# The coextension of pomonoids and its application to triangular norms

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**Abstract**

Group coextensions of monoids, which generalise Schreier-type extensions of groups, have originally been defined by P. A. Grillet and J. Leech. The present paper deals with pomonoids, that is, monoids that are endowed with a compatible partial order. Following the lines of the unordered case, we define pogroup coextensions of pomonoids. We furthermore generalise the construction to the case that pomonoids instead of pogroups are used as the extending structures.

The intended application lies in fuzzy logic, where triangular norms are those binary operations that are commonly used to interpret the conjunction. We present conditions under which the coextension of a finite totally ordered monoid leads to a triangular norm. Triangular norms of a certain type can therefore be classified on the basis of the presented results.

# 1 Introduction

A partially ordered monoid, or pomonoid for short, is a monoid $(S; \cdot, 1)$ endowed with a compatible order, that is, a partial order $\leqslant$ such that $a \leqslant b$ and $c \leqslant d$ imply $a \cdot c \leqslant b \cdot d$. Pomonoids occur in various contexts. In particular, they have drawn interest in computer science, where, for instance, they are used in the theory of language recognition [RoSa]. Pomonoids furthermore play a major role in non-classical logics and it is in this context that the present work has to be seen. More specifically, the semantics of fuzzy logics is often based on a chain equipped with a binary operation corresponding to the logical conjunction [Háj]. For instance, the well-known fuzzy logic MTL [EsGo] uses the real unit interval $[0, 1]$ as a set of truth values, endowed with a left-continuous binary operation that makes $[0, 1]$ into a commutative, negative totally ordered monoid. The present work is meant to be applicable, at first place, in the context of the theory of these operations, called triangular norms [KMP].

Our topic are coextensions of pomonoids of a particular kind. By a coextension of a pomonoid $S$, we mean a pomonoid $E$ such that there is a surjective, order-determining homomorphism from $E$ to $S$. Our starting point is the unordered case. Methods of constructing coextensions of monoids have been proposed by several authors; see, e.g., [Réd, ClPr, AzSe]. In particular, $\mathcal{H}$-coextensions were studied by P. A. Grillet [Gri1] and J. Leech [Lee]. In this case, the congruence on $E$ is contained in $\mathcal{H}$ and with each congruence class a group can be associated, called the Schützenberger group. $\mathcal{H}$-coextensions can be subsumed under the broader framework of group coextensions. A group coextension of a monoid $S$ can be constructed in a quite transparent way. To this end, we assign groups $G_a$ to each $a \in S$ as well as homomorphisms $\varphi^a_b \colon G_a \to G_b$ to each pair $a, b \in S$ such that $b \leqslant_{\mathcal{R}} a$, and under certain natural assumptions the set $\{(a, x) \colon a \in S,\ x \in G_a\}$ can be made in into a monoid.

It seems reasonable to ask if this idea can be adapted to the ordered case. The monoid under consideration being endowed with a compatible order, the question is if a group coextension can be made into a pomonoid such that the original structure is its homomorphic image. We will in fact see that we can define the coextension of a pomonoid by what we call a preordered system of pogroups. Regretfully, relatively strong assumptions are necessary to ensure the compatibility of the order. In a further step, we will replace the pogroups as extending structures with pomonoids and we are led to a considerably more flexible framework.

The coextension of a negative, commutative pomonoid can give rise to a triangular norm. In this case all involved structures must be totally ordered; we have to require that the extended universe is order-isomorphic to the real unit interval. The idea of improving our understanding of triangular norms by studying coextensions of this kind has been guiding in the present work as well as already in the second author's previous work [Vet2]. The paper [Vet2], however, differs in style from the present one considerably. The procedure relies to some extent on a real-analytic approach and the construction consists partly of data with no clear algebraic interpretation. In this respect the present work represents a progress as we proceed solely within our algebraic framework. We must admit, however, that the approach of [Vet2], although less systematic, is more flexible and the class of covered triangular norms is wider.

We proceed as follows. We begin by fixing the basic terminology on pomonoids in Section 2. In Section 3, we recall the technique of group coextensions of monoids. We show in the subsequent Section 4 that an analogous procedure is possible in the ordered case. Our next step is to relax the assumptions about the extending structures. In Section 5, we describe coextensions of monoids by a preordered system of monoids and we adapt the approach to the ordered case in Section 6. In Section 7, we finally turn to the intended application and we explain in which respect our results provide a progress for the theory of triangular norms. Some concluding remarks are contained in Section 8.

# 2 Partially ordered monoids

Our paper is devoted to the following partially ordered algebras.

**Definition 2.1.** We call the structure $(S; \cdot, \leqslant, 1)$ a *partially ordered monoid*, or *pomonoid* for short, if (i) $(S; \cdot, 1)$ is a monoid, that is, a semigroup with an identity, and (ii) $\leqslant$ is a compatible partial order on $S$, that is, $x \leqslant y$ implies $x \cdot z \leqslant y \cdot z$ and $z \cdot x \leqslant z \cdot y$ for any $x, y, z \in S$.

A pomonoid $S$ is called *commutative* if $x \cdot y = y \cdot x$ for any $x, y \in S$. Moreover, $S$ is called *negative* if $x \leqslant 1$ for any $x \in S$. Finally, if $\leqslant$ is a total order, we refer to $S$ as a *totally ordered monoid*, or *tomonoid* for short.

As usual, we will mostly replace the symbol for the monoidal product by juxtaposition. Moreover, in this paper, we exclusively consider the commutative case and hence we drop this attribute. That is, by a monoid we mean from now on a commutative monoid, and similarly for all structures whose reducts are monoids, like pomonoids, tomonoids, or groups.

By the *trivial* pomonoid, we mean the pomonoid consisting of the identity alone.

A *homomorphism* $\varphi\colon E \to S$ between monoids $(E; \cdot_E, 1_E)$ and $(S; \cdot_S, 1_S)$ is a mapping such that $\varphi(x \cdot_E y) = \varphi(x) \cdot_S \varphi(y)$ for any $x, y \in E$ and $\varphi(1_E) = 1_S$. A homomorphism $\varphi$ from a pomonoid $(E; \cdot_E, \leqslant_E, 1_E)$ to another one $(S; \cdot_S, \leqslant_S, 1_S)$ is a monoid homomorphism such that, for any $x, y \in E$, $x \leqslant_E y$ implies $\varphi(x) \leqslant_S \varphi(y)$. A homomorphism of pomonoids $\varphi$ is moreover called *order-determining* if, for any $x, y \in E$, $\varphi(x) <_S \varphi(y)$ implies $x <_E y$. If, in this case, $\varphi$ is surjective, $S$ is called a *homomorphic image* of $E$.

Note that there is at least one homomorphism between (po)monoids $E$ and $S$: the mapping $\varphi\colon E \to S,\ x \mapsto 1_S$. We will refer to it as the *trivial* homomorphism.

It is the aim of this paper to construct from a given pomonoid $S$ pomonoids $E$ such that $S$ is a homomorphic image of $E$. We use the notion of a coextension in this context [Gri2]. Namely, a monoid $E$ together with a surjective homomorphism $\pi$ from $E$ to a further monoid $S$ is called a *coextension* of $S$. Likewise, a pomonoid $E$ together with an order-determining, surjective homomorphism (of pomonoids) from $E$ to $S$ is called a *coextension* of $S$.

Associated with any monoid $(S; \cdot, 1)$, there is Green's preorder $\leqslant_{\mathcal{H}}$: for $x, y \in S$, we have $x \leqslant_{\mathcal{H}} y$ if $x = y\,z$ for some $z \in S$. Moreover, Green's relation $\mathcal{H}$, where $x \mathrel{\mathcal{H}} y$ if $x \leqslant_{\mathcal{H}} y$ and $y \leqslant_{\mathcal{H}} x$, is a congruence of $S$. Note that $S$ has trivial $\mathcal{H}$-classes, that is, $\mathcal{H}$ is the equality if and only if $\leqslant_{\mathcal{H}}$ is a partial order. For instance, if $S$ is a negative pomonoid, the $\mathcal{H}$-classes of $S$ are trivial and $\leqslant_{\mathcal{H}}$ is a partial order contained in $\leqslant$.

# 3 Group coextension of monoids

We are concerned in this paper with the coextension of pomonoids and we are particularly interested in the case that the extending structures are related to groups. In this section, we begin by reviewing well-known results on the unordered case. Our starting point is the technique of group coextensions of monoids, which is due to P. A. Grillet [Gri1] and J. Leech [Lee].

Let $E$ be a monoid and let $\mathcal{C}$ be a congruence contained in $\mathcal{H}$. Then $E$ is called an $\mathcal{H}$-coextension of the quotient $S = E/\mathcal{C}$. We shall review how $E$ can be constructed from $S$. We follow mainly the lines of [Gri2]; minor differences concern the notation and we also make the straightforward adaptations for the presence of an identity.

With any $\mathcal{C}$-class $C$ of $E$, let us associate the set $T(C)$ of those $t \in E$ such that $C$ is closed under multiplication with $t$, that is, such that $t\,C \subseteq C$. Then the set of all mappings $\lambda_t^C \colon C \to C,\; c \mapsto t\,c$, where $t \in T(C)$, is called the *Schützenberger group* of $C$ and denoted by $\Gamma(C)$. We will write $\lambda_t^C \,.\, c$ instead of $\lambda_t^C(c)$. The name is justified by the fact that $\Gamma(C)$ is indeed a group of permutations of $C$. Moreover, $\Gamma(C)$ is simply transitive, that is, for each $c, d \in C$ there is exactly one $\lambda_t^C \in \Gamma(C)$ such that $c = \lambda_t^C \,.\, d$.

Let furthermore $A, B$ be two $\mathcal{C}$-classes of $E$ such that $B \leqslant_{\mathcal{H}} A$. Then there is a unique homomorphism $\varphi_B^A \colon \Gamma(A) \to \Gamma(B)$ such that $(\lambda^A \,.\, c)\, u = \varphi_B^A(\lambda^A) \,.\, (c\, u)$ for any $\lambda^A \in \Gamma(A)$, $c \in A$, and $u \in E$ such that $c\,u \in B$. Indeed, $T(A) \subseteq T(B)$ and $\varphi_B^A(\lambda_t^A) = \lambda_t^B$, where $t \in T(A)$. We have that $\varphi_A^A = \mathrm{id}_A$ and $\varphi_C^B \circ \varphi_B^A = \varphi_C^A$ for any $\mathcal{C}$-classes $A$, $B$, and $C$.

