The semantics of fuzzy logics: two approaches to finite tomonoids

Thomas Vetterlein and Milan Petrík

Abstract Fuzzy logic generalises classical logic; in addition to the latter's truth values "false" and "true", the former allows also intermediary truth degrees. The conjunction is, accordingly, interpreted by an operation acting on a chain, making the set of truth degrees into a totally ordered monoid. We present in this chapter two different ways of investigating this type of algebras. We restrict to the finite case.

1 Introduction

The idea on which fuzzy logic is build is best understood in relationship with the canonical way in which reasoning is formalised: with classical propositional logic. The latter is the logic of "false" and "true" and propositions are evaluated in this two-element set. Among the connectives we find the logical "and", "or", and "not", interpreted in the well-known way. In addition to the two classical truth values, fuzzy logic uses intermediary degrees of truth. Usually, "false" and "true" are identified with the real numbers 0 and 1, respectively; the remaining real numbers serve as further truth degrees and may express relative tendencies.

The difficulty of this approach is that there is no straightforward way to tell how the logical connectives should be interpreted. We rather have to make a decision, for instance, about the interpretation of the conjunction. Different interpretations will in general lead to different logics. As a consequence of this situation, fuzzy logic has in fact emerged as a family of many-valued logics, each of which may bring its own

Thomas Vetterlein
Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz, Austria, e-mail: Thomas.Vetterlein@jku.at

Milan Petrík
Department of Mathematics, Faculty of Engineering, Czech University of Life Sciences, Prague, Czech Republic, e-mail: petrik@cs.cas.cz

challenges. According to a common agreement, the binary operation on the real unit interval taken for this purpose should be a t-norm: associative, commutative, possessing 1 as an identity, and monotone in each argument. If the set of truth values is not taken to be an uncountable set but for instance a finite chain, the operation should still fulfil the same algebraic conditions. It is natural to assume that the chain of truth degrees is a negative, commutative totally ordered monoid.

The present work is to be seen among the efforts of classifying these algebraic structures. A considerable amount of work has been done on this topic during recent years. In line with the given background, residuation has usually been additionally assumed and MTL-algebras were considered [12, 31]. Our paper [36] is devoted to MTL-algebras based on the real unit interval. For residuated lattices in general, see [3, 16]. MTL-chains fulfilling certain additional properties were considered in several works as well. For instance, MTL-chains with the weak cancellation property are the topic of [30] as well as [20]. Idempotent residuated chains are studied in [4]. The paper [22] deals with finite MTL-chains and their relationship to Abelian totally ordered groups.

The present chapter is devoted to the finite case. The tomonoids considered are assumed to be either finite, or at least to be finitely generated. We present two different approaches based on our work [37] and [35], respectively. We provide an introduction to the main ideas; further details can be found in the indicated papers. We note that our handbook chapter [40] offers two further approaches to the structures under consideration and thus can be seen to be continued by the present overview.

2 Totally ordered monoids

We investigate in this chapter the following structures.

Definition 1. An algebra $(L;+,0)$ is a *monoid* if (i) $+$ is an associative binary operation and (ii) 0 is an identity for $+$. A monoid $(L;+,0)$ is called *commutative* if $+$ is commutative.

A partial order $\leqslant$ on a monoid L is called *compatible* if, for any $a,b,c,d \in L$, $a \leqslant b$ and $c \leqslant d$ imply $a+c \leqslant b+d$. A structure $(L;\leqslant,+,0)$ such that $(L;+,0)$ is a monoid and $\leqslant$ is a compatible total order on L is called a *totally ordered monoid*, or *tomonoid* for short.

Moreover, a tomonoid $(L;\leqslant,+,0)$ is called *commutative* if so is its monoidal reduct. L is called *positive* if 0 is the bottom element. L is called *finitely generated* if L, as a monoid, is generated by finitely many elements.

For instance, let $[0,1]$ be the real unit interval and let $\oplus\colon [0,1]^2 \to [0,1]$ be a t-conorm, that is, associative, commutative, behaving neutrally w.r.t. 0, and monotone in each argument [28]. Then $([0,1];\leqslant,\oplus,0)$ is commutative, positive tomonoid. Similarly, let $L \subset [0,1]$ be a finite subset of $[0,1]$ containing 0 and 1 and let $\oplus\colon L^2 \to L$ be a discrete t-conorm [10]. This is equivalent to say that $(L;\leqslant,\oplus,0)$ is a finite, commutative, positive tomonoid.

We have written tomonoids in the additive way; alternatively, we may deal with the dual structures. In this case, the order is reversed and the multiplicative notation is used. In particular, the monoidal operation is then denoted by a product-like symbol and the monoidal identity by 1. The aforementioned examples would in this case become tomonoids based on a t-norm or a discrete t-norm, respectively. The choice of order and notation is not solely a matter of taste. In many-valued logics, a larger value corresponds to a higher degree of presence and hence the multiplicative notation is common. In the context of free monoids, in contrast, the additive notation is predominant. Within the present chapter, both possibilities will be made use of.

A tomonoid consisting of the monoidal identity alone is called *trivial*. We will tacitly assume throughout this paper that all tomonoids are non-trivial. A set of generators of a (non-trivial) tomonoid L will be understood to be a non-empty, finite set of elements distinct from 0 that generate L as a monoid.

Congruences of tomonoids are defined as follows; cf. [13]. Recall that a subset C of a poset is called *convex* if $a, c \in C$ and $a \leqslant b \leqslant c$ imply $b \in C$.

Definition 2. Let $(L; \leqslant, +, 0)$ be a tomonoid. A *tomonoid congruence* on L is a congruence $\approx$ of L as a monoid such that all $\approx$-classes are convex. On the quotient $\langle L \rangle_{\approx}$, we then denote the operation induced by $+$ again by $+$ and, for $a, b \in L$, we let $\langle a \rangle_{\approx} \leqslant \langle b \rangle_{\approx}$ if $a \approx b$ or $a < b$.

We immediately check that this definition is as intended.

Lemma 1. *Let $\approx$ be a tomonoid congruence on a tomonoid $(L; \leqslant, +, 0)$. Then the quotient $(\langle L \rangle_{\approx}; \leqslant, +, \langle 0 \rangle_{\approx})$ is a tomonoid again. Furthermore, if L is commutative, positive, finitely generated, then so is $\langle L \rangle_{\approx}$, respectively.*

It is difficult to classify the congruences of tomonoids. There are, however, certain special types that allow an easy description. For instance, an ideal of a commutative, positive tomonoid induces a congruence in a natural way [3]. For a discussion of this type of congruences, see, e.g., [36]. Moreover, there is an order-theoretic analogue of a Rees quotient; this type of congruences will be central in the second part of this chapter.

3 Representation of tomonoids by direction cones

The first part of the present chapter is devoted to finitely generated, positive, commutative tomonoids; we will write "fg.p.c. tomonoids" for short. In particular, the finite, positive, commutative tomonoids, which correspond to the so-called discrete t-norms [10], are included in the discussion.

We investigate a particular way of representing such tomonoids. We are guided by the following ideas. First of all, any monoid can be identified with a congruence on a free monoid. Similarly, we may describe tomonoids by what we call monomial preorders. Second, the order of totally ordered Abelian groups is characterised by their cone. We introduce for tomonoids an analogous object; the so-called direction cones are certain subsets of $\mathbb{Z}^n$ that describe tomonoids and each fg.p.c. tomonoid is a quotient of a tomonoid arising in this way.

The results of this section originate from the paper [37], to which we refer for further details. A continuation of this work, in which the finite case is especially emphasised, can be found in [39].

3.1 Congruences and monomial preorders

Free commutative monoids play a central role in what follows. We identify the free commutative monoid over $n \geqslant 1$ elements with $\mathbb{N}^n$. The addition is defined pointwise and the identity is $\bar{0} = (0,\ldots,0)$, the n-tuple consisting of zeros only. We also define $u_i = (0,\ldots,0,1,0,\ldots,0)$, "1" being at the i-th position. Clearly then, $U(\mathbb{N}^n) = \{u_1,\ldots,u_n\}$ is a set of generators of $\mathbb{N}^n$.

We endow $\mathbb{N}^n$ with the componentwise natural order. That is, for $(a_1,\ldots,a_n)$, $(b_1,\ldots,b_n) \in \mathbb{N}^n$, we put

$$(a_1,\ldots,a_n) \trianglelefteq (b_1,\ldots,b_n) \quad \text{if} \quad a_1 \leqslant b_1,\ \ldots,\ a_n \leqslant b_n. \tag{1}$$

Clearly, $\trianglelefteq$ is a lattice order on $(\mathbb{N}^n;+,\bar{0})$ and $\trianglelefteq$ is compatible with the addition.

Fg.p.c. tomonoids can be conveniently described on the basis of the free commutative monoid $\mathbb{N}^n$ as follows.

We call a reflexive and transitive binary relation $\preccurlyeq$ on a set A a *preorder*. We write $a \prec b$ if $a \preccurlyeq b$ but not $b \preccurlyeq a$. Any preorder $\preccurlyeq$ gives rise to an equivalence relation $\approx$, called its *symmetrisation*, where $a \approx b$ if $a \preccurlyeq b$ and $b \preccurlyeq a$. We call the equivalence class of some a w.r.t. $\approx$ a $\preccurlyeq$*-class* and we denote it by $\langle a \rangle_{\preccurlyeq}$. The preorder $\preccurlyeq$ induces on the quotient $\langle A \rangle_{\preccurlyeq}$ a partial order, which we denote by $\preccurlyeq$ again.

We call a preorder $\preccurlyeq$ *total* if $a \preccurlyeq b$ or $b \preccurlyeq a$ for any pair $a, b \in A$. Moreover, we call $\preccurlyeq$ *positive* if $0 \prec a$ for all $a \neq 0$. Finally, if $\preccurlyeq$ is defined on a monoid $(L;+,0)$, we call $\preccurlyeq$ *compatible* if $a \preccurlyeq b$ implies $a + c \preccurlyeq b + c$.

In computational mathematics, the notion "monomial ordering" refers to compatible, positive, total orders on $\mathbb{N}^n$; see, e.g., [9]. Analogously, we call a preorder $\preccurlyeq$ on $\mathbb{N}^n$ *monomial* if $\preccurlyeq$ is compatible, positive, and total. The significance of monomial preorders becomes clear in the following proposition.

Proposition 1. *Let $\preccurlyeq$ be a monomial preorder on $(\mathbb{N}^n;+,\bar{0})$. Then its symmetrisation is a monoid congruence whose classes are convex and such that $\langle \bar{0} \rangle_{\preccurlyeq} = \{\bar{0}\}$. Moreover, $(\langle \mathbb{N}^n \rangle_{\preccurlyeq};\preccurlyeq,+,\{\bar{0}\})$ is a fg.p.c. tomonoid.*

Conversely, let $(L;\leqslant,+,0)$ be a fg.p.c. tomonoid; assume that the $n \geqslant 1$ elements $g_1,\ldots,g_n \in L\backslash\{0\}$ generate L. Let $\iota\colon \mathbb{N}^n \to L$ be the surjective monoid homomorphism determined by $\iota(u_i) = g_i$, $i = 1,\ldots,n$. For $a, b \in \mathbb{N}^n$ define

$$a \preccurlyeq b \quad \textit{if} \quad \iota(a) \leqslant \iota(b). \tag{2}$$

Then $\preccurlyeq$ *is a monomial preorder of* $\mathbb{N}^n$*, and* ι *induces an isomorphism between* $(\langle\mathbb{N}^n\rangle_{\preccurlyeq}; \preccurlyeq, +, \{\bar{0}\})$ *and* $(L; \leqslant, +, 0)$.

