BL-ALGEBRAS AND EFFECT ALGEBRAS

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Abstract. Although the notions of a BL-algebra and of an effect algebra arose in rather different contextes, both types of algebras have certain structural properties in common. To clarify their mutual relation, we introduce weak effect algebras, which generalize effect algebras in that the order is no longer necessarily determined by the partial addition. A subclass of the weak effect algebras is shown to be identifiable with the BL-algebras.

Moreover, weak D-posets are defined, being based on a partial difference rather than a partial addition. They are equivalent to weak effect algebras.

Finally, it is seen to which subclasses of the weak effect algebras certain subclasses of the BL-algebras, namely the MV-, product, and Gödel algebras, correspond.

1. Introduction

Basic Logic, or BL for short, has been introduced by Hájek in order to provide a general framework for formalizing statements of fuzzy nature [Haj]. It is an adequate calculus when we have to do with statements about which we may say that they are true principally only to a certain degree, that is, to which it is in general unreasonable or impossible to assign a sharp yes or no.

Formulas of propositional BL may be interpreted by means of BL-algebras [Haj, Got]. With respect to a semantics defined in this way, BL is complete: formulas proved by BL are exactly those valid in any BL-algebra.

The situation is different in quantum logics; in this case, we have to do with statements of a probabilistic character. For instance, the logic

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UPaQL from [DaGi], which could be called the Logic of Effects, is aimed at formalizing statements arising in connection with a quantum mechanical experiment. A typical such statement would say that a certain quantum-mechanical yes-no experiment leads to a positive result. Again we have to do with statements to which it is principally impossible to assign a sharp truth value – but this time for the reason that a certain test result is unpredictable.

The interpretation of formulas of UPaQL is accomplished by effect algebras [FoBe]. Also in this case, a completeness theorem has been proved.

So it is clear that we have to distinguish sharply between statements of fuzzy character, treated in the framework of BL, and statements of probabilistic character, as they arise in quantum logics. Nevertheless, the algebraic structures on which the models of both mentioned logics may be based, clearly have structural properties in common, and we feel free to examine the question to what extent both types of algebras may be brought onto a common line. That is, what we do in this article is to compare BL-algebras and effect algebras and to see that they are not as distinct as one could expect.

It is certainly easy to see that effect and BL-algebras have nothing in common but MV-algebras, which may be considered as rather small subclasses of both types of algebras. This is not our subject. We shall rather show that by a slight generalization of effect algebras, we arrive at a structure which includes not only all effect algebras, but also all BL-algebras.

We proceed as follows. We review in the following Section 2 some facts about BL-algebras, summarizing results from a preceding paper [Vet]. In particular, we show how to axiomatize (the dual of) a BL-algebra on the base of a single addition-like operation. In Section 3, we turn to effect algebras, and we introduce the notion of a weak effect algebra. This new structure is a poset endowed with a partial addition which is compatible with the order. The only difference to effect algebras lies in the fact that the order is no longer necessarily determined by the addition.

We then see in Section 4 how BL-algebras may be understood as certain weak effect algebras. Namely, we may restrict the addition of (the dual of) a BL-algebra to a partial, but cancellative operation; then we arrive at a weak effect algebra with certain further properties.

Moreover, effect algebras are equivalent to D-posets. In analogy to this fact, we will also define a version of our weak effect algebras based on a difference rather than an addition. This is done in Sections 5 and 6.

Finally, we establish in Section 7 which subclasses of weak effect algebras correspond to the best known subclasses of BL-algebras: the MV-algebras, product algebras, and Gödel algebras.

2. BL-algebras

BL-algebras were introduced as those algebras which are appropriate to model formulas of Hájek's Basic Logic [Haj, Got]. They may be understood as being residuated lattices fulfilling some further natural requirements [Haj, Vet].

Definition 2.1. A *BL-algebra* is a structure $(L; \leq, \odot, \Rightarrow, 0, 1)$ such that the following holds:

(BL1) $(L; \leq, 0, 1)$ is a lattice with a smallest element 0 and a largest element 1.

(BL2) $(L; \odot, 1)$ is a commutative monoid, that is, $\odot$ is an associative and commutative binary operation, and 1 is a neutral element with respect to $\odot$.

(BL3) $\odot$ is compatible with $\wedge$ and $\vee$, that is, we have $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$ and $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$ for any $a, b, c \in L$.

(BL4) For any $a, b \in L$, $a \Rightarrow b$ is the maximal element x such that $a \odot x \leq b$.

(BL5) $\Rightarrow$ is compatible with $\wedge$ and $\vee$, that is, we have $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$ and $a \Rightarrow (b \vee c) = (a \Rightarrow b) \vee (a \Rightarrow c)$ as well as $(b \wedge c) \Rightarrow a = (b \Rightarrow a) \vee (c \Rightarrow a)$ and $(b \vee c) \Rightarrow a = (b \Rightarrow a) \wedge (c \Rightarrow a)$ for any $a, b, c \in L$.

(BL6) L is naturally ordered with respect to $\odot$, that is, for $a, b \in L$ we have $a \leq b$ if and only if $a = b \odot x$ for some $x \in L$.

Rather than considering BL-algebras themselves, we will work with their duals; they arise from BL-algebras simply by reversing the order and by changing the order of the arguments of $\Rightarrow$.

Definition 2.2. A structure $(L; \leq, \oplus, \ominus, 0, 1)$ is called a *dual BL-algebra* if $(L; \leq_{\rm BL}, \odot, \Rightarrow, 0_{\rm BL}, 1_{\rm BL})$ is a BL-algebra, where

$$a \leq b \text{ iff } b \leq_{\rm BL} a,$$
$$a \oplus b = a \odot b, \qquad a \ominus b = b \Rightarrow a,$$
$$0 = 1_{\rm BL}, \qquad 1 = 0_{\rm BL}.$$

We note that dual BL-algebras are the same as Iorgulescu's reversed right-BL algebras [Ior2].

Now, we intend to relate BL-algebras to effect algebras; and effect algebras are based on a single (partial) addition. So what we will do first, is to axiomatize BL-algebras on the base of a single (total) addition. The following results are from [Vet, Theorem 3.3].

