi$  is a binary operation on  $\Delta^+$  such that  $\Phi = \prod_{\varphi}$  for some binary operation  $\varphi$  from  $[0,1]^2$  into [0,1] and that it is derivable from a function f on random variables defined on a common probability space. Then precisely one of the following holds:

- (1)  $f = \max and \varphi$  is a quasi-copula;
- (2)  $f = \min and \varphi$  is the dual of a quasi-copula;
- (3) f and  $\varphi$  are trivial, i.e., in the sense that they are the projections onto either the first or the second coordinate.

Next we turn to some operations on  $\Delta^+$  following (4.2) introduced in Section 4.

## 7. Operations involving semicopulas

**Definition 7.1** (compare also [41, Definition 7.2.1]). Let f be a mapping from  $[0,1]^2$  to [0,1] and let L belong to  $\mathfrak{L}$ ; the function  $\tau_{f,L} : \Delta^+ \times \Delta^+ \to [0,1]^{\overline{\mathbb{R}}_+}$  is defined, for every pair F and G of d.d.f.'s and for every  $x \in \overline{\mathbb{R}}_+$ , by

$$\tau_{f,L}(F,G)(x) = \sup\{f(F(u),G(v)) \mid L(u,v) = x\}.$$
(7.1)

If L denotes the sum, then its index is usually omitted, and we simply write  $\tau_f$ .

Again, we have provided the definition in its most general form. As will be seen later, the appropriate choice for f will be a semicopula. In [41] these operations have in fact been introduced for semicopulas and have been denoted by  $\tau_{T,L}$  (with T being a semicopula, not necessarily a *t*-norm).

Notice that the operation  $\tau_{f,L}$ , where f is a function from  $[0,1]^2$  to [0,1] and L is in  $\mathfrak{L}$ , may be represented according to the structure introduced in Section 3, i.e.,

$$\tau_{f,L} = \Theta_{f,=L,\sup}$$

where by  $((u, v) =_L x)$  we mean (L(u, v) = x).

## 7.1. Operations on $\Delta^+$

The following sufficient conditions on f and L ensure that  $\tau_{f,L}$  is a binary operation on  $\Delta^+$ ; they were presented in [41, Lemma 7.2.3], however, we state it an slightly different way.

**Lemma 7.1.** Let S be a left-continuous semicopula and let  $L \in \mathfrak{L}$  satisfy condition (LS) of Definition 3.5. Then  $\tau_{S,L}$  is a binary operation on  $\Delta^+$ .

It is easily seen that  $\tau_{S,\max} = \Pi_S$  for any semicopula S. Following Theorem 6.1 we know that S has to be left-continuous in order to guarantee that  $\tau_{S,\max}$  is a binary operation on  $\Delta^+$ . Moreover, it has to be a left-continuous *t*-norm if  $\tau_{S,\max}$  is to be a triangle function. Thus one might be lead to conjecture that the semicopula involved must necessarily be left-continuous; however, as the following example shows, this need not be the case.

**Example 7.1** ([41, p. 100]). Consider the drastic product D, which is not left-continuous, and the standard summation, then

$$\tau_D(F,G)(x) = \max\left\{F(\max\{0, x - G^{\wedge}(1)\}), G(\max\{0, x - F^{\wedge}(1)\})\right\}.$$
 (7.2)

It is a triangle function and therefore, in particular, a binary operation on  $\Delta^+$ .

Moreover,  $\tau_D$  is not continuous on  $(\Delta^+, d_S)$ . If  $F_n$  is the exponential distribution of parameter 1/n,  $F_n \sim \Gamma(1, 1/n)$ , namely, for  $x \in \mathbb{R}_+$ ,

$$F_n(x) = 1 - e^{-x/n},$$

then  $(F_n)_{n\in\mathbb{N}}$  converges weakly to  $\varepsilon_0$ , but  $(\tau_D(F_n, F_n))_{n\in\mathbb{N}}$  does not converge weakly to  $\varepsilon_0 = \tau_D(\varepsilon_0, \varepsilon_0)$ , since  $\tau_D(F_n, F_n) = \varepsilon_\infty$  for every  $n \in \mathbb{N}$ .

Further, it might be conjectured that f in Definition 7.1 has to be a semicopula in order to guarantee that  $\tau_{f,L}$  given by (7.1) is an operation on  $\Delta^+$ . However, this also need not be the case as the following example shows.

**Example 7.2.** Consider the function  $f: [0, 1]^2 \to [0, 1], f(x, y) = \max\{x - y, 0\}$ , which is increasing in its first and decreasing in its second argument, and thus not a semicopula. Choose  $L = \max$ . Then, for every  $x \in \mathbb{R}_+$ , the set  $L(x) = \{(u, v) \mid L(u, v) = x\}$  contains at least the points (x, 0), (0, x). Since f attains its maximum whenever the first argument becomes as large as possible and the second as small as possible, we can conclude that for any d.d.f.s  $F, G \in \Delta^+$ 

 $\tau_{f,L}(F,G)(x) = \sup\{f(F(u), G(v)) \mid \max\{u, v\} = x\} = f(F(x), G(0)) = F(x),$ 

i.e.,  $\tau_{f,L}(F,G) = F$  is nothing else than the projection on the first coordinate. Notice that the same result is achieved if, instead of  $L = \max$ , any other operation  $L \in \mathfrak{L}$  fulfilling (L0) and (LS) is chosen. Clearly,  $\tau_{f,L}(F,G)$  is again a d.d.f. and is continuous on  $(\Delta^+, d_S)$ .

If  $\tau_{f,L}$  is a binary operation on  $\Delta^+$  for an arbitrary  $L \in \mathfrak{L}$ , the boundary conditions for a d.d.f. suffice to imply some necessary conditions on f.

**Lemma 7.2.** Let f be a function from  $[0,1]^2$  to [0,1] and let  $L \in \mathfrak{L}$ . If  $\tau_{f,L}$  is a binary operation on  $\Delta^+$ , then f(0,0) = 0 and  $\sup\{f(1,0), f(0,1), f(1,1)\} = 1$ .

Let us next turn to the question whether, whenever  $\tau_{f,L}$  is a binary operation on  $\Delta^+$ , the continuity of some of the d.d.f.'s affects the continuity of  $\tau_{f,L}(F,G)$ as a d.d.f.. Moynihan gave the following result on the continuity of the triangle function  $\tau_T(F,G)$ , where L is replaced by standard addition and T is a t-norm.

**Theorem 7.3** ([28, Theorem 1.1]). If T is a continuous t-norm and if either F or G is continuous, then  $\tau_T(F,G)$  is continuous.

We now extend this theorem for functions  $\tau_{S,L}$  where S is a continuous semicopula and L is commutative and associative, fulfilling (LS) and (L0) of Definition 3.5. The choice of S guarantees that  $\tau_{S,L}$  is indeed a binary operation on  $\Delta^+$  (see Lemma 7.1). The proof is adapted from that of Moynihan's theorem, the main difference being that, while the sum on  $\mathbb{R}_+$  is the restriction of a group operation, and hence subtraction and a limited cancellativity are allowed, this is not so with a generic semigroup operation like L. For all A and B subsets of  $\overline{\mathbb{R}}_+$  and for every  $L \in \mathfrak{L}$ , let L(A, B) denote the set

$$L(A, B) := \{ L(u, v) \mid u \in A, v \in B \}.$$

We first show a relationship between the set of points of discontinuity of all d.d.f.s involved.

**Theorem 7.4.** Let S be a continuous semicopula, let L be commutative, associative and let it fulfil both (LS) and (L0) of Definition 3.5. For any given F and G in  $\Delta^+$ , let H be the d.d.f. defined by  $H := \tau_{S,L}(F,G)$ . Then

$$D(H) \subset L\left(D(F), D(G)\right). \tag{7.3}$$

*Proof.* Let  $\varepsilon > 0$  be given arbitrarily. In view of the uniform continuity of S on the unit square  $[0, 1]^2$ , there exists  $\eta = \eta(\varepsilon) > 0$  such that

$$|S(a_1, b_1) - S(a_2, b_2)| < \varepsilon,$$

whenever  $|a_1 - a_2| < \eta$  and  $|b_1 - b_2| < \eta$ .

Since what we are going to say below is true if both D(F) and D(G) are finite (or empty), there is no loss of generality in assuming that both D(F) and D(G)contain a countable infinity of points; then there exist two subsets  $\widetilde{A}$  and  $\widetilde{B}$  of  $\mathbb{N}$ such that  $D(F) = \{x_i \mid i \in \widetilde{A}\}$  and  $D(G) = \{y_j \mid j \in \widetilde{B}\}$ . Since both

$$\sum_{i \in \widetilde{A}} \left[ \ell^+ F(x_i) - F(x_i) \right] \le 1 \quad \text{and} \quad \sum_{i \in \widetilde{B}} \left[ \ell^+ G(y_j) - G(y_j) \right] \le 1,$$

there are finite subsets  $A \subset \widetilde{A}$  and  $B \subset \widetilde{B}$  such that

$$\sum_{i \in \widetilde{A} \setminus A} \left[ \ell^+ F(x_i) - F(x_i) \right] \le \frac{\eta}{2} \quad \text{and} \quad \sum_{i \in \widetilde{B} \setminus B} \left[ \ell^+ G(y_j) - G(y_j) \right] \le \frac{\eta}{2}.$$

Thus, if x > y and  $F_d(x) \ge F_d(y) + \eta/2$ , then there exists  $x_i \in D(F)$  with  $i \in A$  such that  $y \le x_i < x$ . Similarly, if  $G_d(x) > G_d(y) + \eta/2$ , then there exists  $y_j \in D(G)$  with  $j \in B$  such that  $y \le y_j < x$ .

Assume, now,  $x \notin L(D(F), D(G))$ . As A and B are finite sets, there exists  $\delta > 0$  such that the closed interval  $[x - \delta, x + \delta]$  contains no point of the type  $L(x_i, y_j)$  with  $i \in A$  and  $j \in B$ . The continuous parts  $F_c$  and  $G_c$  are uniformly continuous; thus, there exist  $\gamma > 0$  such that

$$|F_c(t) - F_c(s)| < \frac{\eta}{2}$$
 and  $|G_c(t) - G_c(s)| < \frac{\eta}{2}$ 

whenever  $|t-s| < \gamma$ .

Since L is continuous, there exists  $\rho > 0$  such that  $|L(u', v') - L(u, v)| < \delta$ whenever  $|u' - u| < \rho$  and  $|v' - v| < \rho$ .

Take now h with  $0 < h < \min\{\gamma, \delta, \rho\}$  and choose u and v such that L(u, v) = x. If

$$F_d(u+h) - F_d(u) < \eta/2,$$
 (7.4)

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then

$$|F(u+h) - F(u)| \le |F_c(u+h) - F_c(u)| + |F_d(u+h) - F_d(u)| < \eta_{2}$$

from which

$$|S(F(u+h),G(v)) - S(F(u),G(v))| < \varepsilon.$$

Therefore,

$$H(x) \ge S(F(u), G(v)) > S(F(u+h), G(v)) - \varepsilon.$$

$$(7.5)$$

If, on the other hand,

$$F_d(u+h) - F_d(u) \ge \eta/2,$$
 (7.6)

it was seen above that there is a point  $x_i \in D(F)$  with  $i \in A$  and  $x_i \in [u, u + h[$ . But then, one would necessarily have  $G_d(v + h) - G_d(v) < \eta/2$ , for, otherwise, there would be  $y_j \in B$  with  $j \in B$  and  $y_j \in [v, v + h[$ . The isotony of L would then yield

$$L(u, v) \le L(x_i, y_j) < L(u+h, v+h),$$

or, because of the continuity of L,

$$x \le L(x_i, y_j) < L(u, v) + \delta = x + \delta,$$

which contradicts the definition of  $\delta$ . Therefore,

$$|G(v+h) - G(v)| \le |G_c(v+h) - G_c(v)| + |G_d(v+h) - G_d(v)| < \eta,$$

whence

$$|S(F(u), G(v+h)) - S(F(u), G(v))| < \varepsilon,$$

so that,

H

$$H(x) \ge S(F(u), G(v)) > S(F(u), G(v+h)) - \varepsilon.$$

$$(7.7)$$

Finally, there are  $h' < \delta$  and  $h'' < \delta$  such that L(u+h,v) = L(u,v) + h' = x + h'and L(u,v+h) = L(u,v) + h'' = x + h''. Therefore, it follows from (7.5) that

$$\begin{aligned} &(x) \ge \sup \left\{ S(F(u+h), G(v)) \mid L(u, v) = x \right\} - \varepsilon \\ &\ge \sup \left\{ S(F(u+h), G(v)) \mid L(u, v) = x \text{ and } (7.4) \text{ holds} \right\} - \varepsilon \\ &= \sup \left\{ S(F(s), G(t)) \mid L(s, t) = x + h' \right\} - \varepsilon = H(x + h') - \varepsilon, \end{aligned}$$

while it follows from (7.7) that

$$\begin{split} H(x) &\geq \sup \left\{ S(F(u), G(v+h)) \mid L(u,v) = x \right\} - \varepsilon \\ &\geq \sup \left\{ S(F(u), G(v+h)) \mid L(u,v) = x \text{ and } (7.6) \text{ holds} \right\} - \varepsilon \\ &= \sup \left\{ S(F(s), G(t)) \mid L(s,t) = x + h'' \right\} - \varepsilon = H(x+h'') - \varepsilon. \end{split}$$

The last two inequalities yield, for every  $t \leq \min\{h', h''\}$ ,

$$H(x) \ge H(x+t) - \varepsilon.$$

This implies that H is right-continuous and, hence, continuous at x, since it is also left-continuous, like any d.d.f.. Therefore x does not belong to D(H); this completes the proof.

The following corollary is an immediate consequence of the previous theorem.

**Corollary 7.5.** In the same assumptions as in Theorem 7.4, if F and G are in  $\Delta^+$  and if at least one of them is continuous, then so is also  $\tau_{S,L}(F,G)$ .

On the other hand, if  $\tau_{S,L}(F,G)$  is a continuous d.d.f. independently of the choice of F and G, then, for some operations L, also the semicopula S involved is continuous.

**Lemma 7.6.** Let S be a semicopula and let  $L \in \mathfrak{L}$  satisfy conditions (LS) and (L0) of Definition 3.5. Then if  $\tau_{S,L}(F,G)$  is a continuous d.d.f. for arbitrary  $F, G \in \Delta^+$ , then S is also continuous.

*Proof.* Let S be a semicopula. Let x be any strictly positive real number and let  $(x_n)_{n\in\mathbb{N}}$  be any sequence of positive real numbers converging to x. By recourse to Lemma 5.1 it suffices to prove that for every  $y_0 > 0$  one has

$$\lim_{n \to +\infty} S(x_n, y_0) = S(x, y_0).$$
(7.8)

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$$z_n := L(x_n, y_0)$$
 and  $z := L(x, y_0).$ 

Notice that, as a consequence of properties (LS) and (L0) of Definition 3.5, one has

$$\sup \left\{ u \mid \exists v : L(u, v) = x \right\} = x.$$

Therefore, if U is the uniform d.f. on (0, 1),

$$\tau_{S,L}(U, V_{y_0})(z_n) = \sup_{\substack{(u,v): L(u,v) = z_n \\ (u,v): L(u,v) = z_n}} S\left(U(u), V_{y_0}(v)\right)$$
$$= \sup_{\substack{(u,v): L(u,v) = z_n \\ (u,v): L(u,v) = z_n}} S\left(U(u), V_{y_0}(y_0)\right) = S\left(U(x_n), y_0\right) = S(x_n, y_0).$$

A similar argument yields

$$\tau_{S,L}(U, V_{y_0})(z) = S(U(x), V_{y_0}(y_0)) = S(x, y_0).$$

Thus, since  $\tau_{S,L}(U, V)$  is continuous, one has

$$\lim_{n \to +\infty} S(x_n, y_0) = \lim_{n \to +\infty} \tau_{S,L} (U, V_{y_0}) (z_n) = \tau_{S,L} (U, V_{y_0}) (z) = S(x, y_0).$$

This concludes the proof.

Finally, we quote a result showing that continuous semicopulas provide a sufficient condition for the continuity of the binary operations  $\tau_{S,L}$  on  $\Delta^+$  on the metric space  $(\Delta^+, d_S)$ .

**Theorem 7.7** ([41, Theorem 7.2.8]). Let S be a function from  $[0,1]^2$  to [0,1] and let  $L \in \mathfrak{L}$  satisfies condition (LB) of Definition 3.5. If S is a continuous semicopula, then  $\tau_{S,L}$  is uniformly continuous on the metric space  $(\Delta^+, d_S)$ .

#### 7.2. Triangle functions

From now on we assume that  $\tau_{f,L}$  is indeed an operation on  $\Delta^+$ . Requiring additional properties for  $\tau_{f,L}$  restricts the choice of the function f. The cases when  $L \in \mathfrak{L}$  fulfils the conditions (L0) and (LS) will first be examined; later on L will be assumed to be commutative and associative.

**Theorem 7.8.** Let S be a function from  $[0,1]^2$  to [0,1] and let  $L \in \mathfrak{L}$  satisfy conditions (LS) and (L0) of Definition 3.5. If  $\tau_{S,L}$  is binary operation on  $\Delta^+$  that fulfils (TF3) and (TF4), then S is a semicopula.

*Proof.* For arbitrary s and t in [0,1] and for all u and v in  $[0,\infty]$ , one has

$$S(s,t) = S\left(V_s(u), V_t(v)\right)$$

Moreover, since conditions (LS) and (L0) of Definition 3.5 together imply that, for any x > 0, the set  $L(x) = \{(u, v) \mid L(u, v) = x\}$  is a curve that connects the points (x, 0) and (0, x), which, in its turn, implies that this curve contains points different from its end-points, the set L(x) contains some point  $(u^*, v^*)$  with  $u^* > 0$  and  $v^* > 0$ . Therefore, for all  $0 < x < \infty$ ,

$$\begin{aligned} \tau_{S,L}(V_s, V_t)(x) &= \sup\{S(V_s(u), V_t(v)) \mid L(u, v) = x\} \\ &= \sup\{S(0, t), S(s, 0), S(s, t) \mid L(u, v) = x\} \\ &= \max\{S(0, t), S(0, s), S(s, t)\} \end{aligned}$$

is again a d.d.f. of type (2.1).

Choose t and x arbitrarily in [0, 1] and  $]0, \infty[$  respectively, and s = 0. Then,  $V_s = \varepsilon_{\infty}$ . Moreover, since  $\tau_{S,L}$  fulfils (TF3) and (TF4) and has therefore  $\varepsilon_{\infty}$  as null element,

$$0 \le S(0,t) = S(V_0(0), V_t(x)) = S(\varepsilon_{\infty}(0), V_t(x))$$
  
$$\le \sup\{S(\varepsilon_{\infty}(u), V_t(v)) \mid L(u, v) = x\} = \tau_{S,L}(\varepsilon_{\infty}, V_t)(x) = \varepsilon_{\infty}(x) = 0,$$

so that S(0,t) = 0 for all  $t \in [0,1]$ . An analogous argument proves that also S(t,0) = 0 for all  $t \in [0,1]$ . In particular S(0,1) = S(1,0) = 0.

Next choose s = 1, so that  $V_1 = \varepsilon_0$ , and let t and x be arbitrary points from [0,1] and  $[0,\infty[$  respectively. As 0 is the null element of S and  $\tau_{S,L}$  has, because of (TF4),  $\varepsilon_0$  as its neutral element, the following equalities hold

$$\tau_{S,L}(V_1, V_t)(x) = \sup\{S(0, 1), S(0, t), S(1, t)\} = \sup\{0, S(1, t)\}$$
  
$$\tau_{S,L}(V_1, V_t)(x) = \tau_{S,L}(\varepsilon_0, V_t)(x) = V_t(x) = t.$$

Since  $S(1,t) \ge 0$ , it follows that S(1,t) = t for all  $t \in [0,1]$ . By applying an analogous argument and by taking into account that S(0,1) = S(1,0) = 0, it follows that 1 is the neutral element of S.

It remains to show that S is increasing in each place; this will be proved explicitly only for the first place, the proof for the other one being completely analogous. If S(s,t) = 0 then  $S(s',t) \ge S(s,t)$  for all s' with  $s' \ge s$ . Therefore, assume S(s,t) > 0 and let  $s' \ge s$ . The isotony of  $\tau_{S,L}$  yields, for all x with  $0 < x \le 1$ ,

$$S(s,t) = \tau_{S,L}(V_s, V_t)(x) \le \tau_{S,L}(V_{s'}, V_t)(x) = S(s', t),$$

which proves that  $s \mapsto S(s,t)$  is increasing for every  $t \in [0,1]$ . Therefore, S is a semicopula.

In view of this result one can immediately conclude that f must be a semicopula whenever  $\tau_{f,L}$  is a triangle function.

**Corollary 7.9.** Let S be a function from  $[0,1]^2$  to [0,1] and let  $L \in \mathfrak{L}$  satisfy conditions (LS) and (L0) of Definition 3.5. If  $\tau_{S,L}$  is a triangle function, then S is a semicopula.

Moreover, the associativity and commutativity of  $\tau_{S,L}$  induce the corresponding properties on the semicopula S involved so that S has to be a t-norm.

**Lemma 7.10.** Let T be a function from  $[0,1]^2$  to [0,1] and let  $L \in \mathfrak{L}$  satisfy the conditions (LS) and (L0) of Definition 3.5. If  $\tau_{T,L}$  is a triangle function, then T is a t-norm.

*Proof.* Let  $s \in [0, 1[$  be arbitrary. As a consequence of Theorem 7.8 T is a semicopula; furthermore, for all  $x \in [0, 1[$ ,

$$\tau_{T,L}(V_s, V_t)(x) = T(s, t).$$

Fix s and t in ]0,1[. Since  $\tau_{T,L}$  is commutative, one has, for all  $x \in [0, \infty[$ ,

$$T(s,t) = \tau_{T,L}(V_s, V_t)(x) = \tau_{T,L}(V_t, V_s)(x) = T(s,t),$$

which establishes the commutativity of T.

As for its associativity, consider s, t, w and x in ]0, 1[. Then simple calculations lead to

$$\tau_{T,L}(\tau_{T,L}(V_s, V_t), V_w)(x) = T(T(s, t), w), \tau_{T,L}(V_s, \tau_{T,L}(V_t, V_w))(x) = T(T(t, w), s).$$

Thus T(T(t, w), s) = T(t, T(w, s)). If 1 or 0 appears among the inputs s, t and w, then the associativity condition holds trivially on account of the boundary conditions of T. Therefore, T is a *t*-norm.

When  $\tau_{T,L}$  is a triangle function T is necessarily a *t*-norm. It is natural to ask for sufficient conditions. It is already known from Lemma 7.1 that the left-continuity of the *t*-norm T ensures that  $\tau_{T,L}$  is a binary operation on  $\Delta^+$ . When  $L \in \mathfrak{L}$  fulfils (L0) and (LS) and is also associative and commutative,  $\tau_{T,L}$  is a triangle function.

**Theorem 7.11** ([41, Theorem 7.2.4]). Let T be a function from  $[0,1]^2$  to [0,1]and let  $L \in \mathfrak{L}$  satisfy conditions (LS) and (L0) of Definition 3.5. If T is a leftcontinuous t-norm and if, in addition, L is commutative and associative, then  $\tau_{T,L}$ is a triangle function.

As was seen by Example 7.1 the left-continuity of a semicopula S is not necessary for  $\tau_S$  to be a triangle function, but it may affect the continuity of the triangle function  $\tau_S$  on the metric space  $(\Delta^+, d_S)$ .

Since condition (L0) implies that L also fulfils condition (LB), the following corollary on the continuity of triangle functions is an immediate consequence of Theorems 7.7 and 7.11.

**Corollary 7.12** ([41, Corollary 7.2.9]). Let T be a function from  $[0,1]^2$  to [0,1]and let  $L \in \mathfrak{L}$  fulfil conditions (LS) and (L0) of Definition 3.5. If T is a continuous t-norm and L is commutative and associative, then the triangle functions  $\tau_{T,L}$  is uniformly continuous on the metric space  $(\Delta^+, d_S)$ .

When L is commutative and associative and satisfies (LS) and (L0), one can prove a result analogous to Theorem 5.2 thus strengthening the result of Theorem 7.11. Notice that similar results under conditions for L that are slightly different from those listed in Definition 3.5 were obtained by Ying in [53].

**Theorem 7.13.** Let T be a function from  $[0,1]^2$  to [0,1] and let  $L \in \mathfrak{L}$ . If L is commutative, associative and satisfies condition (L0) of Definition 3.5, the following statements are equivalent:

- (a)  $\tau_{T,L}$  is a continuous triangle function;
- (b) L satisfies condition (LS) and T is a continuous t-norm.

*Proof.* The proof of the implication (b)  $\implies$  (a) is a direct consequence of Theorem 7.11 and Corollary 7.12. Thus only the converse implication has to be proved. We first prove that L satisfies condition (LS).

Assume, if possible, that there exist numbers  $u_1$ ,  $u_2$ ,  $v_1$ ,  $v_2$  in  $\mathbb{R}_+$ , with  $0 < u_1 < u_2$  and  $0 < v_1 < v_2$ , such that  $L(u_1, v_1) = L(u_2, v_2)$ . Then, by definition of  $\tau_{T,L}$ ,

$$\tau_{T,L} \left( \varepsilon_{u_1}, \varepsilon_{v_1} \right) \left( L(u_1, v_1) \right) = \sup \left\{ T \left( \varepsilon_{u_1}(u), \varepsilon_{v_1}(v) \right) \mid L(u, v) = L(u_1, v_1) \right\}$$
$$\geq T \left( \varepsilon_{u_1}(u_2), \varepsilon_{v_1}(v_2) \right) = T(1, 1) = 1.$$

Therefore  $L(u_1, v_1) > 0$ , since, otherwise  $\tau_{T,L}(\varepsilon_{u_1}, \varepsilon_{v_1})(0) = 0$ .

For every  $y \in [0, L(u_1, v_1)]$ , the relationship L(u, v) = y implies either  $u < u_1$ or  $v < v_2$ , or both. Since

 $\tau_{T,L}\left(\varepsilon_{u_{1}},\varepsilon_{v_{1}}\right)\left(y\right) = \sup\left\{T\left(\varepsilon_{u_{1}}(u),\varepsilon_{v_{1}}(v)\right) \mid L(u,v) = y\right\},\$ 

one has  $\tau_{T,L}(\varepsilon_{u_1}, \varepsilon_{v_1})(y) = 0$  and, as a consequence,

$$\sup_{y < L(u_1, v_1)} \tau_{T,L} \left( \varepsilon_{u_1}, \varepsilon_{v_1} \right) \left( y \right) = 0 < 1 = \tau_{T,L} \left( \varepsilon_{u_1}, \varepsilon_{v_1} \right) \left( L(u_1, v_1) \right),$$

so that  $\tau_{T,L}(\varepsilon_{u_1}, \varepsilon_{v_1})$  is not left-continuous. Thus L satisfies (LS).

As a consequence of Lemma 7.10, T is known to be a *t*-norm. It remains to show that it is also continuous; this is done in the following lemma which provides a slightly more general result since fewer restrictions are imposed on L.

**Lemma 7.14.** Let S be a commutative semicopula and let  $L \in \mathfrak{L}$  satisfy conditions (LS) and (L0) of Definition 3.5. If  $\tau_{S,L}$  is a continuous triangle function, then S is also continuous.

*Proof.* Let S be a commutative semicopula. Let x be any strictly positive real number and let  $(x_n)_{n \in \mathbb{N}}$  be any sequence of positive real numbers converging to x. By recourse to Lemma 5.1 it suffices to prove that, for every  $y_0 > 0$ , one has that, for every  $\varepsilon > 0$ , some  $n \in \mathbb{N}$  exists such that, for all  $m \geq n$ ,

$$|S(x, y_0) - S(x_m, y_0)| < \varepsilon.$$

$$(7.9)$$

Notice that since S takes values in the unit interval, the inequality is trivially fulfilled for any  $\varepsilon \geq 1$ . We shall therefore restrict our considerations to  $\varepsilon < 1$  only.

Next we associate with each  $x_n$  the d.d.f.  $V_{x_n}$  defined by (2.1) and select similarly  $V_x$  and  $G = V_{y_0}$  through (2.1). Therefore, we get a sequence of d.d.f.'s  $(V_{x_n})_{n \in \mathbb{N}}$  which converges to  $V_x$  pointwise, i.e.,  $V_{x_n}(u) \longrightarrow V_x(u)$  for all  $u \in \mathbb{R}_+$  and, as a consequence, converges also weakly, i.e. for any  $\delta > 0$  there exists some  $n \in \mathbb{N}$  such that, for all  $m \geq n$ ,  $|V_x(u) - V_{x_m}(u)| < \delta$ , and, even more,  $d_S(V_{x_m}, V_x) < \delta$ .

Now choose  $\varepsilon$  such that  $0 < \varepsilon < 1$ . Since  $\tau_{S,L}$  is uniformly continuous on the metric space  $(\Delta^+, d_S)$ , it follows that there exists some  $\gamma > 0$  such that

 $d_S(\tau_{S,L}(V_{x_n},G),\tau_{S,L}(V_x,G)) < \varepsilon, \quad \text{whenever} \quad d_S(V_{x_n},V_x) < \gamma.$ 

Note that  $d_S(G,G) < \gamma$  is trivially fulfilled. Since  $(V_{x_n})_{n \in \mathbb{N}}$  is a convergent sequence of d.d.f.'s, one can choose  $n^* \in \mathbb{N}$  such that  $|V_x(u) - V_{x_m}(u)| < \gamma$  for all  $u \in \mathbb{R}_+$  and also  $d_S(V_{x_m}, V_x) < \gamma$  for all  $m \ge n^*$ . Choose such an  $m \ge n^*$  arbitrarily and fix it for the rest of the proof.

Assume first that  $d_S(\tau_{S,L}(V_{x_m}, G), \tau_{S,L}(V_x, G)) > 0$ ; then there exists some  $0 < h^* < \varepsilon$  such that, for all  $u \in [0, 1/h^*[$ ,

$$\tau_{S,L}(V_{x_m},G)(u) \le \tau_{S,L}(V_x,G)(u+h^*) + h^*, \tau_{S,L}(V_x,G)(u) \le \tau_{S,L}(V_{x_m},G)(u+h^*) + h^*$$

are fulfilled. Now choose  $u^*$  with  $0 < u^* \leq \min\{\frac{1}{\varepsilon}, 1-\varepsilon\}$ . Then necessarily  $u^* + h^* < 1$ . Because of properties (L0) and (LS) and because of the fact that  $u^* > 0$ , there exists a pair (u, v) with  $u^* \geq u > 0$  and  $u^* \geq v > 0$  such that

 $L(u, v) = u^*$ . Therefore,

$$\tau_{S,L}(V_{x_m}, G)(u^*) = \sup\{S(V_{x_m}(u), G(v)) \mid L(u, v) = u^*\}$$
  
= sup{S(0, y\_0), S(x\_m, 0), S(x\_m, y\_0)}  
= S(x\_m, y\_0) = \tau\_{S,L}(V\_{x\_m}, G)(u^\* + h^\*),  
$$\tau_{S,L}(V_x, G)(u^*) = S(x, y_0) = \tau_{S,L}(V_x, G)(u^* + h^*),$$

and, as a consequence,

 $S(x_m, y_0) - \varepsilon < S(x_m, y_0) - h^* \le S(x, y_0) \le S(x_m, y_0) + h^* < S(x_m, y_0) + \varepsilon$ , namely

$$|S(x, y_0) - S(x_m, y_0)| < \varepsilon.$$

When  $d_S(\tau_{S,L}(V_{x_m}, G), \tau_{S,L}(V_x, G)) = 0$ ,  $\tau_{S,L}(V_{x_m}, G) = \tau_{S,L}(V_x, G)$  since  $d_S$  is a metric on  $\Delta^+$ . Choosing an arbitrary  $v^* \in [0, 1[$  leads to

$$S(x_m, y_0) = \tau_{S,L}(V_{x_m}, G)(v^*) = \tau_{S,L}(V_x, G)(v^*) = S(x, y)$$

so that  $|S(x, y_0) - S(x_m, y_0)| < \varepsilon$  is trivially fulfilled. Therefore, for arbitrary  $\varepsilon > 0$ there exists  $n = n(\varepsilon) \in \mathbb{N}$  such that for any  $m \ge n$ ,  $|S(x, y_0) - S(x_m, y_0)| < \varepsilon$ , which proves the continuity of the semicopula S.

# 8. Operations involving co-semicopulas

**Definition 8.1** (compare also [41, Definition 7.3.1]). Let  $S^*$  be a co-semicopula and let L belong to  $\mathfrak{L}$ ; a function  $\tau^*_{S^*,L} : \Delta^+ \times \Delta^+ \to [0,1]^{\mathbb{R}_+}$  is defined for every pair F and G of d.d.f.'s and for every  $x \in \overline{\mathbb{R}}_+$  by

$$\tau_{S^*,L}^*(F,G)(x) = \inf\{S^*(F(u),G(v)) \mid L(u,v) = x\}.$$
(8.1)

If L is the sum, then this index is usually omitted, and we write simply  $\tau_{S^*}^*$ .

The notation adopted here is slightly different from that of the book [41]. The choice of a co-semicopula  $S^*$  in the above definition guarantees that, for all F and G in  $\Delta^+$ , the function  $\tau^*_{S^*,L}(F,G)$  is increasing on  $\mathbb{R}_+$ , satisfies  $\tau^*_{S^*,L}(F,G)(0) = 0$  and at least  $\tau^*_{S^*,L}(F,G)(\infty) \leq 1$ . Since every co-semicopula is an increasing function, also  $\tau^*_{S^*,L}$  is increasing on  $\Delta^+$  in each argument (see also [41, Lemma 7.3.2]). It was shown in [41, Lemma 7.3.7] that, for every co-semicopula  $S^*$ ,  $\tau^*_{S^*,\min} = \Pi_{S^*}$ . Theorem 6.1 ensures that  $S^*$  has to be left-continuous in order to guarantee that  $\tau_{S^*,\min}$  is indeed a binary operation on  $\Delta^+$ .

We present the proof of the following results, since they do not appear explicitly in [41]; our Lemma 8.1 slightly differs from Lemma 7.3.3 in [41]. It demands an additional condition on L namely (L0), which, in our opinion, is essential for proving the result. There is no proof of this result in [41] where the reader is only referred to a modification of the proof of Theorem 5.1 in [34], and this, in its turn, deals only with the special case L = +. **Lemma 8.1.** If  $S^*$  is a left-continuous co-semicopula and L satisfies conditions (LS) and (L0) of Definition 3.5, then  $\tau^*_{S^*,L}$  is a binary operation on  $\Delta^+$ .

Proof. Let S be a left-continuous semicopula. Then  $S(u, v) := 1 - S^*(1 - u, 1 - v)$ is the right-continuous semicopula associated with  $S^*$ . We show that  $\tau_{S^*,L}$  sends  $\Delta^+ \times \Delta^+$  into  $\Delta^+$ . To this end, let F and G be arbitrary d.d.f.'s and define

$$V_{F,G}(x) := \sup \left\{ S(1 - F(u), 1 - G(v) \mid L(u, v) = x \right\};$$

then

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$$\tau^*_{S^*L}(F,G)(x) = 1 - V_{F,G}(x).$$

We now have to show that  $1 - V_{F,G}$  is a d.d.f., namely, that (i)  $V_{F,G}(x)$  is in [0,1] for every x > 0, (ii) V(0) = 1, (iii)  $V_{F,G}$  is decreasing, and (iv)  $V_{F,G}$  is left-continuous. The proof of (i) and (ii) is trivial, since S is an operation on [0,1] and L is increasing in each coordinate and fulfils L(0,0) = 0. As for (iii), let  $\delta > 0$  be given and let  $x_1$  and  $x_2$  be such that  $x_2 > x_1$ . Then, there exist  $u_2$  and  $v_2$  with  $L(u_2, v_2) = x_2$  such that

$$0 \le V_{F,G}(x_2) - S(1 - F(u_2), 1 - G(v_2)) < \delta.$$

Because of properties (LS) and (L0) of Definition 3.5, there exist  $u_1$  and  $v_1$  such that

$$L(u_1, v_1) = x_1, \quad u_1 \le u_2, \quad v_1 \le v_2.$$

Therefore,

$$V_{F,G}(x_1) \ge S (1 - F(u_1), 1 - G(v_1))$$
  

$$\ge S (1 - F(u_2), 1 - G(v_2)) > V_{F,G}(x_2) - \delta.$$

The arbitrariness of  $\delta > 0$  implies  $V_{F,G}(x_1) \ge V_{F,G}(x_2)$ .

In order to establish (iv), the left-continuity of  $\tau_{S^*,L}^*(F,G)$  at an arbitrary point  $x_0 > 0$ , notice that there is nothing to prove if  $V_{F,G}(x_0) = 1$ ; therefore assume  $V_{F,G}(x_0) < 1$ , and suppose, if possible, that  $V_{F,G}$  is not left-continuous at  $x_0$ ; then, there exist  $\eta > 0$  and a sequence of points  $(x_n)_{n \in \mathbb{N}}$  increasing to  $x_0$  such that, for all  $n \in \mathbb{N}$ ,

$$V_{F,G}(x_n) \ge V_{F,G}(x_0) + 2\eta.$$

It follows from the definition of  $\tau^*_{S^*,L}(F,G)$ , that, for every  $n \in \mathbb{N}$ , there exist some  $u_n$  and  $v_n$  such that  $L(u_n, v_n) = x_n$  and

$$V_{F,G}(x_n) \ge S\left(1 - F(u_n), 1 - G(v_n)\right) \ge V_{F,G}(x_0) + \eta.$$
(8.2)

Thus, since  $x_n \leq x_0$  for every n, condition (L0) of Definition 3.5 implies that the sequences  $(u_n)_{n\in\mathbb{N}}$  and  $(v_n)_{n\in\mathbb{N}}$  are bounded and, as a consequence, contain convergent subsequences  $(u_{n(k)})_{k\in\mathbb{N}}$  and  $(v_{n(k)})_{k\in\mathbb{N}}$ . We may choose these sequences to be monotone. Set

$$u_0 := \lim_{k \to +\infty} u_{n(k)}$$
 and  $v_0 := \lim_{k \to +\infty} v_{n(k)}$ .

Since  $L(u_{n(k)}, v_{n(k)}) = x_{n(k)}$ , the continuity of the function L yields

$$L(u_0, v_0) = x_0. (8.3)$$

We now distinguish three cases:

Case 1. The sequences  $(u_{n(k)})_{k\in\mathbb{N}}$  and  $(v_{n(k)})_{k\in\mathbb{N}}$  increase to  $u_0$  and  $v_0$  respectively. Since F and G are left-continuous and S is right-continuous, the sequence

$$\left(S\left(1-F(u_{n(k)}),1-G(v_{n(k)})\right)\right)_{k\in\mathbb{N}}$$

decreases to  $S(1 - F(u_0), 1 - G(v_0))$ . But then the definition of  $\tau^*_{S^*,L}$  and equations (8.2) and (8.3) yield

$$V_{F,G}(x_0) \ge S (1 - F(u_0), 1 - G(v_0)) \ge V_{F,G}(x_0) + \eta,$$

which is impossible.

Case 2. The sequences  $(u_{n(k)})_{k\in\mathbb{N}}$  and  $(v_{n(k)})_{k\in\mathbb{N}}$  decrease to  $u_0$  and  $v_0$  respectively; this cannot happen since  $L(u_{n(k)}, v_{n(k)}) = x_{n(k)} < x_0$ .

Case 3. One of the two sequences, say  $(u_{n(k)})_{k\in\mathbb{N}}$ , increases, while the other one,  $(v_{n(k)})_{k\in\mathbb{N}}$ , decreases. Then the sequence  $(1 - F(u_{n(k)}))_{k\in\mathbb{N}}$  decreases to  $1 - F(u_0)$ , while the sequence  $(1 - G(v_{n(k)}))_{k\in\mathbb{N}}$  increases to  $1 - \ell^+ G(v_0)$ . Since, for every  $k \in \mathbb{N}$ ,

$$1 - \ell^+ G(v_{n(k)}) \le 1 - \ell^+ G(v_0) \le 1 - G(v_0),$$

we have

$$S\left(1 - F(u_{n(k)}), 1 - G(v_0)\right) \ge S\left(1 - F(u_{n(k)}), 1 - G(v_{n(k)})\right) \ge V_{F,G}(x_0) + \eta.$$

Let k go to  $+\infty$  in order to obtain

$$V_{F,G}(x_0) \ge S (1 - F(u_0), 1 - G(v_0)) \ge V_{F,G}(x_0) + \eta.$$

This is again a contradiction, so that  $V_{F,G}$  is indeed left-continuous at  $x_0$ . Therefore  $\tau^*_{S^*,L}(F,G) = 1 - V_{F,G}$  belongs to  $\Delta^+$ , i.e.  $\tau^*_{S^*,L}$  is indeed an operation on  $\Delta^+$ .

We can now state the formal theorem that gives conditions under which  $\tau^*_{S^*,L}$  is a triangle function.

**Theorem 8.2.** If  $T^*$  is a continuous t-conorm and if  $L \in \mathfrak{L}$  is commutative, associative and satisfies properties (LS) and (L0) of Definition 3.5, then  $\tau^*_{T^*,L}$  is a triangle function.

*Proof.* First, because of Lemma 8.1 we know that  $\tau^*_{T^*,L}$  is already a binary operation on  $\Delta^+$ . Therefore, only the properties of triangle functions remain to be checked. Let F be an arbitrary d.d.f.; then

$$\tau_{T^*,L}^*(F,\varepsilon_0)(x) = \inf \left\{ T^*(F(u),\varepsilon_0(v)) \mid L(u,v) = x \right\} = T^*(F(x),0) = F(x),$$

so that  $\tau^*_{T^*,L}(F,\varepsilon_0) = F$  for every  $F \in \Delta^+$  and property (TF4) is satisfied.

Properties (TF1) and (TF3) are easy consequences of the properties of  $T^*$ and L. Before proving the associativity of  $\tau^*_{T^*,L}$ , we notice that a t-conorm is also

associative, as is easily checked. Let F, G and H be in  $\Delta^+$ ; then

$$\begin{split} \tau^*_{T^*,L} \left( \tau^*_{T^*,L}(F,G), H \right)(x) &= \inf \left\{ T^* \left( \tau^*_{T^*,L}(F,G)(u), H(v) \right) \mid L(u,v) = x \right\} \\ &= \inf_{L(u,v)=x} T^* \left( \inf_{L(s,t)=u} T^* \left( F(s), G(t) \right), H(v) \right) \\ &= \inf_{L(u,v)=x} \inf_{L(s,t)=u} T^* \left( T^* \left( F(s), G(t) \right), H(v) \right) \\ &= \inf_{L(L(s,t),v)=x} T^* \left( F(s), T^* \left( G(t), H(v) \right) \right) \\ &= \inf_{L(s,L(t,v))=x} T^* \left( F(s), \inf_{L(t,v)=w} T^* \left( G(t), H(v) \right) \right) \\ &= \inf_{L(s,w)=x} T^* \left( F(s), \pi^*_{T^*,L}(G,H)(w) \right) = \tau^*_{T^*,L} \left( F, \tau^*_{T^*,L}(G,H) \right)(x), \end{split}$$

where in the second line we have used the continuity of  $T^*$ , in the fourth one the associativity of  $T^*$ , in the fifth line that of L. This concludes the proof.

Finally, we mention a result showing that under the assumptions of Theorem 8.2  $\tau_{S^*,L}^*$  is not only a triangle function, but even more, it is continuous on the metric space  $(\Delta^+, d_S)$ .

**Lemma 8.3** ([41, Corollary 7.3.9]). If  $T^*$  is a continuous t-conorm and if  $L \in \mathfrak{L}$  is commutative, associative and fulfils both (LS) and (L0) of Definition 3.5, then the triangle function  $\tau^*_{T^*,L}$  is uniformly continuous on the metric space  $(\Delta^+, d_S)$ .

## 9. Operations involving quasi-copulas

**Definition 9.1** ([41, Definition 7.5.1]). Let Q be a quasi-copula and let L belong to  $\mathfrak{L}$ ; a function  $\rho_{Q,L} : \Delta^+ \times \Delta^+ \to [0,1]^{\mathbb{R}_+}$  is defined, for every pair F and G of d.d.f.'s and for every  $x \in \mathbb{R}_+$ , by

$$\rho_{Q,L}(F,G)(x) = \inf\{\overline{Q}(F(u),G(v)) \mid L(u,v) = x\}$$

where  $\overline{Q}$  is defined by (3.4).

Also in this case we have abstained from introducing  $\rho$  for a more general class of operation, since the resemblance of this definition with Definition 8.1 is obvious. More precisely, we can state that functions of type  $\rho_{Q,L}$  are completely covered by Definition 8.1. To see this, notice that every quasi-copula Q fulfils the

1-Lipschitz property. As a consequence,  $\overline{Q}$  is increasing in each argument (see also Lemma 3.3). Moreover,  $\overline{Q}$  has neutral element 0, since

$$\overline{Q}(u,0) = u + 0 - Q(u,0) = u = \overline{Q}(0,u).$$

Therefore,  $\overline{Q}$  is a co-semicopula. Now it immediately follows that, for every quasicopula Q,

$$\rho_{Q,L} = \tau_{\overline{Q},L}^*. \tag{9.1}$$

Although, the operations  $\rho_{Q,L}$  can be subsumed as particular cases of operations of the type  $\tau_{S^*,L}^*$ , it is reasonable to consider them separately. In particular, for those  $L \in \mathfrak{L}$  that are so called "composition laws", the operations  $\rho_{C,L}$  are of importance in the generalized theory of information of Kampé de Fériet and Forte (see also [14, 39]).

We are now in a position to rephrase Theorem 7.5.2 and Corollary 7.5.3 in [41]. We shall also give the proof of this latter result (see Corollary 9.2, below)<sup>4</sup> since the proof in [41] refers to a modification of the proof of Theorem 5.1 in [34].

**Lemma 9.1** ([41, Theorem 7.5.2]). Let Q be a quasi-copula and let  $L \in \mathfrak{L}$  satisfy (LS). Then  $\rho_{Q,L}$  is a binary operation on  $\Delta^+$ .

**Corollary 9.2** ([41, Corollary 7.5.3]). If Q is a symmetric quasi-copula, if L is also commutative and satisfies properties (LS) and (L0) in Definition 3.5, and if both  $\overline{Q}$  and L are associative, then  $\rho_{Q,L}$  is a triangle function.

*Proof.* Since Q is a symmetric quasi-copula,  $\overline{Q}$  is a symmetric continuous cosemicopula. Moreover, the associativity of  $\overline{Q}$  implies that  $\overline{Q}$  is a continuous *t*conorm; therefore, all the assumptions of Theorem 8.2 are fulfilled.  $\Box$ 

Similarly one proves

**Lemma 9.3.** If Q is a symmetric quasi-copula, if L is commutative and satisfies properties (LS) and (L0) in Definition 3.5, and if both  $\overline{Q}$  and L are associative, then the triangle function  $\rho_{Q,L}$  is uniformly continuous on the metric space  $(\Delta^+, d_S)$ .

As a consequence of previous examples and Equation (9.1), we can immediately conclude (compare also [41, Lemma 7.5.4]) that, for every quasi-copula Q,

 $\rho_{Q,\max} = \tau_{\overline{Q},\max}^* = \Pi_{\min} \quad \text{and} \quad \rho_{Q,\min} = \tau_{\overline{Q},\min}^* = \Pi_{\overline{Q}}.$ 

 $<sup>\</sup>frac{4}{4}$  The reader ought to be alerted to the fact that its correct statement can be found in the *errata* in the Dover edition of the book [41].

### 10. Operations involving copulas

In analogy with convolution \*, i.e., the function  $*: \Delta^+ \times \Delta^+ \to \Delta^+$  defined by  $(F * G)(0) := 0, (F * G)(\infty) := 1$  and

$$(F * G)(x) := \int_{[0,x[} F(x-t) \, dG(t) \tag{10.1}$$

for arbitrary  $F, G \in \Delta^+$ , the following class of operations  $\sigma_{C,L}$  was introduced by Sklar in [51].

**Definition 10.1** ([41, Definition 7.4.1]). Let *C* be a copula and *L* belong to  $\mathfrak{L}$ ; a function  $\sigma_{C,L}: \Delta^+ \times \Delta^+ \to \Delta^+$  is defined, for every pair *F* and *G* of d.d.f.'s and for every  $x \in \mathbb{R}_+$ , by

$$\sigma_{C,L}(F,G)(0) := 0, \qquad \sigma_{C,L}(F,G)(\infty) := 1$$

and

$$\sigma_{C,L}(F,G)(x) := \int_{L(x)} dC(F(u), G(v))$$
(10.2)

for all  $x \in [0, +\infty)$ , where

$$L(x) = \{ (u, v) \mid u, v \in \mathbb{R}_+, L(u, v) < x \}.$$

If L is the sum, then we drop L in  $\sigma_{C,L}$  and simply write  $\sigma_C$ .

The integral in (10.2) is just the Lebesgue–Stieltjes *H*-measure of the set L(x)where H(u, v) = C(F(u), G(v)) for all  $u, v \in \mathbb{R}$ . The operation  $\sigma_{C,L}$  has a probabilistic interpretation (see [40, Theorem 4]): if X and Y are positive real-valued random variables on a probability space  $(\Omega, \mathcal{A}, P)$ , having d.f.'s  $F_X$  and  $F_Y$ , and if C is their copula, then the d.f.  $F_{L(X,Y)}$  of the random variable L(X,Y) is given by

$$F_{L(X,Y)}(t) = \int_{\{(u,v)\in\mathbb{R}_+:L(u,v)\leq t\}} dC \left(F_X(u), F_Y(v)\right) = \sigma_{C,L}(F,G)(t)$$

In order to avoid possible misunderstandings, we notice that if the d.f.'s  $F_X$  and  $F_Y$  of the random variables X and Y are continuous the copula C of X and Y is unique. Otherwise, the given pair of random variables uniquely defines a subcopula on Ran  $F_X \times \text{Ran } F_Y$ , rather than a copula; it is then possible to use a bilinear interpolation (see [6, 31]) in order to single out a unique copula. Therefore one can speak of *the* copula of the random variables X and Y.

For every copula C and for every  $L \in \mathfrak{L}$ ,  $\sigma_{C,L}$  is a binary operation on  $\Delta^+$  (see also Theorem 4.2 in [10] and Theorem 7.4.2 in [41]). Moreover,  $\sigma_{C,L}$  is increasing in each argument on  $\Delta^+ \times \Delta^+$ . Therefore, one of the properties of a triangle function is already satisfied by  $\sigma_{C,L}$ . The other properties will now be examined.

The first ones of these were derived in [11] by Frank who extended his previous results of [10] dealing with the particular case L = +.

**Lemma 10.1** ([11, Theorem 2.1]). Let C be a copula and  $L \in \mathfrak{L}$ . Then  $\sigma_{C,L}$  has  $\varepsilon_0$  as its neutral element if, and only if, L fulfills (L0) of Definition 3.5.

**Lemma 10.2** ([11, Theorem 2.2]). Let C be a copula and  $L \in \mathfrak{L}$ . If  $\sigma_{C,L}$  is commutative, then also L is commutative. Vice versa, if C and L are commutative, then so is  $\sigma_{C,L}$ .

Finally, we turn to associativity. If  $\sigma_{C,L}$  is associative, then also L has necessarily to be associative (compare [11, Theorem 2.3]). Therefore, when  $\sigma_{C,L}$  is a triangle function, L must necessarily have neutral element 0, and must be both commutative and associative. Moreover, by definition, it is increasing and continuous in each of its arguments. Therefore, it is reasonable to consider, in particular, functions  $L \in \mathfrak{L}$  for which there exists some continuous and strictly increasing function  $h: \mathbb{R}_+ \to \mathbb{R}_+$  such that

$$L(u,v) = h^{-1}(h(u) + h(v)).$$
(10.3)

Such functions have no idempotent elements different from 0 and  $\infty$ . Therefore, e.g., max is not covered by this approach, since any element of  $\overline{\mathbb{R}}_+$  is an idempotent element of max. We shall therefore consider this case separately.

The following theorem provides sufficient conditions that ensure that  $\sigma_{C,L}$  is associative.

**Theorem 10.3** ([11, Theorem 4.4]). Let C be a copula and let  $L \in \mathfrak{L}$  have the form  $L(u, v) = h^{-1}(h(u) + h(v))$  where  $h: \mathbb{R}_+ \to \mathbb{R}_+$  is continuous and strictly increasing. Then  $\sigma_{C,L}$  is associative if, and only if, C is a (trivial or non-trivial) ordinal sum of product t-norms, i.e., either C is the minimum or there exist some index set  $I \neq \emptyset$  and some corresponding family  $(]a_i, b_i[]_{i \in I}$  of pairwise disjoint subintervals of the unit interval, such that

$$C(x,y) = \begin{cases} a_i + \frac{1}{b_i - a_i} (x - a_i)(y - a_i), & \text{if } (x,y) \in [a_i, b_i]^2, \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

Ordinal sums of products are not only copulas but also *t*-norms. Therefore, the triangle functions of the preceding theorem coincide with those induced by a specific class of *t*-norms; moreover, the class of admissible copulas is rather small. Choosing *h* to be the identity mapping leads again to the standard convolution (10.1), which is then equal to  $\sigma_{\Pi}$ . When *L* equals the minimum or the maximum  $\sigma_{C,L}$  can be computed directly (see also [41]):

•  $\sigma_{C,\max} = \Pi_C$ : since  $\max(x) = [0, x[ \times [0, x[,$ 

$$\sigma_{C,\max}(F,G)(x) = \int_{[0,x[\times[0,x[} dC(F(u),G(v)) = C(F(x),G(x)) = \Pi_C(F,G)(x)) dC(F(u),G(v)) = \Gamma_C(F,G)(x) dC(F(u),G(v)) = \Gamma_C(F,G)(v) dC(F(u$$

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$$\begin{aligned} \sigma_{C,\min} &= \Pi_{\overline{C}} \colon \text{since } \min(x) = \left( [0, x[\times \overline{\mathbb{R}}_+) \cup \left( \overline{\mathbb{R}}_+ \times [0, x[] \right), \right. \\ \sigma_{C,\min}(F, G)(x) &= \int_{\min(x)} dC(F(u), G(v)) \\ &= P_H \left( [0, x[\times \overline{\mathbb{R}}_+) + P_H \left( \overline{\mathbb{R}}_+ \times [0, x[] \right) - P_H \left( [0, x[\times [0, x[]) \\ &= F(x) + G(x) - C(F(x), G(x)) = \overline{C}(F(x), G(x)) = \Pi_{\overline{C}}(F, G)(x). \end{aligned}$$

**Lemma 10.4.** For a copula C,  $\sigma_{C,\max}$  is a triangle function if, and only if, C is a continuous t-norm.

*Proof.* It follows from Theorem 5.2 that  $\sigma_{C,\max} = \prod_C$  is a triangle function, if, and only if, C is a left-continuous *t*-norm. As a consequence C has to be associative and therefore a continuous *t*-norm.

**Lemma 10.5.** For no copula C is  $\sigma_{C,\min}$  a triangle function.

*Proof.* It follows again from Theorem 5.2 that  $\sigma_{C,\min} = \prod_{\overline{C}}$  is a triangle function, whenever, if possible,  $\overline{C}$  is a left-continuous *t*-norm. However, since 1 is the unique null element of  $\overline{C}$  this is never the case. Therefore,  $\sigma_{C,\min}$ , thanks to the continuity of C (and therefore also of  $\overline{C}$ ), is a binary operation on  $\Delta^+$ , but never a triangle function.

Notice that this result also follows immediately from the fact that min has neutral element 1 and not 0 as demanded by Lemma 10.1. Therefore, any  $\sigma_{C,\min}$  is a binary operation on  $\Delta^+$  but not a triangle function.

#### 11. Inequalities

Interesting inequalities hold among the operations  $\tau_{f,L}$ ,  $\tau^*_{S^*,L}$ ,  $\rho_{Q,L}$  and  $\sigma_{C,L}$  that have been introduced in the previous sections (see [30, 41]).

**Theorem 11.1** ([41, Theorem 7.2.12]). If  $L_1$  and  $L_2$  are functions in  $\mathfrak{L}$  with  $L_1 \leq L_2$ , and  $S_1$  and  $S_2$  are in  $\mathcal{S}$  with  $S_1 \leq S_2$ , then

$$au_{S_1,L_2} \le au_{S_2,L_1}.$$

In particular, if Q is a quasi-copula, i.e.,  $Q \in Q$ , then

$$\tau_{W,L_2} \le \tau_{Q,L_2} \le \tau_{Q,L_1} \le \tau_{M,L_1};$$

if, moreover, L satisfies property (L0) of Definition 3.5, then  $L \ge \max$  and

$$\tau_{S,L} \le \Pi_S \le \Pi_M$$

for every  $S \in \mathcal{S}$ .

**Theorem 11.2** ([41, Theorem 7.5.5]). Let Q be a quasi-copula and let L satisfy condition (LS) of Definition 3.5. Then

$$\tau_{W,L} \le \tau_{Q,L} \le \rho_{Q,L} \le \rho_{W,L};$$

moreover, if C is a copula, then

$$\tau_{C,L} \leq \sigma_{C,L} \leq \rho_{C,L}.$$

**Theorem 11.3** ([41, eq. (7.4.10)]). Let C belong to C and L to  $\mathfrak{L}$ , then  $\sigma_{C,L_2} \leq \sigma_{C,L_1}$  whenever  $L_1 \leq L_2$ .

Some of the previous results can be strengthened if Q = M.

**Theorem 11.4** ([41, Theorem 7.5.6]). If F and G are in  $\mathcal{D}^+$ , namely if they are proper d.d.f.'s, then, for every  $L \in \mathfrak{L}$ ,

$$\tau_{M,L}(F,G) = \rho_{M,L}(F,G).$$
 (11.1)

If, in addition, L has  $+\infty$  as its null element and is continuous on all of  $\overline{\mathbb{R}}^2_+$ , then (11.1) holds for all F and G in  $\Delta^+$ .

We provide an example in order to show that the equality in (11.1) need not hold if F and G are not in  $\mathcal{D}^+$ , and L does not have  $+\infty$  as a null element. To this end, consider the d.d.f.'s  $V_s$  and  $V_t$  defined in (2.1), with s < t and choose  $L = \Pi$ , i.e., the product, for which  $+\infty$  is not a null element, since  $0 \cdot (+\infty) = 0$ . Then, for every  $x \in ]0, +\infty[$ ,

$$\tau_{M,\Pi}(V_s, V_t)(x) = \sup_{uv=x} M(V_s(u), V_t(v)) = M(s, t) = s,$$

while

$$\rho_{M,\Pi}(V_s, V_t)(x) = \inf_{uv=x} \overline{M}\left(V_s(u), V_t(v)\right) = \max\{s, t\} = t > s.$$

The following result is a direct consequence of Theorems 11.2 and 11.4.

**Theorem 11.5** ([41, Corollary 7.5.7]). If  $L \in \mathfrak{L}$  has  $+\infty$  as its null element, is continuous on all of  $\overline{\mathbb{R}}^2_+$  and satisfies condition (LS) of Definition 3.5, then

$$\tau_{M,L} = \rho_{M,L} = \sigma_{M,L}.\tag{11.2}$$

A comparison of Definitions 8.1 and 9.1 yields

**Corollary 11.6** ([41, Corollary 7.5.8]). For every  $L \in \mathfrak{L}$ ,

$$\tau_{W^*,L}^* = \rho_{W,L}, \quad \tau_{\Pi^*,L}^* = \rho_{\Pi,L}, \quad \tau_{M^*,L}^* = \rho_{M,L}.$$
  
Therefore, under the hypotheses of Theorem 11.4,

$$\tau_{M^*,L} = \tau_{M,L}$$

## 12. The stability of $\mathcal{D}^+$

In different questions concerning PN spaces, for instance in the study of boundedness, it is relevant to know when the space  $\mathcal{D}^+$  of proper d.d.f.'s is stable under a triangle function  $\tau$ , or, equivalently, when a triangle function  $\tau$  is a binary operation on the set  $\mathcal{D}^+$  of proper d.d.f.'s, namely when

$$\tau \left( \mathcal{D}^+ \times \mathcal{D}^+ \right) \subseteq \mathcal{D}^+. \tag{12.1}$$

Not every triangle function  $\tau$  satisfies (12.1). For instance, take  $\tau = \tau_D$ , where  $\tau_D$  is given by (7.2). If  $\Phi_+$  is the d.f. of the random variable |X| where X has the standard normal law,  $X \sim N(0, 1)$ , so that  $\Phi_+(0) = 0$  and, if x > 0,

$$\Phi_{+}(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{x} \exp\left(-t^{2}/2\right) dt;$$

thus  $\Phi^{\wedge}_{+}(1) = +\infty$ . Then, for every x > 0,

$$\tau_D(\Phi_+, \Phi_+)(x) = \max\{\Phi_+(0), \Phi_+(0)\} = 0,$$

namely  $\tau_D(\Phi_+, \Phi_+) = \varepsilon_{\infty}$ , which is not in  $\mathcal{D}^+$  while  $\Phi_+$  is.

Sufficient conditions for (12.1) are provided by the following theorem.

**Theorem 12.1.** If one of the following conditions holds:

- (a)  $\tau = \Pi_T$  for some left-continuous t-norm T;
- (b)  $\tau = \tau_{T,L}$  for some left-continuous t-norm T and for some  $L \in \mathfrak{L}$ ;
- (c)  $\tau = \sigma_{C,L}$  for some copula C and for some  $L \in \mathfrak{L}$  such that property (LS) of Definition 3.5 is satisfied;
- (d)  $\tau$  is the convolution \*;
- (e)  $\rho_{C,L}$ , where C is an associative and symmetric copula and L satisfies property (LS) of Definition 3.5,

then the set  $\mathcal{D}^+$  is stable under  $\tau$ , i.e.,  $\tau(\mathcal{D}^+ \times \mathcal{D}^+) \subseteq \mathcal{D}^+$ .

*Proof.* Let F and G be d.f.'s belonging to  $\mathcal{D}^+$ , i.e., such that

$$\lim_{t \to +\infty} F(t) = \lim_{t \to +\infty} G(t) = 1$$

(a) If  $\tau = \Pi_T$  for some left-continuous *t*-norm *T*, then

$$\Pi_T \left( F, G \right) \left( t \right) = T \left( F(t), G(t) \right) \xrightarrow[t \to +\infty]{} T(1, 1) = 1$$

so that also  $\Pi_T(F,G)$  belongs to  $\mathcal{D}^+$ .

(b) In view of the definition of  $\tau_{T,L}$ , one has, for every x > 0, for all u and v such that L(u, v) = x, and for every pair of d.d.f.'s (F, G),

$$\tau_{T,L}(F,G)(x) \ge T\left(F(u),G(v)\right).$$

Now, let x tend to  $+\infty$  in order to obtain, for all u and v in  $\mathbb{R}_+$ ,

$$\ell^{-}\tau_{T,L}(F,G)(+\infty) \ge T\left(F(u),G(v)\right).$$

Letting u and v go to  $+\infty$  yields, because T, F and G are all left-continuous,

$$\ell^{-}\tau_{T,L}(F,G)(+\infty) \ge T\left(F(u),G(v)\right) \xrightarrow[u \to +\infty, v \to +\infty]{} T(1,1) = 1$$

Thus,  $\tau_{T,L}(F,G)$  is in  $\mathcal{D}^+$ .

(c) We have already stated that, if X and Y are positive real-valued random variables on a probability space  $(\Omega, \mathcal{A}, P)$ , having d.f.'s  $F_X$  and  $F_Y$ , if C is their copula, then the d.f.  $F_{L(X,Y)}$  of the random variable L(X,Y) is given by

$$F_{L(X,Y)}(t) = \int_{\{(u,v)\in\mathbb{R}_+:L(u,v)\leq t\}} dC \left(F_X(u), F_Y(v)\right).$$

But, since both X and Y are real-valued, both

$$P(X < +\infty) = 1$$
 and  $P(Y < +\infty) = 1$ ,

or, equivalently,

$$\lim_{t \to +\infty} F_X(t) = 1 \quad \text{and} \quad \lim_{t \to +\infty} F_Y(t) = 1$$

hold. On account of property (LS) of Definition 3.5, also L(X, Y) is a.c. finite, viz.  $P(L(X, Y) < +\infty) = 1$ , or, equivalently,

$$1 = \lim_{t \to +\infty} F_{L(X,Y)}(t) = \lim_{t \to +\infty} \int_{\{(u,v) \in \mathbb{R}_+ : L(u,v) \le t\}} dC \left( F_X(u), F_Y(v) \right),$$

which proves the assertion.

(d) This is a particular case of the previous one, when  $C = \Pi$  and L is the sum, or equivalently, when the two (continuous) random variables X and Y are independent and the operation acting on them is the sum.

(e) Since L satisfies property (LS) of Definition 3.5, the relationship  $L(u, v) = +\infty$  holds if, and only if, at least one between u and v, say v, equals  $+\infty$ . Therefore, for every u > 0,

$$\ell^{-}\rho_{C,L}(F,G)(+\infty) = \inf_{u \in \mathbb{R}_{+}} \left\{ F(u) + 1 - C(F(u),1) \right\} = 1,$$

which concludes the proof.

13. Multiplications

A multiplication is a binary operation on  $\Delta$  rather than on  $\Delta^+$ ; it generalizes the notion of triangle function. The name and the concept itself were introduced by Schweizer in [34].

**Definition 13.1.** A multiplication on  $\Delta$  is a binary operation on  $\Delta$  that is commutative, associative, increasing in each place, and whose restriction to  $\Delta^+$  is a triangle function.

The main properties of multiplications are collected in the results below. In the following Theorems the definition of the operations  $\Pi_T$ , of  $\tau_T$ , and of  $\tau_{T^*}^*$  is extended from  $\Delta^+ \times \Delta^+$  to  $\Delta \times \Delta$  (see [34]).<sup>5</sup>

**Theorem 13.1** ([34]). Let T be a t-norm; the function  $\Pi_T$  defined by (5.1) for all F and G in  $\Delta$  is an order-preserving multiplication on  $\Delta$ . If T is continuous, then  $\Pi_T$  is jointly continuous on the metric space  $(\Delta, d_S)$ .

It should be noticed that, when considered as a binary operation on  $\Delta$ , rather than on  $\Delta^+$ , the d.f.  $\varepsilon_0$  is not an identity for  $\Pi_T$ . The identity in the (continuous) semigroup  $(\Delta, \Pi_T)$  is  $\varepsilon_{-\infty}$ , the d.f. identically equal to 1 on  $\mathbb{R}$ , while the (continuous) semigroup  $(\mathcal{D}, \Pi_T)$  has no identity.

**Theorem 13.2** ([34]). Let T be a t-norm; the function  $\tau_T$  defined by (7.1) for all F and G in  $\Delta$  is an order-preserving multiplication on  $\Delta$  that has  $\varepsilon_0$  as an identity. If T is continuous, then  $\tau_T$  is jointly continuous on the metric space  $(\mathcal{D}, d_S)$ .

It is easily shown that the multiplication  $\tau_T$  is not continuous on  $(\Delta, d_S)$ .

**Example 13.1.** Consider the sequences of d.f.'s  $(F_n)_{n\in\mathbb{N}}$  and  $(G_n)_{n\in\mathbb{N}}$ , where, for every  $n \in \mathbb{N}$ ,  $F_n = \varepsilon_n$  and  $G_n = G$ , this latter being the d.f. defined by  $G(-\infty) = 0$ ,  $G(+\infty) = 1$  and, for  $x \in \mathbb{R}$ , by G(x) = 1/2. Then the two sequences converge weakly to  $\varepsilon_{\infty}$  and to G, respectively, namely

 $\lim_{n \to +\infty} d_S \left( F_n, \varepsilon_\infty \right) = 0, \qquad \text{and} \qquad \lim_{n \to +\infty} d_S \left( G_n, G \right) = 0.$ 

On the other hand, for every  $x \in \mathbb{R}$ ,

$$\tau_T\left(\varepsilon_{\infty}, G\right)(x) = \sup\left\{T\left(\varepsilon_{\infty}(u), 1/2\right) \mid (u, v) : u + v = x\right\} = 0,$$

i.e.,  $\tau_T(\varepsilon_{\infty}, G) = \varepsilon_{\infty}$ , while, for every  $n \in \mathbb{N}$ ,

$$\tau_T(F_n, G_n)(x) = \sup \{ T(\varepsilon_n(u), 1/2) \mid (u, v) : u + v = x \} = 1/2.$$

Thus, the sequence  $(\tau_T(F_n, G_n))_{n \in \mathbb{N}}$  does not converge weakly to  $\varepsilon_{\infty}$ .

**Theorem 13.3** ([34]). Let T be a t-norm; the function  $\tau_{T^*}^*$  defined by (8.1) for all F and G in  $\Delta$  is an order-preserving multiplication on  $\Delta$  that has  $\varepsilon_0$  as an identity. If  $T^*$  is continuous, then  $\tau_{T^*}^*$  is jointly continuous on the metric space  $(\mathcal{D}, d_S)$ .

Like  $\tau_T$ , also the multiplication  $\tau_{T^*}^*$  is not continuous on  $(\Delta, d_S)$ . In order to see this, consider the following example.

<sup>&</sup>lt;sup>5</sup> In this section, whenever we refer to equations (7.1) and (8.1), we take L to be the sum, and then we allow it to be defined on  $\mathbb{R}^2$  rather than on  $\mathbb{R}^2_+$ .
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**Example 13.2.** Consider the sequences  $(F_n)_{n \in \mathbb{N}}$  and  $(G_n)_{n \in \mathbb{N}}$  where, for every  $n \in \mathbb{N}$ ,  $F_n = \varepsilon_{-n}$  while  $G_n$  is the same d.f. as in the previous example. Then the two sequences converge weakly to  $\varepsilon_{-\infty}$  and to G, respectively, namely

$$\lim_{n \to +\infty} d_S \left( F_n, \varepsilon_{-\infty} \right) = 0, \quad \text{and} \quad \lim_{n \to +\infty} d_S \left( G_n, G \right) = 0.$$

But, for every  $x \in \mathbb{R}$ , one has

$$\lim_{n \to +\infty} \tau_{T^*}^* \left( \varepsilon_{-n}, G \right) (x) = 1/2 \neq 1 = \tau_{T^*}^* \left( \varepsilon_{-\infty}, G \right) (x)$$

**Definition 13.2.** The convolution between d.f.'s of  $\Delta$  is defined, for all F and G in  $\Delta$ , and for every  $x \in \mathbb{R}$ , by

$$(F * G)(x) = \int_{\mathbb{R}} F(x - y) \, dG(y).$$
 (13.1)

All the spaces  $\Delta$ ,  $\mathcal{D}$ ,  $\Delta^+$  and  $\mathcal{D}^+$  are stable under convolution (see [34, Theorem 9.1 (i)]).

**Theorem 13.4** ([34]). The convolution defined on  $\Delta \times \Delta$  by (13.1) is an orderpreserving multiplication on  $\Delta$  that has  $\varepsilon_0$  as an identity. It is jointly continuous on the metric spaces  $(\mathcal{D}, d_S)$ ,  $(\Delta^+, d_S)$  and  $(\mathcal{D}^+, d_S)$  but not on  $(\Delta, d_S)$ .

The proof of the continuity of the convolution on the spaces  $(\Delta^+, d_S)$  and  $(\Delta, d_L)$  can be found in Theorems 7.2 and 9.1 (viii) of [34]; that it is continuous on the space  $(\mathcal{D}, d_S)$  follows from the fact that on  $\mathcal{D}$  the Sibley and the Lévy metrics  $d_S$  and  $d_L$  are both topologically equivalent to the topology of weak convergence. Thus, it suffices to prove that \* is not continuous on  $(\Delta, d_S)$ . To this end, consider the same sequences  $(F_n)_{n\in\mathbb{N}}$  and  $(G_n)_{n\in\mathbb{N}}$  of the Example 13.1. Then

 $F_n * G_n = \varepsilon_n * G_n = G_n * \varepsilon_n = G_n = G$  and  $\varepsilon_\infty * G = \varepsilon_\infty$ ;

as a consequence, the sequence  $(F_n * G_n)_{n \in \mathbb{N}}$  does not converge to  $\varepsilon_{\infty}$ .

### 14. The subset of step functions

We consider in this section the subsets of  $\Delta^+$  and of  $\Delta$  consisting of the two-valued d.f.'s, the two values necessarily being 0 and 1; these two subsets are respectively

$$E^+ := \left\{ \varepsilon_a : a \in \overline{\mathbb{R}}_+ \right\} \subset \Delta^+ \qquad \text{and} \qquad E := \left\{ \varepsilon_a : a \in \overline{\mathbb{R}} \right\} \subset \Delta$$

In some questions regarding PM and PN spaces it is of some interest to study the result of the application of a triangle function or of a multiplication  $\tau$  to pairs of the type ( $\varepsilon_a, \varepsilon_b$ ). In the following theorem we summarize the results when the multiplication considered belongs to one of the types studied in the preceding section. The proof consists in a straightforward computation based on the definition. **Theorem 14.1.** Let a and b belong to  $\overline{\mathbb{R}}$ . Then

(a) for every t-norm T,

 $\Pi_T(\varepsilon_a,\varepsilon_b)=\varepsilon_{a\vee b};$ 

(b) if T and L satisfy the assumptions of Theorem 7.13,

$$\tau_{T,L}(\varepsilon_a,\varepsilon_b) = \varepsilon_{L(a,b)};$$
(c) if T<sup>\*</sup> and L satisfy the assumptions of Theorem 8.2,

$$\tau_{T^* L}^*(\varepsilon_a, \varepsilon_b) = \varepsilon_{L(a,b)};$$

(d) if Q and L satisfy the assumptions of Corollary 9.2,

$$\rho_{Q,L}(\varepsilon_a,\varepsilon_b)=\varepsilon_{L(a,b)},$$

(e) if C and L satisfy the assumptions of Theorem 10.3

 $\sigma_{C,L}(\varepsilon_a,\varepsilon_b)=\varepsilon_{L(a,b)},$ 

(f)  $\varepsilon_a * \varepsilon_b = \varepsilon_{a+b}$ .

In all cases,  $E^+$  is stable under the triangle function  $\tau$  considered,

$$\tau(E^+ \times E^+) \subseteq E^+.$$

**Example 14.1.** If D is the drastic *t*-norm, then

 $\tau_D(\varepsilon_a,\varepsilon_b)=\varepsilon_{a+b}.$ 

**Example 14.2.** Let  $\tau_{k,T}$  be the triangle function of (5.4), and assume, without loss of generality, a < b. A simple calculation shows that, if  $k \leq b$ , then  $\tau_{k,T}(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$ , while  $\tau_{k,T}(\varepsilon_a, \varepsilon_b) = \varepsilon_b$  if k > b. In either case,  $E^+$  is stable also under the triangle functions of this family.

In [44] the following result was proved.

**Lemma 14.2.** The spaces E and  $E^+$  are homeomorphic to  $\overline{\mathbb{R}}$  and to  $\overline{\mathbb{R}}_+$  respectively.

**Theorem 14.3.** Let T be a continuous t-norm, let L be commutative, associative and fulfil both (LS) and (L0) of Definition 3.5, then the topological semigroups  $(\overline{\mathbb{R}}_+, L)$  and  $(\Delta^+, d_S, \tau_{T,L})$  are homeomorphic.

*Proof.* The assertion follows from Theorem 14.1, from Lemma 14.2 and the fact that L is a semigroup operation on  $\overline{\mathbb{R}}_+$ .

A similar proof establishes

**Theorem 14.4.** Let  $T^*$  be a continuous t-conorm, let L be commutative, associative and fulfil both (LS) and (L0) of Definition 3.5, then the topological semigroups  $(\Delta^+, d_S, \tau^*_{T^*, L})$  and  $(\overline{\mathbb{R}}_+, L)$  are homeomorphic.

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### 15. A few questions

The reader should refer to the problem section (Section 7.9) of the book [41] by Schweizer and Sklar and to the notes to Chapter 7 in the Dover edition of that book in order to have a list of open problems on triangle functions. The problems presented there touch also aspects not yet covered in the present paper, but which will be the object of the second part.

**1.** Are there triangle functions different from the types we have listed and briefly presented above? Having at one's disposal a great variety of possible triangle functions would be of great theoretical interest, and would also enrich the collection of tools available to researchers for the applications.

2. To the best of the authors' knowledge, triangle functions have hitherto been used and discussed almost exclusively in the theory of PM and PN spaces. Particular examples of triangle functions, especially of the type  $\Pi_T$  and  $\tau_{T,L}$  with L some basic operation on the real line, also appear in, e.g., the treatment of fuzzy numbers or in information theory (compare also [32]), but are not in the focus of the investigation therein. However, even in the theories of PM and PN spaces, the triangle functions considered belong to the family  $\tau_T$  and, in the case of PN spaces, also to the family  $\tau_{T^*}$ , where T is a *continuous t*-norm and  $T^*$  its associated t-conorm. Is there room in these theories for triangle functions of the type  $\tau_{T,L}$ , with  $L \in \mathfrak{L}$  different from the sum? If yes, what is the meaning of the binary operation L?

**3.** It has been shown (see Theorem 12.1) that for all the families of triangle functions considered the inclusion

$$\tau(\mathcal{D}^+ \times \mathcal{D}^+) \subset \mathcal{D}^+$$

holds. The one example we have where this inclusion is not respected is of a discontinuous triangle functions, viz.  $\tau_D$ ; this leads to the following question: If the triangle function  $\tau$  is continuous on  $(\Delta^+, d_S)$ , does the above inclusion hold? Notice, however that the inclusion (12.1) may hold even for a discontinuous triangle function. In fact, let  $a \in \mathbb{R}_+$  and let T be a left-continuous *t*-norm, then the triangle function  $\tau_{a,T}$  of (5.4) is not continuous, but the above inclusion holds for it.

