Finitary prelinear and linear orthosets

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Abstract

An orthoset is a set equipped with a symmetric and irreflexive binary relation. A linear orthoset is an orthoset such that for any two distinct elements $\boldsymbol{e}, \boldsymbol{f}$ there is a third element $\boldsymbol{g}$ such that exactly one of $\boldsymbol{f}$ and $\boldsymbol{g}$ is orthogonal to $\boldsymbol{e}$ and the pairs $\boldsymbol{e}, \boldsymbol{f}$ and $\boldsymbol{e}, \boldsymbol{g}$ have the same orthogonal complement. Linear orthosets naturally arise from Hermitian spaces. In case of a finite rank, they are in a one-to-one correspondence with the irreducible modular ortholattices of finite heigth. We define an orthoset to be prelinear by assuming the above mentioned property for non-orthogonal pairs $\boldsymbol{e}, \boldsymbol{f}$ only. In this paper, we establish some structural properties of prelinear and linear orthosets under the assumption of finiteness or finite rank.

Keywords: orthoset, prelinear orthoset, linear orthoset, finite rank, orthomodular lattice, modular lattice, covering property

1 Introduction

An *orthoset* is a pair $(X, \perp)$, where X is a set and $\perp$ is a symmetric, irreflexive binary relation on X. Elements e and f such that $e \perp f$ are called *orthogonal* and orthosets are sometimes also referred to as *orthogonality spaces*. David Foulis and his collaborators [1, 11] proposed orthosets as an abstract version of the Hilbert space model underlying quantum physics. Our guiding example is $(P(H), \perp)$, where $P(H)$ is the collection of one-dimensional subspaces of a Hilbert space H and $\perp$ is the usual orthogonality relation.

Recently, we introduced and studied the prelinearity and linearity of orthosets, which are simple combinatorial conditions [2, 7, 8]. We established that linear orthosets of finite rank $n \geq 4$ are exactly linear orthosets of the form $(P(H), \perp)$, where H is an n-dimensional Hermitian space [9].

Our main goal in this article is to study prelinear and linear orthosets which are finite or of a finite rank:

(i) Characterize prelinear orthosets of finite rank.
(ii) Study direct product of prelinear orthosets.
(iii) Classify linear orthosets of rank ≤ 3.
(iv) Compute the number of all possible finite prelinear orthosets with a given number of elements.

The paper is organized as follows: Section 2 contains basic notions and facts about lattices, ortholattices, and their direct product decompositions. In Section 3, known properties of orthosets are recalled. A lattice-theoretic representation theorem for prelinear orthosets of finite rank is presented and used to characterize linear orthosets of finite rank. In Section 4, we study direct products of prelinear orthosets and we show that every prelinear orthoset of finite rank can be written as a direct product of finitely many linear orthosets of finite rank.

In Section 5, we obtain new structural results for prelinear and linear orthosets of rank n where $n \in \{1, 2, 3\}$. Moreover, in the final Section 6, we prove that there is only one type of non-trivial finite linear orthoset. Afterwards, we find a formula which allows us to compute the number of all possible prelinear orthosets with a given number of elements.

2 Lattice theoretical background

In this section, we gather some preliminary results on lattices and ortholattices, which will be needed in the following sections. Some of the results of this section are new, and they are related to orthosets, so we state them in this section with short proofs. We assume that the reader is familiar with the basics of lattice theory.

Lattices and ortholattices

We recall some notions and properties about lattices and ortholattices [4, 6].

Let a, b be elements of a lattice. We say that b *covers* a and write $a \lessdot b$ when $a \leq b$ and there is no c such that $a < c < b$.

We say that a lattice L is *trivial* if $|L| = 1$, otherwise we say that it is *non-trivial*.

Let L be a lattice with 0. An element p of L is called an *atom* if $0 \lessdot p$. Moreover, L is called *atomistic* if each element is the join of atoms. We denote the collection of all atoms of L by $\mathcal{A}(L)$. A lattice with 0 said to fulfil *covering property* if for any a and for any atom p, $a \wedge p = 0$ implies $a \lessdot a \vee p$.

The *length* of a finite chain (totally ordered set) C is the $|C| - 1$. A lattice L is said to be of *finite height* if, for some $n \in \mathbb{N}$, all chains contained in L are of length at most n. In this case, the maximum of lengths of chains is referred to as the height of L. We note that any lattice of finite height is complete.

Let a, b be two elements of a bounded lattice L. We say that b is a *complement* of a if $a \vee b = 1$ and $a \wedge b = 0$. If L is distributive then any complement is uniquely determined. L is called *complemented* if each element possesses a complement. A *Boolean sublattice* of L is a bounded sublattice of L which is complemented and distributive.

A bounded lattice L is called an *ortholattice* if L is equipped with a unary operation ${}^{\perp}\colon L \to L$ called *orthocomplementation* such that, for all $a \in L$:

(i) $a^{\perp}$ is a complement of a,
(ii) If $a \leq b$ then $b^{\perp} \leq a^{\perp}$,
(iii) $a^{\perp\perp} = a$.

Also, we write $a \perp b$ if $a \leq b^{\perp}$. In this case we say that a and b are *orthogonal*.

A lattice is called *modular* if $a \leq b$ implies $a \vee (x \wedge b) = (a \vee x) \wedge b$.

An *orthomodular* lattice is an ortholattice such that $a \leq b$ implies $a \vee (a^{\perp} \wedge b) = b$ [4]. A Boolean algebra is a distributive ortholattice. Clearly, any Boolean algebra is a modular ortholattice and any modular ortholattice is orthomodular. We note that any modular ortholattice of finite height is atomistic.

We denote the two-element Boolean algebra by $\mathbf{2}$. A Boolean algebra with n atoms will be denoted by $\mathbf{2}^n$.

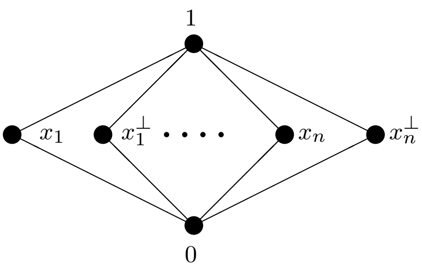
Let L be any lattice with 0 and $a \in L$. The interval $L[0, a] = \{x \in L \mid x \leq a\}$ is a sublattice of L. If L is complete, atomistic or modular, then so is $L[0, a]$, respectively.

The *direct product* of a collection of lattices is defined by the usual componentwise operations. The *horizontal sum* of two non-trivial bounded lattices is basically the non-trivial bounded lattice obtained by glueing the two lattices at their smallest and largest elements.

A *homomorphism (isomorphism)* between two ortholattices is a lattice homomorphism (isomorphism) which preserves the orthocomplementation.

