Categories of orthogonality spaces

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Abstract

An orthogonality space is a set equipped with a symmetric and irreflexive binary relation. We consider orthogonality spaces with the additional property that any collection of mutually orthogonal elements gives rise to the structure of a Boolean algebra. Together with the maps that preserve the Boolean structures, we are led to the category $\mathcal{NOS}$ of normal orthogonality spaces.

Moreover, an orthogonality space of finite rank is called linear if for any two distinct elements e and f there is a third one g such that exactly one of f and g is orthogonal to e and the pairs e, f and e, g have the same orthogonal complement. Linear orthogonality spaces arise from finite-dimensional Hermitian spaces. We are led to the full subcategory $\mathcal{LOS}$ of $\mathcal{NOS}$ and we show that the morphisms are the orthogonality-preserving lineations.

Finally, we consider the full subcategory $\mathcal{EOS}$ of $\mathcal{LOS}$ whose members arise from Hermitian spaces over Euclidean subfields of $\mathbb{R}$. We establish that the morphisms of $\mathcal{EOS}$ are induced by generalised semiunitary mappings.

Keywords: Orthogonality spaces; undirected graphs; categories; Boolean subalgebras; linear orthogonality spaces; generalised semilinear map; generalised semiunitary map

MSC: 81P10; 06C15; 46C05

1 Introduction

In quantum mechanics, physical processes are described in a way assigning an essential role to the observer; rather than predicting on the basis of complete initial conditions the unambiguous development of some physical system, the theory assigns probabilities to pairs consisting of a preparation procedure and the outcome of a subsequent measurement. Why the formalism has proved successful is by and large today still unanswered; we could admit that we rather got used to it. But even at the most basic level, there are unresolved issues. A key ingredient of the model is a certain inner-product space – a Hilbert space over the field of complex numbers –, and the deeper reasons for this choice are a matter of ongoing discussions.

The probably oldest approach aiming to clarify the basic principles on which quantum theory is based is due to Birkhoff and von Neumann [BiNe]. The keyword "quantum logic" is often used in this context but might be misleading. What in our eyes rather matters is the idea of increasing the degree of abstraction: the question is whether the Hilbert space can be recovered from a considerably simpler structure. Numerous types of algebras, including partial ones, have been proposed and investigated, the best-known example being orthomodular lattices, which describe the Hilbert space by means of the inner structure of its closed subspaces. For an overview of possible directions, we may refer, e.g., to the handbooks [EGL1, EGL2].

Increasing the degree of abstraction means to restrict the structure to the necessary minimum. An approach that was proposed in the 1960's by David Foulis and his collaborators goes presumably to the limits of what is possible. They coined the notion of an orthogonality space, which is simply a set endowed with a symmetric and irreflexive binary relation. The prototypical example is the collection of one-dimensional subspaces of a Hilbert space together with the usual orthogonality relation.

The notion of an orthogonality space is in the centre of the present work and the main motivation behind our work is to elaborate on its role within the basic quantum-physical model. We generally deal with the case of a finite rank, meaning that there are only finitely many pairwise orthogonal elements. We should certainly be aware of the fact that orthogonality spaces are as general as undirected graphs, which in turn are rarely put into context with inner-product spaces. As has been shown in [Vet3], however, the relationship between the two types of structures is close. An orthogonality space of finite rank is called *linear* if, for any distinct elements e and f, there is a further one g such that exactly one of f and g is orthogonal to e and the set of elements orthogonal to e and f coincides with the set of elements orthogonal to e and g. Linearity characterises the orthogonality spaces that arise from finite-dimensional Hermitian spaces.

In physics, symmetries of the model generally play a fundamental role. It might thus not come as a surprise that orthogonality spaces associated with complex Hilbert spaces are describable by the particular properties of their automorphisms [Vet1, Vet2]. Here, we further elaborate on this issue, but we adopt a more general perspective than in the previous works.

The present paper is devoted to the investigation of structure-preserving maps between orthogonality spaces. We do so first in a general context, taking into account features inherent to orthogonality spaces, and in a second step, we turn to the narrower class of linear orthogonality spaces. We start with the question how to reasonably define morphisms. It certainly seems to make sense to require nothing more than the preservation of the single binary relation on which the structures are based. We call orthogonality-preserving maps homomorphisms. A simple illustration shows, however, that this notion is inappropriate when the context that we ultimately have in mind is given by inner-product spaces. Indeed, for linear orthogonality spaces, we expect a morphism to preserve, in some sense, linear dependence. The following situation illustrates the difficulties, even though we otherwise deal with the finite-dimensional case only [Šem]. Consider the complex projective space over three dimensions $P(\mathbb{C}^3)$ as well as over $2^{\aleph_0}$ dimensions $P(\mathbb{C}^{2^{\aleph_0}})$; then any injective map from $P(\mathbb{C}^3)$ to $P(\mathbb{C}^{2^{\aleph_0}})$ such that the image consists of mutually orthogonal elements is a homomorphism of orthogonality spaces, but in no way related to the preservation of linear dependence.

Having in mind the Hilbert space model of quantum physics, we have thus found that it is natural to restrict from the outset to a narrower class of orthogonality spaces, ruling out situations that we must consider as inappropriate. In quantum mechanics, observables correspond to Boolean algebras. In a finite-state system, measurement outcomes correspond to mutually orthogonal subspaces, which in turn generate a Boolean subalgebra of the lattice of closed subspaces. We require to have an analogue of this situation in our more abstract setting.

To be more specific, let us first recall that orthogonality spaces lead us straightforwardly to the realm of lattice theory. A subset A of an orthogonality space $(X, \perp)$ is called orthoclosed if $A = B^\perp$ for some $B \subseteq X$, where $B^\perp$ is the set of $e \in X$ orthogonal to all elements of B. The set of orthoclosed subsets form a complete ortholattice $\mathcal{C}(X, \perp)$. Now, consider a collection $E = \{x_1, \ldots, x_k\}$ of mutually orthogonal elements of X. Then the subsets of E generate a subortholattice of $\mathcal{C}(X, \perp)$. This subortholattice is, in general, not isomorphic to the Boolean algebra of subsets of E; in case it always is, we call $(X, \perp)$ *normal*. We moreover name homomorphisms in the same way if they preserve, in a natural sense, Boolean subalgebras of $\mathcal{C}(X, \perp)$. We thus arrive at the category $\mathcal{NOS}$ of normal orthogonality spaces and normal homomorphisms.

We take up in this way an often-discussed issue. Indeed, for the aim of recovering a Hilbert space or, more generally, an orthomodular lattice from suitable substructures, it has been a guiding motive to consider the lattice as being glued together from its Boolean subalgebras; see, e.g., [Nav, Section 4]. Moreover, deep results have been achieved on the question how to reconstruct orthomodular lattices or related quantum structures from the poset of their Boolean subalgebras [HaNa, HHLN].

Any linear orthogonality space is normal and thus our next step is to consider normal homomorphisms between linear orthogonality spaces. That is, we investigate the full subcategory $\mathcal{LOS}$ of $\mathcal{NOS}$, consisting of linear orthogonality spaces. It turns out that the morphisms in $\mathcal{LOS}$ do have the most basic property to be expected: they are maps between projective spaces that preserve the triple relation of being contained in a line, that is, they are lineations. In fact, we show that the morphisms are exactly the orthogonality-preserving lineations.

Our final objective is to describe the morphisms in $\mathcal{LOS}$ as precisely as possible. Generalisations of the fundamental theorem of projective geometry show that any lineation is induced by a generalised semilinear transformation – provides it is non-degenerate [Mach, Fau]. Here, non-degeneracy means two additional conditions to hold: (1) the image is not contained in a two-dimensional subspace, and (2) the image of a line is never two-element. Provided that the rank is at least 3, condition (1) is ensured. Condition (2), however, leads us to an issue dealt with in the discussions around the peculiarities of quantum physics: we show that a violation of (2) implies the existence of two-valued measures. The exclusion of two-valued measures is in turn a consequence of Gleason's Theorem in case that the skew field is $\mathbb{C}$ or $\mathbb{R}$. Although the case of specific further skew fields has been discussed [Dvu], not much seems to be known about the general case. Here, we show that if the skew field of scalars is an Euclidean subfield of the reals, two-valued measures do not exist and it follows that morphisms are represented by generalised semilinear maps.

Moreover, what we deal with lineations that, in addition, preserve an orthogonality relation. It seems natural to ask whether the representing generalised semilinear map can be chosen to preserve in some sense the inner product. We establish that this is the case under particular conditions: the skew field is commutative, that is, a field and there is a basis of vectors of equal length. We conclude that morphisms between Hermitian spaces over Euclidean subfield of the reals are induced by what we call generalised semiunitary maps.

The paper is organised as follows. In the following Section 2, we fix the basic notation used in this paper. Moreover, we introduce and discuss normal orthogonality spaces, in particular we also include a characterisation of normality as an intrinsic property, without reference to the associated ortholattice. In Section 3, we investigate the category $\mathcal{NOS}$ of normal orthogonality spaces and normal homomorphisms. In Section 4, we prepare the ground for the discussion of those orthogonality spaces that arise from inner-product spaces; in particular, we discuss lineations between projective spaces and discuss their representation in the presence of an inner product. Then, in Section 5, we recall the notion of linear orthogonality spaces and show that linearity implies normality. Finally, in Section 6, we study the full subcategory $\mathcal{LOS}$ of $\mathcal{NOS}$ that consists of linear orthogonality spaces, with a focus on the description of the morphisms by means of generalised semilinear maps. Some concluding remarks are found in the final Section 7.

2 Normal orthogonality spaces

We deal in this paper with the following relational structures.

Definition 2.1. An *orthogonality space* is a non-empty set X equipped with a symmetric, irreflexive binary relation $\perp$, called the *orthogonality relation*. The supremum of the cardinalities of sets of mutually orthogonal elements of X is called the *rank* of $(X, \perp)$.

We may observe that orthogonality spaces are essentially the same as undirected graphs, understood such that the edges are two-elements subsets of the set of nodes. The rank of an orthogonality space is under this identification the supremum of the sizes of cliques. The present work, however, is not motivated by graph theory, our guiding example rather originates in quantum physics.

Example 2.2. *Let H be a Hilbert space. Then the set $P(H)$ of one-dimensional* *subspaces of H, together with the usual orthogonality relation, is an orthogonality space, whose rank coincides with the dimension of H.*

The *(orthogonal) complement* of a subset A of an orthogonality space X is

$$A^{\perp} \;=\; \{x \in X \colon x \perp a \text{ for all } a \in A\}.$$

The map $\mathcal{P}(X) \to \mathcal{P}(X),\ A \mapsto A^{\perp\perp}$ is a closure operator on X. We call the closed subsets *orthoclosed* and we denote the collection of orthoclosed subsets by $\mathcal{C}(X, \perp)$. Endowed with the set-theoretical inclusion and the orthocomplementation $^{\perp}$, $\mathcal{C}(X, \perp)$ becomes a complete ortholattice. The ortholattice $(\mathcal{C}(X, \perp); \cap, \vee, ^{\perp}, \varnothing, X)$ will be our primary tool to investigate $(X, \perp)$.

Example 2.3. *Let $(P(H), \perp)$ be the orthogonality space arising from the Hilbert space H according to Example 2.2. Then we may identify $\mathcal{C}(P(H), \perp)$ with the set $\mathcal{C}(H)$ of closed subspaces of H, endowed with the set-theoretical inclusion and the orthocomplementation.*

In this paper, we will focus exclusively on the case of a finite rank. Our guiding example is, accordingly, the orthogonality space associated with a finite-dimensional Hilbert space. From now on, all orthogonality spaces are tacitly assumed to be of finite rank.

We will next introduce a condition on orthogonality spaces that mimics a key feature of the quantum-physical formalism. In quantum mechanics, a physical system is modelled by means of a Hilbert space and observables correspond to Boolean subalgebras of the lattice of its closed subspaces. We will require that orthogonality spaces possess substructures of the corresponding type.

Definition 2.4. An orthogonality space $(X, \perp)$ is called *normal* if, for any mutually orthogonal elements $e_1, \ldots, e_k$ of X, where $k \geqslant 1$, the subalgebra of the ortholattice $\mathcal{C}(X, \perp)$ generated by $\{e_1\}^{\perp\perp}, \ldots, \{e_k\}^{\perp\perp}$ is Boolean.

We may understand normality also as a coherence condition. By a subset A of an orthogonality space to be orthogonal, we mean that A consists of mutually orthogonal elements.