**4.** In Section 14 it was proved that the subset of  $E^+ \subset \Delta^+$  of step functions is stable under the triangle functions  $\Pi_T$ ,  $\tau_{T,L}$ ,  $\tau_{T^*,L}^*$ ,  $\rho_{Q,L}$ ,  $\sigma_{C,L}$ , and, even, under  $\tau_D$  and  $\tau_{a,T}$ , namely under all the triangle functions considered in the present paper. Then

- (a) is  $E^+$  stable under any triangle function  $\tau$ ?
- (b) If  $\tau$  belongs to any of the classes of triangle functions studied in this paper are there other subsets A of  $\Delta^+$  that are stable under  $\tau$  in the sense that

$$\tau(A \times A) \subseteq A?$$

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# Quasi-copulas with a given sub-diagonal section

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### Abstract

As is well known, the Fréchet-Hoeffding bounds are the best possible for both copulas and quasi-copulas: for every (quasi-) copula Q, max $\{x + y - 1, 0\} \le Q(x, y) \le \min\{x, y\}$  for all  $x, y \in [0, 1]$ . Sharper bounds hold when the (quasi-)copulas take prescribed values, e.g., along their diagonal or horizontal resp. vertical sections. Here we pursue two goals: first, we investigate construction methods for (quasi-)copulas with a given sub-diagonal section, i.e., with prescribed values along the straight line segment joining the points  $(x_0, 0)$  and  $(1, 1 - x_0)$  for  $x_0 \in [0, 1]$ . Then, we determine the best-possible bounds for sets of quasi-copulas with a given sub-diagonal section.

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### 1. Introduction

Copulas and quasi-copulas play an important role in many applications. Copulas were first introduced by Sklar in 1959 in [23]. The copula of a random pair (X, Y) completely captures the dependence structure of (X, Y); moreover, every copula is the restriction to the unit square of a bivariate distribution function whose marginals are uniform on [0, 1]. Quasi-copulas were introduced by Alsina et al. in [2] and characterized by Genest et al. in [13]; they characterize operations on distribution functions induced by operations on random variables defined on the same probability space.

Moreover, copulas and quasi-copulas are of considerable interest in other fields also, like e.g., many-valued logics and preference modelling, mainly because associative copulas are also continuous triangular norms (see, e.g., [1,3,4, 9,14,15,21,22]) often, but not only, used for modelling many-valued conjunctions. Therefore, having at one's disposal a large number of examples of (quasi-)copulas is of great practical and theoretical interest. During the last few years several investigations have been devoted to constructing copulas and quasi-copulas with given values along specified sections and to determining the best-possible bounds for the functions thus constructed (see [6–8,10–12,16,19,20]). In this spirit we study quasi-copulas with a given sub-diagonal section.

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### 2. Preliminaries

Before discussing the main results we summarize a few basic notions and properties that will be useful later on.

### 2.1. Basic operations

**Definition 1** ([23]). A copula C is a function from  $[0, 1]^2$  into [0, 1] with the following properties:

(a) C(0, u) = C(u, 0) = 0 and C(1, u) = C(u, 1) = u for every  $u \in [0, 1]$ ; (b) for all u, u', v and v' in [0, 1] with  $u \le u'$  and  $v \le v'$ 

$$\Delta_{u,u'}^{v,v'}(C) := C(u',v') - C(u',v) - C(u,v') + C(u,v) \ge 0.$$
<sup>(1)</sup>

The expression  $\Delta_{u,u'}^{v,v'}(C)$  is called the *C*-volume of the rectangle  $[u, u'] \times [v, v']$ . The set of copulas will be denoted by C.

It follows immediately from the definition that every copula *C* is increasing in each place, i.e., for every  $v \in [0, 1]$ , the functions  $u \mapsto C(u, v)$  and  $u \mapsto C(v, u)$  are both increasing,<sup>1</sup> and that it satisfies the *Lipschitz condition*, i.e., for all u, u', v and v' in [0, 1],

$$|C(u', v') - C(u, v)| \le |u' - u| + |v' - v|.$$
<sup>(2)</sup>

**Definition 2** ([2,13]). A quasi-copula Q is a function from  $[0, 1]^2$  into [0, 1] with the following properties:

- (a) Q(0, u) = Q(u, 0) = 0 and Q(1, u) = Q(u, 1) = u for every  $u \in [0, 1]$ ;
- (b) Q is increasing in each place, i.e.,  $u \mapsto Q(u, v)$  and  $u \mapsto Q(v, u)$  are both increasing functions;
- (c) Q satisfies the Lipschitz condition (2) for all u, u', v, v' in [0, 1].

The set of quasi-copulas will be denoted by Q.

Every copula is also a quasi-copula, but there are quasi-copulas that are not copulas; these latter ones will be called *proper* quasi-copulas. Therefore the inclusion  $C \subset Q$  is strict. A quasi-copula Q is called *symmetric* if Q(u, v) = Q(v, u) for all  $u, v \in [0, 1]$ .

The most important copulas are the minimum M, the product  $\Pi$ , and the copula W, which are given by

 $M(u, v) := \min\{u, v\}, \qquad \Pi(u, v) := u v, \qquad W(u, v) := \max\{0, u + v - 1\}.$ 

The copulas M and W are also known as the *Fréchet–Hoeffding bounds* for copulas and quasi-copulas, since  $W \le Q \le M$  for any quasi-copula Q; and  $\Pi$  is also known as the independence copula (for more details on copulas and quasi-copulas we refer to [18]).

**Definition 3.** Given a quasi-copula Q and  $x_0 \in [0, 1]$ , the sub-diagonal section  $\delta_{x_0}^Q$  of Q at  $x_0$  is the function  $\delta_{x_0}^Q : [0, 1 - x_0] \to [0, 1 - x_0]$  defined by

$$\delta_{x_0}^{\mathcal{Q}}(t) \coloneqq Q(x_0 + t, t). \tag{3}$$

When  $x_0 = 0$  we omit the index and refer to  $\delta^Q$  as the *diagonal section* of Q.

Sub-diagonals may be defined as functions fulfilling particular properties.

**Definition 4.** Given  $x_0 \in [0, 1]$ , a *sub-diagonal*  $\delta_{x_0}$  is a function from  $[0, 1 - x_0]$  into  $[0, 1 - x_0]$  with the following properties:

(DS1)  $\delta_{x_0}(1-x_0) = 1-x_0;$ 

<sup>&</sup>lt;sup>1</sup> We use the term increasing in the weak sense, viz. a function  $\varphi$  is increasing if  $\varphi(u_1) \leq \varphi(u_2)$  whenever  $u_1 < u_2$ ; when the strict inequality holds, we say that the function is *strictly increasing*.



Fig. 1. Sub-domains of  $[0, 1]^2$ .

(DS2)  $0 \le \delta_{x_0}(t) \le t$ , for every  $t \in [0, 1 - x_0]$ ;

(DS3)  $0 \le \delta_{x_0}(t') - \delta_{x_0}(t) \le 2(t'-t)$ , for every  $t, t' \in [0, 1-x_0]$  with  $t \le t'$ .

When  $x_0 = 0$  we again omit the index and simply speak of a *diagonal*.

The sub-diagonal section of any quasi-copula is a sub-diagonal. In fact the conditions specified above, are easy consequences of the properties of a quasi-copula. Condition (DS2) follows immediately from the inequality  $Q(x, y) \le M(x, y)$  that holds for all x and y in [0, 1], whereas condition (DS3) expresses the 2-Lipschitz condition and the isotony of any sub-diagonal section. Since  $W \le Q$ , it follows from (3) that, for all  $t \in [0, 1 - x_0]$ ,

$$\delta_{x_0}^Q(t) \ge \max\{x_0 + 2t - 1, 0\}.$$

The sub-diagonal section at  $x_0$  of a copula *C* has a simple probabilistic meaning. If *C* is the restriction to the unit square of the distribution function of two random variables *X* and *Y* with uniform distribution on (0, 1), then

$$\delta_{x_0}^C(t) = C(x_0 + t, t) = P(\max\{X - x_0, Y\} < t).$$

### 2.2. Further notions and basic properties

For a given  $x_0 \in [0, 1]$  we distinguish the following sub-domains of the unit square (see also Fig. 1):

$$T_U(x_0) \coloneqq \{(u, v) \in [0, 1]^2 \mid u - x_0 \le v\}; \qquad T_L(x_0) \coloneqq \{(u, v) \in [0, 1]^2 \mid u - x_0 \ge v\}; \\ S_1(x_0) \coloneqq [0, x_0] \times [1 - x_0, 1]; \qquad S_2(x_0) \coloneqq [x_0, 1] \times [0, 1 - x_0]; \\ S_L(x_0) \coloneqq [0, x_0]^2; \qquad S_U(x_0) \coloneqq [1 - x_0, 1]^2; \qquad D(x_0) \coloneqq S_L(x_0) \cup S_U(x_0).$$

If  $x_0 \ge 0.5$ , then  $S_L(x_0)$  and  $S_U(x_0)$  are not disjoint, while  $S_1(x_0)$  and  $S_2(x_0)$  always have in common just the single point  $(x_0, 1 - x_0)$  independently of the choice of  $x_0$ .

We further introduce two functions  $m_{x_0}: [0, 1]^2 \rightarrow [0, 1 - x_0]$  and  $M_{x_0}: [0, 1]^2 \rightarrow [0, 1 - x_0]$  through

$$m_{x_0}(u, v) := \max\{\min\{u - x_0, v\}, 0\};$$
(4)

$$M_{x_0}(u, v) := \min\{\max\{u - x_0, v\}, 1 - x_0\}.$$
(5)

One has  $(u, v) \in T_L(x_0)$  if and only if  $m_{x_0}(u, v) = u - x_0$  and  $M_{x_0}(u, v) = v$ . The truncation by 0 resp.  $1 - x_0$  ensures that  $\delta_{x_0}$  (or any function derived from it) may be applied to  $m_{x_0}(u, v)$  resp.  $M_{x_0}(u, v)$  for arbitrary  $(u, v) \in [0, 1]^2$ . Notice that

 $m_{x_0}(u, v) = \min\{u - x_0, v\}, \qquad M_{x_0}(u, v) = \max\{u - x_0, v\},$ 

for all  $(u, v) \in S_2(x_0)$ .

Associated with a sub-diagonal  $\delta_{x_0}$  we consider the function  $\hat{\delta}_{x_0}$  defined by

$$\hat{\delta}_{x_0}: [0, 1 - x_0] \to [0, 1 - x_0], \qquad \hat{\delta}_{x_0}(t) := t - \delta_{x_0}(t).$$
(6)

It is immediate to see that any  $\hat{\delta}_{x_0}$  fulfills the following properties, which are all consequences of (DS1)–(DS3):

- (HD1)  $\hat{\delta}_{x_0}(0) = \hat{\delta}_{x_0}(1 x_0) = 0;$
- (HD2)  $|\hat{\delta}_{x_0}(t') \hat{\delta}_{x_0}(t)| \le |t' t|$  for every  $t, t' \in [0, 1 x_0]$ ;
- (HD3)  $0 \le \hat{\delta}_{x_0}(t) \le \min\{t, 1 x_0 t\}$  for every  $t \in [0, 1 x_0]$ .

Property (HD2) implies that the function  $t \mapsto t + \hat{\delta}_{x_0}(t)$  is increasing.

The following functions will also be needed:

$$q_{x_0}: [0, 1-x_0] \to [0, 1-x_0], \qquad q_{x_0}(u, v) := \max\left\{\hat{\delta}_{x_0}(t) \mid t \in \left[m_{x_0}(u, v), M_{x_0}(u, v)\right]\right\}; \tag{7}$$

$$h_{x_0}: [0, 1 - x_0] \to [0, 1 - x_0], \qquad h_{x_0}(u, v) \coloneqq \min\left\{\hat{\delta}_{x_0}(t) \mid t \in \left[m_{x_0}(u, v), M_{x_0}(u, v)\right]\right\};$$
(8)

$$k_{x_0}:[0,1-x_0] \to [0,1-x_0], \qquad k_{x_0}(u,v) \coloneqq \frac{1}{2}\delta_{x_0}(m_{x_0}(u,v)) + \frac{1}{2}\delta_{x_0}(M_{x_0}(u,v)); \tag{9}$$

allowing to determine the maximum and the minimum of  $\hat{\delta}_{x_0}$  on some sub-domains.

### 2.3. Problem statements and known results

The first and natural problem that arises, and which will be investigated in the next sections, is the following one: Given a sub-diagonal  $\delta_{x_0}$ , does there exist a copula or a quasi-copula Q whose sub-diagonal section  $\delta_{x_0}^Q$  coincides with  $\delta_{x_0}$ , i.e., for which  $\delta_{x_0}^Q = \delta_{x_0}$ ? As will be seen, we shall answer this question in the positive by constructing several quasi-copulas with the required property. A second question is then: Given a sub-diagonal  $\delta_{x_0}$ , if  $Q_{\delta_{x_0}}$  and  $C_{\delta_{x_0}}$  denote, respectively, the set of all the quasi-copulas and of all the copulas whose sub-diagonal sections coincide with  $\delta_{x_0}$ , what are the best-possible bounds for  $Q_{\delta_{x_0}}$ ? We shall show later how to construct lower and upper bounds for  $Q_{\delta_{x_0}}$ .

The answers to the questions put above are already known for the particular case of  $x_0 = 0$ , i.e., for diagonal (quasi-)copulas. We briefly recall the main results (see also [19,20]).

**Theorem 1.** Consider a diagonal  $\delta$  and define  $B_{\delta}$ ,  $K_{\delta}$  and  $A_{\delta}$  as functions from  $[0, 1]^2$  into [0, 1] by

$$B_{\delta}(u, v) := \min\{u, v\} - \min\{\hat{\delta}(t) \mid t \in [\min\{u, v\}, \max\{u, v\}]\};$$
(10)

$$K_{\delta}(u,v) := \min\{u, v, \frac{\delta(u) + \delta(v)}{2}\}; \tag{11}$$

$$A_{\delta}(u, v) := \min\{u, v, \max\{u, v\} - \max\{\hat{\delta}(t) \mid t \in [\min\{u, v\}, \max\{u, v\}]\}\};$$
(12)

for every  $u, v \in [0, 1]$ . Then the following statements hold:

- (a)  $B_{\delta}$  and  $K_{\delta}$  are symmetric copulas;  $A_{\delta}$  is a symmetric quasi-copula.
- (b) For every quasi-copula Q with  $\delta^Q = \delta$  the following inequalities hold:

$$B_{\delta} \leq Q \leq A_{\delta}$$

(c)  $C_{\delta} \leq K_{\delta}$  for every symmetric copula  $C_{\delta} \in C_{\delta}$ .

Therefore, for the case  $x_0 = 0$ , best-possible bounds as well as constructions are already known. For more details on how to determine also non-symmetric (quasi-)copulas we refer to [6,20]. As a consequence, in what follows we shall restrict our considerations to  $x_0 \in [0, 1[$ .

We also mention that, by well-known transformations of (quasi-)copulas, and as a consequence of our results, (quasi-)copulas with prescribed values on other preassigned line segments can be constructed. More precisely, given a (quasi-)copula Q, the functions Q', Q'', and Q''' from  $[0, 1]^2$  into [0, 1] defined by

$$Q'(u, v) := u - Q(u, 1 - v);$$
  

$$Q''(u, v) := v - Q(1 - u, v);$$
  

$$Q'''(u, v) := u + v - 1 + Q(1 - u, 1 - v);$$

are (quasi-)copulas (compare also [16,18]). Therefore, for a given  $x_0 \in [0, 1[$ , a given sub-diagonal  $\delta_{x_0}$ , and a quasicopula  $Q \in Q_{\delta_{x_0}}$ , Q' is a quasi-copula with prescribed values along the straight line segment joining the points  $(x_0, 1)$ and  $(1, x_0)$ , determined by corresponding values of  $\delta_{x_0}$ . Analogously, the values of Q'' (resp. Q''') are prescribed by  $\delta_{x_0}$  for the line segment joining the points  $(1 - x_0, 0)$  and  $(0, 1 - x_0)$  (resp.  $(1 - x_0, 1)$  and  $(0, x_0)$ ). Thus, such (quasi-)copulas, either with given opposite super-diagonal section (Q'), opposite sub-diagonal section (Q''), or superdiagonal section (Q'''), can be constructed, and their respective bounds can be determined.

### 3. From diagonal to sub-diagonal (quasi-)copulas

Bilinear transformations of quasi-copulas with given diagonals may be used to construct quasi-copulas with a prescribed sub-diagonal section. From a geometrical viewpoint the idea is to rescale and shift a given quasi-copula so that it is defined on  $S_2(x_0)$  rather than on the whole unit square and to fill the gaps on  $[0, 1]^2 \setminus S_2(x_0)$  in an appropriate way. Because of the transformation and shifting process, the value of the new quasi-copula at the point  $(x_0, 1 - x_0)$  must be equal to 0. Following the results of [17] such a quasi-copula Q may be represented as a W-ordinal sum, i.e., there exist quasi-copulas  $Q_1$  and  $Q_2$  such that

$$Q(u, v) = \begin{cases} x_0 Q_1\left(\frac{u}{x_0}, \frac{x_0 + v - 1}{x_0}\right), & \text{if } (u, v) \in S_1(x_0), \\ (1 - x_0) Q_2\left(\frac{u - x_0}{1 - x_0}, \frac{v}{1 - x_0}\right), & \text{if } (u, v) \in S_2(x_0), \\ W(u, v), & \text{otherwise.} \end{cases}$$
(13)

Note that Q is indeed again a quasi-copula and will be denoted by  $(\langle 0, x_0, Q_1 \rangle, \langle x_0, 1, Q_2 \rangle)^W$ .

Moreover, if  $Q_1$  and  $Q_2$  are copulas, then Q also is a copula [17].

**Proposition 2.** For  $x_0 \in [0, 1[$ , let  $\delta_{x_0}$  be a sub-diagonal. Consider an arbitrary quasi-copula  $Q_1$  and a quasi-copula  $Q_2 \in Q_{\widetilde{\delta}}, \widetilde{\delta}: [0, 1] \to [0, 1]$  being the diagonal defined by

$$\widetilde{\delta}(t) := \frac{\delta_{x_0}((1-x_0)t)}{1-x_0}.$$
(14)

Then the W-ordinal sum  $Q = (\langle 0, x_0, Q_1 \rangle, \langle x_0, 1, Q_2 \rangle)^W$  defined by (13) is a quasi-copula that fulfills  $\delta_{x_0}^Q = \delta_{x_0}$ , i.e.,  $Q \in Q_{\delta_{x_0}}$ .

**Proof.** It suffices to prove that, for every  $x_0 \in ]0, 1[, \tilde{\delta}]$  is indeed a diagonal and that  $Q(t + x_0, t) = \delta_{x_0}(t)$  for all  $t \in [0, 1 - x_0]$ . Since  $(1 - x_0)\tilde{\delta}(t) = \delta_{x_0}((1 - x_0)t)$  it follows immediately from properties (DS1) and (DS2) that  $\tilde{\delta}(1) = 1$  and  $0 \leq \tilde{\delta}(t) \leq t$  for all  $t \in [0, 1]$ . Moreover, since  $\delta_{x_0}$  fulfills (DS3), it can be easily seen that, for all t and t' in [0, 1] with  $t \leq t'$ ,

$$0 \le \widetilde{\delta}(t') - \widetilde{\delta}(t) = \frac{1}{1 - x_0} \left( \delta_{x_0} \left( (1 - x_0)t' \right) - \delta_{x_0} ((1 - x_0)t) \right) \le 2(t' - t).$$

Now assume that  $t \in [0, 1 - x_0]$ ; then  $(t + x_0, t) \in S_2(x_0)$  and

$$Q(t+x_0,t) = (1-x_0) Q_2\left(\frac{t}{1-x_0},\frac{t}{1-x_0}\right) = (1-x_0) \widetilde{\delta}\left(\frac{t}{1-x_0}\right) = \delta_{x_0}(t)$$

so that indeed Q belongs to  $\mathcal{Q}_{\delta_{x_0}}$ .  $\Box$ 

**Corollary 3.** For every  $x_0 \in ]0, 1[$  and for every sub-diagonal  $\delta_{x_0}$ , there exist a quasi-copula Q and a copula C that belong to  $Q_{\delta_{x_0}}$  and to  $C_{\delta_{x_0}}$ , respectively.

Although bilinear transformations already provide a rich tool for constructing sub-diagonal (quasi-)copulas, a few remarks are in order. If the original (quasi-)copula was symmetric with respect to the main diagonal, then some symmetry is inherited by the (quasi-)copula thus constructed. However, this construction does not necessarily yield symmetric sub-diagonal (quasi-)copulas. Moreover, although, as we shall see in Section 8, this method allows us to

determine the best-possible lower bound of (quasi-)copulas with a given sub-diagonal section, the largest possible sub-diagonal (quasi-)copula cannot be found by means of *W*-ordinal sums. The reason is that transforming the boundary conditions of the (quasi-)copula involved implies rather restrictive conditions on the new function, i.e., it forces the resulting operation to coincide with the lower Fréchet–Hoeffding bound *W* on  $[0, x_0] \times [0, 1 - x_0]$  and on  $[x_0, 1] \times [1 - x_0, 1]$ . To illustrate this fact, let us apply the previous results to sub-diagonals provided by the largest and smallest possible (quasi-)copulas and by choosing  $Q_1$  and  $Q_2$ , respectively, to be the largest and the smallest possible quasi-copulas.

**Example 1.** Given  $x_0 \in [0, 1[$ , consider the smallest sub-diagonal  $\delta_{x_0}^W(t) := \max\{x_0 + 2t - 1, 0\}$ . Then  $\widetilde{\delta}^W$  is given by  $\widetilde{\delta}^W(t) = \max\{2t - 1, 0\}$ , the diagonal section of the quasi-copula W. Choose  $Q_1 = W$  and  $Q_2 = B_{\widetilde{\delta}^W}$  given by (10); this yields

$$Q_{2}(u, v) = \min\{u, v\} - \min\{t - \delta^{W}(t) \mid t \in [\min\{u, v\}, \max\{u, v\}]\}$$
  
= min{u, v} - min{t, 1 - t | t \in [\min\{u, v\}, \max\{u, v\}]}  
= min{u, v} - min{min{u, v}, 1 - max{u, v}}  
= max{0, min{u, v} + max{u, v} - 1}  
= max{0, u + v - 1} = W(u, v).

Therefore, the corresponding *W*-ordinal sum *Q* defined by (13) is the lower Fréchet–Hoeffding bound *W* and belongs to  $Q_{\delta_{X_0}^W}$ .

**Example 2.** Given  $x_0 \in [0, 1[$ , consider the largest sub-diagonal  $\delta_{x_0}^M(t) := t$ . Then  $\widetilde{\delta}^M$  is given by  $\widetilde{\delta}^M(t) = t$ , the diagonal section of the quasi-copula M. Choose  $Q_1 = M$  and  $Q_2 = A_{\widetilde{\delta}^M}$  given by (12); this yields

$$Q_2(u, v) = \min \left\{ u, v, \max\{u, v\} - \max\{t - \widetilde{\delta}^M(t) \mid t \in [\min\{u, v\}, \max\{u, v\}] \right\}$$
  
= min{u, v, max{u, v}} = M(u, v).

Therefore, the W-ordinal sum Q defined by (13) is given by

$$Q(u, v) = \begin{cases} \min\{u, v + x_0 - 1\}, & \text{if } (u, v) \in S_1(x_0), \\ \min\{u - x_0, v\}, & \text{if } (u, v) \in S_2(x_0), \\ W(u, v), & \text{otherwise,} \end{cases}$$

which differs from the upper Fréchet–Hoeffding bound M.

### 4. From sub-diagonal quasi-copulas to new sub-diagonal quasi-copulas

New quasi-copulas and copulas having a prescribed sub-diagonal section may be obtained by adapting the construction introduced in [6] and called *splice* in [20].

**Definition 5.** Let *A* and *B* be quasi-copulas or copulas with the same sub-diagonal section at  $x_0$ , i.e.,  $\delta_{x_0}^A = \delta_{x_0}^B$ . The splice  $A \otimes_{x_0} B$  of *A* and *B* at  $x_0$  is defined via

$$(A \oslash_{x_0} B) (u, v) := \begin{cases} A(u, v), & \text{if } (u, v) \in T_U(x_0), \\ B(u, v), & \text{if } (u, v) \in T_L(x_0). \end{cases}$$
 (15)

**Theorem 4.** Let A and B be two (quasi-)copulas having the same sub-diagonal section  $\delta_{x_0}$  at  $x_0$ , i.e.  $A, B \in Q_{\delta_{x_0}}$ . Then the splice  $A \oslash_{x_0} B$  defined by (15) is a quasi-copula belonging to  $Q_{\delta_{x_0}}, A \oslash_{x_0} B \in Q_{\delta_{x_0}}$ .

**Proof.** Assume that *A* and *B* are (quasi-)copulas in  $Q_{\delta_{x_0}}$ . Then the boundary conditions for a (quasi-)copula are automatically satisfied by  $A \oslash_{x_0} B$ ; it is also clear that its sub-diagonal section coincides with  $\delta_{x_0}$ . Therefore, one has to prove that  $A \oslash_{x_0} B$  is increasing in each place and satisfies the Lipschitz condition; to this end, it is enough to prove, that, if  $u_1 - x_0 \le v$  and  $u_2 - x_0 > v$  for  $v \in [0, 1]$ , then

$$0 \le (A \oslash_{x_0} B)(u_2, v) - (A \oslash_{x_0} B)(u_1, v) \le u_2 - u_1.$$

Now

$$(A \oslash_{x_0} B)(u_2, v) - (A \oslash_{x_0} B)(u_1, v) = B(u_2, v) - A(u_1, v)$$
  
=  $B(u_2, v) - \delta_{x_0}(v + x_0, v) + \delta_{x_0}(v + x_0, v) - A(u_1, v)$   
=  $B(u_2, v) - B(v + x_0, v) + A(v + x_0, v) - A(u_1, v) \ge 0.$ 

The same argument also yields

$$(A \oslash_{x_0} B)(u_2, v) - (A \oslash_{x_0} B)(u_1, v) \le u_2 - (v + x_0) + (v + x_0) - u_1 = u_2 - u_1$$

which proves the assertion.  $\Box$ 

**Theorem 5.** Let A and B be two copulas having the same sub-diagonal section  $\delta_{x_0}$  at  $x_0$ ,  $A, B \in C_{\delta_{x_0}}$  and fulfilling, for all  $(u, v) \in T_L(x_0)$ ,

$$A(u, v) = A(v + x_0, u - x_0)$$
 and  $B(u, v) = B(v + x_0, u - x_0).$ 

Then the splice  $A \otimes_{x_0} B$  defined by (15) is a copula belonging to  $\mathcal{C}_{\delta_{x_0}}$ ,  $A \otimes_{x_0} B \in \mathcal{C}_{\delta_{x_0}}$ .

**Proof.** Assume that *A* and *B* are copulas in  $C_{\delta_{x_0}}$ . We already know from Theorem 4 that  $A \otimes_{x_0} B$  is a quasi-copula. In order to prove that  $A \otimes_{x_0} B$  is also 2-increasing, it suffices to verify this property for every square whose diagonal lies on the segment joining the points  $(u, u - x_0)$  and  $(v, v - x_0)$ , namely for every square  $[u, v] \times [u - x_0, v - x_0]$ ,  $u, v \in [x_0, 1]$  with u < v. Now, we have

$$\begin{aligned} &\Delta_{u,v}^{u-x_0,v-x_0} \left( A \oslash_{x_0} B \right) \\ &= \left( A \oslash_{x_0} B \right) (u, u - x_0) + \left( A \oslash_{x_0} B \right) (v, v - x_0) - \left( A \oslash_{x_0} B \right) (u, v - x_0) - \left( A \oslash_{x_0} B \right) (v, u - x_0) \\ &= A(u, u - x_0) + A(v, v - x_0) - A(u, v - x_0) - B(v, u - x_0) \\ &= B(u, u - x_0) + B(v, v - x_0) - A(u, v - x_0) - B(v, u - x_0) \\ &\geq \min\{V_A([u, v] \times [u - x_0, v - x_0]), V_B([u, v] \times [u - x_0, v - x_0])\} + |B(u, v - x_0) - A(u, v - x_0)| \ge 0 \end{aligned}$$

which concludes the proof.  $\Box$ 

Notice that, even when A and B are symmetric (quasi-)copulas, their splice  $A \oslash_{x_0} B$  need not be symmetric. In fact, if there is a point (u, v) with  $u \in ]x_0, 1[$  and  $v \in ]0, u - x_0[$  such that  $A(u, v) \neq B(u, v)$ , then

$$(A \oslash_{x_0} B)(u, v) = B(u, v) \neq A(u, v) = A(v, u) = (A \oslash_{x_0} B)(v, u).$$

Similarly if (u, v) is such that  $u \in [0, 1 - x_0[, v \in ]u + x_0, 1[$  and  $A(u, v) \neq B(u, v)$ , then

$$\left(A \oslash_{x_0} B\right)(u, v) = A(u, v) \neq B(u, v) = B(v, u) = \left(A \oslash_{x_0} B\right)(v, u).$$

In either case,  $A \oslash_{x_0} B$  is not symmetric.

#### 5. Symmetrization of quasi-copulas

As mentioned above, the (quasi-)copulas constructed by *W*-ordinal sums or by the splicing method are, in general, not symmetric. Below we present a method for obtaining a symmetric quasi-copula starting from a non-symmetric (quasi-)copula. Notice that this method may be applied to arbitrary (quasi-)copulas.

**Proposition 6.** Let Q be any (quasi-)copula. Then the functions  $Q^*$  and  $Q_*$  from  $[0, 1]^2$  into [0, 1] defined by

$$Q^{*}(u, v) := Q(\max\{u, v\}, \min\{u, v\}),$$
(16)

$$Q_*(u, v) \coloneqq Q(\min\{u, v\}, \max\{u, v\})$$

$$\tag{17}$$

are symmetric quasi-copulas.

**Proof.** It is obvious that, for any (quasi-)copula Q,

$$Q^*(u, v) = \begin{cases} Q(u, v), & \text{if } u \ge v, \\ Q(v, u), & \text{otherwise,} \end{cases} \qquad Q_*(u, v) = \begin{cases} Q(u, v), & \text{if } u \le v, \\ Q(v, u), & \text{otherwise,} \end{cases}$$

which immediately shows that  $Q^*$  and  $Q_*$  fulfill the boundary conditions and are increasing since Q is increasing in each argument. The Lipschitz property of  $Q^*$  and  $Q_*$  follows immediately from the corresponding property of Q when u, u', v, v' determine a rectangle completely contained either below or above the main diagonal of  $[0, 1]^2$ . When u = v and u' = v' we can conclude that

$$|Q^*(u', u') - Q^*(u, u)| = |Q_*(u', u') - Q_*(u, u)| = |Q(u', u') - Q(u, u)| \le 2|u' - u|$$

In all other cases the rectangle  $[u, u'] \times [v, v']$  can be decomposed into smaller rectangles belonging to the previous types, so that an application of the triangle inequality establishes the Lipschitz property for  $Q^*$  and  $Q_*$ . Since max and min are symmetric, so are  $Q^*$  and  $Q_*$ ; this concludes the proof.  $\Box$ 

The previous operations can also be obtained through a splicing along the main diagonal for  $x_0 = 0$ . Originally the corresponding approach was introduced in order to obtain non-symmetric operations with given diagonals (see also [6,20]); however, it can also be applied to obtain symmetric ones. More precisely, since for any (quasi-)copula Q also  $Q^t$  defined by  $Q^t(u, v) := Q(v, u)$  is a (quasi-)copula, it immediately follows that, for  $x_0 = 0$ ,

$$Q^*(u, v) = (Q^t \otimes_0 Q)(u, v)$$
 and  $Q_*(u, v) = (Q \otimes_0 Q^t)(u, v).$ 

When Q is a symmetric (quasi-)copula, then  $Q = Q^t$  and therefore  $Q^* = Q = Q_*$ . Moreover, the results in [6,20] allow us to state the following corollaries.

**Corollary 7.** Let C be a copula. Then  $C^*$  is a copula if and only if  $2C(u, v) \le C(u, u) + C(v, v)$  for all  $u, v \in [0, 1]$  with  $u \le v$ , and  $C_*$  is a copula if and only if  $2C(u, v) \le C(u, u) + C(v, v)$  for all  $u, v \in [0, 1]$  with  $u \ge v$ .

**Corollary 8.** Let C be a copula with diagonal section  $\delta$ . If, for all  $u, v \in [0, 1]$ ,

 $\max(C(u, v), C(v, u)) \le K_{\delta}(u, v),$ 

where  $K_{\delta}$  is given by (11), then  $C^*$  and  $C_*$  are copulas.

Before turning to examples of symmetric quasi-copulas with a given sub-diagonal section at  $x_0$ , we introduce some new notions in complete analogy to  $Q^*$ . Therefore, for  $x_0 \in [0, 1[$  and a sub-diagonal  $\delta_{x_0}$ , define functions  $m_{x_0}^*, M_{x_0}^*, q_{x_0}^*, h_{x_0}^*, h_{x_0}^*, h_{x_0}^*$ , from  $[0, 1]^2$  into  $[0, 1 - x_0]$  by

$$\begin{split} m_{x_0}^*(u, v) &:= m_{x_0}(\max\{u, v\}, \min\{u, v\}); \\ M_{x_0}^*(u, v) &:= M_{x_0}(\max\{u, v\}, \min\{u, v\}); \\ q_{x_0}^*(u, v) &:= q_{x_0}(\max\{u, v\}, \min\{u, v\}); \\ h_{x_0}^*(u, v) &:= h_{x_0}(\max\{u, v\}, \min\{u, v\}); \\ k_{x_0}^*(u, v) &:= k_{x_0}(\max\{u, v\}, \min\{u, v\}). \end{split}$$

Notice that  $m_{x_0}^*(u, v) \ge 0$  and  $M_{x_0}^*(u, v) \le 1 - x_0$ , for all (u, v) in  $[0, 1]^2 \setminus D(x_0)$ .

### 6. Examples of symmetric sub-diagonal quasi-copulas

**Proposition 9.** For  $x_0 \in [0, 1[$  and for a sub-diagonal  $\delta_{x_0}$ , the function  $B^*_{\delta_{x_0}}$  from  $[0, 1]^2$  into [0, 1] defined by

$$B_{\delta_{x_0}}^*(u,v) \coloneqq \begin{cases} m_{x_0}^*(u,v) - h_{x_0}^*(u,v), & if(u,v) \in [0,1]^2 \setminus D(x_0), \\ W(u,v), & otherwise, \end{cases}$$
(18)

is a symmetric quasi-copula belonging to  $\mathcal{Q}_{\delta_{\chi_0}}$ .

**Proof.** We shall show that  $B^*_{\delta_{x_0}}$  is the symmetrization of the *W*-ordinal sum  $B_{\delta_{x_0}}$  given by  $(\langle 0, x_0, W \rangle, \langle x_0, 1, B_{\widetilde{\delta}} \rangle)^W$ with  $\delta$  defined by (14). Recall that

$$B_{\widetilde{\delta}}(u,v) = \min\{u,v\} - \min\{t - \widetilde{\delta}(t) \mid t \in [\min\{u,v\}, \max\{u,v\}]\},\$$

and that the *W*-ordinal sum  $B_{\delta_{x_0}}$  is given by

$$B_{\delta_{x_0}}(u, v) = \begin{cases} (1 - x_0) B_{\widetilde{\delta}}\left(\frac{u - x_0}{1 - x_0}, \frac{v}{1 - x_0}\right), & \text{if } (u, v) \in S_2(x_0) \\ W(u, v), & \text{otherwise.} \end{cases}$$

By simple calculations and by taking into account that, for all (u, v) in  $S_2(x_0)$ ,

$$m_{x_0}(u, v) = \min\{u - x_0, v\}, \qquad M_{x_0}(u, v) = \max\{u - x_0, v\}, (1 - x_0)\,\widetilde{\delta}(t) = \delta_{x_0}((1 - x_0)t), \quad h_{x_0}(u, v) = \min\left\{\hat{\delta}_{x_0}(t) \mid t \in [m_{x_0}(u, v), M_{x_0}(u, v)]\right\},$$

it can be shown that indeed, for all  $(u, v) \in S_2(x_0)$ ,

$$(1-x_0) B_{\widetilde{\delta}}\left(\frac{u-x_0}{1-x_0}, \frac{v}{1-x_0}\right) = m_{x_0}(u, v) - h_{x_0}(u, v)$$

and, therefore,

$$B_{\delta_{x_0}}(u, v) = \begin{cases} m_{x_0}(u, v) - h_{x_0}(u, v), & \text{if } (u, v) \in S_2(x_0), \\ W(u, v), & \text{otherwise.} \end{cases}$$
(19)

Moreover, since  $B_{\delta}$  and W are copulas,  $B_{\delta_{x_0}}$  also is a copula. Its sub-diagonal section coincides with  $\delta_{x_0}$  because of **Proposition 2.** 

The symmetrization of  $B_{\delta}$  is given, for all  $(u, v) \in [0, 1]^2$ , by

$$B^*_{\delta_{x_0}}(u, v) = B_{\delta_{x_0}}(\max\{u, v\}, \min\{u, v\}).$$

If  $(\max\{u, v\}, \min\{u, v\})$  belongs to  $S_2(x_0)$ , or, equivalently, if (u, v) is in  $[0, 1]^2 \setminus D(x_0)$ , then  $B^*_{\delta_{x_0}}(u, v) =$  $B_{\delta_{x_0}}(\max\{u, v\}, \min\{u, v\}) = m_{x_0}^*(u, v) - h_{x_0}^*(u, v) \text{ for all } (u, v) \text{ in } [0, 1]^2 \setminus D(x_0).$ Finally, if (u, v) is in  $D(x_0)$ , then  $W(\max\{u, v\}, \min\{u, v\}) = W(u, v)$ . Therefore,  $B_{\delta_{x_0}}^*$  may be rewritten as

$$B^*_{\delta_{x_0}}(u, v) = \begin{cases} m^*_{x_0}(u, v) - h^*_{x_0}(u, v), & \text{if } (u, v) \in [0, 1]^2 \setminus D(x_0), \\ W(u, v), & \text{otherwise.} \end{cases}$$

It follows from Propositions 2 and 6 that  $B^*_{\delta_{x_0}}$  is indeed a quasi-copula whose sub-diagonal section at  $x_0$  coincides with  $\delta_{x_0}$ , i.e.,  $B^*_{\delta_{x_0}} \in \mathcal{Q}_{\delta_{x_0}}$ . 

We show, by means of an example that  $B^*_{\delta_{x_0}}$  is, in general, a proper quasi-copula; to this end, we shall show that one may as well have  $\Delta_{u,u}^{v,v}(B^*_{\delta_{x_0}}) < 0$ , for some *u* and *v* with u < v.

**Example 3.** Let  $x_0 < 1/3$  and consider the diagonal section  $\delta_{x_0}^{\Pi}$  of the product copula  $\Pi$ ; then

$$\delta_{x_0}^{\Pi}(t) = t (x_0 + t)$$
 and  $\hat{\delta}_{x_0}^{\Pi}(t) = t (1 - x_0) - t^2$ .

For the values of  $x_0$  considered, choose u and v greater than  $(1-x_0)/2$  so that  $\hat{\delta}_{x_0}^{\Pi}$  is a decreasing function; this allows to calculate explicitly

$$\begin{aligned} \Delta_{u,u}^{v,v}(B^*_{\delta_{x_0}}) &= u - v - (1 - x_0)v + v^2 - (1 - x_0)u + u^2 + 2(1 - x_0)u - 2u^2 \\ &= u - v - (1 - x_0)(v - u) + v^2 - u^2 \\ &= (v - u) \ (-1 - (1 - x_0) + u + v) = (v - u) \ (-2 + x_0 + u + v) < 0, \end{aligned}$$

since  $u < v < 1 - x_0$ .

By considerations analogous to those carried out in Proposition 9 it is possible to obtain the following two symmetric quasi-copulas, which, as shall be shown later, are again, in general, proper quasi-copulas.

**Proposition 10.** For  $x_0 \in [0, 1[$  and a sub-diagonal  $\delta_{x_0}$ , the functions  $K^*_{\delta_{x_0}}$  and  $A^*_{\delta_{x_0}}$  from  $[0, 1]^2$  into [0, 1] defined by

$$K_{\delta_{x_0}}^*(u,v) \coloneqq \begin{cases} \min\{m_{x_0}^*(u,v), k_{x_0}^*(u,v)\}, & if(u,v) \in [0,1]^2 \setminus D(x_0), \\ W(u,v), & otherwise, \end{cases}$$
(20)

and

$$A_{\delta_{x_0}}^*(u,v) := \begin{cases} \min\{m_{x_0}^*(u,v), M_{x_0}^*(u,v) - q_{x_0}^*(u,v)\}, & if(u,v) \in [0,1]^2 \setminus D(x_0), \\ W(u,v), & otherwise, \end{cases}$$
(21)

are symmetric quasi-copulas, and they both belong to  $\mathcal{Q}_{\delta_{x_0}}$ .

**Proof.** In complete analogy with the proof of Proposition 9, it can be shown that  $K_{\delta_{x_0}}^*$  is the symmetrization of the *W*-ordinal sum  $K_{\delta_{x_0}} = (\langle 0, x_0, W \rangle, \langle x_0, 1, K_{\delta} \rangle)^W$  and that  $A_{\delta_{x_0}}^*$  is the symmetrization of the *W*-ordinal sum  $A_{\delta_{x_0}} = (\langle 0, x_0, W \rangle, \langle x_0, 1, A_{\delta} \rangle)^W$  with  $\delta$  given by (14). It therefore follows from Propositions 2 and 6, that both the operations are indeed quasi-copulas whose sub-diagonal sections at  $x_0$  coincide with  $\delta_{x_0}$ .

While the *W*-ordinal sum  $K_{\delta_{x_0}}$  is not only a quasi-copula but also a copula, the quasi-copula  $K^*_{\delta_{x_0}}$  is, in general, a proper quasi-copula, as is  $A^*_{\delta_{x_0}}$ .

**Example 4.** Let  $x_0$  be in ]0, 1[ and let  $\delta_{x_0}^M$  be the diagonal section at  $x_0$  of the upper Fréchet–Hoeffding bound M, i.e.,  $\delta_{x_0}^M(t) = t$  for all  $t \in [0, 1 - x_0]$ . Given an arbitrary square  $[u, v]^2 \subseteq [0, 1]^2$  with  $u < v < u + x_0$ , a simple calculation leads to

$$\begin{aligned} \Delta_{u,u}^{v,v}(K_{\delta_{x_0}}^*) &= K_{\delta_{x_0}}^*(v,v) + K_{\delta_{x_0}}^*(u,u) - K_{\delta_{x_0}}^*(u,v) - K_{\delta_{x_0}}^*(v,u) \\ &= \min\left\{v - x_0, \frac{1}{2}(\delta_{x_0}(v - x_0) + \delta_{x_0}(v))\right\} + \min\left\{u - x_0, \frac{1}{2}(\delta_{x_0}(u - x_0) + \delta_{x_0}(u))\right\} \\ &- 2\min\left\{v - x_0, \frac{1}{2}(\delta_{x_0}(v - x_0) + \delta_{x_0}(u))\right\} \\ &= \min\left\{v - x_0, v - \frac{x_0}{2}\right\} + \min\left\{u - x_0, u - \frac{x_0}{2}\right\} - 2\min\left\{v - x_0, \frac{u + v - x_0}{2}\right\} \\ &= (v - x_0) + (u - x_0) - 2(v - x_0) = -(v - u) < 0, \end{aligned}$$

since  $v < u + x_0$  implies  $v - x_0 < (u + v - x_0)/2$ .

**Example 5.** As in Example 4, let  $x_0$  be in ]0, 1[, let  $\delta_{x_0}^M$  be the diagonal section at  $x_0$  of the upper Fréchet–Hoeffding bound M,  $\delta_{x_0}^M(t) = t$ , and consider the square  $[u, v]^2 \subseteq [0, 1]^2$  with  $u < v < u + x_0$ . Since  $\hat{\delta}_{x_0}^M(t) = 0$  for every  $t \in [0, 1 - x_0]$  a simple calculation leads to

$$\begin{aligned} \Delta_{u,u}^{v,v}(A_{\delta_{x_0}}^*) &= A_{\delta_{x_0}}^*(v,v) + A_{\delta_{x_0}}^*(u,u) - 2A_{\delta_{x_0}}^*(u,v) \\ &= \min\{v - x_0, v\} + \min\{u - x_0, u\} - 2\min\{v - x_0, u\} \\ &= (v - x_0) + (u - x_0) - 2(v - x_0) = -(v - u) < 0. \end{aligned}$$

The previous examples have shown how to construct symmetric sub-diagonal quasi-copulas. However, as shown above, these quasi-copulas are usually proper. Therefore, it is natural to ask whether there exist symmetric sub-diagonal copulas. The following section answers this question in the positive, at least when  $x_0$  has a restricted range.

### 7. A symmetric sub-diagonal copula

We now turn to the construction of a copula *C*, rather than a quasi-copula, with a prescribed sub-diagonal section  $\delta_{x_0}^C$  at  $x_0$  with  $x_0 \in [0, 1/2]$ . While the copula thus constructed is symmetric, it is not a symmetrization of a *W*-ordinal sum; as a consequence, the properties of a copula have to be proved directly.

**Theorem 11.** For  $x_0 \in [0, 1/2]$  and for a sub-diagonal  $\delta_{x_0}$ , such that the function  $f:[x_0, 1-x_0] \rightarrow [0, 1-x_0]$ ,  $f(u) := \delta_{x_0}(u) - \delta_{x_0}(u - x_0)$  is increasing, the function  $C_{\delta_{x_0}}$  from  $[0, 1]^2$  into [0, 1] defined by

$$C_{\delta_{x_0}}(u,v) := \begin{cases} \min\{u, v, k_{x_0}^*(u,v)\}, & \text{if } (u,v) \in [0,1]^2 \setminus D(x_0), \\ \min\{k_{x_0}^*(x_0,u), k_{x_0}^*(x_0,v)\}, & \text{if } (u,v) \in S_L(x_0), \\ \min\{A_{x_0}(u,v), A_{x_0}(v,u)\}, & \text{if } (u,v) \in S_U(x_0), \end{cases}$$
(22)

where

$$A_{x_0}(u,v) \coloneqq u + k_{x_0}^*(v, 1 - x_0) - (1 - x_0), \tag{23}$$

is a symmetric copula whose sub-diagonal section at  $x_0$  equals  $\delta_{x_0}$ , namely  $C_{\delta_{x_0}} \in C_{\delta_{x_0}}$ .

**Proof.** Clearly the segment connecting the points  $(x_0, 0)$  and  $(1, 1 - x_0)$  lies entirely in the set  $[0, 1]^2 \setminus D(x_0)$ , so that

$$C_{\delta_{x_0}}(x_0+t,t) = \min\{x_0+t,t,k_{x_0}^*(x_0+t,t)\}$$
  
=  $\min\left\{t,\frac{1}{2}\left(\delta_{x_0}(t)+\delta_{x_0}(t)\right)\right\} = \min\{t,\delta_{x_0}(t)\} = \delta_{x_0}(t),$ 

because of (DS2). Thus, the last assertion in the statement of the theorem is proved.

Now, we shall prove that  $C_{\delta_{x_0}}$  is actually a copula. While the first of the boundary conditions of Definition 1 is easily verified, the second one needs some argument. Assume, first, that  $v < 1 - x_0$ . The 2-Lipschitz condition (DS3) yields

$$\frac{1}{2}\delta_{x_0}(1-x_0) - \frac{1}{2}\delta_{x_0}(v) \le (1-x_0) - v$$

or, equivalently, because of (DS1),

$$v \le \frac{1}{2} (1 - x_0) + \frac{1}{2} \delta_{x_0}(v) = k_{x_0}^*(1, v)$$

thus,  $C_{\delta_{x_0}}(1, v) = v$ . Assume, next, that  $v \ge 1 - x_0$ . Notice that, as a consequence of the 2-Lipschitz condition (DS3),  $A_{x_0}(u, v) \le A_{x_0}(v, u)$  if and only if  $u \le v$ . Therefore,

$$C_{\delta_{x_0}}(1,v) = A_{x_0}(v,1) = v + k_{x_0}^*(1,1-x_0) - (1-x_0) = v.$$

Since  $C_{\delta_{x_0}}$  satisfies the boundary conditions, it remains to prove that it is 2-increasing, namely that it satisfies condition (b) of Definition 1. There are several cases to be considered.

*Case* 1: Consider a rectangle  $[u_1, u_2] \times [v_1, v_2]$  that is entirely contained either in the triangle  $T_L(x_0)$  or in the trapezoidal area bounded by the straight lines  $u = x_0$ ,  $v = 1 - x_0$ , v = u and  $v = u - x_0$ . Then,

$$\begin{aligned} \Delta_{u_1,u_2}^{v_1,v_2}(C_{\delta_{x_0}}) &= C_{\delta_{x_0}}(u_1,v_1) - C_{\delta_{x_0}}(u_1,v_2) + C_{\delta_{x_0}}(u_2,v_2) - C_{\delta_{x_0}}(u_2,v_1) \\ &= \min\{v_1, k_{x_0}^*(u_1,v_1)\} - \min\{v_2, k_{x_0}^*(u_1,v_2)\} \\ &+ \min\{v_2, k_{x_0}^*(u_2,v_2)\} - \min\{v_1, k_{x_0}^*(u_2,v_1)\}, \end{aligned}$$

$$(24)$$

since  $v_i \leq u_j$  for all  $i, j \in \{1, 2\}$ .

We shall consider several subcases:

(a) 
$$v_1 \le k_{x_0}^*(u_1, v_1) \le k_{x_0}^*(u_2, v_1)$$
:

$$\Delta_{u_1,u_2}^{v_1,v_2}(C_{\delta_{x_0}}) = \min\{v_2, k_{x_0}^*(u_2, v_2)\} - \min\{v_2, k_{x_0}^*(u_1, v_2)\} \ge 0$$

since  $\delta_{x_0}$  is increasing.

- (b)  $k_{x_0}^*(u_1, v_1) < v_1 \le k_{x_0}^*(u_2, v_1)$ , then the following possibilities may be distinguished: (b1) Notice that it is not possible to have  $k_{x_0}^*(u_1, v_1) < v_1$  and  $v_2 < k_{x_0}^*(u_1, v_2)$ , at the same time, because these two inequalities together imply

$$v_2 - v_1 < k_{x_0}^*(u_1, v_2) - k_{x_0}^*(u_1, v_1) = \frac{1}{2} \left( \delta_{x_0}(v_2) - \delta_{x_0}(v_1) \right),$$
  
which contradicts (DS3). Therefore, the following cases remain:

(b2) If  $k_{x_0}^*(u_1, v_2) \le v_2 < k_{x_0}^*(u_2, v_2)$ , then, invoking the 2-Lipschitz property (DS3), we have

$$\Delta_{u_{1},u_{2}}^{v_{1},v_{2}}(C_{\delta_{x_{0}}}) = k_{x_{0}}^{*}(u_{1},v_{1}) - k_{x_{0}}^{*}(u_{1},v_{2}) + v_{2} - v_{1}$$

$$= v_2 - v_1 - \frac{1}{2} \left( \delta_{x_0}(v_2) - \delta_{x_0}(v_1) \right) \ge 0$$
  
(b3)  $v_2 \ge k^* (u_2, v_2) \ge k^* (u_1, v_2)$ :

$$\begin{aligned} \Delta_{u_{1},u_{2}}^{v_{1},v_{2}}(C_{\delta_{x_{0}}}) &= k_{x_{0}}^{*}(u_{1},v_{2}) \\ &= \frac{1}{2}\delta_{x_{0}}(u_{2}-x_{0}) + \frac{1}{2}\delta_{x_{0}}(v_{1}) - v_{1} \\ &= k_{x_{0}}^{*}(u_{2}-x_{0}) + \frac{1}{2}\delta_{x_{0}}(v_{1}) - v_{1} \\ &= k_{x_{0}}^{*}(u_{2},v_{1}) - v_{1} \\ &\geq 0. \end{aligned}$$

(c)  $k_{x_0}^*(u_2, v_1) < v_1$ ; then, one necessarily has  $v_2 \ge k_{x_0}^*(u_2, v_2)$ , since, otherwise, the inequalities  $v_1 > k_{x_0}^*(u_2, v_1)$ and  $v_2 < k_{x_0}^*(u_2, v_2)$  would imply

$$v_2 - v_1 < k_{x_0}^*(u_2, v_2) - k_{x_0}^*(u_2, v_1) = \frac{1}{2} \left( \delta_{x_0}(v_2) - \delta_{x_0}(v_1) \right)$$

which contradicts (DS3). Then

$$\Delta_{u_1,u_2}^{v_1,v_2}(C_{\delta_{x_0}}) = k_{x_0}^*(u_1,v_1) - k_{x_0}^*(u_2,v_1) + k_{x_0}^*(u_2,v_2) - k_{x_0}^*(u_1,v_2) = 0.$$

As a consequence, in this case one always has  $\Delta_{u_1,u_2}^{v_1,v_2}(C_{\delta_{x_0}}) \ge 0$ .

*Case* 2: Consider a square  $[u_1, u_2] \times [u_1 - x_0, u_2 - x_0]$ , with  $u_1 \ge x_0 \ge u_2 - u_1$ . Then

$$\begin{aligned} \Delta_{u_1,u_2}^{u_1-x_0,u_2-x_0}(C_{\delta_{x_0}}) &= \delta_{x_0}(u_2-x_0) + \delta_{x_0}(u_1-x_0) - \min\left\{u_1, u_2-x_0, k_{x_0}^*(u_1, u_2-x_0)\right\} \\ &- \min\left\{u_2, u_1-x_0, k_{x_0}^*(u_2, u_1-x_0)\right\} \\ &= \frac{1}{2}\,\delta_{x_0}(u_2-x_0) + \frac{1}{2}\,\delta_{x_0}(u_1-x_0) - \min\left\{u_1-x_0, k_{x_0}^*(u_2, u_1-x_0)\right\} \ge 0, \end{aligned}$$

since  $k_{x_0}^*(u_2, u_1 - x_0) = \frac{1}{2} \delta_{x_0}(u_2 - x_0) + \frac{1}{2} \delta_{x_0}(u_1 - x_0).$ *Case* 3: Consider a square  $[u, v] \times [u, v] \subseteq [0, 1]^2 \setminus D(x_0)$  such that  $v - u \le x_0$ , i.e.,  $x_0 \le u \le v \le 1 - x_0$ . Then

$$\begin{aligned} \Delta_{u,u}^{v,v}(C_{\delta_{x_0}}) &= \min\{v, k_{x_0}^*(v, v)\} + \min\{u, k_{x_0}^*(u, u)\} - 2\min\{u, k_{x_0}^*(u, v)\} \\ &= \frac{1}{2} \left( \delta_{x_0}(v - x_0) + \delta_{x_0}(v) \right) + \frac{1}{2} \left( \delta_{x_0}(u - x_0) + \delta_{x_0}(u) \right) - \delta_{x_0}(v - x_0) - \delta_{x_0}(u) \\ &= \frac{1}{2} \left( \left( \delta_{x_0}(v) - \delta_{x_0}(v - x_0) \right) - \left( \delta_{x_0}(u) - \delta_{x_0}(u - x_0) \right) \right) \ge 0, \end{aligned}$$

since  $u \mapsto \delta_{x_0}(u) - \delta_{x_0}(u - x_0)$  is increasing.

Case 4: Consider a rectangle  $[u_1, u_2] \times [v_1, v_2] \subseteq S_L(x_0)$  such that  $u_1 \ge v_2$ , i.e.,  $0 \le v_1 \le v_2 \le u_1 \le u_2 \le x_0$ . Notice that

$$\min\{k_{x_0}^*(x_0, u), k_{x_0}^*(x_0, v)\} = \frac{1}{2}\,\delta_{x_0}(\min\{u, v\}) = \frac{1}{2}\,\delta_{x_0}(v).$$

Then

$$\Delta_{u_1,u_2}^{v_1,v_2}(C_{\delta_{x_0}}) = \frac{1}{2}\,\delta_{x_0}(v_2) + \frac{1}{2}\,\delta_{x_0}(v_1) - \frac{1}{2}\,\delta_{x_0}(v_2) - \frac{1}{2}\,\delta_{x_0}(v_1) = 0.$$

*Case* 5: Consider a square  $[u, v] \times [u, v] \subseteq S_L(x_0)$ . Then

$$\begin{aligned} \Delta_{u,u}^{v,v}(C_{\delta_{x_0}}) &= k_{x_0}^*(x_0, v) + k_{x_0}^*(x_0, u) - 2 \, k_{x_0}^*(x_0, u) \\ &= k_{x_0}^*(x_0, v) - k_{x_0}^*(x_0, u) = \frac{1}{2} \left( \delta_{x_0}(v) - \delta_{x_0}(u) \right) \ge 0. \end{aligned}$$

*Case* 6: Consider a square  $[u_1, u_2] \times [v_1, v_2] \subset S_U(x_0)$  such that  $u_1 \ge v_2$ , i.e.,  $1 - x_0 \le v_1 \le v_2 \le u_1 \le u_2 \le 1$ . Then

$$\begin{aligned} \Delta_{u_1,u_2}^{v_1,v_2}(C_{\delta_{x_0}}) &= v_2 + \frac{1}{2}\,\delta_{x_0}(u_2 - x_0) + v_1 + \frac{1}{2}\,\delta_{x_0}(u_1 - x_0) \\ &- \left(v_2 + \frac{1}{2}\,\delta_{x_0}(u_1 - x_0)\right) - \left(v_1 + \frac{1}{2}\,\delta_{x_0}(u_2 - x_0)\right) = 0. \end{aligned}$$

*Case* 7: Consider a square  $[u, v] \times [u, v] \subseteq S_U(x_0)$ . Then

$$\begin{aligned} \Delta_{u,v}^{u,v}(C_{\delta_{x_0}}) &= A_{x_0}(v,v) + A_{x_0}(u,u) - 2 A_{x_0}(u,v) \\ &= v - u - \left(k_{x_0}^*(v,1-x_0) - k_{x_0}^*(u,1-x_0)\right) \\ &= v - u - \frac{1}{2} \left(\delta_{x_0}(v-x_0) - \delta_{x_0}(u-x_0)\right) \ge 0 \end{aligned}$$

because of (DS3).

Taking into account that  $C_{\delta_{x_0}}$  is symmetric, any rectangle in  $[0, 1]^2$  having vertices in more than one of the regions considered may be decomposed into the union of a finite number of rectangles considered in the previous cases, one sees that its  $C_{\delta_{x_0}}$ -volume is non-negative. This concludes the proof.

Note that the requirement that the function f in the statement of Theorem 11 is increasing is satisfied by the subdiagonal sections at  $x_0$  of many important copulas. We only mention W,  $\Pi$ , M, the family of Frank copulas, all the copulas introduced by Durante in [5], and hence in particular the family of Cuadras–Augé copulas. However, not every copula has a sub-diagonal section at  $x_0$  for which the above condition is satisfied. For instance, if C is the following shuffle of M,

$$C(u, v) = \begin{cases} \min\{u, v\}, & \text{if } \min\{u, v\} \le \frac{1}{2}, \\ \max\left\{\frac{1}{2}, u + v - 1\right\}, & \text{otherwise,} \end{cases}$$

and  $x_0 = \frac{1}{4}$ , then  $u \mapsto \delta_{x_0}^C(u) - \delta_{x_0}^C(u - x_0)$  is decreasing for  $u \in \left[\frac{1}{2}, \frac{5}{8}\right]$ .

### 8. The set $\mathcal{Q}_{\delta_{x_0}}$ and its bounds

Let us now turn to the second problem stated in Section 2. Given a sub-diagonal  $\delta_{x_0}$ , we have seen that the sets  $Q_{\delta_{x_0}}$  and  $C_{\delta_{x_0}}$  are not empty. Moreover, the following result, whose proof is immediate, can be stated.

**Theorem 12.** The sets  $Q_{\delta_{x_0}}$  and  $C_{\delta_{x_0}}$  are both convex and compact, where compactness is meant in the sense of the natural topology of these spaces, namely the topology of the  $L^{\infty}$ -norm, or, equivalently, of uniform convergence on the unit square  $[0, 1]^2$ .

We now study the best-possible bounds for  $Q_{\delta_{x_0}}$ .

**Theorem 13.** For  $x_0 \in [0, 1[$  and a sub-diagonal  $\delta_{x_0}$ , the copula  $B_{\delta_{x_0}}$  defined by (19) is the smallest (quasi-)copula whose sub-diagonal section at  $x_0$  coincides with  $\delta_{x_0}$ , viz.  $B_{\delta_{x_0}} \leq Q$  for every quasi-copula Q in  $Q_{\delta_{x_0}}$ .

**Proof.** In the proof of Proposition 9 it has already been shown that  $B_{\delta_{x_0}}$  is a copula and, consequently, a quasi-copula, and that the relationship  $B_{\delta_{x_0}} \in C_{\delta_{x_0}} \subset Q_{\delta_{x_0}}$  holds.

Let Q be any (quasi-)copula having  $\delta_{x_0}$  as its sub-diagonal section at  $x_0$ . Obviously, one has  $B_{\delta_{x_0}}(u, v) \leq Q(u, v)$ at every point (u, v) that does not belong to  $S_2(x_0)$ . Then, take  $(u, v) \in S_2(x_0)$  and assume, first, that  $v > u - x_0$ ; thus  $B_{\delta_{x_0}}(u, v) = (u - x_0) - \hat{\delta}_{x_0}(t^*)$ , where  $t^*$  is such that

$$\hat{\delta}_{x_0}(t^*) \coloneqq \min_{t \in [u-x_0,v]} \hat{\delta}_{x_0}(t).$$

We recall (see [13, Proposition 3]) that a quasi-copula satisfies inequality (1) whenever at least one of u, u', v or v' is equal to either 0 or 1. This inequality for Q applied to the rectangle  $[u, t^* + x_0] \times [t^*, 1]$  yields

$$0 \le t^* + x_0 - u + Q(u, t^*) - Q(t^* + x_0, t^*) = t^* - (u - x_0) + Q(u, t^*) - \delta_{x_0}(t^*),$$

so that

$$B_{\delta_{x_0}}(u,v) = (u-x_0) - \hat{\delta}_{x_0}(t^*) = \delta_{x_0}(t^*) - (t^* - (u-x_0)) \le Q(u,t^*) \le Q(u,v).$$

A similar argument holds when  $v \le u - x_0$ . This concludes the proof.  $\Box$ 

**Theorem 14.** For  $x_0 \in [0, 1[$  and a sub-diagonal  $\delta_{x_0}$ , the function  $G_{\delta_{x_0}}$  from  $[0, 1]^2$  into [0, 1] defined by

$$G_{\delta_{x_0}}(u,v) \coloneqq \begin{cases} \min\{u, v, M'_{x_0}(u,v) - q_{x_0}(u,v)\}, & \text{if } (u,v) \in T_U(x_0), \\ \min\{m'_{x_0}(u,v), M'_{x_0}(u,v) - q_{x_0}(u,v)\}, & \text{if } (u,v) \in T_L(x_0), \end{cases}$$
(25)

where  $m'_{x_0}(u, v) = \min\{u - x_0, v\}$ ,  $M'_{x_0}(u, v) = \max\{u - x_0, v\}$ , and  $q_{x_0}$  is defined by (7), is a quasi-copula such that its sub-diagonal section at  $x_0$  coincides with  $\delta_{x_0}$ . Moreover  $Q \leq G_{\delta_{x_0}}$ , for every quasi-copula Q in  $Q_{\delta_{x_0}}$ .

We shall prove this theorem by recourse to a series of propositions and lemmata. But, before doing this, we note that the function  $G_{\delta_{x_0}}$  may be written in the equivalent manner

$$G_{\delta_{x_0}}(u,v) = \begin{cases} \min\{u, v - q_{x_0}(u,v)\}, & \text{if } (u,v) \in T_U(x_0), \\ \min\{v, u - x_0 - q_{x_0}(u,v)\}, & \text{if } (u,v) \in T_L(x_0). \end{cases}$$
(26)

We first prove that  $G_{\delta_{x_0}}$  takes values only in the unit interval.

**Lemma 15.** For  $x_0 \in [0, 1[$ , for a sub-diagonal  $\delta_{x_0}$ , and for all  $(u, v) \in [0, 1]^2$ ,

$$M'_{x_0}(u, v) - q_{x_0}(u, v) \ge 0.$$

**Proof.** Choose (u, v) arbitrarily in  $[0, 1]^2$  and set  $q_{x_0}(u, v) = \hat{\delta}_{x_0}(t^*)$  for some  $t^*$  in  $[m_{x_0}(u, v), M_{x_0}(u, v)]$ . From (HD3) we get

$$q_{x_0}(u,v) = \hat{\delta}_{x_0}(t^*) \le t^* \le M_{x_0}(u,v) \le M'_{x_0}(u,v)$$

namely the assertion.  $\Box$ 

**Lemma 16.** For  $x_0 \in [0, 1[$  and for a sub-diagonal  $\delta_{x_0}$ , the function  $G_{\delta_{x_0}}$  defined by (25) satisfies, for every  $u \in [0, 1]$ ,  $G_{\delta_{x_0}}(u, 0) = G_{\delta_{x_0}}(0, u) = 0$  and  $G_{\delta_{x_0}}(u, 1) = G_{\delta_{x_0}}(1, u) = u$ .

**Proof.** If (u, v) is in  $T_U(x_0)$  and either u = 0 or v = 0, it follows immediately that  $G_{\delta_{x_0}}(u, 0) = G_{\delta_{x_0}}(0, v) = 0$ since  $M'_{x_0}(u, v) - q_{x_0}(u, v) \ge 0$  for arbitrary u and v as shown before. It remains to consider the case  $(u, v) \in T_L(x_0)$ and v = 0, i.e.,  $u \ge x_0$  and v = 0. Then it follows that  $m'_{x_0}(u, 0) = 0$ , so that  $G_{\delta_{x_0}}(u, 0) = 0$ .

Next, take either u = 1 or v = 1. If u = 1 and  $v \ge 1 - x_0$ , then, from (26),  $G_{\delta_{x_0}}(1, v) = v - q_{x_0}(1, v) = v$ , since  $q_{x_0}(1, v) = 0$ . If u = 1 and  $v < 1 - x_0$ , then, from (26),  $G_{\delta_{x_0}}(1, v) = \min\{v, 1 - x_0\} = v$ .

Now assume that  $v \le 1 - x_0$ . Because of (HD3), we know that

 $\hat{\delta}_{x_0}(t) \le \min\{t, 1 - x_0 - t\} \le 1 - x_0 - t$ 

for every  $t \in [v, 1 - x_0]$  and, as a consequence,  $\hat{\delta}_{x_0}(t) \le 1 - x_0 - v$  or, equivalently,  $v \le 1 - x_0 - \hat{\delta}_{x_0}(t)$  for every  $t \in [v, 1 - x_0]$ . Since the inequality holds for all  $t \in [v, 1 - x_0]$ , it can be concluded that

$$v \le 1 - x_0 - \max\{\hat{\delta}_{x_0}(t) \mid t \in [v, 1 - x_0]\} = M_{x_0}(1, v) - q_{x_0}(1, v),$$

which also shows that in this case  $G_{\delta_{x_0}}(1, v) = \min\{v, M_{x_0}(1, v) - q_{x_0}(1, v)\} = v$ .

Next assume that v = 1. Then, for every choice of u, (u, 1) belongs to  $T_U(x_0)$ , and  $M_{x_0}(u, 1) = 1 - x_0$ , and  $m_{x_0}(u, 1) = \max\{0, u - x_0\}$  so that

$$G_{\delta_{x_0}}(u, 1) = \min\{u, 1 - \max\{\hat{\delta}_{x_0}(t) \mid t \in [\max\{0, u - x_0\}, 1 - x_0]\}\}.$$

Consider first  $u \ge x_0$ ; then we also have  $m_{x_0}(u, 1) = u - x_0$ . Because of (HD3), we again know that  $\delta_{x_0}(t) \le 1 - x_0 - t$  for every  $t \in [0, 1 - x_0]$ . We can therefore conclude that, for all  $t \ge u - x_0$ ,

 $\hat{\delta}_{x_0}(t) \le 1 - x_0 - t \le 1 - x_0 - (u - x_0) = 1 - u$ 

or, equivalently, that  $u \le 1 - \hat{\delta}_{x_0}(t)$  for all  $t \in [u - x_0, 1 - x_0]$  and, in particular,

 $u \le 1 - \max\{\hat{\delta}_{x_0}(t) \mid t \in [u - x_0, 1 - x_0]\},\$ 

so that  $G_{\delta_{x_0}}(u, 1) = u$ .

If  $u \le x_0$ , then  $m_{x_0}(u, 1) = 0$  and

$$1 - \max\{\hat{\delta}_{x_0}(t) \mid t \in [0, 1 - x_0]\} \ge 1 - \max\{t \mid t \in [0, 1 - x_0]\} + \min\{\delta_{x_0}(t) \mid t \in [0, 1 - x_0]\}$$
$$= 1 - (1 - x_0) + \delta_{x_0}(0) = x_0 \ge u;$$

thus finally also in this case  $G_{\delta_{x_0}}(u, 1) = u$ .  $\Box$ 

**Proposition 17.** For  $x_0 \in [0, 1[$  and a sub-diagonal  $\delta_{x_0}$ , the function  $G_{\delta_{x_0}}$  defined by (25) is increasing in each place and satisfies the Lipschitz property (2).

**Proof.** In order to show the isotony and the Lipschitz property of  $G_{\delta_{x_0}}$  it suffices to prove that  $G_{\delta_{x_0}}$  is increasing and Lipschitz separately on  $T_U(x_0)$  and  $T_L(x_0)$ . The proof for the other cases can be split into several steps dealing only with the situations mentioned before and, then, by applying either basic summation or the triangle inequality. We shall therefore distinguish four different cases and, for each case, prove that  $G_{\delta_{x_0}}$  is increasing and satisfies the Lipschitz property.

*Case* 1: Let both  $(u_1, v)$  and  $(u_2, v)$  be in  $T_U(x_0)$  with  $u_1 \le u_2$ , i.e.,  $u_1 - x_0 \le u_2 - x_0 \le v$ . Then

$$n_{x_0}(u_1, v) = \max\{0, u_1 - x_0\} \le \max\{0, u_2 - x_0\} = m_{x_0}(u_2, v);$$

$$M_{x_0}(u_1, v) = \min\{1 - x_0, v\} = M_{x_0}(u_2, v).$$

Therefore, one has  $q_{x_0}(u_1, v) \ge q_{x_0}(u_2, v)$ . Also recall that, for every point  $(u, v) \in T_U(x_0)$ ,

 $G_{\delta_{x_0}}(u, v) = \min\{u, v - q_{x_0}(u, v)\}.$ 

We show first that, for every  $v, u \mapsto G_{\delta_{x_0}}(u, v)$  is increasing. If  $G_{\delta_{x_0}}(u_2, v) = u_2$ , then

 $G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) = u_2 - \min\{u_1, v - q_{x_0}(u_1, v)\} \ge u_2 - u_1 \ge 0.$ 

If  $G_{\delta_{x_0}}(u_2, v) = v - q_{x_0}(u_2, v)$ , then

$$G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) = v - q(u_2, v) - \min\{u_1, v - q_{x_0}(u_1, v)\}$$
  
 
$$\geq v - q_{x_0}(u_2, v) - v + q_{x_0}(u_1, v) \geq 0.$$

Now we turn to the Lipschitz property. If  $G_{\delta_{x_0}}(u_1, v) = u_1$ , then

 $G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) = \min\{u_2, v - q_{x_0}(u_2, v)\} - u_1 \le u_2 - u_1.$ 

If  $G_{\delta_{x_0}}(u_1, v) = v - q_{x_0}(u_1, v)$ , then

$$G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) \le v - q_{x_0}(u_2, v) - v + q_{x_0}(u_1, v) = q_{x_0}(u_1, v) - q_{x_0}(u_2, v);$$

thus it remains to show that  $q_{x_0}(u_1, v) - q_{x_0}(u_2, v) \le u_2 - u_1$ .

If  $q_{x_0}(u_1, v) = q_{x_0}(u_2, v)$  there is nothing to prove; therefore, assume that  $q_{x_0}(u_1, v) > q_{x_0}(u_2, v)$ , which in turn means that  $m_{x_0}(u_2, v) > m_{x_0}(u_1, v) \ge 0$  and  $m_{x_0}(u_2, v) = u_2 - x_0$ . This also means that there exist t' and t'' fulfilling

$$m_{x_0}(u_1, v) \le t' < m_{x_0}(u_2, v) \le t'' \le M_{x_0}(u_2, v)$$

so that  $q_{x_0}(u_1, v) = \hat{\delta}_{x_0}(t')$  and  $q_{x_0}(u_2, v) = \hat{\delta}_{x_0}(t'')$ . Moreover  $\hat{\delta}_{x_0}(t'') \ge \hat{\delta}_{x_0}(t)$  for every  $t \in [m_{x_0}(u_2, v), M_{x_0}(u_2, v)]$ , in particular for  $t = m_{x_0}(u_2, v) = u_2 - x_0$ , since  $q_{x_0}(u_2, v)$  is the maximum of  $\hat{\delta}_{x_0}$  on the same interval. Taking into account that  $\hat{\delta}_{x_0}$  also is Lipschitz (see property (HD2)) one has

$$\begin{aligned} q_{x_0}(u_1, v) - q_{x_0}(u_2, v) &= \hat{\delta}_{x_0}(t') - \hat{\delta}_{x_0}(t'') \le \hat{\delta}_{x_0}(t') - \hat{\delta}_{x_0}(u_2 - x_0) \\ &\le u_2 - x_0 - t' \le u_2 - x_0 - m_{x_0}(u_1, v) \le u_2 - x_0 - (u_1 - x_0) \le u_2 - u_1, \end{aligned}$$

which proves the assertion in this case.

Case 2: Let  $(u, v_1)$  and  $(u, v_2)$  belong to  $T_U(x_0)$  with  $v_1 \le v_2$ , i.e.,  $u - x_0 \le v_1 \le v_2$ . In this case we have

$$m_{x_0}(u, v_1) = \max\{0, u - x_0\} = m_{x_0}(u, v_2);$$

 $M_{x_0}(u, v_1) = \min\{1 - x_0, v_1\} \le \min\{1 - x_0, v_2\} = M_{x_0}(u, v_2).$ 

Therefore, one has  $q_{x_0}(u, v_1) \le q_{x_0}(u, v_2)$  and, for all (u, v) in  $T_U(x_0)$ ,  $G_{\delta_{x_0}}(u, v) = \min\{u, v - q_{x_0}(u, v)\}$ . For the isotony of  $G_{\delta_{x_0}}$ , let us first assume that  $G_{\delta_{x_0}}(u, v_2) = u$ ; then

$$G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) = u - G_{\delta_{x_0}}(u, v_1) \ge u - u = 0.$$

If  $G_{\delta_{x_0}}(u, v_2) = v_2 - q_{x_0}(u, v_2)$ , then

$$G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) \ge v_2 - q_{x_0}(u, v_2) - (v_1 - q_{x_0}(u, v_1)) = (v_2 - v_1) - (q_{x_0}(u, v_2) - q_{x_0}(u, v_1)).$$

Therefore it suffices to prove that  $v_2 - v_1 \ge q_{x_0}(u, v_2) - q_{x_0}(u, v_1)$ . This latter inequality is trivially fulfilled when  $q_{x_0}(u, v_2) = q_{x_0}(u, v_1)$ . As in the previous case, if  $q_{x_0}(u, v_2) > q_{x_0}(u, v_1)$ , then  $M_{x_0}(u, v_1) = v_1 < 1 - x_0$  and there exist t' and t'' with  $m_{x_0}(u, v_1) \le t' \le v_1 < t'' \le M_{x_0}(u, v_2)$  such that

$$q_{x_0}(u, v_2) = \hat{\delta}_{x_0}(t'') > \hat{\delta}_{x_0}(t') = q_{x_0}(u, v_1).$$

Then

$$0 \le q_{x_0}(u, v_2) - q_{x_0}(u, v_1) = \hat{\delta}_{x_0}(t'') - \hat{\delta}_{x_0}(t') \le \hat{\delta}_{x_0}(t'') - \hat{\delta}_{x_0}(v_1) \\ \le t'' - v_1 \le M_{x_0}(u, v_2) - v_1 \le v_2 - v_1$$

because of the Lipschitz property (HD2) of  $\hat{\delta}_{x_0}$  and of the fact that  $q_{x_0}(u, v_1)$  is the maximum of  $\hat{\delta}_{x_0}$  on the interval  $[m_{x_0}(u, v_1), M_{x_0}(u, v_1)]$ .

As for the Lipschitz property of  $G_{\delta_{x_0}}$ , if  $G_{\delta_{x_0}}(u, v_1) = u$ , then clearly,

$$G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) = G_{\delta_{x_0}}(u, v_2) - u \le u - u = 0 \le v_2 - v_1.$$

For  $G_{\delta_{x_0}}(u, v_1) = v_1 - q_{x_0}(u, v_1)$  it follows that

$$\begin{aligned} G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) &\leq v_2 - q_{x_0}(u, v_2) - (v_1 - q_{x_0}(u, v_1)) \\ &= v_2 - v_1 - (q_{x_0}(u, v_2) - q_{x_0}(u, v_1)) \leq v_2 - v_1, \end{aligned}$$

since  $q_{x_0}(u, v_2) - q_{x_0}(u, v_1) \ge 0$ .