Example 1

Let n be any arbitary non-zero cardinal number. The horizontal sum of n copies of the four-element Boolean algebra $\mathbf{2}^2$ is a modular ortholattice which is denoted by $\mathbf{MO}_n$. For a finite n, it has $2n + 2$ elements and its Hasse diagram is shown on the right.



Central elements of lattices

Let L denote a bounded lattice throughout the remainder of this section. The results and definitions mentioned in this section are based mainly on [6] so we omit the repeated references.

Definition 1 An element $a \in L$ is called *central* when there exist two bounded lattices L_1 and L_2 and an isomorphism $\Phi\colon L \to L_1 \times L_2$ such that $\Phi(a) = (1_{L_1}, 0_{L_2})$.

We denote the collection of all central elements of L by $Z(L)$.

Assuming the context of Definition 1, let us put $a' = \Phi^{-1}(0_{L_1}, 1_{L_2})$. Since $L[0,a]$ is isomorphic to L_1 by $\Phi\ |_{L[0,a]}\colon L[0,a] \to L_1 \times \{0_{L_2}\}$ and $L[0,a']$ is isomorphic to L_2 by $\Phi\ |_{L[0,a']}\colon L[0,a'] \to \{0_{L_1}\} \times L_2$ we see that $a \in L$ is central if and only if there is an element $a' \in L$ such that

$$\Phi\colon L \to L[0,a] \times L[0,a'], \quad x \mapsto (x \wedge a, x \wedge a') \tag{1}$$

is a lattice isomorphism.

Proposition 1 *Let a be a central element of L. Then the element a' fulfilling (1) is uniquely determined. Moreover a' is the unique complement of a which is also central. Furthermore, if a, b are central, then the elements $a \vee b$ and $a \wedge b$ are central as well.*

Note that the elements 0 and 1 are always central and we call them *trivial central elements*. There are some other equivalent formulations of a central element as follows [6, Theorem 4.13]. ⋆

Lemma 2 *Let L be a bounded lattice and $a \in L$. Then the following are equivalent:*

(i) *a is a central element.*

(ii) *There is an element $a' \in L$ such that*

$$x = (x \wedge a) \vee (x \wedge a') = (x \vee a) \wedge (x \vee a') \tag{2}$$

for all $x \in L$.

(iii) *a is complemented, and for all $x, y \in L$ we have $(x \vee a) \wedge y = (x \wedge y) \vee (a \wedge y)$ and $(x \wedge a) \vee y = (x \vee y) \wedge (a \vee y)$.*

We note that, as a consequence, if a is central in a bounded lattice, then, for all $x \in L$, $(x \vee a) \wedge a' = x \wedge a'$ and $(x \vee a') \wedge a = x \wedge a$. ⋆

On the other hand, the last item proves that the central elements yield the distributive law in a restricted case, which is called *MacLaren characterization* [4, 5]. Hence $Z(L)$ is a Boolean sublattice of L.

Lemma 3 *Let L be a bounded lattice and $a, b \in L$ such that $a \le b$. If $a \in Z(L)$ then $a \in Z\big(L[0,b]\big)$.*

Proof If $a \in Z(L)$, there is an $a' \in L$ and an isomorphism $\Phi\colon L \to L[0,a] \times L[0,a']$. Let $\Phi(b) = (a, c)$. Then $\Phi\mid_{L[0,b]}\colon L[0,b] \to L[0,a] \times L[0,c]$ is a lattice isomorphism. $\square$

Lemma 4 *Let L be a bounded lattice and $a \in Z(L)$. If $b \in Z\big(L[0,a]\big)$ then $b \in Z(L)$.*

Proof First, we have a lattice isomorphism $\Phi\colon L \to L[0,a] \times L[0,a']$ such that $\Phi(a) = (a, 0)$ and $\Phi(b) = (b, 0)$. Second, since $b \in Z\big(L[0,a]\big)$ we have a lattice isomorphism $\Psi\colon L[0,a] \to L[0,b] \times L[0,b']$ such that $\Psi(b) = (b, 0)$. We conclude that we have a lattice isomorphism $\Gamma\colon L \to L[0,b] \times L[0,b'] \times L[0,a']$ such that $\Gamma = (\Psi \times \mathrm{id}_{L[0,a']}) \circ \Phi$. Hence $\Gamma(b) = (b, (0,0))$, which completes the proof. $\square$

In the sequel, a central element of an ortholattice is meant to be a central element of its lattice reduct. Note that in ortholattices the unique complement of a central element is necessarily its orthocomplement.

Lemma 5 *Let L be an ortholattice and $a \in Z(L)$. Then the sublattice $L[0,a]$ is an ortholattice where the orthocomplement is defined by $x' = x^\perp \wedge a$, for all $x \in L[0,a]$. Moreover, (1) is an ortholattice isomorphism $\Phi\colon L \to L[0,a] \times L[0,a^\perp]$, $x \mapsto (x \wedge a, x \wedge a^\perp)$.*

Proof Take any $x \in L[0,a]$. Then:

$$x \vee (a^\perp \wedge a) = x \vee 0 = x = (x \vee a^\perp) \wedge (x \vee a) = (x \vee a^\perp) \wedge a.$$

From [6, Lemma 29.10] we obtain that $L[0,a]$ is an ortholattice. To verify the second part of the statement we only need to show that Φ preserves the orthocomplementation. By Lemma 2 we have:

$$\begin{aligned}\Phi(x)^\perp &= (x \wedge a, x \wedge a^\perp)^\perp = ((x \wedge a)', (x \wedge a^\perp)') = ((x \wedge a)^\perp \wedge a, (x \wedge a^\perp)^\perp \wedge a^\perp)\\ &= ((x^\perp \vee a^\perp) \wedge a, (x^\perp \vee a) \wedge a^\perp) = (x^\perp \wedge a, x^\perp \wedge a^\perp) = \Phi(x^\perp)\end{aligned}$$

$\square$

Lemma 6 *Let L be any ortholattice and $a \in L$. Then the following are equivalent:*

(i) *a is a central element.*
(ii) *$x = (x \wedge a) \vee (x \wedge a^\perp)$ for all $x \in L$.*
(iii) *L is isomorphic to the direct product of the ortholattices $L[0,a] \times L[0,a^\perp]$.*
(iv) *$(x \vee a) \wedge y = (x \wedge y) \vee (a \wedge y)$ for all $x, y \in L$.*

From Lemma 6, (iv) and Proposition 1 we conclude that, for an ortholattice L, $Z(L)$ is a Boolean subalgebra of L.

⋆

Example 2 For any Boolean lattice L, we have $Z(L) = L$. In particular, $Z(\mathbf{2}) = \mathbf{2}$ and $Z(\mathbf{MO}_1) = \mathbf{MO}_1 \cong \mathbf{2} \times \mathbf{2}$. Moreover, we have $Z(\mathbf{MO}_n) = \{0, 1\}$ for all $n \geq 2$.

