Boolean algebras with an automorphism group: a framework for Łukasiewicz logic

Thomas Vetterlein

European Centre for Soft Computing,
Edificio Científico-Tecnológico, C/ Gonzalo Gutiérrez Quirós s/n,
33600 Mieres, Spain
Thomas.Vetterlein@softcomputing.es

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Abstract

We introduce a framework within which reasoning according to Łukasiewicz logic can be represented. We consider a separable Boolean algebra $\mathcal{B}$ endowed with a (certain type of) group G of automorphisms; the pair $(\mathcal{B}, G)$ will be called a Boolean ambiguity algebra. $\mathcal{B}$ is meant to model a system of crisp properties; G is meant to express uncertainty about these properties.

We define fuzzy propositions as subsets of $\mathcal{B}$ which are, most importantly, closed under the action of G. By defining a conjunction and implication for pairs of fuzzy propositions in an appropriate manner, we are led to the algebraic structure characteristic for Łukasiewicz logic.

Key words: Łukasiewicz logic, MV-algebra, Boolean algebra, Boolean algebra with an automorphism group

1 Introduction

Since the times when many-valued logics were introduced, the question has been raised again and again which kind of properties the propositions of these logics actually express. Numerous suggestions to gain a better understanding were made; for a general overview, see e.g. [13]. We are primarily interested in many-valued logics which are based on a t-norm and its corresponding residuum. A nice compilation of approaches of how to interpret t-norm based logics can be found in J. Paris's article [11]. Among the approaches followed at present, we want to mention the dialogue-game based semantics, going back to R. Giles and later elaborated in particular by C. Fermüller; see e.g. [3, 4].

The present article aims at a better understanding of the meaning of Łukasiewicz logic [2]. It was actually not our original intention to focus on this specific logic, and so we will not justify this choice by referring to the common agreement about its particular importance. As a matter of fact, it simply was this logic which "came out" in the end. We add our conjecture that our approach can be modified so as to cover also weaker logics and in particular Hájek's *Basic Logic* [6].

Generally, our approach to the foundational problem of fuzzy logics is as follows. We intend to provide a framework within which a system of entities suitable to interpret propositions of many-valued calculi can be singled out, and which furthermore allows us to define a conjunction and an implication in a natural way. There should be a clear evidence that our basic understanding of many-valued logics is supported.

In this paper, we will work out a simple idea how such a framework could look like. To understand this idea, consider, for a moment, the case of classical propositional logic. Here, we may choose as the primary semantical notion a system of subsets, representing yes-no properties. Systems of subsets, endowed with the set-theoretical operations, are Boolean algebras and may serve as a framework for classical propositional logic.

Now, we will generalise this framework so as to model statements with which a certain uncertainty is associated. Note that this is in contrast to most approaches to interpret fuzzy logics, where vagueness is the primary notion. What we propose is to model a fuzzy property by a subset of a Boolean algebra rather than a single element. Namely, we assume that we have to do with properties which are perceivable only up to the action of some group of automorphism acting on the Boolean algebra. Accordingly, a subset modelling a fuzzy property is required to be closed under the action of this group.

So our framework will consist of a Boolean algebra and an automorphism group. We note that the group may be the trivial, i.e. the one-element one; in this case we get classical logic. Otherwise, by determining carefully the details of the system of subsets of the Boolean algebra, of the operations on it, and of the automorphism group, we are more or less naturally led to the kind of algebra corresponding to Łukasiewicz logic: MV-algebras.

We should underline that we share our principal aims with the articles [3], [10], and several of those reviewed in [11]; our approach, however, is a rather specific one. At least, we may say that the general idea – to start with an abstract structure and defining from it all what is needed –, is found also elsewhere, in particular in case of the formalism in [10]; but it will be hard to find anything more in common. With respect to the mathematical content of our work, the picture is certainly different. Let us mention the related work of Cignoli, Dubuc, and Mundici, who established that Boolean algebras endowed with one automorphism whose orbits are all finite, correspond to locally finite MV-algebras [1]. Furthermore, one way which we propose here for the interpretation of connectives, is very similar to what appears in Ono and Komori's work [9]. Note finally that the present paper follows the lines of our note [14]. However, the concept of [14] is different and the described formalism is not a special case of the formalism presented here.

Let us summarise our results. We study so-called Boolean ambiguity algebras, which are pairs $(\mathcal{B}, G)$ of a separable Boolean algebra $\mathcal{B}$ and a group G of automorphisms of $\mathcal{B}$. We will first, in Section 2, assume that the Boolean algebra is complete, and we will define fuzzy propositions as the orbits of G, that is, as the sets $\{g(a) : \ g \in G\}$ for $a \in \mathcal{B}$. We require G to be compact – meaning that no element has infinitely many pairwise disjoint images [7] – and full [5]. We will may make use of parts of the theory developed by Kawada in [8]. In particular, we will see that the set of fuzzy propositions is lattice-ordered in a natural way and that we may define a conjunction and an implication in a straightforward manner. The resulting structure is an MV-algebra. Moreover, in the subsequent Section 3, we show that the connectives take a particularly easy form when representing the group orbits in a certain alternative way. In particular, the conjunction becomes simply the pointwise Boolean meet.

In the additional Sections 4 and 5, a variant of this formalism is developed; the difference is that the completeness assumption is dropped. This means that Kawada's results no longer apply and that we are forced to require rather strong conditions for G to be able to derive similar results as before.

Finally, we will see that the MV-algebras which are representable either by means of the basic concept or the variant of it, suffice to generate the whole variety. So we may provide alternative semantics for Łukasiewicz logic based on Boolean ambiguity algebras (Section 6).

2 The orbit algebra of a complete Boolean ambiguity algebra

In this article, we examine pairs $(\mathcal{B}, G)$, where $\mathcal{B}$ is a Boolean algebra and G is a group of automorphisms of $\mathcal{B}$. We should have the following picture in mind. An element of $\mathcal{B}$ should be thought of a sharp property arising in some context. We furthermore assume that, due to principal limitations of our observational capabilities, we may determine any such property only up to the action of G; for any $a \in \mathcal{B}$ and $g \in G$, we are supposed not to be able to distinguish between a and $g(a)$.

This picture motivates us to study the set of all G-orbits of $\mathcal{B}$; a G-orbit is a subset of $\mathcal{B}$ of the form $\{g(a) \in \mathcal{B} \colon\; g \in G\}$ for some $a \in \mathcal{B}$. In a first step, we will consider the case that the Boolean algebra is complete. By this assumption, we may make use of results of Kawada, contained in his famous work [8] to derive the existence of invariant measures.