Case 3: Let  $(u_1, v)$  and  $(u_2, v)$  belong to  $T_L(x_0)$ , with  $u_1 \le u_2$ , i.e.,  $v \le u_1 - x_0 \le u_2 - x_0$ . In this case

$$m_{x_0}(u_1, v) = v = m_{x_0}(u_2, v);$$
  

$$M_{x_0}(u_1, v) = u_1 - x_0 \le u_2 - x_0 = M_{x_0}(u_2, v).$$

Therefore, one has  $q_{x_0}(u_1, v) \le q_{x_0}(u_2, v)$  and, for all  $(u, v) \in T_L(x_0)$ ,  $G_{\delta_{x_0}}(u, v) = \min\{v, u - x_0 - q_{x_0}(u, v)\}$ .

For the isotony, if  $G_{\delta_{x_0}}(u_2, v) = v$ , then  $G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) \ge v - v = 0$ . If  $G_{\delta_{x_0}}(u_2, v) = u_2 - x_0 - q_{x_0}(u_2, v)$ , then

$$G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) \ge u_2 - u_1 - (q_{x_0}(u_2, v) - q_{x_0}(u_1, v)).$$

If  $q_{x_0}(u_2, v) = q_{x_0}(u_1, v)$ , it is immediate that  $G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) \ge 0$ . For the case  $q_{x_0}(u_2, v) > q_{x_0}(u_1, v)$ , there are necessarily t' and t'' such that  $v \le t' \le u_1 - x_0 < t'' \le u_2 - x_0$  and, moreover,  $q_{x_0}(u_2, v) = \hat{\delta}_{x_0}(t'')$  and  $q_{x_0}(u_1, v) = \hat{\delta}_{x_0}(t')$ . By an argument analogous to that of the previous case, we can conclude that

$$q_{x_0}(u_2, v) - q_{x_0}(u_1, v) = \hat{\delta}_{x_0}(t'') - \hat{\delta}_{x_0}(t') \le \hat{\delta}_{x_0}(t'') - \hat{\delta}_{x_0}(u_1 - x_0)$$
  
$$\le t'' - (u_1 - x_0) \le u_2 - x_0 - (u_1 - x_0) = u_2 - u_1,$$

so that indeed  $G_{\delta_{x_0}}$  is increasing in its first argument on  $T_L(x_0)$ .

For the Lipschitz property, if  $G_{\delta_{x_0}}(u_1, v) = v$ , then

$$G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) \le v - v = 0 \le u_2 - u_1.$$

If  $G_{\delta_{x_0}}(u_1, v) = u_1 - x_0 - q_{x_0}(u_1, v)$ , then

 $G_{\delta_{x_0}}(u_2, v) - G_{\delta_{x_0}}(u_1, v) \le u_2 - u_1 - (q_{x_0}(u_2, v) - q_{x_0}(u_1, v)) \le u_2 - u_1.$ 

Case 4: Finally, let  $(u, v_1)$  and  $(u, v_2)$  be in  $T_L(x_0)$ , with  $v_1 \le v_2$ , i.e.,  $v_1 \le v_2 \le u - x_0$ . In this case

 $m_{x_0}(u, v_1) = v_1 \le v_2 = m_{x_0}(u, v_2)$  and  $M_{x_0}(u, v_1) = u - x_0 = M_{x_0}(u, v_2).$ 

Therefore, one has  $q_{x_0}(u, v_1) \ge q_{x_0}(u, v_2)$ . Moreover, if  $q_{x_0}(u, v_1) > q_{x_0}(u, v_2)$ , then there exist t' and t'' with  $v_1 \leq t' < v_2 \leq t'' \leq u - x_0$  such that  $q_{x_0}(u, v_1) = \hat{\delta}_{x_0}(t')$  and  $q_{x_0}(u, v_2) = \hat{\delta}_{x_0}(t'')$ . An argument analogous to those of the previous cases again yields

$$0 \le q_{x_0}(u, v_1) - q_{x_0}(u, v_2) = \hat{\delta}_{x_0}(t') - \hat{\delta}_{x_0}(t'') \le \hat{\delta}_{x_0}(t') - \hat{\delta}_{x_0}(v_2) \le v_2 - t' \le v_2 - v_1.$$

Notice that, for all  $(u, v) \in T_L(x_0)$ ,  $G_{\delta_{x_0}}(u, v) = \min\{v, u - x_0 - q_{x_0}(u, v)\}$ . For the isotony, if  $G_{\delta_{x_0}}(u, v_2) = v_2$ , then  $G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) \ge v_2 - v_1 \ge 0$ . Otherwise,  $G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) \ge v_2 - v_1 \ge 0$ .  $G_{\delta_{x_0}}(u, v_1) \ge q_{x_0}(u, v_1) - q_{x_0}(u, v_2) \ge 0.$ 

Ås for the Lipschitz property, if  $G_{\delta_{x_0}}(u, v_1) = u - x_0 - q_{x_0}(u, v_1)$ , then

$$G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) \le q_{x_0}(u, v_1) - q_{x_0}(u, v_2) \le v_2 - v_1$$

Otherwise,  $G_{\delta_{x_0}}(u, v_2) - G_{\delta_{x_0}}(u, v_1) \le v_2 - v_1$  follows immediately.

In conclusion, in all the cases, we are able to show that  $G_{\delta_{x_0}}$  is increasing and satisfies the Lipschitz property. 

**Lemma 18.** Let  $x_0$  be in ]0, 1[ and let  $\delta_{x_0}$  be a sub-diagonal. Then the sub-diagonal section at  $x_0$  of the quasi-copula  $G_{\delta_{x_0}}$  defined by (25) equals  $\delta_{x_0}$ , i.e.,  $G_{\delta_{x_0}} \in \mathcal{Q}_{\delta_{x_0}}$ .

**Proof.** Let t be in  $[0, 1 - x_0]$ . Then

$$m_{x_0}(t+x_0,t) = m'_{x_0}(t+x_0,t) = t = M_{x_0}(t+x_0,t) = M'_{x_0}(t+x_0,t).$$

As a consequence,  $q_{x_0}(t + x_0, t) = \hat{\delta}_{x_0}(t) = t - \delta_{x_0}(t)$  and

$$G_{\delta_{x_0}}(t+x_0,t) = \min\{t, t-(t-\delta_{x_0}(t))\} = \min\{t, \delta_{x_0}(t)\} = \delta_{x_0}(t),$$

which proves the assertion. 

**Proposition 19.** For a given  $x_0 \in [0, 1[$  and a given sub-diagonal  $\delta_{x_0}$ ,  $G_{\delta_{x_0}}$  is the largest quasi-copula whose subdiagonal section at  $x_0$  coincides with  $\delta_{x_0}$ , namely,  $Q \leq G_{\delta_{x_0}}$  for every  $Q \in Q_{\delta_{x_0}}$ .

**Proof.** Let Q be a quasi-copula in  $\mathcal{Q}_{\delta_{x_0}}$ . For  $(u, v) \in [0, 1]^2$  consider the following two cases. Assume first that  $(u, v) \in T_U(x_0)$ , i.e.,  $v \ge u - x_0$ . In this case

$$G_{\delta_{x_0}}(u, v) = \min\{u, v - q_{x_0}(u, v)\};$$
  

$$q_{x_0}(u, v) = \max\left\{\hat{\delta}_{x_0}(t) \mid t \in [m_{x_0}(u, v), M_{x_0}(u, v)]\right\};$$
  

$$m_{x_0}(u, v) = \max\{0, u - x_0\};$$
  

$$M_{x_0}(u, v) = \min\{1 - x_0, v\}.$$

Since  $Q(u, v) \leq \min\{u, v\}$ , it remains to prove that  $Q(u, v) \leq v - q_{x_0}(u, v)$ . For every  $t \in [\max\{0, u - x_0\}, t \in \max\{0, u - x_0\}, t \in [\max\{0, u,$  $\min\{1 - x_0, v\}$ ], the Lipschitz property yields

 $Q(u, v) - Q(u, t) \le v - t$  or, equivalently,  $Q(u, v) \le v - t + Q(u, t)$ .

Since a quasi-copula is increasing,

$$Q(u, v) \le v - t + Q(u, t) \le v - t + Q(x_0 + t, t) = v - t + \delta_{x_0}(t) = v - \hat{\delta}_{x_0}(t)$$

for all  $t \in [m_{x_0}(u, v), M_{x_0}(u, v)]$ , and, hence,

$$Q(u, v) \le v - \max\left\{\hat{\delta}_{x_0}(t) \mid t \in \left[m_{x_0}(u, v), M_{x_0}(u, v)\right]\right\} = v - q_{x_0}(u, v).$$

Combining both the inequalities leads to  $Q(u, v) \leq G_{\delta_{x_0}}(u, v)$ .

Assume, now, that  $(u, v) \in T_L(x_0)$ , and, thus,  $v \le u - x_0$ . Since  $Q(u, v) \le \min\{u, v\} \le v$ , only  $Q(u, v) \le u - x_0 - q_{x_0}(u, v)$  has to be shown.

As in the previous case, the Lipschitz property and the isotony of Q yield, for every  $t \in [m_{x_0}(u, v), M_{x_0}(u, v)] = [v, u - x_0],$ 

$$Q(u, v) \le u - (t + x_0) + Q(t + x_0, v) \le u - x_0 - t + Q(t + x_0, t)$$
  
=  $u - x_0 - t + \delta_{x_0}(t) = u - x_0 - \hat{\delta}_{x_0}(t).$ 

Therefore,  $Q(u, v) \le u - x_0 - \max\{\hat{\delta}_{x_0}(t) \mid t \in [v, u - x_0]\} = u - x_0 - q_{x_0}(u, v)$ , so that  $Q(u, v) \le G_{\delta_{x_0}}(u, v)$ , which concludes the proof.  $\Box$ 

The following examples show that, on the one hand,  $G_{\delta_{x_0}}$  is, in general, a proper quasi-copula, and, on the other hand, that it is the largest possible quasi-copula with a given sub-diagonal section at  $x_0$ .

**Example 6.** Take  $x_0 = 0.3$  and consider the sub-diagonal section at 0.3 of the independence copula  $\Pi$ ,  $\delta_{x_0}^{\Pi}(t) = 0.3 \cdot t + t^2$ , whence  $\hat{\delta}_{x_0}^{\Pi}(t) = 0.7 \cdot t - t^2$ . After an easy calculation, one sees that the  $G_{\delta_{x_0}}$ -volume of the square  $[0.55, 0.65] \times [0.25, 0.35]$  equals

$$\Delta_{0.55,0.65}^{0.25,0.35}(G_{\delta_{x_0}}) = -0.1 < 0.$$

**Example 7.** For  $x_0 \in [0, 1[$ , consider the largest sub-diagonal,  $\delta_{x_0}(t) = t$ , which is the sub-diagonal section at  $x_0$  of the copula M. Then  $\hat{\delta}_{x_0}(t) = 0$  and  $q_{x_0}(u, v) = 0$ . Therefore,

$$G_{\delta_{x_0}}(u, v) = \begin{cases} \min\{u, v\}, & \text{if } (u, v) \in T_U(x_0), \\ \min\{v, u - x_0\}, & \text{if } (u, v) \in T_L(x_0); \end{cases}$$

however, if  $(u, v) \in T_L(x_0)$ , then  $\min\{v, u - x_0\} = v = \min\{u, v\}$ . Therefore,  $G_{\delta_{x_0}}(u, v) = M(u, v)$  for all  $u, v \in [0, 1]$ .

### 9. Comparison with diagonal copulas

What we have presented in the previous sections ought to be compared with the work of Fredricks and Nelsen [10, 11], who introduced the notion of *diagonal copula*, namely of a copula with a given diagonal section. Since the subdiagonal section at  $x_0$  coincides with the diagonal section when  $x_0 = 0$ , namely  $\delta_0 = \delta$ , one can prove the following result.

**Theorem 20.** (a) If  $B^*_{\delta_{x_0}}$  is the quasi-copula given in Proposition 9, then, for all (u, v) in  $[0, 1]^2$ ,

$$\lim_{x_0 \to 0} B^*_{\delta_{x_0}}(u, v) = \min\{u, v\} - \min\{t - \delta(t) \mid t \in [\min\{u, v\}, \max\{u, v\}]\},$$
(27)

namely the Bertino copula. (b) If  $C_{\delta_{\chi_0}}$  is the copula given by (22), then, for all (u, v) in  $[0, 1]^2$ ,

$$\lim_{x_0 \to 0} C_{\delta_{x_0}}(u, v) = \min\left\{ u, v, \frac{1}{2} \left( \delta(u) + \delta(v) \right) \right\}.$$
(28)

**Proof.** Since every (quasi-)copula satisfies the same boundary conditions, in either case, one may consider only points (u, v) of the open unit square  $]0, 1[^2$ . But then, every such point (u, v) belongs to the set  $]0, 1[^2 \setminus D(x_0)$ , when  $x_0$  is small enough, precisely, when  $x_0 < \min\{u, v\}$ ; thus, since every (quasi-)copula is continuous, one has

$$\lim_{x_0 \to 0} B^*_{\delta_{x_0}}(u, v) = \lim_{x_0 \to 0} \left( m_{x_0}(u, v) - \min\left\{ \hat{\delta}_{x_0}(t) \mid t \in [m_{x_0}(u, v), M_{x_0}(u, v)] \right\} \right)$$
  
= min{u, v} - min{t - \delta(t) | t \in [min{u, v}, max{u, v}]},

and

$$\lim_{x_0 \to 0} C_{\delta_{x_0}}(u, v) = \lim_{x_0 \to 0} \min\left\{u, v, A_{x_0}(u, v)\right\} = \min\left\{u, v, \frac{1}{2}\left(\delta(u) + \delta(v)\right)\right\},\$$

which proves the assertion.  $\Box$ 

The limits as  $x_0$  tends to 0 of the quasi-copula  $B^*_{\delta_{x_0}}$  and of the copula  $C_{\delta_{x_0}}$  are respectively equal to the smallest and the largest copula having  $\delta$  as their diagonal section.

For the quasi-copulas of Section 6, one has respectively

$$\lim_{x_0 \to 0} K^*_{\delta_{x_0}}(u, v) = \min\left\{u, v, \frac{1}{2}\delta(\min\{u, v\}) + \frac{1}{2}\delta(\max\{u, v\})\right\} = K_{\delta}(u, v),$$

and

 $\lim_{x_0 \to 0} A^*_{\delta_{x_0}}(u, v) = \min\{u, v, \min\{u, v\} - \max\{t - \delta(t) \mid t \in [\min\{u, v\}, \max\{u, v\}]\}\}$  $= A_{\delta}(u, v).$ 

Finally, for the largest quasi-copula in  $Q_{x_0}$ , one has

$$\lim_{x_0\to 0} G_{\delta_{x_0}}(u,v) = A_{\delta}(u,v).$$

Notice that the limit as  $x_0$  tends to 0 of the quasi-copula  $K^*_{\delta_{x_0}}$  is again the diagonal copula *C* of Fredricks and Nelsen. On the other hand  $A^*_{\delta_{x_0}}$  defines a proper quasi-copula that also has  $\delta_{x_0}$  as its diagonal section.

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# On representations of 2-increasing binary aggregation functions

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### A R T I C L E I N F O

### ABSTRACT

Article history: Received 31 January 2008 Received in revised form 20 May 2008 Accepted 7 August 2008 In this paper, we provide two different representations of 2-increasing binary aggregation functions by means of their lower and upper margins and a suitable copula. © 2008 Elsevier Inc. All rights reserved.

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### 2-Increasingness Supermodularity Copulas

Binary aggregation function

### 1. Introduction

Keywords:

Aggregation is an important tool in any discipline where the fusion of different pieces of information is of interest. Aggregation functions take arbitrarily but finitely many inputs from the unit interval and map them to a representative value in the unit interval [1,3,4,23]. As such they are successfully applied in, e.g., multicriteria decision making where each alternative is evaluated with respect to a fixed set of criteria, each of the single scores expressed by a number from the unit interval, and finally being aggregated to a global score of the alternative [13]. Similar approaches can be found in the fields of preference modelling [10] or utility theory [6]. Further applications comprise, but are clearly not limited to, computer-assisted assessment [18] and flexible database queries [2]. Particular classes of binary aggregation functions have been investigated in the framework of many-valued logics [11,12] or probability distribution functions, in particular copulas [16].

In this contribution we focus on the class of 2-increasing binary aggregation functions and their representations [7]. The most prominent and most studied examples thereof are copulas [16], whose importance in statistics and probability theory is well-known, but which also attracted attention from the fields of aggregation [8]. Further synonyms for the 2-increasing property are the property of supermodularity [17], quasi-monotonicity or lattice superadditivity [14] indicating that 2-increasing functions are of interest also in other fields related to pure and applied mathematics, like the theory of majorization [14], especially stochastic orders [15,20], capacities [5], and several other problems arising in economics [17,22].

The idea of our investigation originated from the celebrated *Sklar's theorem* [21], which allows to represent every bivariate probability distribution function  $F : \mathbb{R}^2 \to [0, 1]$  in the form  $F(x, y) = C(F_1(x), F_2(y))$ , where

- $F_1$  and  $F_2$  are the upper margins of F, obtained as limits of  $F(x_1, x_2)$  when  $x_i$  tends to  $+\infty$  for i = 1, 2,
- *C* is a copula, i.e., a bivariate distribution function on [0,1]<sup>2</sup> whose univariate marginal distribution functions are uniformly distributed on [0,1].

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Along the same lines of thoughts we investigate whether it is possible to use copulas also for the representation of arbitrary, not necessarily (right- or left-) continuous 2-increasing aggregation functions in terms of all their margins, i.e., by allowing also their lower margins to be taken into account.

Basic preliminaries and facts about 2-increasing aggregation functions are summarized in Section 2. Section 3 is devoted to the representation of 2-increasing aggregation functions with given upper margins. Section 4 provides another representation result when both lower as well as upper margins are given.

### 2. Preliminaries

Let us recall some basic notions that we will use in the sequel.

**Definition 1.** A (*binary*) *aggregation function* is a function  $A : [0,1]^2 \rightarrow [0,1]$  satisfying the following properties:

(i) 
$$A(0,0) = 0$$
 and  $A(1,1) = 1$ ,

(ii)  $A(x_1, y_1) \leq A(x_2, y_2)$  for all  $x_1, x_2 \in [0, 1]$  and for all  $y_1, y_2 \in [0, 1]$  with  $x_1 \leq x_2$  and  $y_1 \leq y_2$ .

The class of all aggregation functions will be denoted by *A*.

**Definition 2.** An aggregation function *A* is 2-*increasing* if, for every rectangle  $R \subseteq [0,1]^2$ ,  $R = [x_1, x_2] \times [y_1, y_2]$  with  $x_1, x_2, y_1, y_2 \in [0,1]$ ,  $x_1 \leq x_2$ , and  $y_1 \leq y_2$ ,

$$V_A(R) := A(x_1, y_1) + A(x_2, y_2) - A(x_1, y_2) - A(x_2, y_1) \ge 0.$$
(1)

The class of all 2-increasing aggregation functions will be abbreviated by  $\mathscr{A}_2$ .

Note that the value  $V_A(R)$  is often referred to as the *A*-volume of *R*.

**Definition 3.** A 2-increasing aggregation function  $C : [0, 1]^2 \rightarrow [0, 1]$  is called a *copula* if it has *neutral element* 1, i.e., C(x, 1) = C(1, x) = x for all  $x \in [0, 1]$ . We will denote the set of all copulas by  $\mathscr{C}$ .

Note that every copula *C* also has *annihilator* 0, i.e., C(x, 0) = C(0, x) = 0 for every  $x \in [0, 1]$ . Moreover, for every copula *C* and for every  $(x, y) \in [0, 1]^2$ , we have

$$T_{\mathbf{L}}(x,y) = \max(x+y-1,0) \leqslant C(x,y) \leqslant \min(x,y) = T_{\mathbf{M}}(x,y).$$

$$\tag{2}$$

Both,  $T_L$  and  $T_M$ , are copulas, also referred to as the *Fréchet–Hoeffding bounds*.

**Definition 4.** Given some  $A \in \mathscr{A}_2$ , the *margins* of A are the functions  $h_0^A$ ,  $h_1^A$ ,  $v_0^A$ , and  $v_1^A$  from [0,1] to [0,1] defined by

$$h_0^A(x) := A(x,0), \quad h_1^A(x) := A(x,1),$$
  
 $v_0^A(y) := A(0,y), \quad v_1^A(y) := A(1,y).$ 

We shall refer to  $h_1^A$  and  $v_1^A$  as upper margins, and  $h_0^A$  and  $v_0^A$  as lower margins.

Vice versa, we introduce a set of margins as a set of four increasing functions that are admissible for serving as a set of margins of some aggregation function  $A \in \mathscr{A}_2$ .

**Definition 5.** A set  $M = \{h_0, h_1, v_0, v_1\}$  where  $h_i : [0, 1] \rightarrow [0, 1]$  and  $v_i : [0, 1] \rightarrow [0, 1]$ , i = 1, 2, is called a *set of margins* if the following conditions are fulfilled:

(M1)  $h_0(0) = v_0(0)$ ,  $h_0(1) = v_1(0)$ ,  $h_1(0) = v_0(1)$ ,  $h_1(1) = v_1(1)$ , (M2) for all  $x' \ge x$  with  $x', x \in [0, 1]$  and for all  $y' \ge y$  with  $y', y \in [0, 1]$ ,

```
h_1(x') + h_0(x) \ge h_1(x) + h_0(x'),

v_1(y') + v_0(y) \ge v_1(y) + v_0(y').
```

A few properties of a set of margins can be immediately derived: because of (M2) it holds that

$$h_1(x') - h_0(x') \ge h_1(x) - h_0(x), \quad v_1(y') - v_0(y') \ge v_1(y) - v_0(y)$$

for all  $x' \ge x$  and  $y' \ge y$ , expressing the fact that the functions  $(h_1 - h_0)$  and  $(v_1 - v_0)$  are increasing. Therefore, it also holds that, for all  $x \in [0, 1]$ , and for all  $y \in [0, 1]$ 

$$\begin{split} h_1(1) - h_0(1) &\ge h_1(x) - h_0(x) \ge h_1(0) - h_0(0) = v_0(1) - v_0(0) \ge 0, \\ v_1(1) - v_0(1) &\ge v_1(y) - v_0(y) \ge v_1(0) - v_0(0) = h_0(1) - h_0(0) \ge 0. \end{split}$$

As a consequence we can conclude that for all *x* and *y* in [0,1]

$$\begin{split} h_1(x) &\ge h_0(x) + h_1(0) - h_0(0) = h_0(x) + \nu_0(1) - \nu_0(0) &\ge h_0(x), \\ \nu_1(y) &\ge \nu_0(y) + \nu_1(0) - \nu_0(0) = \nu_0(y) + h_0(1) - h_0(0) &\ge \nu_0(y). \end{split}$$

Clearly, the margins of any  $A \in \mathscr{A}_2$  form a set of margins. The set of all 2-increasing aggregation functions coinciding on a given set of margins  $M = \{h_0, h_1, v_0, v_1\}$  will be denoted by  $\mathscr{A}_2^M$ , i.e.,

$$\mathscr{A}_{2}^{M} := \{ A \in \mathscr{A}_{2} | h_{0}^{A} = h_{0}, h_{1}^{A} = h_{1}, v_{0}^{A} = v_{0}, v_{1}^{A} = v_{1} \}$$

Further, note that  $\mathscr{C} = \mathscr{A}_2^{\{\mathbf{0}, \mathrm{id}, \mathbf{0}, \mathrm{id}\}} \subset \mathscr{A}_2$  with  $\mathbf{0} : [0, 1] \rightarrow [0, 1]$ ,  $\mathbf{0}(x) = \mathbf{0}$  and  $\mathrm{id} : [0, 1] \rightarrow [0, 1]$ ,  $\mathrm{id}(x) = x$ .

**Definition 6.** A function  $A : [0, 1]^2 \rightarrow [0, 1]$  is *modular* if the *A*-volume of each sub-rectangle of the unit square vanishes, i.e.,  $V_A(R) = 0$  for all rectangles  $R \subseteq [0, 1]^2$ .

Clearly, modular aggregation functions form a proper subclass of  $\mathcal{A}_2$ . These functions are known to have a simple representation [7, Proposition 3.6] as the following proposition shows.

**Proposition 7.** Let  $A \in \mathscr{A}_2$ . Then A is modular if and only if  $A(x, y) = h_0^A(x) + v_0^A(y) = h_1^A(x) + v_1^A(y) - 1$  holds for all x, y in [0, 1].

In view of the last result, every modular aggregation function can be represented by means of its upper as well as lower margins. Therefore, the case of modular aggregation functions will not be treated in the sequel.

### 3. 2-Increasing aggregation functions with given upper margins

First, let us note that, by means of some copula *C* and some functions  $f : [0, 1] \rightarrow [0, 1]$  and  $g : [0, 1] \rightarrow [0, 1]$ , we can construct 2-increasing aggregation functions such that their upper margins coincide with *f* and *g*.

**Proposition 8.** Consider a copula C and two increasing functions  $f,g:[0,1] \rightarrow [0,1]$  such that f(1) = g(1) = 1 and C(f(0),g(0)) = 0. Then the function  $A_{C,(f,g)}:[0,1]^2 \rightarrow [0,1]$  defined by

$$A_{C,(f,g)}(x,y) := C(f(x),g(y))$$

is a 2-increasing aggregation function. Moreover, f and g are the upper margins of  $A_{C,(f,g)}$ ,  $h_1^{A_{C,(f,g)}} = f$  and  $v_1^{A_{C,(f,g)}} = g$ .

**Proof.** Since *C* as well as *f* and *g* are increasing,  $A_{C,(f,g)}$  is increasing. Moreover, since *C* is 2-increasing, it is easy to prove that  $A_{C,(f,g)}$  is 2-increasing. Further,  $A_{C,(f,g)}(0,0) = C(f(0),g(0)) = 0$  and  $A_{C,(f,g)}(1,1) = C(f(1),g(1)) = 1$ . Finally, it follows that

$$\begin{aligned} h_1^{A_{\mathcal{C}(f,g)}}(x) &= A_{\mathcal{C},(f,g)}(x,1) = \mathcal{C}(f(x),g(1)) = \mathcal{C}(f(x),1) = f(x), \\ v_1^{A_{\mathcal{C},(f,g)}}(y) &= A_{\mathcal{C},(f,g)}(1,y) = \mathcal{C}(f(1),g(y)) = \mathcal{C}(1,g(y)) = g(y), \end{aligned}$$

for all  $x, y \in [0, 1]$ .  $\Box$ 

Proposition 8 illustrates that, for given functions f and g, the choice of the copula C involved is not arbitrary, but restricted by the values of f and g at 0. However, when either f(0) = 0 or g(0) = 0, every copula C will lead to a 2-increasing aggregation function, because C(f(0), g(0)) = 0 is obviously fulfilled.

**Corollary 9.** Consider two increasing functions  $f, g : [0, 1] \rightarrow [0, 1]$  such that f(1) = g(1) = 1 and f(0) = 0 or g(0) = 0. Then for every copula C,

$$A_{C,(f,g)}(x,y) = C(f(x),g(y))$$

is a 2-increasing aggregation function such that  $h_1^{A_{C,(f,g)}} = f$ ,  $v_1^{A_{C,(f,g)}} = g$ .

More interestingly, every 2-increasing aggregation function, can be represented by means of a suitable copula and its upper margins only.

**Theorem 10.** Let A be a 2-increasing aggregation function, then there exists a copula C such that  $A(x,y) = C(h_1^A(x), v_1^A(y))$  for all  $x, y \in [0, 1]$ .

Before proving this theorem, we need some preliminary results. First, we notice that, for any aggregation function A, it follows immediately that all its margins  $h_0^A$ ,  $h_1^A$ ,  $v_0^A$ ,  $v_1^A$  are increasing. Hence, for every  $s \in \text{Ran}(h_1^A)$  and every  $t \in \text{Ran}(v_1^A)$ , the (non-empty) sets

$$\begin{aligned} &(h_1^A)^{-1}(\{s\}) := \{x \in [0,1] \mid h_1^A(x) = s\}, \\ &(v_1^A)^{-1}(\{t\}) := \{y \in [0,1] \mid v_1^A(y) = t\}, \end{aligned}$$

form intervals. Since they neither need to be open nor closed, their suprema, denoted by  $s^*$ ,  $t^*$ , and their infima, denoted by  $s_*$ ,  $t_*$ , need not be contained in the corresponding sets, however, surely does their arithmetic mean, i.e.,

 $\frac{1}{2}(s_* + s^*) \in (h_1^A)^{-1}(\{s\}) \text{ and } \frac{1}{2}(t_* + t^*) \in (v_1^A)^{-1}(\{t\}). \text{ Therefore, we can define two functions } (h_1^A)^* : \operatorname{Ran}(h_1^A) \to [0, 1], (v_1^A)^* : \operatorname{Ran}(v_1^A) \to [0, 1] \text{ by } (v_1^A)^* : \operatorname{Ran}(v_1^A)^* : \operatorname{Ran}(v_1^A) \to [0, 1] \text{ by } (v_1^A)^* : \operatorname{Ran}(v_1^A)^* : \operatorname{Ran}(v$ 

$$(h_1^A)^*(s) := \frac{1}{2}(s_* + s^*), \quad (v_1^A)^*(t) := \frac{1}{2}(t_* + t^*).$$

**Lemma 11.** Let A be a 2-increasing aggregation function. For arbitrary  $s \in \text{Ran}(h_1^A)$  and  $t \in \text{Ran}(v_1^A)$ , denote by  $s' = (h_1^A)^*(s)$  and  $t' = (v_1^A)^*(t)$ . Then, it holds that

$$A(x, y) = A(s', t')$$

for all  $(x, y) \in (h_1^A)^{-1}(\{s\}) \times (v_1^A)^{-1}(\{t\})$ .

**Proof.** We know that  $s', t' \in [0, 1]$  and, moreover, that they fulfill  $h_1^A(s') = s$  and  $v_1^A(t') = t$ . If s' and t' are unique, then there is nothing further to prove. Therefore, without loss of generality, we first assume that y = t' and that there exist  $x_1, x_2 \in [0, 1]$  with  $x_1 < x_2$  and  $h_1^A(x_1) = h_1^A(x_2) = s$ . Since A is 2-increasing we know that

$$V_A([x_1, x_2] \times [t', 1]) = A(x_2, 1) - A(x_1, 1) + A(x_1, t') - A(x_2, t') = h_1^A(x_2) - h_1^A(x_1) + A(x_1, t') - A(x_2, t') = A(x_1, t') - A(x_2, t') \ge 0.$$

Because *A* is increasing,  $A(x_1, t') = A(x', t')$ . Similar arguments can be applied when x = s' and  $y_1 < y_2$  with  $v_1^A(y_1) = v_1^A(y_2)$ . As a consequence, for all  $(x, y) \in (h_1^A)^{-1}(\{s\}) \times (v_1^A)^{-1}(\{t\})$  we obtain that A(x, y) = A(x, t') = A(s', t'), which concludes the proof.  $\Box$ 

**Proof** (*Proof of Theorem 10*). Let *A* be a 2-increasing aggregation function with upper margins  $h_1 := h_1^A$  and  $v_1 := v_1^A$ . Then  $h_1$  and  $v_1$  are both increasing and fulfill  $h_1(1) = v_1(1) = 1$ .

Define the sets  $S := \operatorname{Ran}(h_1) \cup \{0\}$  and  $T := \operatorname{Ran}(v_1) \cup \{0\}$ ; both are subsets of [0,1] and contain 0 as well as 1. We introduce a function  $C' : S_1 \times S_2 \to [0, 1]$  by

$$C'(s,t) = \begin{cases} A(h_1^*(s), v_1^*(t)) & \text{if } (s,t) \in \operatorname{Ran}(h_1) \times \operatorname{Ran}(v_1), \\ 0 & \text{otherwise.} \end{cases}$$
(3)

Because of Lemma 11, C' is well-defined.

Let us now look at the properties of C'. It can be easily seen that  $\text{Dom}C' = S \times T$ . Further, if  $0 \notin \notin \notin \notin \text{Ran}(h_1)$  (resp.  $0 \notin \notin \text{Ran}(v_1)$ ), then C'(0,t) = 0 (resp. C'(s,0) = 0). If  $0 \in \text{Ran}(h_1)$  (resp.  $0 \in \text{Ran}(v_1)$ ), then  $h_1(0) = 0$  (resp.  $v_1(0) = 0$ ) and A(0,y) = 0 (resp. A(x,0) = 0) for all y (resp. x) in [0,1]. Therefore,  $C'(0,t) = A(0,v_1^*(t)) = 0$  (resp.  $C'(s,0) = A(h_1^*(s),0) = 0$ ). As a consequence, for all  $s \in S$  and all  $t \in T$ ,

$$C'(s,0) = C'(0,t) = 0$$

Moreover, for all  $s \in S$  and for all  $t \in T$ ,

$$C'(s, 1) = A(h_1^*(s), 1) = h_1 \circ h_1^*(s) = s,$$
  

$$C'(1, t) = A(1, v_1^*(t)) = v_1 \circ v_1^*(t) = t.$$

Since *A* is 2-increasing, it follows that  $V_{C'}([s_1, s_2] \times [t_1, t_2]) \ge 0$  for all  $s_1, s_2 \in S$ ,  $s_1 \le s_2$  and for all  $t_1, t_2 \in T$ ,  $t_1 \le t_2$ .

In case  $0 \notin \operatorname{Ran}(h_1) \cap \operatorname{Ran}(v_1)$ , for all  $s_1, s_2 \in S$  and  $t_1, t_2 \in T$ ,  $V_{C'}([0, s_2] \times [t_1, t_2]) \ge 0$  and  $V_{C'}([s_1, s_2] \times [0, t_2]) \ge 0$  because  $h_1$  and  $v_1$  are increasing. The function C' is a subcopula [19, Definition 6.2.2]. Moreover, it holds that for all  $(s, t) \in \operatorname{Ran}(h_1) \times \operatorname{Ran}(v_1)$ ,  $C'(s, t) = A(h_1^*(s), v_1^*(t))$  or equivalently, for all  $(x, y) \in [0, 1]^2$ ,  $A(x, y) = C'(h_1(x), v_1(y))$ .

It is known that, for every subcopula C' there exists a copula C coinciding with C' on DomC' [19, Theorem 6.2.6]. Thus there exists a copula C fulfilling

$$A(x,y) = C(h_1(x), v_1(y))$$

for all x, y in [0, 1].  $\Box$ 

Note that the proof of Theorem 10 adopts in essence the same ideas as that of Sklar's theorem. However, while Sklar's theorem was formulated for distribution functions which are left-continuous (or right-continuous), we have proven the analogous result for 2-increasing aggregation functions which need neither be left- nor right-continuous, and, hence, we have been forced to do several modifications in our proof, especially related to the definition of the functions  $(h_1^A)^*$  and  $(v_1^A)^*$ . Since every copula is continuous, we can state the following corollary.

**Corollary 12.** Let A be a 2-increasing aggregation function. Then A is jointly continuous on  $[0, 1]^2$  if and only if its margins  $h_1^A$  and  $v_1^A$  are continuous.

Note that, given  $A \in \mathscr{A}_2$ , the copula *C* from Theorem 10 is uniquely determined just on  $\text{Ran}(h_1^A) \times \text{Ran}(v_1^A)$ , and, hence, different copulas may represent the same aggregation function *A*, even if *A* is continuous, as the following example illustrates.

Example 13. Consider the 2-increasing aggregation function

$$A(x,y) = \frac{1}{4} \max(3x + 3y - 2, 0).$$

Its upper margins  $h_1^A$ ,  $v_1^A$  can be easily computed as  $h_1^A(x) = v_1^A(x) = \frac{1}{4}(1+3x)$ . For the sake of simplicity we will denote  $h := h_1^A = v_1^A$ . Then

$$A(x,y) = T_{L}(h(x),h(y)) = \max(h(x) + h(y) - 1,0)$$

but also  $A(x, y) = C^*(h(x), h(y))$  with  $C^*$  being the copula given by

$$C^*(s,t) = \begin{cases} \min(s,t-\frac{3}{4}), & \text{if } (s,t) \in [0,\frac{1}{4}] \times [\frac{3}{4},1], \\ \min(s-\frac{3}{4},t), & \text{if } (s,t) \in [\frac{3}{4},1] \times [0,\frac{1}{4}], \\ T_{L}(s,t), & \text{otherwise.} \end{cases}$$

In particular,  $C^*$  is the largest and  $T_L$  is the smallest possible copula such that A(x, y) = C(h(x), h(y)) for some copula C. Note that for all  $(s, t) \in \left[\frac{1}{4}, 1\right] \times \left[\frac{1}{4}, 1\right]$  it holds that  $C^*(s, t) = T_L(s, t)$ .

The uniqueness of the copula prescribed by Theorem 10 can be obtained for continuous aggregation functions with annihilator 0.

**Corollary 14.** Let A be a continuous 2-increasing aggregation function with annihilator 0, then there exists a unique copula C such that

$$A(x,y) = C(h_1^A(x), v_1^A(y))$$

for all  $x, y \in [0, 1]$ .

**Proof.** We know already that there exists a copula *C*. Since *A* has annihilator 0 it follows that  $h_1^A(0) = v_1^A(0) = 0$  and since *A* is continuous also  $h_1^A$  and  $v_1^A$  are continuous, therefore  $\text{Ran}(h_1^A) = \text{Ran}(v_1^A) = [0, 1]$ . As a consequence *C'* defined by Eq. (3) is uniquely defined on  $\text{Ran}(h_1^A) \times \text{Ran}(v_1^A) = [0, 1]^2$ .  $\Box$ 

The representation given by Theorem 10 allows us to provide in an easy way pointwise upper and lower bounds of the set of all 2-increasing aggregation functions with the same upper margins.

**Proposition 15.** For every  $A \in \mathscr{A}_2$  with upper margins  $h_1^A$  and  $v_1^A$ , we have

$$A_{T_{L},(h^{A},v^{A})}(x,y) = \max(h^{A}_{1}(x) + v^{A}_{1}(y) - 1, 0) \leqslant A(x,y) \leqslant \min(h^{A}_{1}(x), v^{A}_{1}(y)) = A_{T_{M},(h^{A},v^{A}_{1})}(x,y) = A_{T_{M$$

**Proof.** The result follows immediately from Theorem 10 by considering the inequalities (2) for copulas.  $\Box$ 

Notice that,  $A_{T_{L},(h_1^A,v_1^A)} \in \mathscr{A}_2$ , whereas  $A_{T_{M},(h_1^A,v_1^A)}$  is not necessarily an aggregation function since  $A_{T_{M},(h_1^A,v_1^A)}(0,0)$  might be different from 0.

### 4. Aggregation functions with given lower and upper margins

Next we consider the set of aggregation functions having prescribed upper and lower margins and we represent every 2-increasing aggregation function by means of a suitable copula and all its margins.

**Lemma 16.** Consider an aggregation function A such that  $\lambda := V_A([0,1]^2) > 0$ . Then the function  $A^V : [0,1]^2 \to \mathbb{R}$  defined by

$$A^{V}(x,y) := \frac{1}{\lambda} V_{A}([0,x] \times [0,y]) = \frac{1}{\lambda} A(x,y) - \frac{1}{\lambda} (h_{0}^{A}(x) + v_{0}^{A}(y))$$

$$\tag{4}$$

is a 2-increasing aggregation function with annihilator 0 if and only if A is 2-increasing.

**Proof.** By definition, the function  $(x, y) \mapsto \frac{1}{\lambda} (h_0^A(x) + v_0^A(y))$  is modular. Therefore, for any rectangle  $R \subseteq [0, 1]^2$  we have  $V_{A^V}(R) = \frac{1}{\lambda} V_A(R)$ . As a consequence,  $A^V$  is 2-increasing if and only if A is so.

Further, note that  $A^{V}(x, y)$  has annihilator 0 and  $A^{V}(1, 1) = 1$ , regardless of further properties of *A*; this follows directly from (4).

Finally, assuming that A is 2-increasing, we have to prove that  $A^V$  is an aggregation function. First note that  $A^V$  is 2-increasing as well. Moreover, 2-increasing binary operations on [0,1] with annihilator 0 are also increasing in each variable [16, Lemma 2.1.4], and so  $A^V$  is.  $\Box$ 

**Theorem 17.** Let A be a 2-increasing aggregation function with margins  $M_A = \{h_0^A, h_1^A, v_0^A, v_1^A\}$  such that  $\lambda_A = V_A([0, 1] \times [0, 1]) > 0$ . Then there exists a copula C such that

$$A(x,y) := \lambda_A C(\varphi_1(x), \varphi_2(y)) + h_0^A(x) + v_0^A(y)$$
(5)

with

$$\begin{split} \varphi_1: [0,1] \to [0,1], \quad \varphi_1(x) := \frac{1}{\lambda_A} (h_1^A(x) - h_0^A(x) - h_1^A(0)), \\ \varphi_2: [0,1] \to [0,1], \quad \varphi_2(y) := \frac{1}{\lambda_A} (v_1^A(y) - v_0^A(y) - v_1^A(0)). \end{split}$$

**Proof.** Let *A* be a 2-increasing aggregation function with a set of margins  $M_A = \{h_0^A, h_1^A, v_0^A, v_1^A\}$  such that  $\lambda_{M_A} = \lambda_A = V_A([0, 1] \times [0, 1]) > 0$ . Then, because of Lemma 16,  $A^V$  defined by (4) is a 2-increasing aggregation function with annihilator 0. Because of Theorem 10 there exists a copula *C* such that

$$A^{V}(x,y) = C(h_{1}^{A^{V}}(x),v_{1}^{A^{V}}(y)),$$

where

$$\begin{split} h_1^{A^{V}}(x) &= \frac{1}{\lambda_A}(h_1^{A}(x) - h_0^{A}(x) - h_1^{A}(0)) = \frac{1}{\lambda_A}V_A([0,x]\times[0,1]),\\ v_1^{A^{V}}(y) &= \frac{1}{\lambda_A}(v_1^{A}(y) - v_0^{A}(y) - v_1^{A}(0)) = \frac{1}{\lambda_A}V_A([0,1]\times[0,y]). \end{split}$$

Therefore,

$$A(x,y) = \lambda_A A^V(x,y) + h_0^A(x) + v_0^A(x) = \lambda_A C \left( \frac{V_A([0,x] \times [0,1])}{\lambda_A}, \frac{V_A([0,1] \times [0,y])}{\lambda_A} \right) + h_0^A(x) + v_0^A(x),$$

which concludes the proof.  $\Box$ 

Given a 2-increasing aggregation function *A*, we can associate two copulas  $C_1$  and  $C_2$  to *A* which are determined respectively by Theorems 10 and 17. When *A* has annihilator 0,  $C_1 = C_2$ ; otherwise, these two copulas can be different.

**Example 18.** Let *A* be the 2-increasing aggregation function given by

$$A(x,y) = \frac{\left(\sqrt{x} + \sqrt{y}\right)^2}{4}$$

having margins  $h_0^A(x) = v_0^A(x) = \frac{x}{4}$  and  $h_1^A(x) = v_1^A(x) = \frac{(\sqrt{x}+1)^2}{4}$ , with  $\lambda_A = \frac{1}{2}$ . Then A can be represented in the form

$$A(x, y) = C_1(h_1^A(x), v_1^A(y)),$$

where  $C_1$  is the copula defined by  $C_1(x, y) = (\max(\sqrt{x} + \sqrt{y} - 1, 0))^2$ . On the other hand, A can be also represented in the form

$$A(x,y) := \lambda_A C_2(\varphi_1(x), \varphi_2(y)) + h_0^A(x) + v_0^A(y),$$

where  $C_2(x, y) = xy$  and

$$\begin{split} \varphi_1(x) &= 2(h_1^A(x) - h_0^A(x) - h_1^A(0)) = \sqrt{x}, \\ \varphi_2(y) &= 2(v_1^A(y) - v_0^A(y) - v_1^A(0)) = \sqrt{y}. \end{split}$$

Observe that Theorem 17 also suggests a way how to construct 2-increasing aggregation functions starting from a suitable set of margins.

**Proposition 19.** Let  $M = \{h_0, h_1, v_0, v_1\}$  be a set of margins such that  $h_0(0) = v_0(0) = 0$ ,  $h_1(1) = v_1(1) = 1$ , and

$$\lambda_M := h_1(1) - h_0(1) - h_1(0) = v_1(1) - v_0(1) - v_1(0) > 0$$

Then, for every copula C, the function  $A^{C}: [0,1]^{2} \rightarrow [0,1]$  defined by

$$A^{C}(x, y) := \lambda_{M}C(\varphi_{1}(x), \varphi_{2}(y)) + h_{0}(x) + \nu_{0}(y)$$

with

$$\begin{split} \varphi_1: [0,1] &\to [0,1], \quad \varphi_1(x) := \frac{1}{\lambda_M} (h_1(x) - h_0(x) - h_1(0)), \\ \varphi_2: [0,1] &\to [0,1], \quad \varphi_2(y) := \frac{1}{\lambda_M} (\nu_1(y) - \nu_0(y) - \nu_1(0)), \end{split}$$

is a 2-increasing aggregation function. Moreover,  $A^{C} \in \mathscr{A}_{2}^{M}$ .

**Proof.** Let  $M = \{h_0, h_1, v_0, v_1\}$  be a set of margins such that  $h_0(0) = v_0(0) = 0$ ,  $h_1(1) = v_1(1) = 1$ , and  $\lambda_M > 0$ . Because of property (M2) it follows that for all  $x' \ge x$  and all  $y' \ge y$ ,

$$\begin{split} h_1(x') - h_0(x') &\ge h_1(x) - h_0(x) \ge h_1(0), \\ \nu_1(y') - \nu_0(y') &\ge \nu_1(y) - \nu_0(y) \ge \nu_1(0), \end{split}$$

and hence  $\varphi_1$  and  $\varphi_2$  are increasing functions. Moreover, they fulfill  $\varphi_1(1) = \varphi_2(1) = 1$  and  $\varphi_1(0) = \varphi_2(0) = 0$ . Therefore, the function  $A^C : [0,1]^2 \rightarrow [0,1]$  given by (6) is increasing in each place, since  $\lambda_M > 0$  and *C* as well as all margins are increasing. Moreover,  $A^C$  fulfills  $A^C(0,0) = h_0(0) + v_0(0) = 0$  and  $A^C(1,1) = \lambda_M + h_0(1) + v_0(1) = 1$ , i.e.,  $A^C$  is an aggregation function. Since *C* is 2-increasing, it follows that  $A^C$  is also 2-increasing. Finally,

$$\begin{aligned} A^{c}(x,0) &= h_{0}(x) + v_{0}(0) = h_{0}(x), \\ A^{c}(x,1) &= \lambda_{M} \varphi_{1}(x) + h_{0}(x) + v_{0}(1) \\ &= h_{1}(x) - h_{0}(x) - h_{1}(0) + h_{0}(x) + h_{1}(0) = h_{1}(x), \\ A^{c}(0,y) &= v_{0}(y), \\ A^{c}(1,y) &= \lambda_{M} \varphi_{2}(y) + h_{0}(1) + v_{0}(y) = v_{1}(y), \end{aligned}$$

i.e.,  $A^{C} \in \mathscr{A}_{2}^{M}$ .  $\Box$ 

The upper and the lower bounds of a class of 2-increasing aggregation functions with given margins were obtained by Durante et al. [7]. Thanks to Theorem 17 this result can be proven in an easier way.

**Proposition 20.** Let A be a 2-increasing aggregation function with margins  $M = \{h_0^A, h_1^A, v_0^A, v_1^A\}$  such that  $\lambda_A := V_A([0, 1]^2) > 0$ . Then the functions  $A_*$ ,  $A^*$  from  $[0, 1]^2$  to [0, 1], defined by

$$\begin{split} A_*(x,y) &:= \max(h_0^A(x) + v_0^A(y), h_1^A(x) + v_1^A(y) - 1), \\ A^*(x,y) &:= \min(h_1^A(x) + v_0^A(y) - h_1^A(0), h_0^A(x) + v_1^A(y) - h_0^A(1)), \end{split}$$

for all  $x, y \in [0, 1]$ , are 2-increasing aggregation functions in  $\mathscr{A}_2^M$ . Further, for every  $\widetilde{A} \in \mathscr{A}_2^M$  it holds that, for all  $x, y \in [0, 1]$ ,

$$A_*(x,y) \leqslant A(x,y) \leqslant A^*(x,y). \tag{7}$$

**Proof.** The operations  $A_*$ ,  $A^*$  are 2-increasing aggregation functions in  $\mathscr{A}_2^M$  because they can be represented by (5) for  $C = T_L$  and  $C = T_M$ , respectively. The bounds (7) are a consequence of Theorem 17 and the inequalities (2).

### 5. Concluding remarks

We have discussed the representation of a 2-increasing binary aggregation function in terms of its upper margins as well as its upper and lower margins. These representations are essentially based on copulas, which have also been used for providing upper and lower bounds for classes of aggregation functions with common upper (and lower) margins.

In the forthcoming manuscript [9], the authors have used the presented results for obtaining new constructions of bivariate copulas by means of the so-called *rectangular patchwork*. These methods have been in particular applied for building statistical models with different dependencies in the tails of the distributions.

The extension of the presented results to the case of *n*-dimensional aggregation functions,  $n \ge 2$ , with additional properties like supermodularity or *n*-increasingness (which are equivalent properties just for n = 2) are clearly of interest. In this context, the most challenging problem might be the representation of an *n*-ary aggregation function by means of its (n - 1)-dimensional margins, similar to the compatibility problem posed for copulas [19].

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# Rectangular patchwork for bivariate copulas and tail dependence

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### Abstract

We present a method for constructing bivariate copulas by changing the values that a given copula assumes on some subrectangles of the unit square. Some applications of this method are discussed, especially in relation to the construction of copulas with different tail dependencies.

**Keywords**: Copula, exchangeability, sections of copulas, tail dependence. **2000 Mathematics Subject Classification**: 62H05; 60E05, 65C05.

## **1** Introduction

Copulas are functions that join bivariate distribution functions (=d.f.'s) with their univariate marginal d.f.'s (see (Joe, 1997) and (Nelsen, 2006) for a thorough exposition). In fact, according to *Sklar's theorem* (Sklar, 1959), for each random vector (X, Y) there is a copula  $C_{X,Y}$  (uniquely defined whenever X and Y are continuous) such that the joint distribution function  $F_{X,Y}$  of (X, Y) may be represented, for all  $x, y \in \mathbb{R}$ , in the form

$$F_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y)),$$

where  $F_X$  and  $F_Y$  are the d.f.'s of X and Y, respectively. Relevant applications of this fact are provided, for instance, in finance (see (Cherubini et al., 2004), (McNeil et al., 2005) and (Malevergne and Sornette, 2006)) and in hydrology (see (Salvadori et al., 2007)).

Specifically, a (bivariate) *copula* is a function  $C: [0,1]^2 \rightarrow [0,1]$  that satisfies the following properties:

- (C1) C(x,0) = C(0,x) = 0 for all  $x \in [0,1]$ ;
- (C2) C(x,1) = C(1,x) = x for all  $x \in [0,1]$ ;

(C3) for all x, x', y, y' in [0, 1] with  $x \le x'$  and  $y \le y'$ ,

$$V_C([x, x'] \times [y, y']) = C(x', y') - C(x, y') - C(x', y) + C(x, y) \ge 0.$$

Conditions (C1) and (C2) express the *boundary properties* of a copula C, (C3) is the 2increasing property of C, and  $V_C([x, x'] \times [y, y'])$  is called the *C*-volume of the rectangle  $[x, x'] \times [y, y']$ . Classical examples of copulas are  $M(x, y) = \min(x, y)$ ,  $W(x, y) = \max(x + y - 1, 0)$  and  $\Pi(x, y) = xy$ , expressing, respectively, comonotone, countermonotone and independence among two random variables.

In the recent literature, several researchers have focused their attention on new methods for constructing families of bivariate copulas with desirable properties and a stochastic interpretation. Some of these constructions are obtained by determining the value of a copula on some subsets of the unit square, like *diagonals* (see the book by Nelsen (2006) and recent papers by Durante et al. (2006, 2007a, 2008b), Durante and Jaworski (2008), Erdely and Gonzáles-Barrios (2006), Klement and Kolesárová (2007), Nelsen et al. (2008), Quesada-Molina et al. (2008)), *horizontal and vertical sections* (Klement et al. (2007), Durante et al. (2007b)), *rectangles* (see the ordinal sum construction of copulas (Nelsen, 2006), De Baets and De Meyer (2007), Siburg and Stoimenov (2007)). Along these lines of investigations, we aim at presenting the so-called "rectangular patchwork" construction, which provides a general frame for all the constructions based on the redefinition of a known copula on some rectangles in the unit square.

The presented method will be, in particular, applied to the construction of copulas with a variety of tail dependencies (see (Joe, 1997) and (Zhang, 2008)). Specifically, we propose a method for constructing copulas with different tail dependencies on the four corners of the unit square. As stressed by Embrechts et al. (2008), this kind of information may be used to determine the sub- or super-additivity of quantile-based risk measures like value-at-risk (see also (McNeil et al., 2005)).

Other two possible applications of the rectangular patchwork include the construction of copulas with given horizontal section (Klement et al., 2007), and the construction of copulas with given diagonal section, especially when they are non-symmetric (Erdely and Gonzáles-Barrios, 2006).

### 2 Rectangular patchwork for copulas

Let C be a copula. Consider a family  $\{S_i\}_{i \in \mathcal{I}}$  of closed and connected subsets of  $[0, 1]^2$  with boundaries  $\partial S_i$  such that  $S_i \cap S_j \subseteq \partial S_i \cap \partial S_j$  whenever  $i \neq j$ , i.e.,  $S_i$  and  $S_j$  have common points just on their boundaries. Moreover, for every  $i \in \mathcal{I}$ , let us consider a continuous mapping  $F_i \colon S_i \to [0, 1]$ , which is increasing in each place, such that  $C = F_i$  on  $\partial S_i$ . We call the function  $F \colon [0, 1]^2 \to [0, 1]$  defined by

$$F(x,y) = \begin{cases} F_i(x,y), & (x,y) \in S_i, \\ C(x,y), & \text{otherwise,} \end{cases}$$
(2.1)

the *patchwork* of  $\{F_i\}_{i \in \mathcal{I}}$  into the copula C.


Figure 1: An illustration of a rectangular patchwork of  $\{F_i\}_{i=1,2,3}$  into C.

Essentially, the patchwork construction allows to define a new function F by rearranging the probability mass distribution of C on the sets  $S_i$  while keeping the mass distribution untouched elsewhere.

One of the oldest patchwork construction for copulas is the *ordinal sum* (see Schweizer and Sklar (1983) and (Nelsen, 2006)), obtained by considering C equal to the copula Mand every  $S_i$  being a square of the type  $[a_i, a_{i+1}]^2$ , where  $0 \le a_i < a_{i+1} \le 1$ . Note that the idea of this method has its roots in the general theory of semigroups (compare also with the books by Schweizer and Sklar (1983), Klement et al. (2000) and the references therein).

Another related construction has been considered by Durante et al. (2007a), under the name *diagonal patchwork*, and by Nelsen et al. (2008), under the name *diagonal splice*. Here, one considers two subsets  $S_1, S_2$ , where  $S_1 = \{(x, y) \in [0, 1]^2 \mid x \ge y\}$  and  $S_2 = \{(x, y) \in [0, 1]^2 \mid x \le y\}$ , and associates to these subsets two functions  $F_1$  and  $F_2$  that are just restrictions of two copulas to  $S_1$  and  $S_2$ , respectively.

In this paper, we consider the subsets  $S_i = R_i$  to be arbitrary rectangles contained in  $[0,1]^2$ . Such construction will be denoted as *rectangular patchwork* of  $\{F_i\}_{i \in \mathcal{I}}$  into C. Note that a rectangular patchwork F satisfies trivially the boundary conditions of a copula. Moreover, due to the fact that, for all rectangles  $R_1, R_2 \subseteq [0,1]^2$  having one edge in common,  $V_F(R_1 \cup R_2) = V_F(R_1) + V_F(R_2)$ , we can formulate the following proposition (see also Proposition 7 in (De Baets and De Meyer, 2007)).

**Proposition 2.1.** Let F be the rectangular patchwork of  $\{F_i\}_{i \in \mathcal{I}}$  into the copula C. Then F is a copula if, and only if,  $F_i$  is 2-increasing on  $R_i$  for every  $i \in \mathcal{I}$ .

Therefore, an important task in applying the rectangular patchwork is the determination of suitable 2-increasing functions  $F_i: R_i \to [0, 1]$  that satisfy  $F_i = C$  on  $\partial R_i$ . In the sequel, we will show that any such function  $F_i$  can be conveniently represented by means of some transformations of a suitable copula.

First, fix some notation. Let  $a_1, a_2, b_1, b_2, c_1, c_2$  be in [0, 1] with  $a_1 < a_2, b_1 < b_2$ and  $c_1 \le c_2$ . Given a function  $F: [a_1, a_2] \times [b_1, b_2] \rightarrow [c_1, c_2]$ , the margins of F are the functions  $h_{b_1}^F$ ,  $h_{b_2}^F$ ,  $v_{a_1}^F$ , and  $v_{a_2}^F$  defined by

$$\begin{split} h_{b_1}^{F} &: \ [a_1, a_2] \to [c_1, c_2] \,, \quad h_{b_1}^{F}(x) := F(x, b_1), \\ h_{b_2}^{F} &: \ [a_1, a_2] \to [c_1, c_2] \,, \quad h_{b_2}^{F}(x) := F(x, b_2), \\ v_{a_1}^{F} &: \ [b_1, b_2] \to [c_1, c_2] \,, \quad v_{a_1}^{F}(y) := F(a_1, y), \\ v_{a_2}^{F} &: \ [b_1, b_2] \to [c_1, c_2] \,, \quad v_{a_2}^{F}(y) := F(a_2, y). \end{split}$$

We shall refer to  $h_{b_2}^F$  and  $v_{a_2}^F$  as upper margins, and  $h_{b_1}^F$  and  $v_{a_1}^F$  as lower margins. We suppose that such an F has full range, i.e.  $F(a_1, b_1) = c_1$  and  $F(a_2, b_2) = c_2$ .

**Theorem 2.1.** Let  $F: [a_1, a_2] \times [b_1, b_2] \rightarrow [c_1, c_2]$  be a 2-increasing function, continuous and increasing in each place, with margins  $h_{b_1}^F$ ,  $h_{b_2}^F$ ,  $v_{a_1}^F$ , and  $v_{a_2}^F$ . Let  $\lambda_F = V_F([a_1, a_2] \times [b_1, b_2])$ . If  $\lambda_F = 0$ , then

$$F(x,y) = h_{b_1}^F(x) + v_{a_1}^F(y) - h_{b_1}^F(a_1).$$

If  $\lambda_F > 0$ , then there exists a unique copula C such that

$$F(x,y) = \lambda_F C\left(\frac{\varphi_1^F(x)}{\lambda_F}, \frac{\varphi_2^F(y)}{\lambda_F}\right) + h_{b_1}^F(x) + v_{a_1}^F(y) - h_{b_1}^F(a_1),$$
(2.2)

with

$$\varphi_1^F(x) = V_F([a_1, x] \times [b_1, b_2]) = h_{b_2}^F(x) - h_{b_2}^F(a_1) - h_{b_1}^F(x) + h_{b_1}^F(a_1),$$
  
$$\varphi_2^F(y) = V_F([a_1, a_2] \times [b_1, y]) = v_{a_2}^F(y) - v_{a_2}^F(b_1) - v_{a_1}^F(y) + v_{a_1}^F(b_1).$$

The proof of the above Theorem is based on the following result, which has been shown by Durante et al. (2008c).

**Lemma 2.1.** Let  $A: [0,1]^2 \to [0,1]$  be a 2-increasing function which is continuous, increasing in each place, satisfying A(0,0) = 0 and A(1,1) = 1. Let  $h_0^A$ ,  $h_1^A$ ,  $v_0^A$ , and  $v_1^A$  be the margins of A. If  $\lambda_A = V_A([0,1] \times [0,1]) > 0$ , then there exists a unique copula C such that

$$A(x,y) = \lambda_A C\left(\frac{\varphi_1^A(x)}{\lambda_A}, \frac{\varphi_2^A(y)}{\lambda_A}\right) + h_0^A(x) + v_0^A(y)$$
(2.3)

with

$$\varphi_1^A(x) = V_A([0,x] \times [0,1]) = h_1^A(x) - h_0^A(x) - h_1^A(0),$$
  
$$\varphi_2^A(y) = V_A([0,1] \times [0,y]) = v_1^A(y) - v_0^A(y) - v_1^A(0).$$

Proof of Theorem 2.1. Let  $F: [a_1, a_2] \times [b_1, b_2] \rightarrow [c_1, c_2]$  be a continuous 2-increasing function such that F is increasing in each place and has full range. Let  $h_{b_1}^F, h_{b_2}^F, v_{a_1}^F$  and  $v_{a_2}^F$  be the margins of F. Let  $\lambda_F = V_F([a_1, a_2] \times [b_1, b_2])$ .

If  $\lambda_F = 0$ , then  $V_F(R) = 0$  for every rectangle  $R \subset [a_1, a_2] \times [b_1, b_2]$ . In particular, for every  $(x, y) \in [a_1, a_2] \times [b_1, b_2]$ ,  $V_F([a_1, x] \times [b_1, y]) = 0$ . As a consequence,

$$F(x,y) = h_{b_1}^F(x) + v_{a_1}^F(y) - h_{b_1}^F(a_1).$$

When  $\lambda_F > 0$ , we define three linear transformations:

$$\begin{aligned} f: & [0,1] \to [a_1,a_2], & f(x) &= a_1 + (a_2 - a_1)x; \\ g: & [0,1] \to [b_1,b_2], & g(x) &= b_1 + (b_2 - b_1)x; \\ k: & [c_1,c_2] \to [0,1], & k(x) &= \frac{x-c_1}{c_2-c_1}. \end{aligned}$$

Let  $A \colon [0,1]^2 \to [0,1]$  be the function defined by

$$A(x,y) = k\left(F(f(x),g(y))\right) = \frac{F(a_1 + (a_2 - a_1)x, b_1 + (b_2 - b_1)y) - c_1}{c_2 - c_1}$$

It is easy to show that A is continuous and increasing in each place with A(0,0) = 0 and A(1,1) = 1, and it satisfies the 2-increasing property, which is in fact preserved by linear transformations. In view of Lemma 2.1, there exists a unique copula C such that A can be represented in the form

$$A(x,y) = \lambda_A C\left(\frac{\varphi_1^A(x)}{\lambda_A}, \frac{\varphi_2^A(y)}{\lambda_A}\right) + h_0^A(x) + v_0^A(y),$$

with  $\lambda_A = \frac{\lambda_F}{c_2 - c_1}$ ,

$$\begin{split} \varphi_1^A(x) &= F(f(x), b_2) - F(f(x), b_1) - F(a_1, b_2) + c_1, \\ \varphi_2^A(y) &= F(a_2, g(y)) - F(a_1, g(y)) - F(a_2, b_1) + c_1, \\ h_0^A(x) + v_0^A(y) &= \frac{1}{c_2 - c_1} \left( F(f(x), b_1) + F(a_1, g(y)) - 2c_1 \right). \end{split}$$

Now, we have that

$$A(f^{-1}(x), g^{-1}(y)) = \frac{\lambda_F}{c_2 - c_1} C\left(\frac{h_{b_2}^F(x) - h_{b_1}^F(x) - h_{b_2}^F(a_1) + c_1}{\lambda_F}, \frac{v_{a_2}^F(y) - v_{a_1}^F(y) - v_{a_2}^F(b_1) + c_1}{\lambda_F}\right) + \frac{1}{c_2 - c_1} \left(h_{b_1}^F(x) + v_{a_1}^F(x) - 2c_1\right)$$

Since  $F(x, y) = k^{-1}A(f^{-1}(x), g^{-1}(y))$ , easy calculations show that F can be explicitly written as in (2.2).

Thanks to Theorem 2.1, we can speak more specifically of the rectangular patchwork of the pairs  $(\langle R_i, C_i \rangle)_{i \in \mathcal{I}}$  into the copula C, where, for every  $i \in \mathcal{I}$ ,  $C_i$  is the copula associated to the 2-increasing function  $F_i \colon R_i \to [0, 1]$ . This fact is underlined in the following result that just follows from above considerations. **Theorem 2.2.** Let  $\{C_i\}_{i \in \mathcal{I}}$  be a family of copulas and let  $\{R_i = [a_1^i, a_2^i] \times [b_1^i, b_2^i]\}_{i \in \mathcal{I}}$ be a family of rectangles in  $[0, 1]^2$  such that  $R_i \cap R_j \subseteq \partial R_i \cap \partial R_j$ , for every  $i \neq j$ . Let C be a copula and put  $\lambda_i := V_C(R_i)$ . Let  $\widetilde{C} : [0, 1]^2 \to [0, 1]$  be defined, for every  $x, y \in [0, 1]$ , by

$$\widetilde{C}(x,y) = \begin{cases} \lambda_i C_i \left( \frac{V_C([a_1^i, x] \times [b_1^i, b_2^i])}{\lambda_i}, \frac{V_C([a_1^i, a_2^i] \times [b_1^i, y])}{\lambda_i} \right) \\ +h_{b_1^i}^C(x) + v_{a_1^i}^C(y) - h_{b_1^i}^C(a_1^i), \quad (x,y) \in R_i \text{ with } \lambda_i \neq 0, \\ C(x,y), \quad otherwise. \end{cases}$$

Then  $\widetilde{C}$  is a copula.

We use the notation  $\widetilde{C} = (\langle R_i, C_i \rangle)_{i \in \mathcal{I}}^C$  for indicating the *rectangular patchwork* of  $(\langle R_i, C_i \rangle)_{i \in \mathcal{I}}$  into the copula C.

As a first and easy consequence, note that, by the construction,  $\tilde{C}$  is absolutely continuous (i.e., admits a density), when C and every  $C_i$  are absolutely continuous for every  $i \in \mathcal{I}$  with  $\lambda_i > 0$ .

Now, note that every copula C can always be represented by means of a rectangular patchwork, for instance by  $(\langle [0,1]^2, C \rangle)^C$ ; but, in general, this representation is not unique. In fact, the copula M can also be represented as the rectangular patchwork of the type  $(\langle [0,a] \times [1-a,1], C_1 \rangle)^M$  for every  $a \in ]0, \frac{1}{2}[$  and for every copula  $C_1$ . This is due to the fact that  $V_M([0,a] \times [1-a,1]) = 0$ .

Copulas M,  $\Pi$  and W have the curious property that they coincide with their own patchworks into themselves, i.e.,  $(\langle R_i, C \rangle)_{i \in \mathcal{I}}^C = C$  holds for any system of rectangles  $\{R_i\}_{i \in \mathcal{I}}$  provided  $C \in \{M, \Pi, W\}$ . However, this property does not hold in general. Consider, for example, the copula C given by the convex combination of the product copula  $\Pi$  and M. Then C is a copula with a singular component just along the main diagonal of the unit square. Let us consider the rectangular patchwork  $\widetilde{C} = (\langle [1 - a, 1] \times [0, a], C \rangle)^C$ , with  $a \in ]0, \frac{1}{2}[$ . Then, contrary to  $C, \widetilde{C}$  has also a probability mass concentrated on the diagonal section of the square  $[1 - a, 1] \times [0, a]$ , so  $\widetilde{C} \neq C$ .

Note that, thanks to the uniqueness of the representation given by Theorem 2.1, *any* possible copula obtained by rectangular patchwork techniques of some 2-increasing functions can be represented in the form given by Theorem 2.2. Therefore, this method can be considered as a general frame that contains the constructions of copulas based on plug-in techniques on rectangles. For instance, an ordinal sum of copulas is simply a rectangular patchwork of the type  $(\langle [a_1^i, a_2^i]^2, C_i \rangle)_{i \in \mathcal{I}}^M$ . In the same way, a *W*-ordinal sum can be represented as a rectangular patchwork of the type  $(\langle [a_1^i, a_2^i]^2, C_i \rangle)_{i \in \mathcal{I}}^M$ . In the same way, a *W*-ordinal sum can be represented as a rectangular patchwork of the type  $(\langle [a_1^i, a_2^i]^2, C_i \rangle)_{i \in \mathcal{I}}^M$  (see also (De Baets and De Meyer, 2004) and Mesiar and Szolgay, 2004)). Other examples are the *orthogonal grid construction* by De Baets and De Meyer (2007) and the (bivariate version of the) gluing method by Siburg and Stoimenov (2007).

For a given copula C, it is of interest to have a general algorithm for generating random variates with distribution function C (Nelsen, 2006). Now, just for simplicity, let consider the rectangular patchwork  $\tilde{C} = (\langle [a_1, a_2] \times [b_1, b_2], C_1 \rangle)^C$ . Provided that



Figure 2: Sampling 1000 points from a rectangular patchwork of type  $\tilde{C} = (\langle [0, 0.5] \times [0, 0.5], C_1 \rangle, \langle [0.5, 1] \times [0.5, 1], C_2 \rangle)^M$  where: (left)  $C_1$  and  $C_2$  are Gaussian copulas with parameters -0.3 and 0.3, respectively; (right)  $C_1$  is a Clayton copula with parameter 3.0 and  $C_2$  is a Gumbel copula with parameter 3.0.

there are efficient algorithms for generating a random sample from the copulas C and  $C_1$ , a simple procedure can also be implemented for sampling from  $\tilde{C}$ . To this end, set  $\varphi_1(x) = \frac{V_C([a_1, x] \times [b_1, b_2])}{V_C([a_1, a_2] \times [b_1, b_2])}$  and  $\varphi_2(y) = \frac{V_C([a_1, a_2] \times [b_1, y])}{V_C([a_1, a_2] \times [b_1, b_2])}$  and suppose that these functions admit inverses.

- ALGORITHM.
- 1. Generate  $(u_1, u_2)$  from the copula C.
- 2. Generate  $(v_1, v_2)$  from the copula  $C_1$ .
- 3. Set  $w_1 = \varphi_1^{-1}(v_1)$  and  $w_2 = \varphi_2^{-1}(v_2)$ .
- 4. If  $(u_1, u_2) \in [a_1, a_2] \times [b_1, b_2]$ , then return  $(w_1, w_2)$ . Otherwise, return  $(u_1, u_2)$ .

Obviously, the above procedure can also be extended to a rectangular patchwork of any finite number of copulas.

# **3** Applications

### **3.1** Copulas with different tail dependencies

Let C be a copula. For every  $a_i \in [0, \frac{1}{2}]$   $(i \in \{1, 2, 3, 4\})$ , consider the rectangles  $R_1, R_2, R_3, R_4$ , given by

$$R_1 = [0, a_1] \times [0, a_1], \quad R_2 = [1 - a_2, 1] \times [0, a_2],$$
$$R_3 = [1 - a_3, 1] \times [1 - a_3, 1], \quad R_4 = [0, a_4] \times [1 - a_4, 1].$$



Figure 3: Rectangular patchwork of Example 3.1

For some copulas  $C_i$ ,  $i \in \{1, 2, 3, 4\}$ , let us consider the rectangular patchwork  $\tilde{C} = (\langle R_i, C_i \rangle)_{i=1,2,3,4}^C$ . Different choices of  $C_i$  produce, in general, different behaviours of the copula  $\tilde{C}$  on the four corners of the unit square. This geometric fact may be used in order to construct copulas with different tail dependencies. These coefficients are defined, for any copula C, in the following way (see, e.g., (Zhang, 2008)):

$$\begin{split} \lambda_U^+(C) &= \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}, \quad \lambda_L^+(C) = \lim_{u \to 0^+} \frac{C(u, u)}{u} \\ \lambda_U^-(C) &= 1 - \lim_{u \to 1^-} \frac{C(1 - u, u)}{1 - u}, \quad \lambda_L^-(C) = 1 - \lim_{u \to 0^+} \frac{C(1 - u, u)}{u} \end{split}$$

Thus, the tail dependence coefficients of  $\widetilde{C}$  depend, in general, on the copulas  $C_i$ , i = 1, 2, 3, 4.

**Example 3.1.** Let us consider the product copula  $\Pi(u, v) = uv$  and let the rectangular patchwork  $\widetilde{C} = (\langle R_i, C_i \rangle)_{i=1,2,3,4}^{\Pi}$  be given by:

$$\widetilde{C}(x,y) = \begin{cases} a^2 C_1\left(\frac{x}{a}, \frac{y}{a}\right), & (x,y) \in R_1, \\ a^2 C_2\left(\frac{x-\overline{a}}{a}, \frac{y}{a}\right) + \overline{a}y, & (x,y) \in R_2, \\ a^2 C_3\left(\frac{x-\overline{a}}{a}, \frac{y-\overline{a}}{a}\right) + \overline{a}x + \overline{a}y - \overline{a}^2, & (x,y) \in R_3, \\ a^2 C_4\left(\frac{x}{a}, \frac{y-\overline{a}}{a}\right) + \overline{a}x, & (x,y) \in R_4, \\ xy, & \text{otherwise}, \end{cases}$$

where  $\overline{a} = 1 - a$ . By considering, the lower positive tail dependence coefficient of  $\widetilde{C}$ , we obtain that

$$\lambda_L^+(\widetilde{C}) = a^3 \lim_{u \to 0^+} \frac{C_1(u, u)}{u} = a^3 \lambda_L^+(C_1).$$



Figure 4: Sampling 1000 points from a rectangular patchwork of type described by Example 3.1 where: (left)  $a = \frac{1}{2}$ ,  $C_2$  and  $C_4$  are Gaussian copulas with parameter -0.8,  $C_1 = C_3 = \Pi$ ; (right)  $a = \frac{1}{2}$ ,  $C_1$  is a Clayton copula with parameter 4.0 and  $C_3$  is a Gumbel copula with parameter 3.0,  $C_2 = C_4 = \Pi$ .

Thus, although  $\Pi$  has lower positive tail dependence coefficient 0, the lower tail dependence coefficient of  $\tilde{C}$  takes values from the interval  $[0, a^3]$ , depending on the lower positive tail dependence coefficient of  $C_1$ . Roughly speaking, the rectangular patchwork "adds" some tail dependencies to the copula  $\Pi$ . Analogous considerations hold for the other coefficients of tail dependence.

Intuitively, for a sufficiently small a > 0, the example shows that it is possible to find a copula C arbitrarily close to  $\Pi$  (with respect to some  $L^p$ -norm) which admits different behaviour on the tails.

# 3.2 Copulas with a given horizontal section

Let  $b \in ]0,1[$  be a fixed number. The horizontal *b*-section of a copula *C* is the function  $h_{C,b}: [0,1] \to [0,1]$  given by  $h_{C,b}(x) = C(x,b)$ . Recently, Klement et al. (2007) studied the class of all possible copulas *C* having the same horizontal section at the level *b*. Specifically, given an increasing and 1-Lipschitz function  $h: [0,1] \to [0,b]$  such that, for every  $t \in [0,1]$ ,

$$\max(t+b-1,0) \le h(t) \le \min(t,b),$$

they investigated the class  $C_h$  of all copulas C such that  $h_{C,b} = h$ . Actually, a full description of all elements of this class can be obtained by rectangular patchwork techniques.

Let us take a copula C with given horizontal diagonal h at the point  $b \in ]0, 1[$ . For example, following Klement et al. (2007), let us consider

$$C(x,y) = \begin{cases} \frac{yh(x)}{b}, & y \le b, \\ \frac{(1-y)h(x)+(y-b)x}{1-b}, & \text{otherwise.} \end{cases}$$
(3.1)

Then, any copula  $\widetilde{C}$  in  $\mathcal{C}_h$  can be obtained as a rectangular patchwork of the type  $(\langle R_i, C_i \rangle)_{i=1,2}^C$ , where  $R_1 = [0,1] \times [0,b]$  and  $R_2 = [0,1] \times [b,1]$ , namely there ex-



Figure 5: Sampling 1000 points from a copula of type (3.2) with horizontal section  $h_{1/3}(t) = t/3$ , where: (left)  $C_1$  and  $C_2$  are Gaussian copulas with parameter 0.8 and -0.8, respectively; (right)  $C_1 = \Pi$  and  $C_2$  is a Gumbel copula with parameter 3.0

ist two copulas  $C_1$  and  $C_2$  such that

$$\widetilde{C}(x,y) = \begin{cases} bC_1\left(\frac{h(x)}{b}, \frac{y}{b}\right), & (x,y) \in R_1, \\ (1-b)C_2\left(\frac{x-h(x)}{1-b}, \frac{y-b}{1-b}\right) + h(x), & \text{otherwise.} \end{cases}$$
(3.2)

For  $C_1 = C_2 = \Pi$ ,  $\tilde{C}$  coincides with C given by (3.1). Moreover, the pointwise lower bound in  $C_h$  can be obtained by taking  $C_1 = C_2 = W$  and the pointwise upper bound in  $C_h$  can be obtained by choosing  $C_1 = C_2 = M$ .

**Example 3.2.** Let us consider the product copula  $\Pi$  and let  $R_1 = [0, 1] \times [0, \theta]$ ,  $R_2 = [0, 1] \times [\theta, 1]$  for some fixed  $\theta \in [0, 1[$ . Now let  $\widetilde{C}_{\theta} = (\langle R_1, M \rangle, \langle R_2, W \rangle)^{\Pi}$ . This copula is given by the expression

$$\widetilde{C}_{\theta}(x,y) = \begin{cases} \theta M\left(x,\frac{y}{\theta}\right), & \text{if}(x,y) \in R_1, \\ (1-\theta)W\left(x,\frac{y-\theta}{1-\theta}\right) + \theta x, & \text{otherwise.} \end{cases}$$

Notice that this family of copulas was treated by Nelsen (2006, Example 3.3) as an example of a family of copulas with given prescribed support. In fact, all copulas from this family distribute the mass along the segments connecting the point (0,0) with  $(1,\theta)$  and  $(1,\theta)$  with (0,1). Observe further that  $\widetilde{C}_{\theta} \in C_{h_{\theta}}$  with  $h_{\theta}(x) = \theta x$ .