Proof. Let $\preccurlyeq$ be a monomial preorder on $\mathbb{N}^n$. Then, for $a,b,c,d \in \mathbb{N}^n$, $a \approx c$ and $b \approx d$ imply $a+b \approx c+d$ by the compatibility of $\preccurlyeq$; hence $\approx$ is a monoid congruence. As $\preccurlyeq$ is also positive, $\preccurlyeq$ extends $\trianglelefteq$, and it follows that the $\preccurlyeq$-classes are convex. Again by the positivity, the $\preccurlyeq$-class of $\bar{0}$ consists of $\bar{0}$ alone.

As $\preccurlyeq$ is compatible, the partial order $\preccurlyeq$ induced on $\langle\mathbb{N}^n\rangle_{\preccurlyeq}$ is compatible as well; that is, $(\langle\mathbb{N}^n\rangle_{\preccurlyeq}; \preccurlyeq, +, \langle\bar{0}\rangle_{\preccurlyeq})$ is a commutative pomonoid. Since, for any $a,b \in \mathbb{N}^n$, $a \preccurlyeq b$ or $b \preccurlyeq a$, $\langle\mathbb{N}^n\rangle_{\preccurlyeq}$ is actually a tomonoid. Moreover, since $\bar{0} \prec a$ for any $a \in \mathbb{N}^n \backslash \{\bar{0}\}$, $\langle\mathbb{N}^n\rangle_{\preccurlyeq}$ is a positive, commutative tomonoid, which is generated by the finitely many elements $\langle u_1\rangle_{\preccurlyeq}, \ldots, \langle u_n\rangle_{\preccurlyeq}$.

For the second part, assume that $(L; \leqslant, +, 0)$ is a fg.p.c. tomonoid and $g_1, \ldots, g_n \in L\backslash\{0\}$ generate L as a monoid. Let furthermore $\iota \colon \mathbb{N}^n \to L$ be as indicated and let $\preccurlyeq$ be defined by (2). By construction, $\preccurlyeq$ is transitive and reflexive, that is, a preorder. $\preccurlyeq$ is compatible because so is $\leqslant$ and ι is a monoid homomorphism. Moreover, $\preccurlyeq$ is positive because L is positive and hence $\iota(a) \leqslant 0$ holds only if $a = \bar{0}$. Hence $\preccurlyeq$ is a monomial preorder. Finally, for $a,b \in \mathbb{N}^n$, we have $a \approx b$ if and only if $a \preccurlyeq b$ and $b \preccurlyeq a$ if and only if $\iota(a) = \iota(b)$; hence ι induces an isomorphism as claimed.

We conclude that any monomial preorder $\preccurlyeq$ on $\mathbb{N}^n$ gives rise to a fg.p.c. tomonoid L. We call L in this case the tomonoid *represented by* $\preccurlyeq$.

Proposition 1 also states that, up to isomorphism, any fg.p.c. tomonoid L arises in this way from a monomial preorder. In other words, describing fg.p.c. tomonoids can be done by describing monomial preorders. This is what we will do in the sequel.

3.2 Tomonoids arising from totally ordered Abelian groups

The positive cones of totally ordered Abelian groups give rise to typical examples of fg.p.c. tomonoids. We will discuss these examples in some detail because they motivate our way of representing fg.p.c. tomonoids in general.

Definition 3. Let $(G; \leqslant, +, 0)$ be a totally ordered Abelian group and let $G^+ = \{g \in G \colon g \geqslant 0\}$ be its positive cone. Assume that G is generated by $g_1, \ldots, g_n \in G^+\backslash\{0\}$, where $n \geqslant 1$. Let L be the submonoid of G generated by $g_1, \ldots, g_n$ and let L be endowed with the total order inherited from G, with the group addition, and with the constant 0. Then we call $(L; \leqslant, +, 0)$ a *group cone tomonoid*.

Clearly, a group cone tomonoid is a fg.p.c. tomonoid. Note that in general we do not deal with the whole positive cone of a totally ordered Abelian group. In fact, the latter is in general not finitely generated even if the group is.

Group cone tomonoids are characterised by the following condition. We say that a fg.p.c. tomonoid L is *cancellative* if, for all $a,b,c \in L$, $a+c = b+c$ implies $a = b$. Note that in this case, for all $a,b,c \in L$, $a \leqslant b$ is equivalent to $a+c \leqslant b+c$.

Proposition 2. *A fg.p.c. tomonoid $(L;\leqslant,+,0)$ is a group cone tomonoid if and only if it is cancellative.*

Proof. The "only if" part follows from the construction of a group cone tomonoid.

To see the "if" part, let L be cancellative. Let G be the group consisting of the differences of elements of L; see, e.g., [14, Chapter II.2]. Viewing L as a subset of G, we introduce a total order on G as follows: for $a,b,c,d \in L$, we define $a-b \leqslant c-d$ if $a+d \leqslant b+c$ in L. Then $(G;\leqslant,+,0)$ is a totally ordered Abelian group, and $(L;\leqslant,+,0)$ is a subtomonoid of $(G^+;\leqslant,+,0)$. The assertion follows.

Group cone tomonoids correspond by Proposition 1 to particular monomial preorders. We call a preorder $\preccurlyeq$ on $\mathbb{N}^n$ *cancellative* if, for any $a,b,c \in \mathbb{N}^n$, $a \preccurlyeq b$ is equivalent to $a+c \preccurlyeq b+c$.

Proposition 3. *Let the fg.p.c. tomonoid L be represented by the monomial preorder $\preccurlyeq$ on $\mathbb{N}^n$. Then L is a group cone tomonoid if and only if $\preccurlyeq$ is cancellative.*

Proof. Let L be a group cone tomonoid. Then $(\langle \mathbb{N}^n \rangle_{\preccurlyeq};\preccurlyeq,+,\{\bar{0}\})$ is cancellative by Proposition 2. Thus, for $a,b,c \in \mathbb{N}^n$, we have $a \preccurlyeq b$ iff $\langle a \rangle_{\preccurlyeq} \preccurlyeq \langle b \rangle_{\preccurlyeq}$ iff $\langle a \rangle_{\preccurlyeq} + \langle c \rangle_{\preccurlyeq} \preccurlyeq \langle b \rangle_{\preccurlyeq} + \langle c \rangle_{\preccurlyeq}$ iff $\langle a+c \rangle_{\preccurlyeq} \preccurlyeq \langle b+c \rangle_{\preccurlyeq}$ iff $a+c \preccurlyeq b+c$, that is, $\preccurlyeq$ is cancellative.

Conversely, let $\preccurlyeq$ be cancellative. Then $(\langle \mathbb{N}^n \rangle_{\preccurlyeq};\preccurlyeq,+,\{\bar{0}\})$ is a cancellative fg.p.c. tomonoid and hence, by Proposition 2, a group cone tomonoid.

Recall next that the order of a partially ordered Abelian group $(G;\leqslant,+,0)$ is uniquely determined by its positive cone G^+. In fact, for any $g,h \in G$, $g \leqslant h$ if and only if $h-g \in G^+$. We may also view the positive cone of a partially ordered group as the set of all differences of elements g and h such that $g \leqslant h$; indeed, $G^+ = \{h-g\colon g,h \in G \text{ such that } g \leqslant h\}$.

We may use the same object to describe group cone tomonoids. We denote by $(\mathbb{Z}^n;+,\bar{0})$ the free Abelian group generated by $n \geqslant 1$ elements. Furthermore, $\trianglelefteq$ will be the partial order on $\mathbb{Z}^n$ defined according to (1): for $a,b \in \mathbb{Z}^n$, we put $a \trianglelefteq b$ if $a+c = b$ for some $c \in \mathbb{N}^n$. Then $(\mathbb{Z}^n;\trianglelefteq,+,\bar{0})$ is a lattice-ordered group.

Definition 4. Let $\preccurlyeq$ be a cancellative monomial preorder on $\mathbb{N}^n$. Then the set

$$P_{\preccurlyeq} \;=\; \{b-a \in \mathbb{Z}^n\colon a,b \in \mathbb{N}^n \text{ such that } a \preccurlyeq b\}$$

is called the *positive cone* of $\preccurlyeq$.

A positive cone determines the preorder from which it is defined as in the case of groups.

Lemma 2. *Let $P \subseteq \mathbb{Z}^n$ be the positive cone of the cancellative monomial preorder $\preccurlyeq$ on $\mathbb{N}^n$. Then we have:*

(GO) *For any* $a,b \in \mathbb{N}^n$, $a \preccurlyeq b$ *if and only if* $b-a \in P$.

Proof. By definition, $a \preccurlyeq b$ implies $b-a \in P$.

Conversely, let $b-a \in P$. Then there are $c,d \in \mathbb{N}^n$ such that $c \preccurlyeq d$ and $d-c = b-a$. It follows $a+d = b+c \preccurlyeq b+d$ and hence $a \preccurlyeq b$.

By Lemma 2, we have for any cancellative monomial preorder $\preccurlyeq$

$$\begin{aligned} P_{\preccurlyeq} &= \{z \in \mathbb{Z}^n \colon a \preccurlyeq b \text{ for some } a,b \in \mathbb{N}^n \text{ such that } z = b-a\} \\ &= \{z \in \mathbb{Z}^n \colon a \preccurlyeq b \text{ for all } a,b \in \mathbb{N}^n \text{ such that } z = b-a\}. \end{aligned} \tag{3}$$

The positive cones of partially ordered Abelian groups possess an intrinsic characterisation: they are exactly the cancellative commutative monoids such that $a+b=0$ implies $a=b=0$ [14]. The positive cones of cancellative monomial preorders can be described in a similar way.

Theorem 1. *A set* $P \subseteq \mathbb{Z}^n$ *is the positive cone of a cancellative monomial preorder on* $\mathbb{N}^n$ *if and only if the following conditions are fulfilled:*

(GC1) *If* $z \in \mathbb{N}^n$, *then* $z \in P$. *Moreover, if* $z \in \mathbb{N}^n \setminus \{\bar{0}\}$, *then* $-z \notin P$.
(GC2) *P is closed under addition.*
(GC3) *For any* $z \in \mathbb{Z}^n$, *at least one of* $z \in P$ *or* $-z \in P$ *holds.*

In this case, $P = P_{\preccurlyeq}$, *where* $\preccurlyeq$ *is given by condition* (GO) *above.*

Proof. Let $\preccurlyeq$ be a cancellative monomial preorder on $\mathbb{N}^n$. Clearly, $0 \in P_{\preccurlyeq}$ then. Furthermore, any $z \in \mathbb{N}^n \backslash \{\bar{0}\}$ is in $P_{\preccurlyeq}$ because $\bar{0} \preccurlyeq z$ holds by the positivity of $\preccurlyeq$. Assume that also $-z \in P_{\preccurlyeq}$. Then there is a $b \in \mathbb{N}^n$ such that $b+z \preccurlyeq b$ and hence by the cancellativity $z \preccurlyeq 0$, in contradiction to the positivity of $\preccurlyeq$. (GC1) is shown.