**Definition 3.1.** Let $(S; \preccurlyeq)$ be a preordered set. For any $a \in S$, let $(G_a; +, 0)$ be a group, and for every pair $a, b \in S$ such that $a \succcurlyeq b$, let $\varphi_b^a \colon G_a \to G_b$ be a group homomorphism. Assume that $\varphi_a^a = \mathrm{id}_{G_a}$ for any $a \in S$, and $\varphi_c^b \circ \varphi_b^a = \varphi_c^a$ for any $a, b, c \in S$ such that $a \succcurlyeq b \succcurlyeq c$. Putting $G = \{(a, G_a) \colon a \in S\}$ and $\varphi = \{((a, b), \varphi_b^a) \colon\; a, b \in S \text{ such that } a \succcurlyeq b\}$, we call the pair $(G, \varphi)$ a *preordered system of groups over* $(S; \preccurlyeq)$.

The Schützenberger groups together with the homomorphisms between them are the motivating example of Definition 3.1. It turns out that $E$ can be reconstructed from $S = E/\mathcal{C}$ and the preordered system of Schützenberger groups in a straightforward way.

Such coextensions belong to a particular type of coextensions of monoids, which we define next. As regards our notation, we will follow the convention to write groups as well as those structures that play in the sequel the role of "extending" structures additively, whereas the multiplicative notation is used for the "coextended" structures.

For a group $(G; +, 0)$, we say that a set $E$ is a *G-act* if there is a map $. \colon G \times E \to E$ such that $0 \,.\, e = e$ and $x \,.\, (y \,.\, e) = (x + y) \,.\, e$ for any $e \in E$. The map $.$ is then called the *(group) action* of $G$ on $E$. We say that the $G$-act is *simply transitive* if, for any $r, s \in E$, there is a unique $x \in G$ such that $x \,.\, r = s$. A systematic discussion of acts (based on the more general definition that we use in Section 5 below) can be found in [KKM].

**Definition 3.2.** Let $(S; \cdot, 1)$ be a monoid and let $(E, \pi)$ be a coextension of $S$. Let $(G, \varphi)$ be a preordered system of groups over $(S; \leqslant_{\mathcal{H}})$. We call $(E, \pi)$ a *group coextension* of $S$ based on $(G, \varphi)$ if $E_a = \{r \in E \colon \pi(r) = a\}$, for each $a \in S$, is a simply transitive $G_a$-act such that, for any $r \in E_a,\ s \in E_b$, and $x \in G_a$, we have

$$(x \,.\, r)\, s = \varphi^a_{ab}(x) \,.\, (r\, s). \tag{1}$$

As above, let $\mathcal{C}$ be a congruence on a monoid $E$ contained in $\mathcal{H}$, and $S = E/\mathcal{C}$. In view of our explanations, we may then observe that $E$ is a group coextension of $S$, based on the preordered system of groups given by the Schützenberger groups and their homomorphisms.

Answering our question how $E$ arises from $S$, the following theorem characterises group coextension in a constructive way. For later reference, we outline the proof.

**Theorem 3.3.** *Let $(S; \cdot, 1)$ be a monoid. Let $(G, \varphi)$ be a preordered system of groups over $(S; \leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in G_{ab}$ be such that the following conditions hold:*

(C0) $\sigma_{1,1} \;=\; 0$;

(C1) $\sigma_{a,b} \;=\; \sigma_{b,a}$ *for any* $a, b \in S$;

(C2) $\varphi^{ab}_{abc}(\sigma_{a,b}) + \sigma_{ab,c} \;=\; \varphi^{bc}_{abc}(\sigma_{b,c}) + \sigma_{a,bc}$ *for any* $a, b, c \in S$.

*Let then*

$$E \;=\; \{(a, x) \colon a \in S,\ x \in G_a\} \tag{2}$$

*and define the product*

$$(a, x)\,(b, y) \;=\; (a\,b,\ \varphi^a_{ab}(x) + \varphi^b_{ab}(y) + \sigma_{a,b}), \tag{3}$$

*where $(a, x), (b, y) \in E$. Then $(E; \cdot, (1, 0))$, together with the surjective homomorphism $\pi \colon E \to S,\ (a, x) \mapsto a$ is a group coextension of $S$ based on $(G, \varphi)$, where, for each $a \in S$, the action of $G_a$ on $E_a$ is given by $y \,.\, (a,\ x) = (a,\ x + y)$.*

*Moreover, up to isomorphism, any group coextension of $S$ arises in this way.*

*Sketch of proof.* Let $(E; \cdot, (1, 0))$ and $\pi$ be as indicated. It is straightforward to verify that the product is by (C2) associative and by (C1) commutative. For any $c \in S$, we have $\sigma_{c,1} = \sigma_{1,c} = \varphi^1_c(\sigma_{1,1}) = \varphi^1_c(0) = 0$ by (C1), (C2), and (C0), hence the element $(1, 0)$ is an identity. Clearly, $\pi$ is a surjective homomorphism and we conclude that $(E, \pi)$ is a coextension of $S$.

Let the group actions be as indicated. Then, for any $(a, x), (a, y) \in E_a$, $z = y - x$ is the unique element of $G_a$ such that $z \,.\, (a, x) = (a, y)$. Thus $E_a$ is a simply transitive $G_a$-act. To verify (1), let $(a, y) \in E_a$, $(b, z) \in E_b$, and $x \in G_a$; we have $(x \,.\, (a, y))\,(b, z) = (a, x + y)\,(b, z) = (a\,b,\ \varphi^a_{ab}(x) + \varphi^a_{ab}(y) + \varphi^b_{ab}(z) + \sigma_{a,b}) = \varphi^a_{ab}(x) \,.\, (a\,b,\ \varphi^a_{ab}(y) + \varphi^b_{ab}(z) + \sigma_{a,b}) = \varphi^a_{ab}(x) \,.\, ((a, y)\,(b, z))$. We have shown that $E$ is a group coextension of $S$ based on $(G, \varphi)$.

Conversely, let $(E, \pi)$ be a group coextension of $S$ based on $(G, \varphi)$. For each $a \in S$, we write $E_a = \{r \in E \colon \pi(r) = a\}$. Let $p_1 = 1 \in E_1$ and for each $a \in S \setminus \{1\}$ choose an arbitrary $p_a \in E_a$. Furthermore, for any $a, b \in S$, let $\sigma_{a,b}$ be the unique element of $G_{ab}$ such that $p_a\, p_b = \sigma_{a,b} \,.\, p_{ab}$.

By assumption, there is for any $r \in E_a$ a unique $x \in G_a$ such that $r = x \,.\, p_a$. Hence we may identify $E$ with $E' = \{(a, x)\colon a \in S,\ x \in G_a\}$ under the correspondence $(a, x) \mapsto x \,.\, p_a$. Under this identification, the mapping $\pi$ is $E' \to S,\ (a, x) \mapsto a$. Moreover, for $a \in S$ and $x, y \in G_a$ we have $y \,.\, (x \,.\, p_a) = (x + y) \,.\, p_a$, hence the action is then $y \,.\, (a, x) = (a, x + y)$.

As $p_1 = 1$, the element $(1, 0)$ of $E'$ corresponds to the identity $1$ of $E$. Furthermore, for any $a, b \in S$, $x \in G_a$, and $y \in G_b$, we calculate $(x \,.\, p_a)\,(y \,.\, p_b) = \varphi^a_{ab}(x) \,.\, (p_a \cdot y \,.\, p_b) = \varphi^a_{ab}(x) \,.\, \varphi^b_{ab}(y) \,.\, (p_a\, p_b) = \varphi^a_{ab}(x) \,.\, \varphi^b_{ab}(y) \,.\, \sigma_{a,b} \,.\, p_{ab} = (\varphi^a_{ab}(x) + \varphi^b_{ab}(y) + \sigma_{a,b}) \,.\, p_{ab}$ by (1), and we conclude that the multiplication in $E'$ is done according to (3).

Finally, conditions (C0) and (C1) are immediate. (C2) is a consequence of the fact that the product in $E'$ of three elements of the form $(a, 0)$, $(b, 0)$, $(c, 0)$ is associative. $\square$

**Remark 3.4.** With respect to the notation of Theorem 3.3, note that $\sigma_{1,a} = \sigma_{a,1}$ is, for any $a \in S$, always the element $0$ of $G_a$. This fact is shown at the beginning of the above proof.

We conclude the section by noting that group coextensions are actually more general than $\mathcal{H}$-coextensions; see [Gri2, Prop. 4.8]. Indeed, with respect to the notation of Theorem 3.3, assume that there is an $a \in S \setminus \{1\}$ such that $a\, c = a$ implies $c = 1$, and let moreover $G_1 = \{0\}$. Then $(a, x) \leqslant_{\mathcal{H}} (a, y)$ means that there is a $(c, z) \in E$ such that $(a, x) = (a, y)\,(c, z) = (a\, c, \varphi^a_{ac}(y) + \varphi^c_{ac}(z) + \sigma_{a,c}) = (a, y + \varphi^1_a(0) + \sigma_{1,a}) = (a, y)$, where we have made use of Remark 3.4. Thus $x = y$ and, provided that $G_a$ is non-trivial, $E_a$ is not contained in an $\mathcal{H}$-class.

# 4 Pogroup coextension of pomonoids

The aim of this section is to adapt the idea of group coextensions of monoids to the case that a compatible order is present. We will show an analogous version of Theorem 3.3 for the ordered case. The extending structures are this time partially ordered groups.

A *partially ordered group*, or a *pogroup* for short, is an (Abelian) group together with a compatible total order. We will write pogroups again additively. For a pogroup $(G; +, \leqslant, 0)$, the *negative cone* of $G$ is the set $G^- = \{x \in G\colon x \leqslant 0\}$. A homomorphism between pogroups is a homomorphism of the monoid reducts that also preserves the order.

We define a *preordered system of pogroups* $(G, \varphi)$ over a preordered set $(S; \preccurlyeq)$ as in Definition 3.1, but replacing groups and their homomorphisms by pogroups and their homomorphisms. For the sake of clarity, we will denote, for $a \in S$, the partial order in $G_a$ by $\leqslant_a$. Moreover, for a pogroup $(G; +, \leqslant, 0)$, we say that a poset $E$ is a *$G$-poset* if there is a map $.\colon G \times E \to E$ making $E$ into a $G$-act such that, for any $x, y \in G$, (i) the map $E \to E,\ e \mapsto x \,.\, e$ is order-preserving and (ii) $x \leqslant y$ implies $x \,.\, e \leqslant y \,.\, e$ for all $e \in E$. For a discussion of $G$-posets (again from a more general perspective), see, e.g., [Fakh, BuMa].