Definition 2.3. A *naturally ordered abelian monoid*, or *NAM* for short, is a structure $(L; \leq, \oplus, 0)$ with the following properties:

(NAM1) $(L; \leq, 0)$ is a poset with a smallest element 0.

(NAM2) $\oplus$ is a binary operation such that for any $a, b, c \in L$
- (a) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
- (b) $a \oplus 0 = a$;
- (c) $a \oplus b = b \oplus a$.

(NAM3) We have $a \leq b$ for $a, b \in L$ if and only if $a \oplus x = b$ for some $x \in L$.

Moreover, a *bounded NAM* is a structure $(L; \leq, \oplus, 0, 1)$ such that

(NAM1') $(L; \leq, 0, 1)$ is a poset with a smallest element 0 and a largest element 1,

and such that, with respect to $(L; \leq, \oplus, 0)$, the axioms (NAM2) and (NAM3) are fulfilled.

BL-algebras may be understood as bounded NAMs; the characteristic properties are the following.

Definition 2.4. A bounded NAM $(L; \leq, \oplus, 0, 1)$ is called *of type BL* if the following conditions hold:

(NAM4) For any $a, b \in L$ such that $a \leq b$ there is a smallest element $x \in L$ such that $a \oplus x = b$.

(NAM5) For any $a, b, c \in L$ such that $c \leq a \oplus b$ there are $a_1 \leq a$ and $b_1 \leq b$ such that (α) $c = a_1 \oplus b_1$ and (β) $a_1 = a$ in case $c \geq a$.

(NAM6) For any $a, b \in L$, there are $a_1, b_1, c \in L$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$, and $a_1 \wedge b_1 = 0$.

Moreover, if L is of type BL, we define the *residuum* of L to be the binary operation $\ominus$, where

$$b \ominus a \stackrel{\text{def}}{=} \min \{x\colon\ a \oplus x = a \vee b\} \quad \text{for } a, b \in L. \tag{1}$$

We note that bounded NAMs of type BL are distributive lattices, which in particular makes possible the definition (1). Furthermore, $b \ominus a$ may be shown to be the smallest element x such that $a \oplus x \geq b$, which justifies the notion "residuum".

Concerning terminology, we further note that our bounded NAMs of type BL are a dual analogue of the left-X-Hájek(R)-(X-BL) algebras from [Ior2].

Theorem 2.5. *Let $(L; \leq, \oplus, \ominus, 0, 1)$ be a dual BL-algebra. Then $(L; \leq, \oplus, 0, 1)$ is a bounded NAM of type BL, whose residuum is $\ominus$.*

Conversely, let $(L; \leq, \oplus, 0, 1)$ be a bounded NAM of type BL. Let $\ominus$ be the residuum of L. Then $(L; \leq, \oplus, \ominus, 0, 1)$ is the unique expansion of L to a dual BL-algebra.

Note that in a bounded NAM of type BL L, the supremum and infimum of a pair $a, b \in L$ calculates according to

$$a \vee b = (b \ominus a) \oplus a, \tag{2}$$

$$a \wedge b = [b \ominus (b \ominus a)] \vee [a \ominus (a \ominus b)]. \tag{3}$$

We furthermore note that we have for any $a, b, c \in L$

$$(c \ominus a) \ominus b = c \ominus (a \oplus b), \tag{4}$$

$$(c \ominus b) \oplus (b \ominus a) = c \ominus a \quad \text{if } a \leq b \leq c, \tag{5}$$

$$(a \ominus b) \wedge (b \ominus a) = 0. \tag{6}$$

We conclude this section by noting that BL-algebras may also be axiomatized on the basis of their implication-like operation only [Ior1, Vet]; namely, we may identify BL-algebras with a subclass of the BCK-algebras. For the notion of a BCK-algebra and the additional properties which we use here, see e.g. [MeJu]. For the sake of a consistent notation within this article, we will write $\ominus$ for the basic operation of a BCK-algebra instead of the usual $\star$.

Definition 2.6. A bounded BCK-algebra $(L; \leq, \ominus, 0, 1)$ is called *of type BL* if (i) it is with condition (S), (ii) for any $a, b, c \in L$ such that

$c \leq a, b$ we have $a \leq b$ iff $a \ominus c \leq b \ominus c$, and (iii) for any $a, b \in L$ we have $(a \ominus b) \wedge (b \ominus a) = 0$.

Furthermore, if L is of type BL, we define the *S-function* of L to be the operation $\oplus$, where, for $a, b \in L$, $a \oplus b$ is the maximal element y such that $y \ominus a \leq b$.

Theorem 2.7. *Let $(L; \leq, \oplus, \ominus, 0, 1)$ be a dual BL-algebra. Then $(L; \leq, \ominus, 0, 1)$ is a bounded BCK-algebra of type BL, whose S-function is $\oplus$.*

Conversely, let $(L; \leq, \ominus, 0, 1)$ be a bounded BCK-algebra of type BL. Let $\oplus$ be its S-function. Then $(L; \leq, \oplus, \ominus, 0, 1)$ is the unique expansion of L to a dual BL-algebra.

3. Effect algebras and weak effect algebras

Effect algebras, introduced by Foulis and Bennett [FoBe], are modelled upon the effects in Hilbert space, that is, upon the set of all positive operators smaller than identity. Every such operator corresponds in physics to a yes-no experiment performable at some quantum-physical system.

We recall the definition of effect algebras. For comparative reasons, we will handle the partial ordering $\leq$ as an own relation.

Definition 3.1. An *effect algebra* is a structure $(L; \leq, +, 0, 1)$ with the following properties:

(E1) $(L; \leq, 0, 1)$ is a poset with a smallest element 0 and a largest element 1.

(E2) $+$ is a partial binary operation such that for any $a, b, c \in E$
- (a) $(a + b) + c$ is defined iff $a + (b + c)$ is defined, and in this case $(a + b) + c = a + (b + c)$;
- (b) $a + 0$ is always defined and equals a;
- (c) $a + b$ is defined iff $b + a$ is defined, and in this case $a + b = b + a$.