For $a,b,c,d \in \mathbb{N}^n$, $a \preccurlyeq b$ and $c \preccurlyeq d$ implies $a+c \preccurlyeq b+c \preccurlyeq b+d$. We conclude that if $b-a, d-c \in P_{\preccurlyeq}$, also $(b-a)+(d-c) = (b+d)-(a+c) \in P_{\preccurlyeq}$. This shows (GC2).

For $a,b \in \mathbb{N}^n$, at least one of $a \preccurlyeq b$ or $b \preccurlyeq a$ holds because $\preccurlyeq$ is total. (GC3) follows as well.

Let now $P \subseteq \mathbb{Z}^n$ fulfil (GC1)–(GC3). For $a,b \in \mathbb{N}^n$, let $a \preccurlyeq b$ if $b-a \in P$. We claim that $\preccurlyeq$ is a cancellative monomial preorder. As $0 \in P$ by (GC1), $\preccurlyeq$ is reflexive. By (GC2), $\preccurlyeq$ is transitive. Hence $\preccurlyeq$ is a preorder. $\preccurlyeq$ is total by (GC3) and positive by (GC1). Finally, by construction, $a \preccurlyeq b$ is equivalent to $a+c \preccurlyeq b+c$; the compatibility and cancellativity of $\preccurlyeq$ follows.

It remains to show that P is actually the positive cone $P_{\preccurlyeq}$ of $\preccurlyeq$. By Lemma 2, we have that, for any $a,b \in \mathbb{N}^n$, $b-a \in P_{\preccurlyeq}$ if and only if $a \preccurlyeq b$. But by construction, $a \preccurlyeq b$ if and only if $b-a \in P$. Hence $P = P_{\preccurlyeq}$.

Finally, if $P \subseteq \mathbb{Z}^n$ is the positive cone of any cancellative monomial preorder $\preccurlyeq$, then $\preccurlyeq$ is by Lemma 2 uniquely determined by (GO). The last statement follows.

3.3 Direction cones

Positive cones describe cancellative fg.p.c. tomonoids. In this section we will generalise this notion to cover a wider class of tomonoids. In this case we will not obtain a strict correlation, but we will be led to a Galois correspondence.

Let $\preccurlyeq$ be a monomial preorder on $\mathbb{N}^n$. If $\preccurlyeq$ is cancellative, then for any $a, b \in \mathbb{N}^n$ the question of whether or not $a \preccurlyeq b$ holds depends only on the difference $z = b - a$: we have $a \preccurlyeq b$ if and only if $c \preccurlyeq d$ for any other pair $c, d \in \mathbb{N}^n$ such that $z = d - c$. In fact, the positive cone $P_\preccurlyeq$ consists of these differences; $a \preccurlyeq b$ if and only if $b - a \in P_\preccurlyeq$.

In general, the question of whether or not we have $a \preccurlyeq b$ does not depend on the difference $b - a$ alone. For instance, it may be the case that $a + c \preccurlyeq b + c$ holds for some $c \in \mathbb{N}^n$ but not $a \preccurlyeq b$. However, let $z \in \mathbb{Z}^n$. Then the following lemma implies that still at least one of following possibilities applies: $a \preccurlyeq b$ for all $a, b \in \mathbb{N}^n$ such that $b - a = z$, or $b \preccurlyeq a$ for all $a, b \in \mathbb{N}^n$ such that $b - a = z$.

Lemma 3. *Let $z \in \mathbb{Z}^n$. Then there is a unique pair $a, b \in \mathbb{N}^n$ such that $z = b - a$ and, for any $c, d \in \mathbb{N}^n$ such that $z = d - c$, we have $c = a + t$ and $d = b + t$ for some $t \in \mathbb{N}^n$.*

Proof. Put $a = -z \vee \bar{0}$ and $b = z \vee \bar{0}$. Then $z = b - a$. Moreover, if $c, d \in \mathbb{N}^n$ such that $d - c = z$, we have $c \trianglerighteq \bar{0}$ and $c = d - z \trianglerighteq -z$, thus $c \trianglerighteq a$; similarly, $d \trianglerighteq b$. As $b - a = d - c$, the differences $c - a$ and $d - b$ coincide and hence $c = a + t$ and $d = b + t$ for some $t \in \mathbb{N}^n$. The uniqueness of a, b follows from the $\trianglelefteq$-minimality.

Let $a, b \in \mathbb{N}^n$ be associated with $z \in \mathbb{Z}^n$ according to Lemma 3. Inspecting the proof, we see that b is simply the positive part of $z \in \mathbb{Z}^n$, and a is its (negated) negative part. Let us define

$$\begin{aligned} z^+ &= z \vee \bar{0}, \\ z^- &= -z \vee \bar{0}. \end{aligned}$$

Then we have

$$z = z^+ - z^-$$

and any other pair of elements of $\mathbb{N}^n$ whose difference is z arises from z^+ and z^- by adding a $t \in \mathbb{N}^n$.

For a compatible preorder $\preccurlyeq$ on $\mathbb{N}^n$, the obvious consequence is the following. Let $z \in \mathbb{Z}^n$. If $z^- \preccurlyeq z^+$, we conclude from Lemma 3 and the compatibility of $\preccurlyeq$ that $a \preccurlyeq b$ actually holds for *any* pair $a, b \in \mathbb{N}^n$ such that $b - a = z$. Thus, intuitively, we may view any $z \in \mathbb{Z}^n$ such that $z^- \preccurlyeq z^+$ as being "positively directed"; for, in this case we have $a \preccurlyeq a + z$ for any $a \in \mathbb{N}^n$ such that $a + z \in \mathbb{N}^n$. Our viewpoint is reflected in the following definition.

Definition 5. Let $\preccurlyeq$ be a monomial preorder on $\mathbb{N}^n$. Then the set

$$C_\preccurlyeq = \{z \in \mathbb{Z}^n : z^- \preccurlyeq z^+\}$$

is called the *direction cone* of $\preccurlyeq$.

By Lemma 3 we then have

$$C_{\preccurlyeq} \;=\; \{z \in \mathbb{Z}^n \colon\, a \preccurlyeq b \text{ for all } a, b \in \mathbb{N}^n \text{ such that } z = b - a\}. \tag{4}$$

The natural question is now if there is a characterisation of direction cones similar to the case of positive cones. Comparing with (3), we see that the direction cone of a cancellative monomial preorder is its positive cone. In the general case, we conclude from the positivity of $\preccurlyeq$ that condition (GC1) for positive cones applies here as well, and from the totality of $\preccurlyeq$ also condition (GC3) is immediate: for each $z \in \mathbb{Z}^n$, at least one of z or $-z$ is in $C_{\preccurlyeq}$.

However, a direction cone does not in general fulfil condition (GC2), that is, it is not necessarily closed under addition. The following notion can be used instead. We call a k-tuple $(x_1, \ldots, x_k)$, $k \geqslant 2$, of elements of $\mathbb{Z}^n$ *addable* if

$$(x_1 + \ldots + x_k)^- + x_1 + \ldots + x_i \;\trianglerighteq\; \bar{0} \tag{5}$$

for all $i = 0, \ldots, k$. Note that for addability the order matters.

Lemma 4. *The direction cone of a monomial preorder on $\mathbb{N}^n$ is a set $C \subseteq \mathbb{Z}^n$ fulfilling the following conditions:*

(C1) *Let $z \in \mathbb{N}^n$. Then $z \in C$ and, if $z \neq \bar{0}$, $-z \notin C$.*
(C2) *Let $(x_1, \ldots, x_k)$, $k \geqslant 2$, be an addable k-tuple of elements of C. Then $x_1 + \ldots + x_k \in C$.*
(C3) *Let $z \in \mathbb{Z}^n$. Then $z \in C$ or $-z \in C$.*

Proof. (C1) We have $\mathbb{N}^n \subseteq C$ because $\preccurlyeq$ is positive. Assume that $-z \in C$, where $z \in \mathbb{N}^n$. Then $z = (-z)^- \preccurlyeq (-z)^+ = \bar{0}$ and the positivity of $\preccurlyeq$ implies $z = \bar{0}$.

Recall next that, by (4), $a \preccurlyeq b$ for any $a, b \in \mathbb{N}^n$ such that $b - a \in C$.

To see (C2), let $(x_1, \ldots, x_k)$ be as indicated, and put $z = x_1 + \ldots + x_k$. Then $z^-, z^- + x_1, \ldots, z^- + x_1 + \ldots + x_k \in \mathbb{N}^n$. By assumption, $x_1, \ldots, x_k \in C$; thus $z^- \preccurlyeq z^- + x_1 \preccurlyeq \ldots \preccurlyeq z^- + x_1 + \ldots + x_k = z^- + z = z^+$.

(C3) holds because $\preccurlyeq$ is total.

Our next aim is to show that conditions (C1)–(C3) characterise direction cones.

A preorder gives rise to a direction cone, which fulfils (C1)–(C3). Conversely, we can assign a preorder to a set fulfilling (C1)–(C3).

Definition 6. Let $C \subseteq \mathbb{Z}^n$ fulfil (C1)–(C3). Let $\preccurlyeq_C$ be the smallest preorder on $\mathbb{N}^n$ such that

(O) $a \preccurlyeq_C b$ for any $a, b \in \mathbb{N}^n$ such that $b - a \in C$.

Then we call $\preccurlyeq_C$ the monomial preorder *induced by* C.

In other words, for a subset C of $\mathbb{Z}^n$ fulfilling (C1)–(C3) and $a, b \in \mathbb{N}^n$, we have $a \preccurlyeq_C b$ if and only if there are $k \geqslant 1$ elements $z_1, \ldots, z_k \in C$ such that a, $a + z_1$, $a + z_1 + z_2$, $\ldots, a + z_1 + \ldots + z_k \trianglerighteq \bar{0}$ and $a + z_1 + \ldots + z_k = b$. We note that this is not the same as to say that $b - a$ is a sum of elements of C.

Lemma 5. *Let $C \subseteq \mathbb{Z}^n$ fulfil* (C1)–(C3)*. Then $\preccurlyeq_C$, the monomial preorder induced by C, is in fact a monomial preorder.*

Proof. By construction, $\preccurlyeq_C$ is a preorder, and by (C3), $\preccurlyeq_C$ is total. It is furthermore clear that $\preccurlyeq_C$ is compatible with the addition.

Assume next that, for some $a \in \mathbb{N}^n$, $a \preccurlyeq_C \bar{0}$ holds according to the prescription (O). Then $a = \bar{0}$ by (C1). It follows that $\bar{0} \prec_C a$ for all $a \in \mathbb{N}^n \backslash \{\bar{0}\}$, that is, $\preccurlyeq_C$ is positive. This completes the proof that $\preccurlyeq_C$ is a monomial preorder.

Theorem 2. *A set $C \subseteq \mathbb{Z}^n$ is the direction cone of a monomial preorder if and only if C fulfils* (C1)–(C3)*. In this case, C is the direction cone of $\preccurlyeq_C$.*

Proof. A direction cone fulfils (C1)–(C3) by Lemma 4.