**Definition 4.1.** Let $(S; \cdot, \leqslant, 1)$ be a pomonoid and let $(E, \pi)$ be a coextension of $S$. Let $(G, \varphi)$ be a preordered system of pogroups over $(S; \leqslant_{\mathcal{H}})$. We call $(E, \pi)$ a *pogroup coextension* of $S$ based on $(G, \varphi)$ if $E_a = \{r \in E \colon \pi(r) = a\}$, for each $a \in S$, is a simply transitive $G_a$-poset such that (i) for any $r \in E_a,\ s \in E_b$, and $x \in G_a$, (1) holds and (ii) for any $r, s \in E_a$, we have $r \leqslant s$ if and only if there is a $x \in G_a^-$ such that $r = x \,.\, s$.

**Theorem 4.2.** *Let $(S; \cdot, \leqslant, 1)$ be a pomonoid. Let $(G, \varphi)$ be a preordered system of pogroups over $(S; \leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in G_{ab}$ be such that* (C0)–(C2) *and the following condition hold:*

(C3) *if, for $a, b, c \in S$, $a < b$ and $a\,c = b\,c$, then for any $x \in G_a$ and $y \in G_b$, $\varphi^a_{ac}(x) + \sigma_{a,c} \leqslant_{ac} \varphi^b_{bc}(y) + \sigma_{b,c}$.*

*On*

$$E \;=\; \{(a, x) \colon a \in S,\ x \in G_a\},$$

*we define a product by* (3) *and a partial order $\leqslant_E$ as follows:*

$$(a, x) \leqslant_E (b, y) \;\text{ if } a < b \text{, or } a = b \text{ and } x \leqslant_a y, \tag{4}$$

*where $(a, x), (b, y) \in E$. Then $(E; \cdot, \leqslant_E, (1, 0))$, together with the order-determining, surjective homomorphism $\pi \colon E \to S,\ (a, x) \mapsto a$, is a pogroup coextension of $S$ based on $(G, \varphi)$, where, for each $a \in S$, the action of $G_a$ on $E_a$ is given by $y \,.\, (a, x) = (a, x + y)$.*

*Moreover, up to isomorphism, any pogroup coextension of $S$ based on $(G, \varphi)$ arises in this way.*

*Proof.* Let $(E; \cdot, \leqslant_E, (1, 0))$ and $\pi$ be as indicated. Disregarding the partial order, we have by Theorem 3.3 that $E$, together with $\pi \colon E \to S,\ (a, x) \mapsto a$, is a group coextension of the monoid $S$ based on the preordered system of group reducts $(G, \varphi)$, where the actions are as indicated.

Clearly, $\leqslant_E$, which is defined by (4), is a partial order. In order to verify that $\leqslant_E$ is compatible, let $(a, x), (b, y), (c, z) \in E$ such that $(a, x) \leqslant_E (b, y)$; our aim is to show $(a, x)\,(c, z) \leqslant_E (b, y)\,(c, z)$. We distinguish by cases:

*Case 1.* $a < b$ and $a\,c < b\,c$. Then the claim is immediate.

*Case 2.* $a < b$ and $a\,c = b\,c$. Then, by (C3), $(a, x)\,(c, z) = (a\,c,\ \varphi^a_{ac}(x) + \varphi^c_{ac}(z) + \sigma_{a,c}) \leqslant_E (b\,c,\ \varphi^b_{bc}(y) + \varphi^c_{ac}(z) + \sigma_{b,c}) = (b, y)\,(c, z)$.

*Case 3.* $a = b$ and $x \leqslant_a y$. By the monotonicity of $\varphi^a_{ac}$, we have $(a, x)\,(c, z) = (a\,c, \varphi^a_{ac}(x) + \varphi^c_{ac}(z) + \sigma_{a,c}) \leqslant_{ac} (a\,c, \varphi^a_{ac}(y) + \varphi^c_{ac}(z) + \sigma_{a,c}) = (a, y)\,(c, z)$.

We conclude that $(E; \cdot, \leqslant_E, (1, 0))$ is a pomonoid. It is furthermore immediate from (4) that the surjective homomorphism of monoids $\pi$ is order-preserving and order-determining.

Finally, we already know that the actions $y \,.\, (a,x) = (a, x+y)$ make each $E_a = \{(a,x) \colon x \in G_a\}$, $a \in S$, into a $G_a$-act fulfilling (1). By (4), we have that, for any $a \in S$ and $x, y \in G_a$, $(a,x) \leqslant_E (a,y)$ is equivalent to $x \leqslant_a y$. It follows that $E_a$ is actually a $G_a$-poset. We moreover conclude that $(a,x) \leqslant_E (a,y)$ if and only if $x = y + z$ for some $z \in G_a^-$ if and only if $(a,x) = z \,.\, (a,y)$ for some $z \in G_a^-$. The proof is complete that $(E, \pi)$ is a pogroup coextension of $S$ based on $(G, \varphi)$.

Conversely, let $(E; \cdot, \leqslant_E, 1)$, together with $\pi$, be a pogroup coextension of $S$ based on $(G, \varphi)$. Then $(E; \cdot, 1)$ is a group coextension of the monoid $S$ by the preordered system of group reducts $(G, \varphi)$. Hence, by Theorem 3.3, we can identify $E$ with the set $\{(a,x) \colon a \in S,\ x \in G_a\}$ and $\pi$ with the projection to the first component. The group actions are then as indicated. Furthermore, there are $\sigma_{a,b} \in G_{ab}$, $a, b \in S$, such that conditions (C0)–(C2) are fulfilled, the multiplication is given by (3), and the identity is $(1, 0)$.

We claim that the partial order $\leqslant_E$ on $E$ is the lexicographic order given by (4). Let $(a,x), (b,y) \in E$. Let $(a,x) \leqslant_E (b,y)$. By assumption, $\pi$ is monotone, hence we have $a \leqslant b$. If $a = b$, there is by Definition 4.1 a $z \in G_a^-$ such that $z \,.\, (b,y) = (a, y+z) = (a,x)$, that is, $x = y + z$ and thus $x \leqslant_a y$. Conversely, assume that either $a < b$, or $a = b$ and $x \leqslant_a y$. Since $\pi$ is order-determining, it follows from $a < b$ that $(a,x) \leqslant_E (b,y)$. Furthermore, if $a = b$ and $x \leqslant_a y$, then $(a,x) = z \,.\, (a,y) = z \,.\, (b,y)$ for some $z \in G_a^-$ and hence we have, again by Definition 4.1, that $(a,x) \leqslant_E (b,y)$.

It remains to verify (C3). Let $a, b, c \in S$ such that $a < b$ and $a\,c = b\,c$, and let $x \in G_a$ and $y \in G_b$. By (4), we have $(a,x) \leqslant_E (b,y)$ and hence $(a,x)\,(c,0) \leqslant_E (b,y)\,(c,0)$. Applying (3) and again (4), the assertion follows. $\square$

We are particularly interested in the case that the order is total as well as in the case that the pomonoid is negative.

**Corollary 4.3.** *Let the pomonoid $E$ be a coextension of the pomonoid $S$ by a preordered system of pogroups $(G, \varphi)$ over $(S; \leqslant_{\mathcal{H}})$. Then we have:*

(i) *$E$ is totally ordered if and only if $S$ is totally ordered and, for any $a \in S$, $G_a$ is totally ordered.*

(ii) *$E$ is negative if and only if $S$ is negative and $G_1 = \{0\}$, i.e., $G_1$ is the trivial pogroup.*

*Proof.* By Theorem 4.2, we may identify $E$ with $\{(a,x) \colon a \in S,\ x \in G_a\}$.

(i) This is immediate from the fact that $E$ is, according to (4), ordered lexicographically.

(ii) By (4), $(a,x) \leqslant_E (1,0)$ holds for all $(a,x) \in E$ if and only if $a \leqslant 1$ for all $a \in S$ and $x \leqslant 0$ for all $x \in G_1$, that is, if and only if $S$ is negative and $G_1 = \{0\}$. $\square$

We next consider further special cases of Theorem 4.2.

First of all, the cumbersome condition (C3) can be simplified if the pogroups are Archimedean. We call a pogroup $(G;+,0)$ *Archimedean* if the following condition holds: for any $x, y \in G$, if $k\,x \leqslant y$ for all $k \in \mathbb{Z}$, then $x = 0$.

**Corollary 4.4.** *Let $(S;\cdot,\leqslant,1)$ be a pomonoid. Let $(G,\varphi)$ be a preordered system of Archimedean pogroups over $(S;\leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in G_{ab}$ be such that* (C0)–(C2) *as well as the following condition hold:*

(C3') *if, for $a, b, c \in S$, $a < b$ and $a\,c = b\,c$, then the homomorphisms $\varphi^a_{ac}$ and $\varphi^b_{bc}$ are trivial and $\sigma_{a,c} \leqslant_{ac} \sigma_{b,c}$.*

*Then $E$ as specified in Theorem 4.2 is a pogroup coextension of $S$ based on $(G,\varphi)$.*

*Moreover, up to isomorphism, any pogroup coextension of $S$ based on $(G,\varphi)$ arises in this way.*

*Proof.* By Theorem 4.2, we have to show that, under the condition of Archimedeanicity, (C3) is equivalent to (C3'). Clearly, (C3') implies (C3).

For the converse direction, assume that (C3) holds. Let $a, b, c \in S$ such that $a < b$ and $a\,c = b\,c$ and let $x \in G_a$. Then $\varphi^a_{ac}(k\,x) + \sigma_{a,c} \leqslant_{ac} \varphi^b_{bc}(0) + \sigma_{b,c} = \sigma_{b,c}$ and hence $k\,\varphi^a_{ac}(x) \leqslant_{ac} -\sigma_{a,c} + \sigma_{b,c}$ for any $k \in \mathbb{Z}$. By the Archimedean property, we conclude that $\varphi^a_{ac}(x) = 0$. Similarly, we see that $\varphi^b_{bc}(y) = 0$ for any $y \in G_b$. $\square$

With reference to Corollary 4.4, we note that (C3') is in particular fulfilled if, for $a, b, c \in S$ such that $a < b$ and $a\,c = b\,c$, we have that $G_a = \{0\}$ or $G_{ac} = G_{bc} = \{0\}$.

We next turn to those cases in which condition (C3) can be dropped.

