Chapter I: Algebraic semantics: the structure of chains

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1 Introduction

Chapter IV of Volume 1 of this Handbook is devoted to the algebraic semantics of substructural logics. Its central topic are FL-algebras, also called pointed residuated lattices, which represent an algebraic counterpart of FL, the Full Lambek logic.

In the present chapter, we resume this topic, adopting however a somewhat narrower point of view. We are interested in the generalised semantics of fuzzy logics and we focus to this end on semilinear, integral residuated lattices. Semilinearity means that the subdirect irreducible algebras are totally ordered. In fact we will restrict the discussion to residuated chains, in accordance with the fact that the most basic property characterising standard semantics in fuzzy logic is linearity. Integrality means that the monoidal identity is the top element. We generally require also this condition to hold, in accordance with the fact that in fuzzy logic it is usually assumed that a fully true proposition behaves in conjunctions neutrally.

We are thus interested in a classification of integral residuated chains. We seek ways to reduce these algebras to simpler ones, or ways to construct these algebras from structures that are better understood. Our point of view is in this sense "constructive" and we will not much develop universal-algebraic aspects. However, our analysis is based on established algebraic procedures; constructions methods that are incompatible with our algebraic framework will not be taken into account.

The problem that we address is, in the general case, far from a solution. No single approach is known at present that is expected to have the potential to cover eventually all structures in which we are interested. Here, we select two approaches, which are totally different in nature, and demonstrate their capabilities and limits. We shall see that each approach leads to an insight into, certainly not all but, quite a range of residuated structures.

The first approach takes up the apparently central role played by lattice-ordered groups within the variety of residuated lattices. Indeed, residuated lattices can be built from lattice-ordered groups in many ways, the best-known examples being MV-algebras or the more comprehensive BL-algebras.

A natural candidate for representations by means of groups are cancellative residuated lattices. For this topic we refer to [22]. Here, we will exploit the property of divisibility, a condition fulfilled, e.g., by BL-algebras. We will, however, not assume commutativity. As the main result, we will prove a representation theorem for divisible, integral residuated chains, also known as totally ordered pseudohoops [8]. This result generalises the well-known structure theorem for BL-algebras [1]. What makes our exposition special is the employed method, which differs, e.g., from the one employed in [8]. The idea is to represent residuated structures by means of partial algebras, as has been proposed in the context of quantum structures [9].

Our second approach deals with the commutative case, but this time we will not assume a property that is as particular as divisibility. Our focus will thus be on integral, commutative residuated chains. The example that we have in mind is the standard semantics of the fuzzy logic MTL. Recall that MTL is the logic of left-continuous t-norms together with their residua; a standard MTL-algebra is the real unit interval endowed with these two operations. In accordance with this example, we will focus on almost complete chains, where almost completeness is, roughly speaking, the same as completeness except for the possible absence of a bottom element.

The residuated chains considered in this second part possess a totally ordered set of quotients induced by filters. Our main concern is the question how to construct coextensions, that is, structures whose quotient is a given one. Under the condition that the congruence classes are isomorphic to real intervals, we can describe all those coextensions that are, in a natural sense, indivisible. In particular, all standard MTL-algebras that possess a quotient induced by an Archimedean filter are fully specifiable in terms of this quotient and the order type of the congruence classes.

We note that for the specification as well as for the visual representation of integral, commutative residuated chains, their quotients, and their coextensions we have a simple, yet efficient tool to our disposal: the regular representation of monoids [7], adapted to our context in the straightforward way. On this basis, we can, for instance, specify a coextension in a "modular" way. We will in fact do so when formulating our main result on real Archimedean coextensions.

We will proceed as follows. The following Section 2 puts up our favourite algebraic framework, providing the basic definitions around totally ordered monoids. The subsequent Section 3 is devoted to the partial-algebra method for the representation of residuated chains, and Section 4 discusses quotients and coextension of residuated chains. We conclude with some more background information as well as hints for further reading in Section 5.

2 Residuated totally ordered monoids

We begin by recalling the algebraic notions relevant for this chapter; cf. Chapter IV of Volume 1. A *residuated lattice* is an algebra $\langle L, \wedge, \vee, \odot, /, \backslash, 1\rangle$ such that (i) $\langle L, \wedge, \vee\rangle$ is a lattice, (ii) $\langle L, \odot, 1\rangle$ is a monoid, and (iii) $/, \backslash$ are the left and right residuals of $\odot$, respectively. The latter condition means that, for $a, b, c \in L$ we have

$$a \odot b \leq c \quad \text{if and only if} \quad b \leq a\backslash c \quad \text{if and only if} \quad a \leq c/b.$$

In other words, for any $a \in L$, the right translation $\cdot \odot a$ and the mapping $\cdot/a$ form a Galois connection; similarly for the left translation $a \odot \cdot$ and the mapping $a\backslash\cdot$.

Moreover, we call a residuated lattice *commutative* if so is the monoidal operation $\odot$, and we call it *integral* if the monoidal identity 1 is the top element of the lattice. Finally, a *chain* is meant to be a totally ordered set, and we refer to a residuated lattice whose underlying order is a chain as a *residuated chain*.

In fuzzy logic, the implications are usually considered as the most basic connectives. Accordingly, we could consider the residuals $/$ and $\backslash$ as the primary operations of a residuated lattice. We will, however, not do so. The residuals on the one hand and the monoidal operation on the other hand are, given the lattice order, mutually uniquely determined. To explore the structure of residuated lattices it is thus not necessary to deal with all three operations. The monoidal operation may be viewed as a product, the residuals may be viewed as divisions; we will describe the structure of residuated lattices on the basis of the former, the product-like operation alone.