Lemma 2.5. *For an orthogonality space $(X, \perp)$, the following are equivalent:*

(1) *$(X, \perp)$ is normal.*

(2) *For any maximal orthogonal set $\{e_1, \ldots, e_n\} \subseteq X$, there is a finite Boolean subalgebra of $\mathcal{C}(X, \perp)$ whose atoms are $\{e_1\}^{\perp\perp}, \ldots, \{e_n\}^{\perp\perp}$.*

(3) *For any maximal orthogonal set $\{e_1, \ldots, e_n\} \subseteq X$ and any $1 \leqslant k < n$, if $f \perp e_1, \ldots, e_k$ and $g \perp e_{k+1}, \ldots, e_n$, then $f \perp g$.*

Proof. (1) $\Rightarrow$ (2): Let $(X, \perp)$ be normal and let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of X. By normality, the subalgebra $\mathcal{B}$ of $\mathcal{C}(X, \perp)$ generated by $\{e_1\}^{\perp\perp}, \ldots,$

$\{e_n\}^{\perp\perp}$ is Boolean. Moreover, $\{e_1\}^{\perp\perp}, \ldots, \{e_n\}^{\perp\perp}$ are mutually orthogonal elements and we have $\{e_1\}^{\perp\perp} \vee \ldots \vee \{e_n\}^{\perp\perp} = \{e_1, \ldots, e_n\}^{\perp\perp} = \varnothing^{\perp} = X$. Thus $\mathcal{B}$ is a finite Boolean subalgebra of $\mathcal{C}(X, \perp)$, its atoms being $\{e_1\}^{\perp\perp}, \ldots, \{e_n\}^{\perp\perp}$.

(2) $\Rightarrow$ (3): Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of X and assume that $\{e_1\}^{\perp\perp}, \ldots, \{e_n\}^{\perp\perp}$ are the atoms of a finite Boolean subalgebra of $\mathcal{C}(X, \perp)$. Let $1 \leqslant k < n$. Then $f \perp e_1, \ldots, e_k$ means $f \in \{e_1, \ldots, e_k\}^{\perp} = \{e_{k+1}, \ldots, e_n\}^{\perp\perp}$, and similarly, $g \perp e_{k+1}, \ldots, e_n$ means $g \in \{e_1, \ldots, e_k\}^{\perp\perp}$. If both $f \perp e_1, \ldots, e_k$ and $g \perp e_{k+1}, \ldots, e_n$ holds, we hence conclude $f \perp g$.

(3) $\Rightarrow$ (1): Let $D = \{e_1, \ldots, e_k\}$, $k \geqslant 1$, be an orthogonal subset of X. Then we may extend D to a maximal orthogonal subset $E = \{e_1, \ldots, e_n\}$ of X, where $n \geqslant k$. For any $A \subseteq E$, we have $\bigvee\{\{e\}^{\perp\perp} \colon e \in A\} = A^{\perp\perp}$; for any $A, B \subseteq E$, we have $A^{\perp\perp} \vee B^{\perp\perp} = (A \cup B)^{\perp\perp}$; and $E^{\perp\perp} = X$. Let $\varnothing \neq A \subsetneq E$. Then $(E \setminus A)^{\perp\perp} \subseteq A^{\perp}$. Moreover, if $f \in A^{\perp}$ and $g \in (E \setminus A)^{\perp}$, we have by assumption $f \perp g$; hence $f \in (E \setminus A)^{\perp\perp}$. We conclude that $A^{\perp\perp} = (E \setminus A)^{\perp}$. We have shown that $\{e_1\}^{\perp\perp}, \ldots, \{e_n\}^{\perp\perp}$ generate a Boolean subalgebra of $\mathcal{C}(X, \perp)$; hence so do $\{e_1\}^{\perp\perp}, \ldots, \{e_k\}^{\perp\perp}$. $\square$

The following notation will be useful. Let $e_1, \ldots, e_k$ be mutually orthogonal elements of a normal orthogonality space $(X, \perp)$. Then the closure of $\{\{e_1\}^{\perp\perp}, \ldots, \{e_k\}^{\perp\perp}\}$ under joins in $\mathcal{C}(X, \perp)$ has the structure of a Boolean algebra, whose top element is $\{e_1, \ldots, e_k\}^{\perp\perp}$. We will denote this Boolean algebra by $\mathcal{B}(e_1, \ldots, e_k)$.

The property of normality applies to our canonical example. We write $[x_1, \ldots, x_k]$ for the linear hull of non-zero vectors $x_1, \ldots, x_k$ of a linear space.

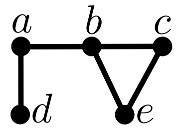
Example 2.6. *Let $x_1, \ldots, x_k$, $k \geqslant 1$, be mutually orthogonal non-zero vectors of a Hilbert space H. Then the subalgebra of $\mathcal{C}(H)$ generated by $[x_1], \ldots, [x_k]$ consists of the joins of subspaces among $[x_1], \ldots, [x_k], [x_1, \ldots, x_k]^{\perp}$. This algebra is Boolean and we conclude that $(P(H), \perp)$ is normal.*

For later considerations, we introduce a further, particularly simple example.

Example 2.7. *For $n \in \mathbb{N} \setminus \{0\}$, we denote by $\mathbf{n}$ an n-element set and we consider the binary relation $\neq$ on $\mathbf{n}$. Then $(\mathbf{n}, \neq)$ is an orthogonality space and $\mathcal{C}(\mathbf{n}, \neq)$ is the powerset of $\mathbf{n}$. Since $\mathcal{C}(\mathbf{n}, \neq)$ is Boolean, we have that $(\mathbf{n}, \neq)$ is normal.*

In general, however, an orthogonality space need not be normal. The subsequent examples of finite orthogonality spaces will be graphically depicted as follows: the elements of the space are represented by points, and two elements are orthogonal if the points are connected by a straight line. For instance, in the Example 2.8 below we have that a, b, c are mutually orthogonal and moreover $d \perp a$ as well as $e \perp b, c$. We note that this representation might remind of Greechie diagrams. It must be kept in mind, however, that an element of an orthogonality space does not necessarily represent an atom of the associated ortholattice. In Example 2.8, for instance, $\{e\}^{\perp\perp}$ properly contains $\{a\}^{\perp\perp}$.

Example 2.8. *Consider the orthogonality space $X = \{a, b, c, d, e\}$ given by the following scheme:*



$\{a, b, c\}$ is a maximal orthogonal set. Furthermore, we have $\{a\}^{\perp\perp} = \{a\}$ and $\{b\}^{\perp\perp} \vee \{c\}^{\perp\perp} = \{b, c\}^{\perp\perp} = \{b, c\}$. Since $\{b, c\}^{\perp} = \{a, e\}$, X is not normal.

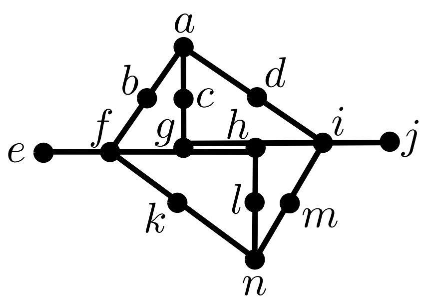
Given a normal orthogonality space $(X, \perp)$, we call an orthoclosed subset A of X together with the inherited orthogonality relation, which we usually still denote by $\perp$, a *subspace* of $(X, \perp)$.

The following proposition and example show that a subspace of a normal orthogonality space is not in general normal, but a subspace that is the closure of any maximal orthogonal subset is so.

Proposition 2.9. *Let $(X, \perp)$ be a normal orthogonality space and let $A \in \mathcal{C}(X, \perp)$ be such that, for any maximal orthogonal subset D of A, we have $D^{\perp\perp} = A$. Then the subspace $(A, \perp)$ is normal.*

Proof. We shall use criterion (3) of Lemma 2.5. Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of A, let $1 \leqslant k < n$, and assume that there are $f, g \in A$ such that $f \perp e_1, \ldots, e_k$ and $g \perp e_{k+1}, \ldots, e_n$. Then we may choose $e_{n+1}, \ldots, e_m \in X$ such that $\{e_1, \ldots, e_m\}$ is a maximal orthogonal subset of X. By assumption, $A = \{e_1, \ldots, e_n\}^{\perp\perp}$, hence $g \perp e_{n+1}, \ldots, e_m$. Thus we have $f \perp e_1, \ldots, e_k$ and $g \perp e_{k+1}, \ldots, e_m$ and the normality of X implies $f \perp g$. We conclude that $(A, \perp)$ is normal. $\square$

Example 2.10. *Let $(X, \perp)$ be the 14-element orthogonality space given as follows:*



(Here, the sets $\{e, f, g, h\}$ and $\{g, h, i, j\}$ are meant to be orthogonal; but, none of e or f is orthogonal to i or j.)

By criterion (3) of Lemma 2.5, we may check that X is normal. However, the subspace $\{f, i\}^{\perp} = \{a, g, h, n\}$ is not.

We might expect that normality of an orthogonality space is closely related to the orthomodularity of the associated ortholattice. This is indeed the case but the two properties do not coincide.

A *Dacey space* is an orthogonality space $(X, \perp)$ such that $\mathcal{C}(X, \perp)$ is an orthomodular lattice. We have the following characterisation of Dacey spaces [Dac, Wlc].

Lemma 2.11. *An orthogonality space $(X, \perp)$ is a Dacey space if and only if, for any $A \in \mathcal{C}(X, \perp)$ and any maximal orthogonal subset D of A, we have that $D^{\perp\perp} = A$.*

Example 2.12. *Let H be a Hilbert space. Then $\mathcal{C}(H)$ is an orthomodular lattice and hence $(P(H), \perp)$ a Dacey space.*

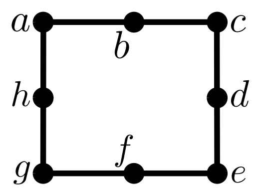
Example 2.13. *By means of Lemma 2.11, we observe that the orthogonality space $(X, \perp)$ from Example 2.8 is not a Dacey space. Indeed, $A = \{b, c, d\} \in \mathcal{C}(X, \perp)$, $\{b, c\}$ is a maximal orthogonal subset of A, and $\{b, c\}^{\perp\perp} = \{b, c\} \subsetneq A$.*

The following proposition and example show that the Dacey spaces form a strict subclass of the normal orthogonality spaces.

Proposition 2.14. *A Dacey space is a normal orthogonality space.*

Proof. Let $(X, \perp)$ be a Dacey space and $\{e_1, \ldots, e_k\}$ be an orthogonal subset of X. Then $\{e_i\}^{\perp\perp}$, $i = 1, \ldots, k$, are pairwise orthogonal and hence pairwise commuting elements of the orthomodular lattice $\mathcal{C}(X, \perp)$. It follows that they generate a Boolean subalgebra [BrHa, Prop. 2.8]. □

Example 2.15. *Consider the following orthogonality space $(X, \perp)$:*



The maximal orthogonal subsets are the elements along a straight line, e.g., $\{a, b, c\}$. By criterion (3) *of Lemma 2.5, we observe that $(X, \perp)$ is normal. We may also check that each subspace of $(X, \perp)$ is normal.*

Moreover, the set $\{a, e\}$ is orthoclosed. But $\{a\}$ is a maximal orthogonal subset of $\{a, e\}$ and $\{a\}^{\perp\perp} = \{a\}$. Hence by Lemma 2.11, $(X, \perp)$ is not a Dacey space.

3 The category $\mathcal{NOS}$ of normal orthogonality spaces

We discuss in this section structure-preserving maps between orthogonality spaces. We shall introduce a category consisting of normal orthogonality spaces and investigate its basic properties.

For orthogonality spaces X and Y, we call a map $\varphi\colon X \to Y$ a *homomorphism* if φ is orthogonality-preserving, that is, if, for any $e, f \in X$, $e \perp f$ implies $\varphi(e) \perp \varphi(f)$. In this case, φ induces the map

$$\bar{\varphi}\colon \mathcal{C}(X, \perp) \to \mathcal{C}(Y, \perp),\ A \mapsto \{\varphi(a)\colon a \in A\}^{\perp\perp}.$$

Obviously, $\bar{\varphi}$ is order- and orthogonality-preserving. It seems that in general, however, we cannot say much more about $\bar{\varphi}$. We will be interested in homomorphisms fulfilling the following additional condition.

Definition 3.1. Let $\varphi\colon X \to Y$ be a homomorphism between the normal orthogonality spaces X and Y. We will call φ *normal* if, for any orthogonal set $e_1, \ldots, e_k \in X$, $k \geqslant 1$, $\bar{\varphi}$ maps $\mathcal{B}(e_1, \ldots, e_k)$ isomorphically to $\mathcal{B}(\varphi(e_1), \ldots, \varphi(e_k))$.

The following lemma might help to elucidate the condition of normality for homomorphisms.