(E3) If, for $a, b, c \in L$, $a + c$ and $b + c$ are defined, then $a \leq b$ if and only if $a + c \leq b + c$.

(E4) We have $a \leq b$ for $a, b \in L$ if and only if $a + x = b$ for some $x \in L$.

We note that usually only (E2) and a version of cancellativity, implied here by (E3), is chosen as axioms for effect algebras, the partial order being defined according to (E4).

Furthermore, our axioms are not without redundancies; one half of the last axiom, (E4), is already contained in the other ones; namely, (E1)–(E3) prove $a \leq a + x$ if the latter term exists.

In the sequel, any statement involving partial operations to hold will mean as usual that all the operations are performable and that the statement is true.

We will now generalize the notion of an effect algebra by weakening the axiom (E4), which states that the ordering of an effect algebra is the natural one. It will be no longer assumed that the partial addition $+$ determines the order.

Definition 3.2. A *weak effect algebra* is a structure $(L; \leq, +, 0, 1)$ such that the axioms (E1), (E2), and (E3) hold as well as the following one:

(E4') If, for $a, b \in L$, $a \leq b$, then there is a largest $\bar{a} \leq a$ such that $\bar{a} + x = b$ for some $x \in L$.

We evidently have the following.

Proposition 3.3. *Any effect algebra is a weak effect algebra.*

A weak effect algebra L is an effect algebra if and only if for any $a, b \in L$ such that $a \leq b$, there is an $x \in L$ such that $a + x = b$.

We now introduce those conditions for weak effect algebras which will prove characteristic in connection with BL-algebras. Note on the one hand that these axioms contain analogues of (NAM5) and (NAM6); on the other hand, these conditions, as we will see, characerize MV-algebras among effect algebras.

Definition 3.4. A weak effect algebra $(L; \leq, +, 0, 1)$ is called a *weak MV-effect algebra* if the following conditions hold:

(E5) For any $a_1, a_2, b_1, b_2 \in L$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in L$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(E6) For any $a, b, c \in L$ such that $c \leq a + b$, there are $a_1 \leq a$ and $b_1 \leq b$ such that (α) $c = a_1 + b_1$ and (β) $a_1 = a$ in case $c \geq a$.

(E7) For any $a, b \in L$, there are a_1, a_2, b_1, b_2 such that $a = a_1 + a_2$, $b = b_1 + b_2$ and $a_1 \leq b$, $b_1 \leq a$ and $a_2 \wedge b_2 = 0$.

Note that property (E5) is the *Riesz interpolation property*, or (RIP), which is definable for any poset; see also [DvVe]. Moreover, for effect algebras the *Riesz decomposition property*, or (RDP_0), has been defined [Rav, DvVe]: it holds if for any three elements $a, b, c \in L$ such that

$c \leq a + b$, there are $a_1 \leq a$ and $b_1 \leq b$ such that $c = a_1 + b_1$; thus, when assuming (E4), (RDP_0) is equivalent to (E6). Finally, (E7) is the analogue of (NAM6), which was called *mutual compatibility* in [Vet].

To justify our terminology, recall that MV-algebras are in a one-to-one correspondence to MV-effect algebras, which in turn are those effect algebras that are lattice-ordered and fulfil (RDP_0). We have also the following.

Proposition 3.5. *An effect algebra is an MV-effect algebra if and only if it fulfils the axioms* (E6) *and* (E7)*. In this case, also* (E5) *holds.*

Proof. Let L be an MV-effect algebra. Then L fulfils (RDP_0), hence by (E3) and (E4) also (E6). L is furthermore lattice-ordered and in particular fulfils (E5). L also fulfils (E7), because for some $a, b \in L$, we may set $a_1 = b_1 = a \wedge b$, and by assigning appropriate values to a_2 and b_2, we easily see that $a_2 \wedge b_2 = 0$.

Conversely, let L be an effect algebra fulfilling (E6) and (E7). (E6) clearly implies (RDP_0). By [DvVe, Proposition 3.3(iii)], then also (RIP), that is, (E5) holds. For some $a, b \in L$, let, according to (E7), $a_1, a_2, b_1, b_2 \in L$ be such that $a = a_1 + a_2,\ b = b_1 + b_2$ and $a_1 \leq b,\ b_1 \leq a$ and $a_2 \wedge b_2 = 0$. Using (E5), choose c such that $a_1, b_1 \leq c \leq a, b$. Then it is not difficult to infer that $c = a \wedge b$. Because L has an involutive complement function, it follows that L is lattice-ordered. □

4. BL-algebras as weak effect algebras

We are now ready to compare BL-algebras to effect algebras. What we will see is the following. Dual BL-algebras whose addition is restricted in a natural manner to a partial one, are proved to be weak effect algebras. Because dual BL-algebras and BL-algebras may be considered the same, this means that weak effect algebras are a common generalization of BL-algebras and effect algebras.

Furthermore, it is clear which condition characterizes effect algebras among weak effect algebras. It is less trivial to find an appropriate one for BL-algebras. We offer here one possibility of how to characterize the subclass of those weak effect algebras which correspond to BL-algebras.

Let us now, as a first step, consider the interrelation of a partial and total addition defined on some poset. For, dual BL-algebras, or bounded NAMs of type BL, are algebras based on a total addition, whereas weak effect algebras are based on a partial addition. How to derive one type of addition from the other one in a natural way is the subject of the following definition.

Definition 4.1. (i) Let $(L; \leq, \oplus, 0, 1)$ be a bounded NAM of type BL. Let $a, b \in L$; we say that a is *minimal in the sum* $a \oplus b$ if a is the smallest element x such that $x \oplus b = a \oplus b$.

Define $a + b \stackrel{\text{def}}{=} a \oplus b$ if a as well as b are minimal in $a \oplus b$, and let else $a + b$ undefined. Then the operation $+$ is called the *partial addition belonging to* $\oplus$.