Irreducibility

In this part, we shall compile basic definitions and facts concerning irreducibility in lattices and ortholattices.

Definition 2 A bounded lattice L is called *irreducible* if $Z(L) = \{0, 1\}$.

In other words, L is irreducible if it is not isomorphic to a product of two non-trivial lattices where both have more than one element. Otherwise, we say that L is *reducible*.

Theorem 7 *Let L be any bounded lattice and $x \in Z(L)$. Then $L[0, x]$ is irreducible if and only if $x \in \mathcal{A}\big(Z(L)\big)$.*

Proof Let $L[0, x]$ be reducible. There is a non-trivial central element $y \in Z\big(L[0, x]\big)$. From Lemma 4 we know that $y \in Z(L)$. Since $0 < y < x$, x is not an atom of $Z(L)$.

Conversely, suppose that x is not an atom of $Z(L)$. Then there exists $y \in Z(L)$ such that $0 < y < x$. Then by Lemma 3, $y \in Z\big(L[0, x]\big)$. This means that $L[0, x]$ is reducible. □

By Lemma 5, an ortholattice L is irreducible if and only if its lattice reduct is irreducible. We need to mention also the following important theorem [3].

Theorem 8 *The only non-trivial finite irreducible modular ortholattices are the two-element Boolean algebra* $\mathbf{2}$ *and* $\mathbf{MO}_n$ *for* $n \geq 2$.

Since every modular ortholattice of finite height satisfies the assumptions of [6, Theorem 16.6] and of Lemma 5 we obtain the following theorem.

Theorem 9 *Let L be a modular ortholattice of finite height. Then L is isomorphic to the direct product of finitely many irreducible modular ortholattices $L[0, z]$ of height at most 2, where $z \in \mathcal{A}(Z(L))$ and the orthocomplementation $'$ in $L[0, z]$ is defined by $x' = x^{\perp} \wedge z$, for all $x \in L[0, z]$.*

3 Orthosets

The central issue in this paper is the study of orthosets of finite rank. In this part, we recall some notions and properties mainly from [9].

Definition 3 An *orthoset* (or *orthogonality space*) is a non-empty set X equipped with a symmetric, irreflexive binary relation $\perp$, called the *orthogonality relation*.

A subset of an orthoset $(X, \perp)$ consisting of mutually orthogonal elements is called a $\perp$-*set*. The supremum of the cardinalities of $\perp$-sets is called the *rank* of $(X, \perp)$. We put $\not\perp = (X \times X) \setminus \perp$.

For any $A, B \subseteq X$, we write $A \perp B$ if $a \perp b$ for all $a \in A$ and $b \in B$.

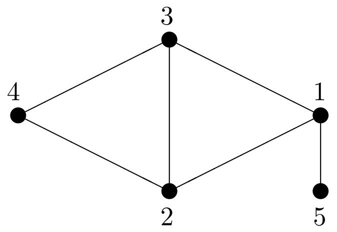
Notation 10 *Any orthoset $(X, \perp)$ is obviously uniquely determined by the set of its maximal $\perp$-sets. We denote this set by $M(X, \perp)$.*

Example 3 Let $X = \{1, 2, 3, 4, 5\}$. We have an orthoset $(X, \perp)$ such that $M(X, \perp) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 5\}\}$. We can visualize $(X, \perp)$ as follows:

Throughout the paper (whenever it is possible) we will picture an orthoset as a hypergraph in which vertices are the elements of an orthoset and edges are the straight lines corresponding to the elements of $M(X, \perp)$.

For orthosets $(X, \perp)$ and $(Y, \perp)$, we call a map $f \colon X \to Y$ a *homomorphism* if f is orthogonality preserving. Moreover, f is called an *isomorphism* if f is bijective and both f and f^{-1} are homomorphisms.

From graph theoretical point of view, orthosets can be considered the same as undirected graphs where adjacents are orthogonal elements, (maximal) cliques are (maximal) $\perp$-sets, and homomorphisms (isomorphisms) of orthosets are graph homomorphisms (isomorphisms). Thus the orthoset given in Example 3 can also be drawn as an undirected graph on the right.



If L is a complete atomistic ortholattice we have an orthoset $\big(\mathcal{A}(L), \perp\big)$ where the orthogonality relation is inherited from L.

The *orthogonal complement* of a subset $A \subseteq X$ of an orthoset $(X, \perp)$ is defined to be

$$A^{\perp} = \{x \in X \colon x \perp a, \text{ for all } a \in A\}.$$

We then have a closure operator on $\mathcal{P}(X)$ that maps any $A \subseteq X$ to $A^{\perp\perp}$. We call the subsets closed with respect to $^{\perp\perp}$ *orthoclosed* and denote the collection of all orthoclosed subsets by $\mathcal{C}(X, \perp)$. An orthoset is called *point-closed* if $\{x\}^{\perp\perp} = \{x\}$, for all $x \in X$.

Lemma 11 ([9]) *If $(X, \perp)$ is an orthoset then $\mathcal{C}(X, \perp)$ is a complete ortholattice.*

As shown in [10, Proposition 2] we have the following.

Lemma 12 *Let L be a complete atomistic ortholattice. Then $\mathcal{A}\big(\mathcal{C}(X, \perp)\big)$ is a point-closed orthoset. Moreover, $\omega\colon L \to \mathcal{C}(\mathcal{A}(L))$, $a \mapsto \{p \in \mathcal{A}(L) \mid p \leq a\}$ is an isomorphism of ortholattices. Conversely, let $(X, \perp)$ be a point-closed orthoset. Then $\mathcal{C}(X, \perp)$ is a complete atomistic ortholattice. The map $X \to \mathcal{A}(\mathcal{C}(X, \perp)), e \mapsto \{e\}$ is an isomorphism of orthosets.*

Hence there is a one-to-one correspondence between point-closed orthosets and complete atomistic ortholattices.

Prelinear and linear orthosets

The following condition was introduced in two equivalent ways in [9, §3] and [7, §5].

Definition 4 An orthoset $(X, \perp)$ is called *prelinear* if it satisfies the following condition for any two distinct elements $e, f \in X$:

(L1) if $e \not\perp f$, there exists a $g \perp e$ such that $\{e, f\}^\perp = \{e, g\}^\perp$.

Lemma 13 ([9]) *If an orthoset $(X, \perp)$ is prelinear, then it is point-closed. In particular, $\mathcal{C}(X, \perp)$ is atomistic, and the atoms are singletons $\{e\}$, for all $e \in X$.*

Moreover:

Theorem 14 *Let $(X, \perp)$ be a prelinear orthoset of finite rank m. Then $\mathcal{C}(X, \perp)$ is a modular ortholattice of height m. Conversely, if L is any modular ortholattice of finite height m, then the orthoset $\big(\mathcal{A}(L), \perp\big)$ is a prelinear orthoset of rank m.*

Proof The first part of the proof is given in [9, Lemma 3.5]. Conversely, let L be a modular ortholattice of finite height m.