Moreover, we focus in this chapter exclusively on residuated chains. Consequently, there is no need to use both the infimum and the supremum. It is, for the sake of forming quotients, moreover no serious obstacle to replace the lattice operations by the total order relation.

DEFINITION 2.0.1. *A structure* $\boldsymbol{L} = \langle L, \leq, \odot, 1 \rangle$ *is a* totally ordered monoid*, or* tomonoid *for short, if* (i) $\langle L, \odot, 1 \rangle$ *is a monoid,* (ii) $\langle L, \leq \rangle$ *is a chain, and* (iii) $\leq$ *is compatible with* $\odot$*, that is,* $a \leq b$ *implies* $a \odot c \leq b \odot c$ *and* $c \odot a \leq c \odot b$.

A tomonoid $\boldsymbol{L}$ *is called* residuated *if, for any* $a, b \in L$*, there is a largest element* c *such that* $a \odot c \leq b$ *and there is a largest element* d *such that* $d \odot a \leq b$*. Moreover,* $\boldsymbol{L}$ *is called* commutative *is so is* $\odot$*, and* $\boldsymbol{L}$ *is called* negative *if* 1 *is the top element.*

It is clear that residuated tomonoids are in a one-to-one correspondence with residuated chains. Under this correspondence, the properties of commutativity coincide, and so do the properties of integrality and negativity, respectively.

In semigroup theory, there are two competing ways of denoting the associative operation. Instead of the multiplicative notation $\langle L, \odot, 1 \rangle$ we may equally well use the additive one $\langle L, \oplus, 0 \rangle$. In logics, the former notation is clearly preferred because $\odot$ typically models the conjunction, 1 stands for the full truth, and the set of propositions is traditionally ordered such that stronger statements are modelled by smaller elements. However, the additive notation is often more practical, and authors indeed regularly switch to the additive notation when it comes to free commutative monoids. In case of the two approaches that we are going to present, we will stick to the notation that is in each case more practical and supports easiest understanding: the first approach will be presented additively, the second one multiplicatively.

We will certainly not use extra expressions like "dual tomonoids" or similarly; we will simply talk about tomonoids $\langle L, \leq, \odot, 1 \rangle$ or $\langle L, \leq, \oplus, 0 \rangle$, respectively. Only one aspect needs to taken into account. Negativity will be applied only to multiplicatively written tomonoids. Analogously, we call an additively written tomonoid $\langle L, \leq, \oplus, 0 \rangle$ *positive* if 0 is the bottom element.

In what follows, a *subtomonoid* of a tomonoid $\boldsymbol{L}$ is a submonoid of L together with the inherited total order. Moreover, a *homomorphism* between tomonoids is defined as usual. Finally, a homomorphism χ between tomonoids $\boldsymbol{K}$ and $\boldsymbol{L}$ is called *sup-preserving* if, whenever the supremum of elements a_ι, $\iota \in I \neq \emptyset$, exists in $\boldsymbol{K}$, then $\chi(\bigvee_\iota a_\iota)$ is the supremum of $\chi(a_\iota)$, $\iota \in I$, in $\boldsymbol{L}$.

3 The partial-algebra method for the representation of totally ordered monoids

3.1 The idea

The typical aim of a representation theorem is to describe the structure of an algebra by means of algebras of a simpler type. In the case of residuated lattices the probably most often considered candidate for the basic constituents are lattice-ordered groups, or ℓ-groups for short. By means of ℓ-groups it will most likely never be possible to fully understand the structure of residuated lattices. However, for certain subclasses the idea has turned out to be fruitful, as will be exemplified by the results of the first part of the present chapter.

To understand the key idea of what we have in mind, let us see on the basis of a simple example how ℓ-groups can be used to represent residuated tomonoids. Let $\star\colon [0,1]^2 \to [0,1], \;\; \langle a,b\rangle \mapsto (a+b-1) \vee 0$ be the Łukasiewicz t-norm and consider the tomonoid $\langle [0,1], \leq, \star, 1\rangle$. This is an MV-algebra, and by Mundici's representation theorem we certainly know how it is related to the totally ordered group of reals. Here, however, we want to explain on the basis of the simple case a procedure that is applicable to a class of residuated structures that is more comprehensible than MV-algebras.

Switching to the additive picture, we are led to the algebra $\langle [0,1], \leq, \oplus, 0\rangle$, where $\oplus$ is the truncated sum, that is,

$$a \oplus b \;=\; (a+b) \wedge 1, \quad a,b \in [0,1]. \tag{1}$$

From $\oplus$, we will now define a partial operation, which we denote by $+$. The partial operation $+$ will, where defined, coincide with the total one $\oplus$; in this sense, $+$ will be a restriction of $\oplus$. Our definition goes as follows:

$$a + b \;=\; \begin{cases} a \oplus b & \text{if } a \text{ is the smallest } x \text{ such that } x \oplus b = a \oplus b, \\ & \text{and } b \text{ is the smallest } y \text{ such that } a \oplus y = a \oplus b; \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{2}$$

Consider now the partial algebra $\langle [0,1], \leq, +, 0\rangle$. If the usual sum of two reals $a, b \in [0,1]$ is strictly greater than 1, their sum $a+b$, according to (2), is undefined. If, however, the usual sum of a and b is at most 1, then $a+b$ stands for what it commonly denotes: the sum of a and b as reals. In short, the partial operation $+$ is the restriction of the usual addition of reals to those pairs whose sum is in $[0,1]$.

Remarkably, we do not lose information when switching to the partial operation $+$. In fact, we can easily recover the original operation: $a \oplus b$ is the maximal element among all defined sums $a' + b'$ such that $a' \leq a$ and $b' \leq b$.