Note, however, that we will not use Kawada's main result; we will not rely on the existence of an invariant measure. If we wanted, we would have to require additionally either the ergodicity of the group or the measurability of the invariant Boolean subalgebra. For an approach which does rely on measures, see [14].

A Boolean algebra is a structure $(\mathcal{B}; \wedge, \vee, \neg, 0, 1)$ such that $(\mathcal{B}; \wedge, \vee, 0, 1)$ is a distributive $0, 1$-lattice and $\neg$ a complementation function. We write $a \setminus b = a \wedge \neg b$ and $a \to b = \neg a \vee b$ for $a, b \in \mathcal{B}$. For general information about Boolean algebras, we recommend [12]; for automorphism groups, see e.g. [5, Section 381].

A Boolean algebra with a countable dense subset is called separable. Furthermore, for an automorphism g of a Boolean algebra and one of its elements a, we denote by $g|_a$ the restriction of g to the interval $[0, a]$.

Definition 2.1 Let $\mathcal{B}$ be a separable Boolean algebra, and let G be a group of automorphisms of $\mathcal{B}$. Then we call the pair $(\mathcal{B}, G)$ a *Boolean ambiguity algebra*.

For $a, b \in \mathcal{B}$, we write $a \sim b$ if $b = g(a)$ for some $g \in G$. We furthermore write $a \perp b$ if there are no non-zero $a_0 \leq a$ and $b_0 \leq b$ such that $a_0 \sim b_0$. Given $(\mathcal{B}, G)$, we introduce the following notions.

(i) We call G *compact* if for all non-zero $a \in B$, every set of pairwise disjoint elements of the form $g(a)$, where $g \in G$, is finite.

(ii) Let $\mathcal{B}$ be σ-complete. We call G *full* if for any two partitions of unity $(a_i)_{i \leq \lambda}$ and $(b_i)_{i \leq \lambda}$, where $\lambda \leq \omega$, and a system $g_i \in G$, $i \leq \lambda$, such that $g_i(a_i) = b_i$, the automorphism g defined by $g|_{a_i} = g_i|_{a_i}$, belongs to G as well.

The Boolean ambiguity algebra $(\mathcal{B}, G)$ will be called *complete* if $\mathcal{B}$ is σ-complete and G is compact and full.

In view of the intended interpretation of our notions, the relation $\sim$ means the indistinguishability of elements, and $\perp$ means distinguishability in a strict sense.

Compactness of an automorphism group is a variant of the notion "weakly wandering" in [7]. It expresses that the uncertainty about a proposition a multiplies the extent of a by a finite number only. The notion of fullness is taken from [5]. It means that each automorphism which we may define piecewise from those existing in G, is in G as well.

The relation $\sim$ is a specialised version of the equally denoted notion in [8]. In certain proofs, we will need the original one as well, here denoted by $\sim_\infty$. Namely, for elements a, b of a Boolean ambiguity algebra, $a \sim_\infty b$ means that there are two countable sets $a_1, \ldots$ and $b_1, \ldots$ of pairwise disjoint elements as well as $g_1, \ldots \in G$ such that $a = \bigvee_i a_i$ and $b = \bigvee_i b_i$ and $g_j(a_j) = b_j$ for every j.

We shall make use of the following facts [8, Lemmas 5, 6].

Lemma 2.2 *Let $(\mathcal{B}, G)$ be a Boolean ambiguity algebra, and let $\mathcal{B}$ be complete. Call $a \in \mathcal{B}$ finite if $b \leq a$ and $a \sim_\infty b$ imply $a = b$.*

(i) *$a \in \mathcal{B}$ is finite if and only if for any countably infinite set $a_1, a_2, \ldots$ of pairwise disjoint elements such that $a_i \leq a$ for all $i < \omega$ and $a_i \sim_\infty a_j$ for all $i, j < \omega$, we have $a_1 = a_2 = \ldots = 0$.*

(ii) *Let $a, b, c, d \in \mathcal{B}$ be finite. If $a \leq b$, $c \leq d$, $a \sim_\infty c$, $b \sim_\infty d$, then $b \setminus a \sim_\infty d \setminus c$.*

Lemma 2.3 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra.*

(i) *Let $a_1, \ldots$ and $b_1, \ldots$ be two countable sets of pairwise disjoint elements such that $a = \bigvee_i a_i$ and $b = \bigvee_i b_i$; let $g_1, \ldots \in G$ such that $g_j(a_j) = b_j$ for all j. Then there is a $g \in G$ such that $g|_{a_j} = g_j|_{a_j}$ for each j.*

In particular, for any $a, b \in \mathcal{B}$, $a \sim_\infty b$ holds if and only if $a \sim b$.

(ii) *If $a \sim b$ and $a \leq b$, then $a = b$.*

Proof. By construction, $a \sim b$ implies $a \sim_\infty b$.

Assume next $a \wedge b = 0$ and $a \sim_\infty b$; let $a_1, \ldots$, $b_1, \ldots$, and $g_1, \ldots \in G$ as specified in (i). Define the automorphism g as follows: Let $g|_{a_i} = g_i|_{a_i}$ and $g|_{b_i} = g_i^{-1}|_{b_i}$ for every i, and let $g|_{\neg(a \vee b)} = id|_{\neg(a \vee b)}$. By fullness, g is in G, and $g(a) = b$. So we have proved that $a \sim b$ is equivalent to $a \sim_\infty b$ for orthogonal pairs a, b.

Let now $a \leq b$ and $a \sim_\infty b$; we shall show that then $a = b$. In particular, assertion (ii) will then be proved. So let $e_1, \ldots \leq b$ be such that, for $i \neq j$, $e_i \wedge e_j = 0$ and $e_i \sim_\infty e_j$. By the preceding paragraph, $e_i \sim e_j$ for $i \neq j$ then, and by the compactness of G, it follows $e_1 = \ldots = 0$. So by Lemma 2.2(i), we conclude $a = b$.