Suppose that we have $p, q \in \mathcal{A}(L)$ such that $p \neq q$. We need to find $p \perp p'$ such that $p \vee p' = p \vee q$.

Since L is modular we obtain from $p \leq p \vee q$ that $p \vee \big(p^\perp \wedge (p \vee q)\big) = p \vee q$. Let us put $p' = p^\perp \wedge (p \vee q)$. Then $p \vee p' = p \vee q$ and $0 < p' < p \vee q$. Since the height of $L[0, p \vee q]$ is 2, we conclude that p' is an atom of L such that $p \perp p'$.

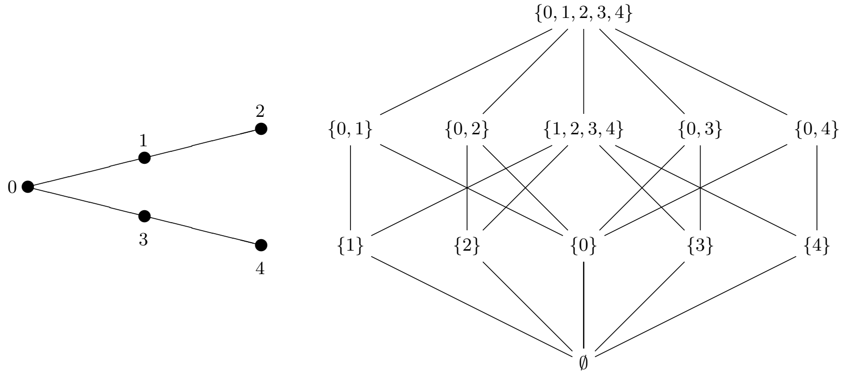
Let Y be any maximal orthogonal subset of $\big(\mathcal{A}(L), \perp\big)$. Then Y has at most m elements, because otherwise we could construct a chain of length $> m$ in L. Suppose that $Y = \{y_1, y_2, \ldots, y_n\}$. By the covering property, we get a chain

$$0 \lessdot y_1 \lessdot y_1 \vee y_2 \lessdot (y_1 \vee y_2) \vee y_3 \lessdot \cdots \lessdot (\ldots(y_1 \vee y_2) \vee \ldots) \vee y_n = 1$$

and by [6, Remark 8.6], we get $n = m$. □

Hence an orthoset $(X, \perp)$ is a prelinear orthoset of finite rank m, if and only if $\mathcal{C}(X, \perp)$ is an atomistic, modular ortholattice of height m. We therefore obtain a one-to-one correspondence between prelinear orthosets of finite rank and modular ortholattices of finite height.

Example 4 Let $(X, \perp)$ be an orthoset with maximal $\perp$-sets $\{0, 1, 2\}$ and $\{0, 3, 4\}$. Note that $(X, \perp)$ is prelinear. $(X, \perp)$ and the corresponding ortholattice $\mathcal{C}(X, \perp)$ are given as follows:



Definition 5 A prelinear orthoset $(X, \perp)$ is called *linear* if it satisfies the following condition:

(L2) if $e \perp f$, there exists a $g \neq e$, $g \not\perp e$ such that $\{e, f\}^{\perp} = \{e, g\}^{\perp}$.

In other words, for an orthoset $(X, \perp)$ to be linear means the following: for any two distinct elements $e, f \in X$, there is a third element g such that $\{e, f\}^{\perp} = \{e, g\}^{\perp}$ and exactly one of f and g is orthogonal to e.

Theorem 15 *Let $(X, \perp)$ be a linear orthoset of finite rank m. Then $\mathcal{C}(X, \perp)$ is an irreducible modular ortholattice of height m. Conversely, let L be an irreducible modular ortholattice of finite height m. Then the orthoset $\big(\mathcal{A}(L), \perp\big)$ is a linear orthoset of rank m.*

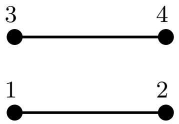
Proof Let $(X, \perp)$ be a linear orthoset of rank m. By Theorem 14, $\mathcal{C}(X, \perp)$ is a modular ortholattice of height m. Moreover, for any distinct elements $e, f \in X$, $\{e, f\}^{\perp\perp}$ contains a third atom. That is, the join of two distinct atoms of $\mathcal{C}(X, \perp)$ contains a third atom. Hence, by [6, Lemma 16.6], $\mathcal{C}(X, \perp)$ is irreducible.

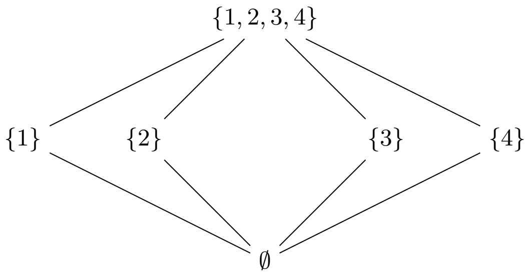
Conversely, let L be an irreducible modular ortholattice of height m. Then by Theorem 14 we obtain that $(\mathcal{A}(L), \perp)$ is a prelinear orthoset of finite rank m. Again by [6, Lemma 16.6], the join of two atoms of L contains a third one. This means that for any orthogonal atoms e and f of L, $e \vee f$ contains a third atom. It then follows $e \vee f = e \vee g$. We conclude that $(\mathcal{A}(L), \perp)$ is linear. □

We established a one-to-one correspondence between linear orthosets of finite ranks and irreducible modular ortholattices of finite height.

Remark 1 Summarily, for an orthoset $(X, \perp)$ of finite rank, the condition (L2) corresponds to the irreducibility, while the condition (L1) corresponds to the modularity and atomisticity of $\mathcal{C}(X, \perp)$.

Example 5 Let $(X, \perp)$ be an orthoset with maximal $\perp$-sets $\{1, 2\}$ and $\{3, 4\}$ which is linear. $(X, \perp)$ and the corresponding ortholattice $\mathcal{C}(X, \perp)$ are given below:





In fact, $\mathcal{C}(X, \perp)$ is isomorphic to $\mathbf{MO}_2$ which is irreducible, modular, and of height two.

4 The structure theorem for prelinear orthosets

In this section, we give a lattice theoretical approach to prelinear orthosets.

Direct product of prelinear orthosets

For this subsection only, $\mathbf{X}_i$ denotes an orthoset $(X_i, \perp_i)$, $i \in \{1, 2\}$. For the sake of simplicity, throughout the subsection, we assume that the orthogonality spaces in question are pairwise disjoint.