Our aim is to embed $\langle [0,1], \leq, +, 0\rangle$ into $\langle \mathsf{R}, \leq, +, 0\rangle$. How can we construct the totally ordered group of reals from our partial algebra? We first determine the monoid freely generated by $[0,1]$ subject to the condition $a + b = c$ if this holds in the partial algebra. The result is the monoid $\langle \mathsf{R}^+, +, 0\rangle$, the positive reals endowed with the usual addition. Second, we make R^+ the positive cone of a totally ordered Abelian group. The result is $\langle \mathsf{R}, \leq, +, 0\rangle$, and our embedding is complete.

The question remains how the algebra $\langle [0,1], \leq, \oplus, 0\rangle$ with which we started is represented by $\langle \mathsf{R}, \leq, +, 0\rangle$. The situation is as follows. The base set $[0,1]$ is an *interval* of R, consisting of all positive elements below the positive element 1. Furthermore, the total order is inherited from R. Finally, the monoidal operation is given according to (1).

The aim of this section is to show that we can proceed analogously to this example under rather general assumptions.

3.2 D.p.r. tomonoids and R-chains

Let us delimit the class of tomonoids to which our method is at present known to apply. Our main assumption is that the order of the tomonoid is the natural one: we require divisibility.

DEFINITION 3.2.1. *A residuated tomonoid $\langle L, \leq, \oplus, 0\rangle$ is called* divisible *if, for any $a, b \in L$ such that $a \leq b$, there are $c, d \in L$ such that $a \oplus c = d \oplus a = b$.*

Hence the structures that we are going to discuss are divisible, positive, residuated tomonoids; we will abbreviate these three properties with "d.p.r.". Note that d.p.r. tomonoids are in a one-to-one correspondence with divisible, integral residuated chains.

We will denote the residuals corresponding to the monoidal operation $\oplus$ of a d.p.r. tomonoid by $\oslash$ and $\oslash$, respectively, and in accordance with our additive notation we will write them in analogy to differences:

$$a \leq b \oplus c \quad \text{iff} \quad a \,\backslash\!\!\!\bigcirc\, b \leq c \quad \text{iff} \quad a \oslash c \leq b.$$

Then a positive, residuated tomonoid L is divisible if, for any $a, b \in L$ such that $a \leq b$,

$$a \oplus (b \,\backslash\!\!\!\bigcirc\, a) \;=\; (b \oslash a) \oplus a \;=\; b.$$

With a d.p.r. tomonoid, we associate a partial algebra as follows.

DEFINITION 3.2.2. *Let $\langle L, \leq, \oplus, 0\rangle$ be a d.p.r. tomonoid. For $a, b \in L$, let $a + b = a \oplus b$ if $a = (a \oplus b) \oslash b$ and $b = (a \oplus b) \,\backslash\!\!\!\bigcirc\, a$; otherwise, let $a + b$ be undefined. Then we call $\langle L, \leq, +, 0\rangle$ the* partial algebra associated with $\langle L, \leq, \oplus, 0\rangle$.

Note that this definition is in accordance with the specification (2) in our informal introduction. Namely, let a and b be elements of a d.p.r. tomonoid. Then $(a \oplus b) \oslash b$ is, by definition, the smallest element x such that $x \oplus b = a \oplus b$. Similarly, $(a \oplus b) \,\backslash\!\!\!\bigcirc\, a$ is the smallest element y such that $a \oplus y = a \oplus b$.

From the partial algebra that we have associated with a d.p.r. tomonoid in Definition 3.2.2, we can recover the original structure in the following way.

LEMMA 3.2.3. *Let $\langle L, \leq, \oplus, 0\rangle$ be a d.p.r. tomonoid, and let $\langle L, \leq, +, 0\rangle$ its associated partial algebra. Then*

$$a \oplus b \;=\; \max\,\{a' + b' \colon\; a' \leq a \text{ and } b' \leq b \text{ such that } a' + b' \text{ is defined}\} \tag{3}$$

for any $a, b \in L$.

Proof. Assume that $a' \leq a,\ b' \leq b$, and $a'+b'$ is defined. Then $a'+b' = a' \oplus b' \leq a \oplus b$.

Let now $c = a \oplus b;\ a' = c \oslash b$, and $b' = c \otimes a'$. Then $a' \oplus b' = a' \oplus (c \otimes a') = c$. Moreover, $c \oslash b' = c \oslash (c \otimes a') = c \oslash (c \otimes (c \oslash b)) = c \oslash b = a'$. Hence $a' + b'$ exists and equals $a \oplus b$. □

What kind of partial algebras do we get here? The following definition compiles their properties.

As regards the existence of partially defined sums, we will follow, whenever reasonable, the usual convention: $a + b = c$ means that $a + b$ exists and equals c.

DEFINITION 3.2.4. *An* R-chain *is a structure* $\langle L, \leq, +, 0 \rangle$ *such that*

(E1) $\langle L, \leq, 0 \rangle$ *is a chain with the bottom element* 0*,*

and such that $+$ *is a partial binary operation fulfilling, for any* $a, b, c \in L$*, the following conditions:*

(E2) $(a+b)+c$ *is defined if and only if* $a+(b+c)$ *is defined, and in this case* $(a+b)+c = a + (b + c)$.

(E3) $a + 0 = 0 + a = a$.

(E4) *If* $a + c$ *and* $b + c$ *are defined, then* $a \leq b$ *if and only if* $a + c \leq b + c$.

If $c + a$ *and* $c + b$ *are defined, then* $a \leq b$ *if and only if* $c + a \leq c + b$.

(E5) *If* $a + b$ *is defined, there are* $x, y \in L$ *such that* $a + b = x + a = b + y$.

(E6) *Let* $a \leq b$*. Then there is a largest element* $\bar{a} \leq a$ *such that* $b = \bar{a} + x$ *for some* $x \in L$.