(ii) Let $(L; \leq, +, 0, 1)$ be a weak effect algebra. Let for any $a, b \in L$ the maximum

$$a \oplus b \stackrel{\text{def}}{=} \max \{a' + b' \colon\ a' \leq a,\ b' \leq b \text{ and } a' + b' \text{ is defined}\} \tag{7}$$

exist. Then the operation $\oplus$ is called the *total addition belonging to* $+$.

Theorem 4.2. *Let $(L; \leq, \oplus, 0, 1)$ be a bounded NAM of type BL, and let $+$ be the partial addition belonging to $\oplus$. Then $(L; \leq, +, 0, 1)$ is a weak MV-effect algebra, and the total addition belonging to $+$ exists and coincides with $\oplus$.*

Proof. (E1) is identical to (NAM1').

Note next that for any $a, b \in L$, $a+b$ is defined and equals c iff $a = c \ominus b$ and $b = c \ominus a$. Here, $\ominus$ is the residuum of L as defined by (1). Note that the minimality of the summands in a sum $a+b$ implies that cancellation holds for the partial addition $+$.

Let now $a \oplus b = c$ for some $a, b, c \in L$. Then we may replace a and b by smaller elements which are summable by the partial addition $+$ and whose sum is still c. This in particular implies (7). Indeed, let $a' = c \ominus (c \ominus a)$ and $b' = c \ominus a$. Then we have by (3) $a' \leq a$ and by (1) $b' \leq b$ and $a' \oplus b' = c$. Clearly, $a' = c \ominus b'$. Also $b' = c \ominus a'$ holds; indeed, we have $b' = c \ominus a \leq c \ominus a'$ from $a' \leq a$, and $c \ominus a' = c \ominus (c \ominus b') \leq b'$ from (3). So $a' + b'$ is defined.

Let now $a, b, c \in L$ such that $(a + b) + c$ is defined. We will show that then $a + (b + c)$ is also defined; then (E2)(a) follows by (NAM2)(a). Let $d = a + b$ and $e = (a + b) + c$. We see by (4) that $e \ominus (b \oplus c) = (e \ominus c) \ominus b = d \ominus b = a$. Furthermore, since $b = d \ominus a \leq e \ominus a$, we have $e \ominus a = b \oplus ((e \ominus a) \ominus b) = b \oplus (e \ominus (a \oplus b)) = b \oplus c$. So we have proved that the sum $a + (b \oplus c)$ is defined. Let $f = b \oplus c$. We have $f \ominus c = (e \ominus a) \ominus c = e \ominus (a \oplus c) = (e \ominus c) \ominus a = d \ominus a = b$. Finally, $f \ominus b = (c \oplus b) \ominus b = [(e \ominus (a \oplus b)) \oplus b] \ominus b = [((e \ominus a) \ominus b) \oplus b] \ominus b = (e \ominus a) \ominus b = e \ominus (a \oplus b) = c$. It follows that also the sum $b + c$ is defined.

(E2)(b) is evident, and (E2)(c) is clear by the definition of $+$.

Let now $a, b, c \in L$ such that $a + c$ and $b + c$ are defined. If then $a \leq b$, we have $a + c = a \oplus c \leq b \oplus c = b + c$. Conversely, from $a + c \leq b + c$ it follows $a = (a + c) \ominus c \leq (b + c) \ominus c = b$. This proves (E3).

To see (E4'), let $a, b \in L$ such that $a \leq b$. Let $\bar{a} = b \ominus (b \ominus a)$ and $x = b \ominus a$. Then we have by our above considerations $\bar{a} \leq a$ and $\bar{a} + x = b$. If now $a' \leq a$ and $a' + y = b$ for some $y \in L$, then $a' = b \ominus y = b \ominus (b \ominus a') \leq b \ominus (b \ominus a) = \bar{a}$. So $\bar{a}$ is the largest element which sums up with some element to b by using the addition $+$.

(E5) holds, because L is lattice-ordered.

To see (E6), let $a, b, c \in L$ such that $c \leq a+b$. By (NAM5), we conclude from $c \leq a + b = a \oplus b$ that $c = a_1 \oplus b_1$ for some $a_1 \leq a$, $b_1 \leq b$. Then by the above considerations, $c = a_1' + b_1'$ for some $a_1' \leq a_1 \leq a$ and $b_1' \leq b_1 \leq b$.

Let us now consider the case $a \leq c \leq a + b$. Let $a_1 = c \ominus (c \ominus a)$ and $b_1 = c \ominus a$; then $c = a_1 + b_1$ and $b_1 \leq (a + b) \ominus a = b$. We claim $a_1 = a$. Setting $d = a + b$, we get, using (5) and (4), $a = d \ominus (d \ominus a) = d \ominus [(d \ominus c) \oplus (c \ominus a)] = (d \ominus (d \ominus c)) \ominus (c \ominus a) \leq c \ominus (c \ominus a) = a_1 \leq a$.

It remains to prove (E7). Let $a, b \in L$. Set $a_2 = a \ominus b,\ b_2 = b \ominus a,\ a_1 = a \ominus a_2,\ b_1 = b \ominus b_2$. Then $a = a_1' + a_2'$ for some $a_1' \leq a_1 = a \ominus (a \ominus b) \leq b$ and $a_2' \leq a_2$; similarly, we have $b = b_1' + b_2'$ for some $b_1' \leq b_1 \leq a$ and $b_2' \leq b_2$. Moreover, by (6), $a_2 \wedge b_2 = 0$, whence $a_2' \wedge b_2' = 0$. □

We next see which kind of weak MV-effect algebras may be identified with bounded NAMs of type BL and thus with BL-algebras. It is simply those for which the total addition belonging to the partial one exists.

Theorem 4.3. *Let $(L; \leq, +, 0, 1)$ be a weak MV-effect algebra such that the total addition $\oplus$ belonging to $+$ exists. Then $(L; \leq, \oplus, 0, 1)$ is a bounded NAM of type BL, and the partial addition belonging to $\oplus$ coincides with $+$.*

Proof. (NAM1') is identical to (E1).