Note that the same procedure can be applied in order to describe the class of all copulas with a given horizontal *and* vertical section (Durante et al., 2007b) as well as to extend a given sub-copula to a copula in all possible ways (see (Carley, 2002) and (Genest and Nešlehová, 2007)).

# 3.3 Copulas with given diagonal section

As already mentioned in the introduction, several methods have been introduced for constructing copulas with a given value on the diagonal section. Particular attention has been



Figure 6: Example of a rectangular patchwork described in Corollary 3.1

laid on constructions that preserve the absolute continuity and/or allow to obtain copulas which are not necessarily symmetric (see (Erdely and Gonzáles-Barrios, 2006), (Durante et al., 2007a), and (Durante and Jaworski, 2008)).

Note that, by using the rectangular patchwork, it is not difficult to obtain a large class of absolutely continuous copulas with the same diagonal section, as the following easy consequence of Theorem 2.2 shows (compare with Theorem 2.1 by Erdely and Gonzáles-Barrios (2006)).

**Corollary 3.1.** Let C be an absolutely continuous copula with diagonal section  $\delta_C(t) = C(t,t)$ . Let  $\widetilde{C}$  be a rectangular patchwork  $\widetilde{C} = (\langle R_i, C_i \rangle)_{i \in \mathcal{I}}^C$ , where every  $C_i$  is a copula and every  $R_i$  is contained completely either in  $\{(x,y) \in [0,1]^2 \mid x \geq y\}$  or in  $\{(x,y) \in [0,1]^2 \mid x \leq y\}$ . Then the diagonal section of  $\widetilde{C}$  equals to  $\delta_C$ . Moreover,  $\widetilde{C}$  is absolutely continuous if, and only if,  $C_i$  is absolutely continuous for every  $i \in \mathcal{I}$  such that  $\lambda_C(R_i) > 0$ .

The copula  $\widetilde{C}$  just obtained are, in general, non-symmetric in the sense that  $\widetilde{C}(x,y) \neq \widetilde{C}(y,x)$  for some  $(x,y) \in [0,1]^2$ . Constructions of non-symmetric copulas are of particular relevance in view of possible application for building non-exchangeable models (see (Klement and Mesiar, 2006), (Nelsen, 2007), (Durante et al., 2008a)). In this context the mapping  $\mu$  from the set C of all copulas to [0,1] has been considered as a measure of the non-symmetry of a copula C,

$$\mu_{+\infty}(C) = 3 \cdot \max_{(x,y) \in [0,1]^2} |C(x,y) - C(y,x)|.$$

**Example 3.3.** Let us consider the product copula  $C = \Pi$  and let  $a \in [0, \frac{1}{2}]$ . Consider the rectangular patchwork  $\widetilde{C} = (\langle [0, a] \times [1 - a, 1], C_1 \rangle, \langle [1 - a, 1] \times [0, a], C_2 \rangle)^{\Pi}$ , where  $\Pi \notin \{C_1, C_2\}$ . For  $C_1 \neq C_2$ ,  $\widetilde{C}$  is not symmetric and its expression is given by:

$$\widetilde{C}(x,y) = \begin{cases} a^2 C_1\left(\frac{x}{a}, \frac{y-\overline{a}}{a}\right) + \overline{a}x, & (x,y) \in [0,a] \times [1-a,1], \\ a^2 C_2\left(\frac{x-\overline{a}}{a}, \frac{y}{a}\right) + \overline{a}y, & (x,y) \in [1-a,1] \times [0,a], \\ xy, & \text{otherwise}, \end{cases}$$

where  $\overline{a} = 1 - a$ . The measure of non-symmetry for  $\widetilde{C}$  is, hence, given by

$$\mu_{+\infty}(\widetilde{C}) = 3a^2 \max_{\substack{(x,y) \in [0,a] \times [1-a,1] \\ (x,y) \in [0,1]^2}} \left| C_1\left(\frac{x}{a}, \frac{y-\overline{a}}{a}\right) - C_2\left(\frac{y-\overline{a}}{a}, \frac{x}{a}\right) \right|$$
  
=  $3a^2 \max_{\substack{(x,y) \in [0,1]^2 \\ (x,y) \in [0,1]^2}} \left| C_1(x,y) - C_2(y,x) \right|.$ 

Maximum asymmetry for such a  $\tilde{C}$  is, hence, obtained when  $C_1$  and  $C_2$  are, respectively, equal to W and M. For such a case,  $\mu_{+\infty}(\tilde{C}) = \frac{3a^2}{2}$ .

# **Concluding remarks**

We have characterized all copulas that can be constructed by means of a rectangular patchwork, i.e., by redefining the values that a copula assumes on some subrectangles of the unit square which are disjoint up to their boundaries. We have illustrated how other constructions of copulas, like e.g., ordinal sums or bivariate gluing, are particular cases of rectangular patchwork copulas. The presented results are formulated for the class of bivariate copulas only. Extensions to *n*-dimensional copulas ( $n \ge 3$ ) need further investigations. Note that for the particular case of ordinal sums some multivariate extension have been provided recently by Mesiar and Sempi (2008).

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# Part II

# Aggregation Functions: Dominance — A Functional Inequality

#### Dominance between ordinal sums on the (non-)transitivity of dominance of t-norms

- A07. S. Saminger, B. De Baets, H. De Meyer. On the dominance relation between ordinal sums of conjunctors. *Kybernetika*, vol. 42, no. 2, pp. 337–350, 2006.
- A08. S. Saminger, P. Sarkoci, B. De Baets. The dominance relation on the class of continuous ordinal sum t-norms. In H.C.M. de Swart, E. Orłowska, M. Roubens, and G. Schmidt, editors, Theory and Applications of Relational Structures as Knowledge Instruments II, Springer, pp. 337–357, 2006.

#### Dominance between continuous Archimedean t-norms easy-to-check conditions

- A09. S. Saminger-Platz, B. De Baets, H. De Meyer. A generalization of the Mulholland inequality for continuous Archimedean t-norms. *Journal of Mathematical Analysis and Applications*, vol. 345, pp. 607–614, 2008.
- A10. S. Saminger-Platz, B. De Baets, H. De Meyer. Differential inequality conditions for dominance between continuous Archimedean t-norms. (accepted for publication in *Mathematical Inequalities & Applications*).
- A11. S. Saminger-Platz. The dominance relation in some families of continuous Archimedean tnorms and copulas. (accepted for publication in *Fuzzy Sets and Systems*, doi:10.1016/j.fss.2008.12.009).

## ON THE DOMINANCE RELATION BETWEEN ORDINAL SUMS OF CONJUNCTORS

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This contribution deals with the dominance relation on the class of conjunctors, containing as particular cases the subclasses of quasi-copulas, copulas and t-norms. The main results pertain to the summand-wise nature of the dominance relation, when applied to ordinal sum conjunctors, and to the relationship between the idempotent elements of two conjunctors involved in a dominance relationship. The results are illustrated on some wellknown parametric families of t-norms and copulas.

Keywords: conjunctor, copula, dominance, ordinal sum, quasi-copula, t-norm

AMS Subject Classification: 26B99, 60E05, 39B62

#### 1. INTRODUCTION

The dominance relation was introduced in the framework of probabilistic metric spaces as a binary relation on the class of all triangle functions [25], and was soon generalized to operations on a partially ordered set [24]. It plays an important role in the construction of Cartesian products of probabilistic metric spaces (see, e. g. [24, 25]), but also in the preservation of several properties, most of them expressed by some inequality, during (dis-)aggregation processes [3, 4, 7, 9, 22, 23]. Therefore, the dominance property was also introduced in the framework of aggregation operators where it enjoyed further development [19, 22, 23].

In this paper, we restrict ourselves to a broad class of aggregation operators, namely those with neutral element 1. They are known as conjunctors and encompass all quasi-copulas, copulas and t-norms. Our emphasis lies on the dominance relation between ordinal sums of conjunctors.

In Section 2, we review the various classes of conjunctors considered in this work and extend the ordinal sum construction and the dominance relation to conjunctors. In the following section, we briefly discuss the dominance relation between ordinally irreducible conjunctors. In Section 4, we lay bare the summand-wise nature of the dominance relation. Finally, we identify interesting properties of the sets of idempotent elements of two conjunctors connected through the dominance relation and illustrate the results on some parametric families of t-norms/copulas.

#### 2. THE DOMINANCE RELATION ON THE CLASS OF CONJUNCTORS

#### 2.1. Conjunctors

In recent years, various classes of binary operators on the unit interval have gained interest in fuzzy set theory and probability theory. Triangular norms, originally introduced in the field of probabilistic metric spaces, now live a second life as models for the pointwise intersection of fuzzy sets or as models for the many-valued conjunction in fuzzy logic. Copulas, and in particular 2-copulas as considered here, connect the marginal distributions of a random vector into the joint distribution. Weaker operators, such a quasi-copulas, are appearing frequently in probability theory, as well as in fuzzy set theory. All of the operators mentioned have two properties in common: neutral element 1 and monotonicity. We now state the formal definitions.

**Definition 1.** ([6, 13]) A binary operation  $C : [0, 1]^2 \rightarrow [0, 1]$  is called a *conjunctor* if it satisfies:

- (i) Neutral element 1: for any  $x \in [0, 1]$  it holds that C(x, 1) = C(1, x) = x.
- (ii) Monotonicity: C is increasing in each variable.

Note that any conjunctor C coincides on  $\{0,1\}^2$  with the Boolean conjunction and satisfies:

(i') Absorbing element 0: for any  $x \in [0, 1]$  it holds that C(x, 0) = C(0, x) = 0.

The comparison of two conjunctors  $C_1$  and  $C_2$  is done pointwisely, i.e. if for all  $x, y \in [0, 1]$  it holds that  $C_1(x, y) \leq C_2(x, y)$ , then we say that  $C_1$  is *weaker* than  $C_2$ , or that  $C_2$  is *stronger* than  $C_1$ , and denote it by  $C_1 \leq C_2$ . For any conjunctor C it holds that  $T_{\mathbf{D}} \leq C \leq T_{\mathbf{M}}$ , with

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2 \\ \min(x,y), & \text{otherwise}, \end{cases}$$

known as the *drastic product*, and  $T_{\mathbf{M}}(x, y) = \min(x, y)$ .

For a conjunctor C and an order isomorphism  $\varphi : [0,1] \to [0,1]$ , i.e. an increasing bijection, its isomorphic transform is the conjunctor  $C_{\varphi} : [0,1]^2 \to [0,1]$  defined by  $C_{\varphi}(x,y) = \varphi^{-1}(C(\varphi(x),\varphi(y)))$ . The conjunctors C and  $C_{\varphi}$  are then referred to as *isomorphic operations*, or also as being isomorphic to each other.

In this paper, we are mainly interested in three particular classes of conjunctors: the class of triangular norms (t-norms), the class of copulas and the class of quasicopulas. Where t-norms have the additional properties of associativity and commutativity, copulas have the property of moderate growth, while quasi-copulas have the 1-Lipschitz property. Note that conjunctors are also known as semi-copulas [11].

**Definition 2.** ([12]) A conjunctor  $C : [0,1]^2 \to [0,1]$  is called a *quasi-copula* if it satisfies:

(iii) 1-Lipschitz property: for any  $x_1, x_2, y_1, y_2 \in [0, 1]$  it holds that:

$$|C(x_1, y_1) - C(x_2, y_2)| \le |x_1 - x_2| + |y_1 - y_2|.$$

**Definition 3.** ([20]) A conjunctor  $C : [0,1]^2 \to [0,1]$  is called a 2-copula (copula for short) if it satisfies:

(iv) Moderate growth: for any  $x_1, x_2, y_1, y_2 \in [0, 1]$  such that  $x_1 \leq x_2$  and  $y_1 \leq y_2$  it holds that:

$$C(x_1, y_2) + C(x_2, y_1) \le C(x_1, y_1) + C(x_2, y_2).$$

As implied by the terminology used, any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the opposite is, of course, not true.

**Definition 4.** ([15, 24]) A conjunctor  $C : [0, 1]^2 \rightarrow [0, 1]$  is called a *t*-norm if it satisfies:

(v) Commutativity: for any  $x, y \in [0, 1]$  it holds that:

$$C(x,y) = C(y,x) \,.$$

(vi) Associativity: for any  $x, y, z \in [0, 1]$  it holds that: C(x, C(y, z)) = C(C(x, y), z).

It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz (see, e.g. [15, 20]). The three main continuous t-norms are the minimum operator  $T_{\mathbf{M}}$ , the algebraic product  $T_{\mathbf{P}}$  and the Lukasiewicz t-norm  $T_{\mathbf{L}}$  (defined by  $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$ ); they are at the same time associative and commutative copulas. For any quasi-copula C it holds that  $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$  (see, e.g. [12]).

#### 2.2. The ordinal sum construction

The ordinal sum construction appears quite frequently, e.g. in the framework of partially ordered sets [2] and in the context of algebraic operations and structures (ordinal sums of semigroups [5], in particular t-norms [14, 16, 21], as well as copulas [20], and aggregation operators [8]). The aim is always the same, namely the preservation of properties of the summand operations into the resulting ordinal sum. Here, we follow a particular approach known as the *id-lower ordinal sum* [8].

**Definition 5.** Let  $(]a_i, b_i]_{i \in I}$  be a family of non-empty, pairwise disjoint open subintervals of [0, 1] and let  $(C_i)_{i \in I}$  be a family of conjunctors. Then the *ordinal* sum  $C = (\langle a_i, b_i, C_i \rangle)_{i \in I} : [0, 1]^2 \to [0, 1]$  is the conjunctor defined by

$$C(x,y) = \begin{cases} a_i + (b_i - a_i) C_i(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}), & \text{if } (x, y) \in [a_i, b_i]^2, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Note that each conjunctor  $C_i$  is squeezed into the corresponding square  $[a_i, b_i]^2$  by a linear transformation. The triplets  $\langle a_i, b_i, C_i \rangle$  are called the *summands* of the

ordinal sum. The intervals  $[a_i, b_i]$  are called the *summand carriers*, the conjunctors  $C_i$  the *summand operations*. A conjunctor C that has no ordinal sum representation different from  $(\langle 0, 1, C \rangle)$  is called *ordinally irreducible*. Obviously,  $T_{\mathbf{M}}$  is not ordinally irreducible.

The ordinal sum construction is powerful as it preserves a lot of properties, such as commutativity, (1-Lipschitz) continuity, etc. For instance, an ordinal sum is continuous if and only if all its summand operations are continuous. Combining various properties, it holds that the classes of quasi-copulas, copulas and triangular norms are all closed under the ordinal sum construction. The ordinal sum construction even allows for the full characterization of continuous t-norms [17].

**Proposition 1.** A binary operation  $T: [0,1]^2 \to [0,1]$  is a continuous t-norm if and only if it is uniquely representable as an ordinal sum of t-norms that are either isomorphic to the Łukasiewicz t-norm  $T_{\mathbf{L}}$  or to the product  $T_{\mathbf{P}}$ .

#### 2.3. The dominance relation

The dominance relation was introduced in the framework of probabilistic metric spaces as a relation between triangle functions which ensures that the Cartesian product of two probabilistic metric spaces is again a probabilistic metric space of the same type ([24, 25]). It was generalized to operations on a partially ordered set [24] and introduced into the framework of t-norms (see also [15]). The dominance relation is indispensable when refining fuzzy partitions and when building Cartesian products of fuzzy equivalence and fuzzy order relations [3, 7]. Moreover, it plays an important role in the preservation of T-transitivity of fuzzy relations involved in a (dis-)aggregation process [9, 23], giving way to its generalization to aggregation operators [23].

**Definition 6.** Consider two conjunctors  $C_1$  and  $C_2$ . We say that  $C_1$  dominates  $C_2$ , denoted by  $C_1 \gg C_2$ , if for all  $x, y, u, v \in [0, 1]$  it holds that

$$C_1(C_2(x,y), C_2(u,v)) \ge C_2(C_1(x,u), C_1(y,v)).$$
(1)

For any two conjunctors  $C_1$  and  $C_2$  and any order isomorphism  $\varphi \colon [0,1] \to [0,1]$ , it holds that  $C_1 \gg C_2$  if and only if  $(C_1)_{\varphi} \gg (C_2)_{\varphi}$  (see also [22, 23]). We will refer to this relationship as the *isomorphism property of dominance*.

Due to the fact that 1 is the common neutral element of all conjunctors, dominance of one conjunctor by another conjunctor implies their comparability:  $C_1 \gg C_2$ implies  $C_1 \ge C_2$  (see also [22]). Obviously, the converse does not hold. Consequently, the dominance relation is antisymmetric on the class of all conjunctors. A conjunctor C for which  $C \gg C$  is said to be *self-dominant*. Self-dominance is evidently equivalent with the bisymmetry property [1]

$$C(C(x,y),C(u,v)) = C(C(x,u),C(y,v)).$$

Commutativity and associativity clearly imply bisymmetry. Moreover, bisymmetry together with 1 being the neutral element imply commutativity and associativity. Hence any t-norm is self-dominant and on the class of all t-norms the dominance relation is not only antisymmetric, but also reflexive. This is, however, not the case for the class of copulas.

**Example 1.** Consider the family of copulas  $(C_{\theta})_{\theta \in [0,1]}$  defined by

$$C_{\theta}(x,y) = \begin{cases} \min(x,y-\theta), & \text{if } (x,y) \in [0,1-\theta] \times [\theta,1], \\ \min(x+\theta-1,y), & \text{if } (x,y) \in [1-\theta,1] \times [0,\theta], \\ T_{\mathbf{L}}(x,y), & \text{otherwise.} \end{cases}$$

The copula  $C_{0.5}$  is the only commutative member of this family (see also [20]). As it is not associative, it is also not bisymmetric, and does therefore not dominate itself (choose, e.g. x = 0.5, y = 1, u = v = 0.75).

Before turning to ordinal sums of conjunctors let us recall some basic results about dominance between (ordinally irreducible) conjunctors, in particular involving the extreme elements of various subclasses of conjunctors.

#### 3. DOMINANCE BETWEEN (ORDINALLY IRREDUCIBLE) CONJUNCTORS

#### 3.1. Conjunctors

Due to their monotonicity, it is immediately clear that any conjunctor C is dominated by  $T_{\mathbf{M}}$ . Conversely, since dominance implies comparability,  $T_{\mathbf{M}}$  is the only conjunctor dominating  $T_{\mathbf{M}}$ . On the other hand, it is easily verified that any conjunctor C dominates the weakest conjunctor  $T_{\mathbf{D}}$ .

In [23], several methods for constructing dominating aggregation operators from given ones have been proposed. As a consequence, we can immediately pose the following lemma.

**Lemma 1.** Consider conjunctors  $C_1$ ,  $C_2$ ,  $C_3$  and C. If  $C_i \gg C$ , for any  $i \in \{1,2,3\}$ , then also the binary operation  $C^* : [0,1]^2 \to [0,1]$  defined by

$$C^*(x,y) = C_3(C_1(x,y), C_2(x,y))$$

dominates C. Moreover,  $C^*$  is a conjunctor if and only if  $C_3 = T_{\mathbf{M}}$ .

#### 3.2. Quasi-copulas and copulas

The strongest (quasi-)copula  $T_{\mathbf{M}}$  dominates all other conjunctors, in particular all (quasi-)copulas. However, not all (quasi-)copulas dominate the weakest (quasi-) copula  $T_{\mathbf{L}}$ , as the following example demonstrates.

**Example 2.** Consider the copula  $C: [0,1]^2 \rightarrow [0,1]$  defined by

$$C(x,y) = \begin{cases} \frac{1}{2}T_{\mathbf{L}}(2x,2y), & \text{if } (x,y) \in \left[0,\frac{1}{2}\right]^2, \\ T_{\mathbf{M}}(x,y), & \text{otherwise.} \end{cases}$$

Putting  $x = y = u = v = \frac{5}{8}$  yields

 $0 = C\left(\frac{1}{4}, \frac{1}{4}\right) = C\left(T_{\mathbf{L}}\left(\frac{5}{8}, \frac{5}{8}\right), T_{\mathbf{L}}\left(\frac{5}{8}, \frac{5}{8}\right)\right) < T_{\mathbf{L}}\left(C\left(\frac{5}{8}, \frac{5}{8}\right), C\left(\frac{5}{8}, \frac{5}{8}\right)\right) = T_{\mathbf{L}}\left(\frac{5}{8}, \frac{5}{8}\right) = \frac{1}{4}$ 

and therefore C does not dominate  $T_{\rm L}$ . Note that C is an ordinal sum copula and a member of the Mayor–Torrens family as discussed also later in Section 5.2.2.

However, the 1-Lipschitz property is a necessary condition for a conjunctor to dominate  $T_{\mathbf{L}}$  (see also [9, 19]).

**Proposition 2.** If a conjunctor C dominates  $T_{\mathbf{L}}$ , then it is a quasi-copula.

Proof. Suppose that a conjunctor C dominates  $T_{\mathbf{L}}$ , i.e. for all  $x, y, u, v \in [0, 1]$ it holds that )

$$C(T_{\mathbf{L}}(x,y),T_{\mathbf{L}}(u,v)) \ge T_{\mathbf{L}}(C(x,u),C(y,v)).$$

$$(2)$$

In order to show that C fulfills the 1-Lipschitz property, it suffices, due to its increasingness, to prove that

$$C(a,b) - C(a - \varepsilon, b - \delta) \le \varepsilon + \delta$$

whenever  $0 \le \varepsilon \le a$ ,  $0 \le \delta \le b$  for arbitrary  $a, b \in [0, 1]$ . We first choose x = a,  $y = 1, u = b, v = 1 - \delta$  for some  $0 \le \delta \le b$  with arbitrary but fixed  $a, b \in [0, 1]$ . Then  $T_{\mathbf{L}}(u, v) = \max(u + v - 1, 0) = \max(b - \delta, 0) = b - \delta$  and hence it follows using Eq. (2) that

$$C(a, b - \delta) = C(T_{\mathbf{L}}(a, 1), T_{\mathbf{L}}(b, 1 - \delta))$$
  

$$\geq T_{\mathbf{L}}(C(a, b), C(1, 1 - \delta))$$
  

$$= T_{\mathbf{L}}(C(a, b), 1 - \delta) = \max(C(a, b) - \delta, 0)$$
  

$$\geq C(a, b) - \delta.$$

Analogously, by putting  $x = a, y = 1 - \varepsilon, u = b, v = 1$  with  $0 \le \varepsilon \le a$ , we can conclude that  $C(a - \varepsilon, b) \ge C(a, b) - \varepsilon$ . As a consequence

$$C(a - \varepsilon, b - \delta) \ge C(a - \varepsilon, b) - \delta \ge C(a, b) - \varepsilon - \delta$$
.

Therefore, C is 1-Lipschitz, and thus a quasi-copula.

3.3. Triangular norms

The class of ordinally irreducible continuous t-norms consists of all continuous Archimedean t-norms, i.e. those t-norms that are either isomorphic to the product  $T_{\mathbf{P}}$ (called *strict* t-norms) or to the Lukasiewicz t-norm  $T_{\mathbf{L}}$  (called *nilpotent* t-norms). The following observations are important, as they imply that it suffices to consider the t-norms  $T_{\mathbf{P}}$  and  $T_{\mathbf{L}}$  in order to understand dominance of a continuous Archimedean t-norm T by a conjunctor C:

- (i) If T is strict, there exists an order isomorphism  $\varphi \colon [0,1] \to [0,1]$  such that  $T = (T_{\mathbf{P}})_{\varphi}$ , leading to the equivalence  $C \gg T \Leftrightarrow C_{\varphi^{-1}} \gg T_{\mathbf{P}}$ .
- (ii) If T is nilpotent, there exists an order isomorphism  $\varphi \colon [0,1] \to [0,1]$  such that  $T = (T_{\mathbf{L}})_{\varphi}$ , leading to the equivalence  $C \gg T \Leftrightarrow C_{\varphi^{-1}} \gg T_{\mathbf{L}}$ .

We have already seen in Proposition 2 that being a quasi-copula is a necessary condition for a conjunctor to dominate  $T_{\mathbf{L}}$ . It is remarkable that the same condition applies for a conjunctor to dominate  $T_{\mathbf{P}}$ .

**Proposition 3.** If a conjunctor C dominates  $T_{\mathbf{P}}$ , then it is a quasi-copula.

Proof. Suppose that a conjunctor C dominates  $T_{\mathbf{P}}$ , i.e. for all  $x, y, u, v \in [0, 1]$ it holds that (3)

$$C(xy, uv) \ge C(x, u)C(y, v).$$
(3)

Again it suffices, due to the increasingness of C, to show that

$$C(a,b) - C(a - \varepsilon, b - \delta) \le \varepsilon + \delta$$

whenever  $0 \leq \varepsilon \leq a, 0 \leq \delta \leq b$  for arbitrary  $a, b \in [0, 1]$ . In case that a = 0 (resp. b = 0, it holds that  $\varepsilon = 0$  (resp.  $\delta = 0$ ), and the inequality is trivially fulfilled. Therefore, it remains to prove that it holds for arbitrary  $a, b \in [0, 1]$ . We first choose  $x = a, y = 1 - \frac{\varepsilon}{a}, u = b, v = 1$  with  $0 \le \varepsilon \le a$ . Then it follows from Eq. (3) that

$$C(a - \varepsilon, b) \ge C(a, b)C(1 - \frac{\varepsilon}{a}, 1) = C(a, b)(1 - \frac{\varepsilon}{a})$$

Since  $C \leq T_{\mathbf{M}}$  it then holds for all  $0 < a \leq 1, 0 \leq b \leq 1$  and  $0 \leq \varepsilon \leq a$  that

$$C(a,b) - C(a - \varepsilon, b) \le C(a,b)(1 - (1 - \frac{\varepsilon}{a})) = \frac{\varepsilon}{a}C(a,b) \le \varepsilon.$$

Similarly, we can conclude from Eq. (3), by choosing x = a, y = 1, u = b, v = $1 - \frac{\delta}{b}$  with  $0 \le \delta \le b$ , that for all  $0 \le a \le 1$ ,  $0 < b \le 1$  with  $0 \le \delta \le b$  also  $C(a, b) - C(a, b - \delta) \leq \delta$ . Hence,

$$C(a,b) - C(a - \varepsilon, b - \delta) = C(a,b) - C(a,b - \delta) + C(a,b - \delta) - C(a - \varepsilon, b - \delta)$$
  
$$\leq \varepsilon + \delta$$

whenever  $0 \le \varepsilon \le a$ ,  $0 \le \delta \le b$  for arbitrary  $a, b \in [0, 1]$ . Therefore, C is 1-Lipschitz, and thus a quasi-copula.  $\square$ 

#### 4. DOMINANCE BETWEEN ORDINAL SUM CONJUNCTORS

#### 4.1. Summand-wise dominance

As the ordinal sum construction is generally applicable, it is important to investigate dominance between two ordinal sum conjunctors in order to gain a deeper understanding of the dominance relation. In a first proposition we show that if both ordinal sum conjunctors are based on the same summand carriers, dominance between these conjunctors is based on the dominance between the corresponding summand operations.

**Proposition 4.** Consider two ordinal sum conjunctors  $C_1 = (\langle a_i, b_i, C_{1,i} \rangle)_{i \in I}$  and  $C_2 = (\langle a_i, b_i, C_{2,i} \rangle)_{i \in I}$ . Then  $C_1$  dominates  $C_2$  if and only if  $C_{1,i}$  dominates  $C_{2,i}$  for all  $i \in I$ .

Proof. Suppose that  $C_1 \gg C_2$ , i.e. for all  $x, y, u, v \in [0, 1]$  it holds that

$$C_1(C_2(x,y), C_2(u,v)) \ge C_2(C_1(x,u), C_1(y,v)).$$
(4)

We want to show that for all  $i \in I$  it holds that  $C_{1,i} \gg C_{2,i}$ . Choose arbitrary  $x, y, u, v \in [0, 1]$  and some  $i \in I$ . Since  $\varphi_i : [a_i, b_i] \to [0, 1], x \mapsto \frac{x-a_i}{b_i-a_i}$  is an increasing bijection, there exist unique  $x', y', u', v' \in [a_i, b_i]$  such that  $\varphi_i(x') = x$ ,  $\varphi_i(y') = y, \varphi_i(u') = u$  and  $\varphi_i(v') = v$ . Since Eq. (4) is fulfilled for all  $x, y, u, v \in [0, 1]$  and in particular for  $x', y', u', v' \in [a_i, b_i]$ , it can be equivalently expressed as

$$\varphi_{i}^{-1} \circ C_{1,i}(C_{2,i}(\varphi_{i}(x'),\varphi_{i}(y')), C_{2,i}(\varphi_{i}(u'),\varphi_{i}(v'))) \\ \geq \varphi_{i}^{-1} \circ C_{2,i}(C_{1,i}(\varphi_{i}(x'),\varphi_{i}(u')), C_{1,i}(\varphi_{i}(y'),\varphi_{i}(v'))),$$

taking into account the ordinal sum structure of  $C_1$  and  $C_2$ . The previous inequality is in turn equivalent to

$$\varphi_i^{-1} \circ C_{1,i}(C_{2,i}(x,y), C_{2,i}(u,v)) \ge \varphi_i^{-1} \circ C_{2,i}(C_{1,i}(x,u), C_{1,i}(y,v)).$$

Applying  $\varphi_i$  to both sides of the above inequality yields  $C_{1,i} \gg C_{2,i}$ .

Conversely, suppose that for all  $i \in I$  it holds that  $C_{1,i} \gg C_{2,i}$ , then Eq. (4) is fulfilled for all  $x, y, u, v \in [a_i, b_i]$  due to the isomorphism property. Next, we will make use of the following observation: for any  $p, q \in [0, 1]$  such that  $\min(p, q) \in [a_i, b_i]$  for some  $i \in I$ , it holds that

$$C_1(p,q) = C_1(\min(p,b_i),\min(q,b_i))$$

Now consider arbitrary  $x, y, u, v \in [0, 1]$  and suppose w.l.o.g. that  $x = \min(x, y, u, v)$ , then we can distinguish the following cases.

**Case 1.** Suppose  $x \in [a_i, b_i]$  for some  $i \in I$ . Let  $y^* = \min(y, b_i)$ ,  $u^* = \min(u, b_i)$ and  $v^* = \min(v, b_i)$ . Note that  $C_1(x, u) = C_1(x, u^*)$ . Moreover, if  $\min(y, v) \in [a_i, b_i]$ , then also  $C_1(y, v) = C_1(y^*, v^*)$ . As  $x, y^*, u^*, v^*$  all belong to  $[a_i, b_i]$ , the assumption  $C_{1,i} \gg C_{2,i}$  and the increasingness of  $C_1$  and  $C_2$  imply that

$$C_2(C_1(x, u), C_1(y, v)) = C_2(C_1(x, u^*), C_1(y^*, v^*))$$
  

$$\leq C_1(C_2(x, y^*), C_2(u^*, v^*))$$
  

$$\leq C_1(C_2(x, y), C_2(u, v)).$$

On the other hand, if  $\min(y, v) \notin [a_i, b_i]$ , we know that  $C_1(y, v) \ge b_i$ . Since  $C_1(x, u^*) \le b_i$  it follows that

$$C_2(C_1(x,u), C_1(y,v)) = C_2(C_1(x,u^*), C_1(y,v))$$
  
= min(C\_1(x,u^\*), C\_1(y,v)) = C\_1(x,u^\*).

Due to the increasingness of  $C_1$  it holds that

$$C_{1}(x, u^{*}) = \min(C_{1}(x, u^{*}), C_{1}(x, v), C_{1}(y, u^{*}), C_{1}(y, v))$$
  
$$= C_{1}(\min(x, y), \min(u^{*}, v))$$
  
$$= C_{1}(C_{2}(x, y), C_{2}(u^{*}, v))$$
  
$$\leq C_{1}(C_{2}(x, y), C_{2}(u, v)).$$

**Case 2.** If  $x \notin [a_i, b_i]$  for all  $i \in I$ , then  $C_1(x, \cdot) = C_2(x, \cdot) = T_{\mathbf{M}}(x, \cdot)$ . One easily verifies that  $C_1(y, v) \ge x$  and  $C_2(u, v) \ge x$ . This leads to

$$C_2(C_1(x, u), C_1(y, v)) = C_2(x, C_1(y, v))$$
  
= min(x, C\_1(y, v)) = x = min(x, C\_2(u, v))  
= C\_1(x, C\_2(u, v)) = C\_1(C\_2(x, y), C\_2(u, v)).

This completes the proof that  $C_1$  dominates  $C_2$ .

#### 4.2. Ordinal sums with different summand carriers

In case the structure of both ordinal sum conjunctors is not the same, we are able to provide some necessary conditions which lead to a characterization of dominance between ordinal sum conjunctors in general. Assume that the ordinal sum conjunctors under consideration are based on two at least partially different families of summand carriers, i. e.  $C_1 = (\langle a_{1,i}, b_{1,i}, C_{1,i} \rangle)_{i \in I}$  and  $C_2 = (\langle a_{2,j}, b_{2,j}, C_{2,j} \rangle)_{j \in J}$ . W.l.o.g. we can assume that these representations are the finest possible, i. e. that each summand operation is ordinally irreducible.

Since any conjunctor is bounded from above by  $T_{\mathbf{M}}$  and dominance implies comparability, the following proposition follows immediately.

**Proposition 5.** If a conjunctor  $C_1$  dominates a conjunctor  $C_2$ , then  $C_1(x, y) = T_{\mathbf{M}}(x, y)$  whenever  $C_2(x, y) = T_{\mathbf{M}}(x, y)$ .

Geometrically speaking, if an ordinal sum conjunctor  $C_1$  dominates an ordinal sum conjunctor  $C_2$ , then it must necessarily consist of more regions where it acts as  $T_{\mathbf{M}}$  than does  $C_2$ . Two such cases are displayed in Figure 1 (a) and (c). Note that no dominance relationship between  $C_1$  and  $C_2$  is possible in a case like illustrated in Figure 1 (b). Therefore, we can immediately state the following corollary.

**Corollary 1.** Consider two ordinal sum conjunctors  $C_1 = (\langle a_{1,i}, b_{1,i}, C_{1,i} \rangle)_{i \in I}$  and  $C_2 = (\langle a_{2,j}, b_{2,j}, C_{2,j} \rangle)_{j \in J}$  with ordinally irreducible summand operations only. If  $C_1$  dominates  $C_2$  then

$$(\forall i \in I) (\exists j \in J) ([a_{1,i}, b_{1,i}] \subseteq [a_{2,j}, b_{2,j}]).$$
(5)

Note that each  $[a_{2,j}, b_{2,j}]$  can contain several or even none of the summand carriers  $[a_{1,i}, b_{1,i}]$  (see also Figure 1 (a) and (c)). Hence, for each  $j \in J$  we can consider the



Fig. 1. Examples of two ordinal sum conjunctors  $C_1$  and  $C_2$  differing in their summand carriers.

following subset of I:

$$I_j = \{i \in I \mid [a_{1,i}, b_{1,i}] \subseteq [a_{2,j}, b_{2,j}]\}.$$
(6)

Based on these notions and due to Proposition 4, dominance between two ordinal sum conjunctors can be reformulated in the following way.

**Proposition 6.** Consider two ordinal sum conjunctors  $C_1 = (\langle a_{1,i}, b_{1,i}, C_{1,i} \rangle)_{i \in I}$ and  $C_2 = (\langle a_{2,j}, b_{2,j}, C_{2,j} \rangle)_{j \in J}$  with ordinally irreducible summand operations only. Then  $C_1$  dominates  $C_2$  if and only if

(i) 
$$\cup_{j \in J} I_j = I$$
,  
(ii)  $C_1^j \gg C_{2,j}$  for all  $j \in J$  with  
 $C_1^j = (\langle \varphi_j(a_{1,i}), \varphi_j(b_{1,i}), C_{1,i} \rangle)_{i \in I_j}$  (7)  
and  $\varphi_j : [a_{2,j}, b_{2,j}] \to [0, 1], \varphi_j(x) = \frac{x - a_{2,j}}{b_{2,j} - a_{2,j}}$ .

Proof. Under condition (i) it is easily verified that  $C_1$  can be equivalently expressed as an ordinal sum based on the summand carriers of  $C_2$  in the following way

$$C_1 = (\langle a_{2,j}, b_{2,j}, C_1^j \rangle)_{j \in J}$$

with  $C_1^j$  defined by Eq. (7). With Corollary 1 and Proposition 4, the proposition now follows immediately.

Note that due to Proposition 6, the study of dominance between ordinal sum conjunctors can be reduced to the study of the dominance of a single ordinally irreducible conjunctor by some ordinal sum conjunctor.

#### 5. THE ROLE OF IDEMPOTENT ELEMENTS

#### 5.1. A basic result

Before turning to particular families of ordinal sum conjunctors, we will next discuss the influence of idempotent elements to the property of dominance. We will denote the set of idempotent elements of some conjunctor C by  $\mathcal{I}(C)$ , i.e.

$$\mathcal{I}(C) = \{ x \in [0,1] \mid C(x,x) = x \}.$$

Due to the construction of an ordinal sum conjunctor C, the endpoints of its summand carriers belong to its set of idempotent elements.

**Proposition 7.** If a conjunctor  $C_1$  dominates a conjunctor  $C_2$ , then the following hold:

- (i)  $\mathcal{I}(C_2) \subseteq \mathcal{I}(C_1)$ ,
- (ii)  $\mathcal{I}(C_1)$  is closed under  $C_2$ .

Proof. The inclusion follows immediately from Proposition 5. Next, suppose that  $d_1, d_2 \in \mathcal{I}(C_1)$ , then

$$\begin{array}{rcl} C_2(d_1, d_2) &=& C_2(C_1(d_1, d_1), C_1(d_2, d_2)) \\ &\leq& C_1(C_2(d_1, d_2), C_2(d_1, d_2)) \\ &\leq& T_{\mathbf{M}}(C_2(d_1, d_2), C_2(d_1, d_2)) = C_2(d_1, d_2). \end{array}$$

showing that  $C_1(C_2(d_1, d_2), C_2(d_1, d_2)) = C_2(d_1, d_2)$  and therefore  $C_2(d_1, d_2) \in \mathcal{I}(C_1)$ .

This proposition has some interesting consequences for the boundary elements of the summand carriers. Firstly, all idempotent elements of  $C_2$  are idempotent elements of  $C_1$ , i. e. either boundary elements themselves, elements of some domain where  $C_1$  acts as  $T_{\mathbf{M}}$ , or isomorphic transformations of idempotent elements of some summand operation. Secondly, for any two idempotent elements  $d_1$  and  $d_2$  of  $C_1$ also  $C_2(d_1, d_2)$  is an idempotent element of  $C_1$ . Consequently, if  $C_1$  is some ordinal sum that dominates  $C_2 = T_{\mathbf{P}}$ , resp.  $C_2 = T_{\mathbf{L}}$ , and  $d \in \mathcal{I}(C_1)$  then also  $d^n \in \mathcal{I}(C_1)$ , resp. max $(nd - n + 1, 0) \in \mathcal{I}(C_1)$ , for all  $n \in \mathbb{N}$ .

**Example 3.** Consider a conjunctor C with trivial idempotent elements only, i. e.  $\mathcal{I}(C) = \{0, 1\}$ . We are now interested in constructing ordinal sums  $C_1$  with summands based on C which fulfill the necessary conditions for dominating  $C_2 = C$  as expressed by Proposition 7. Clearly,  $C_1 = (\langle d, 1, C \rangle)$  is a first possibility (see Figure 2 (a)). Adding one further summand to  $C_1$ , i. e. building  $C'_1 = (\langle a, d, C \rangle, \langle d, 1, C \rangle)$ , demands that  $a \geq C_2(d, d)$ , since otherwise  $C_2(d, d) \notin \mathcal{I}(C'_1)$  (see also Figure 2 (b)).



Fig. 2. Illustrations to Example 3.

#### 5.2. Applications to some parametric families

To conclude, we consider two families consisting of conjunctors with only one summand but varying boundary elements. All members of these families are t-norms as well as copulas. We have opted for these families as they involve  $T_{\mathbf{P}}$ , resp.  $T_{\mathbf{L}}$ , only as summand operation.

#### 5.2.1. A family involving $T_{\mathbf{P}}$

The members of the family of Dubois–Prade t-norms [10] are given by  $T_{\lambda}^{\mathbf{DP}}$  =  $(\langle 0, \lambda, T_{\mathbf{P}} \rangle)$  for  $\lambda \in [0, 1]$ . Obviously, they are ordinal sums with the product as

((0,  $\lambda, \mathbf{P})$ ) for  $\lambda \in [0, 1]$ . Obviously, they are ordinar sums with the product as single summand operation. The case  $\lambda = 0$  corresponds to  $T_{\mathbf{M}}$ , the case  $\lambda = 1$  to  $T_{\mathbf{P}}$ . If  $\lambda_1 \leq \lambda_2$ , then  $T_{\lambda_1}^{\mathbf{DP}} \geq T_{\lambda_2}^{\mathbf{DP}}$ . Therefore, if  $T_{\lambda_1}^{\mathbf{DP}} \gg T_{\lambda_2}^{\mathbf{DP}}$  then  $\lambda_1 \leq \lambda_2$ . If  $\lambda_1 = 0$  or  $\lambda_1 = \lambda_2$ , then the dominance property is trivially fulfilled. Therefore, suppose that  $0 < \lambda_1 < \lambda_2$ . For better readability we denote  $T_{\lambda_1}^{\mathbf{DP}}$ , resp.  $T_{\lambda_2}^{\mathbf{DP}}$ , by  $T_1$ , resp.  $T_2$ . Suppose that  $T_1$  dominates  $T_2$ . For each  $T_i, i \in \{1, 2\}$ , its set of idempotent elements is given by  $\mathcal{I}(T_i) = \{0\} \cup [\lambda_i, 1] .$ 

Due to Proposition 7, it holds that  $T_2(\lambda_1, \lambda_1) \in \mathcal{I}(T_1)$ . However,

$$0 \neq T_2(\lambda_1, \lambda_1) = \lambda_2 \cdot T_{\mathbf{P}}(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}) = \frac{\lambda_1}{\lambda_2} \cdot \lambda_1 < \lambda_1$$

due to the strict monotonicity of  $T_{\mathbf{P}}$ . This leads to a contradiction.

Consequently, the only dominance relationships in the family of Dubois–Prade tnorms are  $T_{\mathbf{M}}$  dominating all other members and self-dominance. Hence, there exists no triplet of pairwisely different t-norms  $T_{\lambda_1}^{\mathbf{DP}}$ ,  $T_{\lambda_2}^{\mathbf{DP}}$  and  $T_{\lambda_3}^{\mathbf{DP}}$  fulfilling  $T_{\lambda_1}^{\mathbf{DP}} \gg T_{\lambda_2}^{\mathbf{DP}}$ and  $T_{\lambda_2}^{\mathbf{DP}} \gg T_{\lambda_3}^{\mathbf{DP}}$ , implying that the dominance relation is (trivially) transitive, and therefore a partial order, on this family.

#### 5.2.2. A family involving $T_{\rm L}$

Similarly, the members of the family of Mayor–Torrens t-norms [18] are given by  $T_{\lambda}^{\mathbf{DP}} = (\langle 0, \lambda, T_{\mathbf{L}} \rangle)$  for  $\lambda \in [0, 1]$ . Obviously, they are ordinal sums with  $T_{\mathbf{L}}$  as single summand operation. The case  $\lambda = 0$  corresponds to  $T_{\mathbf{M}}$ , the case  $\lambda = 1$  to  $T_{\mathbf{L}}$ . Again,  $T_{\lambda_1}^{\mathbf{MT}} \gg T_{\lambda_2}^{\mathbf{MT}}$  implies  $\lambda_1 \leq \lambda_2$ . If  $\lambda_1 = 0$  or  $\lambda_1 = \lambda_2$ , then the dominance property is trivially fulfilled. Therefore, suppose that  $0 < \lambda_1 < \lambda_2$ . We denote  $T_{\lambda_1}^{\mathbf{MT}}$ , resp.  $T_{\lambda_2}^{\mathbf{MT}}$ , by  $T_1$ , resp.  $T_2$ . The sets of idempotent elements are of the following form

$$\mathcal{I}(T_i) = \{0\} \cup [\lambda_i, 1] .$$

Due to Proposition 7, it holds that  $T_2(\lambda_1, \lambda_1) \in \mathcal{I}(T_1)$ . Since  $T_2(\lambda_1, \lambda_1) \leq \lambda_1$ , either  $T_2(\lambda_1, \lambda_1) = 0$  or  $T_2(\lambda_1, \lambda_1) = \lambda_1$ . The latter implies that  $\lambda_1 \in \mathcal{I}(T_2)$ , a contradiction. Hence,  $T_2(\lambda_1, \lambda_1) = 0$  or equivalently  $\lambda_1 \leq \frac{\lambda_2}{2}$ . Now choose x such that  $\frac{\lambda_2}{2} < x < \frac{\lambda_2}{2} + \frac{\lambda_1}{4}$ 

and put u = v = y = x, then  $T_1(T_2(x, y), T_2(u, v)) = 0$  and  $T_2(T_1(x, u), T_1(y, v)) = 2x - \lambda_2 > 0$ , a final contradiction.

Therefore, also in the Mayor–Torrens family, there exist no other dominance relationships than  $T_{\mathbf{M}}$  dominating all other members and self-dominance.

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# The Dominance Relation on the Class of Continuous T-Norms from an Ordinal Sum Point of View

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Abstract. This paper addresses the relation of dominance on the class of continuous t-norms with a particular focus on continuous ordinal sum t-norms. Exactly, in this framework counter-examples to the conjecture that dominance is not only a reflexive and antisymmetric, but also a transitive relation could be found. We elaborate the details which have led to these results and illustrate them by several examples. In addition, to this original and comprehensive overview, we provide geometrical insight into dominance relationships involving prototypical Archimedean t-norms, the Lukasiewicz t-norm and the product t-norm.

## 1 Introduction

The dominance property was originally introduced within the framework of probabilistic metric spaces [42] and was soon abstracted to operations on an arbitrary partially ordered set [38]. A probabilistic metric space allows for *imprecise distances*: the distance between two objects p and q is characterized by a cumulative distribution function  $F_{pq} : \mathbb{R} \to [0, 1]$ . The metric in such spaces is defined in analogy to the axioms of (pseudo-)metric spaces, the most disputable axiom being the probabilistic metric spaces known as *Menger spaces* the triangle inequality reads as follows: for any three objects p, q, r and for any  $x, y \geq 0$  it holds that

$$F_{pr}(x+y) \ge T(F_{pq}(x), F_{qr}(y)), \qquad (1)$$

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where  $T: [0, 1]^2 \rightarrow [0, 1]$  is a *t-norm*, *i.e.* a binary operation on the unit interval which is commutative, associative, increasing in both arguments and which has neutral element 1.

The dominance property plays an important role in the construction of *Cartesian products* of probabilistic metric spaces, as it ensures that the triangle inequality holds for the resulting product space provided it holds for all factor spaces involved [38,42]. Similarly, it is responsible for the preservation of the *T*-transitivity property when building *fuzzy equivalence* or *fuzzy order relations* on a product space, *i.e.*  $R: X^2 \to [0,1]$ , defined by  $R(\mathbf{x}, \mathbf{y}) = \mathbf{A}(R_1(x_1, y_1), \ldots, R_n(x_n, y_n))$  with  $X = \prod_{i=1}^n X_i, R_i: X_i^2 \to [0,1]$  fuzzy relations on  $X_i$  being all *T*-transitive, *i.e.* 

$$T(R_i(x,y), R_i(y,z)) \le R_i(x,z)$$

and **A** some aggregation operator, or when intersecting such fuzzy relations on a single space, *i.e.*  $R(x,y) = T(R_1(x,y), \ldots, R_n(x,y))$  [2,3,8,32]. The dominance property was therefore introduced in the framework of aggregation operators where it enjoyed further development, again due its role in the preservation of a variety of properties, most of them expressed by some inequality, during (dis-)aggregation processes (see also [9,29]).

Besides these application points of view, the dominance property turned out to be an interesting mathematical notion *per se*. Due to the common neutral element of t-norms and their commutativity and associativity, the dominance property constitutes a reflexive and antisymmetric relation on the class of all t-norms. Whether it is also transitive has been posed as an open question already in 1983 in [38] and remained unanswered for quite some time. Several particular families of t-norms have been investigated (see, e.g., [17,34,40]) and supported the conjecture that the dominance relation would indeed be transitive, either due to its rare occurrence within the family considered or due to its abundant occurrence, in accordance with the parameter of the family. Several research teams participating in the EU COST action TARSKI have been studying various aspects of the dominance relation over the past few years. Finally, the conjecture was recently rejected [35]: the dominance relation is not transitive on the class of continuous t-norms and therefore also not on the class of t-norms in general. The counterexample was found among continuous ordinal sum t-norms.

In this contribution we discuss the dominance relation on the class of continuous t-norms and elaborate the details which have led to the counterexamples demonstrating the non-transitivity of the dominance relation in the class of t-norms. First, we provide a thorough introduction of all the necessary properties and details about t-norms. We then continue with a brief discussion of the dominance relation on the class of continuous Archimedean t-norms and provide geometrical insight in two prototypical cases. Subsequently, we turn to continuous ordinal sum t-norms and particular families of such ordinal sum t-norms. The present contribution provides a comprehensive and original overview of the stateof-the-art knowledge of the dominance relation on the class of continuous ordinal sum t-norms and as such depends on results also published in [17,30,31,35].

# 2 Triangular Norms

For the reader's convenience we briefly summarize basic properties of t-norms which will be necessary for a thorough understanding of the following parts. Many of the herein included results (including proofs, further details and references) can be found in [18,19,20] or in the monographs [1,17].

# 2.1 Basic Properties

Triangular norms (briefly t-norms) were first introduced in the context of probabilistic metric spaces [36,38,39], based on some ideas already presented in [24]. They are an indispensable tool for the interpretation of the conjunction in fuzzy logics [14] and, as a consequence, for the intersection of fuzzy sets [46]. Further, they play an important role in various further fields like decision making [11,13], statistics [26], as well as the theories of non-additive measures [21,27,41,45] and cooperative games [4].

**Definition 1.** A triangular norm (briefly t-norm) is a binary operation T on the unit interval [0,1] which is commutative, associative, increasing and has 1 as neutral element, i.e. it is a function  $T: [0,1]^2 \rightarrow [0,1]$  such that for all  $x, y, z \in [0,1]$ :

 $\begin{array}{l} (T1) \ T(x,y) = T(y,x), \\ (T2) \ T(x,T(y,z)) = T(T(x,y),z), \\ (T3) \ T(x,y) \leq T(x,z) \qquad \mbox{whenever } y \leq z, \\ (T4) \ T(x,1) = x. \end{array}$ 

It is an immediate consequence that due to the boundary and monotonicity conditions as well as commutativity it follows that, for all  $x \in [0, 1]$ , any t-norm T satisfies

$$T(0,x) = T(x,0) = 0,$$
(2)

$$T(1,x) = x. ag{3}$$

Therefore, all t-norms coincide on the boundary of the unit square  $[0, 1]^2$ .

Example 1. The most prominent examples of t-norms are the minimum  $T_{\mathbf{M}}$ , the product  $T_{\mathbf{P}}$ , the Lukasiewicz t-norm  $T_{\mathbf{L}}$  and the drastic product  $T_{\mathbf{D}}$  (see Figure 1 for 3D and contour plots). They are given by:

$$T_{\mathbf{M}}(x,y) = \min(x,y),\tag{4}$$

$$T_{\mathbf{P}}(x,y) = x \cdot y,\tag{5}$$

$$T_{\mathbf{L}}(x,y) = \max(x+y-1,0),$$
 (6)

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$
(7)

Since t-norms are just functions from the unit square into the unit interval, the comparison of t-norms is done in the usual way, *i.e.* pointwisely.



**Fig. 1.** 3D plots (top) and contour plots (bottom) of the four basic t-norms  $T_{\mathbf{M}}$ ,  $T_{\mathbf{P}}$ ,  $T_{\mathbf{L}}$ , and  $T_{\mathbf{D}}$  (observe that there are no contour lines for  $T_{\mathbf{D}}$ )

**Definition 2.** Let  $T_1$  and  $T_2$  be two t-norms. If  $T_1(x, y) \leq T_2(x, y)$  for all  $x, y \in [0, 1]$ , then we say that  $T_1$  is weaker than  $T_2$  or, equivalently, that  $T_2$  is stronger than  $T_1$ , and we write  $T_1 \leq T_2$ .

Further, t-norms can be transformed by means of an order isomorphism, *i.e.* an increasing  $[0,1] \rightarrow [0,1]$  bijection, preserving several properties (like, e.g., continuity) of the t-norm involved.

**Definition 3.** Let T be a t-norm and  $\varphi$  an order isomorphism. Then the isomorphic transform of T under  $\varphi$  is the t-norm  $T_{\varphi}$  defined by

$$T_{\varphi}(x,y) = \varphi^{-1}(T(\varphi(x),\varphi(y))).$$
(8)

Note that the drastic product  $T_{\mathbf{D}}$  and the minimum  $T_{\mathbf{M}}$  are the smallest and the largest t-norm, respectively. Moreover, they are the only t-norms that are invariant under arbitrary order isomorphisms.

Let us now focus on the continuity of t-norms.

**Definition 4.** A t-norm T is continuous if for all convergent sequences  $(x_n)_{n \in \mathbb{N}}$ ,  $(y_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$  we have

$$T\left(\lim_{n\to\infty}x_n,\lim_{n\to\infty}y_n\right) = \lim_{n\to\infty}T(x_n,y_n).$$

Obviously, the basic t-norms  $T_{\mathbf{M}}, T_{\mathbf{P}}$  and  $T_{\mathbf{L}}$  are continuous, whereas the drastic product  $T_{\mathbf{D}}$  is not. Note that for a t-norm T its continuity is equivalent to the continuity in each component (see also [17,18]), *i.e.* for any  $x_0, y_0 \in$ [0,1] both the vertical section  $T(x_0, \cdot) \colon [0,1] \to [0,1]$  and the horizontal section  $T(\cdot, y_0) \colon [0,1] \to [0,1]$  are continuous functions in one variable. The following classes of continuous t-norms are of particular importance.

**Definition 5.** (i) A t-norm T is called strict if it is continuous and strictly monotone, i.e. it fulfills for all  $x, y, z \in [0, 1]$ 

$$T(x,y) < T(x,z)$$
 whenever  $x > 0$  and  $y < z$ .

(ii) A t-norm T is called nilpotent if it is continuous and if each  $x \in [0, 1[$  is a nilpotent element of T, i.e. there exists some  $n \in \mathbb{N}$  such that

$$T(\underbrace{x,\ldots,x}_{n \ times}) = 0$$

The product  $T_{\mathbf{P}}$  is a strict t-norm whereas the Łukasiewicz t-norm  $T_{\mathbf{L}}$  is a nilpotent t-norm. Both of them are *Archimedean* t-norms, *i.e.* they fulfill for all  $(x, y) \in [0, 1]^2$  that there exists an  $n \in \mathbb{N}$  such that

$$T(\underbrace{x, \ldots, x}_{n \text{ times}}) < y.$$

It is remarkable that continuous Archimedean t-norms can be divided into just two subclasses — the nilpotent and the strict t-norms [17,18]. Moreover, since two continuous Archimedean t-norms are isomorphic if and only if they are either both strict or both nilpotent, we can immediately formulate the following proposition (see also [17,18]).

## **Proposition 1.** Let T be a t-norm.

- -T is a strict t-norm if and only if it is isomorphic to the product  $T_{\mathbf{P}}$ .
- T is a nilpotent t-norm if and only if it is isomorphic to the Lukasiewicz t-norm  $T_{\mathbf{L}}$ .

Besides the above introduced properties, idempotent elements play an important role in the characterization of t-norms.

**Definition 6.** Let T be a t-norm. An element  $x \in [0,1]$  is called an idempotent element of T if T(x,x) = x. We will further denote by  $\mathcal{I}(T)$  the set of all idempotent elements of T. The numbers 0 and 1 (which are idempotent elements for each t-norm T) are called trivial idempotent elements of T, each idempotent element in [0,1] will be called a non-trivial idempotent element of T.

The set of idempotent elements of the minimum  $T_{\mathbf{M}}$  equals [0, 1] (actually,  $T_{\mathbf{M}}$  is the only t-norm with this property) whereas  $T_{\mathbf{P}}$ ,  $T_{\mathbf{L}}$ , and  $T_{\mathbf{D}}$  possess only trivial idempotent elements.

# 2.2 Ordinal Sum T-Norms

Ordinal sum t-norms are based on a construction principle for semigroups which goes back to A.H. Clifford [5] (see also [6,15,28]) based on ideas presented in [7,16]. It has been successfully applied to t-norms in [12,22,37].

**Definition 7.** Let  $(]a_i, b_i[]_{i \in I}$  be a family of non-empty, pairwise disjoint open subintervals of [0,1] and let  $(T_i)_{i \in I}$  be a family of t-norms. The ordinal sum  $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$  is the t-norm defined by

$$T(x,y) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}), & \text{if } (x,y) \in [a_i, b_i]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

We will refer to  $\langle a_i, b_i, T_i \rangle$  as its summands, to  $[a_i, b_i]$  as its summand carriers, and to  $T_i$  as its summand operations or summand t-norms. The index set I is necessarily finite or countably infinite. It may also be empty in which case the ordinal sum is nothing else but  $T_{\mathbf{M}}$ .

Note that by construction, the set of idempotent elements  $\mathcal{I}(T)$  of some ordinal sum  $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$  contains the set  $M = [0, 1] \setminus \bigcup_{i \in I} ]a_i, b_i[$ . Moreover,  $\mathcal{I}(T) = M$  if and only if each  $T_i$  has only trivial idempotent elements. It is clear that an ordinal sum t-norm is continuous if and only if all of its summand t-norms are continuous.

In general, the representation of a t-norm as an ordinal sum of t-norms is not unique. For instance, for each subinterval [a, b] of [0, 1] we have

$$T_{\mathbf{M}} = (\emptyset) = (\langle 0, 1, T_{\mathbf{M}} \rangle) = (\langle a, b, T_{\mathbf{M}} \rangle)$$
.

This gives rise to the following definition.

**Definition 8.** A t-norm T that has no ordinal sum representation different from  $(\langle 0, 1, T \rangle)$  is called ordinally irreducible.

Note that each continuous Archimedean t-norm, in particular also  $T_{\mathbf{P}}$  and  $T_{\mathbf{L}}$ , has only trivial idempotent elements and is therefore ordinally irreducible. Moreover, there are no other ordinally irreducible continuous t-norms.

Based on the above information, we can now turn to the representation of continuous t-norms (see also [17,22,25,38])

**Theorem 1.** A binary operation on the unit interval is a continuous t-norm if and only if it is an ordinal sum of continuous Archimedean t-norms.

Therefore, continuous t-norms are either:

- strict, *i.e.* isomorphic to the product t-norm  $T_{\mathbf{P}}$ ,
- nilpotent, *i.e.* isomorphic to the Łukasiewicz t-norm  $T_{\rm L}$ ,
- the minimum  $T_{\mathbf{M}}$  itself, *i.e.*  $I = \emptyset$ , or
- non-trivial ordinal sums with strict or nilpotent summand operations, *i.e.*  $I \neq \emptyset$  and no  $]a_i, b_i[$  equals ]0, 1[.

## 2.3 The Dominance Property for T-Norms

Let us now focus on the dominance relation on the class of t-norms [38,42,44].

**Definition 9.** We say that a t-norm  $T_1$  dominates a t-norm  $T_2$ , or equivalently, that  $T_2$  is dominated by  $T_1$ , and write  $T_1 \gg T_2$ , if for all  $x, y, u, v \in [0, 1]$ 

$$T_1(T_2(x,y), T_2(u,v)) \ge T_2(T_1(x,u), T_1(y,v)).$$
(9)

Due to the fact that 1 is the common neutral element of all t-norms, dominance of one t-norm by another t-norm implies their comparability (see also [29]), *i.e.*  $T_1 \gg T_2$  implies  $T_1 \ge T_2$ . Similarly to the ordering of t-norms, any t-norm Tis dominated by itself and by  $T_{\mathbf{M}}$ , and dominates  $T_{\mathbf{D}}$ , *i.e.* for any t-norm T it holds that

 $T_{\mathbf{M}} \gg T, \qquad T \gg T, \qquad T \gg T_{\mathbf{D}}.$ 

As a consequence we can immediately state that dominance is a reflexive and antisymmetric relation on the class of all t-norms. We will show later that it is not transitive, not even on the class of continuous t-norms. Hence, the dominance relation is not a partial order on the set of all t-norms.

Finally, we mention that a dominance relationship between two t-norms is preserved under isomorphic transformations [32].

**Proposition 2.** A t-norm  $T_1$  dominates a t-norm  $T_2$  if and only if  $(T_1)_{\varphi}$  dominates  $(T_2)_{\varphi}$  for any order isomorphism  $\varphi$ .

# 3 Continuous Archimedean T-Norms

### **3.1** Isomorphic Transformations

The problem we study here is to determine whether a first continuous Archimedean t-norm  $T_1$  dominates a second such t-norm  $T_2$ . Since dominance is preserved under isomorphic transformations, this problem can be transformed into one of the following prototypical problems. Suppose that  $T_1 \gg T_2$ :

- If  $T_1$  is nilpotent, then  $T_2$  has to be nilpotent as well. In that case, there exist some order isomorphisms  $\varphi$  and  $\psi$  such that  $(T_1)_{\varphi} = T_{\mathbf{L}}$  and  $(T_2)_{\psi} = T_{\mathbf{L}}$ leading to

 $T_1 \gg T_2 \Leftrightarrow (T_1)_{\psi} \gg T_{\mathbf{L}} \Leftrightarrow T_{\mathbf{L}} \gg (T_2)_{\varphi}.$ 

- If  $T_1$  is strict, then  $T_2$  can be either strict or nilpotent. In both cases, there exist order isomorphisms  $\varphi$  and  $\psi$  such that

$$T_1 \gg T_2 \Leftrightarrow (T_1)_{\psi} \gg T_{\mathbf{L}} \Leftrightarrow T_{\mathbf{P}} \gg (T_2)_{\varphi}$$

in case  $T_2$  is nilpotent, and

$$T_1 \gg T_2 \Leftrightarrow (T_1)_{\psi} \gg T_{\mathbf{P}} \Leftrightarrow T_{\mathbf{P}} \gg (T_2)_{\varphi}$$

in case  $T_2$  is strict.

Summarizing, it suffices to investigate the classes of t-norms dominating or being dominated either by  $T_{\mathbf{P}}$  or by  $T_{\mathbf{L}}$ . In the next section, we will provide a geometrical interpretation for these particular cases. Necessary as well as sufficient conditions for aggregation operators (and therefore also t-norms) dominating one of these t-norms can be found, e.g., in [29,30,32,43].

### **3.2** Geometrical Interpretation

The inequality expressing dominance is difficult to grasp since it concerns four variables involved in various compositions of mappings. Providing an insightful geometrical interpretation would be more than welcome. We will present such an interpretation for the two cases discussed above: t-norms dominating or being dominated either by  $T_{\rm L}$  or by  $T_{\rm P}$ .

Note that the inequality expressing dominance trivially holds if at least one of the arguments equals 0. Hence, we can restrict our attention to arguments  $x, y, u, v \in [0, 1]$  only.

**Dominance Relationships Involving**  $T_{\mathbf{L}}$ . Let us consider some t-norm T which dominates  $T_{\mathbf{L}}$ , *i.e.* for all  $x, y, u, v \in [0, 1]$  we have

$$T(T_{\mathbf{L}}(x,u), T_{\mathbf{L}}(y,v)) \ge T_{\mathbf{L}}(T(x,y), T(u,v)).$$

$$(10)$$

For any fixed  $u, v \in [0, 1]$ , we introduce new variables  $a = T_{\mathbf{L}}(x, u)$  and  $b = T_{\mathbf{L}}(y, v)$  ranging over [0, u] and [0, v], respectively. If a = 0 then  $x + u - 1 \leq 0$ ; similarly, if b = 0 then  $y + v - 1 \leq 0$ . In any case, it follows that  $T(x, y) + T(u, v) - 1 \leq 0$  and (10) is satisfied trivially as both sides evaluate to 0. On the other hand, if a, b > 0 then x and y can be recovered from the expressions x = 1 - u + a and y = 1 - v + b. Using these new variables, the dominance inequality is transformed into

$$T(a,b) \ge T_{\mathbf{L}}(T(1-u+a,1-v+b),T(u,v))$$
(11)

for all  $u, v \in [0, 1]$  and all  $a \in [0, u]$ ,  $b \in [0, v]$ . The right-hand side can be interpreted geometrically in the following way:

- First, the graph of T(1 u + a, 1 v + b) as a function of a and b is nothing else but a translation of the original graph such that the point (1, 1, 1) is moved to the point (u, v, 1).
- Using  $T_{\mathbf{L}}$  to combine this function with the value T(u, v) means that this translated graph is subsequently translated along the direction of the z-axis such that the original reference point (1, 1, 1) is now located in the point (u, v, T(u, v)).
- As a consequence, parts of the resulting surface are now located outside the unit cube. Due to the definition of  $T_{\rm L}$ , these parts are simply truncated by 0, *i.e.* they are substituted by the corresponding parts of the *xy*-plane.

The fact that T dominates  $T_{\mathbf{L}}$  means that this translated surface lies below the original one, and this for any choice of u, v. The situation in which a t-norm T is dominated by  $T_{\mathbf{L}}$  has a similar interpretation, the only difference being that the translated surface should now be above the original one.

In Fig. 2, this geometrical interpretation is illustrated for the case  $T_{\mathbf{M}} \gg T_{\mathbf{L}}$ . For any choice of u, v (see Fig. 2 (a)) the box  $[0, u] \times [0, v] \times [0, T_{\mathbf{M}}(u, v)]$  is constructed (see Fig. 2 (b)) and the original graph of  $T_{\mathbf{M}}$  is translated moving


Fig. 2. Geometrical interpretation of  $T_{\mathbf{M}}$  dominating  $T_{\mathbf{L}}$ 

the point (1, 1, 1) to the point  $(u, v, T_{\mathbf{M}}(u, v))$  (see Fig. 2 (c)). Then the translated surface is compared with the original one (see Fig. 2 (d)). One can see immediately that the new surface lies below the original one for any choice of u, v.

**Dominance Relationships Involving**  $T_{\mathbf{P}}$ . The case of a t-norm T dominating  $T_{\mathbf{P}}$  has an even simpler geometrical interpretation. First of all,  $T \gg T_{\mathbf{P}}$  means that for all  $x, y, u, v \in [0, 1]$  it holds

$$T(xu, yv) \ge T(x, y)T(u, v).$$

For any fixed  $u, v \in [0, 1]$ , we introduce new variables a = xu and b = yv ranging over [0, u] and [0, v], respectively. Using these new variables, the dominance inequality is transformed into

$$T(a,b) \ge T(\frac{a}{u}, \frac{b}{v})T(u,v) \tag{12}$$

for all  $u, v \in [0, 1]$  and all  $a \in [0, u]$ ,  $b \in [0, v]$ . The right-hand side can be interpreted geometrically in the following way.

The graph of  $T(\frac{a}{u}, \frac{b}{v})T(u, v)$  as a function of a and b is exactly the graph of T linearly rescaled in order to fit into the box  $[0, u] \times [0, v] \times [0, T(u, v)]$ . This rescaling is obviously different for any u, v. The fact that T dominates  $T_{\mathbf{P}}$  means that this rescaled graph lies below the original graph. The situation in which



Fig. 3. Geometrical interpretation of  $T_{\mathbf{M}}$  dominating  $T_{\mathbf{P}}$ 

a t-norm T is dominated by  $T_{\mathbf{P}}$  has again a similar interpretation, the only difference being again that the rescaled graph should now be above the original one.

In Fig. 3, this geometrical interpretation is illustrated for the case  $T_{\mathbf{M}} \gg T_{\mathbf{P}}$ . For any choice of u, v (see Fig. 3 (a)) the box  $[0, u] \times [0, v] \times [0, T_{\mathbf{M}}(u, v)]$  is constructed (see Fig. 3 (b)) and the original graph of  $T_{\mathbf{M}}$  is rescaled in order to fit into this box (see Fig. 3 (c)). Then the rescaled surface is compared with the original one (see Fig. 3 (d)). One can see immediately that the new surface lies below the original one for any choice of u, v.

# 4 Continuous Non-Archimedean T-Norms

Let us now focus on dominance involving continuous non-Archimedean t-norms, *i.e.* involving non-trivial ordinal sums of continuous Archimedean t-norms.

# 4.1 Summand-wise Dominance

When studying the dominance relationship between two ordinal sum t-norms, we have to take into account the underlying structure of the ordinal sums. In case both ordinal sum t-norms are determined by the same family of non-empty, pairwise disjoint open subintervals, dominance between the ordinal sum t-norms is determined by the dominance between all corresponding summand t-norms [30].

**Proposition 3.** Consider the two ordinal sum t-norms  $T_1 = (\langle a_i, b_i, T_{1,i} \rangle)_{i \in I}$ and  $T_2 = (\langle a_i, b_i, T_{2,i} \rangle)_{i \in I}$ . Then  $T_1$  dominates  $T_2$  if and only if  $T_{1,i}$  dominates  $T_{2,i}$  for all  $i \in I$ .

# 4.2 Ordinal Sum T-Norms with Different Summand Carriers

In case the structure of both ordinal sum t-norms is not the same, we are able to provide some necessary conditions which lead to a characterization of dominance between ordinal sum t-norms in general. Assume that the ordinal sum t-norms  $T_1$ and  $T_2$  under consideration are based on two at least partially different families of summand carriers, *i.e.*  $T_1 = (\langle a_{1,i}, b_{1,i}, T_{1,i} \rangle)_{i \in I}$  and  $T_2 = (\langle a_{2,j}, b_{2,j}, T_{2,j} \rangle)_{j \in J}$ . W.l.o.g. we can assume that these representations are the finest possible, *i.e.* that each summand t-norm is ordinally irreducible.

Since for a continuous t-norm T the existence of a non-trivial idempotent element d is even equivalent to being representable as an ordinal sum  $T = (\langle 0, d, T' \rangle, \langle d, 1, T'' \rangle)$  for some summand t-norms T' and T'' (see also [17]), it is indeed reasonable to assume that the representations of two continuous t-norms  $T_1 = (\langle a_{1,i}, b_{1,i}, T_{1,i} \rangle)_{i \in I}$  and  $T_2 = (\langle a_{2,j}, b_{2,j}, T_{2,j} \rangle)_{j \in J}$  are such that there exists no  $T_{1,i}$ , resp.  $T_{2,j}$ , with a non-trivial idempotent element  $d \in [a_{1,i}, b_{1,i}[$ , resp.  $d \in [a_{2,j}, b_{2,j}[$ . Necessary Conditions Due to the Induced Order. Since any t-norm is bounded from above by  $T_{\mathbf{M}}$  and dominance implies their comparability we immediately can state the following lemma [30].

**Lemma 1.** If a t-norm  $T_1$  dominates a t-norm  $T_2$ , then  $T_1(x,y) = T_{\mathbf{M}}(x,y)$ whenever  $T_2(x,y) = T_{\mathbf{M}}(x,y)$ .

Geometrically speaking, if an ordinal sum t-norm  $T_1$  dominates an ordinal sum t-norm  $T_2$ , then it must necessarily consist of more regions where it acts as  $T_{\mathbf{M}}$  than  $T_2$ . Two such cases are displayed in Fig. 4 (a) and (c). Note that no dominance relationship between  $T_1$  and  $T_2$  is possible in a case like illustrated in Fig. 4 (b).



**Fig. 4.** Examples of two ordinal sum t-norms  $T_1$  and  $T_2$  differing in their summand carriers

Therefore, we can immediately state the following corollary [30].

**Corollary 1.** Consider the two ordinal sum t-norms  $T_1 = (\langle a_{1,i}, b_{1,i}, T_{1,i} \rangle)_{i \in I}$ and  $T_2 = (\langle a_{2,j}, b_{2,j}, T_{2,j} \rangle)_{j \in J}$  with ordinally irreducible summand t-norms only. If  $T_1$  dominates  $T_2$  then

$$\forall i \in I : \exists j \in J : [a_{1,i}, b_{1,i}] \subseteq [a_{2,j}, b_{2,j}] .$$
(13)

Note that each  $[a_{2,j}, b_{2,j}]$  can contain several or even none of the summand carriers  $[a_{1,i}, b_{1,i}]$  (see also Fig. 4 (a) and (c)). Hence, for each  $j \in J$  we can consider the following subset of I:

$$I_j = \{i \in I \mid [a_{1,i}, b_{1,i}] \subseteq [a_{2,j}, b_{2,j}]\}.$$
(14)

Based on these notions and due to Proposition 3, dominance between two ordinal sum t-norms can be reformulated in the following way [30].

**Proposition 4.** Consider two ordinal sum t-norms  $T_1 = (\langle a_{1,i}, b_{1,i}, T_{1,i} \rangle)_{i \in I}$ and  $T_2 = (\langle a_{2,j}, b_{2,j}, T_{2,j} \rangle)_{j \in J}$  with ordinally irreducible summand operations only. Then  $T_1$  dominates  $T_2$  if and only if

(i) 
$$\cup_{j \in J} I_j = I$$
,  
(ii)  $T_1^j \gg T_{2,j}$  for all  $j \in J$  with  
 $T_1^j = (\langle \varphi_j(a_{1,i}), \varphi_j(b_{1,i}), T_{1,i} \rangle)_{i \in I_j}$  (15)  
and  $\varphi_j \colon [a_{2,j}, b_{2,j}] \to [0, 1], \ \varphi_j(x) = \frac{x - a_{2,j}}{b_{2,j} - a_{2,j}}.$ 

Note that due to Proposition 4, the study of dominance between ordinal sum tnorms can be reduced to the study of dominance of a single ordinally irreducible t-norm by some ordinal sum t-norm. In particular, if all ordinal sum t-norms involved are just based on a single t-norm  $T^*$  as summand operation, it suffices to investigate the dominance of  $T^*$  by ordinal sum t-norms  $T = (\langle a_i, b_i, T^* \rangle)_{i \in I}$ .

Example 2. Let us now briefly elaborate the three different cases of ordinal sum t-norms displayed in Fig. 4 in more detail :

- Consider the ordinal sum t-norms  $T_1$  and  $T_2$  as displayed in Fig. 4 (a). Due to Proposition 3,  $T_1 \gg T_2$  is equivalent to showing that  $T_{1,1} \gg T_{2,1}$  and  $T_1^2 \gg T_{2,2}$ , where  $T_1^2$  is the ordinal sum t-norm defined by

$$T_1^2 = \left( \langle \varphi_2(a_{1,2}), \varphi_2(b_{1,2}), T_{1,2} \rangle, \langle \varphi_2(a_{1,3}), \varphi_2(b_{1,3}), T_{1,3} \rangle \right),$$

with  $\varphi_2: [a_{2,2}, b_{2,2}] \to [0, 1], \varphi_2(x) = \frac{x - a_{2,2}}{b_{2,2} - a_{2,2}}.$ - Having a look at the ordinal sum t-norms  $T_1$  and  $T_2$  as displayed in Fig. 4 (b), we immediately see that  $[a_{1,1}, b_{1,1}] \not\subseteq [a_{2,1}, b_{2,1}]$  and vice versa, so that

$$T_1(x,y) = T_{\mathbf{M}}(x,y) \neq T_2(x,y) \text{ for some } x, y \in [a_{2,1}, a_{1,1}],$$
  
$$T_2(x,y) = T_{\mathbf{M}}(x,y) \neq T_1(x,y) \text{ for some } x, y \in [b_{2,1}, b_{1,1}].$$

Hence, due to Lemma 1, in this case a dominance relationship is impossible. - On the other hand, for the ordinal sum t-norms  $T_1$  and  $T_2$  as displayed in Fig. 4 (c), the dominance of  $T_2$  by  $T_1$  is still possible. Again, due to Proposition 3,  $T_1 \gg T_2$  is equivalent to  $T_1^2 \gg T_{2,2}$ , where  $T_1^2$  is the ordinal sum t-norm defined by

$$T_1^2 = (\langle \varphi_2(a_{1,1}), \varphi_2(b_{1,1}), T_{1,1} \rangle)$$

with  $\varphi_2: [a_{2,2}, b_{2,2}] \to [0,1], \ \varphi_2(x) = \frac{x - a_{2,2}}{b_{2,2} - a_{2,2}}.$ 

Necessary Conditions Due to Idempotent Elements. The idempotent elements play an important role in dominance relationships, as is expressed by the following proposition [30].

**Proposition 5.** If a t-norm  $T_1$  dominates a t-norm  $T_2$ , then the following observations hold:

(i)  $\mathcal{I}(T_1)$  is closed under  $T_2$ ; (ii)  $\mathcal{I}(T_2) \subseteq \mathcal{I}(T_1)$ .



Fig. 5. Illustrations to Example 3

Note that for the representation of a continuous ordinal sum t-norm  $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$  in terms of ordinally irreducible summand t-norms  $T_i$ , the set of idempotent elements is given by  $\mathcal{I}(T) = [0, 1] \setminus \bigcup_{i \in I} ]a_i, b_i[$ . Therefore, this proposition has some interesting consequences for the boundary elements of the summand carriers. Firstly, all idempotent elements of  $T_2$  are idempotent elements of  $T_1$ , *i.e.* either endpoints of summand carriers of  $T_1$  or elements of some domain where  $T_1$  acts as  $T_{\mathbf{M}}$ . Secondly, for any idempotent elements  $d_1, d_2$  of  $T_1$  we know that also  $T_2(d_1, d_2)$ , is an idempotent element of  $T_1$ . Consequently, if  $T_1$  is some ordinal sum t-norm that dominates  $T_2 = T_{\mathbf{P}}$ , resp.  $T_2 = T_{\mathbf{L}}$ , and  $d \in \mathcal{I}(T_1)$  then also  $d^n \in \mathcal{I}(T_1)$ , resp.  $\max(nd - n + 1, 0) \in \mathcal{I}(T_1)$ , for all  $n \in \mathbb{N}$ .