Conversely, let C fulfil (C1)–(C3). Let $\preccurlyeq_C$ be the induced preorder. By Lemma 5, $\preccurlyeq_C$ is a monomial preorder.

It remains to show that $C_{\preccurlyeq_C}$, the direction cone of $\preccurlyeq_C$, coincides with C, that is, for $z \in \mathbb{Z}^n$, $z^- \preccurlyeq_C z^+$ if and only if $z \in C$. The "if" part holds by construction. For the "only if" part, assume that $z^- \preccurlyeq_C z^+ = z^- + z$. Then $z = x_1 + \ldots + x_k$ for some $x_1, \ldots, x_k \in C$ such that $z^- + x_1 + \ldots x_i \geqslant \bar{0}$ for $i = 0, \ldots, k$. Then $(x_1, \ldots, x_k)$ is addable, hence $z \in C$ by (C2).

In the sequel, when speaking about direction cones without reference to a monomial preorder, we mean a subset of $\mathbb{Z}^n$ that fulfils the conditions (C1)–(C3).

A direction cone induces a preorder. As seen next, any preorder contains a preorder arising in this way.

Theorem 3. *Let $\preccurlyeq$ be a monomial preorder. Then $\preccurlyeq$ extends $\preccurlyeq_{C_\preccurlyeq}$, the monomial preorder induced by the direction cone of $\preccurlyeq$.*

Moreover, the direction cone of $\preccurlyeq_{C_\preccurlyeq}$ is $C_\preccurlyeq$ again.

Proof. Let $a, b \in \mathbb{N}^n$ and assume that $a \preccurlyeq_{C_\preccurlyeq} b$ holds according to the prescription (O). Then $b - a \in C_\preccurlyeq$, that is, $z^- \preccurlyeq z^+$, where $z = b - a$. In view of Lemma 3, it follows $a \preccurlyeq b$. We conclude that $\preccurlyeq_{C_\preccurlyeq} \subseteq \preccurlyeq$.

The second part holds by Theorem 2.

We apply the shown facts to tomonoids.

Definition 7. Let $C \subseteq \mathbb{Z}^n$ be a direction cone. Then we call the tomonoid represented by $\preccurlyeq_C$ a *cone tomonoid*.

Theorem 4. *Each fg.p.c. tomonoid L is the quotient of a cone tomonoid.*

Proof. This follows from Theorem 3.

3.4 A Galois connection

We have seen that there is a mutual correspondence between monomial preorders and direction cones. This correspondence is not one-to-one, some monomial preorders are proper extensions of those that are induced by direction cones. However, we can established a Galois correspondence between the two sets.

Let us fix an $n \geqslant 1$. Let $\mathscr{P}$ be the set of all monomial preorders on $\mathbb{N}^n$ and let $\mathscr{C}$ be the set of all direction cones in $\mathbb{Z}^n$. We partially order the two sets by means of the set-theoretic inclusion. We then readily check that the two mappings

$$\begin{aligned} &\mathscr{P} \to \mathscr{C}, \quad \preccurlyeq \mapsto C_{\preccurlyeq}, \\ &\mathscr{C} \to \mathscr{P}, \quad C \mapsto \preccurlyeq_C \end{aligned}$$

are order-preserving. The mappings are not one-to-one; in fact, the former is surjective but not injective, and the latter is injective but not surjective. From Theorems 2 and 3 we conclude what results when applying the mappings successively: any $\preccurlyeq \in \mathscr{P}$ is an extension of $\preccurlyeq_{C_{\preccurlyeq}}$; and any $C \in \mathscr{C}$ is equal to $C_{\preccurlyeq_C}$. Hence there is the following Galois connection between $\mathscr{P}$ and $\mathscr{C}$: for any $\preccurlyeq \in \mathscr{P}$ and $C \in \mathscr{C}$,

$$\preccurlyeq_C \subseteq \preccurlyeq \quad \text{if and only if} \quad C \subseteq C_{\preccurlyeq}.$$

3.5 Example

We conclude by presenting an example illustrating the results of this section. Let L be the 9-element fg.p.c. tomonoid specified as follows. Let L be generated by its two elements a and b and assume that

$$\begin{gathered} 0 \;<\; a \;<\; b \;<\; 2a \;<\; a+b \;<\; 2b \;<\; 3a \;< \\ 2a+b \;=\; a+2b \;=\; 4a \;<\; 2a+2b \;=\; 3a+b \;=\; 5a \;=\; 3b \end{gathered}$$

and that the last indicated element is the top element. In accordance with Proposition 1, let $\iota\colon \mathbb{N}^2 \to L$ be the surjective monoid homomorphism such that $\iota((1,0)) = a$ and $\iota((0,1)) = b$, and endow $\mathbb{N}^2$ with the preorder $\preccurlyeq$ according to (2). Then we have

$$\begin{aligned} &(0,0) \prec (1,0) \prec (0,1) \prec (2,0) \prec (1,1) \prec (0,2) \prec \\ &(3,0) \prec (2,1) \approx (1,2) \approx (4,0) \prec (m,n), \end{aligned}$$

where (m,n) is any of the remaining elements of $\mathbb{N}^2$. A graphical representation of $(L; \leqslant, +, 0)$ can be found in Figure 1.

According to Definition 5, the direction cone is

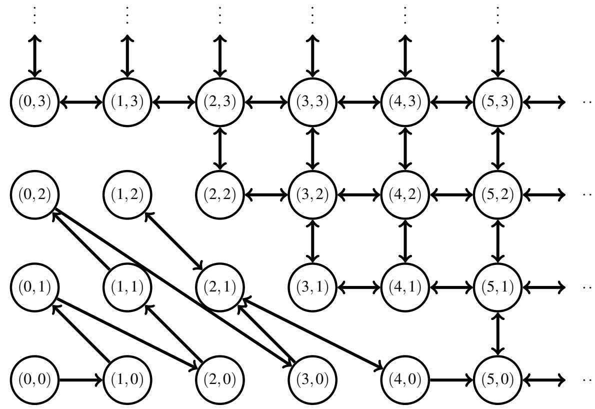

Fig. 1 The example tomonoid L. The simple arrows indicate the immediate-successor relation w.r.t. $\preccurlyeq$; the double arrows indicate $\preccurlyeq$-equivalence.

$$\begin{aligned}
C_{\preccurlyeq} &= \{(p,q) \in \mathbb{Z}^2 \colon (-p \vee 0, -q \vee 0) \preccurlyeq (p \vee 0,\, q \vee 0)\} \\
&= \{(p,q) \in \mathbb{Z}^2 \colon p,q \geqslant 0\} \cup \\
&\quad \{(-2,2),(-1,1),(-1,2),(2,-1),(3,-2),(3,-1),(4,-2),(4,-1)\} \cup \\
&\quad \{(p,q) \in \mathbb{Z}^2 \colon p \leqslant 0 \text{ and } q \geqslant 3\} \cup \\
&\quad \{(p,q) \in \mathbb{Z}^2 \colon p \geqslant 5 \text{ and } q \leqslant 0\}.
\end{aligned}$$

This set is depicted in Figure 2.

Finally, we calculate $\preccurlyeq_{C_{\preccurlyeq}}$, the preorder representing a cone tomonoid whose quotient is L. The preorder $\preccurlyeq_{C_{\preccurlyeq}}$ can most easily be read off directly from Figure 1. Namely, we collect the order relations that hold between elements of the form $(m,0)$ and $(0,n)$, where $m,n \geqslant 1$; then we translate and concatenate them. The result is depicted in Figure 3. From $\preccurlyeq_{C_{\preccurlyeq}}$, we get L by requiring the elements $(2,1)$, $(1,2)$, and $(4,0)$ of $\mathbb{N}^2$ to be equivalent.

4 One-element Rees coextensions of finite negative tomonoids

In the second part of this chapter, we develop a much different point of view on tomonoids. To begin with, we switch to the dual order and the multiplicative notation, as is common in fuzzy logic.

We will again assume the property called "positive" in the previous part. In the present context, however, positivity means that 1 is the top element; accordingly, we will refer to this property as "negative". Furthermore, we will restrict to the finite

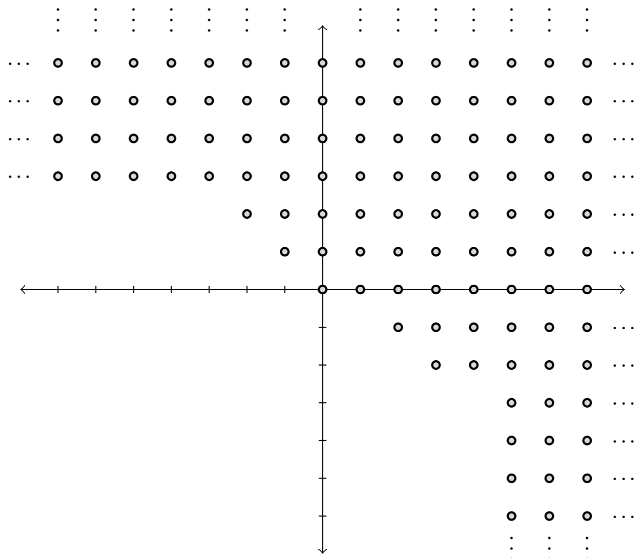

Fig. 2 The direction cone $C_{\preccurlyeq}$ of the monomial preorder $\preccurlyeq$ representing L. Each element of $C_{\preccurlyeq}$ is depicted as a circle in the $\mathbb{Z}^2$ plane.

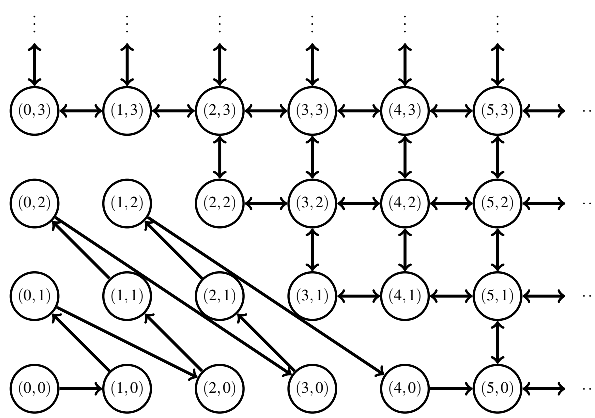

Fig. 3 The cone tomonoid represented by $\preccurlyeq_{C_{\preccurlyeq}}$, whose quotient is L.

case. Finally, our considerations do not rely on the commutativity of the monoidal product and hence we will not assume this condition here.

We shall write "f.n." for "finite, negative". That is, a f.n. tomonoid is a structure $(L; \leqslant, \odot, 1)$ such that $(L; \odot, 1)$ is a finite monoid and $\leqslant$ is a compatible total order whose top element is 1.