We have the following immediate statement (see also [8, Proposition 4.7.]).

Proposition 16 *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two disjoint orthosets. We put*

$$\perp = \perp_1 \cup \perp_2 \cup X_1 \times X_2 \cup X_2 \times X_1 \ .$$

Then $\perp$ defines an orthogonality relation on the set $X_1 \cup X_2$.

Definition 6 The orthoset $(X_1 \cup X_2, \perp)$ is called the *direct product* of orthosets and denoted by $\mathbf{X}_1 \sqcup \mathbf{X}_2$.

Note that the construction given in Definition 6 is associative. The following statements directly follow from Definition 6.

Lemma 17 *The set of all maximal $\perp$-sets of $\mathbf{X}_1 \sqcup \mathbf{X}_2$ is:*

$$M(\mathbf{X}_1 \sqcup \mathbf{X}_2) := \{D_1 \cup D_2 \mid D_1 \in M(\mathbf{X}_1), D_2 \in M(\mathbf{X}_2)\}$$

and $|M(\mathbf{X}_1 \sqcup \mathbf{X}_2)| = |M(\mathbf{X}_1)| \cdot |M(\mathbf{X}_2)|$. *Moreover, if* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are of finite ranks* m *and* n*, then* $\mathbf{X}_1 \sqcup \mathbf{X}_2$ *is of rank* $m+n$.

Proposition 18 *Let* $a, b \in X_1 \cup X_2$. *Then the following are equivalent:*

(i) $a \not\perp b$,

(ii) $a, b \in X_i$ *and* $a \not\perp_i b$*, where* $i \in \{1,2\}$.

Moreover, $\{a\}^{\perp} = X_j \cup \{a\}^{\perp_i}$ *where* $a \in X_i$, $i \neq j$ *and* $i, j \in \{1,2\}$.

Lemma 19 *Considering* $\mathbf{X}_1 \sqcup \mathbf{X}_2$*, we have:*

(i) $Y^{\perp} = (Y \cap X_1)^{\perp_1} \cup (Y \cap X_2)^{\perp_2}$*, for all* $Y \subseteq X_1 \cup X_2$.

(ii) $\{a,b\}^{\perp} = X_j \cup \{a,b\}^{\perp_i}$*, if* $a, b \in X_i$*, where* $i, j \in \{1,2\}$ *and* $i \neq j$.

(iii) $\{a,b\}^{\perp} = \{a\}^{\perp_i} \cup \{b\}^{\perp_j}$*, if* $a \in X_i$ *and* $b \in X_j$*, where* $i, j \in \{1,2\}$ *and* $i \neq j$.

Corollary 20 *Considering* $\mathbf{X}_1 \sqcup \mathbf{X}_2$*,* $A \subseteq X_1$ *and* $B \subseteq X_2$*, we have:*

(i) $(A \cup B)^{\perp} = A^{\perp_1} \cup B^{\perp_2}$.

(ii) $(A \cup B)^{\perp\perp} = A^{\perp_1\perp_1} \cup B^{\perp_2\perp_2}$.

(iii) *If* $A \in \mathcal{C}(X_1, \perp_1)$ *and* $B \in \mathcal{C}(X_2, \perp_2)$*, then* $(A \cup B) \in \mathcal{C}(X_1 \cup X_2, \perp)$.

The following theorem establishes a connection between direct products of orthosets and direct products of lattices of orthoclosed subsets.

Theorem 21 *We have*

$$\mathcal{C}(X_1 \cup X_2, \perp) \cong \mathcal{C}(X_1, \perp_1) \times \mathcal{C}(X_2, \perp_2).$$

Moreover, if $\mathcal{C}(X, \perp) \cong \mathcal{C}(X_1, \perp_1) \times \mathcal{C}(X_2, \perp_2)$ *then* $(X, \perp) \cong \mathbf{X}_1 \sqcup \mathbf{X}_2$.

Proof Let $Y \in \mathcal{C}(X_1 \cup X_2, \perp)$. From Corollary 20 we can write

$$Y = Y^{\perp\perp} = \big((Y \cap X_1) \cup (Y \cup X_2)\big)^{\perp\perp} = (Y \cap X_1)^{\perp_1\perp_1} \cup (Y \cap X_2)^{\perp_2\perp_2}$$

from which we get $(Y \cap X_i)^{\perp\perp} = (Y \cap X_i)$ and therefore $(Y \cap X_i) \in \mathcal{C}(X_i, \perp_i)$ for $i \in \{1,2\}$. Now let us define

$$\begin{array}{rcl} f\colon \mathcal{C}(X_1 \cup X_2, \perp) & \to & \mathcal{C}(X_1, \perp_1) \times \mathcal{C}(X_2, \perp_2) \\ Y & \mapsto & (Y \cap X_1, Y \cap X_2) \end{array}$$

It is a transparent procedure to check that f is bijective and order preserving.

Suppose now that $\mathcal{C}(X, \perp) \cong \mathcal{C}(X_1, \perp_1) \times \mathcal{C}(X_2, \perp_2)$. We can identify X with $X_1 \cup X_2$. Clearly, any atom of $\mathcal{C}(X_1, \perp_1)$ is orthogonal to any atom of $\mathcal{C}(X_2, \perp_2)$. Let a and b are atoms of $\mathcal{C}(X_1, \perp_1)$. Then they are orthogonal in $\mathcal{C}(X_1, \perp_1)$, if and only if, they are orthogonal in $(X_1, \perp_1)$. The same arguments are valid for atoms a and b of $\mathcal{C}(X_2, \perp_2)$. Hence $(X, \perp) \cong \mathbf{X}_1 \sqcup \mathbf{X}_2$. □

Theorem 22 $\mathbf{X}_1 \sqcup \mathbf{X}_2$ *is prelinear if and only if* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are prelinear.*

Proof Assume first that $\mathbf{X}_1$ and $\mathbf{X}_2$ are prelinear. Suppose that $e \not\perp f$. From Proposition 18 we obtain that there is $i \in \{1,2\}$ such that $e, f \in X_i$ and $e \not\perp_i f$. Since $(X_i, \perp_i)$ is a prelinear orthoset, there exists $g \in X_i$ and $e \perp_i g$ such that $\{e,f\}^{\perp_i} = \{e,g\}^{\perp_i}$. From Lemma 19 we conclude for $j \neq i$

$$\{e,f\}^{\perp} = X_j \cup \{e,f\}^{\perp_i} = X_j \cup \{e,g\}^{\perp_i} = \{e,g\}^{\perp}.$$

Recall that $e \perp g$ follows from $e \perp_i g$ in the sense of Definition 6.