Assume finally $a \sim_\infty b$. From Lemma 2.2(ii), we conclude $\neg a \sim_\infty \neg b$. By fullness, assertion (i) follows. □

Lemma 2.4 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra.*

(i) *Let $a < b$, $c < d$, $g(a) = c$ for some $g \in G$, and $b \sim d$. Then there is a $\bar{g} \in G$ such that $\bar{g}|_a = g|_a$ and $\bar{g}(b) = d$. In particular, $b \setminus a \sim d \setminus c$.*

(ii) *Let $a, b \in \mathcal{B}$ such that $a \sim b$. Let $c = (a \wedge b) \vee \neg(a \vee b)$, $d = a \setminus b$, $e = b \setminus a$. Then there is a $g \in G$ such that $g|_c = id|_c$, $g(d) = e$, $g(e) = d$.*

Proof. (i) By assumption, g maps a to c and there is a $h \in G$ mapping $\neg b$ to $\neg d$. So by Lemma 2.3(i), there is a $\bar{g}$ such that $\bar{g}|_a = g|_a$ and $\bar{g}|_{\neg b} = h|_{\neg b}$, and $\bar{g}$ obviously has the required property.

(ii) By part (i), there is a $g \in G$ such that $g(d) = e$, implying also $g^{-1}(e) = d$. So the assertion follows from fullness. □

We will now exhibit the basic properties of the set of G-orbits in an Boolean algebra.

Definition 2.5 Let $(\mathcal{B}, G)$ be a Boolean ambiguity algebra. For $a \in \mathcal{B}$, let $[a] = \{b \in \mathcal{B} \colon\ b \sim a\}$, and let $\mathit{0} = [0]$ and $\mathit{1} = [1]$. Let $\mathcal{O}_{(\mathcal{B},G)} = \{[a] \colon\ a \in \mathcal{B}\}$.

For $a, b \in \mathcal{B}$, let $a \precsim b$ if $a' \leq b'$ for some $a' \sim a$ and $b' \sim b$. Furthermore, let $a \prec b$ if $a' < b'$ for some $a' \sim a$ and $b' \sim b$.

For $\varphi, \psi \in \mathcal{O}_{(\mathcal{B},G)}$, we define $\varphi \leq \psi$ if $a \leq b$ for some $a \in \varphi$ and $b \in \psi$. We call $(\mathcal{O}_{(\mathcal{B},G)}; \leq, \mathit{0}, \mathit{1})$ the *orbit preordered set* or, if $\leq$ happens to be a lattice order, the *orbit lattice* of $(\mathcal{B}, G)$.

Note that, for $a, b \in \mathcal{B}$, $a \precsim b$ is equivalent to $[a] \leq [b]$.

Lemma 2.6 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra.*

(i) *Let $a, b, c \in \mathcal{B}$ such that $a \precsim b \precsim c$ and $a \leq c$. Then there is a $b' \sim b$ such that $a \leq b' \leq c$.*

(ii) *Let $a \precsim b$ and $a, b \leq c$. Then $c \setminus b \precsim c \setminus a$.*

Proof. (i) Let $b_1, b_2 \sim b$ such that $a \leq b_1$ and $b_2 \leq c$. By Lemma 2.4(ii), $b_1 \setminus c \leq b_1 \setminus b_2 \sim b_2 \setminus b_1 \leq c \setminus b_1$. It follows $b_1 \setminus c \sim d$ for some $d \leq c \setminus b_1$; so $b' = (c \wedge b_1) \vee d$ fulfils the requirements.

(ii) Let $a' \sim a$ be such that $a' \leq b$. By Lemma 2.4(ii), $a \setminus a' \sim a' \setminus a$. So $c \setminus a = (c \setminus (a \vee a')) \vee (a' \setminus a) \sim (c \setminus (a \vee a')) \vee (a \setminus a') = c \setminus a' \geq c \setminus b$. □

Let us cite one more fact from [8, Lemma 16].

Lemma 2.7 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. Let $a, b \in \mathcal{B}$. Then there is a pair $e, f \in B$ of disjoint elements which are invariant under G such that (α) $a \wedge e \prec b \wedge e$ or $a \wedge e = b \wedge e = 0$, (β) $b \wedge f \prec a \wedge f$ or $a \wedge f = b \wedge f = 0$, and (γ) $a \wedge \neg(e \vee f) \sim_\infty b \wedge \neg(e \vee f)$.*

Proposition 2.8 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. Then $\sim$ is an equivalence relation, and $(\mathcal{O}_{(\mathcal{B},G)}; \leq, 0, 1)$ is a bounded lattice. $0 = \{0\}$ is the smallest and $1 = \{1\}$ is the largest element. Moreover, for any $a, b \in \mathcal{B}$, there is a $b' \sim b$ such that $[a \wedge b'] = [a] \wedge [b]$ and $[a \vee b'] = [a] \vee [b]$.*

Proof. Clearly, $\sim$ is an equivalence relation, and $\precsim$ is reflexive and transitive. So $(\mathcal{O}_{(\mathcal{B},G)}; \precsim)$ is a preordered set. It moreover follows from Lemma 2.6(i) that $\precsim$ is antisymmetric. So $(\mathcal{O}_{(\mathcal{B},G)}; \precsim)$ is a poset with the smallest element $0 = [0]$ and the largest element $1 = [1]$. Evidently $[0] = \{0\}$ and $[1] = \{1\}$.

Let now $a, b \in \mathcal{B}$. By Lemma 2.7, there is an $e \in \mathcal{B}$ which is invariant under G and such that $a \wedge e \precsim b \wedge e$ and $b \wedge \neg e \precsim a \wedge \neg e$. Because G is full, there is a $b' \sim b$ such that $a \wedge e \leq b' \wedge e$ and $b' \wedge \neg e \leq a \wedge \neg e$.

We claim that $[a \wedge b']$ is the infimum of $[a]$ and $[b]$. Clearly, $a \wedge b' \precsim a, b$, that is, $[a \wedge b'] \leq [a], [b]$. Let $x \in \mathcal{B}$ be such that $x \precsim a$ and $x \precsim b$. Then $g_1(x \wedge e) = g_1(x) \wedge e \leq a \wedge e \leq b' \wedge e$ and $g_2(x \wedge \neg e) \leq b' \wedge \neg e \leq a \wedge \neg e$ for some $g_1, g_2 \in G$. Let $g_3 \in G$ such that $g_3|_e = g_1|_e$ and $g_3|_{\neg e} = g_2|_{\neg e}$, and it follows $g_3(x) = g_1(x \wedge e) \vee g_2(x \wedge \neg e) \leq a \wedge b'$, that is, $x \precsim a \wedge b'$.