We claim that, for any $a, b, c \in L$, $(a \oplus b) \oplus c = \max \{a' + b' + c' \colon\ a' \leq a,\ b' \leq b,\ c' \leq c\}$; (NAM2)(a) then follows. By definition (7), we have $(a \oplus b) \oplus c = \max \{d' + c' \colon\ d' \leq a \oplus b,\ c' \leq c\}$. Because $a' \leq a$ and $b' \leq b$ implies $a' + b' \leq a \oplus b$ if $a' + b'$ exists, any element from the former set the maximum is taken over, is contained in the latter one. On the other hand, $d' \leq a \oplus b$ means by (7) and (E6) that $d' = a' + b'$ for some $a' \leq a$ and $b' \leq b$, so if then $c' \leq c$ and $d' + c'$ exists, we have

$d' + c' = a' + b' + c'$, and so any element from the latter set is in the former one.

(NAM2)(b) is evident from (E2)(b), and (NAM2)(c) follows directly from the definition of $\oplus$ and (E2)(c).

For any $a, x \in L$, we have $a \oplus x = \max\,\{a' + x'\colon\ a' \leq a,\ x' \leq x\} \geq a + 0 = a$. This shows one half of (NAM3).

We next note that for $a, b, c, d \in L$, $a + b = c + d$ and $a \leq c$ implies $b \geq d$. Indeed, from $a \leq c \leq a + b$ we have by (E6) that $c = a + b'$ for some $b' \leq b$, whence $a + b = a + b' + d$ and by (E3) $b = b' + d \geq d$.

Let now $a, b \in L$ such that $a \leq b$. Let, according to (E4'), $\bar{a} \leq a$ be the maximal element such that $\bar{a} + x = b$ for some $x \in L$. We claim that then $a \oplus x = b$; this proves the second part of (NAM3). We have $a \oplus x = \max\,\{a' + x'\colon\ a' \leq a,\ x' \leq x\} \geq \bar{a} + x = b$. Let $a' \leq a$ and $x' \leq x$ such that $a' + x' = a \oplus x$. From $a' \leq b \leq a' + x'$ it follows by (E6) that $b = a' + x''$ for some $x'' \leq x'$. Because of $b = \bar{a} + x$ and the maximality of $\bar{a}$, we have $a' \leq \bar{a}$ and by the result of the preceding paragraph $x \leq x''$, that is, $x = x' = x''$. So finally, $b \leq a' + x' \leq \bar{a} + x = b$, and thus $b = a' + x' = a \oplus x$.

We claim further that x is the smallest element such that $a \oplus x = b$, which shows (NAM4). Indeed, let $a \oplus y = b$ for any other $y \in L$. Then $\bar{a} + x = b = a' + y'$ for some $a' \leq a$ and $y' \leq y$, and from the maximality of $\bar{a}$ we conclude $a' \leq \bar{a}$ and further $x \leq y' \leq y$.

We next show (NAM5); so let $a, b, c \in L$ be such that $c \leq a \oplus b$. It follows from (4) and (E6) that $c = a_1 + b_1 = a_1 \oplus b_1$ for some $a_1 \leq a$ and $b_1 \leq b$.

Assume now $a \leq c \leq a \oplus b$. Let $a \oplus b = a' + b'$, where a' is chosen the maximal element below a and summing up with some element to $a \oplus b$. Then by what was proved before, $b' \leq b$. From $a' \leq c \leq a' + b'$, we have by (E6) $c = a' + b''$ for some $b'' \leq b'$. Let now $c = a'' + b'''$, where a'' is chosen the maximal element below a summing up with some element to c. Then again according to what we saw above, $c = a \oplus b'''$, and by the maximality of a'', $a' \leq a''$, hence $b''' \leq b'' \leq b$.

To see (NAM6), let $a, b \in L$, and let, according to (E7), $a_1, a_2, b_1, b_2 \in L$ be such that $a = a_1 + a_2,\ \ b = b_1 + b_2,\ \ a_1 \leq b,\ \ b_1 \leq a,\ \ a_2 \wedge b_2 = 0$. According to (E5), let furthermore $c \in L$ be such that $a_1, b_1 \leq c \leq a, b$. Let $a = a_1' + a_2'$, where a_1' is the largest element below c adding up with some element to a; then we have $a = c \oplus a_2'$, and moreover $a_1 \leq a_1'$, hence $a_2' \leq a_2$. Similarly, we have $b = c \oplus b_2'$ for some $b_2' \leq b_2$. So $a_2' \wedge b_2' = 0$, and the requirements of (NAM6) are fulfilled. □

5. D-posets and weak D-posets

Effect algebras may be easily formulated on the base of a partial difference rather than a partial addition; the appropriate notion is that of a D-poset, introduced by Kôpka and Chovanec [KoCh].

It is also possible to base weak effect algebras on a difference, although this is not as straightforward as in the case of the more special effect algebras. We recall first the notion of a D-poset.

Definition 5.1. A *D-poset* is a structure $(L; \leq, -, 0, 1)$ with the following properties:

(D1) $(L; \leq, 0, 1)$ is a poset with a smallest element 0 and a largest element 1.

(D2) $-$ is a partial binary operation such that for any $a, b, c \in E$
- (a) If $b - a$ exists, then also $b - (b - a)$ exists and equals a;
- (b) $a - 0$ exists and equals a;
- (c) If $c - b$ and $b - a$ exists, then $(c - a) - (c - b)$ exist and equals $b - a$.

(D3) If, for $a, b, c \in L$, $a - c$ and $b - c$ are defined, then $a \leq b$ if and only if $a - c \leq b - c$.

(D4) $b - a$ is defined if and only if $a \leq b$.

In analogy to the case of effect algebras, we will weaken the axiom (D4) concerning the connection of the partial difference and the partial order.

Definition 5.2. A *weak D-poset* is a structure $(L; \leq, -, 0, 1)$ such that the axioms (D1), (D2), and (D3) hold as well as the following one:

(D4') If, for $a, b \in L$, $a \leq b$, then there is a largest $\bar{a} \leq a$ such that $b - \bar{a}$ is defined.