Lemma 3.2. *Let $\varphi\colon X \to Y$ be a homomorphism between normal orthogonality spaces. Then the following are equivalent:*

(1) *φ is normal.*

(2) *For any orthogonal subset $\{e_1, \ldots, e_k\}$ of X, where $k \geqslant 0$, we have $\bar{\varphi}(\{e_1, \ldots, e_k\}^{\perp\perp}) = \{\varphi(e_1), \ldots, \varphi(e_k)\}^{\perp\perp}$.*

(3) *For any orthogonal subset $\{e_1, \ldots, e_k\}$ of X, where $k \geqslant 0$, we have $\varphi(\{e_1, \ldots, e_k\}^{\perp\perp}) \subseteq \{\varphi(e_1), \ldots, \varphi(e_k)\}^{\perp\perp}$.*

(4) *For any maximal orthogonal subset $\{e_1, \ldots, e_n\}$ of X, we have $\varphi(X)^{\perp\perp} = \{\varphi(e_1), \ldots, \varphi(e_n)\}^{\perp\perp}$.*

(5) *For any maximal orthogonal subset $\{e_1, \ldots, e_n\}$ of X, we have $\varphi(X) \subseteq \{\varphi(e_1), \ldots, \varphi(e_n)\}^{\perp\perp}$.*

Proof. (1) $\Rightarrow$ (2): Let φ be normal and let $\{e_1, \ldots, e_k\} \subseteq X$ be orthogonal. Then $\bar{\varphi}$ maps the top element of $\mathcal{B}(e_1, \ldots, e_k)$ to the top element of $\mathcal{B}(\varphi(e_1), \ldots, \varphi(e_k))$, that is, $\bar{\varphi}(\{e_1, \ldots, e_k\}^{\perp\perp}) = \{\varphi(e_1), \ldots, \varphi(e_k)\}^{\perp\perp}$.

(2) $\Rightarrow$ (1): Let (2) hold and let $\{e_1, \ldots, e_k\} \subseteq X$ be orthogonal. Recall that the Boolean algebra $\mathcal{B}(e_1, \ldots, e_k)$ consists of the elements $A^{\perp\perp} \in \mathcal{C}(X, \perp)$, where $A \subseteq \{e_1, \ldots e_k\}$. By assumption, $\bar{\varphi}(A^{\perp\perp}) = (\varphi(A))^{\perp\perp}$. Thus $\bar{\varphi}$ establishes an isomorphism between $\mathcal{B}(e_1, \ldots, e_k)$ and $\mathcal{B}(\varphi(e_1), \ldots, \varphi(e_k))$.

The equivalence of (2) and (3) as well as the equivalence of (4) and (5) are clear. Moreover, (2) clearly implies (4). We conclude the proof by showing that (5) implies (3).

Assume that (5) holds. Let $\{e_1, \ldots, e_k\}$ be an orthogonal subset of X. We extend it to a maximal orthogonal set $E = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_m\}$. Furthermore, $f_1 = \varphi(e_1), \ldots, f_m = \varphi(e_m)$ are pairwise orthogonal elements of Y. We extend $\varphi(E)$ to a maximal orthogonal subset F of Y.

Let $A \subseteq E$. We shall show that $\varphi(A^{\perp\perp}) \subseteq \varphi(A)^{\perp\perp}$, then in particular (3) will follow. As φ is orthogonality-preserving, $\varphi(A^{\perp\perp}) \perp \varphi(E \setminus A)^{\perp\perp}$. Moreover, by assumption, $\varphi(A^{\perp\perp}) \subseteq \varphi(X) \subseteq \varphi(E)^{\perp\perp} \perp (F \setminus \varphi(E))^{\perp\perp}$. It follows $\varphi(A^{\perp\perp}) \perp \varphi(E \setminus A)^{\perp\perp} \vee (F \setminus \varphi(E))^{\perp\perp} = (F \setminus \varphi(A))^{\perp\perp}$ and hence, by the normality of Y, $\varphi(A^{\perp\perp}) \subseteq (F \setminus \varphi(A))^{\perp} = \varphi(A)^{\perp\perp}$. □

We observe that normal homomorphisms are, in a restricted sense, linearity-preserving.

Lemma 3.3. *Let X and Y be normal orthogonality spaces and let $\varphi\colon X \to Y$ be a normal homomorphism. Let $e, f \in X$ be such that $e \perp f$. If $g \in \{e, f\}^{\perp\perp}$, then $\varphi(g) \in \{\varphi(e), \varphi(f)\}^{\perp\perp}$.*

Proof. The assertion holds by Lemma 3.2, property (3). □

An *automorphism* of an orthogonality space $(X, \perp)$ is a bijection $\varphi\colon X \to X$ such that, for any $e, f \in X$, $e \perp f$ if and only if $\varphi(e) \perp \varphi(f)$. Automorphisms are always normal homomorphisms, in particular the identity is normal.

Lemma 3.4. *Let X be a normal orthogonality space and let $\varphi\colon X \to X$ be an automorphism. Then φ is normal.*

Proof. $\bar{\varphi}$ is an automorphism of $\mathcal{C}(X, \perp)$. □

We see next that normal homomorphisms are closed under composition.

Lemma 3.5. *Let X, Y, and Z be normal orthogonality spaces and let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be normal homomorphisms. Then also $\psi \circ \varphi$ is a normal homomorphism.*

Proof. Clearly, $\psi \circ \varphi$ is orthogonality-preserving. Moreover, the normality follows by means of property (3) in Lemma 3.2. □

We define the category $\mathcal{NOS}$ to consist of the normal orthogonality spaces (of finite rank) and the normal homomorphisms.

We first check whether an inclusion map between normal orthogonality spaces is normal. The following example shows that this is not in general the case.

Example 3.6. *Consider again the orthogonality space $(X, \perp)$ from Example 2.15, which is normal but not Dacey, and let $A = \{a, e\}$. Then $A \in \mathcal{C}(X, \perp)$ and $(A, \varnothing)$ is a subspace of $(X, \perp)$, which is normal. Let now $i_A\colon A \to X$ be the inclusion map. We have that $\{a\}$ is a maximal orthogonal subset of A and*

$$i_A(A)^{\perp\perp} = \{a, e\}^{\perp\perp} = \{a, e\} \neq \{a\} = \{a\}^{\perp\perp} = \{i_A(a)\}^{\perp\perp}.$$

Hence, by Lemma 3.2, property (4), *i_A is not normal.*

Theorem 3.7. *Let $(X, \perp)$ be a normal orthogonality space. The X is a Dacey space if and only if, for any $A \in \mathcal{C}(X, \perp)$, the subspace $(A, \perp)$ is normal and the inclusion map $\iota\colon A \to X$ is a morphism in $\mathcal{NOS}$.*

Proof. Assume first that X is a Dacey space. Let $A \in \mathcal{C}(X, \perp)$. By Lemma 2.11 and Proposition 2.9, $(A, \perp)$ is a normal subspace. Moreover, the inclusion map $\iota\colon A \to X,\ x \mapsto x$ is clearly orthogonality-preserving. Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of A. Then $A = \{e_1, \ldots, e_n\}^{\perp\perp}$ by Lemma 2.11. By Lemma 3.2, property (4), we conclude that ι is actually a normal homomorphism.

Conversely, assume that, for any $A \in \mathcal{C}(X, \perp)$, $(A, \perp)$ is normal and the inclusion map $\iota\colon A \to X,\ x \mapsto x$ is a morphism of $\mathcal{NOS}$. Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of some $A \in \mathcal{C}(X, \perp)$. Then again by Lemma 3.2, property (4), we have that $\iota(A)^{\perp\perp} = \{\iota(e_1), \ldots, \iota(e_n)\}^{\perp\perp}$, that is, $A = \{e_1, \ldots, e_n\}^{\perp\perp}$. By Lemma 2.11, we conclude that X is a Dacey space. $\square$

We note that for a normal orthogonality space to be a Dacey space, it is not enough to assume that all subspaces are normal. Indeed, Example 2.15 provides a counterexample.

We shall next characterise the monomorphisms and epimorphisms in $\mathcal{NOS}$. To this end, we consider a doubling point construction, explained in the following lemma.

To increase clarity, we will occasionally use subscripts for the denotation of orthogonality relations and the associated ortholattice complements.

Lemma 3.8. *Let $(X, \perp_X)$ be a normal orthogonality space and $x \in X$. Let Z arise from X by replacing x with two new elements x_1 and x_2. We define the orthogonality relation $\perp_Z$ on Z as follows: For $e, f \in X \setminus \{x_1, x_2\}$ such that $e \perp_X f$, we let $e \perp_Z f$; and for $e \in X$ such that $e \perp_X x$, we let $x_1, x_2 \perp_Z e$ and $e \perp_Z x_1, x_2$. Then $(Z, \perp_Z)$ is a normal orthogonality space.*

Moreover, we define $f_1, f_2 \colon X \to Z$ as follows: $f_1(z) = f_2(z) = z$ if $z \neq x$; $f_1(x) = x_1$; and $f_2(x) = x_2$. Then f_1, f_2 are morphisms in $\mathcal{NOS}$.

Proof. Note first that, for any $e \in Z$, we have $e \perp_Z x_1$ if and only if $e \perp_Z x_2$. Furthermore, for $e \in Z \cap X$, we have $e \perp_X x$ if and only if $e \perp_Z x_1, x_2$.

For $A \subseteq X$, let $d(A) \subseteq Z$ be such that $d(A) = (A \setminus \{x\}) \cup \{x_1, x_2\}$ if $x \in A$, and $d(A) = A$ if $x \notin A$. Then, for any $A \subseteq X$, we have $d(A^{\perp_X}) = d(A)^{\perp_Z}$. Indeed, if $x \in A$, we have $d(A^{\perp_X}) = A^{\perp_X} = d(A)^{\perp_Z}$; and if $x \notin A$, we likewise have $d(A^{\perp_X}) = A^{\perp_Z} = d(A)^{\perp_Z}$.

We conclude that, for any $A \subseteq X$, we have $d(A^{\perp_X\perp_X}) = d(A)^{\perp_Z\perp_Z}$. In particular, we get a map $\delta \colon \mathcal{C}(X, \perp) \to \mathcal{C}(Z, \perp),\ A \mapsto d(A)$. Clearly, δ is order-preserving; we have seen that δ preserves the orthocomplement; and by construction, δ is injective. Moreover, for any $B \in \mathcal{C}(Z, \perp)$ we have that either none of x_1 and x_2 or both x_1 and x_2 are in B. Hence there is an $A \subseteq X$ such that $B = d(A)$. As $d(A^{\perp_X\perp_X}) = d(A)^{\perp_Z\perp_Z} = d(A)$, we have by the injectivity of d that $A \in \mathcal{C}(X, \perp)$, that is, δ is surjective. We conclude that δ is an isomorphism of ortholattices.

In particular, we have $\delta(\{x\}^{\perp_X\perp_X}) = \{x_1, x_2\}^{\perp_Z\perp_Z} = \{x_1\}^{\perp_Z\perp_Z} = \{x_2\}^{\perp_Z\perp_Z}$. It follows that $(Z, \perp_Z)$ is normal.

We use Lemma 3.2 to show that f_1 is normal. Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of X. Then $\{f_1(e_1), \ldots, f_1(e_n)\}$ is a maximal orthogonal subset of Z and hence $\{f_1(e_1), \ldots, f_1(e_n)\}^{\perp_Z\perp_Z} = Z$. Furthermore, we have $f_1(X)^{\perp_Z\perp_Z} = \varnothing^{\perp_Z} = Z$. By property (4) of Lemma 3.2, we conclude that f_1 is normal, and by similar reasoning we see that so is f_2. $\square$

Proposition 3.9. *Let $\varphi \colon X \to Y$ be a morphism in $\mathcal{NOS}$. Then we have:*

(i) *φ is a monomorphism in $\mathcal{NOS}$ if and only if φ is injective.*

(ii) *φ is an epimorphism in $\mathcal{NOS}$ if and only if φ is surjective.*

Proof. (i): Assume that φ is a monomorphism in $\mathcal{NOS}$. Let $x_1, x_2 \in X$ be such that $\varphi(x_1) = \varphi(x_2)$. Let $(\mathbf{1}, \neq) = (\{p\}, \varnothing)$ be the one-element orthogonality space, cf. Example 2.7. Clearly, $(\{p\}, \varnothing)$ is normal. Then the maps $\widehat{x_1}, \widehat{x_2} \colon \{p\} \to X$, given by $\widehat{x_1}(p) = x_1$ and $\widehat{x_2}(p) = x_2$ are morphisms in $\mathcal{NOS}$. It follows $\varphi \circ \widehat{x_1} = \varphi \circ \widehat{x_2}$ and hence $\widehat{x_1} = \widehat{x_2}$. We conclude $x_1 = x_2$, that is, φ is injective.

The converse direction is evident.

(ii): Assume that φ is an epimorphism in $\mathcal{NOS}$ that is not surjective. Let $y \in Y$ be such that $y \notin \varphi(X)$. Let $Z = (Y \setminus \{y\}) \cup \{y_1, y_2\}$, where y_1, y_2 are new elements, and let $\perp_Z$ be defined as in Lemma 3.8, such that $(Z, \perp_Z)$ becomes a normal orthogonality space. Likewise, let $f_1, f_2 \colon Y \to Z$ be such that $f_1(z) = f_2(z) = z$ if $z \neq y$, $f_1(y) = y_1$, and $f_2(y) = y_2$. By Lemma 3.8, f_1 and f_2 are morphisms in $\mathcal{NOS}$. But from $f_1 \circ \varphi = f_2 \circ \varphi$ we conclude $f_1 = f_2$, a contradiction.