Similarly, we may show that $[a \vee b']$ is the supremum of $[a]$ and $[b]$. So $(\mathcal{O}_{(\mathcal{B},G)}; \leq)$ is a lattice, and the infima and suprema are representable in the asserted way. □

Lemma 2.9 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. Let $a_1, a_2, \ldots, b \in \mathcal{B}$ such that $a_1 \geq a_2 \geq \cdots$ and $b \precsim a_1, a_2, \ldots$. Then also $b \precsim \bigwedge_i a_i$.*

Proof. By Lemma 2.6(i), we see that there are $a_1' \sim a_1$, $a_2' \sim a_2$ such that $a_1' \geq a_2' \geq \cdots \geq b$. So by Lemmas 2.3(i) and 2.4(i), $a_1 \setminus \bigwedge_{i \geq 2} a_i =$

$(a_1 \setminus a_2) \vee (a_2 \setminus a_3) \vee \ldots \sim (a_1' \setminus a_2') \vee (a_2' \setminus a_3') \vee \ldots = a_1' \setminus \bigwedge_{i \geq 2} a_i'$. So by Lemma 2.4(i), $b \leq \bigwedge_i a_i' \sim \bigwedge_i a_i$. $\square$

We now endow the orbit lattice of a complete Boolean ambiguity algebra with two operations, which arise quite naturally.

Lemma 2.10 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. Let $a, b \in \mathcal{B}$. Then $\{[a' \wedge b']\colon\ a' \sim a,\ b' \sim b\}$ has a minimal element, and $\{[a' \to b']\colon\ a' \sim a,\ b' \sim b\}$ has a maximal element.*

Proof. To prove the first part, let $b' \sim b$ be such that $[a \wedge \neg b'] = [a] \wedge [\neg b]$. We claim that then $a \wedge b'$ represents the required minimum, that is, $a \wedge b' \precsim a \wedge b''$ for any $b'' \sim b$. Indeed, let $b'' \sim b$; from $[a \wedge \neg b''] \leq [a] \wedge [\neg b] = [a \wedge \neg b']$, we conclude by Lemma 2.6(ii) that $a \wedge b'' = a \setminus (a \wedge \neg b'') \succsim a \setminus (a \wedge \neg b') = a \wedge b'$.

Similarly, we proceed for the second part. This time, let $b' \sim b$ be such that $[a \wedge b'] = [a] \wedge [b]$. We claim that $\neg a \vee b'$ represents the required maximum, that is, $\neg a \vee b' \succsim \neg a \vee b''$ for any $b'' \sim b$. Indeed, let $b'' \sim b$; from $[a \wedge b''] \leq [a] \wedge [b] = [a \wedge b']$, it follows that $\neg a \vee b'' = (a \wedge b'') \vee \neg a \precsim (a \wedge b') \vee \neg a = \neg a \vee b'$. $\square$

The operations whose existence are assured by Lemma 2.10, are those we are primarily interested in.

Definition 2.11 Let $(\mathcal{B}, G)$ be a Boolean ambiguity algebra. Assume that the following infima and suprema exists for all $a, b \in \mathcal{B}$:

$$\begin{aligned} [a] \odot [b] &= \inf \{[a' \wedge b']\colon\ a' \sim a,\ b' \sim b\}, \\ [a] \to [b] &= \sup \{[a' \to b']\colon\ a' \sim a,\ b' \sim b\}. \end{aligned} \tag{1}$$

Then we call the structure $(\mathcal{O}_{(\mathcal{B},G)}; \leq, \odot, \to, 0, 1)$ the *orbit algebra of* $(\mathcal{B}, G)$. We then moreover define an additional unary operation on L by $\neg[a] = [a] \to 0$ for $a \in \mathcal{B}$.

Proposition 2.12 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. Let $a, b \in \mathcal{B}$ such that $[a \wedge b] = [a] \wedge [b]$. Then $[a] \odot [\neg b] = [a \wedge \neg b]$ and $[a] \to [b] = [a \to b]$.*

Proof. This is clear from Lemma 2.10 and its proof. $\square$

We arrive at the main statement of this section: under the completeness assumption, the orbit algebra is actually an MV-algebra. For unexplained notions and also alternative ways to define MV-algebras, we refer to [2, 6].

Definition 2.13 *A residuated lattice $(L; \leq, \odot, \rightarrow, 0, 1)$ is called an - MV-algebra if L is divisible and the operation $\neg = \cdot \rightarrow 0$ is involutive.*

Theorem 2.14 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. Then the orbit algebra $(\mathcal{O}_{(\mathcal{B},G)}; \leq, \odot, \rightarrow, 0, 1)$ is an MV-algebra.*

Proof. We first prove that $(\mathcal{O}_{(\mathcal{B},G)}; \leq, \odot, 1)$ is an ordered monoid. Clearly, $\odot$ is commutative, and 1 is neutral w.r.t. $\odot$. Furthermore, let $a, b, c \in \mathcal{B}$; let $a' \sim a$ and $b' \sim b$ be such that $[a] \odot [b] = [a' \wedge b']$; then $([a] \odot [b]) \odot [c] = \min \{[d \wedge c']\colon\ d \sim a' \wedge b',\ c' \sim c\} = \min \{[a'' \wedge b'' \wedge c']\colon a'' \sim a,\ b'' \sim b,\ c' \sim c\}$, whence associativity of $\odot$ follows. Finally, it is easily seen that $\odot$ is in both arguments isotone.

We next show the residuation property. $[a] \odot [b] \leq [c]$ obviously holds iff there are $a' \sim a,\ b' \sim b, c' \sim c$ such that $a' \wedge b' \leq c'$; and $[a] \leq [b] \rightarrow [c]$ holds iff there are $a' \sim a,\ b' \sim b,\ c' \sim c$ such that $a' \leq b' \rightarrow c'$. So $[a] \odot [b] \leq [c]$ if and only if $[a] \leq [b] \rightarrow [c]$.

We next prove $[a] \wedge [b] = [a] \odot ([a] \rightarrow [b])$. By Proposition 2.8, there are $a' \sim a$ and $b' \sim b$ such that $[a' \wedge b'] = [a] \wedge [b]$. Then, by Proposition 2.12, $[a] \rightarrow [b] = [a' \rightarrow b']$. Because $\neg(a' \rightarrow b') \leq a'$, we trivially have $[a'] \wedge [\neg(a' \rightarrow b')] = [a' \wedge \neg(a' \rightarrow b')]$, and again by Proposition 2.12 it follows that $[a] \odot ([a] \rightarrow [b]) = [a'] \odot [a' \rightarrow b'] = [a \wedge (a' \rightarrow b')] = [a' \wedge b'] = [a] \wedge [b]$. So L is divisible.