Example 3. Consider a t-norm  $T^*$  with trivial idempotent elements only, *i.e.*  $\mathcal{I}(T^*) = \{0, 1\}$ . We are now interested in constructing ordinal sum t-norms  $T_1$  with summand operations  $T^*$  which fulfill the necessary conditions for dominating  $T_2 = T^*$  as expressed by Proposition 5. Clearly,  $T_1 = (\langle d, 1, T^* \rangle)$  is a first possibility (see Fig. 5 (a)). Adding one further summand to  $T_1$ , *i.e.* building  $T'_1 = (\langle a, d, T^* \rangle, \langle d, 1, T^* \rangle)$ , demands that  $a \geq T_2(d, d)$ , since otherwise  $T_2(d, d) \notin \mathcal{I}(T'_1)$  (see also Fig. 5 (b)).

# 5 Particular Continuous Ordinal Sum T-Norms

We will now focus on particular ordinal sum t-norms with either the Łukasiewicz t-norm or the product t-norm as only summand operation and study the dominance relationship between such t-norms.

# 5.1 Ordinal Sum T-Norms Based on $T_{\rm L}$

According to Proposition 5, the set of idempotent elements of a t-norm  $T_1$  dominating a t-norm  $T_2$  should be closed under  $T_2$  and should contain the idempotent elements of  $T_2$ . If we restrict our attention to ordinal sum t-norms with  $T_{\mathbf{L}}$  as only summand operation, this proposition can be turned into a characterization [33]. **Proposition 6.** Consider two ordinal sum t-norms  $T_1$  and  $T_2$  based on  $T_L$ , i.e.  $T_1 = (\langle a_i, b_i, T_L \rangle)_{i \in I}$  and  $T_2 = (\langle a_j, b_j, T_L \rangle)_{j \in J}$ . Then  $T_1$  dominates  $T_2$  if and only if the following two conditions hold:

(i)  $\mathcal{I}(T_1)$  is closed under  $T_2$ ; (ii)  $\mathcal{I}(T_2) \subseteq \mathcal{I}(T_1)$ .

Now consider the particular case  $T_2 = T_{\mathbf{L}}$ . Clearly, the second condition is trivially fulfilled and can be omitted. In order to be able to apply the above proposition to this case, we need to understand what it means for a set to be closed under  $T_{\mathbf{L}}$  [33].

**Lemma 2.** A subset  $S \subseteq [0,1]$  is closed under  $T_{\mathbf{L}}$  if and only if the set

 $1 - S = \{1 - x \mid x \in S\}$ 

is closed under truncated addition, i.e. whenever  $a, b \in 1-S$  also  $\min(a+b, 1) \in 1-S$ .

Consequently, an ordinal sum t-norm T based on  $T_{\mathbf{L}}$  dominates  $T_{\mathbf{L}}$  if and only if the set of its complemented idempotent elements is closed under truncated addition. Let us apply this insight to some particular families of ordinal sum t-norms based on  $T_{\mathbf{L}}$ .

The Mayor-Torrens Family. The Mayor-Torrens t-norms form a family parameterized by a single real parameter  $\lambda \in [0, 1]$  [23]:

$$T_{\lambda}^{\mathbf{MT}} = (\langle 0, \lambda, T_{\mathbf{L}} \rangle).$$

These t-norms are ordinal sums based on  $T_{\mathbf{L}}$  with a single summand located in the lower left corner of the unit square (see also Fig. 6 (a)). In particular, it holds that  $T_0^{\mathbf{MT}} = T_{\mathbf{M}}$  and  $T_1^{\mathbf{MT}} = T_{\mathbf{L}}$ . Note that  $T_{\lambda_1}^{\mathbf{MT}} \ge T_{\lambda_2}^{\mathbf{MT}}$  if and only if  $\lambda_1 \le \lambda_2$ . Hence,  $T_{\lambda_1}^{\mathbf{MT}} \gg T_{\lambda_2}^{\mathbf{MT}}$  implies  $\lambda_1 \le \lambda_2$ . If  $\lambda_1 = 0$  or  $\lambda_1 = \lambda_2$ , then the dominance relationship trivially holds. Suppose

If  $\lambda_1 = 0$  or  $\lambda_1 = \lambda_2$ , then the dominance relationship trivially holds. Suppose that  $0 < \lambda_1 < \lambda_2$ , then  $T_{\lambda_1}^{\mathbf{MT}}$  dominates  $T_{\lambda_2}^{\mathbf{MT}}$  if and only if  $T_{\lambda^*}^{\mathbf{MT}} = (\langle 0, \lambda^*, T_{\mathbf{L}} \rangle)$ dominates  $T_{\mathbf{L}}$  with  $\lambda^* = \frac{\lambda_1}{\lambda_2}$  (see also Proposition 4). The set of idempotent elements of  $T_{\lambda^*}^{\mathbf{MT}}$  is

$$\mathcal{I}(T_{\lambda^*}^{\mathbf{MT}}) = \{0\} \cup [\lambda^*, 1]$$

and therefore

$$1 - \mathcal{I}(T_{\lambda^*}^{\mathbf{MT}}) = [0, 1 - \lambda^*] \cup \{1\}.$$

For  $a = 1 - \lambda^*$  and  $b = \min(a, \frac{1-a}{2})$  it holds that  $a, b \in 1 - \mathcal{I}(T_{\lambda^*}^{\mathbf{MT}}), a + b < 1$ but  $a + b \notin 1 - \mathcal{I}(T_{\lambda^*}^{\mathbf{MT}})$ . According to Lemma 2 and Proposition 6, there exist no dominance relationships within the Mayor-Torrens family other than  $T_{\mathbf{M}}$ dominating all other members and self-dominance. Hence, there exists no triplet of pairwisely different t-norms  $T_{\lambda_1}^{\mathbf{MT}}, T_{\lambda_2}^{\mathbf{MT}}$  and  $T_{\lambda_3}^{\mathbf{MT}}$  fulfilling  $T_{\lambda_1}^{\mathbf{MT}} \gg T_{\lambda_2}^{\mathbf{MT}}$  and  $T_{\lambda_2}^{\mathbf{MT}} \gg T_{\lambda_3}^{\mathbf{MT}}$ , implying that the dominance relation is (trivially) transitive, and therefore a partial order, on this family. The Hasse-diagram of  $((T_{\lambda}^{\mathbf{MT}})_{\lambda \in [0,1]}, \ll)$ is displayed in Fig. 6 (b).



**Fig. 6.** Examples of Mayor-Torrens t-norms, Hasse-diagram of  $((T_{\lambda}^{\mathbf{MT}})_{\lambda \in [0,1]}, \ll)$ 

The Modified Mayor-Torrens Family. In this paragraph, we consider the family of t-norms parameterized by a single real parameter  $\mu \in [0, 1]$ :

$$T_{\mu} = (\langle \mu, 1, T_{\mathbf{L}} \rangle)$$

Contrary to the Mayor-Torrens family, the summands are located in the upper right corner of the unit square. Hence,  $T_0 = T_{\mathbf{L}}$  and  $T_1 = T_{\mathbf{M}}$  (see also Fig. 7 (a)). Note that  $T_{\mu_1} \ge T_{\mu_2}$  if and only if  $\mu_1 \ge \mu_2$ . Hence,  $T_{\mu_1} \gg T_{\mu_2}$  implies  $\mu_1 \ge \mu_2$ .

If  $\mu_1 = 1$  or  $\mu_1 = \mu_2$ , then the dominance relationship trivially holds. Assume that  $\mu_2 < \mu_1 < 1$ , then  $T_{\mu_1}$  dominates  $T_{\mu_2}$  if and only if  $T_{\mu^*}$  dominates  $T_{\mathbf{L}}$  with  $\mu^* = \frac{\mu_1 - \mu_2}{1 - \mu_2}$ . The set of idempotent elements of  $T_{\mu^*}$  is

$$\mathcal{I}(T_{\mu^*}) = [0, \mu^*] \cup \{1\}$$

and therefore

$$1 - \mathcal{I}(T_{\mu^*}) = \{0\} \cup [1 - \mu^*, 1]$$
.

One easily verifies that the latter set is closed under truncated addition. Hence, within the modified family, it holds that  $T_{\mu_1} \gg T_{\mu_2}$  whenever  $\mu_1 \ge \mu_2$ . In other words, this family is totally ordered by the dominance relation. The Hasse-diagram of  $((T_{\mu})_{\mu \in [0,1]}, \ll)$  is displayed in Fig. 7 (b).

Violation of Transitivity. We can now provide counterexamples to the conjecture that the dominance relation is transitive on the class of t-norms by considering ordinal sum t-norms based on  $T_{\mathbf{L}}$  with two summands. More specifically, we consider the t-norm  $T_{\lambda} = (\langle 0, \lambda, T_{\mathbf{L}} \rangle, \langle \lambda, 1, T_{\mathbf{L}} \rangle)$  with parameter  $\lambda \in [0, 1]$ . We will show that for any  $\lambda \in [0, \frac{1}{2}]$  it holds that

$$T_{\lambda}^{\mathbf{MT}} \gg T_{\lambda}, \qquad T_{\lambda} \gg T_{\mathbf{L}}, \qquad T_{\lambda}^{\mathbf{MT}} \not\gg T_{\mathbf{L}}$$
(16)

violating the transitivity of the dominance relation.

First, both  $T_{\lambda}^{\mathbf{MT}}$  and  $T_{\lambda}$  can be understood as ordinal sum t-norms with the same structure:  $T_{\lambda}^{\mathbf{MT}}$  can be written as  $(\langle 0, \lambda, T_{\mathbf{L}} \rangle, \langle \lambda, 1, T_{\mathbf{M}} \rangle)$ , hence the common summand carriers are  $[0, \lambda]$  and  $[\lambda, 1]$  (see Fig. 8).



**Fig. 7.** Examples modified Mayor-Torrens t-norms, Hasse-diagram of  $((T_{\mu})_{\mu \in [0,1]}, \ll)$ 

Since  $T_{\mathbf{L}} \gg T_{\mathbf{L}}$  and  $T_{\mathbf{M}} \gg T_{\mathbf{L}}$ , Proposition 3 implies that  $T_{\lambda}^{\mathbf{MT}} \gg T_{\lambda}$  for any  $\lambda \in [0, 1]$ . Second, the set of idempotent elements of  $T_{\lambda}$  is given by  $\mathcal{I}(T_{\lambda}) = \{0, \lambda, 1\}$  and thus

$$1 - \mathcal{I}(T_{\lambda}) = \{0, 1 - \lambda, 1\}.$$

This set is closed under truncated addition if and only if  $1 - \lambda \geq \frac{1}{2}$ . Therefore, according to Lemma 2 and Proposition 6, it holds that  $T_{\lambda}$  dominates  $T_{\mathbf{L}}$  if and only if  $\lambda \in [0, \frac{1}{2}]$ . Finally, in the Mayor-Torrens family it does not hold that  $T_{\lambda}^{\mathbf{MT}} \gg T_{\mathbf{L}} = T_0^{\mathbf{MT}}$  for any  $\lambda \in [0, 1]$ . Combining all of the above shows that (16) holds if and only if  $\lambda \in [0, \frac{1}{2}]$ .



**Fig. 8.** Three ordinal sum t-norms based on  $T_{\mathbf{L}}$  violating the transitivity of the dominance relation. From left to right:  $T_{\lambda}^{\mathbf{MT}}$ ,  $T_{\lambda}$  and  $T_{\mathbf{L}}$ . Violation of transitivity occurs if and only if  $\lambda \in \left[0, \frac{1}{2}\right]$ .

# 5.2 Ordinal Sum T-Norms Based on $T_{\rm P}$

We now turn to ordinal sum t-norms with  $T_{\mathbf{P}}$  as only summand operation and start again with a family of t-norms with a single summand in the lower left corner of the unit square.

The Dubois-Prade Family. The Dubois-Prade t-norms form a family parameterized by a single real parameter  $\lambda \in [0, 1]$  [10]:

$$T_{\lambda}^{\mathbf{DP}} = \left( \langle 0, \lambda, T_{\mathbf{P}} \rangle \right).$$

The case  $\lambda = 0$  corresponds to  $T_{\mathbf{M}}$ , the case  $\lambda = 1$  to  $T_{\mathbf{P}}$ . Note that  $T_{\lambda_1}^{\mathbf{DP}} \geq T_{\lambda_2}^{\mathbf{DP}}$ if and only if  $\lambda_1 \leq \lambda_2$ . Hence,  $T_{\lambda_1}^{\mathbf{DP}} \gg T_{\lambda_2}^{\mathbf{DP}}$  implies  $\lambda_1 \leq \lambda_2$ . If  $\lambda_1 = 0$  or  $\lambda_1 = \lambda_2$ , then the dominance relationship trivially holds. There-

If  $\lambda_1 = 0$  or  $\lambda_1 = \lambda_2$ , then the dominance relationship trivially holds. Therefore, suppose that  $0 < \lambda_1 < \lambda_2$ . The set of idempotent elements of  $T_{\lambda_1}^{\mathbf{DP}}$  is given by

$$\mathcal{I}(T_{\lambda_1}^{\mathbf{DP}}) = \{0\} \cup [\lambda_1, 1] .$$

It then holds that

$$0 \neq T_{\lambda_2}^{\mathbf{DP}}(\lambda_1, \lambda_1) = \lambda_2 \cdot T_{\mathbf{P}}(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}) = \frac{\lambda_1}{\lambda_2} \cdot \lambda_1 < \lambda_1$$

due to the strict monotonicity of  $T_{\mathbf{P}}$ . Hence,  $T_{\lambda_2}^{\mathbf{DP}}(\lambda_1, \lambda_1) \notin \mathcal{I}(T_{\lambda_1}^{\mathbf{DP}})$ . According to Proposition 5,  $T_{\lambda_1}^{\mathbf{DP}}$  does not dominate  $T_{\lambda_2}^{\mathbf{DP}}$ . Consequently, the only dominance relationships in the Dubois-Prade family

Consequently, the only dominance relationships in the Dubois-Prade family are  $T_{\mathbf{M}}$  dominating all other members and self-dominance. The dominance relation is again (trivially) transitive, and therefore a partial order, on this family (see Fig. 9).



**Fig. 9.** Examples of Dubois-Prade t-norms, Hasse-diagram of  $((T_{\lambda}^{\mathbf{DP}})_{\lambda \in [0,1]}, \ll)$ 

In contrast to dominance between ordinal sum t-norms based on  $T_{\mathbf{L}}$ , dominance between ordinal sum t-norms based on  $T_{\mathbf{P}}$  is not fully understood. The following lemma provides one way of constructing an ordinal sum t-norm based on  $T_{\mathbf{P}}$ dominating  $T_{\mathbf{P}}$ . It follows immediately from Proposition 5.

**Lemma 3.** Let  $\lambda \in [0, 1[$  and  $m \in \mathbb{N}$ . Then the ordinal sum t-norm  $T_{\lambda,m}$  defined as

$$T_{\lambda,m} = (\langle \lambda^n, \lambda^{n-1}, T_{\mathbf{P}} \rangle)_{n=1,2,\dots,m}$$

dominates  $T_{\mathbf{P}}$ .

This simple lemma allows to construct interesting examples.

The Modified Dubois-Prade Family. Similarly as for the Mayor-Torrens family, we propose a modification of the Dubois-Prade family, by locating the single summand in the upper right corner of the unit square. Explicitly, we consider the family of t-norms parameterized by a single real parameter  $\lambda \in [0, 1]$ :

$$T_{\lambda} = (\langle \lambda, 1, T_{\mathbf{P}} \rangle).$$

Note that these t-norms are special cases of Lemma 3 as  $T_{\lambda} = T_{\lambda,1}$ . In particular,  $T_0 = T_{\mathbf{P}}$  and  $T_1 = T_{\mathbf{M}}$ . Note that  $T_{\lambda_1} \ge T_{\lambda_2}$  if and only if  $\lambda_1 \ge \lambda_2$ . Hence,  $T_{\lambda_1} \gg T_{\lambda_2}$  implies  $\lambda_1 \ge \lambda_2$ .

If  $\lambda_1 = 1$  or  $\lambda_1 = \lambda_2$ , then the dominance relationship again trivially holds. Moreover, due to Lemma 3, the dominance relationship also holds if  $\lambda_2 = 0$ , *i.e.*  $T_{\lambda_1} \gg T_0$ . Consider the case  $0 < \lambda_2 < \lambda_1 < 1$ , then  $T_{\lambda_1}$  dominates  $T_{\lambda_2}$  if and only if  $(\langle \frac{\lambda_1 - \lambda_2}{1 - \lambda_2}, 1, T_{\mathbf{P}} \rangle)$  dominates  $T_{\mathbf{P}}$ . Thanks to Lemma 3, it then follows that the modified Dubois-Prade family is totally ordered by the dominance relation (see Fig. 10 (b)).



**Fig. 10.** Examples modified Dubois-Prade t-norms, Hasse-diagram of  $((T_{\lambda})_{\lambda \in [0,1]}, \ll)$ 

**Violation of Transitivity.** Also ordinal sum t-norms based on  $T_{\mathbf{P}}$  allow us to construct a counterexample demonstrating the non-transitivity of the dominance relation. Consider the ordinal sum t-norms  $T_1 = (\langle \frac{1}{4}, \frac{1}{2}, T_{\mathbf{P}} \rangle, \langle \frac{3}{4}, 1, T_{\mathbf{P}} \rangle)$  and  $T_2 = T_{\frac{1}{2},2}$  (see Lemma 3). It then holds that

$$T_1 \gg T_2, \qquad T_2 \gg T_{\mathbf{P}}, \qquad T_1 \not\gg T_{\mathbf{P}}$$

violating the transitivity of the dominance relation (see Fig. 11).

Note that the t-norm  $T_1$  can also be written as  $T_1 = (\langle \frac{1}{4}, \frac{1}{2}, T_{\mathbf{P}} \rangle, \langle \frac{1}{2}, 1, T^* \rangle)$ with  $T^*$  the member of the modified Dubois-Prade family with parameter  $\lambda = \frac{1}{2}$ . Using Proposition 3 and the dominance relationships within the modified Dubois-Prade family, it follows immediately that  $T_1 \gg T_2$ . The dominance relationship  $T_2 \gg T_{\mathbf{P}}$  is an immediate consequence of Lemma 3. Finally, we consider the set of idempotent elements of  $T_1$ :

$$\mathcal{I}(T_1) = \begin{bmatrix} 0, \frac{1}{4} \end{bmatrix} \cup \begin{bmatrix} \frac{1}{2}, \frac{3}{4} \end{bmatrix}.$$



Fig. 11. Three ordinal sum t-norms based on  $T_{\mathbf{P}}$  violating the transitivity of the dominance relation

It holds that  $\frac{5}{8} \in \mathcal{I}(T_1)$ , while

$$T_{\mathbf{P}}(\frac{5}{8}, \frac{5}{8}) = \frac{25}{64} \notin \mathcal{I}(T_1) \,.$$

Proposition 5 then implies that  $T_1$  does not dominate  $T_{\mathbf{P}}$ .

# 6 Final Remarks

The dominance relation is a reflexive and antisymmetric relation on the class of t-norms. That it is not transitive and therefore not a partial order was illustrated by several examples whereas the particular role of ordinal sums dominating either the Lukasiewicz t-norm or the product t-norm is remarkable. Note that by the isomorphism property of dominance these examples can be transformed into counterexamples involving arbitrary nilpotent resp. strict t-norms. Properties related to idempotent elements and to the induced order heavily determine the occurrence of dominance within particular families of t-norms as shown by the parameterized families in the last section.

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# A generalization of the Mulholland inequality for continuous Archimedean t-norms

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#### ABSTRACT

It is well known that dominance between strict t-norms is closely related to the Mulholland inequality, which can be seen as a generalization of the Minkowski inequality. However, strict t-norms constitute only one part of the class of continuous Archimedean t-norms, the basic elements from which all continuous t-norms are composed. In this paper, dominance between continuous Archimedean t-norms is shown to be related to a generalization of the Mulholland inequality. We provide sufficient and necessary conditions for its fulfillment.

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#### 1. Introduction

In 1950, Mulholland presented a generalization of the Minkowski inequality, which later on became known as the *Mulholland inequality* [13]. In the same contribution, he provided a sufficient condition for its fulfillment by a continuous function that is strictly increasing on its domain. In 1984, Tardiff demonstrated that this inequality plays an essential role in the investigation of dominance between strict *triangular norms* (t-norms for short) and provided a different sufficient condition [24]. In 2002, Jarczyk and Matkowski clarified the relationship between the two sufficient conditions, showing that Tardiff's condition implies that of Mulholland [5].

On the other hand, the dominance relation was originally introduced in the framework of probabilistic metric spaces [22] and was soon abstracted to operations on a partially ordered set (see, e.g. [20]). The dominance relation, in particular between t-norms, plays a profound role in various topics, such as the construction of Cartesian products of probabilistic metric and normed spaces [11,20,22], the construction of many-valued equivalence relations [2,3,25] and many-valued order relations [1], as well as in the preservation of various properties during (dis-)aggregation processes in flexible querying, preference modelling and computer-assisted assessment [2,4,14,16]. These applications instigated the study of the dominance relation in the broader context of aggregation operators [12,14,16].

The dominance relation is an interesting mathematical notion *per se.* As it constitutes a reflexive and antisymmetric relation on the class of t-norms, and counterexamples for its transitivity were not readily found, it remained an intriguing open problem [7,18,20,21,24] for more than 20 years whether or not it was an order relation. Only recently the question was answered to the negative [17,19]. However, due to its relevance in applications, it is still of interest to determine whether or not the dominance relation establishes an order relation on some subclasses of t-norms. Of particular importance are the continuous Archimedean t-norms, as they are the basic elements of which all continuous t-norms are composed. Therefore,

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establishing sufficient conditions for dominance between continuous Archimedean t-norms is of interest and constitutes the main goal of our contribution.

After some brief preliminaries on t-norms, we demonstrate the close relationship between dominance between continuous Archimedean t-norms and a generalization of the Mulholland inequality. A short survey on sufficient conditions for continuous functions which are strictly increasing on the whole domain is followed by appropriate sufficient and necessary conditions in the more general case. This provides the basis for the investigation of dominance between continuous Archimedean t-norms in the last section.

### 2. Continuous Archimedean t-norms

We briefly summarize some basic properties of t-norms for a thorough understanding of this paper (see, e.g. [6-10]).

**Definition 1.** A *t*-norm  $T : [0, 1]^2 \rightarrow [0, 1]$  is a binary operation on the unit interval that is commutative, associative, increasing and has 1 as neutral element.

Well-known examples of t-norms are the minimum  $T_{\mathbf{M}}$ , the product  $T_{\mathbf{P}}$  and the Łukasiewicz t-norm  $T_{\mathbf{L}}$  defined by  $T_{\mathbf{M}}(u, v) = \min(u, v)$ ,  $T_{\mathbf{P}}(u, v) = u \cdot v$  and  $T_{\mathbf{L}}(u, v) = \max(u + v - 1, 0)$ .

Since t-norms are just functions from the unit square to the unit interval, their comparison is done pointwisely:  $T_1 \leq T_2$  if  $T_1(u, v) \leq T_2(u, v)$  for all  $u, v \in [0, 1]$ , expressed as " $T_1$  is *weaker* than  $T_2$ " or " $T_2$  is *stronger* than  $T_1$ ." The minimum  $T_M$  is the strongest of all t-norms. Furthermore, it holds that  $T_P \geq T_L$ .

A continuous t-norm *T* is *Archimedean* if and only if for all  $u \in [0, 1[$  it holds that T(u, u) < u. The class of continuous Archimedean t-norms can be partitioned into two subclasses: the class of *strict t-norms*, which are continuous and strictly increasing, and the class of *nilpotent t-norms*, which are continuous and fulfill that for each  $u \in [0, 1[$  there exists some  $n \in \mathbb{N}$  such that

$$T(\underbrace{u,\ldots,u}_{n \text{ times}}) = \mathbf{0}.$$

The product  $T_{\mathbf{P}}$  is strict, whereas the Łukasiewicz t-norm  $T_{\mathbf{L}}$  is nilpotent.

Note that for a strict t-norm T it holds that T(u, v) > 0 for all  $u, v \in [0, 1]$ , while for a nilpotent t-norm T it holds that for any  $u \in [0, 1[$  there exists some  $v \in [0, 1[$  such that T(u, v) = 0 (each  $u \in [0, 1[$  is a so-called *zero divisor*). Therefore, for a nilpotent t-norm  $T_1$  and a strict t-norm  $T_2$  it can never hold that  $T_1 \ge T_2$ .

Of particular interest in the discussion of continuous Archimedean t-norms is the notion of an additive generator.

**Definition 2.** An *additive generator* of a continuous Archimedean t-norm *T* is a continuous, strictly decreasing function  $t:[0,1] \rightarrow [0,\infty]$  which satisfies t(1) = 0 such that for all  $u, v \in [0,1]$  it holds that

$$T(u, v) = t^{(-1)} (t(u) + t(v))$$
(1)

with

$$t^{(-1)}(u) = t^{-1}(\min(t(0), u))$$
<sup>(2)</sup>

the *pseudo-inverse* of the decreasing function *t*.

An additive generator is uniquely determined up to a positive multiplicative constant. Any additive generator of a strict t-norm satisfies  $t(0) = \infty$ , while that of a nilpotent t-norm satisfies  $t(0) < \infty$ . In the case of strict t-norms, the pseudo-inverse  $t^{(-1)}$  of an additive generator t coincides with its standard inverse  $t^{-1}$ . In any case, the following relationships between an additive generator t and its pseudo-inverse  $t^{(-1)}$  hold

$$t \circ t^{(-1)}|_{\operatorname{Ran}(t)} = \operatorname{id}_{\operatorname{Ran}(t)} \quad \text{and} \quad t^{(-1)} \circ t = \operatorname{id}_{[0,1]}.$$
 (3)

### 3. Dominance and related inequalities

Just as triangular norms, the dominance relation finds its origin in the field of probabilistic metric spaces [20,22]. It was originally introduced for associative operations (with common neutral element) on a partially ordered set [20], and has been further investigated for t-norms [15,17–19,21,24] and aggregation operators [12,14,16]. We state the definition for t-norms only.

**Definition 3.** Consider two t-norms  $T_1$  and  $T_2$ . We say that  $T_1$  *dominates*  $T_2$  (or  $T_2$  is dominated by  $T_1$ ), denoted by  $T_1 \gg T_2$ , if for all  $x, y, u, v \in [0, 1]$  it holds that

$$T_1(T_2(x, y), T_2(u, v)) \ge T_2(T_1(x, u), T_1(y, v)).$$
(4)

Note that any t-norm is dominated by itself and by  $T_{\mathbf{M}}$ . Since all t-norms have neutral element 1, dominance between two t-norms implies their comparability:  $T_1 \gg T_2$  implies  $T_1 \ge T_2$ . The converse does not hold, not even for strict t-norms [24]. Since for a nilpotent t-norm  $T_1$  and a strict t-norm  $T_2$ , it cannot hold that  $T_1 \ge T_2$ , it also cannot hold that  $T_1 \gg T_2$ . Therefore, for a continuous Archimedean t-norm  $T_1$  and a strict t-norm  $T_2$ ,  $T_1 \gg T_2$  implies that also  $T_1$  is strict.

The dominance relation between two continuous Archimedean t-norms can be expressed in terms of their generators. This was shown for strict t-norms in [24] and is generalized below.

**Theorem 1.** Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . Then  $T_1$  dominates  $T_2$  if and only if the function  $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$  fulfills for all  $a, b, c, d \in [0, t_2(0)]$ 

$$h^{(-1)}(h(a) + h(c)) + h^{(-1)}(h(b) + h(d)) \ge h^{(-1)}(h(a+b) + h(c+d))$$
(5)

with  $h^{(-1)}$ :  $[0, \infty] \rightarrow [0, \infty]$  the pseudo-inverse of the increasing function h, given by  $h^{(-1)} = t_2 \circ t_1^{(-1)}$ .

**Proof.** The case of two strict t-norms  $T_1$  and  $T_2$  was treated by Tardiff [24]. Therefore, we suppose that at least one of the t-norms involved is nilpotent.

Note also that (4) is trivially fulfilled when  $0 \in \{x, y, u, v\}$ . Hence, the verification of (5) can be restricted to  $a, b, c, d \in [0, t_2(0)[$  only.

(i) Suppose first that  $T_1 \gg T_2$ . Expressing (4) in terms of generators and applying the decreasing function  $t_2$  to both sides leads to

$$h^{(-1)}[h(t_2(x) + t_2(y)) + h(t_2(u) + t_2(v))] \leq t_2 \circ t_2^{(-1)}[h^{(-1)}(t_1(x) + t_1(u)) + h^{(-1)}(t_1(y) + t_1(v))]$$

for all  $x, y, u, v \in [0, 1]$ . Consider  $a, b, c, d \in [0, t_2(0)]$ , then the continuity of  $t_2$  implies the existence of  $x = t_2^{-1}(a) = t_2^{(-1)}(a)$ ,  $y = t_2^{-1}(b) = t_2^{(-1)}(b)$ ,  $u = t_2^{-1}(c) = t_2^{(-1)}(c)$ ,  $v = t_2^{-1}(d) = t_2^{(-1)}(d)$ . It then follows that

$$h^{(-1)}(h(a+b)+h(c+d)) \leq t_2 \circ t_2^{(-1)}[h^{(-1)}(h(a)+h(c))+h^{(-1)}(h(b)+h(d))].$$

Denote  $K = h^{(-1)}(h(a) + h(c)) + h^{(-1)}(h(b) + h(d))$ . If  $K \ge t_2(0)$ , then

$$h^{(-1)}(h(a+b)+h(c+d)) \leq t_2 \circ t_2^{(-1)}(K) = t_2 \circ t_2^{(-1)} \circ t_2(0) = t_2(0) \leq K.$$

Otherwise, it holds that

$$h^{(-1)}(h(a+b)+h(c+d)) \leq t_2 \circ t_2^{(-1)}(K) = t_2 \circ t_2^{-1}(K) = K.$$

This shows that (5) is fulfilled for all  $a, b, c, d \in [0, t_2(0)]$ .

(ii) Conversely, suppose that *h* fulfills (5) for all  $a, b, c, d \in [0, t_2(0)]$ , then

$$t_{2} \circ t_{1}^{(-1)} (t_{1} \circ t_{2}^{(-1)}(a) + t_{1} \circ t_{2}^{(-1)}(c)) + t_{2} \circ t_{1}^{(-1)} (t_{1} \circ t_{2}^{(-1)}(b) + t_{1} \circ t_{2}^{(-1)}(d))$$
  
$$\geq t_{2} \circ t_{1}^{(-1)} (t_{1} \circ t_{2}^{(-1)}(a+b) + t_{1} \circ t_{2}^{(-1)}(c+d)).$$

Consider  $x, y, u, v \in [0, 1]$  and let  $a = t_2(x)$ ,  $b = t_2(y)$ ,  $c = t_2(u)$  and  $d = t_2(v)$ . It then follows that

$$t_{2} \circ t_{1}^{(-1)}(t_{1}(x) + t_{1}(u)) + t_{2} \circ t_{1}^{(-1)}(t_{1}(y) + t_{1}(v)) \ge t_{2} \circ t_{1}^{(-1)}(t_{1} \circ t_{2}^{(-1)}(t_{2}(x) + t_{2}(y)) + t_{1} \circ t_{2}^{(-1)}(t_{2}(u) + t_{2}(v))).$$

Applying the decreasing function  $t_2^{(-1)}$  to both sides leads to

$$T_2(T_1(x, u), T_1(y, v)) \leq T_1(T_2(x, y), T_2(u, v)).$$

Hence,  $T_1$  dominates  $T_2$ .  $\Box$ 

#### 4. The Mulholland inequality

Using the notations of Theorem 1, if  $T_1$  and  $T_2$  are strict, then  $t_2(0) = \infty$ , h is strictly increasing and thus  $h^{(-1)} = h^{-1}$ . Inequality (5) then simplifies to

$$h^{-1}(h(a) + h(c)) + h^{-1}(h(b) + h(d)) \ge h^{-1}(h(a+b) + h(c+d)),$$
(6)

for all  $a, b, c, d \in [0, \infty[$  (the inequality is trivially fulfilled when  $\infty \in \{a, b, c, d\}$ ). This inequality is known as the *Mulholland inequality* and is a generalization of the *Minkowski inequality* [13].

It is remarkable that functions h fulfilling (6) have been investigated independently from the context of dominance [5,13,23,24]. A brief overview of the most important findings is given next.

**Proposition 2.** (See [13].) Consider a continuous, strictly increasing function  $h : [0, \infty[ \rightarrow [0, \infty[$  such that h(0) = 0. If h fulfills the Mulholland inequality (6), then it is convex on  $]0, \infty[$ .

**Proposition 3.** (See [13].) Consider a continuous, strictly increasing function  $h : [0, \infty[ \rightarrow [0, \infty[$  such that h(0) = 0. If h is convex on  $]0, \infty[$  and  $\log \circ h \circ \exp$  is convex on  $]-\infty, \infty[$ , then h fulfills the Mulholland inequality (6).

**Proposition 4.** (See [24].) Consider a differentiable, strictly increasing function  $h : [0, \infty[ \rightarrow [0, \infty[$  such that h(0) = 0. If h is convex on  $]0, \infty[$  and  $\log \circ h' \circ \exp$  is convex on  $]-\infty, \infty[$ , then h fulfills the Mulholland inequality (6).

It can be shown that for a continuous function  $f : [0, \infty[ \rightarrow [0, \infty[$  such that  $f(]0, \infty[) \subseteq ]0, \infty[$ , it holds that  $\log \circ f \circ \exp$  is convex on  $]-\infty$ ,  $\log(t)[$ , with  $t \in ]0, \infty[$ , if and only if f fulfills

$$f(\sqrt{xy}) \leqslant \sqrt{f(x)f(y)}$$

for all  $x, y \in [0, t[$ . The latter condition is referred to as the *geometric convexity* of f on [0, t[ (*geo-convexity* for short); if f(0) = 0, then the geo-convexity holds on [0, t[. Moreover, if f is increasing, then the convexity of  $\log \circ f$  on [0, t[, called *log-convexity* of f, implies its geo-convexity. Jarczyk and Matkowski [5] have investigated the relationship between the geo-convexity of a function and that of its derivative.

**Proposition 5.** (See [5].) Consider a differentiable function  $f: ]0, \infty[ \rightarrow ]0, \infty[$  such that  $\lim_{x\to 0} f(x) = 0$  and f'(x) > 0 for all  $x \in ]0, \infty[$ . If f' is geo-convex, then so is f.

Combining the above results leads to the following relationships between the sufficient conditions on h for the fulfillment of the Mulholland inequality:

h is convex, fulfills h(0) = 0, and ... h' is geo-convex  $\Leftarrow$  h' is log-convex  $\downarrow$  h is geo-convex  $\Leftarrow$  h is log-convex  $\downarrow$ h fulfills (6)

### 5. A generalization of the Mulholland inequality

In this section, we aim at a generalization of the results of Mulholland and Tardiff in order to guarantee their applicability to the investigation of dominance between two continuous Archimedean t-norms.

#### 5.1. A first sufficient condition

**Theorem 6.** Consider a function  $h: [0, \infty] \to [0, \infty]$  and some fixed value  $t \in [0, \infty]$  such that

```
(h1) h is continuous on [0, t];
```

```
(h2) h is strictly increasing on [0, t] and h(x) \ge h(t) whenever x \ge t;
```

- (h3) h(0) = 0;
- (h4) *h* is convex on ]0, *t*[;
- (h5) h is geo-convex on ]0, t[.

Define the functions  $g: [0, \infty] \to [0, \infty]$  and  $H: [0, \infty]^2 \to [0, \infty]$  by

g(x) :=	$h^{-1}(x),$	$if x \in [0, h(t)],$	(8)
	l t ,	otherwise,	(0)

$$H(x, y) := g(h(x) + h(y)).$$
(9)

Then the following inequality holds for all  $a, b, c, d \in [0, \infty]$ 

$$H(a+b,c+d) \leqslant H(a,c) + H(b,d). \tag{10}$$

**Remark 1.** Clearly, *g* is continuous and increasing. Also *H* is continuous in each argument and increasing. Obviously, it holds that

 $H(t, x) = H(x, t) = t, \quad \text{for all } x \in [0, \infty], \tag{11}$ 

$$H(0, x) = H(x, 0) = x, \quad \text{for all } x \in [0, t].$$
(12)

Further, the convexity of *h* on ]0, *t*[ is equivalent to the concavity of *g* on ]0, *h*(*t*)[. Since *h* is increasing and continuous on [0, *t*], its convexity on ]0, *t*[ implies its convexity on [0, *t*]. As argued before, the geo-convexity of *h* on ]0, *t*[ is equivalent to the convexity of  $\log \circ h \circ \exp$  on  $]-\infty$ ,  $\log(t)$ [, which in its turn is equivalent to the concavity of the function  $\log \circ g \circ \exp$  on  $]-\infty$ ,  $\log(h(t))$ [. It is easy to show that in these cases, it also holds that *g* is concave on ]0,  $\infty$ [ and  $\log \circ g \circ \exp$  is concave on  $]-\infty$ ,  $\infty$ [.

Inspired by Mulholland [13], we introduce another function that will be essential in our proof.

**Lemma 7.** Under the assumptions of Theorem 6, define the function  $\psi : [0, t] \rightarrow [0, \infty]$  by

$$\psi(x) := \begin{cases} \frac{h(x)}{x}, & \text{if } x > 0, \\ \lim_{y \to 0^+} \frac{h(y)}{y}, & \text{if } x = 0. \end{cases}$$
(13)

Then  $\psi$  is increasing on [0, t].

**Proof.** Note that the function  $\psi$  is strictly positive on ]0, t] and continuous on [0, t]. Consider  $0 < x < x + \epsilon < t$ , then we need to show that  $\psi(x) \leq \psi(x + \epsilon)$ . Let  $\alpha = \frac{\epsilon}{x + \epsilon}$  and  $\beta = 1 - \alpha$ , then the convexity of *h* on  $[0, x + \epsilon]$  implies that

$$h(\beta(x+\epsilon)) \leq \alpha h(0) + \beta h(x+\epsilon) = \beta h(x+\epsilon),$$

which can be rewritten as  $h(x) \leq \frac{x}{x+\epsilon}h(x+\epsilon)$ , and hence  $\psi(x) \leq \psi(x+\epsilon)$ . The continuity of  $\psi$  then implies that it is increasing on [0, t].  $\Box$ 

We now turn to the proof of Theorem 6.

Proof of Theorem 6. The proof consists of several cases.

(1) At least one of a, b, c, d belongs to  $[t, \infty]$ .

Since *H* is increasing, it follows from (11) that H(x, y) = t whenever  $x \ge t$  or  $y \ge t$ . This implies that (10) trivially holds when one of the arguments is greater than or equal to *t*.

(2) All of a, b, c, d belong to [0, t] and a + b < t and c + d < t.

If a = b = 0 or c = d = 0, then (10) holds due to (12). We therefore assume that 0 < a + b as well as 0 < c + d. The proof of this case is based on the observation that (10) is a consequence of a more general inequality, namely

$$x\psi(a+b) + y\psi(c+d) \leqslant H(x,y)\frac{h(a+b) + h(c+d)}{H(a+b,c+d)},$$
(14)

for all *x*, *y* such that  $0 \le x \le a+b$  and  $0 \le y \le c+d$ . Indeed, assume that (14) holds, then expressing it for both (x, y) = (a, c) and (x, y) = (b, d) and adding side by side leads to

$$h(a+b) + h(c+d) = a\psi(a+b) + c\psi(c+d) + b\psi(a+b) + d\psi(c+d) \leq (H(a,c) + H(b,d)) \frac{h(a+b) + h(c+d)}{H(a+b,c+d)},$$

which implies (10), since h(a + b) + h(c + d) > 0 and H(a + b, c + d) > 0. We therefore attempt to show (14).

- (a) In case x = y = 0, it is trivially fulfilled.
- (b) In case x = 0 and y > 0, we need to show that

$$\psi(c+d) \leqslant \frac{h(a+b) + h(c+d)}{H(a+b, c+d)}$$

In case  $h(a+b) + h(c+d) \leq h(t)$ , it holds that

$$\frac{h(a+b) + h(c+d)}{H(a+b,c+d)} = \frac{h(g(h(a+b) + h(c+d)))}{H(a+b,c+d)} = \psi(H(a+b,c+d)).$$

Since  $\psi(c + d) = \psi(H(0, c + d))$ , the increasingness of *H* and  $\psi$  (see Remark 1 and Lemma 7) imply that  $\psi(c + d) \leq \psi(H(a + b, c + d))$  and hence

$$\psi(c+d) \leqslant \frac{h(a+b) + h(c+d)}{H(a+b, c+d)}$$

In case h(a + b) + h(c + d) > h(t), it holds that H(a + b, c + d) = t = H(t, c + d) and the increasingness of H and  $\psi$  imply again that

$$\psi(c+d) = \psi(H(0,c+d)) \leq \psi(H(t,c+d)) = \frac{h(H(t,c+d))}{H(t,c+d)} = \frac{h(t)}{H(a+b,c+d)} < \frac{h(a+b) + h(c+d)}{H(a+b,c+d)}$$

(c) The case x > 0 and y = 0 is similar to the previous one.

(d) If x > 0, y > 0, and both are such that  $h(x) + h(y) \ge h(t)$ , then (14) also trivially holds, since H(x, y) = H(a+b, c+d) = t,  $x \le a+b$ ,  $y \le c+d$  and  $\psi$  is positive. If h(x) + h(y) < h(t), then we can transform (14) into

$$\frac{x\psi(a+b) + y\psi(c+d)}{H(x,y)} \leqslant \frac{h(a+b) + h(c+d)}{H(a+b,c+d)} = \frac{(a+b)\psi(a+b) + (c+d)\psi(c+d)}{H(a+b,c+d)}.$$

It is therefore sufficient to show that the function  $G: [0, a + b] \times [0, c + d] \rightarrow [0, \infty]$  defined by

$$G(x, y) := \frac{x\psi(a+b) + y\psi(c+d)}{H(x, y)}$$
(15)

attains its maximum at (a + b, c + d). Since h(x) + h(y) < h(t), it holds that  $H(x, y) = h^{-1}(h(x) + h(y))$ . This question is identical to the one positively answered by Mulholland on a subdomain  $]0, a + b] \times ]0, c + d]$  of  $[0, \infty[^2 [13]]$ . Note that his way of proving this result initially relies on the existence of the derivative of *h*, a condition that is later on removed thanks to the other conditions on *h*, so that we can conclude that (5) holds whenever all *a*, *b*, *c*, *d* belong to [0, t[ and a + b < t, c + d < t.

(3) All of a, b, c, d belong to [0, t[ and  $a + b \ge t$  or  $c + d \ge t$ .

We first assume that a + b = t and consider the sequence  $(b_n)_{n \in \mathbb{N}}$  with  $b_n := t - a - \frac{1}{n}$ . It then holds that  $a + b_n < t$ , yet  $\lim_{n \to \infty} a + b_n = a + b = t$ . However, for any  $n \in \mathbb{N}$ , the previous case implies that

 $H(a+b_n, c+d) \leqslant H(a, c) + H(b_n, d).$ 

Since *H* is continuous in each argument, we can further conclude that

$$H(a + b, c + d) = \lim_{n \to \infty} H(a + b_n, c + d) \leq H(a, c) + \lim_{n \to \infty} H(b_n, d) = H(a, c) + H(b, d).$$

Next we assume that a + b > t. As a consequence, it holds that

H(a + b, c + d) = H(t, c + d) = H(a + (t - a), c + d) = t

and the increasingness of *H* implies that

 $H(a+b, c+d) = H(a+(t-a), c+d) \leq H(a, c) + H(t-a, d) \leq H(a, c) + H(b, d).$ 

The case  $c + d \ge t$  is completely analogous.  $\Box$ 

#### 5.2. A second sufficient condition

A careful inspection of the proof of Proposition 5 as provided in [5] shows that it can be generalized as follows.

**Lemma 8.** Consider a function  $f : ]0, \infty[ \rightarrow ]0, \infty[$  with  $\lim_{x\to 0} f(x) = 0$  and such that f is differentiable on [0, t[ with  $t \in ]0, \infty[$  and f'(x) > 0 for all  $x \in [0, t]$ . If f' is geo-convex on ]0, t[, then so is f.

Based on this result we can immediately generalize the result of Tardiff [23,24].

**Proposition 9.** Consider a function  $h: [0, \infty] \to [0, \infty]$  and some fixed value  $t \in [0, \infty]$  such that

(h1) *h* is continuous on [0, *t*];

- (h2) *h* is strictly increasing on [0, t] and  $h(x) \ge h(t)$  whenever  $x \ge t$ ;
- (h3) h(0) = 0;
- (h4) *h* is convex on ]0, *t*[;

(h6) *h* is differentiable on ]0, t[ and h' is geo-convex on ]0, t[.

Define the function  $g:[0,\infty] \to [0,\infty]$  by (8) and the function  $H:[0,\infty]^2 \to [0,\infty]$  by (9). Then the following inequality holds for all  $a, b, c, d \in [0,\infty]$ ,

 $H(a+b, c+d) \leq H(a, c) + H(b, d).$ 

#### 5.3. A necessary condition

The convexity of h on  $]0, \infty[$  is a necessary condition for the classical Mulholland inequality to hold, and as such it is part of each of the known sets of sufficient conditions. A similar observation holds for the generalized Mulholland inequality, but now the convexity of h on ]0, t[ is a necessary condition.

**Proposition 10.** Consider a function  $h : [0, \infty] \to [0, \infty]$  and some fixed value  $t \in [0, \infty]$  such that

- (h1) *h* is continuous on [0, *t*];
- (h2) *h* is strictly increasing on [0, t] and  $h(x) \ge h(t)$  whenever  $x \ge t$ ;
- (h3) h(0) = 0.

Define the function  $g : [0, \infty] \to [0, \infty]$  by (8) and the function  $H : [0, \infty]^2 \to [0, \infty]$  by (9). If H fulfills (10) for all  $a, b, c, d \in [0, \infty]$ , then h is convex on ]0, t[.

**Proof.** As the convexity of *h* on ]0, t[ is equivalent to the concavity of *g* on ]0, h(t)[, and *g* is continuous, it suffices to show that

$$g\left(\frac{x+y}{2}\right) \ge \frac{1}{2}g(x) + \frac{1}{2}g(y),$$

for all  $x, y \in [0, h(t)[$ . Choose arbitrary  $x, y \in [0, h(t)[$  such that x < y and put a = g(x),  $b = g(\frac{x+y}{2}) - g(x)$ ,  $c = g(\frac{y-x}{2})$  and d = 0. Note that in each of these cases g coincides with  $h^{-1}$  and that  $a, b, c, d \in [0, t[$ . We can therefore compute

$$h(a) + h(c) = \frac{x + y}{2},$$
  

$$h(b) + h(d) = h\left(g\left(\frac{x + y}{2}\right) - g(x)\right)$$
  

$$h(a + b) = \frac{x + y}{2},$$
  

$$h(c + d) = \frac{y - x}{2}.$$

Since *H* fulfills (10) it holds that  $H(a + b, c + d) \leq H(a, c) + H(b, d)$ , or explicitly

$$g(y) = g\left(\frac{x+y}{2} + \frac{y-x}{2}\right) \leqslant g\left(\frac{x+y}{2}\right) + g\left(\frac{x+y}{2}\right) - g(x) = 2g\left(\frac{x+y}{2}\right) - g(x). \quad \Box$$

### 6. Dominance between continuous Archimedean t-norms

Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$  and the corresponding function  $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$ . As mentioned in Section 4, if  $T_1$  and  $T_2$  are strict, then  $t_2(0) = \infty$ , h is strictly increasing,  $h^{(-1)} = h^{-1}$  and dominance between  $T_1$  and  $T_2$  is equivalent to the Mulholland inequality for h. Recall that if  $T_2$  is strict, then  $T_1 \gg T_2$  implies that  $T_1$  is strict as well. In case  $T_2$  is a nilpotent t-norm,  $T_1$  might be a strict or nilpotent t-norm and the parameters of Theorem 6 and Proposition 9 are given by:

(1) If  $T_1$  is strict, then  $h = t_1 \circ t_2^{(-1)}$ ,  $g = t_2 \circ t_1^{-1} = h^{(-1)}$ ,  $t = t_2(0)$ , and  $h(t) = \infty$ . (2) If  $T_1$  is nilpotent, then  $h = t_1 \circ t_2^{(-1)}$ ,  $g = t_2 \circ t_1^{(-1)} = h^{(-1)}$ ,  $t = t_2(0)$ , and  $h(t) = t_1(0)$ .

Note that in any case, *h* is continuous, strictly increasing on  $[0, t_2(0)]$  and fulfills h(0) = 0 as well as  $h(x) = h(t_2(0)) = t_1(0)$  for all  $x \ge t_2(0)$ . Moreover, it holds that  $H(x, y) = h^{(-1)}(h(x) + h(y))$ , in accordance with Theorem 1. As such we can rephrase Theorem 6 and Proposition 9 as well as Proposition 10 for the dominance between continuous Archimedean t-norms.

**Proposition 11.** Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . If the function  $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$  is convex and geo-convex on ]0, t[, then  $T_1$  dominates  $T_2$ .

**Proposition 12.** Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . If the function  $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$  is differentiable and convex on  $]0, t_2(0)[$ , and h' is geo-convex on  $]0, t_2(0)[$ , then  $T_1$  dominates  $T_2$ .

**Proposition 13.** Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . If  $T_1$  dominates  $T_2$ , then the function  $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$  is convex on  $]0, t_2(0)[$ .

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# DIFFERENTIAL INEQUALITY CONDITIONS FOR DOMINANCE BETWEEN CONTINUOUS ARCHIMEDEAN T-NORMS

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(communicated by N. Elezović)

*Abstract.* Dominance between triangular norms (t-norms) is a versatile relationship. For continuous Archimedean t-norms, dominance can be verified by checking one of many sufficient conditions derived from a generalization of the Mulholland inequality. These conditions pertain to various convexity properties of compositions of additive generators and their inverses. In this paper, assuming differentiability of these additive generators, we propose equivalent sufficient conditions that can be expressed as inequalities involving derivatives of the additive generators, avoiding the need of composing them. We demonstrate the powerfulness of the results by the straightforward rediscovery of dominance relationships in the Schweizer-Sklar t-norm family, as well as by unveiling some formerly unknown dominance relationships in the Sugeno-Weber t-norm family. Finally, we illustrate that the results can also be applied to members of different parametric t-norm families.

## 1. Introduction

The dominance relation was originally introduced in the framework of probabilistic metric spaces [23] and was soon abstracted to operations on a partially ordered set [21]. The dominance relation, in particular between t-norms, plays a profound role in various topics, such as the construction of Cartesian products of probabilistic metric and normed spaces [5, 21, 23], the construction of many-valued equivalence relations [2, 3, 26] and many-valued order relations [1], and in the preservation of various properties during (dis-)aggregation processes in flexible querying, preference modelling and computer-assisted assessment [2, 4, 14, 17]. These applications instigated the study of the dominance relation in the broader context of aggregation operators [12, 14, 17].

Additional to these application aspects, the dominance relation is an interesting mathematical notion *per se*. Because of the common neutral element, dominance constitutes a reflexive and antisymmetric relation on the class of t-norms. Since counterexamples for its transitivity were not readily found, it remained an intriguing open problem [8, 19, 21, 22, 25] for more than 20 years whether or not it was an order relation. Only recently the question was answered to the negative [18, 20]. However, due to its relevance in applications, it is still of interest to determine subclasses of t-norms

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on which the dominance relation establishes an order relation. Of particular importance are continuous Archimedean t-norms, as they are the basic elements of which all continuous t-norms are composed. Moreover, they can be represented by means of continuous additive generators.

It was shown in [16], see also [13, 24, 25] for earlier results dealing with strict t-norms only, that dominance between continuous Archimedean t-norms can be equivalently expressed as a functional inequality involving compositions of the additive generators (and their inverses) of the corresponding t-norms. This inequality, being a generalization of the *Minkowski inequality*, is often referred to as the *Mulholland inequality*. Although sufficient and necessary conditions for its fulfilment are already known, see [13, 16, 24, 25], and can be visualized easily for two t-norms, they have hardly ever been used for proving resp. disproving dominance between two arbitrary members of a family or families of t-norms. The aim of the present contribution is to establish easy-to-check conditions that involve directly the additive generators and their derivatives (provided they exist).

After a short introduction on t-norms, we summarize the known sufficient and necessary conditions for dominance. Subsequently, we derive new differential conditions for dominance between continuous Archimedean t-norms and demonstrate their strength by applying them to some parametric families of triangular norms leading to new results on dominance between two continuous Archimedean t-norms.

## 2. Triangular norms and the dominance relation

We briefly summarize some basic properties of t-norms for a thorough understanding of this paper (for further details see, e.g., [7, 8, 9, 10, 11, 15, 17, 18]).

DEFINITION 1. A t-norm  $T : [0,1]^2 \rightarrow [0,1]$  is a binary operation on the unit interval that is commutative, associative, increasing and has 1 as neutral element.

Well-known examples of t-norms are the *minimum*  $T_{\mathbf{M}}$ , the *product*  $T_{\mathbf{P}}$ , the *Łuka-siewicz t-norm*  $T_{\mathbf{L}}$  and the *drastic product*  $T_{\mathbf{D}}$ , defined by  $T_{\mathbf{M}}(u, v) = \min(u, v)$ ,  $T_{\mathbf{P}}(u, v) = u \cdot v$ ,  $T_{\mathbf{L}}(u, v) = \max(u + v - 1, 0)$ , and

$$T_{\mathbf{D}}(u,v) = \begin{cases} \min(u,v), & \text{if } \max(u,v) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Since t-norms are just functions from the unit square to the unit interval, their comparison is done pointwisely:  $T_1 \leq T_2$  if  $T_1(u,v) \leq T_2(u,v)$  for all  $u,v \in [0,1]$ , expressed as " $T_1$  is *weaker* than  $T_2$ " or " $T_2$  is *stronger* than  $T_1$ ". The minimum  $T_M$  is the strongest of all t-norms, the drastic product  $T_D$  is the weakest of all t-norms. Furthermore, it holds that  $T_P \geq T_L$ .

Just as triangular norms, the dominance relation finds its origin in the field of probabilistic metric spaces [21, 23]. It was originally introduced for associative operations (with common neutral element) on a partially ordered set [21], and has been further investigated for t-norms [15, 19, 20, 25] and aggregation operators [14, 17]. We state the definition for t-norms only. DEFINITION 2. Consider two t-norms  $T_1$  and  $T_2$ . We say that  $T_1$  dominates  $T_2$  (or  $T_2$  is dominated by  $T_1$ ), denoted by  $T_1 \gg T_2$ , if for all  $x, y, u, v \in [0, 1]$  it holds that

$$T_1(T_2(x,y), T_2(u,v)) \ge T_2(T_1(x,u), T_1(y,v)).$$
(1)

Note that every t-norm is dominated by itself and by  $T_M$ ; moreover, it dominates  $T_D$ . Since all t-norms have neutral element 1, dominance between two t-norms implies their comparability:  $T_1 \gg T_2$  implies  $T_1 \ge T_2$ . The converse does not hold, not even for strict t-norms [8]. Due to the induced comparability it also follows that dominance is an antisymmetric relation on the class of t-norms.

DEFINITION 3. A t-norm *T* is called *Archimedean* if for all  $u, v \in ]0, 1[$  there exists an  $n \in \mathbb{N}$  such that

$$T(\underbrace{u,\ldots,u}_{n \text{ times}}) < v.$$

DEFINITION 4.

(i) A t-norm *T* is called *strict* if it is continuous and strictly monotone, i.e., for all  $u, v, w \in [0, 1]$  it holds that

$$T(u,v) < T(u,w)$$
 whenever  $u > 0$  and  $v < w$ .

(ii) A t-norm *T* is called *nilpotent* if it is continuous and if each  $u \in [0, 1[$  is a nilpotent element of *T*, i.e., there exists some  $n \in \mathbb{N}$  such that

$$T(\underbrace{u,\ldots,u}_{n \text{ times}}) = 0$$

A continuous t-norm *T* is Archimedean if and only if for all  $u \in [0, 1[$  it holds that T(u, u) < u. The class of continuous Archimedean t-norms can be partitioned into two disjoint subclasses: the class of strict t-norms and the class of nilpotent t-norms. The product  $T_P$  is strict, whereas the Łukasiewicz t-norm  $T_L$  is nilpotent.

Note that for a strict t-norm *T* it holds that T(u,v) > 0 for all  $u, v \in ]0,1]$ , while for a nilpotent t-norm *T* it holds that for every  $u \in ]0,1[$  there exists some  $v \in ]0,1[$  such that T(u,v) = 0 (each  $u \in ]0,1[$  is a so-called *zero divisor*). Therefore, for a nilpotent t-norm  $T_1$  and a strict t-norm  $T_2$  it can never hold that  $T_1 \ge T_2$  and, as a consequence, never that  $T_1 \gg T_2$ .

Of particular interest in the discussion of continuous Archimedean t-norms and dominance between them is the notion of an *additive generator*.

DEFINITION 5. An *additive generator* of a t-norm *T* is a strictly decreasing function  $t: [0,1] \rightarrow [0,\infty]$  which is right-continuous in 0 and satisfies t(1) = 0 such that for all  $u, v \in [0,1]$  it holds that

$$T(u,v) = t^{(-1)}(t(u) + t(v))$$

with

$$t^{(-1)}(u) = t^{-1}(\min(t(0), u))$$

the pseudo-inverse of the decreasing function t.

An additive generator is uniquely determined up to a positive multiplicative constant. A t-norm T with additive generator t is continuous if and only if t is continuous. Continuous Archimedean t-norms are exactly those t-norms with a continuous additive generator. Any additive generator of a strict t-norm satisfies  $t(0) = \infty$ , while that of a nilpotent t-norm satisfies  $t(0) < \infty$ . In the case of strict t-norms, the pseudo-inverse  $t^{(-1)}$  of an additive generator t coincides with its standard inverse  $t^{-1}$ . In any case, the following relationships hold between an additive generator t and its pseudo-inverse  $t^{(-1)}$ 

$$t \circ t^{(-1)}|_{\operatorname{Ran}(t)} = \operatorname{id}_{\operatorname{Ran}(t)}$$
 and  $t^{(-1)} \circ t = \operatorname{id}_{[0,1]}$ .

# 3. The generalized Mulholland inequality and related conditions

The dominance relation between two continuous Archimedean t-norms can be expressed in terms of their generators. This was shown for strict t-norms in [25] and was generalized as follows in [16].

PROPOSITION 1. Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . Then  $T_1$  dominates  $T_2$  if and only if the function  $h = t_1 \circ t_2^{(-1)}$ :  $[0,\infty] \to [0,\infty]$  fulfills for all  $a,b,c,d \in [0,t_2(0)]$  the inequality

$$h^{(-1)}(h(a) + h(c)) + h^{(-1)}(h(b) + h(d)) \ge h^{(-1)}(h(a+b) + h(c+d)),$$
(2)

with  $h^{(-1)} = t_2 \circ t_1^{(-1)}$ :  $[0,\infty] \to [0,\infty]$  the pseudo-inverse of the increasing function h.

Since (1) is trivially fulfilled for arbitrary t-norms  $T_1$  and  $T_2$  as soon as 0 appears among the arguments, it suffices to prove that (2) holds for all  $a, b, c, d \in [0, t_2(0)]$  in order to show dominance between the continuous Archimedean t-norms considered.

In case some function  $f: [0,\infty] \to [0,\infty]$  fulfills (2) for all  $a,b,c,d \in [0,\infty]$ , we say that it fulfills the *generalized Mulholland inequality*. In [16] (see also [6, 13, 25]), sufficient and necessary conditions for the generalized Mulholland inequality to hold for a function  $f: [0,\infty] \to [0,\infty]$ , which is continuous and strictly increasing on some subdomain [0,t], with  $t \in [0,\infty[$ , and for which f(0) = 0 holds, have been investigated. Properties such as the convexity, the geometric convexity, and the logarithmic convexity of a function showed up to be most relevant.

DEFINITION 6. A function  $f: [0,\infty[ \rightarrow [0,\infty[$  is called *geometric convex* (*geoconvex* for short) on ]0,t[, with  $t \in ]0,\infty[$ , if for all  $x, y \in ]0,t[$  it holds that

$$f(\sqrt{xy}) \le \sqrt{f(x)f(y)}$$
.

It is called *logarithmic convex* (*log-convex* for short) on ]0,t[ if the function  $\log \circ f: [0,\infty[ \to [-\infty,\infty[$  is convex on ]0,t[.

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For a continuous function f such that  $f(]0,\infty[) \subseteq ]0,\infty[$ , its geo-convexity on ]0,t[ is equivalent to the convexity of the function  $\log \circ f \circ \exp$  on  $]-\infty, \log(t)[$ . Clearly, if f(0) = 0, then the geo-convexity holds also on [0,t[. Further, if f is increasing, then its log-convexity implies its geo-convexity. Moreover, the relationship between the geo-convexity of a function and that of its derivative can be expressed in the following way.

LEMMA 2. [6, 16] Consider a function  $f: [0,\infty[ \rightarrow ]0,\infty[$  with  $\lim_{x\to 0} f(x) = 0$ and such that f is differentiable on [0,t[, with  $t \in ]0,\infty[$ , and f'(x) > 0 for all  $x \in ]0,t[$ . If f' is geo-convex on ]0,t[, then so is f.

Applying these relationships and the results obtained in [16] to the dominance relation between continuous Archimedean t-norms we can state the following:

PROPOSITION 3. [16] Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$ with additive generators  $t_1$  and  $t_2$ . If  $T_1$  dominates  $T_2$ , then the function  $h = t_1 \circ t_2^{(-1)}$ :  $[0,\infty] \to [0,\infty]$  is convex on  $]0,t_2(0)[$ .

PROPOSITION 4. [16] Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . If the function  $h = t_1 \circ t_2^{(-1)}$ :  $[0,\infty] \to [0,\infty]$  is convex on  $]0,t_2(0)[$  and log- or geo-convex on  $]0,t_2(0)[$ , then h fulfills (2) for all  $a,b,c,d \in [0,t_2(0)]$ , i.e.,  $T_1$  dominates  $T_2$ .

PROPOSITION 5. [16] Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$ with additive generators  $t_1$  and  $t_2$ . If the function  $h = t_1 \circ t_2^{(-1)}$ :  $[0,\infty] \to [0,\infty]$  is differentiable and convex on  $]0,t_2(0)[$ , and h' is log- or geo-convex on  $]0,t_2(0)[$ , then h fulfills (2) for all  $a,b,c,d \in [0,t_2(0)]$ , i.e.,  $T_1$  dominates  $T_2$ .

The relationships between the above sufficient conditions for dominance are summarized in Fig. 3. Corresponding conditions for the subclass of strict t-norms have already been discussed in [25]. Although these sufficient conditions can be visualized easily, concrete proofs might become cumbersome, in particular for two members of a same parametric family, because h is a compound function. In fact, the conditions mentioned above have never been used for (dis-)proving dominance apart from one particular case: for proving dominance between members of a family of t-norms whose additive generators are powers of some basic additive generator. In this case the generalized Mulholland inequality turns into the Minkowski inequality whose solution is well known (see [8] for further details).

However, if the additive generators have derivatives of sufficiently high order, the sufficient conditions expressed as properties of h can be reformulated as equivalent (differential) conditions on the corresponding additive generators. As such we can provide localized conditions that are equivalent to the global ones and allow to (dis-)prove dominance between two continuous Archimedean t-norms.

Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . If the function

$$h = t_1 \circ t_2^{(-1)} \colon [0,\infty] \to [0,\infty]$$

is convex on  $]0, t_2(0)[$  and  $\ldots$ 

$$\begin{array}{ll} h' \text{ exists and} & h' \text{ exists and} \\ h' \text{ is log-convex on } ]0, t_2(0)[ \implies h' \text{ is geo-convex on } ]0, t_2(0)[ \\ & \downarrow \\ h \text{ is log-convex on } ]0, t_2(0)[ \implies h \text{ is geo-convex on } ]0, t_2(0)[ \\ & \downarrow \\ T_1 \gg T_2 \end{array}$$

Figure 1. Sufficient conditions for dominance between two continuous Archimedean t-norms  $T_1$  and  $T_2$ 

## 4. Differential inequality conditions

Throughout this section,  $T_1$  and  $T_2$  are two continuous Archimedean t-norms with continuous additive generators  $t_1$  and  $t_2$ . Then the function

$$h = t_1 \circ t_2^{(-1)} \colon [0,\infty] \to [0,\infty]$$

is continuous and strictly increasing on  $]0, t_2(0)[$ , h(0) = 0 and  $h(]0, t_2(0)[) \subseteq ]0, t_1(0)[$ . Further, we assume that  $t_1$  and  $t_2$  are sufficiently often (i.e., once, twice or three times) differentiable. It then holds in particular that  $t'_1(u) < 0$  and  $t'_2(u) < 0$  for all  $u \in ]0, 1[$ . For every  $x \in ]0, t_2(0)[$ , there exists a unique  $u \in ]0, 1[$  such that  $x = t_2(u)$  and  $t_2^{-1}(x) = u$ . The identity

$$\frac{d}{dx}x = \frac{d}{dx}t_2(t_2^{-1}(x)) = \frac{dt_2(u)}{du}\Big|_{u=t_2^{-1}(x)} \cdot \frac{dt_2^{-1}(x)}{dx} = 1$$

allows to express the derivatives of *h* at *x* in terms of the derivatives of  $t_1$  and  $t_2$  at  $u = t_2^{-1}(x)$ . Explicitly,

$$h(x) = t_1(t_2^{-1}(x)) = t_1(u)|_{u=t_2^{-1}(x)};$$
(3)  

$$h'(x) = \frac{d}{dx}h(x) = \frac{d}{dx}(t_1(t_2^{-1}(x))) = \frac{dt_1(u)}{du}|_{u=t_2^{-1}(x)} \cdot \frac{dt_2^{-1}(x)}{dx}$$

$$= \frac{dt_1(u)}{du} \cdot \frac{1}{\frac{dt_2(u)}{du}}|_{u=t_2^{-1}(x)} = \frac{t'_1(u)}{t'_2(u)}|_{u=t_2^{-1}(x)};$$
(4)  

$$h''(x) = \frac{d}{dx}h'(x) = \frac{d}{dx}\frac{t'_1(u)}{t'_2(u)}|_{u=t_2^{-1}(x)} = \frac{d}{du}\frac{t'_1(u)}{t'_2(u)}|_{u=t_2^{-1}(x)} \cdot \frac{dt_2^{-1}(x)}{dx}$$

$$= \frac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^2(u)} \cdot \frac{1}{\frac{dt_2(u)}{du}} \bigg|_{u = t_2^{-1}(x)} = \frac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^3(u)} \bigg|_{u = t_2^{-1}(x)}.$$
 (5)

Let us now turn to the convexity, the logarithmic and the geometric convexity of h and its derivative.

**PROPOSITION 6.** The function h is convex on  $]0,t_2(0)[$ , i.e.,

$$h''(x) \ge 0 \tag{6}$$

for all  $x \in (0, t_2(0))$ , if and only if

$$t_1'(u)t_2''(u) - t_1''(u)t_2'(u) \ge 0$$
<sup>(7)</sup>

for all  $u \in [0, 1[$ .

*Proof.* Since h''(x) can be expressed by (5) and  $t'_2(u) < 0$  for all  $u \in ]0,1[$ , it follows immediately that

$$\forall x \in ]0, t_2(0)[:h''(x) \ge 0 \quad \Leftrightarrow \quad \forall u \in ]0, 1[:t_1'(u)t_2''(u) - t_1''(u)t_2'(u) \ge 0.$$

**PROPOSITION 7.** The function h is log-convex on  $]0,t_2(0)[$ , i.e.,

$$h(x)h''(x) - h'^{2}(x) \ge 0$$
(8)

for all  $x \in (0, t_2(0))$ , if and only if

$$t_1'^2(u)t_2'(u) + t_1(u)(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)) \ge 0$$
(9)

*for all*  $u \in (0, 1)$ *.* 

*Proof.* The function *h* is log-convex on  $]0, t_2(0)[$  if and only if

$$(\log \circ h)''(x) = \frac{h(x)h''(x) - {h'}^2(x)}{h^2(x)} \ge 0$$

for all  $x \in [0, t_2(0))$ . Since h(x) > 0 for all x > 0, we can write equivalently

$$h(x)h''(x) - {h'}^2(x) \ge 0$$

for all  $x \in [0, t_2(0)]$ . Using (3)–(5), the latter turns out to be equivalent to

$$t_1(u)\frac{t_1''(u)t_2'(u)-t_2''(u)t_1'(u)}{t_2'^{3}(u)}-\frac{t_1'^{2}(u)}{t_2'^{2}(u)} \ge 0$$

or also

$$t_1'^2(u)t_2'(u) + t_1(u)\left(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)\right) \ge 0$$

for all  $u \in (0, 1)$ .

**PROPOSITION 8.** The function h is geo-convex on  $]0, t_2(0)[$ , i.e.,

$$h(x)h'(x) + x\left(h(x)h''(x) - {h'}^2(x)\right) \ge 0$$
(10)

for all  $x \in (0, t_2(0))$ , if and only if

$$\frac{t_1'^2(u) - t_1(u)t_1''(u)}{t_1(u)t_1'(u)} \ge \frac{t_2'^2(u) - t_2(u)t_2''(u)}{t_2(u)t_2'(u)}$$
(11)

*for all*  $u \in (0, 1)$ *.* 

*Proof.* First, we show that the geometric convexity of *h* on  $]0, t_2(0)[$  is equivalent to Eq. (10) for all  $x \in ]0, t_2(0)[$ . The geometric convexity of *h* on  $]0, t_2(0)[$  is equivalent to the convexity of the function  $\chi = \log \circ h \circ \exp: [-\infty, \log(t_2(0))] \rightarrow [-\infty, \log(t_1(0))]$  on  $]-\infty, \log(t_2(0))[$ . Since *h* is twice differentiable, also  $\chi$  is twice differentiable and

$$\begin{split} \chi'(v) &= \frac{h'(e^{v})}{h(e^{v})} e^{v}; \\ \chi''(v) &= \left(\frac{h'(e^{v})}{h(e^{v})} + e^{v} \frac{h(e^{v})h''(e^{v}) - h'(e^{v})^{2}}{h(e^{v})^{2}}\right) e^{v} \\ &= \frac{e^{v}}{h(e^{v})^{2}} \left(h'(e^{v})h(e^{v}) + e^{v}(h(e^{v})h''(e^{v}) - h'(e^{v})^{2})\right). \end{split}$$

Since always  $\frac{e^{v}}{h(e^{v})^{2}} > 0$ ,  $\chi''(v) \ge 0$  is equivalent to  $h'(e^{v})h(e^{v}) + e^{v}(h(e^{v})h''(e^{v}) - h'(e^{v})^{2}) \ge 0$ , or, replacing  $e^{v}$  by x, to

$$h(x)h'(x) + x(h(x)h''(x) - h'^{2}(x)) \ge 0.$$

Using Eqs. (3)–(5), the validity of Eq. (10) for all  $x \in ]0, t_2(0)[$  turns out to be equivalent to

$$t_1(u) \cdot \frac{t_1'(u)}{t_2'(u)} + t_2(u) \left( t_1(u) \cdot \frac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^{3}(u)} - \frac{t_1'^{2}(u)}{t_2'^{2}(u)} \right) \ge 0,$$

or also

$$\frac{t_1'^2(u) - t_1(u)t_1''(u)}{t_1(u)t_1'(u)} \ge \frac{t_2'^2(u) - t_2(u)t_2''(u)}{t_2(u)t_2'(u)}$$

for all  $u \in (0, 1)$ .

REMARK 9. Investigating the differential formulations of the convexity, log-convexity and geo-convexity of h, it becomes evident that the log-convexity of h implies its convexity as well as its geo-convexity. Indeed, if h is log-convex, i.e.

$$t_1'^2(u)t_2'(u) + t_1(u)(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)) \ge 0,$$

it follows that

t

$$t_1(u)(t_1'(u)t_2''(u)-t_1''(u)t_2'(u)) \ge -t_1'^2(u)t_2'(u)\ge 0,$$

since  $t'_{2}(u) < 0$  for all  $u \in [0,1[$ . As  $t_{1}(u) > 0$  for all  $u \in [0,1[$ , it must hold that  $t'_{1}(u)t''_{2}(u) - t''_{1}(u)t'_{2}(u) \ge 0$  for all  $u \in [0,1[$ , i.e., *h* is convex.

Again assume that *h* is log-convex, i.e.  $h(x)h''(x) - {h'}^2(x) \ge 0$  for all  $x \in [0, t_2(0)[$ . Since for any such *x* it holds that *x*, h(x) and h'(x) are positive, it also holds that

$$h(x)h'(x) + x(h(x)h''(x) - h'^{2}(x)) \ge 0$$

for all  $x \in [0, t_2(0)[$ , i.e. *h* is geo-convex on  $[0, t_2(0)[$ .

Similarly as for Eqs. (3)–(5), for all  $x \in ]0, t_2(0)[$ , the third derivative of *h* at *x* can be expressed as

$$\begin{split} h'''(x) &= \frac{d}{dx}h''(x) = \frac{d}{du} \left( \frac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^3(u)} \right) \Big|_{u=t_2^{-1}(x)} \cdot \frac{dt_2^{-1}(x)}{dx} \\ &= \frac{t_2'^3(u)(t_1'''(u)t_2'(u) - t_2'''(u)t_1'(u)) - 3t_2'^2(u)t_2''(u)(t_1''(u)t_2'(u) - t_2''(u)t_1'(u))}{t_2'^6(u)} \cdot \frac{1}{\frac{dt_2(u)}{du}} \Big|_{u=t_2^{-1}(x)} \\ &= \frac{1}{t_2'^5(u)} \left( 3t_1'(u)t_2''^2(u) - t_1'(u)t_2'(u)t_2'''(u) - 3t_1''(u)t_2'(u)t_2''(u) + t_1'''(u)t_2'^2(u) \right) \Big|_{u=t_2^{-1}(x)} \end{split}$$

Substitution in the corresponding formulas and reshuffling the inequalities leads to the following corollaries which we state without their easy but tedious and cumbersome proofs.

COROLLARY 10. The function 
$$h'$$
 is log-convex on  $]0, t_2(0)[$ , i.e.,  
 $h'(x)h'''(x) - h''^2(x) \ge 0$ 
(12)

for all  $x \in [0, t_2(0)]$ , if and only if

$$t_{1}^{\prime 2}(u)\left(2t_{2}^{\prime \prime 2}(u)-t_{2}^{\prime}(u)t_{2}^{\prime \prime \prime}(u)\right) \geq t_{2}^{\prime 2}(u)\left(t_{1}^{\prime \prime 2}(u)-t_{1}^{\prime}(u)t_{1}^{\prime \prime \prime}(u)\right)+t_{1}^{\prime}(u)t_{1}^{\prime \prime}(u)t_{2}^{\prime \prime}(u)t_{2}^{\prime \prime}(u)$$
(13)

*for all*  $u \in (0, 1)$ .

COROLLARY 11. The function h' is geo-convex on 
$$]0,t_2(0)[$$
, i.e.,  
 $h'(x)h''(x) + x(h'(x)h'''(x) - h''^2(x)) \ge 0$  (14)

for all  $x \in (0, t_2(0))$ , if and only if

$$t_{2}(u)\left(t_{1}'(u)t_{2}'(u)\left(t_{1}'''(u)t_{2}'(u)-t_{2}'''(u)t_{1}'(u)\right)\right)$$
$$-\left(t_{1}''(u)t_{2}'(u)-t_{2}''(u)t_{1}'(u)\right)\left(2t_{1}'(u)t_{2}''(u)+t_{1}''(u)t_{2}'(u)\right)\right)$$
$$\geq t_{1}'(u)t_{2}'^{2}(u)\left(t_{1}'(u)t_{2}''(u)-t_{1}''(u)t_{2}'(u)\right)$$
(15)

for all  $u \in [0, 1[$ .

## 5. Dominance within a single parametric family of t-norms

Although the differential inequality conditions look cumbersome at first sight, they often reduce to easy-to-check inequalities when applied to members of parametric families of t-norms, as we will demonstrate in this and the following section. First, we consider the family of Schweizer-Sklar t-norms. Although it is known [22] that dominance within this family is accordance with the ordering of the parameters, we provide an alternative (and easier) proof based on the new differential inequality conditions in order to illustrate their strength. Second, we examine dominance within the family of Sugeno-Weber t-norms, leading to relationships not yet established so far, since most of its members are nilpotent t-norms. We tackle these problems by following the scheme of sufficient conditions displayed in Fig. 3. We will provide the differential inequality for the necessary convexity of h as well as the differential inequality corresponding to the strongest sufficient condition leading to the discovery of a dominance relationship.