The other direction is again evident. □

Let $\varphi \colon X \to Y$ be a morphism in $\mathcal{NOS}$. We call φ *quasi-surjective* if $Y = \varphi(X)^{\perp\perp}$. Clearly, if φ is surjective, φ is also quasi-surjective. Moreover, we call φ *full* if, for any $x_1, x_2 \in X$ such that $\varphi(x_1) \perp \varphi(x_2)$, there are $x_1', x_2' \in X$ such that $x_1' \perp x_2'$ and $\varphi(x_1) = \varphi(x_1')$ and $\varphi(x_2) = \varphi(x_2')$. Finally, we call φ an *embedding* if φ is injective and full.

We may factorise a morphism in $\mathcal{NOS}$ as follows.

Theorem 3.10. *Let $\varphi \colon X \to Y$ be a morphism in $\mathcal{NOS}$. Then there are morphisms $\alpha \colon X \to Z$ and $\beta \colon Z \to Y$ such that $\varphi = \beta \circ \alpha$, where α is quasi-surjective and β is an embedding.*

Proof. In this proof, we mark the ortholattice complement on $\mathcal{C}(Z, \perp)$ by a subscript Z, whereas the unmarked ones refer to $\mathcal{C}(X, \perp)$ or $\mathcal{C}(Y, \perp)$.

We claim that the subspace $Z = \varphi(X)^{\perp\perp}$ of Y is normal. Let $e_1, \ldots, e_m$ be a maximal orthogonal subset of X and let $f_1 = \varphi(e_1), \ldots, f_m = \varphi(e_m), f_{m+1}, \ldots, f_n$ be a maximal orthogonal subset of Y. As φ is normal, we have by Lemma 3.2 that $Z = \{f_1, \ldots, f_m\}^{\perp\perp}$. From the normality of Y, it furthermore follows that $Z = \{f_{m+1}, \ldots, f_n\}^{\perp}$. Let now $G = \{g_1, \ldots, g_l\}$ be a maximal orthogonal subset of Z. We readily see that then $g_1, \ldots, g_l, f_{m+1}, \ldots, f_n$ is a maximal orthogonal subset of Y. By the normality of Y, $\{g_1\}^{\perp\perp}, \ldots, \{g_l\}^{\perp\perp}$, $\{f_{m+1}\}^{\perp\perp}, \ldots, \{f_n\}^{\perp\perp}$ generate a Boolean subalgebra $\mathcal{B}$ of $\mathcal{C}(Y, \perp)$, and we have $Z = \{f_{m+1}, \ldots, f_n\}^{\perp} = G^{\perp\perp}$.

Furthermore, for any $A \subseteq \{g_1, \ldots, g_l\}$, we have $A^{\perp_Z} = A^{\perp} \cap Z = (G \setminus A)^{\perp\perp}$ and $A^{\perp_Z\perp_Z} = (A^{\perp} \cap Z)^{\perp} \cap Z = (A^{\perp\perp} \vee Z^{\perp}) \cap Z = A^{\perp\perp}$, because $A, Z \in \mathcal{B}$. We conclude that the subalgebra of $\mathcal{C}(Z, \perp)$ generated by $\{g_1\}^{\perp_Z\perp_Z}, \ldots, \{g_l\}^{\perp_Z\perp_Z}$ coincides with the Boolean algebra $\mathcal{B}(g_1, \ldots, g_l) \subseteq \mathcal{C}(Y, \perp)$ and is thus Boolean as well. We have shown that Z is indeed a normal orthogonality space.

Let $\alpha \colon X \to Z,\ x \mapsto \varphi(x)$ and let $\beta \colon Z \to Y$ be the inclusion map. Clearly, α and β are orthogonality-preserving and $\varphi = \beta \circ \alpha$. To see that α is normal, let again $e_1, \ldots, e_m$ be a maximal orthogonal subset of X and let $f_1, \ldots, f_n$ as above. Then $\alpha(X)^{\perp_Z\perp_Z} = \varphi(X)^{\perp_Z\perp_Z} = (\varphi(X)^{\perp} \cap Z)^{\perp} \cap Z = (Z^{\perp} \cap Z)^{\perp} \cap Z = Z$

and $\{\alpha(e_1), \ldots, \alpha(e_m)\}^{\perp_Z \perp_Z} = \{f_1, \ldots, f_m\}^{\perp_Z \perp_Z} = (\{f_1, \ldots, f_m\}^{\perp} \cap Z)^{\perp} \cap Z = (Z^{\perp} \cap Z)^{\perp} \cap Z = Z$, hence α is normal by Lemma 3.2.

To see that β is normal, let again $g_1, \ldots, g_l$ be a maximal orthogonal subset of Z and let $f_1, \ldots, f_n$ be as above. Then $\beta(Z)^{\perp\perp} = Z^{\perp\perp} = Z = \{g_1, \ldots, g_l\}^{\perp\perp} = \{\beta(g_1), \ldots, \beta(g_l)\}^{\perp\perp}$, hence the normality follows from Lemma 3.2. The fact that β is an embedding is obvious. $\square$

The next two propositions deal with equalisers as well as with a certain kind of sum in $\mathcal{NOS}$.

Proposition 3.11. *The category $\mathcal{NOS}$ does not have equalisers.*

Proof. Let us consider the normal orthogonality space $(X, \perp_X)$ from Example 2.15. We define $\varphi\colon X \to X,\ a \mapsto a,\ b \mapsto h,\ c \mapsto g,\ d \mapsto f,\ e \mapsto e,\ f \mapsto d,\ g \mapsto c,\ h \mapsto b$. Then φ is an automorphism of X and hence, by Lemma 3.4, a morphism of $\mathcal{NOS}$.

Let us assume that the pair of arrows $X \overset{\varphi}{\underset{\mathrm{id}_X}{\rightrightarrows}} X$ in $\mathcal{NOS}$ possesses an equaliser $\psi\colon Y \to X$. Since the diagram $Y \xrightarrow{\psi} X \overset{\varphi}{\underset{\mathrm{id}_X}{\rightrightarrows}} X$ commutes, the image of ψ must be contained in $\{a, e\}$. We consider two cases.

Case 1. Assume that ψ is a constant map, that is, $\psi(Y) = \{a\}$ or $\psi(Y) = \{e\}$. We assume that $\psi(Y) = \{a\}$; the other case is similar. Let again $(\mathbf{1}, \neq) = (\{p\}, \varnothing)$ be the normal orthogonality space consisting of a single element and consider the morphism $\widehat{e}\colon \{p\} \to X,\ p \mapsto e$. Then $\varphi \circ \widehat{e} = \mathrm{id}_X \circ \widehat{e}$. But there is no map $k\colon \{p\} \to Y$ such that $\widehat{e} = \psi \circ k$.

Case 2. Assume that $\psi(Y) = \{a, e\}$. Let $y \in Y$ be such that $\psi(y) = a$. Because $a \not\perp e$, we have that $\{y\}$ is a maximal orthogonal subset of Y. Moreover, $\psi(Y)^{\perp_X \perp_X} = \{a, e\}^{\perp_X \perp_X} = \{a, e\} \neq \{a\} = \{a\}^{\perp_X \perp_X} = \{\psi(y)\}^{\perp_X \perp_X}$, in contradiction to the normality of ψ.

We conclude that the pair φ, id_X does not possess an equaliser. $\square$

Let $(X_i, \perp_i)$, $i \in I$, be normal orthogonality spaces whose rank is bounded above by some $n \in \mathbb{N}$. In the category $\mathcal{NOS}$, we call an object $(X, \perp_X)$ together with morphisms $\mathrm{in}_i\colon X_i \to X$, $i \in I$, a *finite ranked sum* if the following holds: For any morphisms $\varphi_i\colon X_i \to Y$, $i \in I$ such that $\varphi_i(X_i)^{\perp_Y \perp_Y} = \varphi_j(X_j)^{\perp_Y \perp_Y}$ for all $i, j \in I$, there is a unique morphism $\varphi\colon X \to Y$ such that $\varphi_i = \varphi \circ \mathrm{in}_i$ for every $i \in I$.

Proposition 3.12. *The category $\mathcal{NOS}$ has finite ranked sums.*

Proof. Let $(X_i, \perp_i)$, $i \in I$ be normal orthogonality spaces whose rank is bounded above by $n \in \mathbb{N}$. We assume that the sets X_i, $i \in I$, are mutually disjoint. Let $X = \bigcup_{i \in I} X_i$ and for $e, f \in X$, let $e \perp f$ if there is an $i \in I$ such that $e, f \in X_i$ and $e \perp_i f$.

Clearly, $(X, \perp)$ is an orthogonality space. We claim that $(X, \perp)$ is normal. Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of X. Then there is an $i \in I$ such that $\{e_1, \ldots, e_n\}$ is a maximal orthogonal subset of X_i. For some $1 \leqslant k < n$, let $f, g \in X$ such that $f \perp e_1, \ldots, e_k$ and $g \perp e_{k+1}, \ldots, e_n$. Then $f, g \in X_i$, and since X_i is normal, we have by Lemma 2.5 that $f \perp_i g$. Thus $f \perp g$ and again by Lemma 2.5, we conclude that $(X, \perp)$ is normal.

For each $i \in I$, let $\mathrm{in}_i \colon X_i \to X$ be the inclusion maps. We claim that in_i is a morphism. By construction, in_i is orthogonality-preserving. Moreover, let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of X_i. Then $\{e_1, \ldots, e_n\}$ is also a maximal orthogonal set of X. Hence $\{\mathrm{in}_i(e_1), \ldots, \mathrm{in}_i(e_n)\}^{\perp\perp} = \{e_1, \ldots, e_n\}^{\perp\perp} = \varnothing^{\perp} = X_i^{\perp\perp} = \mathrm{in}_i(X_i)^{\perp\perp}$ and the normality follows from Lemma 3.2.

Let now $(Y, \perp_Y)$ be a further normal orthogonality space and let $\varphi_i \colon X_i \to Y$, $i \in I$, be morphisms such that $\varphi_i(X_i)^{\perp_Y\perp_Y} = \varphi_j(X_j)^{\perp_Y\perp_Y}$ for all $i, j \in I$. We have to show that there exists a unique morphism $\varphi \colon X \to Y$ such that $\varphi_i = \varphi \circ \mathrm{in}_i$ for every $i \in I$. The only map φ fulfilling the latter requirement is defined as follows: for $x \in X$, we let $\varphi(x) = \varphi_i(x)$ for the unique $i \in I$ such that $x \in X_i$. Clearly, φ is a homomorphism and we have to verify that φ is normal. Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal subset of X. Then $\{e_1, \ldots, e_n\}$ is a maximal orthogonal subset of X_j for some $j \in I$. Applying Lemma 3.2, property (4), to φ_j, we get

$$\begin{aligned}\{\varphi(e_1), \ldots, \varphi(e_n)\}^{\perp_Y\perp_Y} &= \{\varphi_j(e_1), \ldots, \varphi_j(e_n)\}^{\perp_Y\perp_Y} = \varphi_j(X_j)^{\perp_Y\perp_Y} \\ = \bigvee_{i \in I} \varphi_i(X_i)^{\perp_Y\perp_Y} &= \Big(\bigcup_{i \in I} \varphi_i(X_i)^{\perp_Y\perp_Y}\Big)^{\perp_Y\perp_Y} \supseteq \varphi(X).\end{aligned}$$

Hence the normality of φ follows from Lemma 3.2, property (5). $\square$

We finally show that normality is preserved under the formation of direct products. Let $(X_i, \perp_i)$, $i \in I$ be orthogonality spaces. On $\prod_{i \in I} X_i$, we define the orthogonality relation componentwise, that is, we let $(e_i)_{i \in I} \perp (f_i)_{i \in I}$ if $e_i \perp f_i$ for all $i \in I$. Then $(\prod_{i \in I} X_i, \perp)$ is called the direct product of the X_i. The projections $\mathrm{p}_j \colon \prod_{i \in I} X_i \to X_j,\ (e_i)_{i \in I} \mapsto e_j$, where $j \in I$, are evidently orthogonality-preserving.

Theorem 3.13. *The direct product of normal orthogonality spaces is normal.*

However, the direct product, together with the projection mappings, is not a categorical product in $\mathcal{NOS}$.

Proof. Let $(X_i, \perp_i)$, $i \in I$, be normal orthogonality spaces. Note that the rank of $(\prod_{i \in I} X_i, \perp)$ is the minimum of the ranks of $(X_i, \perp_i)$, $i \in I$.