Our aim is to describe the construction of f.n. tomonoids in a step-by-step fashion. The main idea is the following. Let $(L; \leqslant, \odot, 1)$ be a non-trivial f.n. tomonoid and let 0 and α be its smallest and second smallest element, respectively. Then the identification of 0 and α is a tomonoid congruence and the quotient is by one element smaller than L. Continuing in the same way, we get a sequence of tomonoids that ends with the trivial one. It seems then natural to ask how to generate such a sequence in the reversed order. That is, given an f.n. tomonoid L, how can we determine all those f.n. tomonoids $\bar{L}$ that are by one element larger and such that the identification of their smallest two elements leads back to L? This is in fact the question that we will answer. We will provide a practical method of determining from L systematically all tomonoids $\bar{L}$ of the indicated type.

The results of the present section are due to [35], where further details can be found.

4.1 Rees congruences

Consider a negative tomonoid $(L; \leqslant, \odot, 1)$ and let q be one of its elements. Then $I_q = \{a \in L \colon a \leqslant q\}$ is an ideal of L, seen as a monoid. Indeed, by the negativity of L, $a \leqslant q$ implies $a \odot b \leqslant q$ and $b \odot a \leqslant q$ for any $b \in L$. Consequently, we may form the Rees quotient of the monoid L by I_q; see, e.g., [23]. Its elements may be identified with the elements that are not in I_q as well as one further element, usually denoted by 0. Obviously, this monoid congruence has only convex classes and hence it is a tomonoid congruence; cf., e.g., [13].

Definition 8. Let $(L; \leqslant, \odot, 1)$ be a f.n. tomonoid and let $q \in L$. For $a, b \in L$, let $a \approx_q b$ if $a = b$ or $a, b \leqslant q$. Then we call $\approx_q$ the *Rees congruence* by q. We denote the quotient by L/q and call it the *Rees quotient* of L by q.

Moreover, we call L a *Rees coextension* of L/q. We call L a *one-element* Rees coextension, or simply a *one-element coextension*, if L is non-trivial and q is the atom of L.

For a finite chain L, we will denote by 0 the bottom element and we write

$$L^\star = L \setminus \{0\}.$$

If L has at least two elements, we furthermore call the second smallest element of L the *atom* of L. We will use in the sequel the symbol α to denote the atom.

Given a non-trivial f.n. tomonoid L, then its Rees quotient L/α by its atom α arises from L by the identification of the smallest two elements. L is in this case a one-element coextension of L/α. Our aim is to determine all one-element coextensions of a f.n. tomonoid. We will then obviously be in the position to construct, starting from the trivial tomonoid, successively all f.n. tomonoids.

4.2 Tomonoid partitions

A binary operation $\odot$ on a set A gives rise to a partition of $A \times A$: the blocks of the partition are the subsets of all those pairs that are mapped by $\odot$ to the same value. The blocks are commonly referred to as the *level sets* of $\odot$. This partition, together with the assignment that associates with each block the respective element of A, specifies $\odot$ uniquely.

The representation of binary operations based on level sets was first applied to the theory of tomonoids in [33]. We note that it comes along with the possibility of representing tomonoids within two dimensions only.

Definition 9. Let $(L; \leqslant, \odot, 1)$ be a tomonoid. We define, for any $(a,b),(c,d) \in L^2$,

$$(a,b) \sim (c,d) \quad \text{if} \quad a \odot b = c \odot d.$$

We call $\sim$ the *level equivalence* of L.

Based on the level equivalence of a tomonoid L, we will endow the set L^2 with a first-order structure as follows.

Definition 10. Let $\leqslant$ be a total order on a set L and let $1 \in L$. We denote the componentwise order on L^2 by $\trianglelefteq$, that is, we put

$$(a,b) \trianglelefteq (c,d) \quad \text{if } a \leqslant c \text{ and } b \leqslant d$$

for $a,b,c,d \in L$. Moreover, let $\sim$ be an equivalence relation on L^2 such that the following conditions hold:

(P1) For any $a,b,c,d,e,f \in L$, if $(1,e) \sim (a,b) \trianglelefteq (c,d) \sim (1,f)$, then $e \leqslant f$.

(P2) For any $(a,b) \in L^2$, there is exactly one $c \in L$ such that $(a,b) \sim (1,c) \sim (c,1)$.

(P3) For any $a,b,c,d,e \in L$, $(a,b) \sim (d,1)$ and $(b,c) \sim (1,e)$ imply $(d,c) \sim (a,e)$.

We then call the structure $(L^2; \trianglelefteq, \sim, (1,1))$ a *tomonoid partition*.

Proposition 4. *Let $(L; \leqslant, \odot, 1)$ be a tomonoid and let $\sim$ be the level equivalence of L. Then $(L^2; \trianglelefteq, \sim, (1,1))$ is a tomonoid partition.*

Proof. Let $a,b,c,d \in L$. By the compatibility of $\leqslant$ with $\odot$, we have that $(a,b) \trianglelefteq (c,d)$ implies $a \odot b \leqslant c \odot d$. (P1) follows. Moreover, as 1 is the monoidal identity, we have that $(a,b) \sim (c,1)$ iff $(a,b) \sim (1,c)$ iff $a \odot b = c$. Hence also (P2) holds. Finally, (P3) is implied by the associativity of $\odot$.

By Proposition 4, each tomonoid L gives rise to a tomonoid partition; we will speak about the tomonoid partition *associated with L*.

We next see that there is a converse of Proposition 4. We will use the following simplified notation. When L is a chain and $1 \in L$, we will identify the elements of the form $(1,c) \in L^2$, where $c \in L$, with c. It will be clear from the context if c denotes an element of L or of L^2. For instance, if $\sim$ is an equivalence relation on L^2, then $(a,b) \sim c$ means $(a,b) \sim (1,c)$. Similarly, the $\sim$-class of some $c \in L$ is meant to be the $\sim$-class containing $(1,c)$.

Proposition 5. *Let $(L^2; \trianglelefteq, \sim, (1,1))$ be a tomonoid partition. Let $\leqslant$ be the underlying total order of L. Moreover, for any $a, b \in L$, let*

$$a \odot b \;=\; \text{the unique } c \text{ such that } (a,b) \sim c. \tag{6}$$

Then $(L; \leqslant, \odot, 1)$ is the unique tomonoid such that $(L^2; \trianglelefteq, \sim, (1,1))$ is its associated tomonoid partition.

Proof. By assumption, L is totally ordered and $\trianglelefteq$ is the induced componentwise order on L^2. Evidently, $\trianglelefteq$ determines the total order $\leqslant$ on L uniquely. It is furthermore clear from (P2) that $\odot$ can be defined by (6).

For $a \in L$, we have $1 \odot a = a$ by construction and $a \odot 1 = 1 \odot a$ by (P2). Furthermore, (P2) and (P3) imply the associativity of $\odot$. Thus $(L; \odot, 1)$ is a monoid. Let $a \leqslant b$. Then $(a,c) \trianglelefteq (b,c)$, and we conclude from (P1) that $a \odot c \leqslant b \odot c$. Similarly, we see that $c \odot a \leqslant c \odot b$. Thus $\leqslant$ is compatible with $\odot$ and $(L; \leqslant, \odot, 1)$ is a tomonoid. It is clear that $\sim$ is the level equivalence of L and we conclude that $(L^2; \trianglelefteq, \sim, (1,1))$ is its associated tomonoid partition.

Let $(L^2; \trianglelefteq, \sim, (1,1))$ be associated to another tomonoid $(L'; \leqslant', \odot', 1')$. Then, by the way in which a tomonoid partition is constructed from a tomonoid, $L' = L$, $\leqslant' = \leqslant$, and $1' = 1$. Furthermore, if for some $a, b, c \in L$ we have $a \odot' b = c$, then $(a,b) \sim (1,c)$ and hence $a \odot b = c$. We conclude $\odot' = \odot$.

By Propositions 4 and 5, tomonoids and tomonoid partitions are in a one-to-one correspondence. We will present our results in the sequel mostly with reference to the latter, that is, with reference to tomonoid partitions.

Let us next devote some remarks to the geometric interpretation of the conditions (P1)–(P3) in Definition 10. Let L be a tomonoid. Then L is a chain and hence L^2 can be viewed as a square array. For elements $(a,b), (c,d) \in L^2$, $(a,b) \trianglelefteq (c,d)$ means that (a,b) is left underneath (c,d). Moreover, for negative tomonoids, 1 is the top element; in this case, $(1,1)$ is located in the upper right corner of L^2. See Fig. Figure 4 for an illustration.

In order to interpret (P1)–(P3), let us view the level equivalence of L as a partition of L^2. Condition (P2) has probably the most straightforward meaning. By (P2), each block contains exactly one element of the form $(1,c)$, $c \in L$. That is, we may index the blocks by the elements of the line indexed by 1. Furthermore, $(c,1)$ and $(1,c)$ are for each $c \in L$ in the same block and hence a similar statement holds also for the column indexed by 1.

By the identification of the blocks with the line indexed by 1, the blocks are totally ordered. Condition (P1) says that the componentwise order on L^2 is in accordance with this total order. Namely, when moving from any element of a block to the right or upwards, we arrive at a block indexed by a larger element.

0	t	u	v	w	x	y	z	1	
0	t	u	v	w	x	y	z	1	1
0	0	t	u	u	v	w	x	z	z
0	0	0	t	t	u	u	v	y	y
0	0	0	t	t	u	u	v	x	x
0	0	0	0	0	t	t	u	w	w
0	0	0	0	0	t	t	u	v	v
0	0	0	0	0	0	0	t	u	u
0	0	0	0	0	0	0	0	t	t
0	0	0	0	0	0	0	0	0	0

Fig. 4 A tomonoid partition associated with an eight-element negative tomonoid L. Rows and columns of the array correspond to the elements of L, thus each square in the array corresponds to a pair $(a,b) \in L^2$, where a is the row index and b is the column index. In order to represent $\sim$, we have indicated in each square (a,b) the product of a and b in L; two squares are $\sim$-equivalent iff they contain the same symbol. For instance, the $\sim$-class of u comprises even elements and the $\sim$-class of 1 just one.

Condition (P3), which accounts for the associativity, possesses an appealing geometric interpretation as well. An illustration is given in Fig. 5. Here, we assume that 1 is the top element of L. Within the square array representing L^2, consider two rectangles such that one hits the upper edge and the other one hits the right edge. Assume that the upper left, upper right, and lower right vertices of these rectangles are in the same blocks, respectively. By (P3), then also the remaining pair, consisting of the lower left vertices, is in the same block. A related property is known from the field of web geometry and called the "Reidemeister condition" [1, 2].

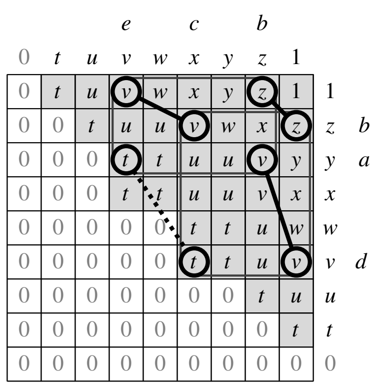

Fig. 5 The "Reidemeister" condition (P3). A (connected or broken) bold line between two elements of the array indicates level equivalence. By (P3), the equivalences of the pairs connected by a solid line imply the equivalence of the pair connected by a broken line.

We conclude the subsection with a characterisation of those tomonoid partitions in which we are actually interested: the finite, negative ones. The slightly optimised characterisation will be useful in subsequent proofs.