**Definition 4.5.** Let $(S;\cdot,\leqslant,1)$ be a pomonoid. We call $S$ *cancellative* if, for any $a, b, c \in S$, $a \cdot c = b \cdot c$ implies $a = b$. Assume, furthermore, that $S$ possesses a smallest element $0$. Then we call $S$ *weakly cancellative* if, for any $a, b, c \in S$, $a \cdot c = b \cdot c \neq 0$ implies $a = b$.

**Corollary 4.6.** *Let $(S;\cdot,\leqslant,1)$ be a cancellative pomonoid. Let $(G,\varphi)$ be a preordered system of pogroups over $(S;\leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in G_{ab}$ be such that* (C0)–(C2) *hold. Then $E$ as specified in Theorem 4.2 is a pogroup coextension of $S$ based on $(G,\varphi)$.*

*Moreover, up to isomorphism, any pogroup coextension of $S$ based on $(G,\varphi)$ arises in this way.*

*Proof.* By cancellativity, the assumption of condition (C3) is never fulfilled. Hence the statement follows from Theorem 4.2. $\square$

Also in the weakly cancellative case, (C3) can be replaced by an easier condition.

**Corollary 4.7.** *Let $(S;\cdot,\leqslant,1)$ be a weakly cancellative pomonoid with the smallest element $0$. Let $(G,\varphi)$ be a preordered system of pogroups over $(S;\leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in G_{ab}$ be such that* (C0)–(C2) *hold and $G_0 = \{0\}$. Then $E$ as specified in Theorem 4.2 is a pogroup coextension of $S$ based on $(G,\varphi)$.*

*Proof.* The assumption of condition (C3) is fulfilled only if $a\,c = b\,c = 0$. In this case the conclusion of (C3) holds trivially because, by assumption, $G_0 = \{0\}$. Hence the statement follows again from Theorem 4.2. $\square$

We conclude that the group coextension of monoids can be adapted to the ordered case without serious problems. As to be expected, however, we may also observe that the addition of a compatible order decreases the flexibility. The limitation is evident from condition (C3) in Theorem 4.2. In the Archimedean case, condition (C3') in Corollary 4.4 applies: if, in the pomonoid to be extended, $a < b$ and $d = ac = bc$, then the homomorphisms $\varphi^a_d$ and $\varphi^b_d$ must be trivial.

Nonetheless, our procedure offers various possibilities of constructing new pomonoids from given ones. Let us consider a first, particularly simple, example.

**Example 4.8.** Let $L$ be the four-element Łukasiewicz chain, that is, the chain $L = \{-3, -2, -1, 0\}$ endowed with the monoidal operation $\oplus$ given by

$$a \oplus b = (a + b) \vee -3,$$

where $+$ is the usual addition of integers. Note that $\leqslant_{\mathcal{H}}$ coincides with the linear order $\leqslant$. Let us define the preordered system $(G, \varphi)$ over $(L, \leqslant_{\mathcal{H}})$ as follows. We put $G_0 = G_{-3} = \{0\}$ and $G_{-1} = G_{-2} = \mathbb{R}$. Here and in the subsequent examples, $\{0\}$ denotes the trivial pomonoid and $\mathbb{R}$ denotes the additive group of reals together with its natural order. Furthermore, let $\varphi^{-1}_{-2} \colon \mathbb{R} \to \mathbb{R},\ x \mapsto 2x$ and note that all remaining homomorphisms $\varphi^a_b$, where $a >_{\mathcal{H}} b$, are trivial.

We have to determine $\sigma_{a,b}$, where $a, b \in L$, in accordance with (C0)–(C2). By Remark 3.4, $\sigma_{0,a} = \sigma_{a,0} = 0$ for all $a \in L$. Moreover, if $a \oplus b = -3$ we have that $\sigma_{a,b} \in G_{-3}$ and hence $\sigma_{a,b} = 0$ as well. It remains to determine $\sigma_{-1,-1}$. Clearly, (C0) and (C1) do not impose any restriction on $\sigma_{-1,-1}$, and a further check shows that this is not the case for (C2) either. Let $\sigma_{-1,-1} = \sigma$ be any real.

The pogroup extension of $L$ obtained in this way is a tomonoid based on the set

$$E \;=\; \{(-3, 0), (0, 0)\} \,\cup\, \{-2, -1\} \times \mathbb{R}.$$

The identity is $(0, 0)$, the order is the lexicographical one, and the monoidal operation is given as follows:

$$\begin{aligned}
(0,0)\,(a,x) \;&=\; (a,x)\,(0,0) \;=\; (a,x),\\
(-1,x)\,(-1,y) \;&=\; (-2,\; 2x+2y+\sigma),\\
(a,x)\,(b,y) \;&=\; (-3,0) \quad \text{if } a \oplus b = -3,
\end{aligned}$$

where $a, b \in L$ and $x, y \in \mathbb{R}$.

A *triangular norm*, or *t-norm* for short, is a binary operation $\odot$ on the real unit interval $[0, 1]$ such that $([0, 1]; \odot, \leqslant, 1)$ is a negative (commutative) tomonoid [KMP]. Note that Example 4.8 is a negative tomonoid order-isomorphic to the subset $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ of the real unit interval $[0, 1]$. In the next example, the base set is order-isomorphic to $[0, 1]$ itself and hence we are led to a t-norm.

**Example 4.9.** On the five-element chain $C = \{-4, -3, -2, -1, 0\}$, we define the following product:

$$a \cdot b = \begin{cases} a & \text{if } b = 0, \\ b & \text{if } a = 0, \\ -3 & \text{if } a = b = -1, \\ -4 & \text{otherwise.} \end{cases}$$

Then $(C; \cdot, \leqslant, 0)$ is a negative tomonoid, which we are going to coextend. Let $G_0 = G_{-2} = G_{-4} = \{0\}$ and let $G_{-1} = G_{-3} = \mathbb{R}$. Let $\varphi_{-3}^{-1} \colon \mathbb{R} \to \mathbb{R},\ x \mapsto x$ and otherwise let $\varphi_b^a$, where $a >_{\mathcal{H}} b$, be the trivial homomorphism from $G_a$ to $G_b$. Furthermore, let $\sigma_{a,b} = 0$ for any $a, b \in S$.

The resulting coextension is based on a set order-isomorphic to the real unit interval $[0, 1]$ and after re-scaling we obtain the following binary operation $\odot \colon [0, 1]^2 \to [0, 1]$:

$$a \odot b = \begin{cases} a & \text{if } b = 1, \\ b & \text{if } a = 1, \\ \frac{1}{2}(2a - 1)^{\frac{1}{2b-1}} & \text{if } \frac{1}{2} < a, b < 1, \\ 0 & \text{if } a \leqslant \frac{1}{2} \text{ or } b \leqslant \frac{1}{2}. \end{cases}$$

This operation is a t-norm, whose usability for fuzzy logic is, however, limited by the fact that it is not left-continuous.

# 5 Monoid coextension of monoids

In Section 4, we have considered ways of coextending pomonoids by pogroups. There are many examples of t-norms, however, that arise from coextensions of finite tomonoids where the extending structures cannot be identified with groups. Yet, the congruence classes can often be identified with subsets of pogroups like their positive cone, their negative cone, or the interval between two of their elements. Such subsets cannot be reasonably endowed with the structure of a pogroup but they may give rise to pomonoids. Hence it seems to be a natural idea to explore possibilities of generalising the construction described in Theorem 4.2 such that the extending structures are just assumed to be pomonoids.

Again, as our first step will consider in this section the unordered case: monoid extensions of monoids. We will adapt the theory to the ordered case in Section 6.

Recall that a group coextension $(E, \pi)$ of some monoid $S$ is characterised by the fact that each preimage $E_a$ of some $a \in S$ w.r.t. $\pi$ can be endowed with the structure of a group. Namely, we can choose an arbitrary element $p_a \in E_a$ as a "base point" and we get to any other element of $E_a$ by the action of a unique group element. Replacing the groups by monoids, we cannot proceed in the same way. Obviously, the role of $p_a$ cannot be taken over by an arbitrary element, but we can still require that for *some* element of $E_a$, we get to any other one by the action of a unique element of a monoid.

In what follows, a *preordered system of monoids* $(M, \varphi)$ over $(S; \preccurlyeq)$ will have the obvious meaning. We will apply conditions (C1) and (C2) to it, which are from now on to be understood as referring to monoids instead of groups.

For a monoid $(M; +, 0)$, we define an *$M$-act* analogously to a $G$-act, that is, as a set $E$ together with a map $. \colon M \times E \to E$ such that $0 \,.\, e = e$ and $x \,.\, (y \,.\, e) = (x + y) \,.\, e$ for any $e \in E$. The map . is then called the *(monoid) action* of $M$ on $E$. Moreover, we say that an $M$-act $E$ is *centred* at an element $p \in E$ if, for each $r \in E$, there is a unique $x \in M$ such that $r = x \,.\, p$.

**Definition 5.1.** Let $(S; \cdot, 1)$ be a monoid and let $(E, \pi)$ be a coextension of $S$. Let $(M, \varphi)$ be a preordered system of monoids over $(S; \leqslant_{\mathcal{H}})$. We call $(E, \pi)$ a *monoid coextension* of $S$ based on $(M, \varphi)$ if $E_a = \{r \in E \colon \pi(r) = a\}$, for each $a \in S$, is an $M_a$-act such that (i) $E_1$ is centred at 1 and $E_a$, for any $a \neq 1$, is centred at some $p_a \in E_a$ and (ii) for any $r \in E_a,\ s \in E_b$, and $x \in M_a$, (1) holds.

**Theorem 5.2.** *Let $(S; \cdot, 1)$ be a monoid. Let $(M, \varphi)$ be a preordered system of monoids over $(S; \leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in M_{ab}$ be such that the following condition holds as well as conditions* (C1)–(C3)*:*

(C0') $\sigma_{1,a} \;=\; 0$ *for any $a \in S$.*

*On*

$$E \;=\; \{(a, x) \colon a \in S,\ x \in M_a\},$$

*we define a product by* (3)*. Then $E$, together with the surjective homomorphism $\pi \colon E \to S,\ (a, x) \mapsto a$, is a monoid coextension of $S$ based on $(M, \varphi)$, where, for each $a \in S$, the action of $M_a$ on $E_a$ is given by $y \,.\, (a,\, x) = (a,\, x + y)$.*

*Moreover, up to isomorphism, any monoid coextension of $S$ based on $(M, \varphi)$ arises in this way.*

*Proof.* We proceed similarly to the proof of Theorem 3.3. Let $(E; \cdot, (1, 0))$ and $\pi$ be as indicated. Then the product is associative and commutative by (C1) and (C2), and $(1, 0)$ is an identity by (C0'). That is, $(E; \cdot, (1, 0))$ is a monoid and $\pi \colon E \to S$ is a surjective homomorphism.