To show that $(\prod_{i \in I} X_i, \perp)$ is normal, let $\{e_1, \ldots, e_n\} \subseteq \prod_{i \in I} X_i$ be a maximal orthogonal set. Let $1 \leqslant k < n$, $f \perp e_1, \ldots, e_k$ and $g \perp e_{k+1}, \ldots, e_n$. This means $\mathrm{p}_i(f) \perp_i \mathrm{p}_i(e_1), \ldots, \mathrm{p}_i(e_k)$ and $\mathrm{p}_i(g) \perp_i \mathrm{p}_i(e_{k+1}), \ldots, \mathrm{p}_i(e_n)$ for all $i \in I$. Hence $\mathrm{p}_i(f) \perp_i \mathrm{p}_i(g)$ for all $i \in I$, that is, $f \perp g$ and the assertion follows.

To verify the second claim, we consider the normal orthogonality spaces $(\mathbf{2}, \perp_{\mathbf{2}}) = (\mathbf{2}, \neq)$ and $(\mathbf{1}, \perp_{\mathbf{1}}) = (\mathbf{1}, \varnothing)$, cf. Example 2.7. Then $\{(1,1)\}$ is a maximal orthogonal subset of $(\mathbf{2} \times \mathbf{1}, \perp)$. As we have $\mathrm{p}_{\mathbf{2}}(\mathbf{2} \times \mathbf{1})^{\perp_{\mathbf{2}}\perp_{\mathbf{2}}} = \mathbf{2}^{\perp_{\mathbf{2}}\perp_{\mathbf{2}}} = \mathbf{2}$ and $\{\mathrm{p}_{\mathbf{2}}((1,1))\}^{\perp_{\mathbf{2}}\perp_{\mathbf{2}}} = \{1\}^{\perp_{\mathbf{2}}\perp_{\mathbf{2}}} = \{1\}$, we observe by Lemma 3.2 that $\mathrm{p}_{\mathbf{2}}$ is not normal. $\square$

4 Hermitian spaces

In the remainder of this paper, we shall study orthogonality spaces arising from inner-product spaces. In this section, we compile the necessary background material.

We first consider linear spaces without any additional structure. We will review the representation of maps between projective spaces that preserve the collinearity of point triples. The most general results in this area are, to our knowledge, due to Faure [Fau]. Here, we will follow the work of Machala [Mach]. The reader is referred to any of these papers for more detailed information.

By an *sfield*, we mean a skew field (i.e., a division ring). Let V be a linear space over an sfield K. We write $V^{\bullet} = V \setminus \{0\}$ and in accordance with Example 2.2, we define $P(V) = \{[x] \colon x \in V^{\bullet}\}$ to be the projective space associated with V. For $x, y, z \in V^{\bullet}$, we write $\ell([x], [y], [z])$ if $[x], [y], [z]$ are on a line of $P(V)$, that is, if x, y, z are linearly dependent.

Let V and V' be linear spaces over the sfields K and K', respectively. We call a map $\varphi \colon P(V) \to P(V')$ a *lineation* if:

(L1) For any $x, y, z \in V^{\bullet}$, $\ell([x], [y], [z])$ implies $\ell(\varphi([x]), \varphi([y]), \varphi([z]))$.

Thus a lineation is a map between projective spaces that preserves the collinearity of point triples. Obviously, (L1) is equivalent to:

(L1') For any $x, y, z \in V^{\bullet}$ such that $\varphi([x]) \neq \varphi([y])$ and $[z] \subseteq [x] + [y]$, we have $\varphi([z]) \subseteq \varphi([x]) + \varphi([y])$.

Thus a lineation can also be understood as follows: if the point $[z]$ lies on the line through $[x]$ and $[y]$ and if $[x]$ and $[y]$ are not mapped to the same point, then $\varphi([z])$ is on the line through $\varphi([x])$ and $\varphi([y])$. It is natural to ask whether a lineation is induced by a suitable map between the underlying linear spaces.

Let K be an sfield. We denote by $K^{\times} = K \setminus \{0\}$ the multiplicative group of K. A *valuation ring* F_K of K is a subring of K such that, for any $\alpha \in K^{\times}$, either $\alpha \in F_K$ or $\alpha^{-1} \in F_K$. In this case, the subgroup $U(F_K)$ of $K^{\times}$ consisting of the units of F_K is called the *group of valuation units*. Obviously, F_K is a local ring, $I_K = F_K \setminus U(F_K) = \{\alpha \in F_K \colon \alpha = 0 \text{ or } \alpha^{-1} \notin F_K\}$ being its unique maximal left (right) ideal. Let K' be a further sfield; then a ring homomorphism $\varrho \colon F_K \to K'$ with kernel I_K is called a *place* from K to K'. Note that in this case, ϱ induces an embedding of the sfield F_K / I_K into K'.

Example 4.1. *Assume that K is an ordered $\star$-sfield (in the sense of Baer). Then the set $F_K = \{\alpha \in K \colon \alpha\alpha^{\star} \leqslant n \text{ for some } n \in \mathbb{N}\}$ of finite elements is a valuation ring. The group of valuation units is $U(F_K) = \{\alpha \in K \colon \frac{1}{n} \leqslant \alpha\alpha^{\star} \leqslant n \text{ for some } n \in \mathbb{N}\}$, containing the so-called medial elements. Moreover, $I_K = \{\alpha \in K \colon \alpha\alpha^{\star} \leqslant \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$ consists of the infinitesimal elements.*

Let now V be a linear space over an sfield K. Let F_K be a valuation ring of K and let F_V be a submodule of V over F_K such that any one-dimensional subspace of V

contains a non-zero element of F_V. Let V' be a further linear space over an sfield K'. Let $\varrho\colon F_K \to K'$ be a place from K to K' and let $A\colon F_V \to V'$ be such that (i) any one-dimensional subspace of V contains a vector in F_V that A does not map to 0, (ii) A is additive, and (iii) for any $x \in F_V$ and $\alpha \in F_K$, we have $A(\alpha x) = \varrho(\alpha)A(x)$. Then A is called a *generalised semilinear map* from V to V'.

Theorem 4.2. *Let $A\colon F_V \to V'$ be a generalised semilinear map between the linear spaces V and V'. Then the prescription*

$$\varphi_A\colon P(V) \to P(V'),\ [x] \mapsto [A(x)], \quad \textit{where } x \in F_V \textit{ and } A(x) \neq 0,$$

defines a lineation.

Sketch of proof; for full details see the proof of [Mach, Satz 5]. Each one-dimensional subspace of V contains by assumption an element $x \in F_V$ such that $A(x) \neq 0$. Moreover, let $y \in [x] \cap F_V$ such that $A(y) \neq 0$. Then either $y = \alpha x$ or $x = \alpha y$ for some $\alpha \in F_K \setminus \{0\}$. In the former case, we have $A(y) = \varrho(\alpha)A(x)$; in the latter case, we have $A(x) = \varrho(\alpha)A(y)$. Here, ϱ is the place associated with A. It follows that $[A(x)] = [A(y)]$. We conclude that we can define φ_A as indicated.

Let now $x, y, z \in F_V$ such that $A(x), A(y), A(z) \neq 0$ and $\ell([x], [y], [z])$. We have to show that $\ell(\varphi_A([x]), \varphi_A([y]), \varphi_A([z]))$. We may assume that $\varphi_A([x])$, $\varphi_A([y])$, and $\varphi_A([z])$ are mutually distinct. Let $\alpha, \beta \in K$ be such that $z = \alpha x + \beta y$. Then $\alpha, \beta \neq 0$. Moreover, either $\alpha^{-1}\beta \in F_K$ or $\beta^{-1}\alpha \in F_K$. In the former case, we set $z' = \alpha^{-1}z = x + \alpha^{-1}\beta\, y$; then $z' \in F_V$ and $A(z') = A(x) + \varrho(\alpha^{-1}\beta)A(y) \neq 0$ because $\varphi_A([x]) = [A(x)]$ and $\varphi_A([y]) = [A(y)]$ are distinct. Hence $\varphi_A([z]) = [A(z')] = [A(x) + \varrho(\alpha^{-1}\beta)A(y)] \subseteq [A(x)] + [A(y)] = \varphi_A([x]) + \varphi_A([y])$ and the assertion follows. In the latter case, we set $z' = \beta^{-1}z$ and proceed similarly. $\square$

Let $A\colon F_V \to V'$ be a generalised semilinear map between the linear spaces V and V'. We will then write $I_V = \{x \in F_V \colon A(x) = 0\}$. Note that, for any $x \in F_V \setminus I_V$, we have $[x] \cap F_V = F_K \cdot x$. Moreover, let $\alpha \in F_K$; then $\alpha x \in I_V$ if and only if $\alpha \in I_K$, or in other words, $\alpha x \in F_V \setminus I_V$ if and only if $\alpha \in U(F_K)$.

For a converse of Theorem 4.2, we need to take into account additional conditions. A lineation $\varphi\colon P(V) \to P(V')$ is called *non-degenerate* if the following conditions hold:

(L2) For any linearly independent vectors $x, y \in V^\bullet$, $\{\varphi([z])\colon z \neq 0,\ z \in [x, y]\}$ contains at least three elements.

(L3) The image of φ is not contained in a two-dimensional subspace of V'.

We arrive at the main theorem of [Mach].

Theorem 4.3. *Every non-degenerate lineation between projective spaces is, in the sense of Theorem 4.2, induced by a generalised semilinear map.*

We shall now consider linear spaces that are equipped with an inner product. We are interested in maps between projective spaces that also preserve an orthogonality relation.

A $\star$-*sfield* is an sfield equipped with an involutorial antiautomorphism *. An *(anisotropic) Hermitian space* is a linear space H over a $\star$-sfield K that is equipped with an anisotropic, symmetric sesquilinear form $(\cdot,\cdot) : H \times H \to K$. For $x, y \in H$, we write $x \perp y$ if $(x, y) = 0$, and for $x, y \in H^\bullet$, $[x] \perp [y]$ means $x \perp y$.

Let H and H' be Hermitian spaces over the $\star$-sfield K and K', respectively. We call $U\colon H \to H'$ a *generalised semiunitary* map if U is a lineation w.r.t. some place ϱ from K to K' and there are a $\lambda \in K$ and a $\lambda' \in K'$ such that

$$(U(x), U(y)) \;=\; \varrho((x, y)\,\lambda)\,\lambda'$$

for any $x, y \in F_H$. The question arises whether orthogonality-preserving lineations are induced by semiunitary maps. Under particular circumstances, we can give an affirmative answer.

Theorem 4.4. *Let H and H' be finite-dimensional Hermitian spaces over the $\star$-fields K and K', respectively. Assume that H possesses an orthogonal basis consisting of vectors of equal length. Let $\varphi\colon P(H) \to P(H')$ be a non-degenerate orthogonality-preserving lineation. Then φ is induced by a generalised semiunitary map.*

Proof. By Theorem 4.3, φ is induced by a generalised semilinear map $U\colon F_H \to H'$ w.r.t. a place $\varrho\colon F_K \to K'$.

We proceed by showing several auxiliary lemmas.

(a) For any $a, b \in F_H \setminus I_H$, $a \perp b$ implies $U(a) \perp U(b)$.

Proof of (a): Assume $a \perp b$. Then $[a] \perp [b]$ and hence $[U(a)] = \varphi([a]) \perp \varphi([b]) = [U(b)]$ as φ is orthogonality-preserving. It follows $U(a) \perp U(b)$.

(b) There is an orthogonal basis $b_1, \ldots, b_n \in F_H \setminus I_H$ of H consisting of vectors of equal length.

Proof of (b): By assumption, H possesses an orthogonal basis $b_1, \ldots, b_n$ consisting of vectors of equal length. In view of condition (i) of the definition of a generalised semilinear map, we may assume that $b_1 \in F_H \setminus I_H$. Let $2 \leqslant i \leqslant n$; we claim that $b_i \in F_H \setminus I_H$ as well. Assume that $b_i \in I_H$. Then $U(b_1+b_i) = U(b_1-b_i) = U(b_1) \neq 0$ but $b_1 + b_i \perp b_1 - b_i$, in contradiction to (a). Assume that $b_i \notin F_H$. Let $\lambda \in K$ be such that $\lambda b_i \in F_H \setminus I_H$. Then $\lambda^{-1} \in F_K$ would imply that $b_i = \lambda^{-1} \cdot \lambda b_i \in F_H$ contrary to the assumption; hence $\lambda \in I_K$. It follows $\lambda b_1, \lambda b_i \in F_H$ and $U(\lambda b_i + \lambda b_1) = U(\lambda b_i - \lambda b_1) = U(\lambda b_i) \neq 0$ but $\lambda b_i + \lambda b_1 \perp \lambda b_i - \lambda b_1$, again a contradiction to (a).

For the rest of the proof, we fix a basis $b_1, \ldots, b_n$ of H as specified in (b).

(c) $U(b_1), \ldots, U(b_n)$ are vectors of equal length.

Proof of (c): Let $2 \leqslant i \leqslant n$. From $b_1 + b_i \perp b_1 - b_i$ it follows by (a) that $U(b_1 + b_i) \perp U(b_1 - b_i)$, that is, $(U(b_1) + U(b_i), U(b_1) - U(b_i)) = 0$. By (a), it follows $(U(b_1), U(b_1)) = (U(b_i), U(b_i))$. The assertion is shown.