Proposition 6. *Let $(L;\leqslant)$ be a finite and at least two-element chain with the top element* 1*. Let* 0 *be the bottom element of L. Then $(L^2;\trianglelefteq,\sim,(1,1))$ is a tomonoid partition if and only if* (P1)*,* (P2)*, and the following condition hold:*

(P3') *For any $a,b,c,d,e \in L \setminus \{0,1\}$, $(a,b) \sim d$ and $(b,c) \sim e$ imply $(d,c) \sim (a,e)$.*

In this case, $(L^2;\trianglelefteq,\sim,(1,1))$ is finite and negative.

Proof. The "only if" part is clear by definition.

To see the "if" part, let $(L^2;\trianglelefteq,\sim,(1,1))$ fulfil (P1), (P2), and (P3'). We next show that the negativity criterion of Lemma 6(i) holds:

($\star$) $(a,b) \sim (1,c)$ implies $c \leqslant a$ and $c \leqslant b$.

Indeed, in this case $(c,1) \sim (1,c) \sim (a,b) \trianglelefteq (a,1)$ by (P2) and the fact that 1 is the top element. Hence, by (P1), $c \leqslant a$. Similarly, we see that $c \leqslant b$.

It remains to prove (P3). Let $a,b,c,d,e \in L$ be such that $(a,b) \sim d$ and $(b,c) \sim e$. We have to show $(d,c) \sim (a,e)$ if one of the five elements equals 0 or 1. We consider certain cases only, the remaining ones are seen similarly.

Let $a = 1$. Then $(1,b) \sim (1,d)$, hence $b = d$ by (P2), and it follows $(d,c) = (b,c) \sim (1,e) = (a,e)$.

Let $d = 1$. Then $(a,b) \sim (1,1)$, and by ($\star$), we conclude $a = b = 1$. From $(b,c) \sim e$ it follows $e = c$. Hence $(d,c) = (a,e)$.

Note next that, for any $f \in L$, $(f,0) \sim 0$. This follows again from ($\star$).

Let $a = 0$. Then $(a,b) = (0,b) \sim 0$ and hence $d = 0$. Hence $(d,c) = (0,c) \sim 0 \sim (0,e) = (a,e)$.

Let $d = 0$. Then $(d,c) = (0,c) \sim 0$. From $(b,c) \sim e$, it follows by ($\star$) that $e \leqslant b$. Hence $(1,0) \sim (0,0) \trianglelefteq (a,e) \trianglelefteq (a,b) \sim (1,0)$ and we conclude from (P1) that $(a,e) \sim 0$. In particular, $(a,e) \sim (d,c)$.

4.3 Properties and constructions for tomonoid partitions

We have seen that tomonoids and tomonoid partitions are in a one-to-one correspondence. Consequently, we can apply properties, constructions, etc. defined for tomonoids to tomonoid partitions as well. We establish in this subsection a few of such correspondences.

For convenience, we will apply to tomonoid partitions the same notions as to tomonoids. For instance, a tomonoid partition will be called negative if the corresponding tomonoid is negative.

Lemma 6. *Let $(L^2;\trianglelefteq,\sim,(1,1))$ be a tomonoid partition.*

(i) *The following statements are pairwise equivalent:*

- *L^2 is negative.*
- *$(1,1)$ is the top element of L^2.*
- *The $\sim$-class of any $c \in L$ is contained in $\{(a,b) \in L^2 \colon a,b \geqslant c\}$.*

(ii) *The following statements are equivalent:*

- *L^2 is commutative.*
- *$(a,b) \sim (b,a)$ for any $a,b \in L$.*

A further property considered in the sequel is Archimedeanicity. In what follows, we write a^n for the n-fold product $a \odot \ldots \odot a$.

Definition 11. We call a negative tomonoid *Archimedean* if, for any $a \leqslant b < 1$, there is an $n \geqslant 1$ such that $b^n \leqslant a$.

Note that negative tomonoids with at most two elements are trivially Archimedean.

Archimedean f.n. tomonoid partitions are characterised as follows.

Lemma 7. *Let $(L^2; \trianglelefteq, \sim, (1,1))$ be a f.n. tomonoid partition. The following statements are pairwise equivalent:*

- *L^2 is Archimedean.*
- *$(b,a) \not\sim (1,a)$ for any $a \in L^\star$ and $b < 1$.*
- *$(a,b) \not\sim (a,1)$ for any $a \in L^\star$ and $b < 1$.*

Proof. Let $(L; \leqslant, \odot, 1)$ be the corresponding f.n. tomonoid and let 0 be the bottom element of L. W.l.o.g., we can assume $0 \neq 1$. We show that (i) and (ii) are equivalent. The equivalence of (i) and (iii) is seen similarly.

Assume that (ii) holds. By the negativity of L, we have $b \odot a < a$ for all $a \neq 0$ and $b < 1$. Let $a < 1$. Then, for any $n \geqslant 1$, either $a^{n+1} < a^n$ or $a^n = 0$. As L is finite, the latter possibility applies for a sufficiently large n. It follows that L is Archimedean.

Assume that (ii) does not hold. Let $a \neq 0$ and $b < 1$ such that $b \odot a = a$. As L is negative, we then have $a \leqslant b$ and it follows $b^n \geqslant b^{n-1} \odot a = a > 0$ for any $n \geqslant 2$. Hence L cannot be Archimedean.

We next see how Rees quotients are formed in our framework.

Proposition 7. *Let $(L^2; \trianglelefteq, \sim, (1,1))$ be a negative tomonoid partition and let $q \in L$. Let $L_q = \{a \in L \colon a > q\} \,\dot\cup\, \{0\}$, where 0 is a new element, and endow L_q with the total order extending the total order on $\{a \in L \colon a > q\}$ such that 0 is the bottom element. Then, for each $c \in {L_q}^\star$, the $\sim$-class of c is contained in $({L_q}^\star)^2$. Let $\sim_q$ be the equivalence relation on ${L_q}^2$ whose classes are the $\sim$-classes of each $c \in {L_q}^\star$ as well as the subset of ${L_q}^2$ containing the remaining elements. Then $({L_q}^2; \trianglelefteq, \sim_q, (1,1))$ is the Rees quotient of L^2 by q.*

Proof. Let $(L; \leqslant, \odot, 1)$ be the corresponding negative tomonoid. Let $\odot_q$ be the binary operation on L_q such that $(L_q; \leqslant, \odot_q, 1)$ is (under the obvious identifications) the Rees quotient of L by q. Let $({L_q}^2; \trianglelefteq, \sim'_q, (1,1))$ be the associated tomonoid partition.

Let $a,b,c \in L$ such that $c > q$ and $(a,b) \sim c$. Then $a,b \geqslant c$ by Lemma 6(i) and consequently $a,b > q$. We conclude that the $\sim$-class of each $c \in {L_q}^\star$ is contained in $({L_q}^\star)^2$.

We have to show $\sim'_q = \sim_q$. Let $a,b,c \in L_q$ such that $c \neq 0$. Then $(a,b) \sim'_q c$ iff $a \odot_q b = c$ iff $a \odot b = c$ iff $(a,b) \sim c$. Hence the $\sim'_q$-class of each $c \in {L_q}^\star$ coincides with the $\sim$-class of c. There is only one further $\sim'_q$-class, the $\sim'_q$-class of 0, which consequently consists of all elements of ${L_q}^2$ not belonging to the $\sim$-class of any $c \in {L_q}^\star$.

We may interpret Proposition 7 once again geometrically. Let L^2 be a finite negative tomonoid partition and let q be an element of the underlying tomonoid L. Then the Rees quotient by q arises from the partition on L^2 by removing all columns and rows indexed by elements $\leqslant q$ and by adding instead a single new column from left and a single new row from below. Moreover, all elements that originally belonged to a class of some $a \leqslant q$ are joined into a single class, which is the class of the new zero. In contrast, the classes of elements strictly larger than q remain unchanged.

Figure 6 shows the chain obtained from a eight-element tomonoid by applying this procedure repeatedly to the respective atom.

0	u	v	w	x	y	z	1	
0	u	v	w	x	y	z	1	1
0	0	u	u	v	w	x	z	z
0	0	0	0	u	u	v	y	y
0	0	0	0	u	u	v	x	x
0	0	0	0	0	0	u	w	w
0	0	0	0	0	0	u	v	v
0	0	0	0	0	0	0	u	u
0	0	0	0	0	0	0	0	0

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0	0	v	w	x	z	z
0	0	0	0	0	v	y	y
0	0	0	0	0	v	x	x
0	0	0	0	0	0	w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

0	w	x	y	z	1	
0	w	x	y	z	1	1
0	0	0	w	x	z	z
0	0	0	0	0	y	y
0	0	0	0	0	x	x
0	0	0	0	0	w	w
0	0	0	0	0	0	0

0	x	y	z	1	
0	x	y	z	1	1
0	0	0	x	z	z
0	0	0	0	y	y
0	0	0	0	x	x
0	0	0	0	0	0

0	y	z	1	
0	y	z	1	1
0	0	0	z	z
0	0	0	y	y
0	0	0	0	0

0	z	1	
0	z	1	1
0	0	z	z
0	0	0	0

0	1	
0	1	1
0	0	0

1	
1	1

Fig. 6 Starting from the eight-element tomonoid shown in Figure 4, the successive formation of Rees quotients by the atom leads eventually to the trivial tomonoid.

4.4 One-element coextensions

Based on the level-set representation, we will in this subsection provide a systematic description of all one-element coextensions of a finite, negative tomonoid. We will restrict to the Archimedean case; for the general case we refer to [35]. That is, we will determine the coextensions of Archimedean f.n. tomonoids that are Archimedean again.

We will proceed, roughly, as follows. We start from a tomonoid partition, seen as a partitioned square array; cf. Figure 4. We enlarge the sides of this square by one element, doubling the lowest row and left-most column. We determine the equivalence relation $\bar{\sim}$ that makes the enlarged square into a tomonoid partition in two steps. We first determine an intermediate equivalence relation $\dot{\sim}$, called the ramification. $\dot{\sim}$ has a universal property: the level equivalence of any Archimedean one-element coextension extends $\dot{\sim}$. Second, we choose the final equivalence relation $\bar{\sim}$, merging certain $\dot{\sim}$-classes such that the part of the square containing the classes of the new tomonoid's bottom element and atom is divided up into exactly two $\bar{\sim}$-classes.

For a chain $(L;\leqslant)$, let us define $\bar{L} = L^\star \dot{\cup} \{0,\alpha\}$, where $0,\alpha$ are new elements, and let us endow $\bar{L}$ with the total order extending the total order on $L^\star$ such that $0 < \alpha < a$ for all $a \in L^\star$. We call $(\bar{L};\leqslant)$ the *zero doubling extension* of L.

Furthermore, let $(L;\leqslant,\odot,1)$ be a f.n. tomonoid. We will assume that any one-element coextension of L is of the form $(\bar{L};\leqslant,\bar{\odot},1)$. In particular, the intersection of L and $\bar{L}$ is exactly $L^\star$ and $a \mathbin{\bar{\odot}} b = a \odot b$ whenever $a,b,a\odot b \in L^\star$.