Finally, we obviously have $\neg[a] = [\neg a]$ for any $a \in \mathcal{B}$; so $\neg$ is involutive. $\square$

We conclude the section with two examples; in one case, the full automorphism group of a Boolean algebra is taken, in the other case the trivial group.

Example 2.15 *Let $\mathcal{B}$ be the finite Boolean algebra with n atoms, and let G be the group of all automorphisms of $\mathcal{B}$. Then $\mathcal{O}_{(\mathcal{B},G)} = L_n$, that is, the $n+1$-element MV-algebra.*

It follows that the class of MV-algebras arising according to Theorem 2.14 from complete Boolean ambiguity algebras is large enough such that we may base the semantics of Łukasiewicz logic on them. The details are given in Section 6.

Example 2.16 *Let $\mathcal{B}$ be a complete Boolean algebra, and let G be the group consisting of the identity of $\mathcal{B}$ only. Then $\mathcal{O}_{(\mathcal{B},G)} = \{\{a\}\colon\ a \in \mathcal{B}\}$; so in this case, the orbit algebra is isomorphic to $\mathcal{B}$.*

3 The filter algebra of a complete Boolean ambiguity algebra

According to the last section's concept, unsharp propositions are modelled by elements of the orbit algebra $\mathcal{O}_{(\mathcal{B},G)}$, which is associated to some complete Boolean ambiguity algebra $(\mathcal{B}, G)$. On $\mathcal{O}_{(\mathcal{B},G)}$, we defined a conjunction-like operation $\odot$ and an implication-like operation $\to$ on $\mathcal{O}_{(\mathcal{B},G)}$ by (1).

The formulas (1) are quite easily comprehensible; still, it would be desirable to interpret the basic operations of many-valued logics in a more intuitive way. The present section is meant to contribute to this aim. We will stay in the same framework as before, which is based on a complete Boolean ambiguity algebra $(\mathcal{B}, G)$. However, we will work with what we call G-filters rather than group orbits. We will prove that the set of G-filters and the set of orbits are order isomorphic. Moreover, the operations $\odot$ and $\to$, when formulated with reference to the set of G-filters, will take a particularly suggestive form.

In what follows, by an order σ-filter of a complete Boolean algebra, we will mean a subset closed under the enlargement of elements and closed under the infima of countable chains.

Definition 3.1 Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. An order σ-filter φ of $\mathcal{B}$ will be called a *G-filter* if (i) for any $a \in \varphi$ and $g \in G$, also $g(a) \in \varphi$, and (ii) for any $a, b \in \varphi$ there is a $g \in G$ such that $a \wedge g(b) \in \varphi$.

Let $\mathcal{F}_{(\mathcal{B},G)}$ be the set of all G-filters of $\mathcal{B}$. For $\varphi, \psi \in \mathcal{F}_{(\mathcal{B},G)}$, let $\varphi \leq \psi$ if $\varphi \supseteq \psi$. Let moreover $\mathit{0} = \mathcal{B}$ and $\mathit{1} = \{1\}$. We will call $(\mathcal{F}_{(\mathcal{B},G)}; \leq, \mathit{0}, \mathit{1})$ the *filter poset* or, if $\leq$ happens to be a lattice order, the *filter lattice* of $(\mathcal{B}, G)$.

So the characteristic properties of a G-filter φ are: φ is closed under the group action; and for $a, b \in \varphi$, not necessarily $a \wedge b \in \varphi$, but $a' \wedge b' \in \varphi$ for some $a' \sim a$ and $b' \sim b$.

Lemma 3.2 *Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. For $a \in \mathcal{B}$, let*

$$\varphi_a = \{x \in \mathcal{B} \colon\; a \precsim x\}. \tag{2}$$

Then φ_a is a G-filter.

Moreover, every G-filter is of this form.

Proof. Let $a \in \mathcal{B}$. φ_a is clearly an order filter and fulfils conditions (i) and (ii) in Definition 3.1. Furthermore, φ_a is closed under the infima of countable chains by Lemma 2.9.

Conversely, let φ be a G-filter. By separability and by Zorn's Lemma, there is a minimal element a in φ. We claim that $\varphi = \varphi_a$. Indeed, by assumption, $a \precsim x$ implies $x \in \varphi$. Conversely, if $x \in \varphi$, then $a \wedge x' \in \varphi$ for some $x' \sim x$, and by the minimality of a, it follows $a \precsim x$. □

In what follows, the expression φ_a, where $a \in \mathcal{B}$, refers to the definition (2).

Lemma 3.3 *Let* $(\mathcal{B}, G)$ *be a complete Boolean ambiguity algebra. Then the mapping*

$$\iota\colon\ \mathcal{O}_{(\mathcal{B},G)} \to \mathcal{F}_{(\mathcal{B},G)},\quad [a] \mapsto \varphi_a \tag{3}$$

is a lattice isomorphism.

Proof. The mapping ι is clearly well-defined. Furthermore, ι is surjective by Lemma 3.2.

If moreover $\varphi_a = \varphi_b$ for $a, b \in \mathcal{B}$, $a \precsim b$ and $b \precsim a$. It follows $a \sim b$ by Proposition 2.8, whence $[a] = [b]$. So ι is also injective.

Evidently, $[a] \leq [b]$ iff $a \precsim b$ iff $\varphi_a \leq \varphi_b$. This completes the proof. □

So G-filters are essentially the same thing as G-orbits. We shall now see which form the operations on the set of G-orbits take when translated to operations on the set of G-filters.

Definition 3.4 Let $(\mathcal{B}, G)$ be a complete Boolean ambiguity algebra. For $a, b \in \mathcal{B}$, we will write $a \bowtie b$ if $a \succsim \neg b$.

For any $\varphi, \psi \in \mathcal{F}_{(\mathcal{B},G)}$, let

$$\begin{aligned} \varphi \odot \psi &= \{x \wedge y\colon\ x \in \varphi,\ y \in \psi\}, \\ \varphi \to \psi &= \{x\colon\ \text{for all } y \in \varphi,\ y \bowtie x \ \text{ and } \ x \wedge y \in \varphi \wedge \psi\}. \end{aligned} \tag{4}$$

Then we call $(\mathcal{F}_{(\mathcal{B},G)}; \leq, \odot, \to, 0, 1)$ the *filter algebra* of $(\mathcal{B}, G)$.