(d) F_K and I_K are closed under *. Moreover, for any $\alpha \in F_K$, we have $\varrho(\alpha^\star) = \varrho(\alpha)^\star$.

Proof of (d): Let $\alpha \in I_K \setminus \{0\}$. Assume that $\alpha^\star \notin F_K$. Then $(\alpha^{-1})^\star = (\alpha^\star)^{-1} \in I_K$. Moreover, $\alpha b_1 - b_2$ and $(\alpha^{-1})^\star b_1 + b_2$ are orthogonal vectors in $F_H \setminus I_H$ and hence $-U(b_2) = U(\alpha b_1 - b_2) \perp U((\alpha^{-1})^\star b_1 + b_2) = U(b_2)$, a contradiction. We conclude that $\alpha^\star \in F_K$. Furthermore, $\alpha b_1 - b_2$ and $b_1 + \alpha^\star b_2$ are orthogonal vectors in $F_H \setminus I_H$ and hence $U(b_2) = -U(\alpha b_1 - b_2) \perp U(b_1 + \alpha^\star b_2) = U(b_1) + \varrho(\alpha^\star)U(b_2)$. We conclude that $\varrho(\alpha^\star) = 0$, that is, $\alpha^\star \in I_K$. We have shown that I_K is closed under *.

Let now $\alpha \in K \setminus \{0\}$ be such that $\alpha^\star \notin F_K$. Then $(\alpha^\star)^{-1} \in I_K$ and hence also $\alpha^{-1} \in I_K$. This means $\alpha \notin F_K$. It follows that also F_K is closed under *.

Finally, let $\alpha \in F_K$. We have that $\alpha b_1 - b_2$ and $b_1 + \alpha^\star b_2$ are orthogonal vectors in $F_H \setminus I_H$. It follows that $\varrho(\alpha)U(b_1) - U(b_2) \perp U(b_1) + \varrho(\alpha^\star)U(b_2)$, that is,

$$\varrho(\alpha)\,(U(b_1), U(b_1)) - \varrho(\alpha^\star)^\star\,(U(b_2), U(b_2)) = 0.$$

Thus, by (c), the assertion follows.

(e) Let $\alpha_1, \ldots, \alpha_n \in K$. Then there is an $\alpha \in K \setminus \{0\}$ such that $\alpha^{-1}\alpha_1, \ldots, \alpha^{-1}\alpha_n \in F_K$ and $\alpha^{-1}\alpha_i \notin I_K$ for at least one i.

The proof of (e) can be found in [Rad, Lemma 6].

(f) Let $x = \alpha_1 b_1 + \ldots + \alpha_n b_n \in F_H$. Then $\alpha_1, \ldots, \alpha_n \in F_K$.

Proof of (f): Assume to the contrary that one of the coefficients is not in F_K. By (e), there is an $\alpha \in K$ such that $\alpha^{-1}\alpha_1, \ldots, \alpha^{-1}\alpha_n \in F_K$ and $\alpha^{-1}\alpha_i \notin I_K$ for some i. Then $\alpha \notin F_K$ and hence $\alpha^{-1} \in I_K$. Hence $0 = \varrho(\alpha^{-1})U(x) = U(\alpha^{-1}x) = \varrho(\alpha^{-1}\alpha_1)U(b_1) + \ldots + \varrho(\alpha^{-1}\alpha_n)U(b_n) \neq 0$, because $\varrho(\alpha^{-1}\alpha_i)U(b_i) \neq 0$ and the summed vectors are mutually orthogonal. The assertion follows.

Let now $x = \alpha_1 b_1 + \ldots + \alpha_n b_n$ and $y = \beta_1 b_1 + \ldots + \beta_n b_n$ be elements of F_H. By (f), $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in F_K$. Using (c) and (d), we get

$$\begin{aligned}
(U(x), U(y)) \;&=\; \varrho(\alpha_1)\,(U(b_1), U(b_1))\,\varrho(\beta_1)^\star + \ldots + \varrho(\alpha_n)\,(U(b_n), U(b_n))\,\varrho(\beta_n)^\star \\
&=\; \varrho(\alpha_1\beta_1^\star + \ldots + \alpha_n\beta_n^\star)\,(U(b_1), U(b_1)) \\
&=\; \varrho\left((\alpha_1\,(b_1, b_1)\,\beta_1^\star + \ldots + \alpha_n\,(b_n, b_n)\,\beta_n^\star)\,(b_1, b_1)^{-1}\right)(U(b_1), U(b_1)) \\
&=\; \varrho((x, y)\,(b_1, b_1)^{-1})\,(U(b_1), U(b_1))\,,
\end{aligned}$$

thus the theorem is proved. □

5 Linear orthogonality spaces

The orthogonality spaces to which we turn now are more special than those discussed so far. We will come a good deal closer to our guiding example.

Definition 5.1. An orthogonality space $(X, \perp)$ is called *linear* if, for any two distinct elements $e, f \in X$, there is a third element g such that $\{e, f\}^\perp = \{e, g\}^\perp$ and exactly one of f and g is orthogonal to e.

In other words, for $(X, \perp)$ to be linear means that (i) for distinct, non-orthogonal elements $e, f \in X$ there is a $g \perp e$ such that $\{e, f\}^\perp = \{e, g\}^\perp$ and (ii) for orthogonal elements $e, f \in X$, there is a $g \not\perp e$ such that $\{e, f\}^\perp = \{e, g\}^\perp$. Note that in both cases g is necessarily distinct from e and f.

Example 5.2. *Let H be a Hilbert space and let $(P(H), \perp)$ again be the orthogonality space arising from H according to Example 2.2. Then we readily check that $(P(H), \perp)$ is linear.*

We start with the following observation. We call an orthogonality space $(X, \perp)$ *irredundant* if, for any $e, f \in X$, $\{e\}^\perp = \{f\}^\perp$ implies $e = f$. Moreover, we call $(X, \perp)$ *strongly irredundant* if, for any $e, f \in X$, $\{e\}^\perp \subseteq \{f\}^\perp$ implies $e = f$. Obviously, strong irredundancy implies irredundancy. We may express strong irredundancy also closure-theoretically; cf., e.g., [Ern]. Indeed, $(X, \perp)$ is strongly irredundant exactly if the specialisation order associated with the closure operator ${}^{\perp\perp}$ is the equality.

Lemma 5.3. *Linear orthogonality spaces are strongly irredundant.*

Proof. Let $(X, \perp)$ be a linear orthogonality space.

We first show that X is irredundant. Let e and f be distinct elements of X. If e and f are orthogonal, then $f \perp e$ but $e \not\perp e$. If not, there is by the linearity some $g \perp e$ such that $\{e, f\}^\perp = \{e, g\}^\perp$. Then $g \notin \{e, g\}^\perp = \{e, f\}^\perp$, hence $g \perp e$ but $g \not\perp f$. Hence $\{e\}^\perp \neq \{f\}^\perp$ in either case.

Let now $e, f \in X$ be such that $\{e\}^\perp \subseteq \{f\}^\perp$. We shall show that then actually $\{e\}^\perp = \{f\}^\perp$; by irredundancy, it will follow that X is strongly irredundant. Assume to the contrary that $\{e\}^\perp \subsetneq \{f\}^\perp$. Then $e \neq f$ and $e \not\perp f$. Hence, by the linearity of X, there is a $g \perp e$ such that $\{e, g\}^\perp = \{e, f\}^\perp$. But this means $g \in \{e, g\}^{\perp\perp} = \{e, f\}^{\perp\perp} = \{e\}^{\perp\perp}$, a contradiction. □

The following correspondence between linear orthogonality spaces and linear spaces was shown in [Vet3].

Theorem 5.4. *Let H be a Hermitian space of finite dimension n. Then $(P(H), \perp)$ is a linear orthogonality space of rank n.*

Conversely, let $(X, \perp)$ be a linear orthogonality space of finite rank $n \geqslant 4$. Then there is a $\star$-sfield K and an n-dimensional Hermitian space H over K such that $(X, \perp)$ is isomorphic to $(P(H), \perp)$.

Clearly, the assumption regarding the rank cannot be omitted in Theorem 5.4. For low ranks, linear orthogonality spaces may be of a much different type than those arising from inner-product spaces.

Example 5.5. *For $n \geqslant 2$, let $D_n = \{0_1, 1_1, \ldots, 0_n, 1_n\}$, endowed with the orthogonality relation such that 0_i and 1_i, for each $i = 1, \ldots, n$, are orthogonal and no further pair. We easily see that $(D_n, \perp)$ is linear. Note that $\mathcal{C}(D_n, \perp)$ is isomorphic to MO_n, the horizontal sum of n four-element Boolean algebras, which is a modular ortholattice with $2n + 2$ elements.*

Each linear orthogonality space is a Dacey space and hence normal. The exact relationship is as follows.

Here, an orthogonality space $(X, \perp)$ is called *irreducible* if X cannot be partitioned into two non-empty subsets A and B such that $e \perp f$ for any $a \in A$ and $b \in B$.

Theorem 5.6. *An orthogonality space $(X, \perp)$ is linear if and only if X is an irreducible, strongly irredundant Dacey space. In particular, X is in this case normal.*

Proof. Let $(X, \perp)$ be linear. By [Vet3, Theorem 3.7], $\mathcal{C}(X, \perp)$ is orthomodular, that is, a Dacey space. By Proposition 2.14, X is hence normal. By Lemma 5.3, X is strongly irredundant. Assume now that $X = A \cup B$, where A and B are disjoint non-empty subsets of X and $e \perp f$ for any $e \in A$ and $f \in B$. By linearity, for any $e \in A$ and $f \in B$, there is a $g \not\perp e$ such that $\{e, f\}^\perp = \{e, g\}^\perp$. Then $g \notin B$ and consequently $g \in A$ and thus $g \perp f$. It follows $\{g\}^{\perp\perp} \subseteq \{e, f\}^{\perp\perp} \cap \{f\}^\perp = \{e\}^{\perp\perp}$ by orthomodularity and hence $f \in \{f\}^{\perp\perp} \subseteq \{e, f\}^{\perp\perp} = \{e\}^{\perp\perp} \vee \{g\}^{\perp\perp} = \{e\}^{\perp\perp}$, in contradiction to $f \in \{e\}^\perp$. We conclude that X is irreducible.

Conversely, let $(X, \perp)$ be an irreducible, strongly irredundant Dacey space. By the strong irredundancy, $\{e\}^{\perp\perp}$ is, for any $e \in X$, an atom of $\mathcal{C}(X, \perp)$ and it follows that $(X, \perp)$ is atomistic. Furthermore, $\mathcal{C}(X, \perp)$ is a complete orthomodular lattice of finite length. It follows that $\mathcal{C}(X, \perp)$ is in fact a modular lattice and hence fulfils the covering property and the exchange property. Moreover, $\mathcal{C}(X, \perp)$ is irreducible. Indeed, if the centre of $\mathcal{C}(X, \perp)$ contained an element $\varnothing \subsetneq A \subsetneq X$, then each atom would be below A or below $A^\perp$, that is, we would have $X = A \cup A^\perp$ and X would not be irreducible.

Let $e, f \in X$ be distinct, non-orthogonal elements. Then $\{e\}^{\perp\perp}$ and $\{f\}^{\perp\perp}$ are distinct atoms and hence $\{e, f\}^{\perp\perp} = \{e\}^{\perp\perp} \vee \{f\}^{\perp\perp}$ covers $\{e\}^{\perp\perp}$. By orthomodularity, there is an element $g \perp e$ such that $\{e, f\}^{\perp\perp} = \{e\}^{\perp\perp} \vee \{g\}^{\perp\perp} = \{e, g\}^{\perp\perp}$, that is, $\{e, f\}^\perp = \{e, g\}^\perp$.

Let $e, f \in X$ be distinct, orthogonal elements. Since $\mathcal{C}(X, \perp)$ is irreducible, the join of $\{e\}^{\perp\perp}$ and $\{f\}^{\perp\perp}$ contains a third atom, that is, there is a $g \neq e, f$ such that $g \in \{e, f\}^{\perp\perp}$. By the exchange property, it follows $\{e, f\}^{\perp\perp} = \{e, g\}^{\perp\perp}$. Thus $\{e, f\}^{\perp\perp} = \{e, g\}^{\perp\perp}$, and $g \not\perp e$ because otherwise $g = f$. The proof of the linearity of X is complete. □

Example 5.7. *We observe from Theorem 5.6 that not every Dacey space is linear. The probably simplest counterexample is $(\mathbf{2}, \neq)$, the orthogonality space consisting of two orthogonal elements, cf. Example 2.7. Obviously, $\mathbf{2}$ is Dacey but not linear. More generally, the same applies, for any $n \geqslant 2$, to $(\mathbf{n}, \neq)$.*

In view of Example 5.5, we may add a description of those linear orthogonality spaces that arise as finite ranked sums.