Similarly, there is a largest element $\bar{\bar{a}} \leq a$ *such that* $b = y + \bar{\bar{a}}$ *for some* $y \in L$.

(E7) *If* $a \leq c \leq a + b$*, there is an* $x \in L$ *such that* $c = a + x$.

Similarly, if $a \leq c \leq b + a$*, there is a* $y \in L$ *such that* $c = y + a$.

The remaining part of this section is devoted to the proof that the partial algebra associated with a d.p.r. tomonoid is in fact an R-chain.

We begin with those properties that are comparably easy to prove.

LEMMA 3.2.5. *Let* $\langle L, \leq, \oplus, 0 \rangle$ *be a d.p.r. tomonoid. Then the associated partial algebra* $\langle L, \leq, +, 0 \rangle$ *fulfils* (E1)*,* (E3)*,* (E4)*,* (E6)*, and* (E7)*.*

Proof. (E1) holds $\leq$ coincides by definition with total order of L as a positive tomonoid.

For the remaining properties, we show only the first half; the second half will follow in each case dually.

It is easily checked that, for any $a \in L$, $a + 0$ and exist and equals a. (E3) follows.

Let now $a, b, c \in L$ such that $a + c$ and $b + c$ are defined. If $a \leq b$, then $a + c = a \oplus c \leq b \oplus c = b + c$. Conversely, if $a + c \leq b + c$, then $a = (a \oplus c) \oslash c = (a + c) \oslash c \leq (b + c) \oslash c = (b \oplus c) \oslash c = b$. We have proved (E4).

Next, let $a, b \in L$ such that $a \leq b$. Let $\bar{a} = b \oslash (b \otimes a)$. Then $\bar{a} \leq a$ and $\bar{a} + x = b$, where $x = b \otimes a$. If $a' \leq a$ and $a' + x' = b$, we have $x' = b \otimes a' \geq b \otimes a = x$ and hence $a' = b \oslash x' \leq b \oslash x = \bar{a}$. This shows (E6).

Finally, let $a, b, c \in L$ such that $a \leq c \leq a+b$. We shall show that $a = c \oslash (c \otimes a)$; it will then follow that $a + x = c$, where $x = c \otimes a$. Putting $d = a + b$, we derive from the divisibility of L that $a = d \oslash b = d \oslash (d \otimes a) \geq c \oslash (c \otimes a) \geq (d \oslash (d \otimes c)) \oslash (c \otimes a) = (d \oslash (d \otimes (a \oplus (c \otimes a)))) \oslash (c \otimes a) = (d \oslash ((d \otimes a) \otimes (c \otimes a))) \oslash (c \otimes a) = d \oslash ((c \otimes a) \oplus ((d \otimes a) \otimes (c \otimes a))) = d \oslash (d \otimes a) = a$. (E7) follows. □

We continue with property (E2), the associativity.

LEMMA 3.2.6. *Let $\langle L, \leq, \oplus, 0 \rangle$ be a d.p.r. tomonoid. Then the associated partial algebra $\langle L, \leq, +, 0 \rangle$ fulfils* (E2).

Proof. Let $a, b, c \in L$ be such that $(a + b) + c$ is defined. Let $e = a + b$ and $d = e + c$. Let $f = d \otimes a$. Then $c = d \otimes e \leq d \otimes a = f \leq d = e + c$, and by (E7), there is a $b' \leq e$ such that $f = b' + c$. Then $b' = f \oslash c = (d \otimes a) \oslash c = (d \oslash c) \otimes a = b$, and we have shown that $f = b + c$.

Next, let $a' = d \oslash f$. Then $d \otimes a' = d \otimes (d \oslash f) = d \otimes (d \oslash (d \otimes a)) = d \otimes a = f$, and it follows $d = a' + f$. Furthermore, $a = (d \oslash c) \oslash b = (d \oslash c) \oslash (f \oslash c) = d \oslash ((f \oslash c) \oplus c) = d \oslash f = a'$. The proof is complete that $(a + b) + c = a + (b + c)$. The other half of (E2) is proved analogously. □

We finally turn to the property (E5). Let us define, for a d.p.r. tomonoid:

$$\begin{aligned} a \preccurlyeq_l b \quad & \text{if there is an } x \in L \text{ such that } b + x \text{ exists and equals } a, \\ a \preccurlyeq_r b \quad & \text{if there is an } y \in L \text{ such that } y + b \text{ exists and equals } a, \end{aligned} \tag{4}$$

where the operation $+$ refers to the associated partial algebra. We shall show that $\preccurlyeq_l$ and $\preccurlyeq_r$ are coinciding partial orders; obviously, (E5) will then follow.

LEMMA 3.2.7. *Let $\langle L, \leq, \oplus, 0 \rangle$ be a d.p.r. tomonoid. Then $\preccurlyeq_l$ and $\preccurlyeq_r$ are partial orders, both being extended by $\leq$.*

Proof. It is clear that $a \preccurlyeq_l b$ or $a \preccurlyeq_r b$ implies $a \leq b$. It further follows that $\preccurlyeq_l$ and $\preccurlyeq_r$ are reflexive and antisymmetric. Finally, the transitivity of $\preccurlyeq_l$ and $\preccurlyeq_r$ follows from (E2). □

LEMMA 3.2.8. *Let $\langle L, \leq, \oplus, 0 \rangle$ be a d.p.r. tomonoid.*

(i) *Let $0 < a \leq b \leq c$. Then $a \preccurlyeq_l b$ and $b \preccurlyeq_l c$ if and only if $a \preccurlyeq_l c$. Similarly, $a \preccurlyeq_r b$ and $b \preccurlyeq_r c$ if and only if $a \preccurlyeq_r c$.*

(ii) *Let $a, b \in L$. Then $a \preccurlyeq_l b$ if and only if $a \preccurlyeq_r b$.*

Proof. (i) We prove the claim for $\preccurlyeq_r$; the assertion follows for $\preccurlyeq_l$ similarly.