Defining the monoid action of $M_a$ on $E_a$ for each $a \in S$ as indicated, $E_a$ becomes an $M_a$-act, which is moreover centred at $(a, 0)$. We verify (1) as in the proof of Theorem 3.3. Hence $E$ is a monoid coextension of $S$ based on $(M, \varphi)$.

Conversely, let $(E; \cdot, 1)$, together with $\pi \colon E \to S$, be a monoid coextension of $S$ based on $(M, \varphi)$. By assumption, the $M_1$-poset $E_1 = \{r \in E \colon \pi(r) = 1\}$ is centred at $p_1 = 1$ and, for any $a \in S \setminus \{1\}$, the $M_a$-poset $E_a = \{r \in E \colon \pi(r) = a\}$ is centred at some $p_a \in E_a$. For each $a, b \in S$, there is furthermore a unique $\sigma_{a,b} \in M_{ab}$ such that $p_a\, p_b = \sigma_{a,b} \cdot p_{ab}$.

We proceed as in the proof of Theorem 3.3 to see that $E$ can be identified with $E' = \{(a, x) \colon x \in M_a\}$ and that then $\pi$ as well as the pomonoid actions are as indicated.

Moreover, $(1, 0) \in E'$ corresponds to $1 \in E$, and multiplication is done according to (3).

Finally, (C0') follows from the fact that $p_1 = 1$. Condition (C1) is immediate and condition (C2) follows again from the associativity of the product defined by (3). $\square$

# 6 Pomonoid coextension of pomonoids

We modify in this section the monoid coextension of monoids in analogy to the procedure in Section 4: we add a compatible order, that is, we replace the monoids with pomonoids.

Again, the meaning of a *preordered system of pomonoids* $(M, \varphi)$ over a preordered set $(S; \preccurlyeq)$ is clear. We will apply to them conditions (C0') and (C1)–(C3); in (C3), the symbols "$G$" have to be replaced by "$M$".

Moreover, for a pomonoid $(M; +, \leqslant, 0)$, we say that a poset $E$ is an $M$*-poset* if there is a map $.\colon M \times E \to E$ making $E$ an $M$-act such that, for any $m, m' \in M$, (i) the map $E \to E,\ e \mapsto m . e$ is order-preserving and (ii) $m \leqslant m'$ implies $m . e \leqslant m' . e$ for all $e \in E$.

**Definition 6.1.** Let $(S; \cdot, \leqslant, 1)$ be a pomonoid and let $(E, \pi)$ be a coextension of $S$. Let $(M, \varphi)$ be a preordered system of pomonoids over $(S; \leqslant_{\mathcal{H}})$. We call $(E, \pi)$ a *pomonoid coextension* of $S$ based on $(M, \varphi)$ if $E_a = \{r \in E \colon \pi(r) = a\}$, for each $a \in S$, is an $M_a$-poset such that (i) $E_1$ is centred at 1 and $E_a$, for any $a \neq 1$, is centred at some $p_a \in E_a$, (ii) for any $r \in E_a,\ s \in E_b$, and $x \in M_a$, (1) holds, and (iii) for any $r, s \in E_a$, we have $r \leqslant s$ if and only if there are $x, y \in M_a$ such that $x \leqslant y,\ r = x . p_a$ and $s = y . p_a$.

**Theorem 6.2.** *Let $(S; \cdot, \leqslant, 1)$ be a pomonoid. Let $(M, \varphi)$ be a preordered system of pomonoids over $(S; \leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in M_{ab}$ be such that* (C0') *and* (C1)–(C3) *hold. On*

$$E = \{(a, x) \colon a \in S,\ x \in M_a\},$$

*define a product by* (3) *and a partial order $\leqslant_E$ by* (4). *Then $(E; \cdot, \leqslant_E, (1, 0))$, together with the order-determining homomorphism $\pi \colon E \to S,\ (a, x) \mapsto a$, is a pomonoid coextension of $S$ based on $(M, \varphi)$, where, for each $a \in S$, the action of $M_a$ on $E_a$ is given by $y . (a, x) = (a, x + y)$.*

*Moreover, up to isomorphism, any pomonoid coextension of $S$ based on $(M, \varphi)$ arises in this way.*

*Proof.* Let $(E; \cdot, \leqslant_E, (1, 0))$ and $\pi$ be as indicated. By Theorem 5.2, the monoid $E$, together with $\pi$, is a monoid coextension of $S$ based on the preordered system of monoid reducts $(M, \varphi)$, where the actions are as indicated.

We proceed as in the proof of Theorem 4.2 to show that the partial order $\leqslant_E$ is compatible. From (4) it is clear that $\pi$ is a surjective, order-determining homomorphism of pomonoids. Thus $(E, \pi)$ is a coextension of $S$.

As $(E, \pi)$ is a monoid coextension of $S$ based on $(M, \varphi)$, the set $E_a = \{(a, x) \colon x \in M_a\}$, for each $a \in S$, is an $M_a$-act fulfilling (1). It is moreover clear that $E_a$ is an $M_a$-poset centred at $p_a = (a, 0)$. Hence parts (i) and (ii) of Definition 6.1 hold. Finally, the "if" part of Definition 6.1 is clear from the fact that $E_a$ is an $M_a$-poset. Furthermore, $(a, x) \leqslant_E (a, y)$ implies $x \leqslant_a y$ by (4), and since $(a, x) = x \,.\, (a, 0) = x \,.\, p_a$ and $(a, y) = y \,.\, p_a$, the "only if" part follows as well. The proof is complete that $(E; \cdot, \leqslant_E, (1, 0))$ is a pomonoid coextension of $S$.

Conversely, let $(E; \cdot, \leqslant_E, 1)$, together with $\pi \colon E \to S$, be a pomonoid coextension of $S$ based on $(M, \varphi)$. Then $(E; \cdot, 1)$ is a monoid coextension of the monoid $S$ based on the monoid reducts $(M, \varphi)$, where $E_a$, for each $a \in S \setminus \{1\}$, is centred at some $p_a$ and $E_1$ is centred at 1. As in the proof of Theorem 5.2, we determine $\sigma_{a,b} \in M_{ab}$, $a, b \in S$, such that conditions (C0'), (C1), and (C2) are fulfilled. We can furthermore identify $E$ with the set $\{(a, x) \colon a \in S, \; x \in M_a\}$ and $\pi$ with the projection to the first component. Multiplication is then given by (3), the identity is $(1, 0)$, and $p_a = (a, 0)$ for each $a \in S$. Finally, the monoid actions are as indicated.

We next show that the partial order $\leqslant_E$ on is the lexicographic one. Let $(a, x), (b, y) \in E$. Assume $(a, x) \leqslant_E (b, y)$. As $\pi$ is a homomorphism of pomonoids, it follows $a \leqslant b$. If $a = b$, we have $x \,.\, p_a = (a, x) \leqslant_E (b, y) = y \,.\, p_a$ and hence $x \leqslant_a y$ by Definition 6.1. Conversely, assume that either $a < b$, or $a = b$ and $x \leqslant_a y$. As $\pi$ is assumed to be order-determining, it follows from $a < b$ that $(a, x) \leqslant_E (b, y)$. Furthermore, if $a = b$ and $x \leqslant_a y$, we have $(a, x) = x \,.\, p_a \leqslant_E y \,.\, p_a = (b, y)$.

Finally, (C3) is seen as in the proof of Theorem 4.2. □

Again, we may specialise Theorem 6.2 to the cases of special interest.

**Corollary 6.3.** *Let $E$ be a pomonoid coextension of the pomonoid $S$ by a preordered system of pomonoids $(M, \varphi)$ over $(S; \leqslant_{\mathcal{H}})$. Then we have:*

(i) *$E$ is totally ordered if and only if $S$ is totally ordered and, for any $a \in S$, $M_a$ is totally ordered.*

(ii) *$E$ is negative if and only if $S$ and $M_1$ are negative.*

*Proof.* This is shown similarly to Corollary 4.3. □

It is moreover clear that condition (C3) can be dropped if the coextended pomonoid is cancellative. Indeed, analogous versions of Corollaries 4.6 and 4.7 hold again.

The main achievement of the transition from pogroup to pomonoids, however, is the fact that we can replace (C3) with the following condition, which is stronger but shows nevertheless that we are, as regards the possibilities of constructing pomonoids, less restricted as before.

**Corollary 6.4.** *Let $(S; \cdot, \leqslant, 1)$ be a pomonoid. Let $(M, \varphi)$ be a preordered system of pomonoids over $(S; \leqslant_{\mathcal{H}})$ and, for each $a, b \in S$, let $\sigma_{a,b} \in M_{ab}$ be such that* (C0'), (C1), (C2)*, and the following condition hold:*

(C3”) *Let, for $a, b, c \in S$, $a < b$ and $a\,c = b\,c$. Then $0$ is the smallest element of $M_{ac}$, $\varphi^a_{ac}$ is trivial, and $\sigma_{a,c} = 0$.*

*Then $E$ as specified in Theorem 6.2 is a pomonoid coextension of $S$.*

*Proof.* Condition (C3”) is obviously stronger than (C3). Hence the assertion is a special case of Theorem 6.2. □

For illustration, we include again two examples.

**Example 6.5.** Let $(L; \oplus, \leqslant, 0)$ be as in Example 4.8, that is, the four-element Łukasiewicz chain. We define a preordered system $(M, \varphi)$ of tomonoids over $(L, \leqslant_{\mathcal{H}})$ as follows. Let $M_0 = (0, 1] \subseteq \mathbb{R}$ be endowed with usual product of reals; let $M_{-1} = M_{-2} = [-1, 0] \subseteq \mathbb{R}$ be endowed with the truncated addition $\oplus$, that is,

$$a \oplus b \;=\; (a + b) \vee -1,$$

for $a, b \in [0, 1]$; and let $M_{-3} = \{0\}$.

We shall determine the pomonoid coextensions of $L$ based on $(M, \varphi)$ according to Theorem 6.2. The homomorphisms are determined by the following choices:

$$\begin{aligned} \varphi^0_{-1}&\colon (0, 1] \to [-1, 0],\; x \mapsto \ln x \,\vee\, -1, \\ \varphi^{-1}_{-2}&\colon [-1, 0] \to [-1, 0],\; x \mapsto \alpha x \,\vee\, -1, \end{aligned}$$

where $\alpha \in \mathbb{R}$ such that $\alpha \geqslant 1$. Moreover, $\sigma_{a,b} = 0$ unless $a = b = -1$. The value $\sigma = \sigma_{-1,-1}$ can be chosen arbitrarily.