Proposition 5.8. *The finite ranked sum of normal orthogonality spaces $(X_i, \perp_i)$, where $i \in I$ and I is at least two elements, is linear if and only if, for all $i \in I$,* (1) *every maximal orthogonal set of X_i has exactly two elements and* (2) *for distinct, non-orthogonal elements $e, f \in X_i$ there is an element $g \perp_i e$ in X_i such that $\{e, f\}^{\perp_i} = \{e, g\}^{\perp_i}$.*

Proof. Let $(X, \perp)$ be the finite ranked sum of $(X_i, \perp_i)$. To see the "only if" part, assume that $(X, \perp)$ is linear. Let $i \in I$. Pick an $e \in X_i$ and an $f \in X_j$, where $i \neq j$. Then $\{e, f\}^\perp = \varnothing$ and $f \not\perp e$. By linearity, there is an element $g \in X$ such that $\{e, f\}^\perp = \{e, g\}^\perp$ and $e \perp g$. It follows that $g \in X_i$, $e \perp_i g$ and $\{e, g\}^{\perp_i} = \varnothing$. Hence $\{e, g\}$ is a maximal orthogonal subset of X_i and hence of X. By [Vet3, Lemma 3.5], $\mathcal{C}(X, \perp)$ is an atomistic modular ortholattice of finite length. By [MaMa, Theorem (8.4)], any maximal orthogonal set of X and hence also of X_i is two-element. (1) is shown.

Let $e, f \in X_i$ be distinct, non-orthogonal elements. Since $(X, \perp)$ is linear there is an element $g \in X$, $g \perp_X e$ such that $\{e, f\}^\perp = \{e, g\}^\perp$. As $g \in X_i$, we have that $g \perp_i e$ and $\{e, f\}^{\perp_i} = \{e, g\}^{\perp_i}$. Also (2) follows.

For the "if" part, assume (1) and (2). Let $e, f \in X$, $e \neq f$. Assume first that $e \in X_i$ and $f \in X_j$, $i \neq j$. We have $\{e, f\}^\perp = \varnothing$ and $f \not\perp e$. Since there is no maximal one-element orthogonal subset of X_i, we obtain that there is an element $g \in X_i$ such that $\{e, g\}^{\perp_i} = \varnothing$ and $e \perp_i g$. We thus conclude $\{e, g\}^\perp = \varnothing = \{e, f\}^\perp$ and $e \perp g$.

Assume now that $e, f \in X_i$. Assume first that $e \perp f$, that is, $e \perp_i f$. Then $\{e, f\}^{\perp_i} = \{e, f\}^\perp = \varnothing$. For any $j \in I$ distinct from i, pick a $g \in X_j$. Then $\{e, g\}^\perp = \varnothing$ and $g \not\perp e$. Assume second that $f \not\perp e$. By assumption, there is an element $g \in X_i$, $g \perp_i e$ such that $\{e, f\}^{\perp_i} = \{e, g\}^{\perp_i} = \varnothing$. Hence there is a $g \perp e$ such that $\{e, f\}^\perp = \{e, g\}^\perp = \varnothing$. The proof of linearity is complete. □

6 The category $\mathcal{LOS}$ of linear orthogonality spaces

We denote by $\mathcal{LOS}$ the full subcategory of $\mathcal{NOS}$ consisting of linear orthogonality spaces.

We start by describing the monomorphisms.

Proposition 6.1. *Let $\varphi\colon X \to Y$ be a morphism in $\mathcal{LOS}$. Then φ is a monomorphism in $\mathcal{LOS}$ if and only if φ is injective.*

Proof. The proof mimics the proof of Lemma 3.9. We make use of the obvious fact that the one-element orthogonality space $(\mathbf{1}, \neq) = (\{p\}, \varnothing)$ is linear. □

We have furthermore the following analogue of Theorem 3.10.

Theorem 6.2. *Let $\varphi\colon X \to Y$ be a morphism in $\mathcal{LOS}$. Then there are morphisms $\alpha\colon X \to Z$ and $\beta\colon Z \to Y$ in $\mathcal{LOS}$ such that $\varphi = \beta \circ \alpha$, where α is quasi-surjective and β is an embedding.*

Proof. We are following the lines of the proof of Theorem 3.10. It remains to check that the orthogonality space $Z = \varphi(X)^{\perp\perp}$ is linear. Again, we mark the ortholattice complement on $\mathcal{C}(Z, \perp)$ by a subscript Z and the unmarked ones refer to $\mathcal{C}(Y, \perp)$.

Let $e, f \in Z$. Then there is an element $g \in Y$ such that $\{e, f\}^\perp = \{e, g\}^\perp$ and exactly one of f and g is orthogonal to e. Then $g \in \{e, g\}^{\perp\perp} = \{e, f\}^{\perp\perp} \subseteq Z$. Moreover, $\{e, f\}^{\perp_Z} = \{e, f\}^\perp \cap Z = \{e, g\}^\perp \cap Z = \{e, g\}^{\perp_Z}$. Hence Z is linear. □

Proposition 6.3. *The category $\mathcal{LOS}$ does not have equalisers.*

Proof. Let us consider the linear orthogonality space $(D_2, \perp)$ from Example 5.5. We define $\varphi\colon D_2 \to D_2$, $0_1 \mapsto 0_2$, $1_1 \mapsto 1_2$, $0_2 \mapsto 0_1$, $1_2 \mapsto 1_1$. Then φ is an automorphism of D_2 and hence, by Lemma 3.4, a morphism of $\mathcal{NOS}$ and hence also of $\mathcal{LOS}$.

Assume that $\psi\colon X \to D_2$ is an equaliser of the pair of arrows $D_2 \overset{\varphi}{\underset{\mathrm{id}_{D_2}}{\rightrightarrows}} D_2$ in $\mathcal{LOS}$. But the diagram $X \xrightarrow{\psi} D_2 \overset{\varphi}{\underset{\mathrm{id}_{D_2}}{\rightrightarrows}} D_2$ cannot commute. We conclude that the pair φ, id_{D_2} does not possess an equaliser. □

Proposition 6.4. *The category $\mathcal{LOS}$ has neither finite ranked sums nor direct products.*

Proof. Let $(X, \perp)$ be the linear orthogonality space arising from a three-dimensional Hermitian space. Then X is normal. The finite ranked sum of X with itself is, by Proposition 3.12, normal but, by Proposition 5.8, not linear.

Furthermore, the direct product of $(\mathbf{1}, \varnothing)$ and any linear orthogonality space with at least two elements has rank 1. But the only linear orthogonality space of rank 1 is, up to isomorphism, $(\mathbf{1}, \varnothing)$. □

The remainder of the section is devoted to a description of the morphisms in $\mathcal{LOS}$. We restrict our considerations to orthogonality spaces that arise from Hermitian spaces; in view of Theorem 5.4, the results hence apply to all linear orthogonality spaces whose rank is at least 4.

Theorem 6.5. *Let H and H' be finite-dimensional Hermitian spaces. Then a map $\varphi\colon P(H) \to P(H')$ is a morphism in $\mathcal{LOS}$ if and only if φ is an orthogonality-preserving lineation.*

Proof. Let $\varphi\colon P(H) \to P(H')$ be a morphism in $\mathcal{LOS}$. Clearly, φ is orthogonality-preserving. Let $x, y, z \in H^{\bullet}$ be such that $\varphi([x]) \neq \varphi([y])$ and $z \in [x, y]$. Let $y' \perp x$ be such that $[x, y] = [x, y']$. By Lemma 3.3, $\varphi([y]), \varphi([z]) \subseteq \varphi([x]) + \varphi([y'])$. By assumption, $\varphi([x]) \neq \varphi([y])$, so that $\varphi([z]) \subseteq \varphi([x]) + \varphi([y']) = \varphi([x]) + \varphi([y])$. By criterion (1'), φ is a lineation.

Conversely, let φ be an orthogonality-preserving lineation. Then φ is a homomorphism of orthogonality spaces. Let $x_1, \ldots, x_n$ be an orthogonal basis of H. We have to verify that $\varphi([x]) \in \{\varphi([x_1]), \ldots, \varphi([x_n])\}^{\perp\perp}$, that is, $\varphi([x]) \subseteq \varphi([x_1]) + \ldots + \varphi([x_n])$ for any $x \in H$; then it will follow by Lemma 3.2, property (5), that φ is normal and hence a morphism.

Assume that $x \in [x_1, x_2]$. We have that $\varphi([x_1]) \perp \varphi([x_2])$ and hence $\varphi([x_1]) \neq \varphi([x_2])$. As φ is a lineation, it follows $\varphi([x]) \subseteq \varphi([x_1]) + \varphi([x_2])$. The assertion follows thus by an inductive argument. □

A morphism of $\mathcal{LOS}$ being a lineation, the question seems natural whether it is non-degenerate. We consider the conditions (L2) and (L3), which define non-degeneracy, separately.

The latter condition is automatic, provided that we assume dimensions of at least 3.

Lemma 6.6. *Let H and H' be Hermitian spaces of finite dimension $\geqslant 3$. Then any morphism in $\mathcal{LOS}$ is a lineation fulfilling* (L3).

Proof. Let $\varphi\colon P(H) \to P(H')$ be a morphism in $\mathcal{LOS}$. By Theorem 6.5, φ is a lineation. Moreover, H is at least 3-dimensional, so that the image of φ contains three mutually orthogonal elements. It follows that φ fulfils (L3). □

In the next lemma, $(\mathbf{3}, \neq)$ is, in accordance with Example 2.7, the orthogonality space consisting of three mutually orthogonal elements.

Lemma 6.7. *Let H and H' be Hermitian spaces of finite dimension $\geqslant 3$ over the $\star$-sfields K and K', respectively. Let $\varphi\colon P(H) \to P(H')$ be a morphism in $\mathcal{LOS}$ that does not fulfil* (L2). *Then there is a 3-dimensional subspace H_3 of H and a morphism in $\mathcal{NOS}$ from $(P(H_3), \perp)$ to $(\mathbf{3}, \neq)$.*

Proof. For convenience, we will formulate this proof in the language of orthogonality spaces rather than linear spaces.

By assumption, there are $e, f \in P(H)$ such that $e \perp f$ and the image of $\{e, f\}^{\perp\perp}$ under φ contains exactly two elements. Thus $\varphi(\{e, f\}^{\perp\perp}) = \{e', f'\}$, where $e' = \varphi(e)$ and $f' = \varphi(f)$. We choose a $g \in P(H)$ be such that $g \perp e, f$. Then e', f', and $g' = \varphi(g)$ are mutually orthogonal.

Let H_3 be the 3-dimensional subspace of H spanned by e, f, and g. Then we have that $\{e, f, g\}^{\perp\perp} = P(H_3)$, the orthogonality relation being induced by the inner product on H_3. Similarly, let H'_3 be the subspace of H spanned by e', f', and g', so that $\{e', f', g'\}^{\perp\perp} = P(H'_3)$. As φ is normal, we conclude from Lemma 3.2, property (3), that the image of $P(H_3)$ under φ is contained in $P(H'_3)$. In other words, $\varphi|_{P(H_3)}$ is an orthogonality-preserving lineation from $P(H_3)$ to $P(H'_3)$.

$(\{e, f, g\}^{\perp\perp}, \perp)$ is a linear orthogonality space. Furthermore, $\{e', f', g'\}$, together with the orthogonality relation of $P(H'_3)$, is an orthogonality space isomorphic with $(\mathbf{3}, \neq)$. In particular, $(\{e', f', g'\}, \perp)$ is normal. Our aim is to show that there is an orthogonality-preserving map $\psi\colon \{e, f, g\}^{\perp\perp} \to \{e', f', g'\}$. Then it will follow that ψ is a morphism of $\mathcal{NOS}$ and the lemma will be proved. For, such a map is a homomorphism of orthogonality spaces, and since the image of any set of three mutually orthogonal elements in $\{e, f, g\}^{\perp\perp}$ is $\{e', f', g'\}$, the normality holds by Lemma 3.2, property (4).

To begin with, we observe that, for any $x \in \{e, f\}^{\perp\perp}$ such that $\varphi(x) = e'$, we have $\varphi(\{g, x\}^{\perp\perp}) \subseteq \{e', g'\}^{\perp\perp}$, and for any $x \in \{e, f\}^{\perp\perp}$ such that $\varphi(x) = f'$, we have $\varphi(\{g, x\}^{\perp\perp}) \subseteq \{f', g'\}^{\perp\perp}$. Furthermore, for any $y \in \{e, f, g\}^{\perp\perp}$, there is an $x \in \{e, f\}^{\perp\perp}$ such that $y \in \{g, x\}^{\perp\perp}$, and from Lemma 3.3 it follows $\varphi(y) \in \{g', \varphi(x)\}^{\perp\perp}$. We conclude that

$$\varphi(\{e, f, g\}^{\perp\perp}) \subseteq \{e', g'\}^{\perp\perp} \cup \{f', g'\}^{\perp\perp}. \tag{1}$$

We now distinguish three cases.