Note that when dropping the axiom (D4), one direction of (D4) is still valid. Namely, if for a pair of elements a, b the difference $b - a$ is defined, then, in view of (D2)(a),(b), $a - a = 0 \leq b - a$, whence by (D3) $a \leq b$.

The following is thus evident.

Proposition 5.3. *Any D-poset is a weak D-poset.*

A weak D-poset L is a D-poset if and only if, for $a, b \in L$, $b - a$ is defined whenever $a \leq b$.

We see that the notions of a weak effect algebra and of a weak D-poset are equivalent.

Proposition 5.4. *Let* $(L; \leq, +, 0, 1)$ *be a weak effect algebra. Then we may define* $b - a = x$ *in case* $a + x = b$ *for some* x*, letting* $b - a$ *else undefined. Then* $(L; \leq, -, 0, 1)$ *is a weak D-poset.*

Moreover, every weak D-poset arises in this way from exactly one weak effect algebra.

Proof. Let $(L; \leq, +, 0, 1)$ be a weak effect algebra. By cancellativity, a consequence of (E3), we may define the operation $-$ in the way shown. (D1) holds by (E1). Let $a, b, c \in L$. If $b - a$ exists, then $a + (b - a) = b$, which means that also $b - (b - a)$ exists and equals a; so (D2)(a) holds. From $a + 0 = a$ we infer that $a - 0 = a$, so (D2)(b) holds. Assume now that $c - b$ and $b - a$ are defined; this implies $[a + (b - a)] + (c - b) = c$, and from associativity, i.e. (E2)(a), we conclude $c - a = (c - b) + (b - a)$ and $(c - a) - (c - b) = b - a$; so (D2)(c) is shown.

If, for some $a, b, c \in L$, $d = a - c$ and $e = b - c$ are defined, then by (E3) $d \leq e$ iff $d + c \leq e + c$ iff $a \leq b$; (D3) follows. If furthermore $a \leq b$, then by (E4') there is a largest $\bar{a} \leq a$ such that $\bar{a} + x = b$ for some $x \in L$, that is, such that $b - \bar{a}$ is defined; so also (D4') holds, and L is proved to be a weak D-poset.

Conversely, let $(L; \leq, -, 0, 1)$ be a weak D-poset. For any $a, b, c \in L$, let $a + b = c$ hold iff $b = c - a$; by (D3), this defines $+$ to be a partial operation. Now, (D1) holds by (E1). Let $a, b, c \in L$. To see (E2)(c), assume $a + b = c$. Then $b = c - a$, and so by (D2)(a) $a = c - (c - a) = c - b$, so $b + a = c$. Now, if $(a + b) + c = d$, then $d - (a + b) = c$ and $(a + b) - a = b$, whence by (D2)(c) $(d - a) - c = b$ and so $d - a = b + c$ and $d = a + (b + c)$; so (E2)(a) holds. (E2)(b) is evident from (D2)(b).

Assume now, for some $a, b, c \in L$, that $d = a + c$ and $e = b + c$ are defined. Then by (D3) $a \leq b$ iff $d - c \leq e - c$ iff $d \leq e$; so (E3) holds. Finally (E4') is easily derived from (D4'); so L is proved to be a weak effect algebra.

Moreover, L as a weak D-poset now arises from L as a weak effect algebra just in the way given in the first part of the Proposition. It is furthermore clear that there is only one weak effect algebra with that property. □

We introduce furthermore the counterpart of a weak MV-effect algebra for weak D-posets.

Definition 5.5. A weak D-poset $(L; \leq, -, 0, 1)$ is called a *weak MV-D-poset* if the following conditions holds:

(D5) For any $a_1, a_2, b_1, b_2 \in L$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in L$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(D6) For $a, b, c \in L$, let $a \leq b$ and let $b - c$ be defined. Then we have $a - c_1 \leq b - c$ for some $c_1 \leq c$. In case $c \leq a$, we may assume $c_1 = c$.

(D7) For any $a, b \in L$, there are $a_1, b_1 \leq a, b$ such that $(a - a_1) \wedge (b - b_1) = 0$.

Proposition 5.6. *Let $(L; \leq, +, 0, 1)$ be a weak effect algebra, and let $(L; \leq, -, 0, 1)$ be the corresponding weak D-poset according to Proposition 5.4. Then L is a weak MV-effect algebra if and only if L is a weak MV-D-poset.*

Proof. It is not difficult to see that (E5) is equivalent to (D5), that (E6) is equivalent to (D6), and that (E7) is equivalent to (D7). We drop the details. □

6. BL-algebras as weak D-posets

In a similar way as we may identify BL-algebras, understood as bounded NAMs of type BL, with a subclass of the weak effect algebras, we may understand BL-algebras as bounded BCK-algebras of type BL and identify them with a subclass of the weak D-posets.

Definition 6.1. (i) Let $(L; \leq, \ominus, 0, 1)$ be a bounded BCK-algebra of type BL. For $a, b \in L$, define $b - a \stackrel{\text{def}}{=} b \ominus a$ in case that a is the smallest $x \in L$ such that $b \ominus x = b \ominus a$, and let else $b - a$ undefined. Then the operation $-$ is called the *partial difference belonging to $\ominus$.*

(ii) Let $(L; \leq, -, 0, 1)$ be a weak D-poset. Let for any $a, b \in L$, $b \ominus a \stackrel{\text{def}}{=} b - \bar{a}$, where $\bar{a}$ is the largest element below a such that $b - \bar{a}$ is defined. Then the operation $\ominus$ is called the *total difference belonging to $-$.*

Theorem 6.2. *Let $(L; \leq, \ominus, 0, 1)$ be a bounded BCK-algebra of type BL, and let $-$ be the partial difference belonging to $\ominus$. Then $(L; \leq, -, 0, 1)$ is a weak MV-D-poset, and the total difference belonging to $-$ coincides with $\ominus$.*

Proof. By Theorem 2.7, there is a unique operation $\oplus$ on L such that $(L; \leq, \oplus, \ominus, 0, 1)$ is a dual BL-algebra; and by Theorem 2.5, $(L; \leq, \oplus, 0, 1)$ is a bounded NAM of type BL. $+$ being the partial addition belonging to $\oplus$, we have from Theorem 4.2 that $(L; \leq, +, 0, 1)$ is a weak MV-effect algebra; and by Propositions 5.4 and 5.6, $(L; \leq, -', 0, 1)$ is a weak MV-D-poset, where for any $a, b, c \in L$ we have $b -' a = c$ iff $a + c = b$.