## 5.1. The family of Schweizer-Sklar t-norms

The family of Schweizer-Sklar t-norms  $(T_{\lambda}^{SS})_{\lambda \in [-\infty,\infty]}$  is given by

$$T_{\lambda}^{\mathbf{SS}}(u,v) = \begin{cases} T_{\mathbf{M}}(u,v), & \text{if } \lambda = -\infty, \\ T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{D}}(u,v), & \text{if } \lambda = \infty, \\ \max(u^{\lambda} + v^{\lambda} - 1, 0)^{1/\lambda}, & \text{if } \lambda \in \left] -\infty, 0[\cup] 0, \infty\right[ \end{cases}$$

For  $\lambda \in ]-\infty,\infty[$ ,  $T_{\lambda}^{SS}$  is a continuous Archimedean t-norm with additive generator

$$t_{\lambda}^{\mathbf{SS}}(u) = \frac{1-u^{\lambda}}{\lambda}, \text{ if } \lambda \in ]-\infty, 0[\cup]0, \infty[, \text{ and } t_0^{\mathbf{SS}}(u) = -\log u, \text{ if } \lambda = 0,$$

for all  $u \in [0,1]$ ; parameters  $\lambda \in ]-\infty,0]$  lead to strict t-norms, while parameters  $\lambda \in ]0,\infty[$  lead to nilpotent t-norms.

In the sequel of this section, we omit the superscript indicating the family when discussing properties of additive generators. Since we deal with Schweizer-Sklar t-norms only, no ambiguity can occur.

Clearly, the derivatives of the additive generators exist and are given, for all  $\lambda \in ]-\infty,\infty[$  and all  $u \in ]0,1[$ , by:

$$\begin{split} t'_{\lambda}(u) &= -u^{\lambda-1}, \\ t''_{\lambda}(u) &= -(\lambda-1)u^{\lambda-2}, \\ t'''_{\lambda}(u) &= -(\lambda-1)(\lambda-2)u^{\lambda-3}. \end{split}$$

The family of Schweizer-Sklar t-norms is ordered according to its parameter:  $T_{\lambda}^{SS} \ge T_{\mu}^{SS}$  if and only if  $\lambda \le \mu$ . Moreover, since  $T_{M}$  dominates every t-norm, and every t-norm dominates itself as well as  $T_{D}$ , it suffices to investigate dominance between two Schweizer-Sklar t-norms  $T_{\lambda}^{SS}$  and  $T_{\mu}^{SS}$  with parameters  $-\infty < \lambda < \mu < \infty$ .

Note that the function  $h = t_{\lambda} \circ t_{\mu}^{(-1)}$ :  $[0, \infty] \to [0, \infty]$  is continuous, strictly increasing and differentiable on  $]0, t_{\mu}(0)[$  and fulfills h(0) = 0. If h is convex on  $]0, t_{\mu}(0)[$  and if either h or h' is log- or geo-convex on  $]0, t_{\mu}(0)[$ , then  $T_{\lambda}^{SS}$  dominates  $T_{\mu}^{SS}$ .

**Convexity of** *h*. The function *h* is convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in ]0, 1[$ ,

$$\begin{split} t'_{\lambda}(u)t''_{\mu}(u) - t''_{\lambda}(u)t'_{\mu}(u) &\geq 0 \quad \Leftrightarrow \\ (\mu - 1)u^{\lambda + \mu - 3} - (\lambda - 1)u^{\lambda + \mu - 3} &\geq 0 \quad \Leftrightarrow \\ (\mu - \lambda)u^{\lambda + \mu - 3} &\geq 0 \quad \Leftrightarrow \\ \mu &\geq \lambda \, . \end{split}$$

**Geo-convexity of** h'. Substituting the expressions for the derivatives of the additive generators in (15) shows that the function h' is geo-convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in ]0, 1[$ ,

$$\begin{split} t_{\mu}(u) \Big( u^{\lambda+\mu-2} \big( (\lambda-1)(\lambda-2)u^{\lambda+\mu-4} - (\mu-1)(\mu-2)u^{\lambda+\mu-4} \big) \\ &- \big( (\lambda-1)u^{\lambda+\mu-3} - (\mu-1)u^{\lambda+\mu-3} \big) \big( 2(\mu-1)u^{\lambda+\mu-3} + (\lambda-1)u^{\lambda+\mu-3} \big) \Big) \\ &\geq -u^{\lambda+2\mu-3} \left( (\mu-1)u^{\lambda+\mu-3} - (\lambda-1)u^{\lambda+\mu-3} \right), \end{split}$$

with rearrangements and simple calculations leading to

$$t_{\mu}(u)\mu(\mu-\lambda) \geq -u^{\mu}(\mu-\lambda).$$

In case  $\mu = 0$ , the latter condition reduces to  $0 \ge \lambda$ , or, equivalently,  $\mu \ge \lambda$ . In case  $\mu \ne 0$ , the condition reads explicitly

$$\begin{aligned} & (\frac{1-u^{\mu}}{\mu})\mu(\mu-\lambda) \geq -u^{\mu}(\mu-\lambda) & \Leftrightarrow \\ & (\mu-\lambda)(1-u^{\mu}+u^{\mu}) \geq 0 & \Leftrightarrow \\ & \mu \geq \lambda \, . \end{aligned}$$

Hence, neither the convexity of h nor the geo-convexity of h' imposes further restrictions on  $\lambda$  and  $\mu$ .

COROLLARY 12. Consider the family of Schweizer-Sklar t-norms  $(T_{\lambda}^{SS})_{\lambda \in [-\infty,\infty]}$ . For all  $\lambda, \mu \in [-\infty,\infty]$  it holds that  $T_{\lambda}^{SS}$  dominates  $T_{\mu}^{SS}$  if and only if  $\lambda \geq \mu$ .

We stress that this result is obtained here much more economically than in [22].

## 5.2. The family of Sugeno-Weber t-norms

The second family we consider is the family of Sugeno-Weber t-norms. Two arguments support its consideration: first, dominance relationships between two members of this family have not yet been laid bare; second, it is of particular interest as all but two of its menbers are nilpotent t-norms.

The family of Sugeno-Weber t-norms  $(T_{\lambda}^{SW})_{\lambda \in [0,\infty]}$  is given by

$$T_{\lambda}^{\mathbf{SW}}(u,v) = \begin{cases} T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{D}}(u,v), & \text{if } \lambda = \infty, \\ \max(0,(1-\lambda)uv + \lambda(u+v-1)), & \text{if } \lambda \in \left]0,\infty\right[. \end{cases}$$

For  $\lambda \in [0, \infty[, T_{\lambda}^{SW}]$  is a continuous Archimedean t-norm with additive generator

$$t_{\lambda}^{\mathbf{SW}}(u) = \begin{cases} -(1-\lambda)\log(\lambda+(1-\lambda)u), & \text{if } \lambda \in [0,\infty[\setminus\{1\}, \\ 1-u, & \text{if } \lambda = 1, \end{cases} \end{cases}$$

for all  $u \in [0,1]$ ; parameters  $\lambda \in ]0,\infty[$  lead to nilpotent t-norms (with  $T_1^{SW} = T_L$  as special case), while  $T_0^{SW} = T_P$  is the only strict member. Note that, for better readability, we again omit the superscript indicating the family when discussing properties of additive generators.

Clearly, the derivatives of the additive generators exist and are, for all  $\lambda \in [0,\infty[ \setminus$  $\{1\}$  and all  $u \in [0, 1[$ , given by:

$$t'_{\lambda}(u) = -\frac{(1-\lambda)^2}{\lambda + (1-\lambda)u},$$
  
$$t''_{\lambda}(u) = \frac{(1-\lambda)^3}{(\lambda + (1-\lambda)u)^2},$$
  
$$t'''_{\lambda}(u) = -\frac{2(1-\lambda)^4}{(\lambda + (1-\lambda)u)^3};$$

in case  $\lambda = 1$ , it holds that  $t'_1(u) = -1$  and  $t''_1(u) = t'''_1(u) = 0$  for all  $u \in ]0,1[$ . The family of Sugeno-Weber t-norms is ordered according to its parameter:  $T_{\lambda}^{SW} \ge 1$  $T_{\mu}^{SW}$  if and only if  $\lambda \leq \mu$ . Moreover, since every t-norm dominates itself as well as  $T_{\mathbf{D}}$ , it suffices to investigate dominance between two Sugeno-Weber t-norms  $T_{\lambda}^{SW}$  and  $T_{\mu}^{SW}$ with parameters  $0 < \lambda < \mu < \infty$ .

Note that the function  $h = t_{\lambda} \circ t_{\mu}^{(-1)}$ :  $[0, \infty] \to [0, \infty]$  is continuous, strictly increasing and differentiable on  $]0, t_{\mu}(0)[$  and fulfills h(0) = 0. If h is convex on  $]0, t_{\mu}(0)[$  and if either h or h' is log- or geo-convex on  $]0, t_{\mu}(0)[$ , then  $T_{\lambda}^{SW}$  dominates  $T_{\mu}^{SW}$ .

**Convexity of** *h*. The function *h* is convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in [0, 1[$ ,

$$t'_{\lambda}(u)t''_{\mu}(u) - t''_{\lambda}(u)t'_{\mu}(u) \ge 0$$

In case  $\lambda \neq 1 \neq \mu$ , the latter inequality is equivalent to

$$\frac{(1-\lambda)^3}{(\lambda+(1-\lambda)u)^2}\frac{(1-\mu)^2}{\mu+(1-\mu)u} \ge \frac{(1-\lambda)^2}{\lambda+(1-\lambda)u}\frac{(1-\mu)^3}{(\mu+(1-\mu)u)^2} \quad \Leftrightarrow \quad$$

$$\frac{1-\lambda}{\lambda+(1-\lambda)u} \ge \frac{1-\mu}{\mu+(1-\mu)u} \qquad \Leftrightarrow \\ (1-\lambda)(\mu+(1-\mu)u) \ge (1-\mu)(\lambda+(1-\lambda)u) \qquad \Leftrightarrow \\ \mu \ge \lambda \,.$$

In case  $\lambda = 1$ , the condition reduces to  $-t''_{\mu}(u) \ge 0$  being equivalent to  $\mu \ge 1 = \lambda$ . In case  $\mu = 1$ , the condition becomes  $t''_{\lambda}(u) \ge 0$ , i.e.,  $\lambda \le 1 = \mu$ . Summarizing, in all cases *h* is convex if and only if  $\mu \ge \lambda$ .

**Log-convexity of** *h'*. Substituting the expressions for the derivatives of the additive generators in (13) and applying basic transformations shows that for all  $\lambda \neq 1 \neq \mu$  the function *h'* is log-convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in ]0, 1[$ ,

$$\frac{(1-\lambda)^5(1-\mu)^4(\mu-\lambda)}{(\lambda+(1-\lambda)u)^4(\mu+(1-\mu)u)^3} \ge 0 \quad \Leftrightarrow \quad (\mu-\lambda)(1-\lambda) \ge 0.$$

This inequality clearly holds whenever  $\lambda < 1$  and  $\mu > \lambda$ . In case  $\lambda = 1 < \mu$ , we obtain in a similar way the condition

$$2t_{\mu}^{\prime\prime 2}(u) - t_{\mu}^{\prime}(u)t_{\mu}^{\prime\prime\prime}(u) = 2\frac{(1-\mu)^{6}}{(\mu+(1-\mu)u)^{4}} - \frac{2(1-\mu)^{6}}{(\mu+(1-\mu)u)^{4}} \ge 0$$

which trivially holds. Finally, in case  $\lambda < \mu = 1$ , we end up with the following equivalent inequality

$$t_{\lambda}^{\prime\prime 2}(u) - t_{\lambda}^{\prime}(u)t_{\lambda}^{\prime\prime\prime}(u) = -\frac{(1-\lambda)^{6}}{(\lambda+(1-\lambda)u)^{4}} \leq 0,$$

which is also obviously fulfilled.

The above results can be summarized as follows.

COROLLARY 13. Consider the family of Sugeno-Weber t-norms  $(T_{\lambda}^{SW})_{\lambda \in [0,\infty]}$ . For all  $\lambda, \mu \in [0,\infty]$  such that

$$\lambda \leq \min(1,\mu)$$

it holds that  $T_{\lambda}^{SW} \gg T_{\mu}^{SW}$ .

This means in particular that any Sugeno-Weber t-norm greater than or equal to the Łukasiewicz t-norm dominates any other, but smaller Sugeno-Weber t-norm. Naturally, this raises the question whether dominance is also in accordance with the ordering of the parameters when both t-norms are smaller than the Łukasiewicz t-norm, i.e, when  $1 < \lambda < \mu$ . However, in general this need not be the case as the following example demonstrates.

EXAMPLE 1. Consider the Sugeno-Weber t-norms  $T_{51}^{SW}$  and  $T_{101}^{SW}$  and let  $x = y = u = v = \frac{975}{1000}$ . Then  $T_{51}^{SW}(x,x) = \frac{147}{160}$  and  $T_{101}^{SW}(x,x) = \frac{142}{160}$  such that

$$T_{51}^{\mathbf{SW}}(T_{101}^{\mathbf{SW}}(x,x),T_{101}^{\mathbf{SW}}(x,x)) = \frac{182}{1280} < \frac{227}{1280} = T_{101}^{\mathbf{SW}}(T_{51}^{\mathbf{SW}}(x,x),T_{51}^{\mathbf{SW}}(x,x)),$$

showing that  $T_{51}^{SW}$  does not dominate  $T_{101}^{SW}$ , although  $\lambda = 51 \le 101 = \mu$ .

So far, we have only exploited the log-convexity of h'. Of course, the remaining sufficient conditions can still be applied. We provide them in two forms: first, after substituting the expressions for the derivatives of the additive generators, and second, in their simplest form after applying several transformations. Further, we discuss the case  $1 < \lambda < \mu$  only in order to gain additional insight into dominance between two Sugeno-Weber t-norms.

**Geo-convexity of** *h'*. The function *h'* is geo-convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in ]0, 1[$ ,

$$\begin{aligned} &-\frac{(1-\lambda)^5(1-\mu)^5(\mu-\lambda)\log(\mu+(1-\mu)u)}{(\lambda+(1-\lambda)u)^4(\mu+(1-\mu)u)^3} \geq -\frac{(1-\lambda)^4(1-\mu)^6(\mu-\lambda)}{(\lambda+(1-\lambda)u)^3(\mu+(1-\mu)u)^4} &\Leftrightarrow \\ &(\mu-\lambda)(1-\lambda)\left(\frac{\mu+(1-\mu)u}{1-\mu}\log(\mu+(1-\mu)u)-\frac{\lambda+(1-\lambda)u}{1-\lambda}\right) \leq 0. \end{aligned}$$

In case  $1 < \lambda < \mu$ , we need to show that, for all  $u \in [0, 1[$ ,

$$\frac{\mu + (1-\mu)u}{\mu - 1} \log(\mu + (1-\mu)u) \le \frac{\lambda + (1-\lambda)u}{\lambda - 1}.$$
(16)

Note that for all  $u \in [0, 1]$ , it holds that the function  $f_u: [1, \infty[ \rightarrow ]0, \infty[, f_u(t) = \frac{t+(1-t)u}{t-1}$  is decreasing, since  $\frac{df_u}{dt}(t) = -\frac{1}{(t-1)^2} < 0$ . Therefore, for  $\mu \ge \lambda$  it holds that

$$\frac{\mu + (1-\mu)u}{\mu - 1} \le \frac{\lambda + (1-\lambda)u}{\lambda - 1}.$$

Hence, as long as the factor  $\log(\mu + (1 - \mu)u)$ , which is always positive for  $\mu > 1$ , is upper bounded by 1, also (16) follows. This requires that  $\mu + (1 - \mu)u \le e$  for all  $u \in [0, 1[$ , i.e.  $\mu \le e$ . We conclude that h' is geo-convex at least when  $1 < \lambda < \mu \le e$ .

**Log-convexity of** *h*. The function *h* is log-convex on  $]0,t_{\mu}(0)[$  if and only if, for all  $u \in ]0,1[$ ,

$$\begin{aligned} &-\frac{(1-\lambda)^4(1-\mu)^2}{(\lambda+(1-\lambda)u)^2(\mu+(1-\mu)u)} \geq \frac{(1-\lambda)^3(\mu-\lambda)(1-\mu)^2\log(\lambda+(1-\lambda)u)}{(\lambda+(1-\lambda)u)^2(\mu+(1-\mu)u)^2} & \Leftrightarrow \\ &\mu+(1-\mu)u+\frac{\mu-\lambda}{1-\lambda}\log(\lambda+(1-\lambda)u)\leq 0\,. \end{aligned}$$

As u approaches 1, the left-hand side approaches 1 as well. Therefore, h can never be log-convex.

**Geo-convexity of** *h*. The function *h* is geo-convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in ]0, 1[$ ,

$$\frac{(\lambda-1)(\log(\lambda+(1-\lambda)u)+1)}{(\lambda+(1-\lambda)u)\log(\lambda+(1-\lambda)u)} \le \frac{(\mu-1)(\log(\mu+(1-\mu)u)+1)}{(\mu+(1-\mu)u)\log(\mu+(1-\mu)u)}.$$
(17)

In case  $1 < \lambda < \mu$ , we consider the function  $g_u$ :  $]1, \infty[ \rightarrow ]0, \infty[$ ,

$$g_u(t) = \frac{(t-1)(\log(t+(1-t)u)+1)}{(t+(1-t)u)\log(t+(1-t)u)},$$
which is increasing whenever

$$\frac{dg_u}{dt}(t) = \frac{\log^2(t+(1-t)u) + \log(t+(1-t)u) + (t-1)(u-1)}{\left((t+(1-t)u)\log(t+(1-t)u)\right)^2}$$

is positive for all  $t \in ]1, \infty[$ . Note that for t > 1 it holds that  $\log(t + (1 - t)u) > 0$  for all  $u \in ]0, 1[$ , and hence,  $\frac{dg_u}{dt}(t)$  is positive whenever

$$p(t) = \log^2(t) + \log(t) - t + 1 \ge 0$$

Numerical investigations (using Maple) show that this is the case for  $1 \le t \le 6.00914$  (with 6.00914 denoting the second root of p(t) = 0). Therefore, *h* is geo-convex at least when  $1 < \lambda < \mu \le 6.00914$ .

Of course, this does not contradict the findings on the geo-convexity of h'. Interestingly, the geo-convexity investigation allows us to extend Corollary 13.

COROLLARY 14. Consider the family of Sugeno-Weber t-norms  $(T_{\lambda}^{SW})_{\lambda \in [0,\infty]}$ . For all  $\lambda, \mu \in [0,\infty]$  such that

(*i*) either  $\lambda \leq \min(1, \mu)$ ,

(*ii*) or 
$$1 < \lambda \le \mu \le 6.00914$$

it holds that  $T_{\lambda}^{SW} \gg T_{\mu}^{SW}$ .

Having in mind the geo-convexity study of h, it is intuitively clear that as  $\lambda$  approaches 1 from the right in (17), even larger values of  $\mu$  will do (knowing that for  $\lambda = 1$ ,  $\mu$  can be arbitrarily large). However, this problem becomes numerically intractable.

#### 6. Dominance between two parametric families of t-norms

Finally, we turn to the investigation of dominance between a member of the family of Dombi t-norms and a member of the family of Yager t-norms. Since (apart from the limit cases) the Dombi t-norms are strict and the Yager t-norms are nilpotent, it suffices to investigate when a Dombi t-norm dominates a Yager t-norm. The investigation of such a mixed case (strict versus nilpotent) is possible thanks to the new conditions applicable to all continuous Archimedean t-norms.

The family of Dombi t-norms  $(T_{\lambda}^{\mathbf{D}})_{\lambda \in [0,\infty]}$  is given by

$$T_{\lambda}^{\mathbf{D}}(u,v) = \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{M}}(u,v), & \text{if } \lambda = \infty, \\ \frac{1}{1 + \left(\left(\frac{1-u}{u}\right)^{\lambda} + \left(\frac{1-v}{v}\right)^{\lambda}\right)^{1/\lambda}}, & \text{if } \lambda \in \left]0, \infty\right[.$$

For  $\lambda \in ]0,\infty[$ ,  $T_{\lambda}^{\mathbf{D}}$  is a strict t-norm with generator  $t_{\lambda}^{\mathbf{D}}(u) = (\frac{1-u}{u})^{\lambda}$  for all  $u \in [0,1]$ . The derivatives of the additive generators are, for all  $\lambda \in ]0,\infty[$  and all  $u \in [0,1]$ , given by:

$$(t_{\lambda}^{\mathbf{D}})'(u) = -\frac{\lambda(1-u)^{\lambda-1}}{u^{\lambda+1}},$$

$$(t_{\lambda}^{\mathbf{D}})''(u) = (\lambda + 1 - 2u) \frac{\lambda(1-u)^{\lambda-2}}{u^{\lambda+2}}.$$

Similarly, the family of Yager t-norms  $(T^{\mathbf{Y}}_{\mu})_{\mu \in [0,\infty]}$  is defined by

$$T^{\mathbf{Y}}_{\mu}(u,v) = \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \mu = 0, \\ T_{\mathbf{M}}(u,v), & \text{if } \mu = \infty, \\ \max(1 - ((1-u)^{\mu} + (1-v)^{\mu})^{1/\mu}, 0), & \text{if } \mu \in ]0, \infty[. \end{cases}$$

For  $\mu \in ]0,\infty[$ ,  $T_{\mu}^{\mathbf{Y}}$  is a nilpotent t-norm with additive generator  $t_{\mu}^{\mathbf{Y}}(u) = (1-u)^{\mu}$  for all  $u \in [0,1]$ . The derivatives of the additive generators are, for all  $\mu \in ]0,\infty[$  and all  $u \in ]0,1[$ , given by:

$$(t_{\mu}^{\mathbf{Y}})'(u) = -\mu(1-u)^{\mu-1},$$
  
$$(t_{\mu}^{\mathbf{Y}})''(u) = \mu(\mu-1)(1-u)^{\mu-2}.$$

Note that for both families it holds that the additive generators of the continuous Archimedean members are powers of the basic additive generators  $t_1^{\mathbf{B}}(u) = \frac{1-u}{u}$ and  $t_1^{\mathbf{Y}}(u) = 1 - u$ . Investigating dominance within each of these families then turns the Mulholland inequality into the Minkowski inequality and dominance within each family is in accordance with the ordering of the parameters (see also [8]), i.e.,

$$T_{\lambda_1}^{\mathbf{D}} \gg T_{\lambda_2}^{\mathbf{D}} \quad \Leftrightarrow \quad \lambda_1 \ge \lambda_2 \qquad \text{and} \qquad T_{\mu_1}^{\mathbf{Y}} \gg T_{\mu_2}^{\mathbf{Y}} \quad \Leftrightarrow \quad \mu_1 \ge \mu_2.$$

We will now investigate for which  $\lambda$  and  $\mu$  it holds that the Dombi t-norm  $T_{\lambda}^{\mathbf{D}}$  dominates the Yager t-norm  $T_{\mu}^{\mathbf{Y}}$ . Since for both families the limiting members are  $T_{\mathbf{D}}$  and  $T_{\mathbf{M}}$ , it suffices to consider  $\lambda, \mu \in ]0, \infty[$  only. Note that the function  $h = t_{\lambda}^{\mathbf{D}} \circ (t_{\mu}^{\mathbf{Y}})^{(-1)}$ :  $[0, \infty] \to [0, \infty]$  is continuous, strictly increasing and differentiable on  $]0, t_{\mu}^{\mathbf{Y}}(0)[$  and fulfills h(0) = 0. If h is convex on  $]0, t_{\mu}^{\mathbf{Y}}(0)[$  and if either h or h' is log- or geo-convex on  $]0, t_{\mu}^{\mathbf{Y}}(0)[$ , then  $T_{\lambda}^{\mathbf{D}}$  dominates  $T_{\mu}^{\mathbf{Y}}$ . For the sake of brevity we will further omit the indication of the families;  $\lambda$  and  $\mu$  therefore not only indicate the specific parameter but also the corresponding family.

**Convexity of** *h*. The function *h* is convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in ]0, 1[$ ,

$$\begin{split} t'_{\lambda}(u)t''_{\mu}(u) - t''_{\lambda}(u)t'_{\mu}(u) \geq 0 & \Leftrightarrow \\ -\lambda\mu(\mu-1)\frac{(1-u)^{\lambda+\mu-3}}{u^{\lambda+1}} + (\lambda+1-2u)\lambda\mu\frac{(1-u)^{\lambda+\mu-3}}{u^{\lambda+2}} \geq 0, & \Leftrightarrow \\ \lambda\mu\frac{(1-u)^{\lambda+\mu-3}}{u^{\lambda+2}}\left(\lambda+1-u(\mu+1)\right) \geq 0, & \Leftrightarrow \\ \lambda+1 \geq u(\mu+1). \end{split}$$

This inequality is fulfilled for all  $u \in ]0,1[$  if and only if  $\lambda \ge \mu$ .

**Geo-convexity of** *h*. The function *h* is geo-convex on  $]0, t_{\mu}(0)[$  if and only if, for all  $u \in ]0, 1[$ ,

$$\frac{t_{\lambda}'^{2}(u) - t_{\lambda}(u)t_{\lambda}''(u)}{t_{\lambda}(u)t_{\lambda}'(u)} \geq \frac{t_{\mu}'^{2}(u) - t_{\mu}(u)t_{\mu}''(u)}{t_{\mu}(u)t_{\mu}'(u)}$$

being equivalent to

$$\begin{aligned} (\lambda - (\lambda + 1 - 2u)) \frac{1}{u(u-1)} &\geq \frac{(\mu - (\mu - 1))}{(u-1)} &\Leftrightarrow \\ \frac{2u-1}{u(u-1)} &\geq \frac{1}{(u-1)} &\Leftrightarrow \\ u &\leq 1, \end{aligned}$$

which obviously is fulfilled for all  $u \in [0, 1[$ .

COROLLARY 15. Consider the families of Dombi t-norms  $(T_{\lambda}^{\mathbf{D}})_{\lambda \in [0,\infty]}$  and of Yager t-norms  $(T_{\mu}^{\mathbf{Y}})_{\mu \in [0,\infty]}$ . For all  $\lambda, \mu \in [0,\infty]$  it holds that  $T_{\lambda}^{\mathbf{D}}$  dominates  $T_{\mu}^{\mathbf{Y}}$  if and only if  $\lambda \geq \mu$ .

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# The dominance relation in some families of continuous Archimedean t-norms and copulas

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#### Abstract

The dominance relation in several families of continuous Archimedean tnorms and copulas is investigated. On the one hand, the contribution provides a comprehensive overview on recent conditions and properties of dominance as well as known results for particular cases of families. On the other hand, it contains new results clarifying the dominance relationship in five additional families of continuous Archimedean t-norms and copulas. *Key words:* Dominance, triangular norms, copulas

#### 1. Introduction

The dominance relation had originally been introduced for triangle functions in the framework of probabilistic metric spaces [47], but was soon abstracted to operations on a partially ordered set [43]. It plays an important role in constructing Cartesian products of probabilistic metric and normed spaces (see [24, 43, 47], but also [39] for more recent results on dominance

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between triangle functions resp. operations on distance distribution functions). Dominance, especially between t-norms and copulas, is further crucial in the construction of many-valued equivalence relations [7, 8, 50] and many-valued order relations [4] as well as in the preservation of various properties, most of them expressed by some inequalities, during (dis-)aggregation processes in flexible querying, preference modelling and computer-assisted assessment [7, 11, 32, 37]. These applications initiated the study of the dominance relation in the broader context of aggregation functions [26, 32, 37].

Besides these application points of view, dominance has been and is still an interesting mathematical notion. E.g., because of the common neutral element of t-norms and their commutativity and associativity, dominance constitutes a reflexive and antisymmetric relation on the class of all t-norms. Whether the relation is also transitive was of interest already since 1983 (see also [43]). It has been answered recently to the negative by Sarkoci [41] (see also [38]) by means of ordinal sum t-norms based on the product or Lukasiewicz t-norm. Meaning that the counter examples have been found in the class of continuous t-norms which form an important subclass of all t-norms.

Obtaining a negative answer has, to some extent, been surprising, since the study of dominance within families of t-norms has been of interest since its very beginnings. Several particular families of t-norms, containing also subfamilies of copulas, had been investigated (see, e.g., [19, 33, 38, 40, 44]) and supported the conjecture that the dominance relation would indeed be transitive, either due to its rare occurrence within the family considered or due to its abundant occurrence. Therefore and due to its relevance in applications, it is still of interest to determine whether on some subclasses

of t-norms dominance constitutes a transitive and as such an order relation. Particularly interesting are families containing continuous Archimedean tnorms which in its turn most often contain families of Archimedean copulas as subclasses. Many such single-parametric families of t-norms and copulas are listed in the Table 2.6 in the book on associative functions by Alsina et al. [2], overlapping to a great extent with the families of Archimedean copulas contained in Table 4.2 in the book on copulas by Nelsen [30].

The aim of the present contribution is to provide results on dominance for several of these families. We pursue this goal in two steps — on the one hand by providing a comprehensive survey on those families for which the dominance relation is already clarified, and on the other hand by proving new results on dominance for five additional families.

Note that, in this contribution, we restrict to dominance among members of a single parametric family of t-norms. For results comparing members of two different families, see, e.g., [34, 36].

The article is organized as follows: In Section 2 some necessary basics on t-norms and Archimedean copulas are summarized. Section 3 contains basic properties and relationships on dominance, in particular dominance among continuous Archimedean t-norms. Section 4 covers the survey on results on dominance known for some of the families contained in [2, 30]. Finally, we present new results on dominance for five additional families of t-norms and copulas. We will close the contribution by a short summary.

#### 2. Triangular norms and copulas

We briefly summarize some basic properties of t-norms and copulas for a thorough understanding of this paper (for further details see, e.g., [2, 18, 19, 20, 21, 22, 30, 33, 37, 38, 45]).

**Definition 1.** A t-norm  $T: [0,1]^2 \rightarrow [0,1]$  is a binary operation on the unit interval which is commutative, associative, increasing and has neutral element 1.

Well-known examples of t-norms are the minimum  $T_{\mathbf{M}}$ , the product  $T_{\mathbf{P}}$ , the Lukasiewicz t-norm  $T_{\mathbf{L}}$  and the drastic product  $T_{\mathbf{D}}$ , defined by  $T_{\mathbf{M}}(u, v) = \min(u, v)$ ,  $T_{\mathbf{P}}(u, v) = u \cdot v$ ,  $T_{\mathbf{L}}(u, v) = \max(u + v - 1, 0)$ , and

$$T_{\mathbf{D}}(u,v) = \begin{cases} \min(u,v), & \text{if } \max(u,v) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

T-norms are compared pointwisely:  $T_1 \leq T_2$  if  $T_1(u, v) \leq T_2(u, v)$  for all  $u, v \in [0, 1]$ , expressing that " $T_1$  is weaker than  $T_2$ " or " $T_2$  is stronger than  $T_1$ ". The minimum  $T_{\mathbf{M}}$  is the strongest of all t-norms, the drastic product  $T_{\mathbf{D}}$  is the weakest of all t-norms.

#### **Definition 2.** A t-norm T is called

(i) Archimedean if for all  $u, v \in [0, 1[$  there exists an  $n \in \mathbb{N}$  such that

$$T(\underbrace{u,\ldots,u}_{n \ times}) < v \, .$$

(ii) A t-norm T is called strict if it is continuous and strictly monotone, i.e., for all  $u, v, w \in [0, 1]$  it holds that

 $T(u,v) < T(u,w) \quad \text{whenever} \quad u > 0 \ \text{and} \ v < w \,.$ 

(iii) A t-norm T is called nilpotent if it is continuous and if each  $u \in [0, 1[$ is a nilpotent element of T, i.e., there exists some  $n \in \mathbb{N}$  such that

$$T(\underbrace{u,\ldots,u}_{n \ times}) = 0.$$

Of particular interest in the discussion of continuous Archimedean tnorms is the notion of an *additive generator*.

**Definition 3.** An additive generator of a t-norm T is a strictly decreasing function  $t: [0,1] \rightarrow [0,\infty]$  which is right-continuous in 0 and satisfies t(1) = 0such that for all  $u, v \in [0,1]$  it holds that  $T(u,v) = t^{(-1)}(t(u) + t(v))$  with  $t^{(-1)}(u) = t^{-1}(\min(t(0), u))$  the pseudo-inverse of the decreasing function t.

An additive generator is uniquely determined up to a positive multiplicative constant. A t-norm T with additive generator t is continuous if and only if t is continuous. Continuous Archimedean t-norms are exactly those t-norms with a continuous additive generator. Any additive generator of a strict t-norm satisfies  $t(0) = \infty$ , while that of a nilpotent t-norm satisfies  $t(0) < \infty$ . In the case of strict t-norms, the pseudo-inverse  $t^{(-1)}$  of an additive generator t coincides with its standard inverse  $t^{-1}$ .

**Definition 4.** A (bivariate) copula  $C: [0,1]^2 \to [0,1]$  is a binary operation on the unit interval which has neutral element 1 and annihilator 0 and which is 2-increasing, i.e., for all u, u', v and v' in [0,1] with  $u \le u'$  and  $v \le v'$ 

$$\Delta_{u,u'}^{v,v'}(C) := C(u',v') - C(u',v) - C(u,v') + C(u,v) \ge 0.$$

The expression  $\Delta_{u,u'}^{v,v'}(C)$  is called the C-volume of the rectangle  $[u, u'] \times [v, v']$ .

It follows immediately from the definition that every copula C is increasing in each argument, and that it satisfies the *Lipschitz condition*, i.e., for all u, u', v and v' in [0, 1],

$$|C(u',v') - C(u,v)| \le |u' - u| + |v' - v|.$$
(1)

Note that copulas need not be associative or commutative. However, some t-norms are also copulas and vice versa. More precisely, every associative copula is a continuous t-norm [19] and, on the other hand, a t-norm is a copula if and only if it fulfills the Lipschitz condition [27].

The importance of copulas in applications comes from *Sklar's theorem* [45], which allows to represent every bivariate probability distribution function  $F: \mathbb{R}^2 \to [0,1]$  by  $F(x,y) = C(F_1(x), F_2(y))$ , where  $F_1$  and  $F_2$  are the upper margins of F, obtained as limits of  $F(x_1, x_2)$  when  $x_i$  tends to  $+\infty$  for i = 1, 2, and C is a copula. The representation is unique whenever F is a continuous bivariate probability distribution. Note that a copula can also be seen as a bivariate distribution function whose upper margins are uniformly distributed on [0, 1].

Several methods for constructing copulas based on different principles and/or respecting additional properties are already known (see [30] for an overview, but also [6, 9, 10, 12, 23, 31]). An important subclass are *Archimedean* copulas which are closely related to continuous Archimedean t-norms.

**Proposition 1.** [27, 42] Consider a continuous Archimedean t-norm T with additive generator t. T is a copula if and only if t is a convex function.

Accordingly, the definition of Archimedean copulas reads as follows (see also [30]):

**Definition 5.** Consider a continuous, convex, additive generator  $\varphi$ , i.e., a continuous, convex, strictly decreasing function  $\varphi \colon [0,1] \to [0,\infty]$  with  $\varphi(1) = 0$ . Then a copula  $C \colon [0,1]^2 \to [0,1]$  defined by  $C(u,v) = \varphi^{(-1)}(\varphi(u) + \varphi(v))$  is called an Archimedean copula.

It is immediate that several families of continuous Archimedean t-norms generated by a parameterized family of additive generators contain families of Archimedean copulas as soon as the respective additive generators are convex. As such, in some cases, different names for the families of (continuous Archimedean) t-norm and the corresponding subfamilies of Archimedean copulas can be found in the literature.

#### 3. Dominance — basic properties

The dominance relation has its roots in the field of probabilistic metric spaces [43, 47]. It was originally introduced for associative operations (with common neutral element) on a partially ordered set [43], and has been further investigated for t-norms [33, 40, 41, 49] and aggregation functions [32, 37]. We state the definition for t-norms only, for copulas, it is defined accordingly.

**Definition 6.** Consider two t-norms  $T_1$  and  $T_2$ . We say that  $T_1$  dominates  $T_2$  (or  $T_2$  is dominated by  $T_1$ ), denoted by  $T_1 \gg T_2$ , if, for all  $x, y, u, v \in [0, 1]$ , it holds that

$$T_1(T_2(x,y), T_2(u,v)) \ge T_2(T_1(x,u), T_1(y,v)).$$
(2)

Note that every t-norm and every copula is dominated by  $T_{\mathbf{M}}$ . Moreover, every t-norm dominates itself and  $T_{\mathbf{D}}$ . Since all t-norms and copulas have

neutral element 1, dominance between two t-norms resp. two copulas implies their comparability:  $T_1 \gg T_2$  implies  $T_1 \ge T_2$ . The converse does not hold, not even for strict t-norms [19]. Due to the induced comparability it also follows that dominance is an antisymmetric relation on the class of t-norms and the class of copulas. Associativity and symmetry ensure that dominance is also reflexive on the class of t-norms.

# 3.1. Dominance between continuous Archimedean t-norms resp. Archimedean copulas

It was shown in [35] (see also [28, 48, 49] for earlier results dealing with strict t-norms only) that dominance between continuous Archimedean tnorms can be equivalently expressed as a functional inequality involving compositions of the additive generators (and their inverses) of the corresponding t-norms.

**Proposition 2.** [35] Consider two continuous Archimedean t-norms  $T_1$  and  $T_2$  with additive generators  $t_1$  and  $t_2$ . Then  $T_1$  dominates  $T_2$  if and only if the function  $h: [0, \infty] \rightarrow [0, \infty], h = t_1 \circ t_2^{(-1)}$  fulfills, for all  $a, b, c, d \in [0, t_2(0)]$ , the inequality

$$h^{(-1)}(h(a) + h(c)) + h^{(-1)}(h(b) + h(d)) \ge h^{(-1)}(h(a+b) + h(c+d)),$$
 (3)

with  $h^{(-1)}$ :  $[0,\infty] \to [0,\infty]$ ,  $h^{(-1)} = t_2 \circ t_1^{(-1)}$ , the pseudo-inverse of the increasing function h. Note that Eq. (3) is referred to as the generalized Mulholland inequality

Since Eq. (2) is trivially fulfilled for arbitrary t-norms  $T_1$  and  $T_2$  as soon as 0 appears among the arguments, it suffices to prove that Eq. (3) holds for

all  $a, b, c, d \in [0, t_2(0)]$  in order to show dominance between the continuous Archimedean t-norms considered.

In [35] (see also [17, 28, 49]), sufficient and necessary conditions for the generalized Mulholland inequality to hold for a function  $f: [0, \infty] \to [0, \infty]$ , which is continuous and strictly increasing on some subdomain [0, t], with  $t \in [0, \infty[$ , and for which f(0) = 0 holds, have been investigated. Properties such as the *convexity*, the *geometric convexity*, and the *logarithmic convexity* of a function showed up to be most relevant. We do not discuss these properties in detail but provide, if necessary, corresponding equivalent formulations which will be relevant for later proofs. More detailed investigations can be found in [28, 35, 36, 48, 49].

Note that for two continuous Archimedean t-norms  $T_1$  and  $T_2$  with continuous additive generators  $t_1$  and  $t_2$ , the function  $h: [0, \infty] \to [0, \infty]$ ,  $h = t_1 \circ t_2^{(-1)}$  is also continuous and strictly increasing on  $]0, t_2(0)[$ . Moreover, h(0) = 0 and  $h(]0, t_2(0)[) \subseteq ]0, t_1(0)[$ . Further, we will assume, if necessary, that  $t_1$  and  $t_2$  are sufficiently often (i.e., once, twice or three times) differentiable such that  $t'_1(u) < 0$  and  $t'_2(u) < 0$  for all  $u \in ]0, 1[$ , and that, for every  $x \in ]0, t_2(0)[$ , there exists a unique  $u \in ]0, 1[$  such that  $x = t_2(u)$  and  $t_2^{-1}(x) = u$ .

Summarizing the results on sufficient and necessary conditions for the fulfillment of the generalized Mulholland inequality and applying the corresponding differential conditions involving the additive generators, as obtained in [35, 36], we can state the following about the dominance relation between continuous Archimedean t-norms:

**Proposition 3.** [35, 36] Consider two continuous Archimedean t-norms  $T_1$ 

and  $T_2$  with twice differentiable additive generators  $t_1$  and  $t_2$ . If  $T_1$  dominates  $T_2$ , then the function  $h: [0, \infty] \to [0, \infty]$ ,  $h = t_1 \circ t_2^{(-1)}$  is convex on  $]0, t_2(0)[$ , i.e.,  $h''(x) \ge 0$  for all  $x \in ]0, t_2(0)[$ , or equivalently, for all  $u \in ]0, 1[$ ,

$$t_1'(u)t_2''(u) - t_1''(u)t_2'(u) \ge 0.$$
(4)

**Proposition 4.** [35, 36] Consider two continuous Archimedean t-norms  $T_1$ and  $T_2$  with twice differentiable additive generators  $t_1$  and  $t_2$ .

If the function  $h: [0, \infty] \to [0, \infty], h = t_1 \circ t_2^{(-1)}$  is convex on  $]0, t_2(0)[$ and if either

• h is log-convex on  $]0, t_2(0)[$ , i.e., h fulfills  $h(x)h''(x) - h'^2(x) \ge 0$  for all  $x \in ]0, t_2(0)[$  or equivalently, for all  $u \in ]0, 1[$ ,

$$t_1'^2(u)t_2'(u) + t_1(u)\big(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)\big) \ge 0,$$
(5)

or

•  $h \text{ is geo-convex on } ]0, t_2(0)[, \text{ i.e., } h \text{ fulfills } h(x)h'(x) + x (h(x)h''(x) - h'^2(x)) \ge 0 \text{ for all } x \in ]0, t_2(0)[ \text{ or equivalently, for all } u \in ]0, 1[,$ 

$$\frac{t_1'^2(u) - t_1(u)t_1''(u)}{t_1(u)t_1'(u)} \ge \frac{t_2'^2(u) - t_2(u)t_2''(u)}{t_2(u)t_2'(u)},\tag{6}$$

then  $T_1$  dominates  $T_2$ 

**Proposition 5.** [35, 36] Consider two continuous Archimedean t-norms  $T_1$ and  $T_2$  with three times differentiable additive generators  $t_1$  and  $t_2$ .

If the function  $h: [0, \infty] \to [0, \infty]$ ,  $h = t_1 \circ t_2^{(-1)}$  is differentiable and convex on  $[0, t_2(0)]$ , and if either

 h' is log-convex on ]0, t<sub>2</sub>(0)[, i.e., h fulfills h'(x)h'''(x) - h''<sup>2</sup>(x) ≥ 0 for all x ∈ ]0, t<sub>2</sub>(0)[, or equivalently, for all u ∈ ]0, 1[,

$$t_{1}^{\prime 2}(u)\left(2t_{2}^{\prime \prime 2}(u) - t_{2}^{\prime}(u)t_{2}^{\prime \prime \prime}(u)\right) \geq t_{2}^{\prime 2}(u)\left(t_{1}^{\prime \prime 2}(u) - t_{1}^{\prime}(u)t_{1}^{\prime \prime \prime}(u)\right) + t_{1}^{\prime}(u)t_{1}^{\prime \prime}(u)t_{2}^{\prime}(u)t_{2}^{\prime \prime}(u), \quad (7)$$

• or h' is geo-convex on  $]0, t_2(0)[$ , i.e., h fulfills  $h'(x)h''(x) + x(h'(x)h'''(x) - h''^2(x)) \ge 0$  for all  $x \in ]0, t_2(0)[$  or equivalently, for all  $u \in ]0, 1[$ ,

$$t_{2}(u)\Big(t_{1}'(u)t_{2}'(u)\big(t_{1}'''(u)t_{2}'(u) - t_{2}'''(u)t_{1}'(u)\big) \\ - \big(t_{1}''(u)t_{2}'(u) - t_{2}''(u)t_{1}'(u)\big)\big(2t_{1}'(u)t_{2}''(u) + t_{1}''(u)t_{2}'(u)\big)\Big) \\ \ge t_{1}'(u)t_{2}'^{2}(u)\left(t_{1}'(u)t_{2}''(u) - t_{1}''(u)t_{2}'(u)\right), \quad (8)$$

#### then $T_1$ dominates $T_2$ .

We will now turn, first, to a brief survey on dominance in families including continuous Archimedean t-norms resp. Archimedean copulas. These families are all included in the before mentioned Table 2.6. in the book on associative functions by Alsina et al. [2] as well as in Table 4.2 in the book on copulas by Nelsen [30]. Their properties are discussed in the before mentioned books at several places, but also in the book on triangular norms by Klement et al. [19]. After this survey we will turn to new results on dominance in other families of t-norms resp. copulas as introduced in [2, 30].

#### 4. Dominance in families of t-norms and copulas — a survey

#### 4.1. The family of Schweizer-Sklar t-norms resp. Clayton copulas

The family of Schweizer-Sklar t-norms  $(T_{\lambda}^{\mathbf{SS}})_{\lambda \in [-\infty,\infty]}$  is given by

$$T_{\lambda}^{\mathbf{SS}}(u,v) = \begin{cases} T_{\mathbf{M}}(u,v), & \text{if } \lambda = -\infty, \\ \\ T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ \\ T_{\mathbf{D}}(u,v), & \text{if } \lambda = \infty, \\ \\ \\ \max(u^{\lambda} + v^{\lambda} - 1, 0)^{1/\lambda}, & \text{otherwise.} \end{cases}$$

The family members are continuous Archimedean t-norms for  $\lambda \in ]-\infty, \infty[$ and copulas for  $\lambda \in [-\infty, 1]$ . The copulas in this family are also known as *Clayton copulas* and have been investigated in [5, 15]. The family members form a decreasing sequence of t-norms resp. copulas with respect to their parameter, i.e.,  $T_{\lambda}^{SS} \geq T_{\mu}^{SS}$  if and only if  $\lambda \leq \mu$ .

The same holds for the dominance relationship, i.e.,  $T_{\lambda}^{SS} \gg T_{\mu}^{SS}$  if and only if  $\lambda \leq \mu$ , as was proven first by Sherwood [44] in 1984 invoking general proof techniques not referring to particular properties of the dominance relation between t-norms. For quite some time, his results on the family of Schweizer-Sklar t-norms remained the only one for dominance in a family of parameterized t-norms. Note that a different proof for this particular family, now involving the generalized Mulholland inequality (3), can be found in [36] (see also [34]). Summarizing, we can say that the dominance relation yields a linear order on the set of Schweizer-Sklar t-norms resp. on the set of Clayton copulas.

## 4.2. The family of Yager t-norms, of Dombi t-norms, and of Aczél-Alsina t-norms resp. Gumbel-Hougaard copulas

The family of Yager t-norms  $(T_{\lambda}^{\mathbf{Y}})_{\lambda \in [0,\infty]}$ , of Dombi t-norms  $(T_{\lambda}^{\mathbf{D}})_{\lambda \in [0,\infty]}$ , and of Aczél-Alsina t-norms  $(T_{\lambda}^{\mathbf{AA}})_{\lambda \in [0,\infty]}$  have in common that they are all generated by powers of some basic additive generator. Their definitions and additive generators are, for all  $\lambda \in [0,\infty]$  resp.  $\lambda \in [0,\infty]$ , given as follows:

$$\begin{split} T_{\lambda}^{\mathbf{Y}}(u,v) &= \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{M}}(u,v), & \text{if } \lambda = \infty, \\ \max\left(1 - ((1-x)^{\lambda} + (1-y)^{\lambda})^{1/\lambda}, 0\right), & \text{otherwise}, \end{cases} \\ t_{\lambda}^{\mathbf{Y}}(u) &= (1-u)^{\lambda}, \text{ for } \lambda \in \left]0, \infty\right[, \end{cases} \\ T_{\mathbf{M}}(u,v), & \text{ if } \lambda = 0, \\ T_{\mathbf{M}}(u,v), & \text{ if } \lambda = \infty, \\ \frac{1}{1+\left(\left(\frac{1-u}{u}\right)^{\lambda} + \left(\frac{1-v}{v}\right)^{\lambda}\right)^{1/\lambda}, & \text{ otherwise}, \end{cases} \\ t_{\lambda}^{\mathbf{D}}(u) &= \left\{ \begin{aligned} T_{\mathbf{D}}(u,v), & \text{ if } \lambda = \infty, \\ \frac{1}{1+\left(\left(\frac{1-u}{u}\right)^{\lambda}, \text{ for } \lambda \in \left]0, \infty\right[, \end{cases} \\ T_{\lambda}^{\mathbf{AA}}(u,v) &= \begin{cases} T_{\mathbf{D}}(u,v), & \text{ if } \lambda = 0, \\ T_{\mathbf{M}}(u,v), & \text{ if } \lambda = 0, \\ T_{\mathbf{M}}(u,v), & \text{ if } \lambda = \infty, \\ e^{-((-\log u)^{\lambda} + (-\log v)^{\lambda})^{1/\lambda}} & \text{ otherwise}, \end{cases} \\ t_{\lambda}^{\mathbf{AA}}(u) &= (-\log(u))^{\lambda}, \text{ for } \lambda \in \left]0, \infty\right[. \end{split}$$

The members of the respective families are continuous Archimedean t-norms for  $\lambda \in [0, \infty[$  and copulas for  $\lambda \in [1, \infty]$ . As mentioned in [30] and according to [16] the subfamily of copulas of the family of Aczél-Alsina t-norms is

called the family of *Gumbel-Hougaard copulas*. The family members form an increasing sequence of t-norms resp. copulas, i.e.,  $T_{\lambda} \geq T_{\mu}$  if and only if  $\lambda \geq \mu$ .

As discussed in [19], the standard Minkowski inequality,

$$(a^{p} + c^{p})^{1/p} + (b^{p} + d^{p})^{1/p} \ge ((a+b)^{p} + (c+d)^{p})^{1/p}$$

being true for all  $p \in [1, \infty[$ , can be applied for proving dominance between two members of a family generated by positive powers of a basic additive generators, as it is the case for the family of Yager, Dombi, and Aczél-Alsina t-norms, respectively. In fact, applying the (generalized) Mulholland inequality (3) to two members  $T_{\lambda}$  and  $T_{\mu}$  of any of these families yields the Minkowski inequality with the parameter p equal to  $\mu/\lambda$ . Therefore, we can state the following:

**Proposition 6.** [19] Consider two members  $T_{\lambda}$  and  $T_{\mu}$ ,  $\lambda, \mu \in [0, \infty]$ , of either the family of Yager t-norms  $(T_{\lambda}^{\mathbf{Y}})_{\lambda \in [0,\infty]}$ , of Dombi t-norms  $(T_{\lambda}^{\mathbf{D}})_{\lambda \in [0,\infty]}$ , or Aczél-Alsina t-norms  $(T_{\lambda}^{\mathbf{AA}})_{\lambda \in [0,\infty]}$ . Then  $T_{\lambda}$  dominates  $T_{\mu}$  if and only if  $\lambda \geq \mu$ . The dominance relation constitutes a linear order on each of these families of t-norms resp. copulas.

For results on dominance between members of two different families we refer to [34, 36].

## 4.3. The family of Frank t-norms resp. copulas and of Hamacher t-norms resp. Ali-Mikhail-Haq copulas

The families of Frank  $(T_{\lambda}^{\mathbf{F}})_{\lambda \in [-\infty,\infty]}$  resp. Hamacher t-norms  $(T_{\mu}^{\mathbf{H}})_{\mu \in [-\infty,1]}$ are defined by

$$T_{\lambda}^{\mathbf{F}}(u,v) = \begin{cases} T_{\mathbf{L}}(u,v), & \text{if } \lambda = -\infty, \\ T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{M}}(u,v), & \text{if } \lambda = \infty, \\ -\frac{1}{\lambda} \log \left( 1 + \frac{(e^{-\lambda u} - 1)(e^{-\lambda v} - 1)}{e^{-\lambda} - 1} \right), & \text{otherwise}, \end{cases}$$
$$T_{\mu}^{\mathbf{H}}(u,v) = \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \mu = -\infty, \\ 0, & \text{if } \mu = 1, u = v = 0, \\ \frac{uv}{1 - \mu(1 - u)(1 - v)}, & \text{otherwise}. \end{cases}$$

The family members are continuous Archimedean t-norms for  $\lambda \in [-\infty, \infty[$ resp.  $\mu \in ]-\infty, 1]$ . The Frank t-norms are all also copulas and have been discussed in [13, 14, 29]. The Hamacher t-norms are copulas in case  $\mu \in [-1, 1]$ and they are also called *Ali-Mikhail-Haq copulas* [1]. The members of each family form an increasing sequence of t-norms resp. copulas with respect to their parameter, i.e.,  $T_{\lambda}^{\mathbf{F}} \geq T_{\mu}^{\mathbf{F}}$  resp.  $T_{\lambda}^{\mathbf{H}} \geq T_{\mu}^{\mathbf{H}}$  if and only if  $\lambda \geq \mu$ .

It has been shown by Sarkoci in [40] that very rarely a dominance relationship appears among members of each of the families:

**Proposition 7.** [40] Consider two members of the family of Frank t-norms  $T_{\lambda_1}^{\mathbf{F}}$ ,  $T_{\lambda_2}^{\mathbf{F}}$ . Then  $T_{\lambda_1}^{\mathbf{F}}$  dominates  $T_{\lambda_2}^{\mathbf{F}}$  if and only if one of the following cases holds:  $\lambda_1 = \infty$ , i.e.,  $T_{\lambda_1}^{\mathbf{F}} = T_{\mathbf{M}}$ ,  $\lambda_1 = \lambda_2$ , or  $\lambda_2 = -\infty$ , i.e.,  $T_{\lambda_2}^{\mathbf{F}} = T_{\mathbf{L}}$ .

Consider two members  $T_{\mu_1}^{\mathbf{H}}$ ,  $T_{\mu_2}^{\mathbf{H}}$  of the family of Hamacher t-norms. Then  $T_{\mu_1}^{\mathbf{H}}$  dominates  $T_{\mu_2}^{\mathbf{H}}$  if and only if one of the following cases holds:  $\mu_1 = 1$ ,  $\mu_1 = \mu_2$ , or  $\mu_2 = -\infty$ , i.e.,  $T_{\mu_2}^{\mathbf{H}} = T_{\mathbf{D}}$ .

Note that, because of the rare occurrence of dominance in these families, transitivity is fulfilled and the dominance relation therefore an order relation on the corresponding families of t-norms resp. copulas.

#### 4.4. The family of Sugeno-Weber t-norms

The family of Sugeno-Weber t-norms  $(T_{\lambda}^{\mathbf{SW}})_{\lambda \in [0,\infty]}$  is defined by

$$T^{\mathbf{SW}}_{\lambda}(u,v) = \begin{cases} T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ \\ T_{\mathbf{D}}(u,v), & \text{if } \lambda = \infty, \\ \\ \max(0,(1-\lambda)uv + \lambda(u+v-1)), & \text{if } \lambda \in \left]0,\infty\right[. \end{cases}$$

For  $\lambda \in [0, \infty[$ , the Sugeno-Weber t-norms are continuous Archimedean tnorms [25, 46, 51], for  $\lambda \in [0, 1]$  they are also copulas. The members of the family form a decreasing sequence of t-norms resp. copulas with respect to their parameter, i.e.,  $T_{\lambda}^{\mathbf{SW}} \geq T_{\mu}^{\mathbf{SW}}$  if and only if  $\lambda \leq \mu$ . The results on dominance among the family members, providing sufficient conditions, have been investigated in [36] and are based on properties related to the generalized Mulholland inequality (3).

**Proposition 8.** [36] Consider the family of Sugeno-Weber t-norms  $(T_{\lambda}^{SW})_{\lambda \in [0,\infty]}$ . For all  $\lambda, \mu \in [0,\infty]$  such that

- (i) either  $\lambda \leq \min(1, \mu)$ ,
- (ii) or  $1 < \lambda \leq \mu \leq t^*$ , with  $t^* = 6.00914$  denoting the second root of  $\log^2(t) + \log(t) t + 1 = 0$ ,

it holds that  $T_{\lambda}^{\mathbf{SW}} \gg T_{\mu}^{\mathbf{SW}}$ . On the other hand, if  $T_{\lambda}^{\mathbf{SW}} \gg T_{\mu}^{\mathbf{SW}}$ , then  $\lambda \leq \mu$ .

Note that the members of the family of Sugeno-Weber t-norms are copulas whenever their parameter is less or equal to 1. Therefore, for any copula members  $T_{\lambda}^{\mathbf{SW}}$ ,  $T_{\mu}^{\mathbf{SW}}$  of the family, i.e.,  $\lambda, \mu \leq 1$ , it holds that  $T_{\lambda}^{\mathbf{SW}}$  dominates  $T_{\mu}^{\mathbf{SW}}$  if and only if  $\lambda \leq \mu$ .

#### 5. Dominance in families of t-norms and copulas — new results

We will now turn to new results on dominance for other families of continuous Archimedean t-norms resp. Archimedean copulas. All of these families have been introduced and discussed in the book on associative functions by Alsina et al. [2] as well as in the book on copulas by Nelsen [30]. Unfortunately, not all of them have been named, moreover, the numberings of the families listed in Table 2.6 in [2] and in Table 4.2 in [30] slightly differ. In the sequel we will therefore stick to the following notation scheme: A family of t-norms (and copulas)  $(T^{\mathbf{x}}_{\lambda})_{\lambda \in I}$  refers to the family 2.6.x of t-norms (and copulas) as listed in Table 2.6 in [2] with a given parameter range I.

# 5.1. The family $(T^{\mathbf{8}}_{\lambda})_{\lambda \in [0,\infty]}$

The family of t-norms  $(T_{\lambda}^{\mathbf{8}})_{\lambda \in [0,\infty]}$  and, for  $\lambda \in [0,\infty]$ , the additive generators of its members are given by

$$T_{\lambda}^{\mathbf{8}}(u,v) = \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \lambda = 0, \\\\ \max\left(0, \frac{uv\lambda^2 - (1-u)(1-v)}{\lambda^2 - (1-u)(1-v)(\lambda-1)^2}\right), & \text{if } \lambda \in \left]0, \infty\right[ \\\\ \frac{uv}{u+v-uv}, & \text{if } \lambda = \infty, \end{cases}$$

$$t_{\lambda}(u) = \begin{cases} \frac{1-u}{1+u(\lambda-1)}, & \text{if } \lambda \in \left]0, \infty\right[\\ \frac{1-u}{u}, & \text{if } \lambda = \infty. \end{cases}$$

These t-norms are continuous Archimedean for  $\lambda \in [0, \infty]$ . For  $\lambda \in [1, \infty]$  they are also copulas (see also [2]). Moreover, the family is increasing w.r.t. its parameter [2], i.e.,  $T_{\lambda}^{\mathbf{8}} \geq T_{\mu}^{\mathbf{8}}$  if and only if  $\lambda \geq \mu$ .

Let us now investigate dominance between two members of this family of t-norms. Since every t-norm T dominates  $T_{\mathbf{D}}$ , we can restrict to parameters  $\lambda \in [0, \infty]$  only. In this case its additive generators are continuous and three times differentiable. The derivatives can be computed, for  $\lambda \in [0, \infty]$ , by

$$t'_{\lambda}(u) = \frac{-\lambda}{(1+u(\lambda-1))^2}, \quad t''_{\lambda}(u) = \frac{2\lambda(\lambda-1)}{(1+u(\lambda-1))^3}, \quad t'''_{\lambda}(u) = \frac{-6\lambda(\lambda-1)^2}{(1+u(\lambda-1))^4}$$

and, for  $\lambda = \infty$ , by

$$t'_{\infty}(u) = -\frac{1}{u^2}, \quad t''_{\infty}(u) = \frac{2}{u^3}, \quad t'''_{\infty}(u) = -\frac{6}{u^4}$$

**Proposition 9.** Consider  $\lambda, \mu \in [0, \infty]$ . Then  $T_{\lambda}^{\mathbf{8}}$  dominates  $T_{\mu}^{\mathbf{8}}$ , if and only if  $\lambda \geq \mu$ .

*Proof.* Assume arbitrary, but fixed  $\lambda, \mu \in [0, \infty]$ . If  $T_{\lambda}^{\mathbf{8}}$  dominates  $T_{\mu}^{\mathbf{8}}$ , then, due to the common neutral element,  $T_{\lambda}^{\mathbf{8}} \geq T_{\mu}^{\mathbf{8}}$  being equivalent to  $\lambda \geq \mu$ .

For proving sufficiency, assume that  $\lambda \geq \mu$ . Since in case of equality and for  $\mu = 0$  dominance is trivially fulfilled, we additionally assume that  $\lambda > \mu > 0$  and restrict first to the case of  $\lambda \neq \infty$ . Following Proposition 5, it suffices to show the convexity of h and the log-convexity of h' on ]0,1[ for proving dominance.

Because of Proposition 3, the convexity of h is equivalent to Eq. (4) which reduces to

$$\frac{2\lambda\mu(\lambda-\mu)}{(1+u(\lambda-1))^3(1+u(\mu-1))^3} \ge 0$$

for all  $u \in [0, 1[$ . Since  $\lambda, \mu > 0$  and  $(1 + u(\lambda - 1)), (1 + u(\mu - 1)) > 0$ , this is equivalent to  $\lambda \ge \mu$  which is obviously true.

The log-convexity of h' is equivalent to Eq. (7) for the additive generators involved and in turn equivalent to the following expression

$$\begin{aligned} \frac{2\lambda^2\mu^2(\mu-1)^2}{(1+u(\lambda-1))^4(1+u(\mu-1))^6} \geq \frac{2\lambda^2\mu^2(\lambda-1)}{(1+u(\lambda-1))^6(1+u(\mu-1))^5} \\ & \left(2(\mu-1)(1+u(\lambda-1))-(\lambda-1)(1+u(\mu-1))\right)\end{aligned}$$

for all  $u \in [0, 1[$ . We introduce the abbreviations  $a = (1 + u(\lambda - 1))$  and  $b = (1 + u(\mu - 1))$ . Note that, since  $\lambda, \mu \in [0, \infty[$  and  $u \in [0, 1[, a, b > 0.$ Therefore, the following are equivalent

$$\begin{aligned} \frac{2\lambda^2\mu^2(\mu-1)^2}{a^4b^6} &\geq \frac{2\lambda^2\mu^2(\lambda-1)}{a^6b^5} \Big( 2(\mu-1)a - (\lambda-1)b) \Big) \\ (\mu-1)^2 &\geq \frac{(\lambda-1)b}{a^2} (2(\mu-1)a - (\lambda-1)b)) \\ (\mu-1)^2 - 2(\mu-1)(\lambda-1)\frac{b}{a} + (\lambda-1)^2\frac{b^2}{a^2} &\geq 0 \\ \left( (\mu-1) - \frac{b}{a}(\lambda-1) \right)^2 &\geq 0. \end{aligned}$$

The last inequality, obviously being true for all  $u \in [0, 1[$ , proves the logconvexity of h' and therefore, that  $T^{\mathbf{8}}_{\lambda}$  dominates  $T^{\mathbf{8}}_{\mu}$  for  $\lambda \neq \infty$  and  $\lambda > \mu$ . For  $\lambda = \infty$  and  $\mu < \lambda$ , the convexity of h is equivalent to, using the same abbreviation as above,

$$\frac{2\lambda a - 2\lambda(\lambda - 1)u}{a^3 u^3} \ge 0$$

for all  $u \in [0, 1[$  which reduces to the tautology  $1 \ge 0$ . On the other hand, the log-convexity of h' on [0, 1[ is equivalent to

$$\frac{2\mu^2(\mu-1)^2}{a^6u^4} \ge \frac{-2u^2}{a^4u^6} + \frac{4\mu^2(\mu-1)}{a^5u^5}$$

for all  $u \in [0, 1[$ , which reduces to  $\left((\mu - 1) + \frac{a}{u}\right)^2 \ge 0$  being obviously fulfilled for all  $u \in [0, 1[$ .

Based on the previous result we can immediately state the following result:

**Corollary 10.** The dominance relation is a linear order on the family  $(T_{\lambda}^{\mathbf{8}})_{\lambda \in [0,\infty]}$ of t-norms. It is a linear order on the family  $(T_{\lambda}^{\mathbf{8}})_{\lambda \in [1,\infty]}$  of copulas.

## 5.2. The family $(T^{\mathbf{9}}_{\lambda})_{\lambda \in [0,\infty]}$

The family of t-norms  $(T_{\lambda}^{\mathbf{9}})_{\lambda \in [0,\infty]}$  and, for  $\lambda \in [0,\infty]$ , the additive generators of its members are given by

$$T_{\lambda}^{\mathbf{9}}(u,v) = \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \lambda = 0, \\ uve^{-\lambda \ln(u) \ln(v)}, & \text{if } \lambda \in \left]0, \infty\right[, \\ T_{\mathbf{P}}(u,v), & \text{if } \lambda = \infty, \end{cases}$$
$$t_{\lambda}(u) = \begin{cases} \ln(1-\lambda \ln(u)), & \text{if } \lambda \in \left]0, \infty\right[\\ -\ln(u), & \text{if } \lambda = \infty. \end{cases}$$

For  $\lambda \in [0, \infty]$ , the family members are continuous Archimedean t-norms as well as copulas. Note that according to [16], the family of copulas is also referred to as the family of *Gumbel-Barnett copulas*. The family is decreasing w.r.t. its parameter (see also [2]), i.e.,  $T_{\lambda}^{\mathbf{9}} \geq T_{\mu}^{\mathbf{9}}$  if and only if  $\lambda \leq \mu$ . We now investigate whether for some  $\lambda, \mu, T_{\lambda}^{\mathbf{9}}$  dominates  $T_{\mu}^{\mathbf{9}}$ .

**Proposition 11.** For all  $\lambda, \mu \in [0, \infty)$  with  $\lambda \neq \mu$  it holds that neither  $T^{\mathbf{9}}_{\mu}$  dominates  $T^{\mathbf{9}}_{\lambda}$  nor  $T^{\mathbf{9}}_{\lambda}$  dominates  $T^{\mathbf{9}}_{\mu}$ .

*Proof.* Consider arbitrary, but fixed  $\lambda, \mu \in [0, \infty)$  and assume w.l.o.g that  $\mu > \lambda$ . Since, because of the common neutral element of t-norms, dominance implies the ordering of the operations involved and the ordering property of the family,  $T^{\mathbf{9}}_{\mu}$  cannot dominate  $T^{\mathbf{9}}_{\lambda}$ . Therefore, assume that  $T^{\mathbf{9}}_{\lambda}$  dominates  $T^{\mathbf{9}}_{\mu}$ , i.e.,

$$T_{\lambda}^{\mathbf{9}}(T_{\mu}^{\mathbf{9}}(x,y),T_{\mu}^{\mathbf{9}}(u,v)) \ge T_{\mu}^{\mathbf{9}}(T_{\lambda}^{\mathbf{9}}(x,u),T_{\lambda}^{\mathbf{9}}(y,v))$$

for all  $x, y, u, v \in [0, 1]$ . Now choose  $x = e^{-2/\lambda} \in [0, 1]$ , then simple computations yield

$$\begin{split} T_{\lambda}^{\mathbf{9}}(T_{\mu}^{\mathbf{9}}(x,x),T_{\mu}^{\mathbf{9}}(x,x)) &= (e^{-\frac{4}{\lambda^{2}}(\lambda+\mu)})^{2} \cdot e^{-\lambda(\ln(e^{-\frac{4}{\lambda^{2}}(\lambda+\mu)}))^{2}} \\ &= e^{-\frac{8}{\lambda^{2}}(\lambda+\mu)} \cdot e^{-\lambda(-\frac{4}{\lambda^{2}}(\lambda+\mu))^{2}} = e^{-\frac{8}{\lambda^{3}}(\lambda+\mu)(3\lambda+2\mu)} \\ T_{\mu}^{\mathbf{9}}(T_{\lambda}^{\mathbf{9}}(x,x),T_{\lambda}^{\mathbf{9}}(x,x)) &= (e^{-\frac{8}{\lambda}})^{2} \cdot e^{-\mu(\ln e^{-\frac{8}{\lambda}})^{2}} = e^{-\frac{16}{\lambda^{2}}(\lambda+4\mu)}. \end{split}$$

Since we have assumed that  $T^{\mathbf{9}}_{\lambda}$  dominates  $T^{\mathbf{9}}_{\mu}$  and since  $e^x$  is strictly increasing it follows that

$$\frac{8}{\lambda^3}(\lambda+\mu)(3\lambda+2\mu) \le \frac{16}{\lambda^2}(\lambda+4\mu),$$

being equivalent to

$$(\lambda - \mu)^2 + \mu(\mu - \lambda) \le 0$$

leading to a contradiction with  $\mu > \lambda$ . Therefore,  $T_{\lambda}^{9}$  does not dominate  $T_{\mu}^{9}$ .

**Proposition 12.** For all  $\lambda \in [0, \infty[$  it holds that  $T^{\mathbf{9}}_{\infty} = T_{\mathbf{P}}$  dominates  $T^{\mathbf{9}}_{\lambda}$ .

*Proof.* Consider an arbitrary, but fixed  $\lambda \in [0, \infty[$ . The dominance inequality  $T^{\mathbf{9}}_{\lambda}(u, v) \cdot T^{\mathbf{9}}_{\lambda}(x, y) \geq T^{\mathbf{9}}_{\lambda}(ux, vy)$  is trivially fulfilled whenever  $0 \in \{x, y, u, v\}$ . Therefore, assume that all  $x, y, u, v \in [0, 1]$ . Then the following are equivalent

$$T_{\lambda}^{\mathbf{9}}(u,v) \cdot T_{\lambda}^{\mathbf{9}}(x,y) \ge T_{\lambda}^{\mathbf{9}}(ux,vy),$$
$$uvxy \cdot e^{-\lambda(\ln(u)\ln(v) + \ln(x)\ln(y))} \ge uvxy \cdot e^{-\lambda\ln(ux)\ln(vy)},$$
$$\ln(u)\ln(v) + \ln(x)\ln(y) \le \ln(ux)\ln(vy),$$
$$0 \le \ln(u)\ln(y) + \ln(v)\ln(x)$$

with the latter inequality being true for all  $x, y, u, v \in [0, 1]$ .

Since all t-norms T dominate the weakest t-norm  $T_{\mathbf{D}} = T_0^{\mathbf{9}}$  being the second limiting t-norm of the actual family we can summarize as follows:

**Corollary 13.** Consider two t-norms  $T^{\mathbf{9}}_{\lambda}$ ,  $T^{\mathbf{9}}_{\mu}$  of the family  $(T^{\mathbf{9}}_{\lambda})_{\lambda \in [0,\infty]}$ . If  $T^{\mathbf{9}}_{\lambda}$  dominates  $T^{\mathbf{9}}_{\mu}$ , then either  $\lambda = \infty$ ,  $\mu = 0$ , or  $\lambda = \mu$  only. Dominance is a transitive and therefore an order relation on the family  $(T^{\mathbf{9}}_{\lambda})_{\lambda \in [0,\infty]}$  of t-norms and on the family of Gumbel-Barnett copulas.

5.3. The family  $(T_{\lambda}^{15})_{\lambda \in [0,\infty]}$ 

The family of t-norms  $(T_{\lambda}^{15})_{\lambda \in [0,\infty]}$  and, for  $\lambda \in [0,\infty[$ , the additive generators of its members are given by

$$T_{\lambda}^{15}(u,v) = \begin{cases} uve^{-\ln(u)\ln(v)}, & \text{if } \lambda = 0, \\ e^{1 - \left((1 - \ln(u))^{\lambda} + (1 - \ln(v))^{\lambda} - 1\right)^{\frac{1}{\lambda}}}, & \text{if } \lambda \in \left]0, \infty\right[ \\ T_{\mathbf{M}}(u,v), & \text{if } \lambda = \infty, \end{cases}$$

$$t_{\lambda}(u) = \begin{cases} \ln(1 - \ln(u)), & \text{if } \lambda = 0, \\ (1 - \ln(u))^{\lambda} - 1, & \text{if } \lambda \in \left]0, \infty\right[ \end{cases}$$

For  $\lambda \in [0, \infty[$ , the family members are continuous Archimedean t-norms as well as Archimedean copulas. The family is increasing w.r.t. its parameter (see also [2]), i.e.,  $T_{\lambda}^{15} \geq T_{\mu}^{15}$  if and only if  $\lambda \geq \mu$ . We now investigate whether for some given  $\lambda$  and  $\mu$ ,  $T_{\lambda}^{15}$  dominates  $T_{\mu}^{15}$ . The derivatives of the additive generators are given, for  $\lambda \in ]0, \infty[$ , by

$$t'_{\lambda}(u) = \frac{-\lambda(1 - \ln(u))^{\lambda - 1}}{u}, \quad t''_{\lambda}(u) = \frac{\lambda(1 - \ln(u))^{\lambda - 2}(\lambda - \ln(u))}{u^2}$$

and for  $\lambda = 0$ , by

$$t'_0(u) = \frac{-1}{u(1 - \ln(u))}, \quad t''_0(u) = \frac{-\ln(u)}{u^2(1 - \ln(u))^2}$$

**Proposition 14.** Consider  $\lambda, \mu \in [0, \infty]$ . Then  $T_{\lambda}^{15}$  dominates  $T_{\mu}^{15}$  if and only if  $\lambda \geq \mu$ .

*Proof.* The necessity is readily shown, since dominance implies ordering between the operations involved. Moreover, since  $T_{\mathbf{M}}$  dominates any t-norm and since every t-norm dominates itself, let us assume that  $\infty > \lambda > \mu$ and consider the case  $\mu > 0$ . Following Proposition 4, it suffices to show the convexity and geo-convexity of h on  $]0, \infty[$  for proving dominance. The convexity of h on  $]0, \infty[$  is accordingly equivalent to

$$\frac{\lambda\mu}{u^3}a^{\lambda+\mu-3}(\lambda-\mu) \ge 0$$

for all  $u \in [0, 1[$ , where we have introduced the abbreviation  $a := 1 - \ln(u)$ . Since  $\ln(u) \leq 0$  for all  $u \in [0, 1[$  it holds that  $a \geq 1$  and further since also

 $\lambda, \mu > 0$  it follows that the inequality is true if and only if  $\lambda \ge \mu$  which is given by the assumption.

The geo-convexity of h is equivalent to Eq. (6) for the additive generators involved and can, by simple computations, be reduced to the following expression

$$\frac{\lambda - \ln(u)(1 - a^{\lambda})}{a(1 - a^{\lambda})u} \ge \frac{\mu - \ln(u)(1 - a^{\mu})}{a(1 - a^{\mu})u}$$

for all  $u \in [0, 1[$ , which is equivalent to

$$\frac{\lambda}{a\left(1-a^{\lambda}\right)u} - \frac{\ln(u)}{au} \ge \frac{\mu}{a\left(1-a^{\mu}\right)u} - \frac{\ln(u)}{au}.$$

Since  $\lambda > \mu$ , and since  $a \ge 1$  it holds that  $\frac{1}{1-a^{\lambda}} \ge \frac{1}{1-a^{\mu}}$  and indeed, for all  $u \in ]0, 1[$ ,

$$\frac{\lambda}{a\left(1-a^{\lambda}\right)u} \ge \frac{\mu}{a\left(1-a^{\mu}\right)u}$$

such that h is geo-convex and, therefore,  $T_{\lambda}^{15}$  dominates  $T_{\mu}^{15}$ .

Let us now turn to the case of  $\lambda > \mu = 0$  and prove dominance directly, i.e., we show that for an arbitrary  $\lambda > 0$  and for all  $u, v, x, y \in [0, 1]$ 

$$T_{\lambda}^{15}(T_0^{15}(u,v), T_0^{15}(x,y)) \ge T_0^{15}(T_{\lambda}^{15}(u,x), T_{\lambda}^{15}(v,y)).$$

Since this inequality holds whenever  $0 \in \{u, v, x, y\}$  we assume that  $u, v, x, y \in [0, 1]$ . We use the abbreviations  $a := 1 - \ln(u), b := 1 - \ln(v), c := 1 - \ln(x),$ and  $d := 1 - \ln(y)$  as well as

$$s := 1 - (a^{\lambda} + c^{\lambda} - 1)^{1/\lambda}$$
 and  $t := 1 - (b^{\lambda} + d^{\lambda} - 1)^{1/\lambda}$ .

Then, simple computations yield

$$\begin{split} T_{\lambda}^{\mathbf{15}}(T_{0}^{\mathbf{15}}(u,v),T_{0}^{\mathbf{15}}(x,y)) &= e^{1 - ((ab)^{\lambda} + (cd)^{\lambda} - 1)^{1/\lambda}}, \\ T_{0}^{\mathbf{15}}(T_{\lambda}^{\mathbf{15}}(u,x),T_{\lambda}^{\mathbf{15}}(v,y)) &= e^{1 - ((1-s)(1-t))}, \end{split}$$

such that dominance of  $T_{\lambda}^{\mathbf{15}}$  over  $T_{0}^{\mathbf{15}}$  is equivalent to

$$\left((ab)^{\lambda} + (cd)^{\lambda} - 1\right)^{1/\lambda} \le \left((ab)^{\lambda} + (cd)^{\lambda} + (a^{\lambda} - 1)(d^{\lambda} - 1) + (b^{\lambda} - 1)(c^{\lambda} - 1) - 1\right)^{1/\lambda}$$

Since  $a, b, c, d \ge 1$ , it holds that  $(ab)^{\lambda} + (cd)^{\lambda} - 1 \ge 1$  and  $(a^{\lambda} - 1)(d^{\lambda} - 1) \ge 0$ as well as  $(b^{\lambda} - 1)(c^{\lambda} - 1) \ge 0$  such that the inequality always holds which concludes the proof.

**Corollary 15.** The dominance relation is a linear order on the family  $(T_{\lambda}^{15})_{\lambda \in [0,\infty]}$  of t-norms and copulas.