Case 1. There is an $h \in \{e,g\}^{\perp\perp}$ such that $h' = \varphi(h) \neq e', g'$. Note that $h' \in \{e',g'\}^{\perp\perp}$. We claim that

$$\varphi(\{f,h\}^{\perp\perp}) \;=\; \{f',h'\}. \tag{2}$$

Let $x \in \{f,h\}^{\perp\perp}$. Since $x \neq g$, there is a unique $t \in \{e,f\}^{\perp\perp}$ such that $x \in \{g,t\}^{\perp\perp}$. Depending on whether $\varphi(t) = e'$ or $\varphi(t) = f'$, we have that $\varphi(x) \in \{e',g'\}^{\perp\perp}$ or $\{f',g'\}^{\perp\perp}$. Furthermore, since $\varphi(x) \in \{f',h'\}^{\perp\perp}$, we conclude that either $\varphi(x) = h'$ or $\varphi(x) = f'$. Thus (2) is shown.

We next claim that

$$\varphi(\{e,f,g\}^{\perp\perp}) \subseteq \{e',g'\}^{\perp\perp} \cup \{f'\}. \tag{3}$$

Let $x \in \{e,f,g\}^{\perp\perp}$ such that $\varphi(x) \notin \{e',g'\}^{\perp\perp}$. Note that then $\varphi(x) \in \{f',g'\}^{\perp\perp}$ by (1). Moreover, there is a unique $y \in \{f,h\}^{\perp\perp} \cap \{e,x\}^{\perp\perp}$. Then $x \in \{e,y\}^{\perp\perp}$ and, by (2), either $\varphi(y) = h'$ or $\varphi(y) = f'$. If $\varphi(y) = h'$, then $\varphi(x) \in \{e',h'\}^{\perp\perp} = \{e',g'\}^{\perp\perp}$, in contradiction to our assumption. Hence we have $\varphi(y) = f'$ and $\varphi(x) \in \{e',f'\}^{\perp\perp}$ and we conclude $\varphi(x) = f'$. Thus (3) is shown.

Finally, let $\tau\colon \{e',g'\}^{\perp\perp} \to \{e',g'\}$ be any orthogonality-preserving map. We define

$$\psi\colon \{e,f,g\}^{\perp\perp} \to \{e',f',g'\},\; x \mapsto \begin{cases} \tau(\varphi(x)) & \text{if } \varphi(x) \in \{e',g'\}^{\perp\perp}, \\ f' & \text{if } \varphi(x) = f'. \end{cases}$$

Then ψ is orthogonality-preserving, as desired.

Case 2. There is a $h \in \{e,f\}^{\perp\perp}$ such that $\varphi(h) \neq e', f'$. Then we argue similarly to Case 1.

Case 3. $\varphi(\{e,f,g\}^{\perp\perp}) = \{e',f',g'\}$. Taking $\psi = \varphi$, we again have that ψ is orthogonality-preserving. □

We shall next consider a quite restricted class of orthogonality spaces. Recall an ordered field is called *Euclidean* if any positive element is a square. We denote by $\mathcal{EOS}$ the full subcategory of $\mathcal{LOS}$, and hence of $\mathcal{NOS}$, consisting of orthogonality spaces that arise from (finite-dimensional) positive definite Hermitian spaces over a Euclidean subfield of the reals.

Lemma 6.8. *In a positive definite Hermitian space over a Euclidean subfield of the reals, every one-dimensional subspace possesses a unit vector.*

Proof. Let K be a Euclidean subfield of the reals and let H be a positive definite Hermitian space over K. Then the only automorphism of K is the identity, hence $\star = \mathrm{id}$.

Let $x \in H^{\bullet}$. As H is positive definite, there is an $\alpha \in K$ such that $\alpha^2 = (x,x)^{-1}$. Then $(\alpha x, \alpha x) = \alpha^2\,(x,x) = 1$. □

In the proof of the next lemma, we follow the lines of Piron's proof of Gleason's Theorem [Pir, p. 75–78].

By a *measure* on a finite-dimensional Hermitian space H, we mean a map μ from $\mathcal{C}(H)$ to the real unit interval such that (i) $\mu(A \vee B) = \mu(A) + \mu(B)$ for any orthogonal subspaces A and B of H and (ii) $\mu(H) = 1$.

Theorem 6.9. *A three-dimensional positive definite Hermitian space over a Euclidean subfield of the reals does not possess two-valued measures.*

Proof. Let R be a Euclidean subfield of the reals and let H be a positive definite Hermitian space over R. Let us assume that there is a two-valued measure μ on H, that is, a map $\mu\colon P(H) \to \{0, 1\}$ such that, among any three orthogonal elements $[x], [y], [z] \in P(H)$, exactly one is mapped to 1. Pick $b_3 \in H^\bullet$ such that $\mu([b_3]) = 1$ and let b_1, b_2, b_3 be an orthogonal basis of H. By Lemma 6.8, we can suppose that b_1, b_2, b_3 are unit vectors. We may hence identify H with R^3, endowed with the standard inner product.

We have that $\mu([\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}]) = 1$ and consequently $\mu([\begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}]) = 0$ for any elements $\alpha, \beta \in K$ that are not both 0. The map $\iota\colon R^2 \to P(R^3),\ (\alpha, \beta) \mapsto [\begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}]$ establishes a one-to-one correspondence between R^2 and the set of those elements of $P(R^3)$ that are not orthogonal to b_3. We shall write $\bar{\mu}$ for $\mu \circ \iota$. Let $\bar{0}$ be the origin of R^2; then $\bar{\mu}(\bar{0}) = 1$. We proceed by showing several auxiliary statements.

(a) Let $L \subseteq R^2$ be a line and let $r \in L$ be the element closest to $\bar{0}$. Then $\bar{\mu}(r) \geqslant \bar{\mu}(s)$ for any $s \in L$.

Proof of (a): We may assume that $s \neq r$. Let $[l] \in P(R^3)$ be parallel to L. Then $[l] \perp \iota r$ and $\mu([l]) = 0$. Moreover, let $t \in L$ be such that $\iota t \perp \iota s$. Then ιr and $[l]$ span the same two-dimensional subspace of R^3 as ιs and ιt. Hence $\bar{\mu}(r) = \bar{\mu}(r) + \mu([l]) = \bar{\mu}(s) + \bar{\mu}(t) \geqslant \bar{\mu}(s)$.

We will denote by $\|r\|$ the (Euclidean) distance between $\bar{0}$ and some $r \in R^2$.

(b) For any $r \in R^2$ and $\tau \in K$ such that $0 < \tau \leqslant 1$, we have $\bar{\mu}(r) \leqslant \bar{\mu}(\tau \cdot r)$.

Proof of (b): Let $r^\perp$ arise from rotating r by $\frac{\pi}{2}$. Consider

$$s \;=\; \tau \cdot r + \sqrt{\tau(1 - \tau)} \cdot r^\perp.$$

Then $\sphericalangle\, r\, s\, \bar{0} = \frac{\pi}{2}$, hence $\bar{\mu}(r) \leqslant \bar{\mu}(s)$ by (a). Likewise, we have $\sphericalangle\, \bar{0}\, \tau r\, s = \frac{\pi}{2}$, hence $\bar{\mu}(s) \leqslant \bar{\mu}(\tau r)$ again by (a).

In what follows, $D_\omega\colon R^2 \to R^2$ denotes the rotation by ω.

(c) Let $r \in R^2,\ n \geqslant 1$, and $s = \cos \frac{\pi}{2^n} D_{\frac{\pi}{2^n}} r$. Then $s \in R^2$ and $\bar{\mu}(r) \leqslant \bar{\mu}(s)$.

Proof of (c): From the fact that, for any $x \in \mathbb{R}$, we have $\cos^2 \frac{x}{2} = \frac{1}{2}(1 + \cos x)$, we conclude that $\cos \frac{\pi}{2^n}, \sin \frac{\pi}{2^n} \in R$ for all n. It follows that $s \in R^2$. Moreover, $\sphericalangle\, \bar{0}\, s\, r = \frac{\pi}{2}$. Hence the last assertion follows from (a).

(d) Let $r \in R^2$ such that $\|r\| > 1$. Then $\bar{\mu}(r) \leqslant \bar{\mu}(-\frac{1}{\|r\|^2} r)$.

Proof of (d): Since $\lim_{n \to \infty} \cos^n \frac{\pi}{n} = 1$, we may choose an m large enough such that $\cos^{2^m} \frac{\pi}{2^m} > \frac{1}{\|r\|^2}$. Let $\omega = \frac{\pi}{2^m}$ and define a sequence $r^{(i)},\ i = 0, \ldots, 2^m$, as follows:

$$r^{(0)} = r, \quad r^{(i+1)} = \cos \omega \cdot D_\omega r^{(i)} \text{ for } 0 \leqslant i < 2^m.$$

By (c), $r^{(i)} \in R^2$ for all i, and $\bar{\mu}(r) = \bar{\mu}(r^{(0)}) \leqslant \bar{\mu}(r^{(1)}) \leqslant \ldots \leqslant \bar{\mu}(r^{(2^m)})$. Moreover, $r^{(2^m)} = -\cos^{2^m} \omega \cdot r$ and $-\cos^{2^m} \omega < -\frac{1}{\|r\|^2}$. Hence, by (b), it follows $\bar{\mu}(r^{(2^m)}) \leqslant \bar{\mu}(-\frac{1}{\|r\|^2} r)$.

(e) Let $r \in R^2$ be such that $\|r\| > 1$. Then $\bar{\mu}(r) = 0$.

Proof of (e): Assume that $\bar{\mu}(r) = 1$. By (d), $\bar{\mu}(r) \leqslant \bar{\mu}(-\frac{1}{\|r\|^2} r)$, hence $\bar{\mu}(-\frac{1}{\|r\|^2} r) = 1$. But ιr and $\iota(-\frac{1}{\|r\|^2} r)$ are perpendicular, a contradiction.

(f) There are $r, s, t \in R^2$ such that $\iota r, \iota s, \iota t$ are mutually orthogonal and $\|r\|, \|s\|, \|t\| > 1$.

Proof of (f): Consider $(2, 0)$, $(-\frac{1}{2}, 1)$, and $(-\frac{1}{2}, -\frac{5}{4})$. $\square$

Theorem 6.10. *Let H and H' be positive definite Hermitian spaces of finite dimension $\geqslant 3$ over Euclidean subfield of the reals. Then any morphism in $\mathcal{EOS}$ between $P(H)$ and $P(H')$ is induced by a generalised semiunitary map.*

Proof. Let $\varphi \colon P(H) \to P(H')$ be a morphism in $\mathcal{EOS}$. By Theorem 6.5, φ is an orthogonality-preserving lineation and by Lemma 6.6, φ fulfils (L3).

Assume that φ does not fulfil (L2). By Lemma 6.7, there is a 3-dimensional subspace H_3 of H and an orthogonality-preserving map from $(P(H_3), \perp)$ to $(\mathbf{3}, \neq)$. This means that H_3 possesses a two-valued measure, in contradiction to Theorem 6.9. We conclude that φ does fulfil (L2) and is hence non-degenerate.

By Lemma 6.8, H possesses an orthogonal basis consisting of vectors of equal length. The assertion now follows by Theorem 4.4. $\square$

7 Conclusion

The objective of this paper has been to establish a categorical framework for orthogonality spaces. The latter structures can be identified with undirected graphs and in the context of graph theory, categories have already been studied, e.g., in [Faw]. However, the categories discussed by the graph theorists have turned out to be unsuitable in the present context. Our primary example originates from quantum physics and hence our intention has been to introduce a category whose morphisms, when applied to linear orthogonality spaces, come close to linear mappings. We have therefore introduced normal orthogonality spaces, which are still more general than linear orthogonality spaces. But normality suggests a definition of morphisms such that, when applied in the context of inner-product spaces, not only the orthogonality relation is taken into account but also the linear structure.

We believe that the presented work is a first step into an area that offers numerous issues for further investigations. We have shown that the morphism between specific Hermitian spaces can be represented by generalised semiunitary maps. It has remained open whether a similar statement is possible for a broader class. More generally, whereas generalised semilinear maps have been studied by several authors, there does not seem to exist any detailed account on maps also preserving an inner product, that is, on generalised semiunitary maps. Moreover, we have seen that the existence of two-valued measures plays a role in the discussion. This question as well as Gleason's Theorem have been studied, with some exceptions [Dvu], in the context of classical fields, whereas the present context suggests to take into account further non-classical fields.

To mention finally a further interesting issue, recall that the lattice-theoretic approach has often been criticised for its inability to deal appropriately with common constructions of Hilbert spaces, like direct sums and tensor products. In the framework of orthogonality spaces, the situation is much different and a categorical framework might be particularly useful for these matters.

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