We have to show that $-'$ is the partial difference belonging to $\ominus$, that is, $-' = -$. So let $a, b, c \in L$. Assume first that $c = b -' a$ is defined. Then a and c are minimal in the sum $a \oplus c$, hence $c = b \ominus a$ and $a = b \ominus c = b \ominus (b \ominus a)$. If now $b \ominus x = b \ominus a$ for some $x \in L$, then $x \geq b \ominus (b \ominus x) = a$, whence a is the smallest x such that $b \ominus x = b \ominus a$; so indeed, $c = b - a$. Conversely, assume that $c = b - a$ is defined. Then a is the smallest x such that $b \ominus x = b \ominus a$, and it follows $a \leq b \ominus (b \ominus a)$, so in view of (3) $a = b \ominus (b \ominus a)$. Since $a \leq b$, we have $b = (b \ominus (b \ominus a)) \oplus (b \ominus a) = (b \ominus (b \ominus a)) + (b \ominus a) = a + c$. So $b -' a = c$.

It remains to prove that $\ominus$ is the total difference belonging to $-$. Let $a, b \in L$, and let $\bar{a}$ be the maximal element below $a \wedge b$ such that $b - \bar{a}$ is defined; note that then $\bar{a}$ is also the maximal element below a with that property. So $\bar{a}$ is the maximal element below $a \wedge b$ such that $\bar{a} + x = b$ exists for some $x \in L$. It follows from Theorem 4.3 and its proof that $(a \wedge b) \oplus x = b$ and $x = b \ominus (a \wedge b) = b \ominus a$. So $b \ominus a = x = b - \bar{a}$, which is what we had to show. □

Theorem 6.3. *Let $(L; \leq, -, 0, 1)$ be a weak MV-D-poset, and let $\ominus$ be the total difference belonging to $-$. Let for any $a, b \in L$*

$$a \oplus b \overset{\text{def}}{=} \max \{c\colon\ c - b' \leq a \text{ for some } b' \leq b\} \tag{8}$$

exist. Then $(L; \leq, \ominus, 0, 1)$ is a bounded BCK-algebra of type BL, the partial difference belonging to $\ominus$ coincides with $-$, and $\oplus$ is the S-function of L.

Proof. According to Propositions 5.4 and 5.6, $(L; \leq, +, 0, 1)$ is a weak MV-effect algebra, where, for $a, b, c \in L$, $a+b = c$ holds iff $b = c-a$. The operation $\oplus$ as defined by (8) obviously coincides with the total addition belonging to $+$. So by Theorem 4.3, $(L; \leq, \oplus, 0, 1)$ is a bounded NAM of type BL. $\ominus'$ being the residuum of L, $(L; \leq, \ominus', 0, 1)$ is by Theorems 2.5 and 2.7 a bounded BCK-algebra of type BL with S-function $\oplus$.

In the same way as in the proof of Theorem 6.2, we see that $\ominus'$ is the total difference belonging to $-$, that is, $\ominus' = \ominus$. We also may procede equally as in the proof of Theorem 6.2 to see that $-$ is the partial difference belonging to $\ominus$. □

7. MV-, PL-, and G-algebras

The importance of the BL-algebras has apparently much to do with the fact that they generalize MV-algebras, PL-algebras, and G-algebras, which may be used to model formulas of the Łukasiewicz logic, product logic, or Gödel logic, respectively. Having established the correspondence between BL-algebras and a subclass of the weak effect algebras, we may wonder which kinds of weak effect algebras correspond to the MV-, PL-, and G-algebras.

Recall that a BL-algebra $(L; \leq, \odot, \Rightarrow, 0, 1)$ is called an *MV-algebra* if the complement operation $^\star\colon\ L \to L,\ \ a \mapsto a \Rightarrow 0$ is involutive; that L is called a *PL-algebra* if for any $a, b, c \in L$ we have $a^{\star\star} \leq (a \odot b \Rightarrow a \odot c) \Rightarrow (b \Rightarrow c)$ and $a \wedge a^\star = 0$; and that L is called a *G-algebra* if $a \odot b = a \wedge b$ for all $a, b \in L$.

For an element a of a weak effect algebra, let us define its complement $a^\star$ to be the smallest element x such that $\bar{a} + x = 1$ for some $\bar{a} \leq a$. It is clear that under the correspondence established between weak effect algebras and BL-algebras, this operation coincides with the equally denoted one for BL-algebras. Note that in a weak effect algebra, $a+b = 1$ implies $b = a^\star$.

Proposition 7.1. *Let $(L; \leq_{\rm BL}, \odot, \Rightarrow, 0_{\rm BL}, 1_{\rm BL})$ be a BL-algebra, and let $(L; \leq, +, 0, 1)$ be the corresponding weak effect algebra according to Definition 2.2 and Theorems 2.5 and 4.2.*

- (i) *L as a BL-algebra is an MV-algebra if and only if for L as a weak effect algebra the following holds: for any $a \in L$ there is some $b \in L$ such that $a + b = 1$.*
- (ii) *L as a BL-algebra is a PL-algebra if and only if for L as a weak effect algebra the following holds: (α) for any $a, b \in L$, $a + b$ is defined if and only if $a \leq b^\star$ and $b \leq a^\star$; (β) for any $a \in L$, $a \vee a^\star = 1$.*
- (iii) *L as a BL-algebra is a G-algebra if and only if for L as a weak effect algebra the following holds: for any $a, b \in L$, $a + b$ is defined if and only if $a \wedge b = 0$.*

Proof. Let us treat L throughout this proof as a weak effect algebra, if not indicated otherwise.

(i) L as a BL-algebra is an MV-algebra iff * is involutive.