Conversely, let $\mathbf{X}_1 \sqcup \mathbf{X}_2$ be prelinear. Assume now that $e, f \in X_1$ and $e \not\perp_1 f$. From prelinearity of $\mathbf{X}_1 \sqcup \mathbf{X}_2$ we find $g \in X_1 \cup X_2$ such that $X_2 \subseteq \{e,f\}^{\perp} = \{e,g\}^{\perp}$ and $e \perp g$. Hence $g \in X_1$, $e \perp_1 g$ and $\{e,f\}^{\perp_1} = \{e,g\}^{\perp_1}$. The case $e, f \in X_2$ and $e \not\perp_2 f$ follows by similar considerations. □

Definition 7 A prelinear orthoset is called *irreducible* if it is not isomorphic to a direct product of two non-trivial prelinear orthosets.

Lemma 23 *A prelinear orthoset* $(X, \perp)$ *of finite rank is irreducible, if and only if,* $\mathcal{C}(X, \perp)$ *is irreducible.*

Proof A prelinear orthoset $(X, \perp)$ of finite rank is reducible, if and only if, there is an isomorphism $(X, \perp) \cong \mathbf{X}_1 \sqcup \mathbf{X}_2$, and both $\mathbf{X}_1$ and $\mathbf{X}_2$ are prelinear orthosets of finite rank, if and only if, $\mathcal{C}(X_1 \cup X_2, \perp) \cong \mathcal{C}(X_1, \perp_1) \times \mathcal{C}(X_2, \perp_2)$, and both $\mathcal{C}(X_1, \perp_1)$ and $\mathcal{C}(X_2, \perp_2)$ are modular ortholattices of finite height, if and only if, $\mathcal{C}(X, \perp)$ is reducible. □ ⋆

Example 6 Let $X_1 = \{1,2,3,4\}$ and $X_2 = \{5,6,7,8\}$. Suppose that we have linear orthosets $(X_1, \perp_2)$ and $(X_2, \perp_2)$ which are both isomorphic to the orthoset from Example 5.

Consider the direct product $\mathbf{X_1} \sqcup \mathbf{X_2}$. We have $1 \perp 5$ and $\{1,5\}^{\perp} = \{2,6\}$. However, there is no $g \in X_1 \cup X_2$ such that $1 \not\perp g$ and $\{1,g\}^{\perp} = \{2,6\}$.

Remark 2 From Example 6 we obtain that Theorem 22 is not true for linear orthosets. Indeed, the direct product of two linear non-trivial orthosets is never linear.

Theorem 24 *Let* $(X, \perp)$ *be a prelinear orthoset. The following are equivalent:*

(i) *The orthoset* $(X, \perp)$ *is irreducible.*
(ii) *The ortholattice* $\mathcal{C}(X, \perp)$ *is irreducible.*
(iii) $(X, \perp)$ *is linear.*

Proof (i)⇔(ii). Transparent by Lemma 23.

(ii)⇔(iii). Obvious from Theorem 15. □

The core of a point-closed orthoset

Lemma 25 *Let $(X, \perp)$ be a point-closed orthoset and $B \in \mathcal{C}(X, \perp)$ a central element. Then*

$$B \vee D = B \cup D$$

for all $D \in \mathcal{C}(X, \perp)$.

Proof We use MacLaren characterization. Let $D \in \mathcal{C}(X, \perp)$ and $y \in B \vee D$. Since $\{y\} \in \mathcal{C}(X, \perp)$ is an atom, we can write:

$$\{y\} = \{y\} \wedge \Big(B \vee D\Big) = \Big(\{y\} \wedge B\Big) \vee \Big(\{y\} \wedge D\Big).$$

which follows that $y \in B \cup D \subseteq B \vee D$ and therefore we have $B \vee D = B \cup D$. □

The statement of Lemma 25 is not true for arbitrary $B \in \mathcal{C}(X, \perp)$.

Example 7 Let $(X, \perp)$ be the prelinear orthoset given in Example 4. We have:

$$\{1\} \vee \{2\} = \{1, 2, 3, 4\} \neq \{1, 2\},$$

since both $\{1\}$ and $\{2\}$ are not cores of $(X, \perp)$.

Definition 8 Let $(X, \perp)$ be an orthoset. A subset $B \subseteq X$ is called a *core* if $B^{\perp} = X \setminus B$.

Clearly, every core is orthoclosed, its complement is a core, and the union and intersection of two cores is again a core. Moreover, if the singleton $\{a\}$ is a core of $(X, \perp)$ then any $D \subseteq X$ with $D \not\subseteq \{a\}^{\perp}$ contains a.

There is a one-to-one correspondence between the cores of a point-closed orthoset $(X, \perp)$ and the central elements of $\mathcal{C}(X, \perp)$.

Lemma 26 *Let $(X, \perp)$ be a point-closed orthoset and $B \subseteq X$. The following are equivalent:*

(i) *B is a core of $(X, \perp)$.*

(ii) *B is a central element of $\mathcal{C}(X, \perp)$.*

Proof (i) $\Rightarrow$ (ii): Let B be a core of $(X, \perp)$ and $Y \in \mathcal{C}(X, \perp)$. Then we can write $B \cup B^{\perp} = X$ by definition. It follows:

$$Y = Y \wedge X = Y \wedge (B \cup B^{\perp}) = (Y \wedge B) \cup (Y \wedge B^{\perp}) \leq (Y \wedge B) \vee (Y \wedge B^{\perp}) \leq Y$$

from which we get $Y = (Y \wedge B) \vee (Y \wedge B^{\perp})$ and hence B is a central element.

(ii) $\Rightarrow$ (i): Let B be a central element of $\mathcal{C}(X, \perp)$. From Lemma 25 we conclude $B \vee D = B \cup D$ for all $D \in \mathcal{C}(X, \perp)$. It follows that $B^{\perp} = X \setminus B$, i.e., B is a core of $(X, \perp)$. □

The structure theorem

Theorem 27 *Any prelinear orthoset of finite rank is isomorphic to a direct product of finitely many linear orthosets of finite rank.*

Proof Let $(X, \perp)$ be a prelinear orthoset of finite rank. Then we have the corresponding complete modular ortholattice $L := \mathcal{C}(X, \perp)$ of finite height. We know from Theorem 9 that L is the direct sum of finitely many irreducible sublattices $\{L[0, z_\alpha] \mid \alpha \in I\}$ of finite height where $z_\alpha \in Z(L)$ for $\alpha \in I$. Hence $(X, \perp)$ is isomorphic to a direct product of finitely many linear orthosets $(\mathcal{A}(L[0, z_\alpha]), \perp)$ by Theorem 21. □

5 Orthosets with low rank

In this section we will apply some lattice theoretical properties to orthosets and obtain some structural results for prelinear and linear spaces of rank n where $n \in \{1, 2, 3\}$.