Note at this point the slight similarity of these definitions to what appears in [9]. However, in [9], an arbitrary monoid is used rather than a Boolean algebra endowed with the infimum; furthermore, the definition of $\to$ is kept easier than ours.

The fact that $\odot$ and $\to$ are indeed operations on the set of G-filters, is implied by the following lemma.

Lemma 3.5 *Let* $\mathcal{B}$ *be a complete Boolean ambiguity algebra. For any* $\varphi, \psi \in \mathcal{F}_{(\mathcal{B},G)}$, $\varphi \odot \psi$ *and* $\varphi \to \psi$ *are* G*-filters as well.*

Moreover, let $a, b, c, d \in \mathcal{B}$ *such that* $[c] = [a] \odot [b]$ *and* $[d] = [a] \to [b]$. *Then* $\varphi_c = \varphi_a \odot \varphi_b$ *and* $\varphi_d = \varphi_a \to \varphi_b$.

Proof. Let $a, b \in \mathcal{B}$. Let c be such that $[c] = [a] \odot [b]$; then c is a $\precsim$-minimal element in $\{x \wedge y \colon\ x \sim a,\ y \sim b\}$ according to Lemma 2.10. So $c = a' \wedge b'$ for an appropriate $a' \sim a$ and $b' \sim b$, and it follows that $c \in \varphi_a \odot \varphi_b$ and $c \precsim x$ for all $x \in \varphi_a \odot \varphi_b$. So the assertions concerning $\odot$ follow.

Next, let d be such that $[d] = [a] \to [b]$. According to Proposition 2.12, choose $a' \sim a$ and $b' \sim b$ such that $[a' \wedge b'] = [a] \wedge [b]$ and $d = \neg a' \vee b'$. Then $\varphi_{a' \wedge b'} = \varphi_a \wedge \varphi_b$ by Lemma 3.3. We have to prove that the sets $\varphi_d = \{x : \ d \precsim x\}$ and $\varphi_a \to \varphi_b = \{x : x \succsim \neg a'$ and, for all $a'' \sim a'$, $\ a'' \wedge x \succsim a' \wedge b'\}$ coincide; the assertions concerning $\to$ will then follow as well.

Let us show first that $d \in \varphi_a \to \varphi_b$, which implies that $\varphi_d \subseteq \varphi_a \to \varphi_b$. Clearly, $d \succsim \neg a'$. Furthermore, for $a'' \sim a'$, $(a' \wedge d) \setminus a'' \precsim \neg a' \precsim d$, whence $a'' \wedge d \succsim a' \wedge b'$.

Conversely, let $x \geq \neg a'$ and $a' \wedge x \succsim a' \wedge b'$. Then $x \setminus d \succsim d \setminus x$, whence $x \succsim d$. It follows that $\varphi_a \to \varphi_b \subseteq \varphi_d$. □

This finishes the proof of the equivalence of this and the last section's concepts.

Theorem 3.6 *Let $\mathcal{B}$ be a complete Boolean ambiguity algebra. Then ι as defined by* (3) *is an isomorphism between the MV-algebras* $(\mathcal{O}_{(\mathcal{B},G)};$ $\leq, \odot, \to, 0, 1)$ *and* $(\mathcal{F}_{(\mathcal{B},G)}; \leq, \odot, \to, 0, 1)$.

4 The orbit algebra of a normal Boolean ambiguity algebra

This and the next section contain results which supplement those of the last two sections. We show that we may proceed along similar lines with assumptions of a more algebraic character. We will provide proofs only as far as necessary, otherwise referring to the proofs of the analogous statements above.

Our considerations so far were based on pairs of complete Boolean algebras and full automorphism groups. As the idea is to base semantics for fuzzy logics on these objects, it seems desirable to avoid infinitary conditions. For this reason, we shall drop the assumption that the Boolean algebra is complete.

In this case, the apparatus developed by Kawada in [8] no longer applies, since it heavily relies on the completeness assumption. We will not put extra assumptions on the Boolean algebra instead; rather the conditions involving the group will be sharpened. We will still assume G to be compact, and the notion of fullness will be generalised in the straightforward way. It is a certain additional condition – called (DP) in the sequel – which will ensure that the set of orbits is still lattice-ordered and that we may define the conjunction and implication as before.

Definition 4.1 Let $(\mathcal{B}, G)$ be a Boolean ambiguity algebra. Given $(\mathcal{B}, G)$, we introduce the following notions.

(i) We call G *f-full* if for any two partitions of unity $(a_i)_{i<l}$ and $(b_i)_{i<l}$, where $l < \omega$, and a system $g_i \in G$, $i < l$, such that $g(a_i) = b_i$, the automorphism g defined by $g|_{a_i} = g_i|_{a_i}$ for each $i < l$, belongs to G as well.

(ii) We say that G has the *decomposition property*, or (DP) for short, if for any $a, b \in \mathcal{B}$, there are $c \leq a$ and $d \leq b$ such that $c \sim d$ and $a \setminus c \perp b \setminus d$.

The Boolean ambiguity algebra $(\mathcal{B}, G)$ will be called *normal* if G is compact, f-full, and fulfils (DP).

In analogy to Section 2, we introduce an auxiliary notion as follows. For elements a, b of a Boolean ambiguity algebra, $a \sim_f b$ means that there are two finite sets $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ of pairwise disjoint elements as well as $g_1, \ldots, g_k \in G$ such that $a = \bigvee_i a_i$ and $b = \bigvee_i b_i$ and $g_j(a_j) = b_j$ for all j.