5.4. The family  $(T_{\lambda}^{22})_{\lambda \in [0,\infty]}$ 

The family of t-norms  $(T_{\lambda}^{22})_{\lambda \in [0,\infty]}$  and, for  $\lambda \in [0,\infty[$ , the additive generators of its members are given by

$$T_{\lambda}^{22}(u,v) = \begin{cases} \frac{uv}{u+v-uv}, & \text{if } \lambda = 0, \\ \frac{\lambda}{\ln\left(e^{\lambda/u}+e^{\lambda/v}-e^{\lambda}\right)}, & \text{if } \lambda \in \left]0,\infty\right[\\ T_{\mathbf{M}}(u,v), & \text{if } \lambda = \infty, \end{cases}$$
$$t_{\lambda}(u) = \begin{cases} \frac{1-u}{u}, & \text{if } \lambda = 0, \\ e^{\lambda/u}-e^{\lambda}, & \text{if } \lambda \in \left]0,\infty\right[.\end{cases}$$

For  $\lambda \in [0, \infty[$  the members are continuous Archimedean t-norms and Archimedean copulas. The family is increasing w.r.t. its parameter (see also [2]), i.e.,  $T_{\lambda}^{22} \geq T_{\mu}^{22}$  if and only if  $\lambda \geq \mu$ . We now investigate whether for some given  $\lambda$  and  $\mu$ ,  $T_{\lambda}^{22}$  dominates  $T_{\mu}^{22}$ . The derivatives of the additive generators are, for  $\lambda \in ]0, \infty[$ , given by

$$t'_{\lambda}(u) = -\frac{\lambda}{u^2} e^{\frac{\lambda}{u}}, \quad t''_{\lambda}(u) = \frac{\lambda}{u^4} e^{\frac{\lambda}{u}} (2u + \lambda), \quad t'''_{\lambda}(u) = -\frac{\lambda}{u} e^{\frac{\lambda}{u}} \left( 6u^2 + 6\lambda u + \lambda^2 \right),$$

and, for  $\lambda = \infty$ , by

$$t'_{\infty}(u) = -\frac{1}{u^2}, \quad t''_{\infty}(u) = \frac{2}{u^3}, \quad t'''_{\infty}(u) = -\frac{6}{u^4}.$$

**Proposition 16.** Consider  $\lambda, \mu \in [0, \infty]$ . Then  $T_{\lambda}^{22}$  dominates  $T_{\mu}^{22}$  if and only if  $\lambda \geq \mu$ .

Proof. Consider arbitrary  $\lambda, \mu \in [0, \infty]$ . If  $T_{\lambda}^{22}$  dominates  $T_{\mu}^{22}$ , it follows that  $T_{\lambda}^{22} \geq T_{\mu}^{22}$  and equivalently  $\lambda \geq \mu$ . Since  $T_{\mathbf{M}}$  dominates every t-norm and since every t-norm dominates itself, let us first assume that  $\infty > \lambda > \mu > 0$ . Following Proposition 5, it suffices to show the convexity of h and the geo-convexity of h' on  $]0, \infty[$  for proving dominance. The convexity of h is equivalent to Eq. (4) which reduces to

$$\frac{\lambda\mu}{u^6}e^{\frac{\lambda+\mu}{u}}(\lambda-\mu) \ge 0$$

for all  $u \in [0, 1[$  which is true whenever  $\lambda \ge \mu$ .

The geo-convexity of h' is equivalent to Eq. (8) for the additive generators involved and is in turn equivalent to the conditions

$$\frac{\lambda^2 \mu^3(\mu-\lambda)}{u^{12}} e^{\frac{2(\lambda+\mu)}{u}} \left(e^{\frac{\mu}{u}} - e^{\mu}\right) \ge \frac{\lambda^2 \mu^3(\mu-\lambda)}{u^{12}} e^{\frac{2\lambda+3\mu}{u}}$$
$$e^{\frac{\mu}{u}} - e^{\mu} \le e^{\frac{\mu}{u}}$$
$$-e^{\mu} \le 0$$

for all  $u \in [0,1[$  which holds independently of u.

It remains to show that, for all  $\lambda \in [0, \infty[, T_{\lambda}^{22} \text{ dominates } T_{0}^{22}]$ . Also in this case the convexity of h and geo-convexity of h' can be equivalently expressed by the additive generators involved. The convexity of h is equivalent to

$$\frac{\lambda^2}{u^6}e^{\frac{\lambda}{u}} \ge 0$$

and the geo-convexity of h is equivalent to the condition

$$0 \ge -\frac{\lambda^3}{u^{12}}e^{\frac{2\lambda}{u}}$$

for all  $u \in [0, 1[$ , both expressions being true for any  $\lambda \in [0, \infty[$  and any  $u \in [0, 1[$ .

**Corollary 17.** The dominance relation is a linear order on the family  $(T_{\lambda}^{22})_{\lambda \in [0,\infty]}$ of t-norms and copulas.

5.5. The family  $(T_{\lambda}^{23})_{\lambda \in [0,\infty]}$ 

The family of t-norms  $(T_{\lambda}^{23})_{\lambda \in [0,\infty]}$  and, for  $\lambda \in [0,\infty[$ , the additive generators of its members are given by

$$T_{\lambda}^{23}(u,v) = \begin{cases} T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ \left(\ln(e^{u^{-\lambda}} + e^{v^{-\lambda}} - e)\right)^{-\frac{1}{\lambda}}, & \text{if } \lambda \in \left]0, \infty\right[, \\ T_{\mathbf{M}}(u,v), & \text{if } \lambda = \infty, \end{cases}$$
$$t_{\lambda}(u) = \begin{cases} -\ln(u), & \text{if } \lambda = 0, \\ e^{u^{-\lambda}} - e, & \text{if } \lambda \in \left]0, \infty\right[. \end{cases}$$

For  $\lambda \in [0, \infty[$  the members are continuous Archimedean t-norms and Archimedean copulas. The family is increasing w.r.t. its parameter (see also [2]), i.e.,  $T_{\lambda}^{23} \geq T_{\mu}^{23}$  if and only if  $\lambda \geq \mu$ . The derivatives of the additive generators are, for  $\lambda \in ]0, \infty[$ , given by

$$t'_{\lambda}(u) = -e^{u^{-\lambda}}u^{-\lambda-1}\lambda, \quad t''_{\lambda}(u) = e^{u^{-\lambda}}u^{-2(\lambda+1)}\lambda\left((\lambda+1)u^{\lambda}+\lambda\right),$$
$$t'''_{\lambda}(u) = -e^{u^{-\lambda}}u^{-3(\lambda+1)}\lambda\left(3\lambda(\lambda+1)u^{\lambda}+(\lambda+1)(\lambda+2)u^{2\lambda}+\lambda^2\right)$$

and for  $\lambda = 0$ , by  $t'_0(u) = -\frac{1}{u}$ ,  $t''_0(u) = \frac{1}{u^2}$ ,  $t''_0(u) = -\frac{2}{u^3}$  for all  $u \in [0, 1]$ .

**Proposition 18.** Consider  $\lambda, \mu \in [0, \infty]$ . Then  $T_{\lambda}^{23}$  dominates  $T_{\mu}^{23}$  if and only if  $\lambda \geq \mu$ .

*Proof.* The necessity is readily shown, since dominance implies ordering between the operations involved. Moreover, since  $T_{\mathbf{M}}$  dominates any t-norm and since every t-norm dominates itself, let us first assume that  $\infty > \lambda > \mu >$ 0. Following Proposition 4, it suffices to show the convexity and geo-convexity of h on  $]0, \infty[$  for proving dominance. The convexity of h is equivalent to the condition

$$e^{(u^{-\lambda}+u^{-\mu})}u^{-2\lambda-2\mu-3}\lambda\mu\left((\lambda-\mu)u^{\lambda+\mu}+\lambda u^{\mu}-\mu u^{\lambda}\right)\geq 0$$

for all  $u \in [0, 1[$ . Since  $\lambda > \mu$  and, for all  $u \in [0, 1[$ ,  $u^{\mu} > u^{\lambda}$  it follows that all summands and factors of the above expression are positive, such that the inequality holds for all  $u \in [0, 1[$ .

The geo-convexity of h is reduced to the equivalent condition

$$\lambda + 1 + \frac{e^{\lambda}}{u^{\lambda + 1}(e - e^{u^{-\lambda}})} \ge \mu + 1 + \frac{e^{\mu}}{u^{\mu + 1}(e - e^{u^{-\mu}})}$$

for all  $u \in [0, 1[$ . Since  $\lambda > \mu$  it also holds that  $e^{\lambda} > e^{\mu}$ ,  $\frac{1}{(e-e^{u-\lambda})} > \frac{1}{(e-e^{u-\mu})}$ , and  $\frac{1}{u^{\lambda+1}} > \frac{1}{u^{\mu+1}}$  such that all summands resp. factors of the left-hand side expression exceed the corresponding summands resp. factors on the righthand side. Therefore, h is geo-convex since  $\lambda > \mu$ .

It remains to show that, for all  $\lambda \in [0, \infty[, T_{\lambda}^{23} \text{ dominates } T_0^{23} = T_{\mathbf{P}}$ . For this case the convexity of h reduces to

$$\lambda^2 u^{-2\lambda-3} e^{u^{-\lambda}} (u^{\lambda} + 1) \ge 0$$

which is true for all  $u \in [0, 1]$ . The log-convexity of h' can be equivalently

expressed by the additive generators involved and is equivalent to the condition

$$0 \ge -e^{2u^{-\lambda}}u^{-3(\lambda+2)}\lambda^4$$

which is clearly true for all  $u \in [0, 1[$  and all  $\lambda \in [0, \infty[$ . Therefore,  $T_{\lambda}^{23}$  dominates  $T_{\mu}^{23}$  if and only if  $\lambda \geq \mu$ .

Therefore, we can state the following

**Corollary 19.** The dominance relation is a linear order on the family  $(T_{\lambda}^{23})_{\lambda \in [0,\infty]}$ of t-norms and copulas.

#### 6. Conclusion

We have discussed dominance in several families of continuous Archimedean t-norms resp. Archimedean copulas. New results have been achieved for five additional families. It is remarkable, that although dominance is not a transitive relation on the set of all (continuous) t-norms resp. copulas, it constitutes an order relation for (nearly) all of the families mentioned and discussed here — either because of its very rare or its abundant occurrence between the family members involved.

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# Part III

# Aggregation and Decision Modelling: Two Case Studies

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Decision Support

# Representation and construction of self-dual aggregation operators

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#### Abstract

Two different characterizations of self-dual aggregation operators are available in the literature: one based on C(x, y) = x/(x + 1 - y) and one based on the arithmetic mean. Both approaches construct a self-dual aggregation operator by combining an aggregation operator with its dual. In this paper, we fit these approaches into a more general framework and characterize *N*-invariant aggregation operators, with *N* an involutive negator. Various binary aggregation operators, fulfilling some kind of symmetry w.r.t. *N* and with a sufficiently large range, can be used to combine an aggregation operator and its dual into an *N*-invariant aggregation operator. Moreover, using aggregation operators to construct *N*-invariant aggregation operators seems rather restrictive. It suffices to consider *n*-ary operators fulfilling some weaker conditions. Special attention is drawn to the equivalence classes that arise as several of these *n*-ary operators can yield the same *N*-invariant aggregation operator.

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## 1. Introduction

In many decision problems, the question arises how to determine a collective decision, preference or opinion, based on several individual decisions, preferences or opinions. Different techniques can be applied to achieve that goal. One possible strategy is simply to carry out an aggregation process based on the

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experts' decisions, preferences or opinions. This is usually done by some aggregation operator which maps arbitrarily but countably many input values to a single output value. Inputs and outputs belong to the same domain and the output should be representative for the input data or at least for some of its aspects.

In preference modeling for example, [0, 1]-valued relations R can be used to render the individual intensity of preference. Consider a finite set of alternatives  $A = \{a_1, \ldots, a_m\}$  and n experts. The opinion of expert k is represented by a relation  $R_k : A^2 \to [0, 1]$ , such that  $R_k(a_i, a_j)$  expresses the degree to which expert k prefers alternative  $a_i$  to alternative  $a_j$  (see e.g. [4,13,14]). In order to rule out incomparability, it is often required that the degree to which  $a_i$  is preferred to  $a_j$  is in some sense complementary to the degree to which  $a_j$  is preferred to  $a_i$ . This naturally leads to the use of *reciprocal preference relations*  $R_k$ , i.e.  $R_k(a_i, a_j) + R_k(a_j, a_i) = 1$ . In this setting, two alternatives  $a_i$  and  $a_j$  are indifferent if  $R_k(a_i, a_j) = R_k(a_j, a_i) = 1/2$ . These individual preferences can be merged by means of an aggregation operator **A**. The relation R is defined by  $R(a_i, a_j) = \mathbf{A}(R_1(a_i, a_j), \ldots, R_n(a_i, a_j))$  and represents the collective preference. Besides some aggregation operator-specific results [4,5], it was soon noticed [13,15] that R is reciprocal provided **A** is a self-dual aggregation operator, i.e. fulfills  $1 - \mathbf{A}(x_1, \ldots, x_n) = \mathbf{A}(1 - x_1, \ldots, 1 - x_n)$  for every  $(x_1, \ldots, x_n) \in [0, 1]^n$ .

Another application of self-dual aggregation operators is situated in multicriteria decision making. Consider a finite set of criteria  $C = \{c_1, \ldots, c_n\}$ . To each alternative  $a_i \in A$  is associated a profile  $(P_1(a_i), \ldots, P_n(a_i))$  where unary functions  $P_k : A \to [0, 1]$  are used to expresses the score of alternative  $a_i$  on criterion k. Aggregating the different partial scores by means of an aggregation operator A yields a global score  $P(a_i) = \mathbf{A}(P_1(a_i), \ldots, P_n(a_i))$  for alternative  $a_i$  [18]. These global scores can be used to rank the alternatives. Self-dual aggregation operators ensure that complementary profiles result in complementary global scores.

So far two characterizations of self-dual aggregation operators have been presented [3,15]. A deeper look into the structure of both results inspired us to extend them to a class of theorems characterizing aggregation operators that are invariant under an involutive negator N. We organized this paper as follows. First we recall the known results concerning self-dual aggregation operators. In Section 3 we present our general framework for characterizing N-invariant aggregation operators by means of aggregation operators. However, it is restrictive to use only aggregation operators to construct N-invariant aggregation operators. Section 4 tackles this problem for the two known characterizations [3,15]. In both cases we determine the minimal conditions on an *n-ary operator* (i.e. a  $[0,1]^n \rightarrow [0,1]$  mapping) such that it generates an N-invariant aggregation operator. For each characterization, multiple *n*-ary operators lead to the same N-invariant aggregation operator. The equivalence classes that arise as such are discussed in Section 5.

#### 2. Aggregation operators

Aggregation comprises any process where arbitrarily but countably many inputs are mapped to a single output value. It is natural to require that all inputs as well as all outputs are from the same domain. Usually also some monotonic behaviour is required and some boundary conditions must be satisfied [3].

**Definition 1.** A mapping  $\mathbf{A} : [0,1]^n \to [0,1]$ ,  $n \in \mathbb{N} \setminus \{0,1\}$ , is called an *n*-ary aggregation operator if it satisfies the following properties:

(AO1)  $A(x_1,...,x_n) \leq A(y_1,...,y_n)$  whenever  $x_i \leq y_i$  for all  $i \in \{1,...,n\}$ , (AO2) A(0,...,0) = 0 and A(1,...,1) = 1.

If the arity *n* of *n*-ary aggregation operators is clear from the context, we will briefly call them *aggregation operators*. An aggregation operator **A** can also be defined to act on any closed interval  $[a,b] \subseteq [-\infty,\infty]$ . Only the boundary conditions have to be modified accordingly:  $\mathbf{A}(a,\ldots,a) = a$  and

 $A(b, \ldots, b) = b$ . We then speak of an aggregation operator acting on [a, b]. Such aggregation operators can easily be transformed, by means of monotone bijections, into aggregation operators acting on some other interval [c, d].

**Proposition 1** [3]. Let  $\mathbf{A} : [a,b]^n \to [a,b]$  be an aggregation operator on [a,b] and let  $\Phi : [c,d] \to [a,b]$ be a monotone bijection. Then the mapping  $\mathbf{A}_{\Phi} : [c,d]^n \to [c,d]$ , defined by

$$\mathbf{A}_{\Phi}(x_1,\ldots,x_n)=\Phi^{-1}(\mathbf{A}(\Phi(x_1),\ldots,\Phi(x_n)))$$

is an aggregation operator on [c, d].

We call  $A_{\Phi}$  the  $\Phi$ -transform of A. For [a,b] = [c,d] the aggregation operator A is called  $\Phi$ -invariant if  $A_{\Phi} = A$  (see also [19–21]). Every  $\Phi$  is in fact an order-preserving or order-reversing bijection from [c, d]to [a, b]. In case [a, b] = [c, d] = [0, 1] we talk about [0, 1]-automorphisms, respectively strict negators. We use the exponential notation  $x^N$  to denote the image of x under a negator N. For [0, 1]-automorphisms  $\phi$  we stick to the common notation  $\phi(x)$ . A strict negator N that fulfills  $(x^N)^N = x$  is called *involutive*. The standard negator  $\mathcal{N}$ , defined by  $x^{\mathcal{N}} = 1 - x$ , is the prototype of an involutive negator. Trillas [23] has shown that the class of involutive negators N consists of all  $\phi$ -transforms of the standard negator:

$$x^{N} = \phi^{-1}(1 - \phi(x)) = \phi^{-1}(\phi(x)^{N})$$

for some [0, 1]-automorphism  $\phi$ . In the literature, the  $\mathcal{N}$ -transform  $\mathbf{A}_{\mathcal{N}}$  of  $\mathbf{A}$  is known as the *dual* of  $\mathbf{A}$  [3].

**Definition 2.** An aggregation operator A is called *self-dual* if it is  $\mathcal{N}$ -invariant.

Several other terms are used for expressing self-duality: *neutrality* [14], *reciprocity* [13,15], etc. Examples of self-dual aggregation operators are [3]:

- (1) quasi-arithmetic means  $\mathbf{M}_f(x_1, \dots, x_n) = f^{-1}(\sum_{i=1}^n f(x_i)/n)$  for which the strictly monotone continuous
- (1) quasi animitie inclusion  $(x_1, \dots, x_n) = \int_{i=1}^n (x_i) f(x_i) dx_i$  function  $f: [0, 1] \to [-\infty, \infty]$  is reciprocal (i.e. f(1 x) = 1 f(x)), (2) weighted means  $\mathbf{W}(x_1, \dots, x_n) = \sum_{i=1}^n w_i \cdot x_i$ , where  $\sum_{i=1}^n w_i = 1$  and  $w_i \ge 0$ , (3) OWA operators  $\mathbf{W}'(x_1, \dots, x_n) = \sum_{i=1}^n w_i \cdot x'_i$ , with  $(x'_1, \dots, x'_n)$  an increasing permutation of  $(x_1, \dots, x_n), \sum_{i=1}^n w_i = 1, w_i \ge 0$  and  $(w_1, \dots, w_n) = (w_n, \dots, w_1)$ .

A self-dual and commutative binary aggregation operator A necessarily satisfies A(x, 1 - x) = 1/2 for every  $x \in [0, 1]$ . This rules out all *uninorms* U (i.e. commutative, associative, increasing binary operators with neutral element  $e \in [0,1]$  [24]) since U(0,1)  $\in \{0,1\}$  [12]. Consequently, no t-norm (uninorm with e = 1) and no t-conorm (uninorm with e = 0) is self-dual. Nullnorms V on the other hand are operators of the type med(a, T, S), with  $a \in [0, 1]$ , T a t-norm and S a t-conorm [2,3]. They are commutative, associative, increasing binary operators with annihilator  $a \in [0, 1]$  (V(x, a) = a for all  $x \in [0, 1]$ ). It is easily verified that a nullnorm is self-dual if and only if a = 1/2 and S is the N-transform of T.

### 3. A characterization of *N*-invariant aggregation operators

Self-dual aggregation operations have already been studied and characterized in [3,15]. In each of these works, a self-dual aggregation operator is constructed by means of an arbitrary aggregation operator and its dual. We will fit the existing characterizations into a more general framework, providing several new theorems for characterizing N-invariant aggregation operators, with N an involutive negator. Results concerning self-dual aggregation operators can be easily retrieved by putting  $N = \mathcal{N}$ .

Silvert has investigated operations that allow to merge two fuzzy sets  $F_1$  and  $F_2$  by means of some rule of combination C into a new fuzzy set such that the complement of the combination is the combination of the complements, i.e.  $1 - C(F_1, F_2) = C(1 - F_1, 1 - F_2)$ , expressing the self-duality of C [22]. Such a binary operator C is called a *symmetric sum* if it is a continuous, commutative, self-dual aggregation operator [10,11,22]. Dombi [8] has investigated strictly increasing associative symmetric sums on ]0, 1[. Idempotent symmetric sums have been discussed by Dubois [9]. Symmetric sums were the source of inspiration for the following proposition due to Calvo et al.

**Proposition 2** [3]. An aggregation operator **A** is self-dual if and only if there exists an aggregation operator **B** such that

$$\mathbf{A}(x_1,\ldots,x_n) = \frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(x_1,\ldots,x_n) + \mathbf{B}(1-x_1,\ldots,1-x_n)}$$
(1)

with  $\frac{0}{0+0} := \frac{1}{2}$ .

To verify the sufficient condition it is enough to take  $\mathbf{B} = \mathbf{A}$ . As indicated in [3], the convention  $\frac{0}{0+0} := \frac{1}{2}$  can be replaced by some other convention, leading to aggregation operators that are not self-dual. For example, if  $\mathbf{B}(x_1, \ldots, x_n) := T_{\mathbf{P}}(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n$  and we assume that  $\frac{0}{0+0} := 0$ , then  $\mathbf{A}$  is the 3 $\Pi$ -operator, a well-known representable uninorm [6,8,12,17]. Besides this characterization, García-Lapresta and Marques Pereira provided an alternative characterization based on the arithmetic mean.

**Proposition 3** [15]. An aggregation operator **A** is self-dual if and only if there exists an aggregation operator **B** such that

$$\mathbf{A}(x_1,\ldots,x_n) = \frac{\mathbf{B}(x_1,\ldots,x_n) + \mathbf{B}_{\mathcal{N}}(x_1,\ldots,x_n)}{2}.$$
(2)

For each self-dual **A** we can again choose  $\mathbf{B} = \mathbf{A}$ . Rewriting Eq. (1) as

$$\mathbf{A}(x_1,\ldots,x_n) = \frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(x_1,\ldots,x_n) + 1 - \mathbf{B}_{\mathcal{N}}(x_1,\ldots,x_n)},\tag{3}$$

it strikes that both expressions Eqs. (3) and (2) are of the form

$$\mathbf{A}(x_1,\ldots,x_n) = \mathcal{C}(\mathbf{B}(x_1,\ldots,x_n),\mathbf{B}_N(x_1,\ldots,x_n))$$
(4)

for some binary operator C and an involutive negator N. The first two plots of Fig. 1 illustrate this binary operator C for Eqs. (3) and (2). As will be shown later, also the third plot in the figure is a valid choice for C. We now intend to sift out those binary operators C that allow to characterize the class of N-invariant aggregation operators, i.e. we want to solve the following problem:

Consider an involutive negator N. Find a binary operator C such that an n-ary operator A is an N-invariant aggregation operator if and only if there exists an aggregation operator **B** such that Eq. (4) holds for every  $(x_1, \ldots, x_n) \in [0, 1]^n$ .

We then say that C enables a full characterization of all N-invariant aggregation operators. Explicitly, the N-invariance of an aggregation operator A means that

$$\mathbf{A}(x_1,\ldots,x_n) = \mathbf{A}(x_1^N,\ldots,x_n^N)^N$$
(5)

for every  $(x_1, \ldots, x_n) \in [0, 1]^n$ . Let  $a_N$  be the unique fixpoint of N (i.e.  $(a_N)^N = a_N$ ). From a geometrical point of view, Eq. (5) enforces some kind of point symmetry w.r.t.  $(a_N, \ldots, a_N)$  upon the aggregation operator **A**. For the point of symmetry  $(a_N, \ldots, a_N)$  it holds that  $\mathbf{A}(a_N, \ldots, a_N) = a_N$ . Once  $\mathbf{A}(x_1, \ldots, x_n)$  is known, Eq. (5) fixes  $\mathbf{A}(x_1^N, \ldots, x_n^N)$ . Before continuing the search for suitable  $\mathcal{C}$  we would like to remark that our starting point slightly differs from Propositions 2 and 3 as we do not assume **A** to be an aggregation operator from the beginning.





Let  $C_B$  be the *n*-ary operator determined by the right-hand side of Eq. (4):

 $\mathcal{C}_{\mathbf{B}}: [0,1]^n \to [0,1]: (x_1,\ldots,x_n) \mapsto \mathcal{C}(\mathbf{B}(x_1,\ldots,x_n),\mathbf{B}_N(x_1,\ldots,x_n)),$ 

then C enables a full characterization of all *N*-invariant aggregation operators if and only if the following assertions hold:

- (1)  $C_B$  is an aggregation operator for every aggregation operator **B**.
- (2)  $C_{\mathbf{B}}$  is *N*-invariant for every aggregation operator **B**.
- (3) For every N-invariant aggregation operator A there exists an aggregation operator B such that  $A = C_B$ .

The following three lemmata tackle these assertions.

**Lemma 1.**  $C_B$  is an aggregation operator for every aggregation operator **B** if and only if C is a binary aggregation operator.

**Proof.** The necessary condition trivially holds. Suppose that  $C_B$  is an aggregation operator for every aggregation operator **B**. By definition it then holds that

 $\mathcal{C}(0,0) = \mathcal{C}(\mathbf{B}(0,\ldots,0),\mathbf{B}_N(0,\ldots,0)) = \mathcal{C}_{\mathbf{B}}(0,\ldots,0) = 0.$ 

Analogously  $\mathcal{C}(1,1) = 1$ . Moreover, due to the rather limited conditions an aggregation operator must fulfill, it is not difficult to see that for every  $(x, y, u, v) \in [0, 1]^4$ , with  $x \leq u$  and  $y \leq v$ , one can construct an

aggregation operator **B** for which there exist two *n*-tuples  $(x_1, \ldots, x_n)$  and  $(y_1, \ldots, y_n)$  such that  $x_i \leq y_i$  for every *i*,  $x = \mathbf{B}(x_1, \ldots, x_n)$ ,  $y = \mathbf{B}_N(x_1, \ldots, x_n)$ ,  $u = \mathbf{B}(y_1, \ldots, y_n)$  and  $v = \mathbf{B}_N(y_1, \ldots, y_n)$ . Since  $C_{\mathbf{B}}(x_1, \ldots, x_n) \leq C_{\mathbf{B}}(y_1, \ldots, y_n)$ , this means that  $C(x, y) \leq C(u, v)$  for every  $(x, y, u, v) \in [0, 1]^4$  with  $x \leq y$  and  $u \leq v$ , expressing that C must be increasing in both its arguments.  $\Box$ 

**Lemma 2.**  $C_{\rm B}$  is N-invariant for every aggregation operator **B** if and only if

$$\mathcal{C}(x,y) = \mathcal{C}(y^N, x^N)^N \tag{6}$$

for every  $(x, y) \in [0, 1]^2$ .

**Proof.** The necessary condition trivially holds. Suppose that  $C_{\mathbf{B}}$  is *N*-invariant for every aggregation operator **B**. For every  $(x, y) \in [0, 1]^2$ , there exists an aggregation operator **B** and an *n*-tuple  $(x_1, \ldots, x_n)$  such that  $x = \mathbf{B}(x_1, \ldots, x_n)$  and  $y = \mathbf{B}_N(x_1, \ldots, x_n)$ . Expressing the *N*-invariance of  $C_{\mathbf{B}}$  then leads to

$$\mathcal{C}(y^N, x^N) = \mathcal{C}(\mathbf{B}_N(x_1, \dots, x_n)^N, \mathbf{B}(x_1, \dots, x_n)^N) = \mathcal{C}_{\mathbf{B}}(x_1^N, \dots, x_n^N) = \mathcal{C}_{\mathbf{B}}(x_1, \dots, x_n)^N$$
$$= \mathcal{C}(\mathbf{B}(x_1, \dots, x_n), \mathbf{B}_N(x_1, \dots, x_n))^N = \mathcal{C}(x, y)^N. \quad \Box$$

Putting  $y = x^N$  in Eq. (6), we see that  $\mathcal{C}(x, x^N) = a_N$ . The black solid lines in Fig. 1 reflect this property. Geometrically, Eq. (6) expresses a kind of symmetry of  $\mathcal{C}$  w.r.t. the negator N. Once  $\mathcal{C}(x, y)$  is known, Eq. (6) fixes the value of  $\mathcal{C}$  in  $(y^N, x^N)$ , the point symmetrical to (x, y) w.r.t. the graph of N. If  $\mathcal{C}$  is not symmetrical in its arguments, Eq. (6) substantially differs from Eq. (5) (n = 2). In case  $\mathcal{C}$  is symmetrical, both equations are identical and hence Eq. (6) will be trivially fulfilled when considering an N-invariant operator  $\mathcal{C}$ .

**Lemma 3.** For every N-invariant aggregation operator **A** there exists an aggregation operator **B** such that  $\mathbf{A} = C_{\mathbf{B}}$  if and only if there exists an increasing function  $f : [0, 1] \rightarrow [0, 1]$  satisfying f(0) = 0, f(1) = 1 and

$$\mathcal{C}(f(x), f(x^N)^N) = x \tag{7}$$

for every  $x \in [0,1]$ .

**Proof.** Suppose that for every *N*-invariant aggregation operator **A** it is possible to find an aggregation operator **B** such that  $\mathbf{A} = C_{\mathbf{B}}$ . Based on the geometrical interpretation of Eq. (5), it is easy to see that there exists an idempotent (i.e.  $\mathbf{A}(x, ..., x) = x$  for every  $x \in [0, 1]$ ) *N*-invariant aggregation operator **A**. In case  $N = \mathcal{N}$ , one could for instance consider the arithmetic mean. Further, consider an arbitrary aggregation operator **B** such that  $\mathbf{A} = C_{\mathbf{B}}$ . Since for every  $x \in [0, 1]$  it holds that

 $x = \mathbf{A}(x,\ldots,x) = \mathcal{C}(\mathbf{B}(x,\ldots,x),\mathbf{B}(x^N,\ldots,x^N)^N),$ 

it suffices to define  $f(x) := \mathbf{B}(x, ..., x)$  for every  $x \in [0, 1]$ . Clearly *f* is increasing with f(0) = 0, f(1) = 1, and fulfills Eq. (7).

Conversely, suppose that there exists an increasing function f, fulfilling the conditions of this lemma. For each **A** it is then sufficient to define the *n*-ary operator **B** as follows

$$\mathbf{B}(x_1,\ldots,x_n)=f(\mathbf{A}(x_1,\ldots,x_n))$$

The boundary conditions and the increasingness of both f and  $\mathbf{A}$  ensure that  $\mathbf{B}$  is an aggregation operator. Replacing x by  $\mathbf{A}(x_1, \ldots, x_n)$  in Eq. (7) and taking into account that  $\mathbf{A}$  is *N*-invariant, immediately leads to  $\mathbf{A} = C_{\mathbf{B}}$ .  $\Box$ 

The dashed black lines in Fig. 1 visualize  $C(f(x), f(x^N)^N) = x$  for some suitable increasing function f. It is worthwhile noting that, for every self-dual aggregation operator A, **B** = A fulfills Eqs. (1) and (2). In



Fig. 2. Binary aggregation operators **B** generating the arithmetic mean  $C_{\mathbf{B}}(x,y) = \frac{x+y}{2}$  by means of the resp. binary aggregation operators  $\mathcal{C}$  from Fig. 1 ( $N = \mathcal{N}$ ).

general  $\mathbf{A} = \mathcal{C}_{\mathbf{A}}$  holds for every N-invariant aggregation operator A if and only if  $\mathcal{C}$  is idempotent (i.e.  $\mathcal{C}(x,x) = x$  for every  $x \in [0,1]$ ). In that case it is sufficient to choose  $f = id_{[0,1]}$ . The proof of Lemma 3 also ensures that, for every suitable f and every self-dual aggregation operator A,  $f(\mathbf{A}(x_1,\ldots,x_n))$ defines an aggregation operator **B** that generates **A**. The three binary aggregation operators in Fig. 2 were created as such and generate the arithmetic mean. They correspond to the different settings in Fig. 1.

Joining the previous lemmata finally leads to the following theorem.

**Theorem 1.** A binary operator  $\mathcal{C}$  enables a full characterization of all N-invariant aggregation operators if and only if the following assertions hold

- (1) C is a binary aggregation operator.
- (2)  $\mathcal{C}(x,y) = \mathcal{C}(y^N,x^N)^N$  for every  $(x,y) \in [0,1]^2$ . (3) There exists an increasing function  $f:[0,1] \to [0,1]$  such that f(0) = 0, f(1) = 1  $\mathcal{C}(f(x), f(x^N)^N) = x$  for every  $x \in [0,1]$ . and

It is now easily checked that the third plot in Fig. 1 indeed enables a full characterization of all N-invariant aggregation operators. In Theorem 1, the symmetry of  $\mathcal{C}$  w.r.t. N, expressed by the second assertion, contributes to the construction of a suitable f in the third assertion. It suffices to find an increasing (w.r.t. the three space coordinates) curve on C, not necessarily continuous, that reaches every number  $x \in [0, a_N[$ . Since  $\mathcal{C}(x, x^N) = a_N$  whenever  $x \in [0, 1]$ , this mathematically amounts to finding two increasing  $[0, a_N[ \rightarrow [0, 1]]$  mappings f(x) and g(x) such that  $g(x) \leq f(x)^N$  and  $\mathcal{C}(f(x), g(x)) = x$  for every  $x \in [0, a_N[$ . If we define  $f(x) := g(x^N)^N$  for every  $x \in [a_N, 1]$ , then the second assertion in the theorem assures that  $\mathcal{C}(f(x), f(x^N)^N) = x$  holds for every  $x \in [0, 1] \setminus \{a_N\}$ . Note that we can choose arbitrarily

$$f(a_N) \in \left[\lim_{x \to a_N} f(x), \lim_{x \to a_N} f(x)\right].$$

Every appropriate function f is constructed as described above. Moreover, due to the first assertion of Theorem 1, C must be increasing. Therefore, we can find an increasing (w.r.t. the three space coordinates) curve on C that reaches every number  $x \in [0, a_N[$  if and only if C reaches every number  $x \in [0, a_N[$ . Theorem 1 can be adjusted accordingly.

**Theorem 2.** A binary operator C enables a full characterization of all N-invariant aggregation operators if and only if the following assertions hold:

- (1) C is a binary aggregation operator.
- (2)  $\mathcal{C}(x, y) = \mathcal{C}(y^N, x^N)^N$  for every  $(x, y) \in [0, 1]^2$ .
- (3) C reaches every element of  $[0, a_N]$ .

It is now natural to wonder whether there exists a binary operator C that enables a full characterization of all *N*-invariant aggregation operators for every involutive negator *N*. This question is answered negatively.

**Theorem 3.** There does not exist a binary operator C that enables a full characterization of all N-invariant aggregation operators for every involutive negator N.

**Proof.** For example, consider the two involutive negators  $N_1$  and  $N_2$  defined by

$$x^{N_1} = \sqrt{1 - x^2}$$
 and  $x^{N_2} = \begin{cases} -x/3 + 1, & x \in [0, 3/4], \\ -3x + 3, & x \in [3/4, 1] \end{cases}$ 

and with fixpoints  $a_{N_1} = \sqrt{1/2}$  and  $a_{N_2} = 3/4$ . Obviously,  $(3/5)^{N_1} = (3/5)^{N_2} = 4/5$  and therefore  $\mathcal{C}(3/5, (3/5)^{N_1}) = \mathcal{C}(3/5, (3/5)^{N_2})$ . The second assertion of Theorem 2, however, implies that

$$\mathcal{C}(3/5, (3/5)^{N_1}) = a_{N_1} = \sqrt{1/2} < 3/4 = a_{N_2} = \mathcal{C}(3/5, (3/5)^{N_2}),$$

a contradiction.  $\Box$ 

Comparing Eq. (1) with Eq. (2), García-Lapresta and Marques Pereira [15] argue that their approach (Eq. (2)), in contrast to Eq. (1), preserves shift-invariance. An aggregation operator **A** is called *shift-invariant* if for all  $t \in [-1, 1]$  and all  $(x_1, \ldots, x_n) \in [0, 1]^n$  it holds that

$$\mathbf{A}(x_1+t,\ldots,x_n+t) = \mathbf{A}(x_1,\ldots,x_n) + t \tag{8}$$

whenever  $(x_1 + t, ..., x_n + t) \in [0, 1]^n$  and  $\mathbf{A}(x_1, ..., x_n) + t \in [0, 1]$ . Interpreting the translations in question as increasing bijections

$$\Phi_t : [\max(-t, 0), \min(1-t, 1)] \to [\max(0, t), \min(1, t+1)] : x \mapsto x+t,$$

with  $t \in [-1, 1]$ , we can consider Eq. (8) as the  $\Phi_t$ -invariance of **A**. Hence, an aggregation operator **A** is shift-invariant if it is  $\Phi_t$ -invariant for every  $t \in [-1, 1]$ . If the aggregation operator  $C_B$  is shift-invariant whenever **B** is shift-invariant we say that C preserves shift-invariance.

**Theorem 4.** The arithmetic mean is the only binary operator that fulfills (6) for N = N and preserves shift-invariance.

**Proof.** Obviously C preserves shift-invariance if and only if C itself is shift-invariant. Aczél [1] showed that the general solution of Eq. (8) (n = 2) is given by C(x, y) = x + f(y - x), for some function  $f: [0, 1] \to [0, 1]$  such that  $x + f(y - x) \in [0, 1]$ . Expressing that Eq. (6) must hold for N = N leads to f(y - x) = (y - x)/2. Consequently, C must be the arithmetic mean.  $\Box$ 

Due to Theorem 2, we now know that the arithmetic mean is the only good choice if we want to preserve shift-invariance.

**Corollary 1.** The arithmetic mean is the only binary operator that enables a full characterization of all selfdual aggregation operators and preserves shift-invariance.

Consequently, shift-invariant self-dual aggregation operators can be created by means of Eq. (2), with **B** a shift-invariant aggregation operator. Recently, shift-invariant self-dual aggregation operators, related to quasi-arithmetic means, were used in a real case study by García-Lapresta and Meneses [16].

#### Remarks 1

. . . .

- (1) In essence we are determining N-invariant aggregation operators for a given involutive negator N. In [20], Mesiar and Rückschlossová tackle a strongly related problem. They characterize those aggregation operators that are invariant under any [0, 1]-automorphism or strict negator. Their work complements the results of Ovchinnikov and Dukhovny [21], who characterized those continuous aggregation operators that are invariant under any [0, 1]-automorphism (see also [19]). All these characterizations are based on the Choquet integral w.r.t. {0, 1}-valued fuzzy measures.
- (2) An aggregation operator A can also be N-invariant on

$$I = [0, 1]^n \setminus \{(x_1, \ldots, x_n) \mid \min(x_1, \ldots, x_n) = 0 \land \max(x_1, \ldots, x_n) = 1\}.$$

For example, as already mentioned, the  $3\Pi$ -operator belongs to the class of representable uninorms. These uninorms are strictly increasing and continuous on  $]0, 1[^2$  and *N*-invariant on *I* [7,12].

## 4. n-Ary operators generating self-dual aggregation operators

So far we have been looking for those operators C that enable a full characterization of all N-invariant aggregation operators. Once C is fixed in accordance with Theorem 2, every aggregation operator **B** will provide an N-invariant aggregation operator **A** and, conversely, with every N-invariant aggregation operator **A** there corresponds at least one aggregation operator **B** such that  $\mathbf{A} = C_{\mathbf{B}}$ . However, **B** itself often does not need to be an aggregation operator. The minimal conditions on an *n*-ary operator **B** such that Eq. (4) yields an N-invariant aggregation operator are inextricably bound up with the choice of C and N. Therefore, general results are not to be expected. Here, we focus on Eqs. (1) and (2), with N = N, and try to generalize Propositions 2 and 3.

In order to generalize Proposition 2 we have to figure out under which conditions Eq. (1) defines a selfdual aggregation operator.

**Proposition 4.** An n-ary operator  $\mathbf{A}$  is a self-dual aggregation operator if and only if there exists an n-ary operator  $\mathbf{B}$  such that

(1) 
$$\mathbf{B}(0,...,0) = 0$$
 and  $\mathbf{B}(1,...,1) > 0$ ,  
(2)  $\frac{\mathbf{B}(x_1,...,x_n)}{\mathbf{B}(1-x_1,...,1-x_n)}$  is increasing, with  $\frac{0}{0} := 1$ ,  
(3)  $\mathbf{A}(x_1,...,x_n) = \frac{\mathbf{B}(x_1,...,x_n)}{\mathbf{B}(x_1,...,x_n) + \mathbf{B}(1-x_1,...,1-x_n)}$ , with  $\frac{0}{0+0} := \frac{1}{2}$ .

**Proof.** In case **B** is an aggregation operator, the first two conditions are trivially fulfilled. Taking into account Proposition 2 immediately leads to the necessary conditions. Conversely, let **B** be an *n*-ary operator fulfilling the first two conditions and define an *n*-ary operator **A** by means of the third condition. It then suffices to prove that **A** is a self-dual aggregation operator. The self-duality of **A** does not depend on the

choice of **B**, but is immediately ensured by the definition of **A** and the convention  $\frac{0}{0+0} := \frac{1}{2}$ . It is also easily verified that **A** fulfills the boundary conditions A(0, ..., 0) = 0 and A(1, ..., 1) = 1 if and only if  $\mathbf{B}(0,\ldots,0) = 0$  and  $\mathbf{B}(1,\ldots,1) > 0$ . This leaves us to prove that A is increasing if and only if

$$\overline{\mathbf{B}}: [0,1]^n \to [0,\infty]: (x_1,\ldots,x_n) \mapsto \frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(1-x_1,\ldots,1-x_n)}$$

with  $\frac{0}{0} := 1$ , is increasing. We have to guarantee that

$$\frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(x_1,\ldots,x_n) + \mathbf{B}(1-x_1,\ldots,1-x_n)} \leq \frac{\mathbf{B}(y_1,\ldots,y_n)}{\mathbf{B}(y_1,\ldots,y_n) + \mathbf{B}(1-y_1,\ldots,1-y_n)}$$
(9)

whenever  $(x_1, ..., x_n) \le (y_1, ..., y_n)$ .

(1) If 
$$\mathbf{B}(1 - x_1, \dots, 1 - x_n) \ge 0$$
 and  $\mathbf{B}(1 - y_1, \dots, 1 - y_n) \ge 0$ , then Eq. (9) is equivalent to  
 $\mathbf{B}(x_1, \dots, x_n) \cdot \mathbf{B}(1 - y_1, \dots, 1 - y_n) \le \mathbf{B}(y_1, \dots, y_n) \cdot \mathbf{B}(1 - x_1, \dots, 1 - x_n),$ 

leading to  $\overline{\mathbf{B}}(x_1,\ldots,x_n) \leq \overline{\mathbf{B}}(y_1,\ldots,y_n)$ .

(2a) If  $\mathbf{B}(1 - x_1, \dots, 1 - x_n) = 0$  and  $\mathbf{B}(x_1, \dots, x_n) > 0$ , then Eq. (9) becomes

$$1 \leq \frac{\mathbf{B}(y_1, \dots, y_n)}{\mathbf{B}(y_1, \dots, y_n) + \mathbf{B}(1 - y_1, \dots, 1 - y_n)}$$

The latter can only occur if  $\mathbf{B}(y_1, \ldots, y_n) > 0$  and  $\mathbf{B}(1 - y_1, \ldots, 1 - y_n) = 0$ , leading to  $\overline{\mathbf{B}}(x_1, \dots, x_n) = \overline{\mathbf{B}}(y_1, \dots, y_n) = +\infty.$ (2b) If  $\mathbf{B}(1 - x_1, \dots, 1 - x_n) = 0$  and  $\mathbf{B}(x_1, \dots, x_n) = 0$ , then Eq. (9) becomes

$$\frac{1}{2} \leqslant \frac{\mathbf{B}(y_1, \dots, y_n)}{\mathbf{B}(y_1, \dots, y_n) + \mathbf{B}(1 - y_1, \dots, 1 - y_n)}$$

which is equivalent to  $\mathbf{B}(1-y_1,\ldots,1-y_n) \leq \mathbf{B}(y_1,\ldots,y_n)$ . Therefore,  $\overline{\mathbf{B}}(x_1,\ldots,x_n) = 1 \leq \mathbf{B}(x_1,\ldots,x_n) = 1$  $\overline{\mathbf{B}}(y_1,\ldots,y_n).$ 

- (3a) If  $\mathbf{B}(1-y_1,\ldots,1-y_n)=0$  and  $\mathbf{B}(y_1,\ldots,y_n)>0$  then immediately  $\overline{\mathbf{B}}(x_1,\ldots,x_n)\leqslant$  $\overline{\mathbf{B}}(y_1,\ldots,y_n)=+\infty.$
- (3b) If **B** $(1 y_1, ..., 1 y_n) = 0$  and **B** $(y_1, ..., y_n) = 0$ , then Eq. (9) becomes

$$\frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(x_1,\ldots,x_n)+\mathbf{B}(1-x_1,\ldots,1-x_n)}\leqslant \frac{1}{2},$$

which is equivalent to  $\mathbf{B}(x_1,\ldots,x_n) \leq \mathbf{B}$   $(1-x_1,\ldots,1-x_n)$ . Therefore,  $\overline{\mathbf{B}}(x_1,\ldots,x_n) \leq 1 =$  $\overline{\mathbf{B}}(y_1,\ldots,y_n).$ 

This completes the proof.  $\Box$ 

It is clear that the conditions on **B** in Proposition 4 are minimal. Note that for every *n*-ary operator **B** generating a self-dual aggregation operator **A** by means of Eq. (1), the *n*-ary operator **B'** defined by

$$\mathbf{B}'(x_1,\ldots,x_n)=\frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(1,\ldots,1)}$$

also generates A and fulfills the conditions of the proposition. Without loss of generality, we may require that **B**(1, ..., 1) = 1.

Next, we carry out an optimal generalization of Proposition 3. The proof is elementary and therefore left out.

**Proposition 5.** An n-ary operator **A** is a self-dual aggregation operator if and only if there exists an n-ary operator **B** such that

- (1)  $\mathbf{B}(0,...,0) = 0$  and  $\mathbf{B}(1,...,1) = 1$ ,
- (2)  $\mathbf{B}(x_1, \dots, x_n) \mathbf{B}(1 x_1, \dots, 1 x_n)$  is increasing, (3)  $\mathbf{A}(x_1, \dots, x_n) = \frac{\mathbf{B}(x_1, \dots, x_n) + \mathbf{B}_{\mathcal{N}}(x_1, \dots, x_n)}{\mathbf{B}(x_1, \dots, x_n)}$ .

(3) 
$$\mathbf{A}(x_1,\ldots,x_n) = \frac{\mathbf{B}(x_1,\ldots,x_n) + \mathbf{B}_{\mathcal{N}}(x_1,\ldots,x_n)}{2}$$

The first plot in Fig. 3 visualizes a non-monotonic binary operator **B** that fulfills the conditions of Proposition 4 and generates the arithmetic mean by means of Eq. (1). Similarly, the second plot in the figure fulfills the conditions of Proposition 5 and generates the arithmetic mean by means of Eq. (2). As announced at the beginning of this section, we only attempted to generalize Propositions 2 and 3. An analogous reasoning can be done for other specific choices of C and N. As an example, the third plot of Fig. 3 presents a binary operator **B** for which  $C_{\rm B}$ , based on the third aggregation operator C in Fig. 1 and with  $N = \mathcal{N}$ , once again is the arithmetic mean. However, **B** itself is not an aggregation operator.

#### 5. Equivalence classes

We have seen that, given the binary aggregation operator C, each self-dual aggregation operator A can be built from an *n*-ary operator **B** fulfilling some extra conditions. Usually, several suchlike operators **B** can generate the same A. The set of all suitable operators **B** is partitioned into equivalence classes, each containing



Fig. 3. Binary operators **B** generating the arithmetic mean  $C_{\mathbf{B}}(x,y) = (x+y)/2$  by means of the resp. binary aggregation operators Cfrom Fig. 1 ( $N = \mathcal{N}$ ).

those *n*-ary operators **B** determining the same **A**. In this section, we concentrate on the content of these equivalence classes. We also suggest how to pick out a (maximal) representative in each equivalence class. Define for every  $x \in [0, 1]$  the set  $C_x^{-1}$  as follows:

$$\mathcal{C}_x^{-1} := \{(u,v) \in [0,1]^2 \mid \mathcal{C}(u,v) = x\}.$$

Due to Eq. (7), we know that  $C_x^{-1} \neq \emptyset$  and therefore  $\bigcup_{x \in [0,1]} C_x^{-1}$  defines a partition of  $[0,1]^2$ . On the other hand, due to Eq. (6), we know that  $(u, v) \in C_x^{-1}$  if and only if  $(v^N, u^N) \in C_{x^N}^{-1}$ . Hence, the partition in question is totally determined by  $\{C_x^{-1} \mid x \in [0, a_N]\}$ .

**Theorem 5.** Consider a binary aggregation operator C fulfilling the conditions of Theorem 2, and a partition  $I \cup I_N$  of  $[0,1]^n \setminus \{(a_N,\ldots,a_N)\}$  such that

$$(x_1,\ldots,x_n)\in I\iff (x_1^N,\ldots,x_n^N)\in I_N.$$

For every N-invariant aggregation operator A and every n-ary operator B, it holds that  $C_B = A$  if and only if

$$(\mathbf{B}(x_1,\ldots,x_n),\mathbf{B}_N(x_1,\ldots,x_n))\in \mathcal{C}_{\mathbf{A}(x_1,\ldots,x_n)}^{-1}$$
(10)

for every  $(x_1, ..., x_n) \in I \cup \{(a_N, ..., a_N)\}.$ 

**Proof.** The sufficient conditions hold by definition. Suppose that Eq. (10) holds, then also

$$\mathcal{C}_{\mathbf{B}}(x_1,\ldots,x_n) = \mathcal{C}(\mathbf{B}(x_1,\ldots,x_n),\mathbf{B}_N(x_1,\ldots,x_n)) = \mathbf{A}(x_1,\ldots,x_n)$$
(11)

for every  $(x_1, \ldots, x_n) \in I \cup \{(a_N, \ldots, a_N)\}$ . Moreover, Eq. (10) also implies that

$$\left(\mathbf{B}_{N}(x_{1},\ldots,x_{n})^{N},\mathbf{B}(x_{1},\ldots,x_{n})^{N}\right)\in\mathcal{C}_{\mathbf{A}(x_{1},\ldots,x_{n})^{N}}^{-1}$$

for every  $(x_1, \ldots, x_n) \in I$ , which can be rewritten as

$$\left(\mathbf{B}(x_1^N,\ldots,x_n^N),\mathbf{B}_N(x_1^N,\ldots,x_n^N)\right)\in \mathcal{C}_{\mathbf{A}(x_1^N,\ldots,x_n^N)}^{-1}.$$

If we then replace  $(x_1, \ldots, x_n)$  by  $(x_1^N, \ldots, x_n^N)$ , it is clear that Eq. (11) also holds for every  $(x_1, \ldots, x_n) \in I_N$ . We can conclude that  $C_{\mathbf{B}}(x_1, \ldots, x_n) = \mathbf{A}(x_1, \ldots, x_n)$  for every  $(x_1, \ldots, x_n) \in I \cup I_N \cup \{(a_N, \ldots, a_N)\} =$  $[0,1]^n$ .  $\Box$ 

By means of Eq. (10) we can construct an appropriate *n*-ary operator **B**. Take for every  $(x_1, \ldots, x_n) \in [0, 1]^n$  an arbitrary point  $(u, v) \in C^{-1}_{\mathbf{A}(x_1, \ldots, x_n)}$  and put  $\mathbf{B}(x_1, \ldots, x_n) := u$  and  $\mathbf{B}(x_1^N, \ldots, x_n^N) := v^N$ . The theorem also enables us, given an *n*-ary operator  $\mathbf{D}$  for which  $C_{\mathbf{D}}$  is a self-dual aggregation operator (e.g. every aggregation operator D will do), to construct the equivalence class containing D. It simply suffices to take  $A := C_D$ . Remark that, in contrast to the previous section, we do not need to verify whether **B** indeed provides an aggregation operator  $C_{\rm B}$ . We start here with a self-dual aggregation operator A or  $C_{\rm D}$ . This is in contrast to Section 4 where we wanted to construct a suchlike self-dual aggregation operator. However, since any aggregation operator must fulfill the same boundary conditions, we derive from Eq. (10) that  $(\mathbf{B}(0,\ldots,0),\mathbf{B}(1,\ldots,1)^N) \in \mathcal{C}_0^{-1}$ . The boundary conditions in Propositions 4 and 5 can now be read immediately from Fig. 1. A similar reasoning for the monotonicity conditions fails in its intentions as for that purpose we need to know in advance which self-dual aggregation operator we are constructing.

To illustrate Theorem 5, we apply it to  $\mathcal{C}(x, y) = x/(x+1-y)$  and  $\mathcal{C}(x, y) = (x+y)/2$ .

**Proposition 6.** Consider a partition  $I \cup I_N$  of  $[0,1]^n \setminus \{(1/2,\ldots,1/2)\}$  such that

$$(x_1,\ldots,x_n)\in I\iff (1-x_1,\ldots,1-x_n)\in I_{\mathcal{N}}.$$

For every self-dual aggregation operator A and every n-ary operator B, Eq. (1) holds if and only if

$$\frac{\mathbf{B}(x_1, \dots, x_n)}{\mathbf{B}(1 - x_1, \dots, 1 - x_n)} = \frac{\mathbf{A}(x_1, \dots, x_n)}{\mathbf{A}(1 - x_1, \dots, 1 - x_n)}$$
(12)

for every  $(x_1, \ldots, x_n) \in I \cup \{(1/2, \ldots, 1/2)\}$ , with  $\frac{0}{0} := 1$ .

**Proof.** Relying on Theorem 5, we only need to show that Eq. (12) is equivalent to Eq. (10) for C(x, y) = x/(x + 1 - y). Since **A** is self-dual, it suffices to prove that  $(x, y) \in C_z^{-1}$  can be rewritten as x/(1 - y) = z/(1 - z), for every  $(x, y, z) \in [0, 1]^3$ . To avoid singularities, we use the conventions  $\frac{0}{0+0} := 1/2$  and  $\frac{0}{0} := 1$ . Note that, by definition,  $(x, y) \in C_z^{-1}$  means that x/(x + 1 - y) = z. We distinguish the following cases:

(1) If x + 1 - y = 0 + 0, then x = 0, y = 1 and z = 1/2, by convention. Consequently

$$\frac{x}{1-y} = \frac{0}{0} = 1 = \frac{1/2}{1/2} = \frac{z}{1-z}.$$

- (2) If x + 1 y > 0, then x = (x + 1 y)z, which is equivalent to (1 z)x = z(1 y). We need to consider four subcases:
  - (a) If 1 z > 0 and 1 y > 0, then necessarily x/(1 y) = z/(1 z).
  - (b) If 1 z = 0 and 1 y > 0, then necessarily z = 0, a contradiction.
  - (c) If 1 z > 0 and 1 y = 0, then necessarily x = 0, a contradiction.
  - (d) If 1-z=0 and 1-y=0, then necessarily z=1 and x > 0, leading to  $x/(1-y) = z/(1-z) = +\infty$ .

This completes the proof.  $\Box$ 

Based on Eq. (12) we can construct **B** as follows. Let  $(x_1, ..., x_n)$  be an arbitrary point in  $I \cup \{(1/2, ..., 1/2)\}$ . If  $A(1 - x_1, ..., 1 - x_n) = 0$ , put  $B(1 - x_1, ..., 1 - x_n) := 0$  and take  $B(x_1, ..., x_n)$  arbitrarily in [0, 1]. If  $A(1 - x_1, ..., 1 - x_n) = 1/2$ , choose  $B(1 - x_1, ..., 1 - x_n) = B(x_1, ..., x_n)$  arbitrarily in [0, 1]. If  $A(1 - x_1, ..., 1 - x_n) \notin \{0, 1/2\}$ , we can take  $B(1 - x_1, ..., 1 - x_n)$  arbitrarily in [0, 1], fixing  $B(x_1, ..., x_n)$  as follows:

$$\mathbf{B}(x_1,\ldots,x_n):=\frac{\mathbf{A}(x_1,\ldots,x_n)\cdot\mathbf{B}(1-x_1,\ldots,1-x_n)}{\mathbf{A}(1-x_1,\ldots,1-x_n)}$$

Repeating this procedure for every  $(x_1, ..., x_n) \in I \cup \{(1/2, ..., 1/2)\}$ , totally determines **B**. For a particular partition  $I \cup I_N$ , we can construct in each equivalence class a unique maximal element that can be used to represent the equivalence class in question. Just put  $\mathbf{B}(1 - x_1, ..., 1 - x_n) := 1$  or  $\mathbf{B}(x_1, ..., x_n) := 1$  whenever possible. A minimal element w.r.t.  $I_N \cup \{(1/2, ..., 1/2)\}$  is usually out of the question. If we are allowed to choose  $\mathbf{B}(1 - x_1, ..., 1 - x_n)$  arbitrarily in ]0, 1], minimizing  $\mathbf{B}(1 - x_1, ..., 1 - x_n)$  becomes impossible.

**Proposition 7.** Consider a partition  $I \cup I_N$  of  $[0,1]^n \setminus \{(1/2,\ldots,1/2)\}$  such that

$$(x_1,\ldots,x_n)\in I\iff (1-x_1,\ldots,1-x_n)\in I_{\mathcal{N}}.$$

For every self-dual aggregation operator A and every n-ary operator B, Eq. (2) holds if and only if

$$\mathbf{B}(x_1, \dots, x_n) - \mathbf{B}(1 - x_1, \dots, 1 - x_n) = 2\mathbf{A}(x_1, \dots, x_n) - 1$$
(13)

for every  $(x_1, \ldots, x_n) \in I \cup \{(1/2, \ldots, 1/2)\}.$ 

**Proof.** Follows immediately from Theorem 5 and Eq. (2).

Similarly to the previous case, we can construct every suitable **B** by means of Eq. (13). For every  $(x_1, \ldots, x_n) \in I \cup \{(1/2, \ldots, 1/2)\}$  we choose  $\mathbf{B}(1 - x_1, \ldots, 1 - x_n)$  arbitrarily in

 $[\max(0, 1 - 2\mathbf{A}(x_1, \ldots, x_n)), \min(2\mathbf{A}(1 - x_1, \ldots, 1 - x_n), 1)].$ 

**B** is now totally fixed because

 $\mathbf{B}(x_1,\ldots,x_n) = 2\mathbf{A}(x_1,\ldots,x_n) - 1 + \mathbf{B}(1-x_1,\ldots,1-x_n).$ 

The upper and lower bound for  $\mathbf{B}(1 - x_1, ..., 1 - x_n)$  ensure that  $\mathbf{B}(x_1, ..., x_n) \in [0, 1]$ . For every partition  $I \cup I_N$ , each equivalence class has some maximal, resp. some minimal element. Indeed, it suffices to let  $\mathbf{B}(1 - x_1, ..., 1 - x_n)$  be the upper, resp. the lower, bound of its delimiting interval.

The plots in Fig. 4 give examples of binary operators **B** that are maximal w.r.t.  $\{(x, y) \in [0, 1]^2 | x + y \ge 1\}$  and generate the arithmetic mean. It is worthwhile noting that the three operators pictured in this figure are aggregation operators. However, it is not difficult to see that a more exotic choice of *I* does not guarantee the monotonicity of **B**.

Furthermore, it strikes that the first two conditions in Proposition 4 are just a consequence of Eq. (12). A similar correlation exists between Proposition 5 and Eq. (13). Given the duality, reflected in the choice of the partition  $I \cup I_N$ , we can also derive necessary and sufficient conditions determining whether two operators **B** and **D** belong to the same equivalence class. We only need to replace **A** in Propositions 6 and 7 by  $C_{\rm D}$ .



Fig. 4. Binary aggregation operators **B** maximal w.r.t.  $\{(x, y) \in [0, 1]^2 | x + y \ge 1\}$  and generating the arithmetic mean  $C_{\mathbf{B}}(x, y) = (x + y)/2$  by means of the resp. binary aggregation operators C from Fig. 1 (N = N).

**Proposition 8.** Consider the binary aggregation operator C, defined by C(x, y) = x/(x + 1 - y). For every two *n*-ary operators **B** and **D**, fulfilling the first two conditions in Proposition 4, it holds that  $C_{\mathbf{B}} = C_{\mathbf{D}}$  if and only if

$$\frac{\mathbf{B}(x_1,\ldots,x_n)}{\mathbf{B}(1-x_1,\ldots,1-x_n)} = \frac{\mathbf{D}(x_1,\ldots,x_n)}{\mathbf{D}(1-x_1,\ldots,1-x_n)}$$

for every  $(x_1, ..., x_n) \in [0, 1]^n$ , with  $\frac{0}{0} := 1$ .

**Proposition 9.** Consider the binary aggregation operator C, defined by C(x, y) = (x + y)/2. For every two *n*-ary operators **B** and **D**, fulfilling the first two conditions in Proposition 5, it holds that  $C_{\mathbf{B}} = C_{\mathbf{D}}$  if and only if

 $\mathbf{B}(x_1,...,x_n) - \mathbf{B}(1-x_1,...,1-x_n) = \mathbf{D}(x_1,...,x_n) - \mathbf{D}(1-x_1,...,1-x_n)$ 

for every  $(x_1, ..., x_n) \in [0, 1]^n$ , with  $\frac{0}{0} := 1$ .

Throughout the previous discussion, we considered *n*-ary operators **B**. However, as pointed out in Section 3, it is enough to work with aggregation operators (see Theorem 2). It is natural to wonder which aggregation operators **B** generate the same *N*-invariant aggregation operator **A**. In other words, we are looking for those increasing *n*-ary operators **B** that fulfill Eq. (10) and satisfy the boundary conditions  $\mathbf{B}(0, \ldots, 0) = 0$  and  $\mathbf{B}(1, \ldots, 1) = 1$ . Unfortunately, it is not clear how to characterize those operators. Suppose we know which *N*-invariant aggregation operator **A** we intend to construct. Following the procedures in the proof of Lemma 3, it can be shown that every increasing function *f*, satisfying f(0) = 0, f(1) = 1 and  $\mathcal{C}(f(x), f(x^N)^N) = x$  for every *x* that is reached by **A**, defines an aggregation operator **B** that generates **A**:

$$\mathbf{B}(x_1,\ldots,x_n):=f(\mathbf{A}(x_1,\ldots,x_n)).$$

However, not every appropriate aggregation operator **B** can be constructed in this way. For example, let **A** be the (binary) arithmetic mean and consider the standard negator  $\mathcal{N}$ , then

$$\mathbf{B}(x,y) = \begin{cases} \frac{x+y}{2}, & y \leq 1-x \land x \neq \frac{1}{2}, \\ 1, & \text{elsewhere,} \end{cases}$$

generates A when using the third aggregation operator C in Fig. 1. In this case it is impossible to define B by means of an appropriate function f since

$$C(\mathbf{B}(1/2, 1/2), 1 - \mathbf{B}(1/2, 1/2)) = C(1, 0) = 1/2 = \mathbf{A}(1/2, 1/2)$$

and

$$C(\mathbf{B}(0,1), 1 - \mathbf{B}(1,0)) = C(1/2, 1/2) = 1/2 = \mathbf{A}(0,1).$$

#### 6. Conclusions

Considering involutive negators N, we have investigated how N-invariant aggregation operators can be characterized. Inspired by the approach of Calvo et al. [3] and the approach of García-Lapresta and Marques Pereira [15], we have been able to develop a general framework where N-invariant aggregation operators  $(n \ge 2)$  are constructed by combining aggregation operators **B** with their dual aggregation operator **B**<sub>N</sub>. For this merge we have used binary aggregation operators C that fulfill  $C(x, y) = C(y^N, x^N)^N$  and that reach every element of [0, 1]. We have identified, for C(x, y) = x/(x + 1 - y), C(x, y) = (x + y)/2 and N = N, the minimal conditions on a general *n*-ary operator **B** such that it generates a self-dual aggregation operator. Similar results can be derived for any other choice of C and N. Finally, we have studied which *n*-ary operators **B** generate the same N-invariant aggregation operator. As an example we had a deeper look on

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C(x,y) = x/(x+1-y) and C(x,y) = (x+y)/2. Also, we have briefly pointed out what happens if we restrict to aggregation operators **B**.

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# Aggregation Operators and Commuting

Susanne Saminger-Platz, Radko Mesiar, and Didier Dubois

Abstract—Commuting is an important property in any two-step information merging procedure where the results should not depend on the order in which the single steps are performed. We investigate the property of commuting for aggregation operators in connection with their relationship to bisymmetry. In case of bisymmetric aggregation operators we show a sufficient condition ensuring that two operators commute, while for bisymmetric aggregation operators with neutral element we even provide a full characterization of commuting n-ary operators by means of unary distributive functions. The case of associative operations, especially uninorms, is considered in detail.

*Index Terms*—Aggregation operators, bisymmetry, commuting operators, consensus.

#### I. INTRODUCTION

N VARIOUS applications where information fusion or multifactorial evaluation is needed, an aggregation process is carried out as a two-stepped procedure whereby several local fusion operations are performed in parallel and then the results are merged into a global result. It may happen that in practice the two steps can be exchanged because there is no reason to perform either of the steps first. For instance, in a multi-person multi-aspect decision problem, each alternative is evaluated by a matrix of ratings where the rows represent evaluations by persons and the columns represent evaluations by criteria. One may, for each row, merge the ratings according to each column with some aggregation operation  $\mathbf{A}$  and form as such the global rating of each person, and then merge the persons' opinions using another aggregation operation  $\mathbf{B}$ . On the other hand, one may decide first to merge the ratings in each column using the aggregation operation **B**, thus forming the global ratings according to each criterion, and then merge these social evaluations across the criteria with aggregation operation A. The problem is that it is not guaranteed that the results of the two procedures will be the same, while one would expect them to be so in any sensible approach. When the two procedures yield the same results operations A and B are said to commute.

This paper is devoted to a mathematical investigation of commuting aggregation operators which are used, e.g., in utility

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theory [15], but also in extension theorems for functional equations [33]. Very often, the commuting property is instrumental in the preservation of some property during an aggregation process, like transitivity when aggregating preference matrices or fuzzy relations (see, e.g., [13] and [34]), or some form of additivity when aggregating set functions (see, e.g., [15]). In fact, early examples of commuting appear in probability theory for the merging of probability distributions. Suppose two joint probability distributions are merged by combining degrees of probability point-wisely. It is natural that the marginals of the resulting joint probability function are the aggregates of the marginals of the original joint probabilities. To fulfill this requirement the aggregation operation must commute with the addition operation involved in the derivation of the marginals. It enforces a weighted arithmetic mean as the only possible aggregation operation for probability functions [31]. This result is closely related to the theory of probabilistic mixtures that plays a key-role in the axiomatic derivation of expected utility theory [22]. In [15], the same question is solved for more general set functions, where the addition is replaced by a t-conorm and the consequences for generalized utility theory are pointed out.

In this paper, the problem of commuting operators is considered with more generality. After a section presenting necessary definitions and background, Section III considers the case of commuting unary operations, called distributive functions, that play a key role in the representation of commuting operators. Section IV provides characterization results concerning bisymmetric operations, i.e., aggregation operations that commute with themselves. Sections V and VI focus on functions distributive over continuous t-(co)norms and particular uninorms, respectively.

#### **II. PRELIMINARIES**

#### A. Aggregation Operators

Aggregation by itself is an important task in any discipline where the fusion of information is of vital interest. It comprehends the transformation of several items of input data into a single output value which is characteristic for the input data itself or some of its aspects. In case of aggregation operators it is assumed that a finite number of inputs from the same (numerical) scale, most often the unit interval, are being aggregated. Moreover, interpreting the inputs as evaluation results of objects according to some criterion, the monotonicity and boundary conditions of its formal definition look very natural:

Definition 1: A function  $\mathbf{A} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$  is called an *aggregation operator* if it fulfills the following properties [10]:

(A01)  $A(x_1,...,x_n) \leq A(y_1,...,y_n)$  whenever  $x_i \leq y_i$ for all  $i \in \{1,...,n\}$ ; (A02) A(x) = x for all  $x \in [0,1]$ ; (A03) A(0,...,0) = 0 and A(1,...,1) = 1. Each aggregation operator **A** can be represented by a family  $(\mathbf{A}_{(n)})_{n \in \mathbb{N}}$  of *n*-ary operations, i.e., functions  $\mathbf{A}_{(n)}$  :  $[0,1]^n \to [0,1]$  given by

$$\mathbf{A}_{(n)}(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_n)$$

In that case,  $\mathbf{A}_{(1)} = \mathrm{id}_{[0,1]}$  and, for  $n \geq 2$ , each  $\mathbf{A}_{(n)}$  is nondecreasing and satisfies  $\mathbf{A}_{(n)}(0,\ldots,0) = 0$  and  $\mathbf{A}_{(n)}(1,\ldots,1) = 1$ . Usually, the aggregation operator  $\mathbf{A}$  and the corresponding family  $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$  of *n*-ary operations are identified with each other. Note that, *n*-ary operations  $\mathbf{A}_{(n)} : [0,1]^n \to [0,1], n \geq 2$ , which fulfill properties (**AO1**) and (**AO3**) are referred to as *n*-ary aggregation operators.

Depending on the requirements applied to the aggregation process several properties for aggregation operators have been introduced. We only mention those few which are relevant for our further investigations. For more elaborated details on aggregation operators we refer to, e.g., [10].

Definition 2: Consider some aggregation operator  $\mathbf{A}: \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1].$ 

i) A is called *symmetric* if for all  $n \in \mathbb{N}$  and for all  $x_i \in [0,1], i \in \{1,\ldots,n\}$ 

$$\mathbf{A}(x_1,\ldots,x_n) = \mathbf{A}\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right)$$

for all permutations  $\alpha = (\alpha(1), \dots, \alpha(n))$  of  $\{1, \dots, n\}$ . ii) **A** is called *bisymmetric* if for all  $n, m \in \mathbb{N}$  and all

 $x_{i,j} \in [0,1]$  with  $i \in \{1, ..., m\}$  and  $j \in \{1, ..., n\}$ 

$$\mathbf{A}_{(m)}(\mathbf{A}_{(n)}(x_{1,1},\dots,x_{1,n}),\dots,\mathbf{A}_{(n)}(x_{m,1},\dots,x_{m,n})) =$$

$$\mathbf{A}_{(n)}(\mathbf{A}_{(m)}(x_{1,1},\ldots,x_{m,1}),\ldots,\mathbf{A}_{(m)}(x_{1,n},\ldots,x_{m,n})).$$

iii) A is called *associative* if for all  $n, m \in \mathbb{N}$  and all  $x_i \in [0,1]$  and all  $y_j \in [0,1]$  with  $i \in \{1,\ldots,m\}$  and  $j \in \{1,\ldots,n\}$ 

$$\mathbf{A}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \mathbf{A}(\mathbf{A}(x_1,\ldots,x_n),\mathbf{A}(y_1,\ldots,y_m)).$$

iv) An element  $e \in [0, 1]$  is called *neutral element* of **A** if for all  $n \in \mathbb{N}$  and for all  $x_i \in [0, 1], i \in \{1, \dots, n\}$  it holds that if  $x_i = e$  for some  $i \in \{1, \dots, n\}$  then

$$\mathbf{A}(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n).$$

v) An element  $d \in [0, 1]$  is called an *idempotent element* of **A** if  $\mathbf{A}(d, \dots, d) = d$  for all  $n \in \mathbb{N}$ . We will abbreviate the set of idempotent elements by

$$\mathcal{I}(\mathbf{A}) = \{ d \in [0,1] \mid \mathbf{A}(d,\ldots,d) = d \}.$$

In case that  $\mathcal{I}(\mathbf{A}) = [0, 1]$ , the aggregation operator is called *idempotent*.

Associative aggregation operators  $\mathbf{A}$  are completely characterized by their binary operators  $\mathbf{A}_{(2)}$  since all *n*-ary, n > 2, aggregation operators  $\mathbf{A}_{(n)}$  can be constructed by the recursive application of the binary operator  $\mathbf{A}_{(2)}$ .

Depending on the additional properties, several subclasses of aggregation operators can and have been distinguished, like, e.g., symmetric and associative operators with some neutral element e: For e = 1, they are referred to as *triangular norms*  (*t-norm* for short), for e = 0, they are called *t-conorms*, for  $e \in ]0, 1[$  we will refer to them as *uninorms* (see also [6], [17], and [24]).

Note that associative and symmetric aggregation operators are also bisymmetric. On the other hand, bisymmetric aggregation operators with some neutral element are associative. Therefore, as just mentioned, the class of all associative and symmetric and, therefore, bisymmetric, aggregation operators with neutral element e consists of all t-norms, t-conorms and uninorms.

Note that not all aggregation processes are carried out on input data from the unit interval, therefore, aggregation operators on other intervals as well as methods for transforming input data are needed to model the required aggregation process. Aggregation operators can be defined as acting on any closed interval  $I = [a, b] \subseteq [-\infty, \infty]$ . We will then speak of an *aggregation operator acting on I*. While (AO1) and (AO2) basically remain the same, only (AO3), expressing the preservation of the boundaries, has to be modified accordingly

$$(AO3')A(a,...,a) = a \text{ and } A(b,...,b) = b.$$

Such aggregation operators can also be achieved from standard aggregation operators by means of isomorphic transformations. By such transformations many of the before mentioned properties are being preserved.

For an isomorphic transformation  $\varphi : [a,b] \to [0,1]$ , i.e., a monotone bijection, the *isomorphic transformation*  $\mathbf{A}_{\varphi}$  of an aggregation operator  $\mathbf{A}$  is given by

$$\mathbf{A}_{\varphi}(x_1,\ldots,x_n) = \varphi^{-1}(\mathbf{A}(\varphi(x_1),\ldots,\varphi(x_n)))$$

and is an aggregation operator on [a, b]. If for two aggregation operators **A**, **B** on (possibly) different intervals, there exists a monotone bijection  $\varphi$  such that  $\mathbf{A} = \mathbf{B}_{\varphi}$  or  $\mathbf{A}_{\varphi} = \mathbf{B}$  we refer to **A** and **B** as *isomorphic aggregation operators*.

By means of increasing bijections, we can introduce t-norms T and t-conorms S on arbitrary interval [a, b] preserving the boundary elements as the corresponding neutral elements. We will denote such t-norms, respectively, t-conorms as t-(co)norms on the corresponding interval I.

#### B. Commuting and Dominance

Definition 3: Consider two aggregation operators A and B. We say that A dominates B  $(A \gg B)$  if for all  $n, m \in \mathbb{N}$  and for all  $x_{i,j} \in [0, 1]$ , with  $i \in \{1, \ldots, m\}$  and  $j \in \{1, \ldots, n\}$ , the following property holds:

$$\mathbf{B}_{(m)} \left( \mathbf{A}_{(n)}(x_{1,1}, \dots, x_{1,n}), \dots, \mathbf{A}_{(n)}(x_{m,1}, \dots, x_{m,n}) \right) \leq (1) \\
\mathbf{A}_{(n)} \left( \mathbf{B}_{(m)}(x_{1,1}, \dots, x_{m,1}), \dots, \mathbf{B}_{(m)}(x_{1,n}, \dots, x_{m,n}) \right).$$

Definition 4: Consider an *n*-ary aggregation operator  $\mathbf{A}_{(n)}$ and an *m*-ary aggregation operator  $\mathbf{B}_{(m)}$ . Then, we say that  $\mathbf{A}_{(n)}$  commutes with  $\mathbf{B}_{(m)}$  if for all  $x_{i,j} \in [0,1]$  with  $i \in \{1,\ldots,m\}$  and  $j \in \{1,\ldots,n\}$ , the following property holds:

$$\mathbf{B}_{(m)} \left( \mathbf{A}_{(n)}(x_{1,1}, \dots, x_{1,n}), \dots, \mathbf{A}_{(n)}(x_{m,1}, \dots, x_{m,n}) \right) = (2) \\
\mathbf{A}_{(n)} \left( \mathbf{B}_{(m)}(x_{1,1}, \dots, x_{m,1}), \dots, \mathbf{B}_{(m)}(x_{1,n}, \dots, x_{m,n}) \right).$$

Two aggregation operators **A** and **B** commute with each other if  $\mathbf{A}_{(n)}$  commutes with  $\mathbf{B}_{(m)}$  for all  $n, m \in \mathbb{N}$ . We will also refer to **A** and **B** as commuting aggregation operators.

Observe that the property of commuting as expressed by (2) is a special case of the so called *generalized bisymmetry* equation as introduced and discussed in [4] and [5] and plays a key role in consistent aggregation.

It is an immediate consequence of the definition of commuting that two aggregation operators commute if and only if they dominate each other; further, that any aggregation operator commuting with itself is bisymmetric and vice versa. Note that in case of two associative aggregation operators, commuting between the binary operators is a necessary and sufficient condition for their commuting in general.

Because of the preservation properties of dominance during isomorphic transformations (see also [34]) we immediately can state the following result:

*Corollary 5:* Let  $\mathbf{A}$  and  $\mathbf{B}$  be two aggregation operators. Then, the following are equivalent:

- i) A commutes with B;
- ii)  $\mathbf{A}_{\varphi}$  commutes with  $\mathbf{B}_{\varphi}$  for some isomorphic transformation  $\varphi$ ;
- iii)  $\mathbf{A}_{\varphi}$  commutes with  $\mathbf{B}_{\varphi}$  for all isomorphic transformations  $\varphi$ .

*Example 6:* The projections to the first coordinate resp. to the last coordinate, i.e.,

$$\mathbf{P}_F(x_1, \dots, x_n) = x_1$$
$$\mathbf{P}_L(x_1, \dots, x_n) = x_n$$

commute with arbitrary aggregation operator A.