Orthosets of rank 1

Let X be a singleton. There is a unique orthoset based on X. In that case the orthogonality relation $\perp$ is the empty set and we call $(X, \perp)$ *trivial*. Note that $(X, \perp)$ is both prelinear and linear.

Orthosets of rank 2

Lemma 28 *Let L be a modular ortholattice of height two. Then L is isomorphic to $\mathbf{MO}_n$, where n is a non-zero cardinal number.*

Proof Let $p \in L \setminus \{0, 1\}$. Then $p \in \mathcal{A}(L)$. Since L has height two, $p^\perp \in \mathcal{A}(L)$. Thus we can write

$$\mathcal{A}(L) = \bigcup_{i \in I} \{x_i, x_i^\perp\}, \quad L = \mathcal{A}(L) \cup \{0, 1\}.$$

We conclude that L is a horizontal sum of I copies of $\mathbf{MO}_1$. Hence $L \cong \mathbf{MO}_n$, where $n = |I|$. □

Theorem 29 *An orthoset $(X, \perp)$ of rank 2 is prelinear if and only if the pairs of orthogonal elements form a partition of X into 2-element subsets.*

Proof Let $(X, \perp)$ be a prelinear orthoset of rank 2. By Lemma 28 $\mathcal{C}(X, \perp)$ is isomorphic to $\mathbf{MO}_n$. Then according to Lemma 12 we get an isomorphism between $(X, \perp)$ and the underlying orthoset $\big(\mathcal{A}(\mathbf{MO}_n), \perp\big)$ with the inherited orthogonality, in which the pairs of orthogonal elements form a partition into 2-element subsets.

Conversely, let $(X, \perp)$ be an orthoset in which the pairs of orthogonal elements form a partition of X into 2-element subsets. Suppose that $e \not\perp f$. Take $e^\perp \in X$ such that $e \perp e^\perp$. Since the pairs of orthogonal elements form a partition, we obtain $\{e, e^\perp\}^\perp = \emptyset = \{e, f\}^\perp$ which proves that $(X, \perp)$ is prelinear. □

Theorem 30 *An orthoset $(X, \perp)$ of rank 2 is linear if and only if the pairs of orthogonal elements form a partition of X into 2-element subsets and $|X| \geq 4$.*

Proof Let $(X, \perp)$ be a linear orthoset of rank 2. Then $\mathcal{C}(X, \perp)$ is an irreducible modular ortholattice of height two. We also know from Theorem 9 that the only irreducible modular ortholattice of height two are lattices isomorphic to $\mathbf{MO}_n$ where $n \geq 2$. The remaining part follows from Theorem 29. □

Orthosets of rank 3

Lemma 31 *Let L be a modular ortholattice of height three. If L is reducible, then it is isomorphic to $\mathbf{2} \times \mathbf{MO}_n$, where n is a non-zero cardinal number.*

Proof Let L be reducible. From Theorem 9 and Lemma 7, L is a direct product of irreducible sublattices $L[0, z_\alpha]$ where $z_\alpha \in Z(L)\setminus\{0, 1\}$ and $z_\alpha \in \mathcal{A}(Z(L))$. Hence we have exactly two non-trivial central elements $a, a^\perp$ which are an atom and a coatom, respectively. We conclude that L is isomorphic to $L[0, a] \times L[0, a^\perp]$. Clearly $L[0, a] \cong \mathbf{2}$, and from Lemma 28 we have $L[0, a^\perp] \cong \mathbf{MO}_n$. Therefore L is isomorphic to $\mathbf{2} \times \mathbf{MO}_n$. □

Lemma 32 *Let L be a modular ortholattice of height three with three atoms. Then L is isomorphic to $Z(L)$. Consequently L is isomorphic to the Boolean algebra $\mathbf{2}^3$.*

Proof From Lemma 31 we have that $L \cong \mathbf{2} \times \mathbf{MO}_n$. We conclude that $n = 1$. Since $\mathbf{MO}_1 \cong \mathbf{2} \times \mathbf{2}$ we obtain that $L \cong \mathbf{2}^3$. □

Lemma 33 *Let L be a modular ortholattice of height three with at least four atoms. Then there is at most one atom of L in $Z(L)$.*

Proof Let $p, q \in \mathcal{A}(L) \cap Z(L)$ be distinct. Then $L \cong L[0, p] \times L[0, p^\perp]$ and $q \leq p^\perp$. Since $L[0, p^\perp]$ has height 2 and $q \in Z(L[0, p^\perp])$ by Lemma 3 we know that $L[0, p^\perp]$ is reducible. There is an atom $r \in Z(L[0, p^\perp])$ such that $L[0, p^\perp] \cong L[0, q] \times L[0, r] \cong \mathbf{MO}_1$ by Lemma 28. Hence L has exactly 3 atoms, a contradiction. □

Example 8 [2] Let A and B be disjoint sets (with the same cardinality) with a bijection $\varphi\colon A \to B$, and fix an element $\bar{x} \notin A \cup B$. We denote by $\mathbf{3}(A, B, \varphi)$ an orthoset in which the maximal orthogonal subsets are $\{\bar{x}, a, \varphi(a)\}, a \in A$. Then $\mathbf{3}(A, B, \varphi)$ is prelinear, has rank 3, and it is not linear.

We call an irreducible modular ortholattice of height 3 a *projective plane with orthogonality*.

Theorem 34 *An orthoset $(X, \perp)$ is prelinear of rank 3 if and only if one of the following holds:*

(i) *There is a bijection $\phi\colon A \to B$ between disjoint sets such that $(X, \perp)$ is isomorphic to $\mathbf{3}(A, B, \varphi)$.*

(ii) $\mathcal{C}(X, \perp)$ *is a projective plane with orthogonality.*

Proof Let $(X, \perp)$ be prelinear of rank 3. If $(X, \perp)$ is linear then $\mathcal{C}(X, \perp)$ is a projective plane with orthogonality.

Suppose now that $(X, \perp)$ is not irreducible. Let $|X| = 3$. From [2, Remark 26] we have $M(X, \perp) = \{X\}$ and the statement is valid.

Assume now that $|X| \geq 4$. Then there are disjoint non-empty orthoclosed subsets U and V such that $U \cup V = X$ and $U = V^{\perp}$ and $V = U^{\perp}$. Moreover, U and V are prelinear.

Note that either U or V is a singleton. Namely, if $|U| \geq 2$ and $|V| \geq 2$ then by prelinearity of U and V we obtain that there are $a, b \in U$, $c, d \in V$ such that $a \perp b$ and $c \perp d$, a contradiction with rank 3.