Lemma 4.2 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra. Assume that $c_1, c_2, \ldots$ are pairwise disjoint and $c_1 \sim_f c_2 \sim_f \cdots$. Then $c_1 = c_2 = \cdots = 0$.*

Proof. As in the proof of Lemma 2.3, we construct automorphisms $g_1, \ldots \in G$ such that $g_1(c_1) = c_2, \ldots$, using the f-fullness. Then the assertion follows from compactness. □

Lemma 4.3 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra.*

(i) *Let $a < b$, $c < d$, $g(a) = c$ for some $g \in G$, and $b \sim d$. Then there is a $\bar{g} \in G$ such that $\bar{g}|_a = g|_a$ and $\bar{g}(b) = d$. In particular, $b \setminus a \sim d \setminus c$.*

(ii) *Let $a, b \in \mathcal{B}$ such that $a \sim b$. Let $c = (a \wedge b) \vee \neg(a \vee b)$, $d = a \setminus b$, $e = b \setminus a$. Then there is a $g \in G$ such that $g|_c = id|_c$, $g(d) = e$, $g(e) = d$.*

(iii) *Let $a, b, c \in \mathcal{B}$ such that $a \precsim b \precsim c$ and $a \leq c$. Then there is a $b' \sim b$ such that $a \leq b' \leq c$.*

Proof. (i) According to (DP), let $a' \leq b \setminus a$ and $c' \leq d \setminus c$ be such that $a' \sim c'$ and $b \setminus (a \vee a') \perp d \setminus (c \vee c')$. Let $a_0 = b,\;\; a_1 = a \vee a'$, $u_0 = b \setminus a_1$, and similarly $c_0 = d,\;\; c_1 = c \vee c',\;\; v_0 = d \setminus c_1$. Then we have $a_0 \sim c_0,\;\; a_1 \sim_f c_1$ and $u_0 \perp v_0$.

Let $u_1 \leq a_1$ and $v_1 \leq c_1$ such that $u_1 \sim v_0$, $v_1 \sim u_0$, and $a_2 \sim_f c_2$, where $a_2 = a_1 \setminus u_1$ and $c_2 = c_1 \setminus v_1$. So then we have $a_1 \sim_f c_1$, $a_2 \sim_f c_2$, and $u_1 \perp v_1$.

In the same way, we may proceed to decompose $a_2 = a_3 \vee u_2$ and $c_2 = c_3 \vee v_2$ such that $a_2 \sim_f c_2,\;\; a_3 \sim_f c_3$, and $u_2 \perp v_2$, and so on. We get a sequence $u_0 \sim_f v_1 \sim_f u_2 \sim_f v_3 \sim_f \cdots$. Since $u_0, u_1, \ldots$ are pairwise disjoint, $u_0 = 0$ by Lemma 4.2; similarly, we have $v_0 = 0$.

So $a' = b \setminus a$ and $c' = d \setminus c$. Since $\neg b \sim \neg d$, it follows from the f-fullness of G that there is a mapping $\bar{g} \in G$ as requested.

(ii) follows easily from part (i) and the f-fullness of G.

(iii) is proved like Lemma 2.6(i); the last step requires an application of the f-fullness of G. □

Lemma 4.4 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra.*

(i) *For $a, b \in \mathcal{B}$, $a \sim_f b$ holds if and only if $a \sim b$.*

(ii) *If $a \sim b$ and $a \leq b$, then $a = b$.*

Proof. (i) Let $a = a_1 \vee \ldots \vee a_k,\;\; b = b_1 \vee \ldots \vee b_k$, and $g_1(a_1) = b_1, \ldots, g_k(a_k) = b_k$. Then, by Lemma 4.3(i), $\neg a_2 \sim \neg b_2$ and there is a $g_2' \in G$ such that $g_2'|_{a_1} = g_1|_{a_1}$ and $g_2'(\neg a_2) = \neg b_2$, that is, $g_2'(a_1 \vee a_2) = b_1 \vee b_2$. In the same way, we may determine g_3' such that $g_3'(a_1 \vee a_2 \vee a_3) = b_1 \vee b_2 \vee b_3$, and so forth, so as to get finally a $g \in G$ such that $g(a) = b$.

(ii) $a \sim b$ and $a < b$ imply $1 > \neg(b \setminus a) \sim_f 1$, which is a contradiction by part (i). □

We next consider the orbit preordered set of a normal Boolean ambiguity algebra; cf. Definition 2.5.

Proposition 4.5 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra. Then $(\mathcal{O}_{(\mathcal{B},G)}; \leq, 0, 1)$ is a bounded lattice. For any $a, b \in \mathcal{B}$, there is a $b' \sim b$ such that $[a \wedge b'] = [a] \wedge [b]$ and $[a \vee b'] = [a] \vee [b]$.*

Proof. As above for Proposition 2.8, we see that $\mathcal{O}_{(\mathcal{B},G)}$ is a bounded poset.

Let $a, b \in \mathcal{B}$. By (DP), there is a $b' \sim b$ such that $a \setminus b' \perp b' \setminus a$. We claim that $a \wedge b' \gtrsim a \wedge b''$ for all $b'' \sim b$, implying that $[a \wedge b'] = [a] \wedge [b]$.

Indeed, $b'' \sim b'$ implies by Lemma 4.3(ii) that $a \wedge \neg b' \wedge b'' \precsim b' \setminus b''$, and because $a \setminus b' \perp b' \setminus a$, it follows $a \wedge \neg b' \wedge b'' \precsim a \wedge b' \wedge \neg b''$; so $a \wedge b'' \precsim a \wedge b'$.

By similar reasoning, we conclude that $[a \vee b'] = [a] \vee [b]$. □

So we have established that, for a normal Boolean ambiguity algebra $(\mathcal{B}, G)$, we may call $(\mathcal{O}_{(\mathcal{B},G)}; \leq, 0, 1)$ the orbit lattice.

We next turn to the conjunction-like and the implication-like operation on the orbit lattice. Because we have proved all the necessary technical lemmas, we can proceed from this point on just like in Section 2.

Lemma 4.6 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra. Let $a, b \in \mathcal{B}$. Then $\{[a' \wedge b'] \colon\ a' \sim a,\ b' \sim b\}$ has a minimal element, and $\{[a' \to b'] \colon\ a' \sim a,\ b' \sim b\}$ has a maximal element.*

Proof. This is proved similarly like Lemma 2.10. □

Due to this lemma, $\odot$ and $\to$ is defined for Boolean ambiguity algebras according to Definition 2.11. Note further that also the same representation of $\odot$ and $\to$ holds:

Proposition 4.7 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra. Let $a, b \in \mathcal{B}$ such that $[a \wedge b] = [a] \wedge [b]$. Then $[a] \odot [\neg b] = [a \wedge \neg b]$ and $[a] \to [b] = [a \to b]$.*

We arrive at the main statement of this section.