By Lemma 3.2.7, $a \preccurlyeq_r b$ and $b \preccurlyeq_r c$ imply $a \preccurlyeq_r c$.

Let $a \preccurlyeq_r c$. From (E7), we conclude $a \preccurlyeq_r b$. This implies $a = b \oslash (b \otimes a) \geq (c \oslash (c \otimes b)) \oslash (b \otimes a) = c \oslash (c \otimes a) = a$, that is, $(c \oslash (c \otimes b)) \oslash (b \otimes a) = b \oslash (b \otimes a) > 0$. By divisibility, we conclude $c \oslash (c \otimes b) = b$, and it follows $b \preccurlyeq_r c$.

(ii) If $a = 0$ or $a = b$, the claim is clear. Assume $0 < a < b$ and $a \preccurlyeq_r b$.

Let x be such that $b = a + x$. If $x \leq a$, we conclude from $x \leq a < b$ and $x \preccurlyeq_l b$ by part (i) that $a \preccurlyeq_l b$.

Assume $a < x$. From $a < x \leq b$ and $a \preccurlyeq_r b$, we have by part (i) that $x \preccurlyeq_r b$. Let s be such that $b = x + s$. If $s \leq a$, we conclude from $s \preccurlyeq_l b$ and $s \leq a < b$ that $a \preccurlyeq_l b$.

Assume $a < s$. Then $a < s \leq x + s = b = a + x$. Thus there is an $r \leq x$ such that $s = a + r$, and we have $b = x + s = x + a + r$. In particular, $r \preccurlyeq_l b$. If then $r \leq a$, we conclude $a \preccurlyeq_l b$.

Assume $a < r$. Then $a < r \leq x \leq x + a$, and it follows that $r = u + a$ for some u. We conclude $b = x + a + r = x + a + u + a$, that is $a \preccurlyeq_l b$.

Analogously, we show that $a \preccurlyeq_l b$ implies $a \preccurlyeq_r b$. □

In what follows, we will denote the coinciding partial orders $\preccurlyeq_l$ and $\preccurlyeq_r$ of the R-chain associated with a d.p.r. tomonoid simply by $\preccurlyeq$.

We have shown:

THEOREM 3.2.9. *Let $\langle L, \leq, \oplus, 0 \rangle$ be a d.p.r. tomonoid. Then the associated algebra $\langle L, \leq, +, 0 \rangle$ is an R-chain.*

3.3 Ordinal sum decomposition of R-chain

Having seen that the partial algebras associated with d.p.r. tomonoids are R-chains, we will prove in this section that these partial algebras are ordinal sums of R-chains that are naturally ordered. The latter notion is defined in the expected way.

DEFINITION 3.3.1. *Let $\langle L, \leq, +, 0 \rangle$ be an R-chain. We say that L is* naturally ordered *if, for any $a, b \in L$, $a \leq b$ if and only if there is an $x \in L$ such that $b = a + x$ if and only if there is a $y \in L$ such that $b = y + a$.*

In other words, to be naturally ordered means for R-chains arising from d.p.r. tomonoids that the total order $\leq$ coincides with $\preccurlyeq$.

The notion of an ordinal sum of R-chain is defined as usual.

DEFINITION 3.3.2. *Let $\langle I, \leq \rangle$ be a chain, and for every $i \in I$, let $\langle L_i, \leq, +, 0_i \rangle$ be an R-chain. Put $L = \dot{\bigcup}_{i \in I} (L_i \backslash \{0_i\}) \cup \{0\}$, where 0 is a new element and $\dot{\cup}$ denotes the disjoint union. For $a, b \in L$, put $a \leq b$ if either $a = 0$, or $a \in L_i$ and $b \in L_j$ such that $i < j$, or $a, b \in L_i$ for some i and $a \leq b$ holds in L_i. Similarly, define $a + b$ if either $a = 0$, in which case $a + b = b$, or $b = 0$, in which case $a + b = a$, or $a, b \in L_i$ for some i and $a + b$ is defined in L_i, in which case $a + b$ is mapped to the same value as in L_i. Then $\langle L, \leq, +, 0 \rangle$ is called the* ordinal sum *of the R-chains L_i w.r.t. $\langle I, \leq \rangle$.*

We easily check:

LEMMA 3.3.3. *An ordinal sum of R-chains is again an R-chain.*

THEOREM 3.3.4. *Let $\langle L, \leq, \oplus, 0\rangle$ be a d.p.r. tomonoid, and let $\langle L, \leq, +, 0\rangle$ be the associated R-chain. Then L is the ordinal sum of naturally ordered R-chains.*

Proof. By Lemma 3.2.8(i), $L\backslash\{0\} = \dot{\bigcup}_{i\in I} C_i$ for pairwise disjoint convex subsets C_i, $i \in I$, of L such that, for $a, b \in L\backslash\{0\}$, the following holds: $a \preccurlyeq b$ if and only if there is an $i \in I$ such that $a, b \in C_i$ and $a \leq b$.

Let $i \in I$. If $a, b \in C_i$ such that $a + b$ exists, we have $a + b \in C_i$ by construction. Consider $C_i \cup \{0\}$ endowed with the restriction of $\leq$ and $+$ to $C_i \cup \{0\}$ as well as with the constant 0. Then it is easily checked that $\langle C_i \cup \{0\}, \leq, +, 0\rangle$ fulfils (E1)–(E7). Hence $C_i \cup \{0\}$ is an R-algebra, which by construction is naturally ordered. Furthermore, L is the ordinal sum of the R-algebras $C_i \cup \{0\}$. □

3.4 Naturally ordered R-chains

We next turn to the characterisation of naturally ordered R-chains. We will show that any such R-chain can be embedded into the positive cone of totally ordered Abelian group.