Let * be involutive. Let $\bar{a} \leq a$ be maximal such that $\bar{a} + b = 1$ for some $b \in L$. It follows $b = a^\star = \bar{a}^\star$ and so $a = \bar{a}$ and $a + b = 1$.

Conversely, let $a \in L$, and assume that there is always some b such that $a + b = 1$. It then follows $b = a^\star$ and $a = b^\star$, whence $a^{\star\star} = b^\star = a$. So * is involutive.

(ii) Let $\oplus$ and $\ominus$ be addition and difference of the dual BL-algebra corresponding to L.

Let us first show that for any $a, b \in L$ such that $a \wedge b = 0$, we have $a \vee b = a + b$. Indeed, we then have $b \ominus a = b \ominus (a \wedge b) = b$ and similarly $a \ominus b = a$. Furthermore, $a \vee b = a \oplus (b \ominus a) = [(a \vee b) \ominus (b \ominus a)] + (b \ominus a) = (a \ominus b) + (b \ominus a) = a + b$.

We further claim that for any $a, b, c \in L$ such that $a \wedge b = 0$ and such that $a + c$ and $b + c$ exists, also $a + b + c$ exists. Indeed, $a \vee b = a + b$, and $\oplus$ and $\ominus$ are compatible with the lattice operations; so $c \geq ((a + b) \oplus c) \ominus (a + b) = ((a \vee b) \oplus c) \ominus (a \vee b) = [((a \oplus c) \ominus a) \vee ((b \oplus c) \ominus a)] \wedge [((a \oplus c) \ominus b) \vee ((b \oplus c) \ominus b)] \geq c$, because $(a \oplus c) \ominus a = (b \oplus c) \ominus b = c$. Furthermore, $a + b \geq ((a + b) \oplus c) \ominus c = ((a \vee b) \oplus c) \ominus c = ((a \oplus c) \ominus c) \vee ((b \oplus c) \ominus c) = a \vee b = a + b$. It follows that $(a + b) + c$ is defined.

Now, from [Vet, Proposition 3.4(ii)], we see that L as a BL-algebra is a PL-algebra iff the following holds: for any $a, b, c \in L$, $a^\star = 1$ and $a \oplus b = a \oplus c$ implies $b = c$, and for any $a \in L$ we have $a \vee a^\star = 1$.

Let these conditions be fulfilled. Since (β) is included in them, it is just (α) that we have to show. Let $a, b \in L$.

If $a + b$ is defined, then $a = (a + b) \ominus b \leq 1 \ominus b = b^\star$, and similarly we get $b \leq a^\star$.

Assume now $a \leq b^\star$ and $b \leq a^\star$. Let us set $a = a_0 \vee a_1$, where $a_0 = a \wedge a^\star$ and $a_1 = a \wedge a^{\star\star} = a^{\star\star}$. Because $a^\star \wedge a^{\star\star} = 1 \ominus (a \vee a^\star) = 0$, we have $a_0 \wedge a_1 = 0$ and so, by what was proved above, $a = a_0 + a_1$. Besides, ${a_0}^\star = 1 \ominus (a \wedge a^\star) = a^\star \vee a^{\star\star} = 1$. In a similar manner, we may split up $b = b_0 + b_1$. Now, $a_0 \oplus b_0 = a_0 + b_0$; indeed, e.g. a_0 is minimal in the sum $a_0 \oplus b_0$, because ${b_0}^\star = 1$ and $a_0 \oplus b_0 = ((a_0 \oplus b_0) \ominus b_0) \oplus b_0$ implies $a_0 = (a_0 \oplus b_0) \ominus b_0$. Moreover, we have $a_1 \wedge b \leq a^{\star\star} \wedge a^\star = 0$ and similarly $b_1 \wedge a = 0$. It follows by (E6) that $a_1 \wedge (a_0 + b_0) = b_1 \wedge (a_0 + b_0) = 0$, so $a_1 + a_0 + b_0$ and $b_1 + a_0 + b_0$ exist. Since $a_1 \wedge b_1 = 0$, it follows from what we proved above that $a_1 + b_1 + a_0 + b_0 = a + b$ exists. This completes the proof of one half of (ii).

Assume now that for any $a, b \in L$ the existence of $a + b$ is equivalent to $a \leq b^\star$ and $b \leq a^\star$, and that for any $a \in L$ $a \vee a^\star = 1$. Let $a, b, c \in L$, and assume $a^\star = 1$ and $a \oplus b = a \oplus c$. We have to prove $b = c$; then the second half of (ii) follows. We have $b^\star = 1 \ominus b = a^\star \ominus b = 1 \ominus (a \oplus b) = 1 \ominus (a \oplus c) = c^\star$. Let us set $a' = a \wedge b^\star = a \wedge c^\star$; then it follows $a' \oplus b = a' \oplus c$. We may now conclude from our assumption that $a' + b = a' + c$; and by cancellation, $b = c$.

(iii) L as a BL-algebra is a PL-algebra iff for the weak effect algebra we have $a \oplus b = a \vee b$ for all $a, b \in L$.

Let this condition hold, and let $a, b \in L$. If then $a + b$ is defined, then $a = (a+b) \ominus b = (a \oplus b) \ominus b = (a \vee b) \ominus b = a \ominus b$ and similarly $b = b \ominus a$, so $a \wedge b = 0$ by condition (iii) in Definition 2.6. If $a \wedge b = 0$, then we know from the proof of (ii) that $a + b$ is defined.

Conversely, let, for any $a, b \in L$, $a + b$ be defined iff $a \wedge b = 0$. For any $a, b \in L$, $a \oplus b = [(a \oplus b) \ominus ((a \oplus b) \ominus a)] + [(a \oplus b) \ominus a]$; so by assumption $0 = [(a \oplus b) \ominus ((a \oplus b) \ominus a)] \wedge [(a \oplus b) \ominus a] = (a \oplus b) \ominus [((a \oplus b) \ominus a) \vee a]$, so $a \oplus b \leq ((a \oplus b) \ominus a) \vee a \leq a \vee b \leq a \oplus b$. □

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