Theorem 4.8 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra. Then $(\mathcal{O}_{(\mathcal{B},G)}; \leq, \odot, \to, 0, 1)$ is an MV-algebra.*

Proof. This is proved in the same way as Theorem 2.14. □

We readily see that mutatis mutandis, Example 2.15 still applies. This means that for a proper choice of the Boolean ambiguity algebra, the MV-algebra appearing in Theorem 4.8 can be any finite linearly ordered MV-algebra.

5 The filter algebra of a normal Boolean ambiguity algebra

Certainly also in the case that we are concerned with a normal and not necessarily complete Boolean ambiguity algebra, it is desirable that the connectives $\odot$ and $\rightarrow$, which are to interpret the conjunction and implication in many-valued logics, take a form like in (4). We will propose a possibility here, which, however, is less elegant than in the case of complete Boolean ambiguity algebras.

Definition 5.1 Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra. We call any set of the form

$$\varphi_a = \{x \in \mathcal{B}\colon\ a \precsim x\}. \tag{5}$$

a *principal G-filter* of $\mathcal{B}$.

Let $\mathcal{F}_{(\mathcal{B},G)}$ be the set of all principal G-filters of $\mathcal{B}$. For $\varphi, \psi \in \mathcal{F}_{(\mathcal{B},G)}$, let $\varphi \leq \psi$ if $\varphi \supseteq \psi$. Let moreover $\mathit{0} = \mathcal{B}$ and $\mathit{1} = \{1\}$. We will call $(\mathcal{F}_{(\mathcal{B},G)}; \leq, \mathit{0}, \mathit{1})$ the *filter poset* or, if $\leq$ happens to be a lattice order, the *filter lattice* of $(\mathcal{B}, G)$.

By this definition, we have a one-to-one correspondence between orbits and filters trivially.

Lemma 5.2 *Let $(\mathcal{B}, G)$ be a normal Boolean ambiguity algebra. Then the mapping*

$$\iota\colon\ \mathcal{O}_{(\mathcal{B},G)} \to \mathcal{F}_{(\mathcal{B},G)}, \quad [a] \mapsto \varphi_a$$

is a lattice isomorphism.

We next define the operations $\odot$ and $\rightarrow$ on the filter lattice of a normal Boolean ambiguity algebra. Let us adopt Definition 3.4 to the present context, to define the filter algebra $(\mathcal{F}_{(\mathcal{B},G)}; \leq, \odot, \rightarrow, \mathit{0}, \mathit{1})$.

Lemma 5.3 *Let $\mathcal{B}$ be a normal Boolean ambiguity algebra. For any $\varphi, \psi \in \mathcal{F}_{(\mathcal{B},G)}$, $\varphi \odot \psi$ and $\varphi \rightarrow \psi$ are principal G-filters as well.*

Moreover, let $a, b, c, d \in \mathcal{B}$ such that $[c] = [a] \odot [b]$ and $[d] = [a] \rightarrow [b]$. Then $\varphi_c = \varphi_a \odot \varphi_b$ and $\varphi_d = \varphi_a \rightarrow \varphi_b$.

Proof. The proof is similar to the proof of Lemma 3.5. $\square$

So we get again the equivalence of this and the preceding section's concepts.

Theorem 5.4 *Let $\mathcal{B}$ be a normal Boolean ambiguity algebra. Then $(\mathcal{O}_{(\mathcal{B},G)}; \leq, \odot, \rightarrow, \mathit{0}, \mathit{1})$ and $(\mathcal{F}_{(\mathcal{B},G)}; \leq, \odot, \rightarrow, \mathit{0}, \mathit{1})$ are isomorphic algebras.*

6 Semantics for Łukasiewicz logic

In this section, we apply our results to provide alternative semantics for Łukasiewicz logic.

A *formula of Łukasiewicz logic*, or *Ł-formula* for short, is built up from propositional variables $\varphi_1, \varphi_2, \ldots$ and the constant 0 by the binary connectives $\odot$ and $\rightarrow$. An Ł-formula φ is called *Ł-valid* if, under an arbitrary interpretation of the variables by values from the real unit interval $[0, 1]$, of the constant 0 by the real number 0, and of the connectives $\odot$ and $\rightarrow$ by the Łukasiewicz t-norm $\odot_L$ and the corresponding residuum $\rightarrow_L$, respectively, φ is assigned the value 1.

Theorem 6.1 *Call an Ł-formula φ cBAA-valid if for all complete Boolean ambiguity algebras $(\mathcal{B}, G)$ and for all interpretations of the variables by elements from $\mathcal{O}_{(\mathcal{B},G)}$, of the constant* 0 *by 0 and of the connectives according to Definition 2.11, φ is assigned 1. Then φ is cBAA-valid if and only it is a Ł-valid.*

Proof. Let φ be cBAA-valid. From Theorem 2.14 and Example 2.15, it follows that φ is valid in all finite linearly ordered MV-algebras. Because the variety of MV-algebras is generated by its finite members, this means that φ is valid in all MV-algebras, hence in particular in $([0, 1]; \leq, \odot_L, \rightarrow_L, 0, 1)$. So φ is Ł-valid.

Conversely, let φ be Ł-valid. Then φ is valid in all MV-algebras; for this fact, see e.g. [2]. So in particular, φ is cBAA-valid by Theorem 2.14. $\square$

Theorem 6.2 *Call an Ł-formula φ BAA-valid if for all normal Boolean ambiguity algebras $(\mathcal{B}, G)$ and for all interpretations of the variables by elements from $\mathcal{O}_{(\mathcal{B},G)}$, of the constant* 0 *by 0 and of the connectives according to Definition 2.11, φ is assigned 1. Then φ is BAA-valid if and only it is a Ł-valid.*

Proof. This follows from Theorem 4.8 and Example 2.15 in the same way as Theorem 6.1. $\square$

It is clear that by Theorems 3.6 and 5.4, we could have used in both cases the filter algebra rather than the orbit algebra of the respective Boolean ambiguity algebra.

7 Conclusion

Based on Boolean ambiguity algebras, which are Boolean algebras endowed with a certain kind of automorphism group, we have presented an alternative semantics for Łukasiewicz logic. We actually explained two different ways how to do so, and in each case there are two variants.

The Łukasiewicz logic is a common, but otherwise rather particular member of the family of multi-valued logics. The present work can be seen as a confirmation of its significance.

It would moreover be desirable to generalise our scheme so as to treat further logics. We intend to focus next on the *Basic Logic* introduced by P. Hájek [6]. Our idea is to generalise the underlying structure: the Boolean algebra should be replaced by a bounded distributive lattice.

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