LEMMA 3.4.1. *Let $\langle L, \leq, +, 0\rangle$ be a naturally ordered R-chain. For any $a, b, c \in L$, the following holds:*

(i) *If $a + b$ exists, $a_1 \leq a$, and $b_1 \leq b$, then also $a_1 + b_1$ exists.*

(ii) *If $a \leq b + c$, there are $b_1 \leq b$ and $c_1 \leq c$ such that $a = b_1 + c_1$ exists.*

Proof. (i) Let $a + b$ exists, and let $a_1 \leq a$ and $b_1 \leq b$. Then $a_1 \preccurlyeq a$ and $b_1 \preccurlyeq b$, hence there are x, y such that $a = x + a_1$ and $b = b_1 + y$. Thus $a + b = x + a_1 + b_1 + y$, and the claim follows from (E2).

(ii) If $a \leq b$, we put $b_1 = a$ and $c_1 = 0$. If $b < a$, there is by (E7) a c_1 such that $a = b + c_1$. By (E4), $c_1 \leq c$; thus we put $b_1 = b$ and we are done. □

We note that the statement of Lemma 3.4.1(ii) is usually refer to as a Riesz decomposition property.

By a scheme of the form (5) in the following lemma to hold, we mean that the sum of any row and any column exists and equals the element to which the respective arrow points to; the order of addition is from left to right or from top to bottom, respectively.

LEMMA 3.4.2. *Let $\langle L, \leq, +, 0\rangle$ be a naturally ordered R-chain. Let $a_1, \ldots, a_m, b_1, \ldots, b_n \in L$ be such that $a_1 + \ldots + a_m = b_1 + \ldots + b_n$, where $n, m \geq 1$. Then there are $d_{11}, \ldots, d_{mn} \in L$ such that*

$$\begin{array}{ccccc} d_{11} & \ldots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \ldots & b_n & & \end{array} \tag{5}$$

and

$$d_{ik} \wedge d_{jl} = 0 \text{ for every } 1 \leq i < j \leq m \text{ and } 1 \leq l < k \leq n. \tag{6}$$

Proof. If $m = 1$ or $n = 1$, the assertion is trivial. Let $m = n = 2$; then our assumption is $a_1 + a_2 = b_1 + b_2$. Assume that $a_1 \leq b_1$. Let d be such that $a_1 + d = b_1$; then we put $d_{11} = a_1$, $d_{12} = 0$, $d_{21} = d$, and $d_{22} = b_2$, and we are done. Similarly, we proceed in case $b_1 \leq a_1$.

Assume next that $m \geq 3$ and $n \geq 2$, and that the assertion holds for any pair $m' < m$ and $n' \leq n$. Then, obviously, the assertion holds for the pair m and n as well. □

From a naturally ordered R-chain $\langle L, \leq, +, 0 \rangle$ we will now construct the free monoid with the elements of L as its generators and with the conditions $a + b = c$, where $a, b, c \in L$ such that $a + b = c$ holds in L. We will have to show that the free monoid does not "collapse": the natural embedding of the R-chain is injective.

DEFINITION 3.4.3. *Let $\langle L, \leq, +, 0 \rangle$ be a naturally ordered R-chain. We call a sequence $\langle a_1, \ldots, a_n \rangle$ of $1 \leq n < \omega$ elements of L a* word *of L. We denote the set of words of L by $W(L)$, and we define $+ : W(L)^2 \to W(L)$ by concatenation.*

Moreover, let $\sim$ be the smallest equivalence relation on $W(L)$ such that

$$\langle a_1, \ldots, a_p, a_{p+1}, \ldots, a_n \rangle \sim \langle a_1, \ldots, a_p + a_{p+1}, \ldots, a_n \rangle$$

holds for any two words in $W(L)$ of the indicated form, where $1 \leq p < n$. We denote the equivalence class of some $\langle a_1, \ldots, a_n \rangle \in W(L)$ by $\langle\!\langle a_1, \ldots, a_n \rangle\!\rangle$ and the set of all equivalence classes by $C(L)$.

As seen in the next lemma, $C(L)$ is a semigroup under elementwise concatenation, into which L, as a semigroup, naturally embeds.

LEMMA 3.4.4. *Let $\langle L, \leq, +, 0 \rangle$ be a naturally ordered R-chain.*

(i) *The equivalence relation $\sim$ on $W(L)$ is compatible with $+$. $+$ being the induced operation, $\langle C(L), +, \langle\!\langle 0 \rangle\!\rangle \rangle$ is a monoid.*

(ii) *Let $a_1, \ldots, a_n, b \in L$, where $n \geq 1$. Then $\langle a_1, \ldots, a_n \rangle \sim \langle b \rangle$ if and only if $a_1 + \ldots + a_n = b$.*

(iii) *Let*

$$\iota \colon L \to C(L), \quad a \mapsto \langle\!\langle a \rangle\!\rangle$$

be the natural embedding of L into $C(L)$. Then ι is injective.

Furthermore, for $a, b \in L$, $a + b$ is defined and equals c if and only if $\iota(a) + \iota(b) = \iota(c)$.

Proof. (i) is evident.

(ii) For any word $\langle a_1, \ldots, a_n \rangle$ the sum of whose elements exists and equals b, the same is true for any word equivalent to $\langle a_1, \ldots, a_n \rangle$.

(iii) The injectivity of ι follows from part (ii).