#### **III.** DISTRIBUTIVE FUNCTIONS

#### A. Basic Property

There is a close relationship between commuting aggregation operators and unary functions being distributive over one of the two aggregation operators involved. On the one hand, such functions can be constructed from commuting aggregation operators, on the other hand — as we will show in the next section — they can be used for constructing commuting operators. Note that such distributive functions are in fact commuting with the involved aggregation operator.

Proposition 7: For any *n*-ary aggregation operator  $\mathbf{A}_{(n)}$ and any *m*-ary aggregation operator  $\mathbf{B}_{(m)}$ ,  $n, m \in \mathbb{N}$ , it holds that if  $\mathbf{A}_{(n)}$  commutes with  $\mathbf{B}_{(m)}$ , then the function  $f_{d,i,\mathbf{A}_{(n)}} : [0,1] \to [0,1]$  defined by

$$i - \text{th position}$$
  
$$f_{d,i,\mathbf{A}_{(n)}}(x) = \mathbf{B}_{(m)}(d,\ldots,d,\overset{\downarrow}{x},d,\ldots,d)$$
(3)

with  $i \in \{1, ..., m\}$  and d some idempotent element of  $\mathbf{A}_{(n)}$ , is distributive over  $\mathbf{A}_{(n)}$ , i.e., it fulfills for all  $i \in \{1, ..., m\}$  and all  $x_j \in [0, 1]$  with  $j \in \{1, ..., n\}$ 

$$f_{d,i,\mathbf{A}_{(n)}}\left(\mathbf{A}_{(n)}(x_1,\ldots,x_n)\right)$$
  
=  $\mathbf{A}_{(n)}\left(f_{d,i,\mathbf{A}_{(n)}}(x_1),\ldots,f_{d,i,\mathbf{A}_{(n)}}(x_n)\right)$ .

Moreover,  $f_{d,i,\mathbf{A}_{(n)}}$  is nondecreasing.

*Proof:* Consider some *n*-ary aggregation operator  $\mathbf{A}_{(n)}$ , one of its idempotent elements d, e.g., 0 or 1, and some *m*-ary aggregation operator  $\mathbf{B}_{(m)}$  such that  $\mathbf{A}_{(n)}$  commutes with  $\mathbf{B}_{(m)}$ . Then, it holds for  $f_{d,i,\mathbf{A}_{(n)}}$ :  $[0,1] \rightarrow [0,1]$  defined by (3) with arbitrary  $i \in \{1, \ldots, n\}$  that

$$\begin{aligned} f_{d,i,\mathbf{A}_{(n)}} \left( \mathbf{A}_{(n)}(x_1,\ldots,x_n) \right) \\ &= \mathbf{B}_{(m)} \left( d,\ldots,d,\mathbf{A}_{(n)}(x_1,\ldots,x_n),d\ldots,d \right) \\ &= \mathbf{B}_{(m)} \left( \mathbf{A}_{(n)}(d,\ldots,d),\ldots,\mathbf{A}_{(n)}(d,\ldots,d), \mathbf{A}_{(n)}(x_1,\ldots,x_n), \mathbf{A}_{(n)}(d,\ldots,d), \ldots,\mathbf{A}_{(n)}(d,\ldots,d), \mathbf{A}_{(n)}(d,\ldots,d), \mathbf{A}_{(n)}(d,\ldots,d) \right) \\ &= \mathbf{A}_{(n)} \left( \mathbf{B}_{(m)}(d,\ldots,d,x_1,d,\ldots,d),\ldots, \mathbf{B}_{(m)}(d,\ldots,d,x_n,d,\ldots,d) \right) \\ &= \mathbf{A}_{(n)} \left( f_{d,i,\mathbf{A}_{(n)}}(x_1),\ldots,f_{d,i,\mathbf{A}_{(n)}}(x_n) \right). \end{aligned}$$

The nondecreasingness of  $f_{d,i,\mathbf{A}_{(n)}}$  follows immediately from the monotonicity of **B**.

Analogously, we can define nondecreasing functions  $f_{d',i,\mathbf{B}_{(m)}}$  which are distributive over  $\mathbf{B}_{(m)}$  with d' some idempotent element of  $\mathbf{B}_{(m)}$ .

#### B. Distributive Functions and Lattice Polynomials

We will denote by  $\mathcal{F}_{\mathbf{A}_{(n)}}$  the set of all nondecreasing functions  $f : [0,1] \to [0,1]$  that are distributive over the *n*-ary aggregation operator  $\mathbf{A}_{(n)}$ , i.e.,

$$\mathcal{F}_{\mathbf{A}_{(n)}} = \{ f : [0,1] \to [0,1] \mid f \text{ is nondecreasing}, \\ f \left( \mathbf{A}_{(n)}(x_1,\dots,x_n) \right) = \mathbf{A}_{(n)}(f(x_1),\dots,f(x_n)) \}.$$

Observe that  $\mathbf{A}_{(1)}$  is the identity function and thus  $\mathcal{F}_{\mathbf{A}_{(1)}}$  contains all nondecreasing functions  $f : [0,1] \rightarrow [0,1]$ . For the readers' convenience we will abbreviate this set simply by  $\mathcal{F}$ , i.e.,

$$\mathcal{F} = \{f : [0,1] \to [0,1] \mid f \text{ is nondecreasing}\} = \mathcal{F}_{\mathbf{A}_{(1)}}.$$

Evidently,  $\mathcal{F}_{\mathbf{A}} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_{\mathbf{A}_{(n)}}$  is the set of all functions  $f \in \mathcal{F}$  that are distributive over the aggregation operator  $\mathbf{A}$ . Note that  $\mathcal{F}_{\mathbf{A}_{(n)}}$  as well as  $\mathcal{F}_{\mathbf{A}}$  contain at least the following functions:

$$\begin{array}{ll} \mathbf{0} : [0,1] \to [0,1], & x \mapsto 0, \\ \mathbf{1} : [0,1] \to [0,1], & x \mapsto 1, \\ \mathrm{id} : [0,1] \to [0,1], & x \mapsto x \end{array}$$

and are, therefore, not empty for arbitrary aggregation operator **A**. The following proposition shows that  $\mathcal{F}_{\mathbf{A}}$  is maximal in case of lattice polynomials only, i.e., **A** can be expressed by  $\land$ ,  $\lor$  and its arguments only [8], compare also, e.g., [29] and [30].

*Proposition 8:* Consider an aggregation operator **A**. Then the following holds:

 $\forall n \in \mathbb{N} : \mathbf{A}_{(n)} \text{ is a lattice polynomial. } \Leftrightarrow \mathcal{F}_{\mathbf{A}} = \mathcal{F}.$ 

*Proof:* If all  $\mathbf{A}_{(n)}$  with  $n \in \mathbb{N}$  are lattice polynomials, it follows immediately from the nondecreasingness of all  $f \in \mathcal{F}$  and the definition of  $\mathcal{F}_{\mathbf{A}_{(n)}}$  that  $\mathcal{F} \subseteq \mathcal{F}_{\mathbf{A}} \subseteq \mathcal{F}$ .

Before showing the sufficiency, note that any n-variable lattice polynomial  $L: [0,1]^n \to [0,1]$  can be put in the following *disjunctive normal form* [8]:

$$L(x_1, \dots, x_n) = \bigvee_{\substack{I \subseteq N, \\ m(I)=1}} \bigwedge_{i \in I} x_i$$
(4)

where  $N = \{1, \ldots, n\}$  and  $m : 2^N \rightarrow \{0, 1\}$  is a nondecreasing set function fulfilling  $m(\emptyset) = 0$  and m(N) = 1. Therefore, in order to show that some n-ary aggregation operator  $A_{(n)}$  is a lattice polynomial, we have to show that a set function  $m: 2^N \to \{0,1\}$  fulfilling the above conditions exists and that  $A_{(n)}$  can be written in the form of (4). For better readability, we will use in the sequel of this proof A instead of  $\mathbf{A}_{(n)}$ , as well as the additional notations  $\mathbf{1}_I = (x_1, \dots, x_n)$ where  $x_i = 1$  if  $i \in I$  and  $x_i = 0$  otherwise, and  $A_I = A(1_I)$ . Now assume that  $\mathcal{F}_{\mathbf{A}} = \mathcal{F}$ .

• First, we show that  $\mathbf{A}(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$  for all  $x_i \in [0,1], i \in \{1, ..., n\}$ . In case  $A(x_1, ..., x_n) =$  $c \notin \{x_1, \ldots, x_n\}$ , depending on the value of c, one of the following functions  $f_j \in \mathcal{F}, j \in \{1, 2, 3\}$ 

$$f_1(a) = \begin{cases} x_*, & \text{if } a \in [0, x_*] \\ a, & \text{otherwise} \end{cases}$$

$$f_2(a) = \begin{cases} x_*, & \text{if } a \in ]x_*, x^*[\\ a, & \text{otherwise} \end{cases}$$

$$f_3(a) = \begin{cases} x^*, & \text{if } a \in [x^*, 1] \\ a, & \text{otherwise,} \end{cases}$$

with  $x_* = \min(x_1, ..., x_n)$  and  $x^* = \max(x_1, ..., x_n)$ , contradicts  $f_j(\mathbf{A}(x_1,\ldots,x_n)) = \mathbf{A}(f_j(x_1),\ldots,f_j(x_n)).$ Therefore, in particular A is idempotent, i.e.,  $\mathbf{A}(x,\ldots,x) = x$  and  $\mathbf{A}_I \in \{0,1\}$  for all  $I \subseteq N$ .

Since for all  $x \in [0,1]$  the functions  $\varphi_x, \psi_x : [0,1] \rightarrow$  $[0,1], \varphi_x(a) = x \cdot a$  resp.  $\psi_x(a) = a(1-x) + x$  fulfill  $\varphi_x, \psi_x \in \mathcal{F}$  we can conclude the following for all  $I \subseteq N$ 

$$\mathbf{A}(\varphi_x(1_I)) = \varphi_x(\mathbf{A}_I) = \mathbf{A}_I \wedge x, \mathbf{A}(\psi_x(1_I)) = \psi_x(\mathbf{A}_I) = \mathbf{A}_I \vee x,$$

since  $\varphi_x(0) = 0, \varphi_x(1) = x, \psi_x(0) = x, \psi_x(1) = 1$ , and  $\mathbf{A}_I \in \{0, 1\}.$ 

Due to the monotonicity of A we can further conclude that for arbitrary  $x_i \in [0, 1], i \in \{1, ..., n\}$ 

$$\mathbf{A}(x_1,\ldots,x_n) \ge \bigwedge_{i\in I} x_i \cdot \mathbf{A}_I = \mathbf{A}_I \wedge (\bigwedge_{i\in I} x_i),$$

by replacing each  $x_i$  either by 0, if  $i \notin I$ , or by  $\bigwedge_{i \in I} x_i$ , if  $i \in I$ , for arbitrary choice of  $I \subseteq N$ . Therefore, also

$$\mathbf{A}(x_1, \dots, x_n) \ge \bigvee_{I \subseteq N} \mathbf{A}_I \land (\bigwedge_{i \in I} x_i) \ge \mathbf{A}_N \land (\bigwedge_{i \in N} x_i)$$
$$= 1 \land (\bigwedge_{i \in N} x_i) = x_*.$$

We abbreviate by  $y^* = \bigvee_{I \subseteq N} \mathbf{A}_I \wedge (\bigwedge_{i \in I} x_i)$  such that the previous inequality can be written as

$$\mathbf{A}(x_1,\ldots,x_n) \ge y^* \ge x_*.$$

Since  $x_* = \min(x_1, \ldots, x_n)$  it is clear that the set J = $\{j \in N \mid x_j \leq y^*\}$  is not empty. Moreover, the following holds for its complement  $N \setminus J$ 

$$y^* = \bigvee_{I \subseteq N} \mathbf{A}_I \land (\bigwedge_{i \in I} x_i) \ge \mathbf{A}_{N \setminus J} \land (\bigwedge_{i \in N \setminus J} x_i)$$

so that necessarily  $\mathbf{A}_{N\setminus J} = 0$ .

If we replace each  $x_i$  in  $\mathbf{A}(x_1,\ldots,x_n)$  either by  $y^*$  in case that  $i \in J$  or by 1 in case that  $i \notin J$ , we can also conclude, due to the monotonicity of A and the properties shown before, that

$$\mathbf{A}(x_1,\ldots,x_n) \le \mathbf{A}(\psi_{y^*}(1_{N\setminus J})) = \mathbf{A}_{N\setminus J} \lor y^* = y^*,$$

showing that

$$\mathbf{A}(x_1,\ldots,x_n) = \bigvee_{I \subseteq N} \mathbf{A}_I \wedge (\bigwedge_{i \in I} x_i).$$

Finally, we define a set function  $m: 2^N \to \{0, 1\}$  by  $m(I) = \mathbf{A}_{I}$ , then it is immediate to show that it is nondecreasing and fulfills  $m(\emptyset) = 0$  m(N) = 1, and that A is indeed a lattice polynomial.

Let us now focus on additional properties of  $\mathcal{F}_A$  in case of particular properties of the aggregation operator A involved.

#### C. Distributive Functions for Bisymmetric and Associative Aggregation Operators

Proposition 9: Let A be a bisymmetric aggregation operator and fix some  $n \in \mathbb{N}$ . If we choose some  $f_i \in \mathcal{F}_{\mathbf{A}_{(n)}}$  $i \in \{1, \ldots, n\}$ , not necessarily different, then also  $g: [0, 1] \rightarrow$ [0,1] defined by

$$g(x) = \mathbf{A}_{(n)}(f_1(x), \dots, f_n(x)) \tag{5}$$

belongs to  $\mathcal{F}_{\mathbf{A}_{(n)}}$ , i.e.,  $\mathcal{F}_{\mathbf{A}_{(n)}}$  is closed under  $\mathbf{A}_{(n)}$ . *Proof:* Consider some bisymmetric aggregation operator A and fix some arbitrary  $f_i \in \mathcal{F}_{\mathbf{A}_{(n)}}, i \in \{1, \dots, n\}$ , for some  $n \in \mathbb{N}$ . Define a function  $g: [0,1] \to [0,1]$  by (5) then the following holds for arbitrary  $x_1, \ldots, x_n \in [0, 1]$  due to the bisymmetry of **A** and the distributivity of all  $f_i$  over  $\mathbf{A}_{(n)}$ :

$$g \left( \mathbf{A}_{(n)}(x_1, \dots, x_n) \right)$$
  
=  $\mathbf{A}_{(n)} \left( f_1 \left( \mathbf{A}_{(n)}(x_1, \dots, x_n) \right), \dots, f_n \left( \mathbf{A}_{(n)}(x_1, \dots, x_n) \right) \right)$   
=  $\mathbf{A}_{(n)} \left( \mathbf{A}_{(n)} \left( f_1(x_1), \dots, f_1(x_n) \right) \right), \dots, \mathbf{A}_{(n)} \left( f_n(x_1), \dots, f_n(x_n) \right)$   
=  $\mathbf{A}_{(n)} \left( \mathbf{A}_{(n)} \left( f_1(x_1), \dots, f_n(x_1) \right) \right), \dots, \mathbf{A}_{(n)} \left( f_1(x_n), \dots, f_n(x_n) \right)$   
=  $\mathbf{A}_{(n)} \left( g(x_1), \dots, g(x_n) \right).$ 

Corollary 10: If A is a bisymmetric aggregation operator and additionally fulfills for all  $n, m \in \mathbb{N}$  and all  $x_{i,i} \in [0, 1]$ ,  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ 

$$\begin{aligned} \mathbf{A}_{(n \cdot m)}(x_{1,1}, \dots, x_{1,n}, \dots, x_{m,1}, \dots, x_{m,n}) \\ &= \mathbf{A}_{(m)} \left( \mathbf{A}_{(n)}(x_{1,1}, \dots, x_{1,n}), \dots, \mathbf{A}_{(n)}(x_{m,1}, \dots, x_{m,n}) \right) \\ \text{then } g \ : \ [0,1] \to [0,1] \text{ defined by} \end{aligned}$$

$$g(x) = \mathbf{A}_{(m)}(f_1(x), \dots, f_m(x))$$

also belongs to  $\mathcal{F}_{\mathbf{A}_{(n)}}$  for arbitrary  $m \in \mathbb{N}$  and arbitrary  $f_i \in \mathcal{F}_{\mathbf{A}_{(n)}}, i \in \{1, \ldots, m\}$ , i.e.,  $\mathcal{F}_{\mathbf{A}(n)}$  is closed under any  $\mathbf{A}_{(m)}$   $m \in \mathbb{N}$ .

Moreover, in case of an associative aggregation operator  $\mathbf{A}$  the relationship can be generalized, expressing that it is sufficient (and necessary) to characterize all functions distributive over the binary aggregation operator  $\mathbf{A}_{(2)}$  only in order to characterize the set  $\mathcal{F}_{\mathbf{A}}$  of all unary mappings distributive over  $\mathbf{A}$  with arbitrary arity.

*Proposition 11:* Let  $\mathbf{A}$  be an associative aggregation operator, then the following holds:

$$\forall f \in \mathcal{F} : \quad f \in \mathcal{F}_{\mathbf{A}} \Leftrightarrow f \in \mathcal{F}_{\mathbf{A}_{(2)}}.$$

**Proof:** Consider an associative aggregation operator **A**. If some nondecreasing function  $f:[0,1] \rightarrow [0,1]$  fulfills  $f \in \mathcal{F}_{\mathbf{A}}$ , it is distributive over all *n*-ary aggregation operators  $\mathbf{A}_{(n)}$ ,  $n \in \mathbb{N}$ , in particular over the binary aggregation operator  $\mathbf{A}_{(2)}$ . On the other hand if  $f \in \mathcal{F}_{\mathbf{A}_{(2)}}$  the property follows directly from the associativity of **A**, i.e., the fact that for all  $n \in \mathbb{N}$  with  $n \geq 2$  it holds that  $\mathbf{A}_{(n)}(x_1,\ldots,x_n) = \mathbf{A}_{(2)}(x_1,\mathbf{A}_{(n-1)}(x_2,\ldots,x_n))$ .

ficient condition for  $\mathcal{F}_{\mathbf{A}} = \mathcal{F}_{\mathbf{A}_{(2)}}$ . However, as the following example will demonstrate, it is not necessary.

*Example 12:* Consider the arithmetic mean  $\mathbf{M} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1], \mathbf{M}(x_1,\ldots,x_n) = \frac{1}{n} \sum_{i=1}^n x_i$ . Then (compare also [2] and [3])

$$\begin{split} \mathcal{F}_{\mathbf{M}_{(2)}} &= \mathcal{F}_{\mathbf{M}} = \{ f : [0,1] \to [0,1] \mid \\ f(x) &= a + bx, a, b \in [0,1], a + b \in [0,1] \} \end{split}$$

although clearly the arithmetic mean is not associative.

*Example 13:* Examples of associative and symmetric and therefore bisymmetric aggregation operators are a-medians med<sub>a</sub>

$$\operatorname{med}_a(x, y) = \operatorname{med}(x, y, a)$$

with  $a \in [0,1]$  [18]. The set  $\mathcal{F}_{\text{med}_a}$  of distributive functions is characterized in the following way: Some nondecreasing function  $f : [0,1] \to [0,1]$  is distributive over  $\text{med}_a$ , i.e.,  $f \in \mathcal{F}_{\text{med}_a}$  if and only if either f(a) = a or f(a) = f(1) < aor f(a) = f(0) > a.

Besides associativity and bisymmetry, the possibility of building isomorphic aggregation operators leads to further insight to relationships between sets of distributive functions.

#### D. Distributive Functions and Isomorphisms

Proposition 14: Consider an aggregation operator **A** and some bijection  $\varphi : [a,b] \to [0,1]$ . Then for all  $f \in \mathcal{F}_{\mathbf{A}}$  it holds that  $f_{\varphi} : = \varphi^{-1} \circ f \circ \varphi \in \mathcal{F}_{\mathbf{A}_{\varphi}}$  where

$$\mathcal{F}_{\mathbf{A}_{\varphi}} = \{ f : [a, b] \to [a, b] \mid f \text{ is nondecreasing and} \\ \text{distributive over } \mathbf{A}_{\varphi} \}.$$

*Proof:* Consider the isomorphic aggregation operators **A** and  $\mathbf{A}_{\varphi}$  with  $\varphi : [a, b] \rightarrow [0, 1]$  some bijection. Further assume  $f \in \mathcal{F}_{\mathbf{A}}$ , then the following are equivalent since for all  $x_i \in$ 

 $[a,b], i \in \{1,\ldots,n\}, n \in \mathbb{N}$ , there exists a unique  $y_i \in [a,b]$  with  $\varphi(y_i) = x_i$ 

$$f \circ \mathbf{A}(x_1, \dots, x_n) = \mathbf{A}(f(x_1), \dots, f(x_n)),$$
  

$$\varphi^{-1} \circ f \circ \mathbf{A}(\varphi(y_1), \dots, \varphi(y_n))$$
  

$$= \varphi^{-1} \circ \mathbf{A}(f \circ \varphi(y_1), \dots, f \circ \varphi(y_n)),$$
  

$$\varphi^{-1} \circ f \circ \varphi \circ \varphi^{-1} \circ \mathbf{A}(\varphi(y_1), \dots, \varphi(y_n))$$
  

$$= \varphi^{-1} \circ \mathbf{A}(\varphi \circ \varphi^{-1} \circ f \circ \varphi(y_1), \dots, \varphi(y_n));$$
  

$$f_{\varphi} \circ \varphi^{-1} \circ \mathbf{A}(\varphi(y_1), \dots, \varphi(y_n))$$
  

$$= \varphi^{-1} \circ \mathbf{A}(\varphi \circ f_{\varphi}(y_1), \dots, \varphi \circ f_{\varphi}(y_n)),$$
  

$$f_{\varphi} \circ \mathbf{A}_{\varphi}(y_1, \dots, y_n) = \mathbf{A}_{\varphi}(f_{\varphi}(y_1), \dots, \varphi(y_n))$$

showing that  $f_{\varphi} \in \mathcal{F}_{\mathbf{A}_{\omega}}$ .

*Example 15:* Following Aczél [1], [3], the class of all continuous, strictly monotone, bisymmetric, and idempotent aggregation operators on the unit interval are just weighted quasi-arithmetic means

$$\mathbf{W}_{\varphi}(x_1,\ldots,x_n) = \varphi^{-1}\left(\sum_{i=1}^n w_i\varphi(x_i)\right)$$

with  $\varphi : [0,1] \to [0,1]$  some monotone nondecreasing bijection and weights  $w_i$  with  $w_i > 0$  for all  $i \in \{1, \ldots, n\}$  and  $\sum_{i=1}^{n} w_i = 1$ . It is immediate that weighted quasi-arithmetic means are isomorphic transformations of weighted arithmetic means **W** with corresponding weights. Due to Proposition 14, the set of distributive functions  $\mathcal{F}_{\mathbf{W}_{\varphi}}$  is, therefore, given by

$$\mathcal{F}_{\mathbf{W}_{\varphi}} = \{ f \in \mathcal{F} \mid f(x) = \varphi^{-1}(a + b\varphi(x))$$
  
and  $a, b, a + b \in [0, 1] \}$ 

since

$$\mathcal{F}_{\mathbf{W}} = \{ f \in \mathcal{F} \mid f(x) = a + bx \text{ and } a, b \in [0, 1] \\ \text{such that } a + b \in [0, 1] \}$$

in case that  $w_i > 0$  for all  $i \in \{1, \dots, n\}$  and  $\sum_{i=1}^n w_i = 1$ .

*Example 16:* For invariant aggregation operators  $\mathbf{A}$ , i.e., aggregation operators fulfilling  $\mathbf{A}_{\varphi} = \mathbf{A}$  for all bijections  $\varphi : [0,1] \rightarrow [0,1]$ , it immediately holds that all nondecreasing bijections are included in  $\mathcal{F}_{\mathbf{A}}$  (see also, e.g., [29], [30] for characterizations of aggregation operators invariant under nondecreasing bijections). This is, e.g., the case for the drastic product  $T_{\mathbf{D}}$  and the weakest aggregation operator  $\mathbf{A}_w$  being defined by

$$T_{\mathbf{D}}(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1\\ 0, & \text{otherwise} \end{cases}$$
$$\mathbf{A}_w(x_1,\dots,x_n) = \begin{cases} 1, & \text{if } x_1 = \dots = x_n = 1\\ 0, & \text{otherwise.} \end{cases}$$

However, their set of distributive functions does not only contain all nondecreasing bijections, but is even much richer, namely

$$\begin{aligned} \mathcal{F}_{T_{\mathbf{D}}} &= \mathcal{F}_{\mathbf{A}_w} \\ &= \{ f \in \mathcal{F} \mid f(x) = 1 \Leftrightarrow x = 1 \text{ and } f(0) = 0 \} \cup \{ \mathbf{0}, \mathbf{1} \}. \end{aligned}$$

Similarly, lattice polynomials are invariant aggregation operators and we know already their sets of distributive functions equal the set of all nondecreasing functions.

However, for arbitrary aggregation operators A at least the following relationship between a bijective distributive function and its inverse can be stated.

*Lemma 17:* Consider an aggregation operator **A**. If  $f \in \mathcal{F}_{\mathbf{A}}$  is bijective then also  $f^{-1} \in \mathcal{F}_{\mathbf{A}}$ .

#### IV. OPERATORS COMMUTING WITH BISYMMETRIC AGGREGATION OPERATORS

After discussing unary operators being distributive over some aggregation operator and as such commuting, let us now turn to more general commuting operators.

Proposition 18: Let A be a bisymmetric aggregation operator. Then any *n*-ary operator  $B, n \in \mathbb{N}$ , on [0, 1] defined by

$$B(x_1,\ldots,x_n) = \mathbf{A}(f_1(x_1),\ldots,f_n(x_n))$$
(6)

with  $f_i \in \mathcal{F}_{\mathbf{A}}$  for  $i \in \{1, \ldots, n\}$  commutes with  $\mathbf{A}$ .

**D**(1)

*Proof:* Consider some bisymmetric aggregation operator A, choose some  $m, n \in \mathbb{N}$  and arbitrary  $f_i \in \mathcal{F}_A$ ,  $i \in \{1, \ldots, n\}$ . Then, the following holds for arbitrary  $x_{i,j} \in [0,1]$  with  $i \in \{1, \ldots, n\}$  and  $j \in \{1, \ldots, m\}$ 

\ • *(* 

$$B(\mathbf{A}(x_{1,1},...,x_{1,m}),...,\mathbf{A}(x_{n,1},...,x_{n,m})) = \mathbf{A}(f_1 \circ \mathbf{A}(x_{1,1},...,x_{1,m}),..., f_n \circ \mathbf{A}(x_{n,1},...,x_{n,m})) = \mathbf{A}(\mathbf{A}(f_1(x_{1,1}),...,f_1(x_{1,m})),... \mathbf{A}(f_n(x_{n,1}),...,f_n(x_{n,m}))) = \mathbf{A}(\mathbf{A}(f_1(x_{1,1}),...,f_n(x_{n,m}))) = \mathbf{A}(B(x_{1,1},...,x_{n,1}),...,B(x_{1,m},...,x_{n,m})).$$

Note that the involved operator B need not be an aggregation operator, e.g., choose  $f_i(x) = \mathbf{1}(x) = 1$  for all  $i \in \{1, \dots, n\}$ , then

$$B(x_1,\ldots,x_n)=1$$

for arbitrary  $x_i \in [0, 1]$ ,  $i \in \{1, ..., n\}$ , and therefore the boundary conditions (AO2) and (AO3) are not fulfilled.

*Remark 19:* Note that the previous proposition provides a sufficient, but not a necessary condition for an operation B to commute with **A**. As mentioned above, any aggregation operator **A** commutes with the projection to the first coordinate  $\mathbf{P}_F$  which is a bisymmetric aggregation operator. However, using  $\mathbf{P}_F(f_1(x_1), \ldots, f_n(x_n)) = f_1(x_1)$ , only aggregation operators depending just on the first coordinate can be obtained although we have that  $\mathcal{F}_{\mathbf{P}_F} = \mathcal{F}$ , since  $\mathbf{P}_F$  is a lattice polynomial.

#### A. Commuting Aggregation Operators

Let us briefly focus on the restrictions which additionally have to be applied to the selected functions  $f_i \in \mathcal{F}_A$  such that the constructed operator B also fulfills the requirements of an aggregation operator. If n = 1, the corresponding  $f_1 \in \mathcal{F}_A$ must be the identity function in order to guarantee B(x) = x. For n > 1, the functions  $f_i \in \mathcal{F}_{\mathbf{A}}, i \in \{1, ..., n\}$ , must be chosen accordingly to **A** such that

$$B(0,...,0) = \mathbf{A}(f_1(0),...,f_n(0)) = 0,$$
  
$$B(1,...,1) = \mathbf{A}(f_1(1),...,f_n(1)) = 1$$

are both fulfilled at the same time. This is for sure guaranteed if for all  $f_i$  it holds that  $f_i(0) = 0$  and  $f_i(1) = 1$ , but it need not be the case as the following example shows.

*Example 20:* The class of all aggregation operators commuting with the minimum

$$\mathcal{D}_{\min}^{(n)} = \{\min(f_1(x_1), \dots, f_n(x_n)) \mid f_i \in \mathcal{F} \text{ with } f_i(1) = 1 \text{ for all } i \in \{1, \dots, n\}, f_i(0) = 0 \text{ for at least one } i \in \{1, \dots, n\}\},\$$

is also the class of all aggregation operators dominating the minimum in the sense of Definition 3 (see also [34]).

#### B. The Role of Neutral Elements

Let us now consider for which bisymmetric aggregation operators  $\mathbf{A}$ , operators B defined by (6) are the only commuting operators, i.e., if (6) does provide a sufficient as well as a necessary condition. For better readability, we will briefly restrict ourselves to binary operators only. Since the projections commute with any aggregation operator  $\mathbf{A}$ , they particularly commute also with such operators  $\mathbf{A}$  for which (6) indeed is necessary and sufficient. In this case, there necessarily exist  $f_i, g_i \in \mathcal{F}_{\mathbf{A}}$ , i = 1, 2, such that for all  $x, y \in [0, 1]$ 

$$\begin{aligned} x &= \mathbf{P}_F(x, y) = \mathbf{A}(f_1(x), f_2(y)) = \mathbf{A}(f_1(x), f_2(0)) \\ &= \mathbf{A}(f_1(x), f_2(1)), \\ y &= \mathbf{P}_F(x, y) = \mathbf{A}(g_1(x), g_2(y)) = \mathbf{A}(g_1(0), g_2(y)) \\ &= \mathbf{A}(g_1(1), g_2(y)). \end{aligned}$$

If there exists some  $x_0, y_0 \in [0, 1]$  such that  $f_1(x_0) \in [g_1(0), g_1(1)]$  and  $g_2(y_0) \in [f_2(0), f_2(1)]$  it follows from the monotonicity of **A** that

$$\begin{aligned} x_0 &= \mathbf{A}(f_1(x_0), f_2(0)) \leq \mathbf{A}(f_1(x_0), g_2(y_0)) \\ &\leq \mathbf{A}(g_1(0), g_2(y_0)) = y_0 \\ y_0 &= \mathbf{A}(g_1(0), g_2(y_0)) \leq \mathbf{A}(f_1(x_0), g_2(y_0)) \\ &\leq \mathbf{A}(f_1(x_0), f_2(1)) = x_0. \end{aligned}$$

Therefore, independently of  $x_0, y_0$ , we have that

$$\mathbf{A}(f_1(x_0), g_2(y_0)) = x_0 = y_0,$$

i.e., such an element is unique. A typical candidate fulfilling the last property is a neutral element e. In such a case, it suffices to choose  $f_1 = g_2 = \text{id}$  and  $f_2(x) = g_1(x) = e$  for all  $x \in [0, 1]$ .

Indeed, we obtain a necessary and sufficient condition if the involved aggregation operator A is bisymmetric and possesses a neutral element e.

Proposition 21: Let  $\mathbf{A}$  be a bisymmetric aggregation operator with neutral element e. An n-ary operator  $B, n \in \mathbb{N}$ , commutes with  $\mathbf{A}$  if and only if there exist  $f_i \in \mathcal{F}_{\mathbf{A}}, i \in \{1, \dots, n\}$ , such that

$$B(x_1,\ldots,x_n) = \mathbf{A}(f_1(x_1),\ldots,f_n(x_n)).$$
(7)

**Proof:** Consider some bisymmetric aggregation operator  $\mathbf{A}$  with neutral element e. If B is defined by (7) for some  $f_i \in \mathcal{F}_{\mathbf{A}}$  then it commutes with  $\mathbf{A}$  due to Proposition 18. In order to show the necessity assume that B commutes with  $\mathbf{A}$ , i.e., especially for all  $x_1, \ldots, x_n \in [0, 1]$  it holds that

$$B(x_1, \dots, x_n) = B(\mathbf{A}(x_1, e, \dots, e), \dots, \mathbf{A}(e, \dots, e, x_n))$$
  
=  $\mathbf{A}(B(x_1, e, \dots, e), \dots, B(e, \dots, e, x_n))$   
=  $\mathbf{A}(f_{e,1,\mathbf{A}}(x_1), \dots, f_{e,n,\mathbf{A}}(x_n))$ 

with  $f_{e,i,\mathbf{A}}$  defined by (3), thus fulfilling  $f_{e,i,\mathbf{A}} \in \mathcal{F}_{\mathbf{A}}$  and proving that B can be expressed as in (7).

Recall once again that any bisymmetric aggregation operator with neutral element is also associative and symmetric and therefore is either a t-norm, a t-conorm or a uninorm. However, note that it is impossible that commuting operators having neutral elements are different operators.

Proposition 22: Consider two aggregation operators A and B with neutral elements  $e_a$ , respectively,  $e_b$ . If A commutes with B, then  $e_a = e_b$ . Moreover, also A = B.

*Proof:* Assume that A and B are commuting aggregation operators with neutral elements  $e_a$ , respectively,  $e_b$ . Therefore

$$e_a = \mathbf{A}(e_a, e_a) = \mathbf{A}(\mathbf{B}(e_a, e_b), \mathbf{B}(e_b, e_a))$$
$$= \mathbf{B}(\mathbf{A}(e_a, e_b), \mathbf{A}(e_b, e_a)) = \mathbf{B}(e_b, e_b) = e_b$$

and

$$\mathbf{A}(x_1, \dots, x_n) = \mathbf{A}(\mathbf{B}(x_1, e, \dots, e), \dots, \mathbf{B}(e, \dots, e, x_n))$$
  
=  $\mathbf{B}(\mathbf{A}(x_1, e, \dots, e), \dots, \mathbf{A}(e, \dots, e, x_n))$   
=  $\mathbf{B}(x_1, \dots, x_n)$ 

for all  $x_1, \ldots, x_n \in [0, 1]$  and arbitrary  $n \in \mathbb{N}$ .

As a consequence commuting does not work between t-norms, t-conorms, or uninorms respectively. The only operators commuting with such bisymmetric operators with neutral element are, besides the operator itself, aggregation operators with no neutral element.

*Example 23:* As mentioned before the projection to the first coordinate  $\mathbf{P}_F$  commutes with any aggregation operator and therefore also, e.g., with the product t-norm  $T_{\mathbf{P}}$ . Observe that  $\mathbf{P}_F$  is bisymmetric but has no neutral element, while  $T_{\mathbf{P}}$  is a bisymmetric aggregation operator with neutral element 1. According to Proposition 21, corresponding functions  $f_i \in \mathcal{F}_{T_{\mathbf{P}}}$ ,  $i \in \{1, \ldots, n\}, n \in \mathbb{N}$ , can be chosen such that

$$\mathbf{P}_F(x_1,\ldots,x_n)=T_{\mathbf{P}}(f_1(x_1),\ldots,f_n(x_n)),$$

namely  $f_1 = \text{id}$  and all other  $f_j = \mathbf{1}$  for  $j \in \{2, \dots, n\}$ . However, for any  $g_1, \dots, g_n \in \mathcal{F}_{\mathbf{P}_F} = \mathcal{F}$  the operator  $\mathbf{P}_F(g_1(x_1), \dots, g_n(x_n)) = g_1(x_1)$  can never represent the product  $T_{\mathbf{P}}$ .

#### C. Consequences

Since Proposition 21 provides a full characterization of commuting operators in case that one of them is bisymmetric with some neutral element and further shows that these operators can be attained through functions distributive over the bisymmetric aggregation operator with neutral element involved, we will now focus on the set of such functions. Note that a full characterization of all bisymmetric aggregation operators with neutral element, in particular if the neutral element is from the open interval, is still missing. Since the characterization of the set of unary functions distributing with such operators is heavily influenced by the structure of the underlying operator, we will later on focus on special subclasses of bisymmetric aggregation operators with neutral element only, namely on

- continuous t-norms;
- continuous t-conorms;
- particular classes of uninorms.

Therefore, consider \* to be some continuous t-norm T, some continuous t-conorm S, or some uninorm U. Note that  $f \in \mathcal{F}_*$  is equivalent to the fact that f fulfills a Cauchy like equation, i.e., for all  $x, y \in [0, 1]$ 

$$f(x * y) = f(x) * f(y).$$
 (8)

Observe that besides  $\mathbf{0}(x) = 0$ ,  $\mathbf{1}(x) = 1$  and  $\mathrm{id}(x) = x$  also the constant function  $f_e(x) = e$  is included in  $\mathcal{F}_*$ .

Lemma 24: If  $d \in \mathcal{I}(*)$ , then  $f_d : [0,1] \to [0,1], f_d(x) = d$  for all  $x \in [0,1]$  fulfills  $f_d \in \mathcal{F}_*$ .

#### V. CHARACTERIZATION OF $\mathcal{F}_{\mathbf{A}}$ for Continuous T-(CO)norms

For the case of continuous t-conorms (8) has been solved by Benvenuti *et al.* in [7] and as such by duality also for continuous t-norms. Continuous t-(co)norms are particularly important subclasses of t-(co)norms. We briefly recall a few basic facts and properties, but refer the interested reader for more details to the monographs [6], [24] and the articles [25]–[27].

The class of continuous t-(co)norms exactly consists of all so called continuous Archimedean t-(co)norms and of ordinal sums of such continuous Archimedean t-(co)norms. Let us first turn to continuous Archimedean t-(co)norms T resp. S. They are in turn characterized as being generated by some continuous additive generator t resp. s, i.e., they can be written as

$$T(x,y) = t^{(-1)}(t(x) + t(y)),$$
  

$$S(x,y) = s^{(-1)}(s(x) + s(y)).$$

In case of (continuous) t-norms, the additive generator  $t : [0, 1] \rightarrow \mathbb{R}$  is a strictly decreasing (continuous) function which fulfills t(1) = 0 and for which  $t^{(-1)}(x) = t^{-1}(\min(t(0), x))$ . In case of (continuous) t-conorms, the additive generator  $s : [0, 1] \rightarrow \mathbb{R}$  is a strictly increasing (continuous) function which fulfills s(0) = 0 and for which  $s^{(-1)}(x) = s^{-1}(\min(s(1), x))$ . Note that in both cases additive generators are unique up to a positive multiplicative constant. For continuous Archimedean t-(co)norms two subclasses can be further distinguished, namely *nilpotent* t-(co)norms for which  $t(0) < \infty$  resp.  $s(1) < \infty$ , and *strict* t-(co)norms with  $t(0) = \infty$  resp.  $s(1) = \infty$ .

Let us now turn to ordinal sum t-(co)norms, a concept applicable to all kinds of t-(co)norms. The main properties are based on results in the framework of semigroups, however, the basic idea of ordinal sums can be described the following way: Define a t-(co)norm T, respectively, S by t-(co)norms on pairwise nonoverlapping subsquares along the diagonal of the unit square and choose for all other cases min in case of t-norms and max in case of t-conorms. Formally, consider a family  $(]a_k, b_k[]_{k \in K}$ of pairwise disjoint open subintervals of the unit-interval and a corresponding family of t-(co)norms  $(T_k)_{k \in K}$  resp.  $(S_k)_{k \in K}$ , then the ordinal sums  $T = (\langle a_k, b_k, T_k \rangle)_{k \in k} : [0, 1]^2 \to [0, 1],$ respectively,  $S = (\langle a_k, b_k, S_k \rangle)_{k \in k} : [0, 1]^2 \to [0, 1]$  are given by (9), resp., (10), shown at the bottom of the page, and are indeed a t-norm resp. a t-conorm. The ordinal sum t-norm T as well as the ordinal sum t-conorm S are continuous if and only if all  $T_k$  resp.  $S_k$  are continuous. Based on these facts let us now briefly recall the main results of [7] which will be further relevant for the investigation of particular classes of uninorms.

#### A. Continuous T-Conorms

Theorem 25 ([7]): Consider a continuous t-conorm S. Then  $[0,1] \setminus \mathcal{I}(S) = \bigcup_{k \in K} a_k, b_k$  for some index set K and there exists a family of continuous strictly increasing mappings  $s_k : [a_k, b_k] \to [0, \infty]$  with  $s(a_k) = 0$  such that (11), shown at the bottom of the page, holds. Let  $f \in \mathcal{F}_S$  and denote by  $f_k$  its restriction to the interval  $]a_k, b_k[.$ 

i) If  $s_k(b_k) = \infty$ , then one of the following holds:

- (ssi)  $f_k(x) = i_k$  with  $i_k \in \mathcal{I}(S)$  and  $f(a_k) \leq i_k \leq$  $f(b_k);$ (ssg)  $f_k(x) = s_h^{-1}(\min(\lambda_k s_k(x), s_h(\beta_h)))$  for some
- $\lambda_k \in ]0,\infty[$  and some  $h \in K$  such that  $f(a_k) \leq a_h$ and  $f(b_k) \geq b_h$ .

ii) If  $s_k(b_k) < \infty$ , then one of the following holds:

(sni)  $f_k(x) = f(b_k) \in \mathcal{I}(S);$ 

(sing)  $f_k(x) = f_h(x) \in \mathcal{I}(\lambda_k)$  (sing)  $f_k(x) = s_h^{-1}(\min(\lambda_k s_k(x), s_h(\beta_h)))$  for some  $h \in K$  so that  $f(a_k) \leq a_h, f(b_k) = b_h, s_h(b_h)$  is finite and  $(s_h(b_h)/s_k(b_k)) \le \lambda_k < \infty$ .

Note that in case of (ssi) and (sni), f is constant on the whole corresponding interval  $]a_k, b_k[$ , respectively,  $]a_k, b_k]$  attaining its value at an idempotent element of S. In case of (ssg) and (sng), there exists at least one  $x_0 \in ]a_k, b_k[$  such that  $f(x_0) \notin$  $\mathcal{I}(S)$  so that necessarily there exists some  $h \in K$  fulfilling  $f(x_0) \in ]a_h, b_h[.$ 

The previous theorem already indicates how all distributive functions  $f \in \mathcal{F}_S$  for some continuous t-conorm S can be obtained:

Theorem 26 ([7]): Consider some continuous t-conorm Sand use the notations as introduced in Theorem 25. Any  $f \in \mathcal{F}_S$ is obtained from a generic function  $f^+:\mathcal{I}(S)\to\mathcal{I}(S)$  which is monotone nondecreasing and from its restrictions  $f_k$  for every interval  $]\alpha_k, \beta_k[$  whereas each restriction  $f_k$  is chosen either by expression (ssi), respectively, (ssg) in case that  $s(b_k) = \infty$  or by expression (sni), respectively, (sng) in case that  $s(b_k) < \infty$ .

*Example 27:* Consider the basic t-conorm  $S_{\mathbf{P}}(x, y) = x + y - y$ *xy*. It is continuous with  $\mathcal{I}(S) = \{0, 1\}$  and  $s : [0, 1] \to [0, \infty]$ ,  $s(x) = -\ln(1-x)$ . Its set of distributive functions is given by

$$\mathcal{F}_{S_{\mathbf{P}}} = \{\mathbf{0}, \mathbf{0}_{[0,1[}, \mathbf{0}_{\{0\}}, \mathbf{1}\} \cup \{f : [0,1] \to [0,1] \mid f(x) = 1 - (1-x)^{\lambda}, \lambda \in ]0, \infty[\}$$

where  $\mathbf{0}_A : [0,1] \rightarrow [0,1]$  is defined by

$$\mathbf{0}_A(x) = \begin{cases} 0, & \text{if } x \in A\\ 1, & \text{otherwise.} \end{cases}$$

*Example 28:* Consider the basic t-conorm  $S_{\rm L}(x,y) =$  $\min(x + y, 1)$ . It is continuous with  $\mathcal{I}(S) = \{0, 1\},\$  $s: [0,1] \to [0,\infty], s(x) = x$ , and

$$\mathcal{F}_{S_{\mathbf{L}}} = \{\mathbf{0}, \mathbf{0}_{\{0\}}, \mathbf{1}\} \cup \{f : [0, 1] \to [0, 1] \mid f(x) = \min(\lambda x, 1), \lambda \in [1, \infty[\}.$$

#### B. Continuous T-Norms

Since t-norms are dual to t-conorms we can get analogous results for functions f being distributive over some continuous t-norm T.

Corollary 29: Consider a continuous t-norm T. Then  $[0,1] \setminus \mathcal{I}(T) = \bigcup_{k \in K} ]a_k, b_k[$  for some index set K and there exists a family of continuous strictly decreasing mappings  $t_k : [a_k, b_k] \to [0, \infty]$  with  $t(b_k) = 0$  such that (12), shown at the bottom of the page, holds. Let  $f \in \mathcal{F}_T$  and denote by  $f_k$  its restriction to the interval  $]a_k, b_k[.$ 

i) If  $t_k(a_k) = \infty$ , then one of the following holds: (tsi)  $f_k(x) = i_k$  with  $i_k \in \mathcal{I}(T)$  and  $f(a_k) \leq i_k \leq$  $f(b_k);$ (tsg)  $f_k(x) = t_h^{-1}(\min(\lambda_k t_k(x), t_h(a_h)))$  for some  $\lambda_k \in [0, \infty[$  and some  $h \in K$  such that  $f(a_k) \le a_h$ and  $f(b_k) \ge b_h$ .

ii) If  $t_k(a_k) < \infty$ , then one of the following holds:

(tni)  $f_k(x) = f(a_k) \in \mathcal{I}(S);$ 

(ing)  $f_k(x) = t_h^{-1}(\min(\lambda_k t_k(x), t_h(a_h)))$  for some  $h \in K$  such that  $f(a_k) = a_h, f(b_k) \ge b_h, t_h(a_h)$  is finite and  $(t_h(a_h)/t_k(a_k)) \leq \lambda_k < \infty$ .

Analogous to the case of continuous t-conorms all functions fbeing distributive over some continuous t-norm T can be found.

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right), & \text{if } (x,y) \in [a_k, b_k]^2\\ \min(x,y), & \text{otherwise} \end{cases}$$
(9)

$$S(x,y) = \begin{cases} a_k + (b_k - a_k)S_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right), & \text{if } (x,y) \in [a_k, b_k]^2\\ \max(x,y), & \text{otherwise} \end{cases}$$
(10)

$$S(x,y) = \begin{cases} s_k^{-1}(\min(s_k(x) + s_k(y), s_k(b_k))), & \text{if } (x,y) \in [a_k, b_k]^2\\ \max(x,y), & \text{otherwise} \end{cases}$$
(11)

$$T(x,y) = \begin{cases} t_k^{-1}(\min(t_k(x) + t_k(y), t_k(a_k))), & \text{if } (x,y) \in [a_k, b_k]^2\\ \min(x, y), & \text{otherwise} \end{cases}$$
(12)

Corollary 30: Consider some continuous t-norm t and use the notations as introduced in Corollary 29. Any  $f \in \mathcal{F}_T$  is obtained from a generic function  $f^+ : \mathcal{I}(T) \to \mathcal{I}(T)$  which is monotone nondecreasing and from its restrictions  $f_k$  for every interval  $]\alpha_k, \beta_k[$  whereas each restriction  $f_k$  is chosen either by expression (tsi), respectively, (tsg) in case that  $t(a_k) = \infty$  or by expression (tni), respectively, (tng) in case that  $t(a_k) < \infty$ .

*Example 31:* Consider the two basic t-norms  $T_{\mathbf{P}}(x, y) = xy$ and  $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$ . For both we have that  $\mathcal{I}(T_{\mathbf{P}}) = \mathcal{I}(T_{\mathbf{L}}) = \{0, 1\}$  and their additive generators are given by  $t_{T_{\mathbf{P}}}(x) = -\ln(x)$  and  $t_{T_{\mathbf{L}}} = 1 - x$ , respectively. Further, we get that

$$\mathcal{F}_{T_{\mathbf{P}}} = \{\mathbf{0}, \mathbf{0}_{[0,1[}, \mathbf{0}_{\{0\}}, \mathbf{1}\} \cup \{f : [0,1] \to [0,1] \mid f(x) = x^{\lambda} \text{ with } \lambda \in ]0, \infty[\}, \\ \mathcal{F}_{T_{\mathbf{L}}} = \{\mathbf{0}, \mathbf{0}_{[0,1[}, \mathbf{1}\} \cup \{f : [0,1] \to [0,1] \mid f(x) = \max(\lambda x + 1 - \lambda, 0) \text{ with } \lambda \in [1,\infty[]\}.$$

# VI. CHARACTERIZATION OF $\mathcal{F}_{\mathbf{A}}$ for (Particular Classes of) Uninorms

Let us now turn to the last class of bisymmetric aggregation operators with some neutral element, namely uninorms whose neutral elements e fulfill  $e \in ]0, 1[$  (see also [11], [17]). Note that uninorms U can be interpreted as combination of some t-norm and some t-conorm, i.e.,

$$U_{(n)}(x_1, \dots, x_n) = U_{(2)}(T(\min(x_1, e), \dots, \min(x_n, e)),$$
  
$$S(\max(x_1, e), \dots, \max(x_n, e)))$$

with T some t-norm acting on [0, e] and S some t-conorm acting on [e, 1]. To express explicitly that some uninorm U is related to some t-norm T and some t-conorm S, we will use the notation  $U_{T,S}$ .

Such created uninorms cover a quite large class of aggregation operators since on the remainder of their domains they can be chosen such that the monotonicity and associativity condition are not violated but otherwise arbitrarily. However, due to its properties any uninorm U fulfills

$$\min(x, y) \le U(x, y) \le \max(x, y)$$

whenever  $\min(x, y) \leq e$  and  $e \leq \max(x, y)$  for all  $x, y \in [0, 1]$ , giving rise to the particular classes  $U_{T,S,\min}, U_{T,S,\max}$  of uninorms. Note further, there exists no uninorm which is continuous on the whole domain [17]. Generated uninorms, which we will discuss later in more detail, therefore, form another important subclass of uninorms, since they are continuous on the whole domain up to the case where  $\{x, y\} = \{0, 1\}$ .

As the next section will show, functions f distributing with some uninorm U heavily depend on the structure of the uninorm. Therefore, since a full characterization of all uninorms is still missing, we restrict the discussion of  $\mathcal{F}_U$  to two particular subclasses of uninorms—namely to uninorms which are either acting as the minimum or as the maximum on their remainders and to generated uninorms.

#### A. Distributive Functions on Uninorms

First of all let us investigate necessary and sufficient conditions for some nondecreasing function  $f : [0,1] \rightarrow [0,1]$  being distributive over some uninorm U, i.e., for all  $x, y \in [0,1]$ 

$$U(f(x), f(y)) = f(U(x, y)).$$

If we choose x = e we see that U(f(e), f(y)) = f(y) for all  $y \in [0, 1]$ , expressing that f(e) acts as a neutral element of U on the range of f. Moreover, U(f(e), f(e)) = f(e) so that necessarily  $f(e) \in \mathcal{I}(U)$ .

From this, we see already, that the set of idempotent elements as well as the range of  $f \in \mathcal{F}_U$  will play a crucial role in characterizing  $\mathcal{F}_U$ .

*Lemma 32:* Consider some  $f \in \mathcal{F}_U$ . Then, the following holds:

- i) if  $e \in \operatorname{Ran}_f$ , then f(e) = e;
- ii) if  $d \in \mathcal{I}(U)$ , then also  $f(d) \in \mathcal{I}(U)$ .

*Proof:* Consider some  $f \in \mathcal{F}_U$ . If  $e \in \operatorname{Ran}_f$  then there exists some  $x_0 \in [0, 1]$ , such that  $f(x_0) = e$  and

$$f(e) = U(e, f(e)) = U(f(x_0), f(e))$$
  
=  $f(U(x_0, e)) = f(x_0) = e.$ 

Moreover, if  $d \in \mathcal{I}(U)$  then also

$$f(d) = f(U(d, d)) = U(f(d), f(d)),$$

i.e.,  $f(d) \in \mathcal{I}(U)$ .

Let us now briefly focus on particular cases where  $e \notin \operatorname{Ran}_{f}$ .

Proposition 33: Consider some uninorm  $U_{T,S}$  with neutral element e and some  $f \in \mathcal{F}$  with either  $\operatorname{Ran}_f \subseteq [0, e[$  or  $\operatorname{Ran}_f \subseteq ]e, 1]$ . Then, the following holds:

In case  $\operatorname{Ran}_f \subseteq [0, e[: f \in \mathcal{F}_{U_{T,S}}]$  if and only if

- i)  $f(e) \in \mathcal{I}(U) \cap [0, e[;$
- ii)  $\forall x \in [e, 1]$  : f(x) = f(e);
- iii)  $f \mid [0,e]$  is distributive over T;
- iv)  $\forall x \in [0,1]$  : f(U(x,1)) = f(x).
- In case  $\operatorname{Ran}_f \subseteq ]e, 1]$  :  $f \in \mathcal{F}_{U_{T,S}}$  if and only if
- i)  $f(e) \in \mathcal{I}(U) \cap ]e, 1];$
- ii)  $\forall x \in [0, e] : f(x) = f(e);$
- iii)  $f \mid [e,1]$  is distributive over S
- iv)  $\forall x \in [0,1]$  : f(U(x,0)) = f(x).

*Proof:* Consider some uninorm  $U = U_{T,S}$  with neutral element e, some  $f \in \mathcal{F}$  with  $\operatorname{Ran}_f \subseteq [0, e]$ . Assume that  $f \in \mathcal{F}_U$ .

Since e is an idempotent element of U and  $\operatorname{Ran}_f \subseteq [0, e]$ , it immediately follows that  $f(e) \in \mathcal{I}(U) \cap [0, e]$ , i.e., f(e) is an idempotent element of the t-norm T involved.

Further, since f(e) acts as a neutral element on  $\operatorname{Ran}_f$  we know that for all  $x \in [e, 1]$  it holds that

$$f(x) = U(f(x), f(e)) = T(f(x), f(e))$$
$$\leq \min(f(x), f(e)) \leq f(e).$$

Moreover, due to the nondecreasingness of  $f, f(e) \le f(x)$  for all  $x \in [e, 1]$  such that indeed f(x) = f(e) for all  $x \in [e, 1]$ .



Fig. 1. Uninorm U and some  $f \in \mathcal{F}_U$  as discussed in Examples 34 and 40.

The fact that  $f \mid [0,e]$  is distributive over T follows immediately from  $f \in \mathcal{F}_{U_{T,S}}$ . Finally, choose arbitrary  $x \in [0,1]$ , then due to property ii)

$$\begin{split} f(U(x,1)) &= U(f(x),f(1)) = U(f(x),f(e)) \\ &= f(U(x,e)) = f(x). \end{split}$$

To prove the sufficiency, assume that  $\operatorname{Ran}_f \subseteq [0, e]$  and that conditions i)-iv) are fulfilled. If both  $x, y \leq e$  then also  $U(x, y) \leq e$ , such that f distributes over U due to condition iii). In case that both  $x, y \geq e$ , also  $U(x, y) \geq e$  such that

$$f(U(x,y)) = f(e) = U(f(e), f(e)) = U(f(x), f(y))$$

due to condition ii) and the fact that f(e) is an idempotent element of U. Finally, let us consider w.l.o.g. some  $x \le e \le y$ . Due to condition iv) and the nondecreasingness of f and U we can conclude that

$$f(U(x,1)) = f(x) = f(U(x,e)) \quad \Rightarrow \quad f(x) = f(U(x,y))$$

Moreover, since  $f \mid [0,e]$  commutes with T resp., U and condition ii), we also know that

$$f(x) = f(U(x, e)) = U(f(x), f(e)) = U(f(x), f(y)),$$

such that  $f \in \mathcal{F}_U$ . Analogously, the remaining case and the characterization of  $f \in \mathcal{F}_U$  in case of  $\operatorname{Ran}_f \subseteq ]e, 1]$  can be shown.

Let us illustrate the previous results by some examples.

*Example 34:* Consider the following uninorm  $U: [0,1]^2 \rightarrow [0,1]$  with neutral element  $e = \frac{1}{2}$ 

$$U(x,y) = \begin{cases} 2xy, & \text{if } (x,y) \in [0,\frac{1}{2}]^2\\ \max(x,y), & \text{otherwise.} \end{cases}$$

Note that  $U = U_{T,S}$  with  $T : [0, e]^2 \to [0, e], T(x, y) = 2xy$  is an isomorphic transformation of the product and  $S : [e, 1]^2 \to [e, 1], S(x, y) = \max(x, y)$  (see also Fig. 1). Its set of idempotent elements  $\mathcal{I}(U)$  is given by  $\{0\} \cup [\frac{1}{2}, 1]$  since the continuous t-norm T has its boundaries as its only trivial idempotent elements.

- Therefore, there is only one function  $f \in \mathcal{F}_U$  with  $\operatorname{Ran}_f \subseteq [0, e]$ , namely the constant function **0**, since  $\mathcal{I}(U) \cap [0, e] = \{0\}$  and f has to be nondecreasing.
- On the other hand, there exist several functions f ∈ F<sub>U</sub> with Ran<sub>f</sub> ⊆ ]e, 1]: We can choose f(e) ∈ ]e, 1] ⊆ I(U) ∩ ]e, 1] arbitrarily and fix as such f(x) for all x ∈ [0, e]. Because S = max is a lattice polynomial, f has just to be nondecreasing on [e, 1] to distribute over S such that condition iii) of Proposition 33 is fulfilled. Finally, condition iv) trivially holds since f(U(x,0)) = f(0) = f(x) in case of x ∈ [0, e] and f(U(x,0)) = f(max(x,0)) = f(x) for all x ∈ ]e, 1]. Therefore, e.g., all functions f<sub>λ</sub> : [0, 1] → [0, 1] with λ ∈ ]<sup>1</sup>/<sub>2</sub>, 1] given by

$$f_{\lambda}(x) = \begin{cases} \lambda, & \text{if } x \in [0, \frac{1}{2}]\\ 2(1-\lambda)x + 2\lambda - 1, & \text{otherwise} \end{cases}$$

distribute over U (see also Fig. 1). *Example 35:* Consider the following uninorm  $U : [0,1]^2 \rightarrow [0,1]$  with neutral element  $e = \frac{1}{2}$ 

$$U(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [\frac{1}{2},1]^2\\ 4xy, & \text{if } (x,y) \in [0,\frac{1}{4}]^2\\ \frac{1}{4}(4x-1)(4y-1) + \frac{1}{4}, & \text{if } (x,y) \in [\frac{1}{4},\frac{1}{2}]^2\\ \min(x,y), & \text{otherwise.} \end{cases}$$



Fig. 2. Uninorm U and some  $f \in \mathcal{F}_U$  as discussed in Examples 35 and 41.

Again  $U = U_{T,S}$  with T on ordinal sum t-norm on  $[0, \frac{1}{2}]$  with twice the product as its summands and  $S = \max$  a basic t-conorm on  $[\frac{1}{2}, 1]$  (see also Fig. 2). The set of idempotent elements  $\mathcal{I}(U)$  is given by  $\{0, \frac{1}{4}\} \cup [\frac{1}{2}, 1]$ .

Let us now focus just on those  $f \in \mathcal{F}_U$  with  $\operatorname{Ran}_f \subseteq [0, \frac{1}{2}]$ , i.e.,  $f(e) \in \{0, \frac{1}{4}\}$ .

- f(e) = 0: Necessarily f = 0 due to the nondecreasingness of f and the necessary properties given in Proposition 33. Therefore, 0 is the only element of *F<sub>U</sub>* for which Ran<sub>f</sub> ⊆ [0, e] and f(e) = 0.
- f(e) = <sup>1</sup>/<sub>4</sub>: Necessarily, we fix f(x) = <sup>1</sup>/<sub>4</sub> for all x ∈ [<sup>1</sup>/<sub>2</sub>, 1] and as such fulfill conditions i), ii), and iv) of Proposition 33 immediately, i.e.,

$$f:[0,1] \to [0,1], \quad f(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in \left[\frac{1}{2},1\right]\\ g(x), & \text{otherwise.} \end{cases}$$

The function  $g: [0, \frac{1}{2}[ \rightarrow [0, \frac{1}{2}[$  and as such also  $f \mid [0,e]$  distributes over the ordinal sum t-norm T if it is one of the following functions (see also Fig. 2):

$$g_{1}(x) = 0, \qquad \forall x \in [0, \frac{1}{2}[$$

$$g_{2,\lambda}(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{4}] \\ \frac{1}{4}(4x-1)^{\lambda}, & \text{otherwise} \end{cases}$$

$$g_{3}(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{4}] \\ \frac{1}{4}, & \text{otherwise} \end{cases}$$

$$g_{4}(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{4}] \\ \frac{1}{4}, & \text{otherwise} \end{cases}$$

$$g_{5,\lambda}(x) = \begin{cases} 4^{\lambda-1}x^{\lambda}, & \text{if } x \in [0, \frac{1}{4}[\\ \frac{1}{4}, & \text{otherwise} \end{cases}$$
$$with \ \lambda \in [1, \infty[, \\g_6(x) = \begin{cases} 0, & \text{if } x = 0\\ \frac{1}{4}, & \text{otherwise} \end{cases}$$
$$g_7(x) = \frac{1}{4}, & \forall x \in [0, \frac{1}{2}[. \end{cases}$$

So far, we have investigated nondecreasing functions f with particular domains being distributive over some uninorm U. However, in case that  $e \in \operatorname{Ran}_f$  the characterization of those  $f \in \mathcal{F}_U$  heavily depends on the structure of the uninorm Uinvolved. Therefore, we will now turn to special subclasses of uninorms.

# B. Special Case: Uninorms $U_{T,S,\min}, U_{T,S,\max}$

We now assume that the uninorm U is such that  $U \mid [0,e]^2$  is some t-norm T on  $[0,e], U \mid [e,1]^2$  some t-conorm S on [e,1] and on the remainder U acts either as the minimum or as the maximum. We will denote such uninorms by  $U_{T,S,\min}$ , respectively,  $U_{T,S,\max}$ . In case that the t-norm T as well as the t-conorm Sinvolved are continuous, we refer to the uninorm  $U_{T,S}$  as weakly continuous t-norm.

We will focus on functions f based on a composition of functions distributive over T, respectively, S, i.e., on functions f:  $[0,1] \rightarrow [0,1]$  defined by

$$f(x) = \begin{cases} f_T(x), & \text{if } x \in [0, e[\\ f_S(x), & \text{if } x \in ]e, 1]\\ e, & \text{if } x = e \end{cases}$$
(13)

with some  $f_T \in \mathcal{F}_T$  and  $f_S \in \mathcal{F}_S$ . We will use the abbreviation  $f = f_T \boxplus_e f_S$ .

Note that not all  $f \in \mathcal{F}_U$  are of the type  $f = f_T \boxplus_e f_S$  as the following example shows.

*Example 36:* Consider some weakly continuous uninorm  $U_{T,S}$ . Then,  $f : [0,1] \rightarrow [0,1]$  defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, e] \\ 1, & \text{if } x \in ]e, 1 \end{cases}$$

fulfills  $f \in \mathcal{F}_{U_{T,S,\min}}$  and  $f \in \mathcal{F}_{U_{T,S,\max}}$ , but  $f \neq f_T \boxplus_e f_S$ .

However, since uninorms can be interpreted as operators acting on a bipolar scale with neutral element e, it is natural to investigate distributive functions f preserving that neutrality level, i.e., fulfilling f(e) = e.

By the construction  $f = f_T \boxplus_e f_S$  provided by (13), it is guaranteed that the restrictions of some  $f \in \mathcal{F}_{U_{T,S}}$  to [0, e] resp. [e, 1] are distributive over the corresponding T, respectively, S. Note that this construction also ensures that, due to the nondecreasingness of f, that  $f(x) \leq e$  for all  $x \in [0, e]$  and  $f(x) \geq e$ for all  $x \in [e, 1]$ . Depending on whether  $U = U_{T,S,\min}$  or  $U = U_{T,S,\max} f$  has to fulfill additional properties for  $f \in \mathcal{F}_U$ .

Proposition 37: Consider some weakly continuous uninorm  $U_{T,S}$ , further some  $f_T \in \mathcal{F}_T$  and  $f_S \in \mathcal{F}_S$  and define f:  $[0,1] \to [0,1]$  by  $f = f_T \boxplus_e f_S$ .

- i)  $f \in \mathcal{F}_{U_{T,S,\min}}$ , if and only if  $\forall x \in [0, e[: f(x) < e \text{ or } \forall y \in [e, 1] : f(y) = e.$
- ii)  $f \in \mathcal{F}_{U_{T,S,\max}}$  if and only if  $\forall x \in [0,e]$  : f(x) = e or  $\forall y \in ]e,1]$  : f(y) > e.

*Proof:* Consider some weakly continuous uninorm  $U_{T,S}$ , further some  $f_T \in \mathcal{F}_T$  and  $f_S \in \mathcal{F}_S$  and define  $f : [0,1] \rightarrow [0,1]$  as  $f = f_T \boxplus_e f_S$  by (13).

Assume that  $f \in \mathcal{F}_{U_{T,S,\min}}$ . Further assume that there exists some  $x_0 \in [0, e[$  with  $f(x_0) = e$  and some  $y_0 \in [e, 1]$  with  $f(y_0) > e$ , then the following holds:

$$f(y_0) = U(e, f(y_0)) = U(f(x_0), f(y_0)) = f(U(x_0, y_0))$$
  
=  $f(\min(x_0, y_0)) = f(x_0) = e$ 

leading to a contradiction. Vice versa, since  $f = f_T \boxplus_e f_S$ it distributes over  $U_{T,S,\min}$  for all  $(x,y) \in [0,e]^2$  and for all  $(x,y) \in [e,1]^2$  due to its construction. Therefore, it suffices to prove that f distributes over U for all  $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$ .

Assume that f additionally fulfills either for all  $x \in [0, e[$ : f(x) < e or for all  $y \in [e, 1]$  : f(y) = e and choose an arbitrary  $x \in [0, e[$  and an arbitrary  $y \in [e, 1]$ . Therefore, either f(x) < e or f(y) = e, in any case  $f(x) \leq f(y)$ , such that

$$f(U(x,y)) = f(\min(x,y)) = f(x) = \min(f(x), f(y))$$
  
= U(f(x), f(y)).

In case that x = e and  $y \in [e, 1]$ , it immediately holds that

$$f(U(x,y)) = f(U(e,y)) = f(y) = U(e, f(y))$$
  
= U(f(e), f(y)) = U(f(x), f(y)).

Analogously, the distributivity of f over  $U_{T,S,\min}$  for some  $(x,y) \in [e,1] \times [0,e]$  can be shown as well as the characterization of all  $f \in U_{T,S,\max}$ .

Based on this result, we can immediately state which functions  $f = f_T \boxplus_e f_S$  are distributive over both  $U_{T,S,\min}$  as well  $U_{T,S,\max}$ .

Lemma 38: Consider some weakly continuous uninorm  $U_{T,S}$ , further some  $f_T \in \mathcal{F}_T$  and  $f_S \in \mathcal{F}_S$  and define  $f : [0,1] \rightarrow [0,1]$  by  $f = f_T \boxplus_e f_S$ .  $f \in \mathcal{F}_{U_{T,S,\min}} \cap \mathcal{F}_{U_{T,S,\max}}$  if and only if either

•  $\forall x \in [0, e[: f(x) < e \text{ and } \forall x \in ]e, 1] : f(x) > e;$ 

•  $\forall x \in [0,1]$  : f(x) = e.

Moreover, due to Proposition 37 and the nondecreasingness of f we can further draw the following conclusions.

Corollary 39: Consider some weakly continuous uninorm  $U_{T,S}$ , further some  $f_T \in \mathcal{F}_T$  and  $f_S \in \mathcal{F}_S$  and define  $f : [0,1] \to [0,1]$  by  $f = f_T \boxplus_e f_S$ .

- i) If  $f \in \mathcal{F}_{U_{T,S,\min}}$  and there exists some  $x_0 \in [0, e[$  such that  $f(x_0) = e$ , then f(x) = e for all  $x \in [x_0, 1]$ .
- ii) If  $f \in \mathcal{F}_{U_{T,S,\max}}$  and there exists some  $y_0 \in ]e, 1]$  such that  $f(y_0) = e$ , then f(x) = e for all  $x \in [0, y_0]$ .

*Example 40:* (Continuation of Example 34) Let us once again consider the uninorm U as introduced in Example 34, i.e.,

$$U(x,y) = \begin{cases} 2xy, & \text{if } (x,y) \in [0,\frac{1}{2}]^2\\ \max(x,y), & \text{otherwise.} \end{cases}$$

It is of the type  $U_{T,S,\max}$  with  $T:[0,e]^2 \rightarrow [0,e]$ , T(x,y) = 2xy an isomorphic transformation of the product and  $S:[e,1]^2 \rightarrow [e,1]$  the maximum. Now, we are looking for those  $f \in \mathcal{F}_U$  which are constructed by  $f = f_T \boxplus_e f_S$ . The sets  $\mathcal{F}_T$  and  $\mathcal{F}_S$  of nondecreasing functions distributing with T, respectively, S are given by

$$\mathcal{F}_T = \left\{ f : \left[0, \frac{1}{2}\right] \to \left[0, \frac{1}{2}\right] \mid \forall x \in \left[0, \frac{1}{2}\right] : \\ f(x) = 2^{(\lambda - 1)} x^{\lambda} \text{ with } \lambda \in \left]0, \infty\right[ \text{ or } \\ f(x) = 0 \text{ or } f(x) = \frac{1}{2} \right\},$$
$$\mathcal{F}_S = \left\{ f : \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right] \mid f \text{ is nondecreasing} \right\}.$$

In accordance with Proposition 37, we now have to choose either  $f_T(x) = \frac{1}{2}$  for all  $x \in [0, \frac{1}{2}[$  or  $f_S(y) > \frac{1}{2}$  for all  $y \in ]\frac{1}{2}, 1]$  such that  $f = f_T \boxplus_e f_S$  fulfills  $f \in \mathcal{F}_U$ , so, e.g.,  $f_i : [0, 1] \to [0, 1]$ ,  $i \in \{1, 2, 3\}$ , (see also Fig. 1)

$$f_1(x) = \begin{cases} 2x^2, & \text{if } x \in [0, \frac{1}{2}] \\ x, & \text{otherwise} \end{cases}$$
$$f_2(x) = \min(2x^2, 1),$$
$$f_3(x) = \max\left(\frac{1}{2}, f_2(x)\right).$$

*Example 41:* (Continuation of Example 35) Note that the uninorm U defined by

$$U(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in \left[\frac{1}{2},1\right]^2 \\ 4xy, & \text{if } (x,y) \in \left[0,\frac{1}{4}\right]^2 \\ \frac{1}{4}(4x-1)(4y-1) + \frac{1}{4}, & \text{if } (x,y) \in \left[\frac{1}{4},\frac{1}{2}\right]^2 \\ \min(x,y), & \text{otherwise} \end{cases}$$

is a uninorm of the type  $U_{T,S,\min}$ . Since T is an ordinal sum t-norm on  $[0, \frac{1}{2}]$ , its set of distributive functions  $\mathcal{F}_T$  is rather large. Some of its elements are already listed in Example 35. Similarly, also  $\mathcal{F}_S$  contains many, namely all nondecreasing functions on  $[\frac{1}{2}, 1]$ . In accordance with Proposition 37 we have to choose functions  $f_T$ ,  $f_S$  such that either  $f_T(x) > \frac{1}{2}$  for all  $x \in [0, \frac{1}{2}[$  or that  $f_S(y) = \frac{1}{2}$  for all  $x \in ]\frac{1}{2}, 1]$ , such that  $f = f_T \boxplus_e f_S \in \mathcal{F}_U$ , e.g.,  $f_i : [0, 1] \to [0, 1], i \in \{1, 2, 3\}$ , (see also Fig. 2)

$$f_1(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}[\\ x, & \text{otherwise} \end{cases}$$
$$f_2(x) = \frac{1}{2},$$
$$f_3(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{4}[\\ \frac{1}{4}(4x-1)^2, & \text{if } x \in [\frac{1}{4}, \frac{1}{2}\\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

#### C. Special Case: Generated Uninorms

An important subclass of uninorms are those generated by some additive generator. They are continuous on the whole domain up to the case where  $\{x, y\} = \{0, 1\}$ .

Definition 42: An operator  $\mathbf{U}: \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$  is an Archimedean uninorm continuous in all points up to  $(x_1,\ldots,x_n), \{0,1\} \in \{x_1,\ldots,x_n\}$ , if and only if there exists a monotone bijection  $h: [0,1] \to [-\infty,\infty]$  such that

$$\mathbf{U}(x_1,\ldots,x_n) = h^{-1}\left(\sum_{i=1}^n h(x_i)\right)$$

with convention  $+\infty + (-\infty) = -\infty$ . The uninorm U is then called a *generated* uninorm with *additive generator* h.

Note that the neutral element e of such a generated uninorm is given by  $h^{-1}(0) = e$ . The increasingness of the additive generator is equivalent to its conjunctive form. Moreover, generated uninorms are related to strict t-norms and strict t-conorms, since t(x) = -h(ex) and s(x) = h(e+(1-e)x) are additive generators of a strict t-norm, respectively, t-conorm, associated with U.

In case of some  $f \in \mathcal{F}_U$  with U generated by the additive generator h, we get

$$\begin{split} f(U(x,y)) &= f \circ h^{-1}(h(x) + h(y)) \\ &= h^{-1}(h \circ f(x) + h \circ f(y)) = U(f(x), f(y)). \end{split}$$

Since h is a bijection this is equivalent to

$$h\circ f\circ h^{-1}(u+v)=h\circ f\circ h^{-1}(u)+h\circ f\circ h^{-1}(v)$$

with h(x) = u and h(y) = v both elements from  $[-\infty, \infty]$  such that for  $h^* = h \circ f \circ h^{-1}$  and arbitrary  $u, v \in [-\infty, \infty]$  it holds that  $h^*(u+v) = h^*(u) + h^*(v)$ . In case that  $h^*$  is continuous the solutions of this equation (see also [3]) are given by

$$h^*(u) = c \cdot u$$

with c > 0. As a consequence

$$f(x) = h^{-1}(c \cdot h(x))$$

leading to the following lemma.

*Lemma 43:* Consider some uninorm U generated by some additive generator h. If  $f \in \mathcal{F}_U$  and f continuous, but not constant, then there exists some c > 0 such that

$$f(x) = h^{-1}(c \cdot h(x))$$

for all  $x \in [0,1]$ .

*Example 44:* Consider some uninorm U generated by some additive generator h and choose  $c_i \ge 0$  for all  $i \in \{1, \ldots, n\}$  and  $c_i > 0$  for at least one  $i \in \{1, \ldots, n\}$ . Then, the operator A defined by

$$A(x_1,\ldots,x_n) = h^{-1}\left(\sum_{i=1}^n c_i h(x_i)\right)$$

commutes with U.

*Example 45:* Consider the additive generator  $h : [0,1] \rightarrow [-\infty,\infty], h(x) = \ln(x/1-x)$ . The generated uninorm  $U^* : [0,1]^2 \rightarrow [0,1]$  is then given by

$$U^*(x,y) = \frac{xy}{xy + (1-x)(1-y)}$$

with neutral element e = 0.5. Note that  $U^*$  is also known as 3–II-operator and has already been discussed by several authors [14], [17], [23], [35], [37]. It is worth remarking that it plays an important role as combining functions of uncertainty factors in expert systems like MYCIN and PROSPECTOR (see also [9], [12], and [21]).

In accordance with the previous example, aggregation operators  $\mathbf{A} : \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$  defined by

$$\mathbf{A}(x_1,\dots,x_n) = \frac{\prod_{i=1}^n x_i^{c_i}}{\prod_{i=1}^n (1-x_i)^{c_i} + \prod_{i=1}^n x_i^{c_i}}$$

with  $c_i \ge 0$  for all  $i \in \{1, \ldots, n\}$  and  $c_i > 0$  for at least one  $i \in \{1, \ldots, n\}$  commute with  $U^*$ .

#### VII. FINAL REMARKS

The issue of commuting aggregations has been considered in the general case and in some important particular cases, especially the one of uninorms, where new nontrivial results are obtained. Finding commuting operations can be a difficult exercise sometimes leading to impossibility results. So, e.g., in the class of OWA operators [36], the set of all aggregation operators commuting with an *n*-ary OWA operator different from min, max, or the arithmetic mean, respectively, is trivial, namely, consisting just of the projections [32]. However, for bisymmetric operations such as the weighted arithmetic mean, results on commuting exist for some 25 years in connection with the problem of consensus functions for probabilities [28], more recently for t-norms and conorms in connection with generalized utility theory [15] or transitivity preservation in the aggregation of fuzzy relations [34]. Commuting operators for uninorms can be relevant in multicriteria decision-making with bipolar scales where bipolar set-functions are used to evaluate the importance of criteria [19], [20]. Indeed the neutral element of the uninorm separates a bipolar evaluation scale in its positive and negative parts [16]. Our results can be instrumental in laying bare consensus functions for multiperson multicriteria decision-making problems on bipolar scales, a topic to be investigated at a further stage.

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