Definition 12. Let $(L^2;\trianglelefteqslant,\sim,(1,1))$ be an Archimedean f.n. tomonoid partition. Let $\bar{L} = L^\star \dot{\cup} \{0,\alpha\}$ be the zero doubling extension of L. We define

$$\begin{aligned} \mathscr{P} &= \{(a,b) \in \bar{L}^2 \colon a,b \in L^\star \text{ and there is a } c \in L^\star \text{ such that } (a,b) \sim c\}, \\ \mathscr{Q} &= \bar{L}^2 \setminus \mathscr{P}. \end{aligned} \tag{7}$$

Let $\dot{\sim}$ be the smallest equivalence relation on $\bar{L}^2$ such that the following conditions hold:

(E1) For any $(a,b),(c,d) \in \mathscr{P}$ such that $(a,b) \sim (c,d)$, we have $(a,b) \mathrel{\dot{\sim}} (c,d)$.

(E2) For any $(a,b),(b,c) \in \mathscr{P}$ and $d,e \in L^\star$ such that $(d,c),(a,e) \in \mathscr{Q}$, $(a,b) \sim d$, and $(b,c) \sim e$, we have $(d,c) \mathrel{\dot{\sim}} (a,e)$.

(E3) For any $a,b,c,e \in L^\star$ such that $(a,b) \in \mathscr{Q}$, $(b,c) \sim e$, and $c < 1$, we have $(a,e) \mathrel{\dot{\sim}} 0$.

Moreover, for any $a,b,c,d \in L^\star$ such that $(b,c) \in \mathscr{Q}$, $(a,b) \sim d$, and $a < 1$, we have $(d,c) \mathrel{\dot{\sim}} 0$.

(E4) We have $(0,1) \mathrel{\dot{\sim}} (1,0) \mathrel{\dot{\sim}} (\alpha,b) \mathrel{\dot{\sim}} (b,\alpha)$ for any $b < 1$, and $(\alpha,1) \mathrel{\dot{\sim}} (1,\alpha)$. Moreover, for any $(a,b),(c,d) \in \mathscr{Q}$ such that $(a,b) \trianglelefteqslant (c,d) \mathrel{\dot{\sim}} 0$, we have $(a,b) \mathrel{\dot{\sim}} 0$.

Then we call the structure $(\bar{L}^2; \trianglelefteq, \dot{\sim}, (1,1))$ the *ramification* of $(L^2; \trianglelefteq, \sim, (1,1))$.

A few remarks might help to clarify the meaning of Definition 12. Let the tomonoid partition $(L^2; \trianglelefteq, \sim, (1,1))$ be given. The subset $\mathscr{P}$ of $\bar{L}^2$ consists of all pairs $(a,b) \in L^{\star 2}$ whose product in L is not the bottom element. That is, $\mathscr{P}$ is the union of the $\sim$-classes of all $c \in L^\star$ and this union lies in $L^{\star 2}$. We note that $\mathscr{P}$ is an upwards closed subset of $\bar{L}^2$ and, consequently, its complement $\mathscr{Q}$ is a downward closed subset of $\bar{L}^2$.

The intermediate equivalence relation $\dot{\sim}$ is determined by successive application of conditions (E1)–(E4). We observe that $\dot{\sim}$-equivalences involving elements of $\mathscr{P}$ are required by condition (E1) only. In fact, all the $\sim$-classes contained in $\mathscr{P}$ are $\dot{\sim}$-classes as well.

The $\dot{\sim}$-classes contained in $\mathscr{Q}$ are determined by conditions (E2)–(E4). In fact, each prescription contained in (E2) and (E3) is of the form that certain $\sim$-equivalences imply that a certain pair of elements of $\mathscr{Q}$ is $\dot{\sim}$-equivalent. Finally, (E4) prescribes that the $\dot{\sim}$-class of 0 is downward closed. We remark that $\mathscr{Q}$ contains the $\dot{\sim}$-class of the bottom element 0, the $\dot{\sim}$-class of the atom α, and possibly further $\dot{\sim}$-classes, which contain neither $(1,c)$ nor $(c,1)$ for any $c \in \bar{L}$.

In the sequel, for two equivalence relations $\sim_1$ and $\sim_2$ on a set A, we say that $\sim_1$ is *coarser* than $\sim_2$ if $\sim_2 \subseteq \sim_1$. In other words, $\sim_1$ coarser than $\sim_2$ if and only if each $\sim_1$-class is a union of $\sim_2$-classes.

Lemma 8. *Let $(L^2; \trianglelefteq, \sim, (1,1))$ be an Archimedean f.n. tomonoid partition and let $(\bar{L}^2; \trianglelefteq, \bar{\sim}, (1,1))$ be an Archimedean one-element coextension of L^2. Furthermore, let $(\bar{L}^2; \trianglelefteq, \dot{\sim}, (1,1))$ be the ramification of L^2. Then $\bar{\sim}$ is coarser than $\dot{\sim}$ and the following holds: the $\bar{\sim}$-class of each $c \in L^\star$ coincides with the $\dot{\sim}$-class of c, the $\bar{\sim}$-class of* 0 *is downward closed, and each $\bar{\sim}$-class contains exactly one element of the form $(1,c)$ for some $c \in \bar{L}$.*

Proof. Let $(L; \leqslant, \odot, 1)$ and $(\bar{L}; \leqslant, \bar{\odot}, 1)$, where $\bar{L} = L^\star \dot{\cup} \{0, \alpha\}$, be the two tomonoids in question.

As noted above, condition (E1) requires $\dot{\sim}$-equivalences only between elements of $\mathscr{P}$ and the remaining conditions require $\dot{\sim}$-equivalences only between elements of $\mathscr{Q}$. Furthermore, $\mathscr{P}$ is the union of the $\sim$-classes of all $c \in L^\star$. By (E1), these $\sim$-classes are also $\dot{\sim}$-classes. Moreover, by Proposition 7, each $\sim$-class of a $c \in L^\star$ is a $\bar{\sim}$-class. We conclude that the $\bar{\sim}$-class of each $c \in L^\star$ coincides with the $\dot{\sim}$-class of c and $\mathscr{P}$ is the union of these subsets.

We next check that any two elements that are $\dot{\sim}$-equivalent according to one of the conditions (E2)–(E4) are also $\bar{\sim}$-equivalent. Since $\dot{\sim}$ is, by assumption, the smallest equivalence relation with the indicated properties, it will then follow that $\dot{\sim} \subseteq \bar{\sim}$.

Ad (E2): Let $(a,b),(b,c) \in \mathscr{P}$, $d,e \in L^\star$, $(a,b) \sim d$, and $(b,c) \sim e$. Then $a,b,c \in L^\star$, hence $a \bar{\odot} b = a \odot b = d$ and $b \bar{\odot} c = b \odot c = e$. Consequently, $d \bar{\odot} c = (a \bar{\odot} b) \bar{\odot} c = a \bar{\odot} (b \bar{\odot} c) = a \bar{\odot} e$, that is $(d,c) \bar{\sim} (a,e)$.

Ad (E3): Let $a,b,c,e \in L^\star$, $(a,b) \in \mathscr{Q}$, $(b,c) \sim e$, and $c < 1$. Then $a \bar{\odot} b \leqslant \alpha$ and hence $a \bar{\odot} e = a \bar{\odot} (b \bar{\odot} c) = (a \bar{\odot} b) \bar{\odot} c \leqslant \alpha \bar{\odot} c$. As L is assumed to be Archimedean,

α is the atom of $\bar{L}$, and $c < 1$, we conclude $\alpha \mathbin{\bar{\odot}} c = 0$. Hence $(a,e) \mathrel{\bar{\sim}} 0$. Similarly, we argue for the second part of (E3).

Ad (E4): As L is Archimedean, we have, for any $b < 1$, $0 \mathbin{\bar{\odot}} 1 = 1 \mathbin{\bar{\odot}} 0 = \alpha \mathbin{\bar{\odot}} b = b \mathbin{\bar{\odot}} \alpha = 0$ by Lemma 7 and hence $(0,1) \mathrel{\bar{\sim}} (1,0) \mathrel{\bar{\sim}} (\alpha,b) \mathrel{\bar{\sim}} (b,\alpha)$. Furthermore, we have $(\alpha,1) \mathrel{\bar{\sim}} (1,\alpha)$. Finally, let $(a,b),(c,d) \in \mathscr{Q}$ and assume $(a,b) \trianglelefteq (c,d) \mathrel{\bar{\sim}} 0$. Then $a \mathbin{\bar{\odot}} b \leqslant c \mathbin{\bar{\odot}} d = 0$ and thus $(a,b) \mathrel{\bar{\sim}} 0$ as well.

It is finally clear that the $\bar{\sim}$-class of 0 is downward closed. The last statement holds by condition (P2) of a tomonoid partition.

The following theorem is the main result of this section.

Theorem 5. *Let $(L^2; \trianglelefteq, \sim, (1,1))$ be an Archimedean f.n. tomonoid partition and let $(\bar{L}^2; \trianglelefteq, \dot{\sim}, (1,1))$ be the ramification of L^2. Let $\bar{\sim}$ be an equivalence relation on L^2 that is coarser than $\dot{\sim}$ and such that the following holds: the $\bar{\sim}$-class of each $c \in L^\star$ coincides with the $\dot{\sim}$-class of c, the $\bar{\sim}$-class of 0 is downward closed, and each $\bar{\sim}$-class contains exactly one element of the form $(1,c)$ for some $c \in \bar{L}$. Then $(\bar{L}^2; \trianglelefteq, \bar{\sim}, (1,1))$ is an Archimedean one-element coextension of L^2.*

Moreover, all Archimedean one-element coextensions of L^2 arise in this way.

Proof. $\mathscr{P}$, defined by (7), is the union of the $\sim$-classes of all $c \in L^\star$. As we have seen in the proof of Lemma 8, these subsets of $\mathscr{P}$ are also $\dot{\sim}$-classes. Recall also that $\mathscr{P}$ is upwards closed and $\mathscr{Q} = \bar{L}^2 \setminus \mathscr{P}$ is downward closed.

By (E4), we have $(1,0) \mathrel{\dot{\sim}} (0,1)$ and $(1,\alpha) \mathrel{\dot{\sim}} (\alpha,1)$. We claim that $(1,0) \mathrel{\not\dot{\sim}} (1,\alpha)$. Indeed, (E1), (E2), and (E3) involve only elements (a,b) such that $a,b \in L^\star$. Hence, none of these prescriptions involves the elements $(1,\alpha)$ or $(\alpha,1)$. Moreover, by (E4), the elements $(a,0)$ and $(0,a)$ for any a as well as (a,α) and (α,a) for any $a \neq 1$ belong to the $\dot{\sim}$-class of $(1,0)$. Again, $(1,\alpha)$ and $(\alpha,1)$ are not concerned. Finally, the $\dot{\sim}$-class of $(1,0)$ is a downward closed set. Also this prescription has no effect on $(1,\alpha)$ or $(\alpha,1)$ because there is no element in $\mathscr{Q}$ that is larger than $(1,\alpha)$ or $(\alpha,1)$. We conclude that $\{(1,\alpha),(\alpha,1)\}$ is an own $\dot{\sim}$-class and our claim is shown.