The coextension is based on the set

$$E \;=\; \{(-3, 0)\} \;\cup\; \{-1, -2\} \times [-1, 0] \;\cup\; \{0\} \times (0, 1].$$

Clearly, the identity is $(0, 0)$ and the order is the lexicographical one. The monoidal operation is given as follows:

$$\begin{aligned} (0, x)\,(0, y) \;&=\; (0, x\,y), \\ (0, x)\,(-1, y) \;&=\; (-1, y)\,(0, x) \;=\; (-1,\, (\ln x + y) \vee -1), \\ (0, x)\,(-2, y) \;&=\; (-2, y)\,(0, x) \;=\; (-2,\, (\alpha \ln x + y) \vee -1), \\ (-1, x)\,(-1, y) \;&=\; (-1, y)\,(-1, x) \;=\; (-2,\, (\alpha x + \alpha y + \sigma) \vee -1), \\ (a, x)\,(b, y) \;&=\; (-3, 0) \quad \text{if } a \cdot b = -3, \end{aligned}$$

where $a, b \in L$ and $x, y \in \mathbb{R}$.

Our next example does not describe a specific structure but actually a known method of constructing t-norms.

**Example 6.6.** Let $(S; \cdot, \leqslant, 1)$ be a negative, weakly cancellative tomonoid; we denote by $0$ its smallest element. Let $(L; +, \leqslant, 0)$ any negative tomonoid, let $M_0 = \{0\}$ and

$M_a = L$ for each $a \in S \setminus \{0\}$. Let $\varphi^a_b$, where $b \leqslant_{\mathcal{H}} a$ and $b \neq 0$, be the identity. Furthermore, let $\sigma_{a,b} = 0$ for any $a, b \in S$.

The resulting tomonoid is based on the set

$$E = \{(0,0)\} \cup (S \setminus \{0\}) \times L,$$

the identity is $(1,0)$ and the order is lexicographic. For $(a,x), (b,y) \in E$ we have

$$(a,x)\,(b,y) = \begin{cases} (a\,b,\ x+y) & \text{if } a\,b \neq 0 \\ (0,0) & \text{otherwise.} \end{cases}$$

If $E$ is order-isomorphic to the real unit interval, we obtain a t-norm. T-norms constructed in this way from cancellative tomonoids have been explored in [Mes], where they are called *H-transforms*.

# 7 Triangular norms

We have already seen several examples of pomonoid extensions that lead to t-norms. We shall give in this section a more systematic view on the construction of t-norms based on the method proposed in this paper.

Let us consider a tomonoid of the form $([0,1]; \odot, \leqslant, 1)$, where $[0,1]$ is the real unit interval and $\leqslant$ is its natural order. We recall that the product $\odot$ is called a t-norm if it makes $[0,1]$ into a negative tomonoid. Consequently, each negative tomonoid order-isomorphic to the real unit interval gives rise to a t-norm.

In fuzzy logic, left-continuous t-norms are of special interest, that is, t-norms $\odot$ such that $(0,1] \to [0,1] \colon x \mapsto x \odot a$ is, for each $a \in [0,1]$, continuous from the left. A corresponding algebraic property is the following one [Vet2]. A pomonoid $(S; \cdot, \leqslant, 1)$ is called *quantic* if (i) all non-empty suprema exist in $S$ and (ii) for any elements $a, b_\iota$, $\iota \in I$, of $S$ we have

$$a \cdot \bigvee_\iota b_\iota = \bigvee_\iota (a \cdot b_\iota) \quad \text{and} \quad (\bigvee_\iota b_\iota) \cdot a = \bigvee_\iota (b_\iota \cdot a).$$

Then a t-norm is left-continuous if and only if $([0,1]; \odot, \leqslant, 1)$ is quantic. We call a tomonoid of the latter form a *t-norm monoid*.

Theorem 7.2 below specifies a class of t-norms that can be systematised on the basis of the present approach. To this end, coextensions are described such that the universe is order-isomorphic to the real unit interval. Accordingly, the involved tomonoids are based on subsets of the reals. The following structures will be used, where "+" denotes the usual addition of reals and $\leqslant$ is always the natural order:

the real line $(\mathbb{R}; +, \leqslant, 0)$;

the negative real cone $(\mathbb{R}^-; +, \leqslant, 0)$, where $\mathbb{R}^- = \{x \in \mathbb{R} \colon x \leqslant 0\}$;

the truncated negative real cone $([-1,0];\leqslant,\oplus,0)$, where

$$x \oplus y \;=\; (x+y) \vee -1;$$

the positive real cone $(\mathbb{R}^+;+,\leqslant,0)$, where $\mathbb{R}^+ = \{x \in \mathbb{R}\colon x \geqslant 0\}$;

the truncated positive real cone $([0,1];\leqslant,\oplus,0)$, where

$$x \oplus y \;=\; (x+y) \wedge 1.$$

Note that the negative real cone is isomorphic to $((0,1];\leqslant,\cdot,1)$, where $(0,1]$ is the left-open real unit interval and $\cdot$ is the usual multiplication of reals. Moreover, the truncated negative real cone is isomorphic to the t-norm monoid based on the Łukasiewicz t-norm.

We will refer to the above-mentioned tomonoids in the sequel simply by their universe. Let us first describe the homomorphisms between two of them.

**Proposition 7.1.** (i) *For a non-trivial homomorphism $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$, $\varphi\colon \mathbb{R}^- \to \mathbb{R}$, $\varphi\colon \mathbb{R}^- \to \mathbb{R}^-$, or $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$, there is a real $c > 0$ such that, for all $x$ of the domain,*

$$\varphi(x) \;=\; c\,x.$$

(ii) *For a non-trivial homomorphism $\varphi\colon \mathbb{R}^- \to [-1,0]$ or $\varphi\colon [-1,0] \to [-1,0]$, one of the following possibilities applies. Either there is a real $c > 0$ or $c \geqslant 1$, respectively, such that, for all $x$ of the domain,*

$$\varphi(x) \;=\; c\,x \vee -1.$$

*Or, for all $x$ of the domain, we have*

$$\varphi(x) \;=\; \begin{cases} 0 & x = 0, \\ -1 & \text{otherwise.} \end{cases}$$

(iii) *For a non-trivial homomorphism $\varphi\colon \mathbb{R}^+ \to [0,1]$ or $\varphi\colon [0,1] \to [0,1]$, one of the following possibilities applies. Either there is a real $c > 0$ or $c \geqslant 1$, respectively, such that, for all $x$ of the domain,*

$$\varphi(x) \;=\; c\,x \wedge 1.$$

*Or, for all $x$ of the domain, we have*

$$\varphi(x) \;=\; \begin{cases} 0 & x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

*In all other cases, a homomorphism between two of the tomonoids $\mathbb{R}$, $\mathbb{R}^-$, $[-1,0]$, $\mathbb{R}^+$, $[0,1]$ is necessarily trivial.*

We now turn to the coextensions based on these tomonoids. Here, for an element $a$ of a finite tomonoid $S$ distinct from the top element, we denote its successor in $S$ by $a'$, that is, $a'$ will be the smallest element strictly greater than $a$. Furthermore, the continuity of homomorphisms will refer to the usual topology on the reals.

**Theorem 7.2.** *Let $(S; \cdot, \leqslant, 1)$ be a finite negative tomonoid and let $(M, \varphi)$ be a pre-ordered system of tomonoids over $(S; \leqslant_{\mathcal{H}})$ such that* (i) *for each $a \in S \setminus \{1\}$, $M_a$ is one of $\mathbb{R}$, $\mathbb{R}^-$, $[-1,0]$, $\mathbb{R}^+$, $[0,1]$ and $M_1$ is one of $\mathbb{R}^-$ or $[-1,0]$ and* (ii) *$\varphi^a_b$ is continuous for each $a \geqslant_{\mathcal{H}} b$. Let $E$ be a pomonoid coextension of $S$ based on $(M, \varphi)$. Then $E$ is a negative tomonoid. Furthermore, $E$ is isomorphic to a t-norm monoid if and only if the following conditions are fulfilled:*

(T1) *$E$ is order-isomorphic to the real unit interval.*

(T2) *For any $a, b \in S$ such that $M_a$ is not upper-bounded, we have:*

- (a) *Let $\varphi^a_{ab}$ be trivial. Then $a\,b = a'\,b$. Moreover, if $M_{a'} \neq [-1,0]$ or $\varphi^{a'}_{ab}$ is trivial, then $\sigma_{a,b} = \sigma_{a',b}$, otherwise $\sigma_{a,b} = -1$.*
- (b) *Let $\varphi^a_{ab}$ be non-trivial and $M_{ab} = [0,1]$. Then $a\,b = a'\,b$ and $\sigma_{a',b} = 1$.*
- (c) *Let $\varphi^a_{ab}$ be non-trivial and let $M_{ab}$ not be upper-bounded. Then $a'\,b = (ab)'$. Moreover, if $M_{(ab)'} \neq [-1,0]$, then $\varphi^b_{(ab)'}$ is trivial and $\sigma_{a',b} = 0$. If $M_{(ab)'} = [-1,0]$ and $\varphi^{a'}_{(ab)'}$ is trivial, then $\sigma_{a',b} = -1$.*

*Proof.* By Corollary 6.3, $E$ is a negative tomonoid.

Let us assume that $E$ is isomorphic to a t-norm monoid. Obviously, (T1) then holds. Our aim is to show (T2). Let $a, b \in S$ be such that $M_a$ is not upper-bounded.

Note that $M_a$ is one of $\mathbb{R}$ or $\mathbb{R}^+$ and hence, by (T1), $M_{a'}$ is the trivial tomonoid $\{0\}$ or one of $[-1,0]$, $\mathbb{R}^+$, $[0,1]$. Let $m_{a'}$ be the least element of $M_{a'}$. As $E$ is quantic, we have, for any $y \in M_b$,

$$\bigvee_{x \in M_a} ((a,x)\,(b,y)) = (a', m_{a'})\,(b,y). \tag{5}$$

By (3), this means

$$\bigvee_{x \in M_a} (a\,b,\ \varphi^a_{ab}(x) + \varphi^b_{ab}(y) + \sigma_{a,b}) \;=\; (a'\,b,\ \varphi^{a'}_{a'b}(m_{a'}) + \varphi^b_{a'b}(y) + \sigma_{a',b}). \tag{6}$$

Ad (a). Assume that $\varphi^a_{ab}$ is trivial. Then, by (6), $a\,b = a'\,b$ and $\varphi^b_{ab}(y) + \sigma_{a,b} = \varphi^{a'}_{ab}(m_{a'}) + \varphi^b_{ab}(y) + \sigma_{a',b}$ for any $y \in M_b$. The latter equation holds in particular for $y = 0$, hence $\sigma_{a,b} = \varphi^{a'}_{ab}(m_{a'}) + \sigma_{a',b}$.