Let moreover $a, b \in L$. If $a + b = c$, then obviously $\iota(a) + \iota(b) = \iota(c)$. Conversely, $\iota(a) + \iota(b) = \iota(c)$ means $\langle a, b \rangle \sim \langle c \rangle$, that is, $a + b = c$ by part (ii). □

Next, we show that the monoid $\langle C(L), +, \langle\!\langle 0 \rangle\!\rangle \rangle$ fulfils the characteristic properties of the positive cone of a partially ordered group. As a preparation, we insert the following generalisation of Lemma 3.4.2.

LEMMA 3.4.5. *Let $\langle L, \leq, +, 0 \rangle$ be a naturally ordered R-chain. Let $a_1, \ldots, a_m, b_1, \ldots, b_n \in L$ such that $\langle a_1, \ldots, a_m \rangle \sim \langle b_1, \ldots, b_n \rangle$, where $n, m \geq 1$. Then there are $d_{11}, \ldots, d_{mn} \in L$ such that* (5) *and* (6) *hold.*

Proof. If $m = n$ and $a_1 = b_1, \ldots, a_m = b_m$, the assertion is trivial. Let $a_1, \ldots, b_n$ be arbitrary, and let $d_{11}, \ldots, d_{mn}$ be such that (5) and (6) hold. We shall show how to modify the scheme (5) to preserve both its correctness and the infimum-zero relations (6) when $\langle b_1, \ldots, b_n \rangle$ is replaced (i) by $\langle b_1, \ldots, b_p + b_{p+1}, \ldots, b_n \rangle$ for some $1 \leq p < n$ or (ii) by $\langle b_1, \ldots, b_p^1, b_p^2, \ldots, b_n \rangle$, where $1 \leq p \leq n$ and $b_{p1} + b_{p2} = b_p$.

Ad (i). We replace, for each $i = 1, \ldots, m$, the neighbouring entries d_{ip} and $d_{i,p+1}$ by their sum, which by Lemma 3.4.1(i) exists. Then the sum of the i-th row is obviously still a_i. To see that the sum of the new column exists and is $b_p + b_{p+1}$, we make repeated use of the fact that two elements one of which is 0 can be interchanged:

$$\begin{aligned} b_p + b_{p+1} &= d_{1p} + \ldots + d_{mp} + d_{1,p+1} + \ldots + d_{m,p+1} \\ &= d_{1p} + d_{1,p+1} + d_{2p} + \ldots + d_{mp} + d_{2,p+1} + \ldots + d_{m,p+1} \\ &= \ldots \\ &= d_{1p} + d_{1,p+1} + d_{2p} + d_{2,p+1} + \ldots + d_{mp} + d_{m,p+1}. \end{aligned}$$

Clearly, the infimum-zero relations still hold.

Ad (ii). We apply Lemma 3.4.2 to the equation $b_p^1 + b_p^2 = d_{1p} + \ldots + d_{mp}$, and replace the column $d_{1p}, \ldots, d_{mp}$ with the new double column. Obviously, in the modified scheme, the rows and columns add up correctly and the required infimum-zero relations hold. □

LEMMA 3.4.6. *Let $\langle L, \leq, +, 0 \rangle$ be a naturally ordered R-chain. Then $\langle C(L), +, \langle\!\langle 0 \rangle\!\rangle \rangle$ is a monoid such that for $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in C(L)$:*

(i) *From $\mathfrak{a} + \mathfrak{b} = \langle\!\langle 0 \rangle\!\rangle$ it follows $\mathfrak{a} = \mathfrak{b} = \langle\!\langle 0 \rangle\!\rangle$.*

(ii) *From $\mathfrak{a} + \mathfrak{b} = \mathfrak{a} + \mathfrak{c}$ or $\mathfrak{b} + \mathfrak{a} = \mathfrak{c} + \mathfrak{a}$ it follows $\mathfrak{b} = \mathfrak{c}$.*

(iii) *There are $\mathfrak{x}, \mathfrak{y} \in C(L)$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{x} + \mathfrak{a} = \mathfrak{b} + \mathfrak{y}$.*

Proof. $C(L)$ is a monoid by Lemma 3.4.4(i).

(i) This follows from Lemma 3.4.4(ii).

(ii) We may restrict to the case that $\mathfrak{a} = \langle\!\langle a \rangle\!\rangle$ for some $a \in L$. Let $\mathfrak{b} = \langle\!\langle b_1, \ldots, b_m \rangle\!\rangle$, $\mathfrak{c} = \langle\!\langle c_1, \ldots, c_n \rangle\!\rangle$, $m, n \geq 1$, and assume $\langle a, b_1, \ldots, b_m \rangle \sim \langle a, c_1, \ldots, c_n \rangle$. We will show that $\mathfrak{b} = \mathfrak{c}$; the second part can be proved analogously.

By Lemma 3.4.5, there are elements in L such that

$$\begin{array}{ccccc} d & d_1 & \ldots & d_n & \to & a \\ e_1 & e_{11} & \ldots & e_{1n} & \to & b_1 \\ \vdots & \vdots & & \vdots & & \vdots \\ e_m & e_{m1} & \ldots & e_{mn} & \to & b_m \\ \downarrow & \downarrow & & \downarrow & & \\ a & c_1 & & c_n, & & \end{array}$$

where any pair of elements one of which is placed further up and further right than the other one, has infimum 0. But the latter condition means $d = a$ and $d_1 = \ldots = d_n = e_1 = \ldots = e_m = 0$. Again using the infimum-zero conditions, we conclude $\langle\!\langle b_1, \ldots, b_m \rangle\!\rangle = \langle\!\langle c_1, \ldots, c_n \rangle\!\rangle$.