Let now $\bar{\sim} \supseteq \dot{\sim}$ be as indicated. Note that, by what we have seen so far, at least one such equivalence relation exists. In accordance with Proposition 6, we will verify (P1), (P2), and (P3').

We have shown that $(1,c) \mathrel{\bar{\sim}} (c,1)$ for all $c \in \bar{L}$. By construction, $\bar{\sim}$ fulfils (P2). Furthermore, the $\bar{\sim}$-class of 0 is downward closed and $\mathscr{Q}$, which is the union of the $\bar{\sim}$-classes of 0 and α, is downward closed as well. We conclude that (P1) holds for $\bar{\sim}$.

It remains to show that $\bar{\sim}$ fulfils (P3'). Let $a,b,c,d,e \in L \setminus \{0,1\}$ such that $(a,b) \mathrel{\bar{\sim}} d$ and $(b,c) \mathrel{\bar{\sim}} e$. We distinguish the following cases.

Case 1. Let $d,e \in L^\star$. Then $(a,b) \sim d$ and $(b,c) \sim e$. As $\sim$ fulfils (P3), we have $(d,c) \sim (a,e)$. In particular, it follows that $(d,c) \in \mathscr{P}$ iff $(a,e) \in \mathscr{P}$. If (d,c) and (a,e) are both in $\mathscr{P}$, we have $(d,c) \mathrel{\bar{\sim}} (a,e)$ because the $\dot{\sim}$-classes contained in $\mathscr{P}$ are $\bar{\sim}$-classes as well. If (d,c) and (a,e) are both in $\mathscr{Q}$, we have $(d,c) \mathrel{\dot{\sim}} (a,e)$ by (E2) and consequently also $(d,c) \mathrel{\bar{\sim}} (a,e)$, because $\bar{\sim}$ extends $\dot{\sim}$.

Case 2. Let $d = \alpha$ and $e \in L^\star$. Then $(d,c) \mathrel{\dot\sim} 0$ by (E4). Furthermore, we have $a \in L^\star$ by (E4), $b,c \in L^\star$ because $(b,c) \in \mathscr{P}$, $(a,b) \in \mathscr{Q}$, and $(b,c) \sim e$. It follows $(a,e) \mathrel{\dot\sim} 0$ by (E3). Consequently, $(d,c) \mathrel{\bar\sim} 0 \mathrel{\bar\sim} (a,e)$.

Case 3. Let $d \in L^\star$ and $e = \alpha$. We argue similarly to Case 2.

Case 4. Let $d = e = \alpha$. Then $(d,c) \mathrel{\dot\sim} (a,e) \mathrel{\dot\sim} 0$ by (E4) and consequently also $(d,c) \mathrel{\bar\sim} (a,e)$.

By Proposition 6, $(\bar{L}^2; \trianglelefteqslant, \bar\sim, (1,1))$ is a f.n. tomonoid partition, which is moreover Archimedean by (E4) and Lemma 7. It is finally clear from Proposition 7 that the Rees quotient of $\bar{L}^2$ by the atom α is L^2.

The final statement follows from Lemma 8.

Let us summarise our construction and add some remarks. In order to determine the one-element coextensions of a f.n. tomonoid L, we start from its associated tomonoid partition $(L^2; \trianglelefteqslant, \sim, (1,1))$. We first determine its ramification $(\bar{L}^2; \trianglelefteqslant, \dot\sim, (1,1))$ according to Definition 12. This is done by means of the conditions (E1)–(E4); note that these prescriptions are largely independent, it is not necessary to apply them in a recursive way. To obtain, second, a coextension of the desired type, the set $\mathscr{Z} = \langle(1,0)\rangle_{\bar\sim}$, i.e. the $\bar\sim$-class of the bottom element, is chosen according to Theorem 5. This is done as simple as follows: $\mathscr{Z}$ is a union of $\dot\sim$-classes contained in $\mathscr{Q}$ including $\langle(1,0)\rangle_{\dot\sim}$ but excluding the $\dot\sim$-class $\{(1,\alpha),(\alpha,1)\}$, and $\mathscr{Z}$ is downward closed. Thus, to determine a specific one-element coextension, all we have to do is to select an arbitrary set of $\dot\sim$-classes different from $\{(\alpha,1),(1,\alpha)\}$ and $\mathscr{Z}$ will then be the smallest downward closed set containing them.

Note that one possible choice is $\mathscr{Z} = \mathscr{Q} \setminus \{(\alpha,1),(1,\alpha)\}$. This means that the explained procedure always leads to a result, that is, every Archimedean, finite, negative tomonoid has at least one Archimedean one-element coextension.

Also in the general case, it is interesting that the explained procedure never requires revisions. At no place decisions are required that lead to an impossible situation, we may always proceed to end up with a coextension as desired.

Acknowledgement

Th. V. was supported by the Austrian Science Fund (FWF): project I 1923-N25 (New perspectives on residuated posets). M. P. was supported by the Czech Science Foundation under Project 15-07724Y.

References

1. J. Aczél, Quasigroups, nets and nomograms, *Advances in Mathematics* **1** (1965), 383–450.
2. W. Blaschke, G. Bol, "Geometrie der Gewebe, topologische Fragen der Differentialgeometrie" (in German), Springer, Berlin 1939.
3. K. Blount, C. Tsinakis, The structure of residuated lattices, *Internat. J. Algebra Comput.* **13** (2003), 437 - 461.

4. W. Chen, X. Zhao, The structure of idempotent residuated chains, *Czech. Math. J.* **59** (2009), 453 - 479.
5. A. Ciabattoni, G. Metcalfe, F. Montagna, Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions, *Fuzzy Sets Syst.* **161**, 369–389 (2010).
6. A. H. Clifford, Extensions of semigroups, *Trans. Amer. Math. Soc.* **68** (1950), 165 - 173.
7. A. H. Clifford, G. B. Preston, *The algebraic theory of semigroups*, vol. 1, American Mathematical Society, Providence 1961.
8. P. Codara, Partitions of a finite partially ordered set, in: E. Damiani, O. D'Antona, V. Marra, F. Palombi (Eds.), "From Combinatorics to Philosophy: The Legacy of G.-C. Rota", Springer, Dordrecht 2009; pp. 45 - 60.
9. D. Cox, J. Little, D. O'Shea, "Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra", 3rd ed, Springer, New York 2007.
10. B. De Baets, R. Mesiar, Discrete triangular norms, in: S.E. Rodabaugh et al. (Eds.), "Topological and algebraic structures in fuzzy sets. A handbook of recent developments in the mathematics of fuzzy sets", Kluwer Academic Publishers, Dordrecht, 2003; pp. 389–400.
11. S. Eilenberg, M. P. Schützenberger, Rational sets in commutative monoids, *J. Algebra* **13**, 173–191 (1969).
12. F. Esteva, Ll. Godo, Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, *Fuzzy Sets Syst.* **124** (2001), 271 - 288.
13. K. Evans, M. Konikoff, J. J. Madden, R. Mathis, G. Whipple, Totally ordered commutative monoids, *Semigroup Forum* **62** (2001), 249 - 278.
14. L. Fuchs, "Partially ordered algebraic systems", Pergamon Press, Oxford (1963).
15. E. Ya. Gabovich, Fully ordered semigroups and their applications, *Russ. Math. Surv.* **31**, 147–216 (1976).
16. N. Galatos, P. Jipsen, T. Kowalski, H. Ono, "Residuated lattices. An algebraic glimpse at substructural logics", Elsevier, Amsterdam 2007.
17. P. A. Grillet, "Semigroups. An introduction to the structure theory", Marcel Dekker, New York 1995.
18. P. A. Grillet, "Commutative semigroups", Kluwer Academic Publishers, Dordrecht 2001.
19. P. Hájek, "Metamathematics of Fuzzy Logic", Kluwer Academic Publisher, Dordrecht (1998).
20. R. Horčík, Structure of commutative cancellative integral residuated lattices on $(0,1]$, *Algebra Univers.* **57** (2007), 303 - 332.
21. R. Horčík, Solutions to some open problems on totally ordered monoids, *J. Log. Comput.* **20** (2010), 977 - 983.
22. R. Horčík, On the structure of finite integral commutative residuated chains, *J. Log. Comput.* **21** (2011), 717 - 728.
23. J. M. Howie, "An introduction to semigroup theory", Academic Press, London 1976.
24. A. J. Hulin, Extensions of ordered semigroups, *Czech. Math. J.* **26** (1976), 1 - 12.
25. J. Ježek, T. Kepka, P. Němec, One-element extensions of commutative semigroups, *Acta Universitatis Carolinae - Mathematica et Physica* **52** (2011), 33 35.
26. L. Kari, G. Thierrin, Languages and compatible relations on monoids. In: Gh. Paun (Ed.), Mathematical Linguistics and Related Topics, Ed. Academiei Romane, Bucuresti (1995), pp. 212–220.
27. N. Kehayopulu, M. Tsingelis, Ideal extensions of ordered semigroups, *Commun. Algebra* **31**, (2003), 4939 - 4969.
28. E. P. Klement, R. Mesiar, E. Pap, "Triangular Norms", Kluwer Acad. Publ., Dordrecht 2000.
29. J. Meakin, K. S. S. Nambooripad, Coextensions of regular semigroups by rectangular bands, *Trans. Am. Math. Soc.* **269** (1982), 197 - 224; **272** (1982), 555 - 568.
30. F. Montagna, C. Noguera, R. Horčík, On weakly cancellative fuzzy logics, *J. Log. Comput.* **16** (2006), 423 - 450.
31. C. Noguera, F. Esteva, Ll. Godo, Generalized continuous and left-continuous t-norms arising from algebraic semantics for fuzzy logics, *Inf. Sci.* **180** (2010), 1354 - 1372.
32. M. Petrich, "Introduction to semigroups", Charles E. Merrill Publishing Company, Columbus 1973.

33. M. Petrík, P. Sarkoci, Associativity of triangular norms characterized by the geometry of their level sets, *Fuzzy Sets Syst.* **202** (2012), 100 - 109.
34. M. Petrík, P. Sarkoci, Convex combinations of nilpotent triangular norms, *J. Math. Anal. Appl.* **350** (2009), 271 - 275.
35. M. Petrík, Th. Vetterlein, Rees coextensions of finite, negative tomonoids, *J. Log. Comput.*, to appear.
36. Th. Vetterlein, Totally ordered monoids based on triangular norms, *Commun. Algebra* **43** (2015), 2643 - 2679.
37. Th. Vetterlein, On positive commutative tomonoids, *Algebra Univers.*, to appear.
38. Th. Vetterlein, Real coextensions as a tool for constructing triangular norms, draft; available at `www.flll.jku.at/sites/default/files/u24/Standard-extensions.pdf`.
39. Th. Vetterlein, A representation of finite, positive, commutative tomonoids, draft; available at `www.flll.jku.at/sites/default/files/u24/Direction-f-cones.pdf`.
40. Th. Vetterlein, Algebraic semantics: the structure of chains; in: P. Cintula, C. Fermüller, C. Noguera (Eds.), "Handbook of Mathematical Fuzzy Logic", vol. 3, to appear.
41. G. W. Whipple, A condition for constructing formally real monoid algebras, *Algebra Univers.* **54**, 53–63 (2005).