If $M_{a'}$ is not $[-1,0]$, then $M_{a'}$ is one of $\{0\}$, $\mathbb{R}^+$, or $[0,1]$, hence $m_{a'} = 0$ and $\sigma_{a,b} = \sigma_{a',b}$, as claimed. If $\varphi^{a'}_{ab}$ is trivial, we draw the same conclusion. Otherwise, $M_{a'} = [-1,0]$ and $\varphi^{a'}_{ab}$ is non-trivial. Then, by Proposition 7.1, $M_{ab} = [-1,0]$ as well and $\varphi^{a'}_{ab}(m_{a'}) = \varphi^{a'}_{ab}(-1) = -1$. Hence $\sigma_{a,b} = -1$ in this case.

Ad (b). Assume that $\varphi^a_{ab}$ is non-trivial and $M_{ab} = [0,1]$. Again by (6), we have $a\,b = a'\,b$. Moreover, $\bigvee_{x \in M_a} \varphi^a_{ab}(x) = 1$ and it follows $\varphi^{a'}_{ab}(m_{a'}) + \varphi^b_{ab}(y) + \sigma_{a',b} = 1$ for any $y \in M_b$, in particular $\varphi^{a'}_{ab}(m_{a'}) + \sigma_{a',b} = 1$. Furthermore, either $\varphi^{a'}_{ab}$ is trivial or else, by Proposition 7.1, $M_{a'}$ cannot be $[-1,0]$ and hence $m_{a'} = 0$. In both cases, we conclude $\sigma_{a',b} = 1$.

Ad (c). Assume that $\varphi^a_{ab}$ is non-trivial and $M_{ab}$ is not upper-bounded, that is, one of $\mathbb{R}$ or $\mathbb{R}^+$. Then, by (6), $a'\,b = (a\,b)'$ and $\varphi^{a'}_{(a\,b)'}(m_{a'}) + \varphi^b_{(a\,b)'}(y) + \sigma_{a',b} = m_{(a\,b)'}$ for any $y \in M_b$, where $m_{(a\,b)'}$ is the least element of $M_{(a\,b)'}$. In particular, we have $\varphi^{a'}_{(a\,b)'}(m_{a'}) + \sigma_{a',b} = m_{(a\,b)'}$.

Assume that $M_{(a\,b)'}$ is not $[-1,0]$ and hence either $\mathbb{R}^+$ or $[0,1]$. This means $m_{(a\,b)'} = 0$. Furthermore, either $\varphi^{a'}_{(a\,b)'}$ is trivial or else, by Proposition 7.1, $M_{a'}$ cannot be $[-1,0]$ and hence $m_{a'} = 0$. We conclude $\varphi^{a'}_{(a\,b)'}(m_{a'}) = 0$. It moreover follows that $\sigma_{a',b} = 0$ and $\varphi^b_{(a\,b)'}$ is trivial.

Assume next that $M_{(a\,b)'} = [-1,0]$ and $\varphi^{a'}_{(a\,b)'}$ is trivial. Then $\sigma_{a',b} = -1$. The proof of (T2) is complete.

For the converse direction, let us assume that (T1) and (T2) holds. By (T1), $E$ is order-isomorphic to the real unit interval.

We have to show that $E$ is quantic. By assumption, all homomorphisms $\varphi^a_b$, where $a \geqslant_{\mathcal{H}} b$, are continuous. Hence for any $(a,x), (b,y) \in E$ such that $x$ is not the least element of $M_a$, we have $\bigvee_{x'<x}((a,x')\,(b,y)) = (a\,b,\; \bigvee_{x'<x} \varphi^a_{ab}(x') + \varphi^b_{ab}(y) + \sigma_{a,b}) = (a\,b,\; \varphi^a_{ab}(x) + \varphi^b_{ab}(y) + \sigma_{a,b}) = (a,x)\,(b,y)$. We conclude that it suffices to show (5) for any $a \in S$ such that $M_a$ is not upper-bounded and any $(b,y) \in E$.

Let $a, b \in S$ and assume that $M_a$ is one of $\mathbb{R}$ or $\mathbb{R}^+$. According to (3), our aim is to show (6), where $y \in M_b$. We distinguish three cases.

*Case 1.* Assume that $\varphi^a_{ab}$ is trivial. Then $a\,b = a'\,b$ by (T2)(a) and we have to show that $\sigma_{a,b} = \varphi^{a'}_{ab}(m_{a'}) + \sigma_{a',b}$. If $M_{a'}$ is not $[-1,0]$, then $m_{a'} = 0$ and, by (T2)(a), $\sigma_{a,b} = \sigma_{a',b}$, hence the assertion follows. We can make the same conclusion if $\varphi^{a'}_{ab}$ is trivial. Otherwise, $M_{a'} = [-1,0]$ and $\varphi^{a'}_{ab}$ is non-trivial. Then, by Proposition 7.1, $M_{ab} = [-1,0]$ and $\varphi^{a'}_{ab}(m_{a'}) = \varphi^{a'}_{ab}(-1) = -1$. As, by (T2)(a), $\sigma_{a,b} = -1$, the assertion is again clear.

*Case 2.* Assume that $\varphi^a_{ab}$ is non-trivial and $M_{ab}$ is upper-bounded. By Proposition 7.1, $M_{ab} = [0,1]$. By (T2)(b), then $a\,b = a'\,b$ and $\sigma_{a',b} = 1$. Hence both sides of (6) equal $(a\,b,\; 1)$.

*Case 3.* Assume that $\varphi^a_{ab}$ is non-trivial and $M_{ab}$ is not upper-bounded. Then $M_{ab} = \mathbb{R}$ or $M_{ab} = \mathbb{R}^+$. The left side of (6) then equals $((a\,b)',\; m_{(a\,b)'})$. By (T2)(c), $a'\,b = (ab)'$ and thus we have to show that $\varphi^{a'}_{(a\,b)'}(m_{a'}) + \varphi^b_{(a\,b)'}(y) + \sigma_{a',b} = m_{(a\,b)'}$ for any $y \in M_b$.

If $M_{(ab)'} \neq [-1,0]$, then $m_{(ab)'} = 0$ and, by (T2)(c), $\varphi^b_{(ab)'}$ is trivial and $\sigma_{a',b} = 0$. Furthermore, either $\varphi^{a'}_{(a\,b)'}$ is trivial or else, by Proposition 7.1, $M_{a'}$ cannot be $[-1,0]$

and hence $m_{a'} = 0$. We conclude $\varphi^{a'}_{(a\,b)'}(m_{a'}) = 0$ and the assertion follows.

If $M_{(ab)'} = [-1, 0]$ and $\varphi^{a'}_{(ab)'}$ is trivial, then $\sigma_{a',b} = -1$ by (T2)(c) and the assertion is clear. If $M_{(ab)'} = [-1, 0]$ and $\varphi^{a'}_{(ab)'}$ is non-trivial, then $M_{a'} = [-1, 0]$ by Proposition 7.1, hence $\varphi^{a'}_{(ab)'}(m_{a'}) = \varphi^{a'}_{(ab)'}(-1) = -1$ and the assertion is clear again. □

We conclude the paper by presenting examples of left-continuous t-norms arising according to Theorem 7.2.

**Example 7.3.** Let $L = \{-4, -3, -2, -1, 0\}$ be the five-element Łukasiewicz chain, the monoidal operation being given by $a \oplus b = (a + b) \vee -4$, where $+$ is the usual sum of integers. Our preordered system $(M, \varphi)$ of pomonoids over $(L, \leqslant_{\mathcal{H}})$ will be the following one. Let $M_a = \mathbb{R}^-$ for any $a \neq -4$ and let $M_{-4} = \{0\}$. As regards the homomorphisms, note again that $\leqslant_{\mathcal{H}}$ coincides with $\leqslant$. For $a \geqslant b > -4$, we define $\varphi^a_b$ to be the identity.

We furthermore choose $\sigma_{a,b}$, where $a, b \in L$, as follows. If $a = 0$ or $b = 0$, or if $a \oplus b = -4$, we necessarily have $\sigma_{a,b} = 0$. In contrast, $\sigma_{-1,-1}$ and $\sigma_{-1,-2}$ can be chosen in an arbitrary way. We put $\sigma_{-1,-1} = \frac{1}{2}$ and $\sigma_{-1,-2} = \frac{1}{4}$.

It is clear that Theorem 7.2 applies because the new base set is order-isomorphic to $[0, 1]$ and all extending tomonoids are upper-bounded. Hence we obtain a left-continuous t-norm, which is in fact the one proposed in [Vet2, Ex. 4.8]:

$$a \odot b = \begin{cases} 4ab - 3a - 3b + 3 & \text{if } a, b > \frac{3}{4}, \\ 4ab - 3a - 2b + 2 & \text{if } \frac{1}{2} < a \leqslant \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a - b + 1 & \text{if } \frac{1}{4} < a \leqslant \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a & \text{if } a \leqslant \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ 2ab - a - b + \frac{3}{4} & \text{if } \frac{1}{2} < a, b \leqslant \frac{3}{4}, \\ ab - \frac{1}{2}a - \frac{1}{4}b + \frac{1}{8} & \text{if } \frac{1}{4} < a \leqslant \frac{1}{2} \text{ and } \frac{1}{2} < b \leqslant \frac{3}{4}, \\ 0 & \text{if } a \leqslant \frac{1}{4} \text{ and } \frac{1}{2} < b \leqslant \frac{3}{4}, \text{ or } a, b \leqslant \frac{1}{2}. \end{cases}$$

(In order to save space, commutativity is to be assumed to complete this as well as the subsequent definitions of t-norms.) See Figure 1 for an illustration.

The application of Theorem 7.2 is particularly easy if, as in Example 7.3, only $\mathbb{R}^-$, $[-1, 0]$, or $[0, 1]$ are used as extending structures because condition (T2) is in this case trivially fulfilled. Our next example demonstrates an application of Theorem 7.2 where this is not so.