Hence we can assume that $U = \{\bar{x}\}$ for a suitable element $\bar{x} \in X$ and $V = \{\bar{x}\}^{\perp}$ (otherwise we interchange U with V). Evidently, V is an orthogonality space of rank 2. From Theorem 29 we obtain subsets $A, B \subseteq V$ and a bijection $\varphi\colon A \to B$ such that $\perp \cap (V \times V)= \{(a, \varphi(a)) \mid a \in A\} \cup \{(\varphi(a), a) \mid a \in A\}$. We conclude that $\{\bar{x}, a, \varphi(a)\}, a \in A$, are all maximal orthogonal subsets $(X, \perp)$, i.e., $(X, \perp) = \mathbf{3}(A, B, \varphi)$.

Since every projective plane with orthogonality and every orthoset of the form $\mathbf{3}(A, B, \varphi)$ are prelinear and of rank 3 the converse direction is evident. □

6 Finite orthosets

In this section we will deal with finite orthosets with an arbitrary rank. We primarily prove that there is only one type of non-trivial finite linear orthosets. Afterwards, we find a natural formula which allows us to compute the number of all possible prelinear orthosets with a given number of elements.

Finite linear orthosets

Lemma 35 *There is no finite irreducible atomistic modular ortholattice of height* ≥ 3.

Proof We know from [4] that the only finite, irreducible, modular ortholattices are the Boolean algebra $\mathbf{2}$ of height one and $\mathbf{MO}_n$ of height two for $n \geq 2$. □ ⋆

Theorem 36 *There is no finite linear orthoset of rank* ≥ 3. *In fact if a linear orthoset* $(X, \perp)$ *is finite, then one of the following holds:*

(i) $(X, \perp)$ *is trivial.*

(ii) *The pairs of orthogonal elements form a partition of* X *into 2-element subsets.*

Finite prelinear orthosets

Consider the two-element Boolean algebra $\mathbf{2}$ which is a modular ortholattice with 1 atom. We have a one-to-one correspondence with the trivial prelinear orthoset. Now consider the finite product lattice $\mathbf{2}^n$ which has n atoms. It is again a modular ortholattice. In this case, we have a one-to-one correspondence with a prelinear orthoset on the set $X = \{x_1, \ldots, x_n\}$ in which every singleton is a core. In other words, there is a one maximal orthogonal subset which is X itself.

Consider now the modular ortholattice $\mathbf{MO}_n$ with $2n+2$ elements. We obtain a one-to-one correspondence between $\mathbf{MO}_n$ and a prelinear orthoset on an n-element set X in which the pairs of orthogonal elements form a partition of X into 2-element subsets.

Let us consider the product lattice $L = \prod_{i=1}^k \mathbf{MO}_{a_i}$ where a_i is a non-zero cardinal number, $1 \leq i \leq k$. We know that $\mathcal{A}(L)$ has $\sum_{i=1}^k 2a_i$ elements. Let us figure out the prelinear orthoset $(X, \perp)$ such that $\mathcal{C}(X, \perp) \cong L$. Following Theorem 21 we obtain a one-to-one correspondence between between L and the orthoset $(X, \perp)$ which is a direct product of prelinear orthosets $(X_i, \perp)$ where X_i has $2a_i$ elements and the pairs of orthogonal elements form a partition of X_i into 2-element subsets.

Notation 37 *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We put: $\mathbf{n} = \{m \in \mathbb{N} \mid 1 \leq m \leq n\}$ and $\lfloor x \rfloor$ for a lower integer part of x.*

A partition of a positive integer *n is a multiset $\{n_1, \ldots, n_k\}$ of positive integers such that $n_1 + \cdots + n_k = n$. The total number of partitions of n is denoted $p(n)$. We call $p(n)$ a* partition function.

From the above considerations we immediately obtain the following.

Theorem 38 *For a given finite modular ortholattice $L = \mathbf{2}^n \times \prod_{i=1}^k \mathbf{MO}_{a_i}$ define a set $X = \mathbf{n} \cup \bigcup_{i=1}^k \mathcal{A}(\mathbf{MO}_{a_i})$ and an orthoset $(X, \perp)$ such that any maximal $\perp$-set is of the form*

$$\mathbf{n} \cup \{x_1, x_1', x_2, x_2', \cdots, x_k, x_k'\} \tag{3}$$

where $x_i \in \mathcal{A}(\mathbf{MO}_{a_i})$ and x_i' is the orthocomplement of x_i in $\mathbf{MO}_{a_i}$. Then $(X, \perp) \cong (\mathcal{A}(L), \perp)$. Moreover, $M(X, \perp)$ has $\prod_{i=1}^k a_i$ elements, and also $|D| = n + 2k$, for all $D \in M(X, \perp)$.

Now we compute the number of all possible prelinear orthosets with a given number of elements.

Theorem 39 *Let $m \in \mathbb{N}$, $m \geq 2$. Then there are $p(\lfloor \frac{m}{2} \rfloor)$ mutually non-isomorphic prelinear orthosets on m elements.*

Proof Let $(X, \perp)$ be a prelinear orthoset such that $|X| = m$. Then $\mathcal{C}(X, \perp) = \mathbf{2}^n \times \prod_{i=1}^k \mathbf{MO}_{a_i}$ for some numbers $a_1, \ldots, a_k \geq 2$ and $n \geq 0$.

Assume first that m is even, that is, $m = 2l$ for some $l \geq 1$. Then $m = n + \sum_1^k 2a_i$. Clearly, $n = 2j$, $j \geq 0$ since m is even. We have a one-to-one correspondence between mutually non-isomorphic prelinear orthosets on m elements and partitions of l. Namely, the multiset

$\{\underbrace{1,\ldots,1}_{j\text{ times}}, a_1,\ldots,a_k\}$ is the respective partition of l such that the corresponding prelinear orthoset is a direct product of $2j$ singletons and k linear orthosets $(X_i,\perp_i)$ of rank 2, where $X_i \subseteq X$. Each X_i is partitioned into $a_i > 1$ many 2-element subsets of orthogonal elements, $1 \leq i \leq k$.

Assume now that m is odd, that is, $m = 2l+1$ for some $l \geq 1$. Then $n = 2j+1$, $j \geq 0$ since m is odd and $m = n + \sum_1^k 2a_i$. Then we have a one-to-one correspondence between mutually non-isomorphic prelinear orthosets on m elements and partitions of l. Here the multiset $\{\underbrace{1,\ldots,1}_{j\text{ times}}, a_1,\ldots,a_k\}$ is the respective partition of l such that the corresponding prelinear orthoset is a direct product of $2j+1$ singletons and k linear orthosets $(X_i,\perp_i)$ of rank 2.

Therefore the number of mutually non-isomorphic prelinear orthosets on m elements equals $p(\lfloor \frac{m}{2} \rfloor)$. □

Corollary 40 *Let $m \in \mathbb{N}$, $m \geq 2$. Then there are $p(\lfloor \frac{m}{2} \rfloor)$ mutually non-isomorphic modular ortholattices with m atoms.*

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