**Example 7.4.** Let $L = \{-3, -2, -1, 0\}$ be the four-element chain, endowed with the drastic product:

$$a \cdot b = \begin{cases} a & \text{if } b = 0, \\ b & \text{if } a = 0, \\ -3 & \text{otherwise.} \end{cases}$$

Note that $a \leqslant_{\mathcal{H}} b$ if and only if $a = -3$ or $b = 0$ or $a = b$. We define a preordered system $(M, \varphi)$ of pomonoids over $(L, \leqslant_{\mathcal{H}})$ as follows. Let $M_0 = \mathbb{R}^-$, $M_{-2} = \mathbb{R}$, and

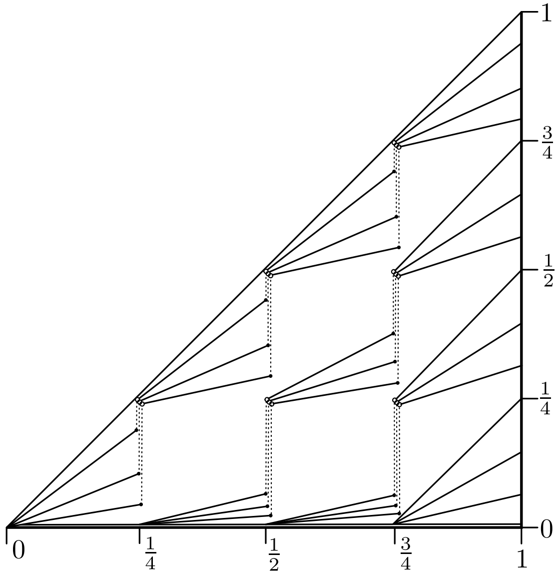


**Figure 1:** The t-norm $\odot$ from Example 7.3. For several $a \in [0, 1]$, the translations by $a$, that is, the mappings $[0, 1] \to [0, 1],\ x \mapsto x \odot a$ are depicted in a schematic way.

$M_{-1} = M_{-3} = \{0\}$. Let $\varphi^0_{-2} \colon M_0 \to M_{-2},\ x \mapsto x$, the other homomorphisms are necessarily trivial. Finally, let $\sigma_{-2,0} = \sigma_{0,-2} = 0$, the other values are necessarily 0 as well.

We claim that Theorem 7.2 applies and thus the coextension of $L$ by $(M, \varphi)$ is isomorphic to a t-norm monoid. Indeed, (T1) is clearly fulfilled. Furthermore, $M_{-2}$ is the only pomonoid that is not upper-bounded. The pair $a = -2$ and $b = 0$ fulfils the assumptions of (T2)(c). We verify its conclusions easily: $(-2)' \cdot 0 = -1 \cdot 0 = -1 = (-2)' = (-2 \cdot 0)'$ and $M_{-1} = \{0\}$. Furthermore, we readily check (T2)(a) for the pairs $a = -2$ and $b < 0$.

The resulting left-continuous t-norm is, again after a re-scaling, the following one, which also occurs in [Vet1, Sec. 8]:

$$a \odot b = \begin{cases} 2ab - a - b + 1 & \text{if } a, b > \frac{1}{2}, \\ \frac{1}{2}(2a)^{\frac{1}{2b-1}} & \text{if } a \leqslant \frac{1}{2} \text{ and } b > \frac{1}{2}, \\ 0 & \text{if } a, b \leqslant \frac{1}{2}, \end{cases}$$

where $a, b \in [0, 1]$. See Figure 2 for an illustration.

Theorem 7.2 certainly does not cover all left-continuous t-norms that arise by means of a coextension of a tomonoid that has, in some sense, a simple structure. A possible generalisation of Theorem 7.2 could concern the case that the coextended tomonoid is infinite. We have in mind examples like the following one, covered by Theorem 6.2.

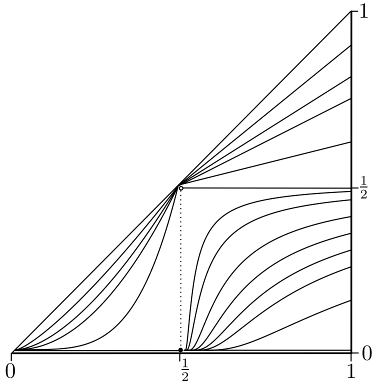


**Figure 2:** The t-norm from Example 7.4.

**Example 7.5.** Let us consider the t-norm monoid based on the product t-norm, that is, $[0,1]$ together with the usual multiplication $\cdot$ of reals. We construct a coextension of $([0,1], \cdot, \leqslant, 1)$ as follows. Let $M_a = \{0\}$ for any $a \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, and let $M_{\frac{1}{2}} = M_1 = [-1, 0]$, the truncated negative real cone. Let $\varphi^1_{\frac{1}{2}} : [-1,0] \to [-1,0],\ x \mapsto x$. Let furthermore $\sigma_{a,b} = 0$ for any $b \geqslant_{\mathcal{H}} a$.

We obtain the following left-continuous t-norm; cf. [Vet2, Ex. 6.12]:

$$a \odot b \;=\; \begin{cases} (a+b-1) \vee \frac{3}{4} & \text{if } a, b > \frac{3}{4}, \\ a & \text{if } \frac{1}{2} < a \leqslant \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ (a+b-1) \vee \frac{1}{4} & \text{if } \frac{1}{4} < a \leqslant \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ a & \text{if } a \leqslant \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ 2^{4(a+b)-7} & \text{if } \frac{1}{2} < a, b \leqslant \frac{3}{4} \text{ and } a+b \leqslant \frac{5}{4}, \\ a+b-\frac{3}{4} & \text{if } \frac{1}{2} < a, b \leqslant \frac{3}{4} \text{ and } a+b > \frac{5}{4}, \\ 2^{4b-5} & \text{if } \frac{1}{4} < a \leqslant \frac{1}{2} \text{ and } \frac{1}{2} < b \leqslant \frac{3}{4}, \\ 2^{4b-3} a & \text{if } a \leqslant \frac{1}{4} \text{ and } \frac{1}{2} < b \leqslant \frac{3}{4}, \\ \frac{1}{8} & \text{if } \frac{1}{4} < a, b \leqslant \frac{1}{2}, \\ \frac{a}{2} & \text{if } a \leqslant \frac{1}{4} \text{ and } \frac{1}{4} < b \leqslant \frac{1}{2}, \\ 2ab & \text{if } a, b \leqslant \frac{1}{4}. \end{cases}$$

An illustration can be found in Figure 3.

Let us finally comment on the limits of the present approach. At first sight, our procedure seems to resemble the approach of [Vet3], where coextensions are described that are based on the chains $\mathbb{R}$, $\mathbb{R}^-$, $\mathbb{R}^+$, $[-1,0]$. The following example shows that not all coextensions discussed in the previous work can be reproduced in the present setting.

**Example 7.6.** Let $L = \{-2, -1, 0\}$ be the three-element Łukasiewicz chain. Let $M_{-2} = \mathbb{R}^+,\ M_{-1} = \{0\}$, and $M_0 = \mathbb{R}^-$. Let

$$E \;=\; \{-2\} \times \mathbb{R}^+ \;\cup\; \{(-1,0)\} \;\cup\; \{0\} \times \mathbb{R}^-$$

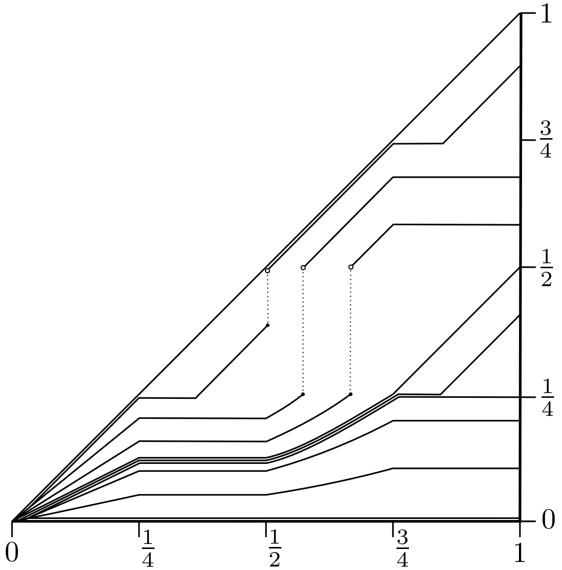


**Figure 3:** The t-norm from Example 7.5.

be ordered lexicographically, and define a product:

$$\begin{aligned}(0,x)\,(0,y) &= (0,x+y),\\ (0,x)\,(-1,0) &= (-1,0),\\ (0,x)\,(-2,y) &= (-2,(x+y)\vee 0),\\ (a,x)\,(b,y) &= (-2,0),\ \text{if } a+b\leqslant -2.\end{aligned}$$

The result is a tomonoid $E$ isomorphic to the t-norm monoid based on what is called the rotated product t-norm [Jen]. $E$ is a coextension of $L$. However, this is not a pomonoid coextension in the sense of Definition 6.1. Indeed, from $\mathbb{R}^-$ to $\mathbb{R}^+$ only the trivial homomorphism exists and hence our procedure would not lead to the indicated product on $E$.

Example 7.6 shows that not all t-norms considered in the framework of the approach of [Vet2] are covered here. We note that, however, the two approaches are not comparable. In Example 4.9 we have constructed a t-norm by means of a coextension of the five-element Łukasiewicz chain. The quotient, however, is not induced by a filter and hence this coextension is of a different type than those considered in [Vet2].

# 8 Conclusion

The study of triangular norms (t-norms), which are significant in fuzzy logic [CHN], has long been dominated by the invention of an increasing number of examples as well as geometrically motivated construction methods. In recent times, a more systematic viewpoint could be developed thanks to a progress in algebraic respects. The relevant structures are residuated chains [GJKO] or, more generally, totally ordered monoids (tomonoids) [EKMMW]. In [Vet2, Vet3], a certain way of coextending tomonoids has been shown to be particularly useful as regards the systematisation of t-norms.

The approach of [Vet2, Vet3] covers a considerable class of the operations in question but is still not free from arbitrariness. Certain features of a coextension of tomonoids have to be added in ad-hoc manner. In this respect, the present work provides a significant progress. Our framework offers a systematic view, not on all t-norm but, on a good part of those discussed in the literature.

We have explored an approach that has been known for a long time and refers to semigroups, that is, to associated structures without an order: the group extensions of Grillet and Leech. We have adapted this theory to the case that a compatible order is present and we have generalised it to include the possibility of using monoids instead of groups as extending structures.

For the sake of a classification of t-norms, we have thus studied coextensions based on tomonoids as extending structures. The future perspective should be directed towards an essential generalisation of this approach: instead of tomonoids, $S$-posets should be used, where $S$ is again a tomonoid and the poset is a chain. The question of defining the suitable algebraic framework will be the topic of subsequent research.

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