(iii) We only prove the first half of the claim. We may furthermore restrict to the case that $\mathfrak{a} = \langle\!\langle a \rangle\!\rangle$ and $\mathfrak{b} = \langle\!\langle b \rangle\!\rangle$ for some $a, b \in L$. If $a \leq b$, then $b = x + a$ for some $x \in L$, hence $\langle\!\langle a, b \rangle\!\rangle = \langle\!\langle a, x \rangle\!\rangle + \langle\!\langle a \rangle\!\rangle$. If $a \geq b$, we have $a = b + x = y + b$ for some $x, y \in L$. If then $y \leq x$, we have $x = z + y$ for some $z \in L$ and thus $\langle\!\langle a, b \rangle\!\rangle = \langle\!\langle b, x, b \rangle\!\rangle = \langle\!\langle b, z, y, b \rangle\!\rangle = \langle\!\langle b, z \rangle\!\rangle + \langle\!\langle a \rangle\!\rangle$. If then $x \leq y$, we have $y = z + x$ for some $z \leq b$ by (E7) and thus $\langle\!\langle a, b \rangle\!\rangle = \langle\!\langle b, x, b \rangle\!\rangle = \langle\!\langle b', z, x, b \rangle\!\rangle = \langle\!\langle b', y, b \rangle\!\rangle = \langle\!\langle b' \rangle\!\rangle + \langle\!\langle a \rangle\!\rangle$, where $b = b' + z$. □

We next establish that the natural order of $C(L)$ is actually a total order.

LEMMA 3.4.7. *Let $\langle L, \leq, +, 0 \rangle$ be a naturally ordered R-chain. Let*

$$\mathfrak{b} \leq \mathfrak{a} \quad \textit{if} \quad \mathfrak{b} + \mathfrak{x} = \mathfrak{a} \textit{ for some } \mathfrak{x} \in C(L)$$

for $\mathfrak{a}, \mathfrak{b} \in C(L)$. Then $\leq$ is a total order.

Proof. Let $\mathfrak{a} = \langle\!\langle a_1, \ldots, a_m \rangle\!\rangle$ and $\mathfrak{b} = \langle\!\langle b_1, \ldots, b_n \rangle\!\rangle$, $m, n \geq 1$. Assume that $a_1 \leq b_1$. Then there is an $x \in L$ such that $b_1 = a_1 + x$, and by Lemma 3.4.6, $\mathfrak{a}$ and $\mathfrak{b}$ are comparable iff so are $\langle\!\langle a_2, \ldots, a_m \rangle\!\rangle$ and $\langle\!\langle x, b_2, \ldots, b_n \rangle\!\rangle$, where the empty word is identified with $\langle\!\langle 0 \rangle\!\rangle$. Similarly, assume $b_1 \leq a_1$. Then there is an $x \in L$ such that $a_1 = b_1 + x$, and $\mathfrak{a}$ and $\mathfrak{b}$ are comparable iff so are $\langle\!\langle x, a_2, \ldots, a_m \rangle\!\rangle$ and $\langle\!\langle b_2, \ldots, b_n \rangle\!\rangle$. From the fact that $\langle\!\langle 0 \rangle\!\rangle$ is comparable with any word, we conclude the assertion by induction. □

We arrive at our main theorem. By an isomorphic embedding of a naturally ordered R-chain $\langle L, \leq, +, 0 \rangle$ into an totally ordered group $\langle G, \leq, +, 0 \rangle$, we mean an injective mapping $\iota \colon L \to G$ such that for $a, b, c \in L$

$a \leq b$ if and only if $\iota(a) \leq \iota(b)$,
$a + b$ is defined and equals c if and only if $\iota(a) + \iota(b) = \iota(c)$,
$\iota(0) = 0$.

THEOREM 3.4.8. *Let $\langle L, \leq, +, 0 \rangle$ be a naturally ordered R-chain. Then there is an isomorphic embedding ι of the R-chain $\langle L, \leq, +, 0 \rangle$ into a totally ordered group $\langle G, \leq, +, 0 \rangle$. The range of ι is a convex subset of G, whose smallest element is $\langle\!\langle 0 \rangle\!\rangle$ and which generates G.*

Proof. By Lemma 3.4.6 and [18, Chapter II, Thm. 4], there is a totally ordered group $\langle G(L), \leq, +, \langle\!\langle 0 \rangle\!\rangle \rangle$ such that $C(L)$ is its positive cone. Furthermore, by Lemma 3.4.4(iii), $\iota\colon L \to C(L),\ \ a \mapsto \langle\!\langle a \rangle\!\rangle$ is an isomorphic embedding of L into $\langle C(L), \leq, +, \langle\!\langle 0 \rangle\!\rangle \rangle$. Consequently, the same mapping, whose range is extended to $G(L)$, is an isomorphic embedding of L into $\langle G(L), \leq, +, \langle\!\langle 0 \rangle\!\rangle \rangle$.

The range of ι in $G(L)$ is then $\{\langle\!\langle a \rangle\!\rangle\colon a \in L\} \subseteq C(L)$. The smallest element of $C(L)$ is $\langle\!\langle 0 \rangle\!\rangle$, which consequently is the smallest element of the range of ι. Moreover, let $\mathfrak{g} \in G(L)$ and $a \in L$ such that $\langle\!\langle 0 \rangle\!\rangle \leq \mathfrak{g} \leq \langle\!\langle a \rangle\!\rangle$. Then $\mathfrak{g} \in C(L)$ and $\mathfrak{g} + \mathfrak{h} = \langle\!\langle a \rangle\!\rangle$ for some $\mathfrak{h} \in C(L)$, and it follows that $\mathfrak{g} = \langle\!\langle b \rangle\!\rangle$ for some $b \in L$. Thus the range of ι is a convex subset of $G(L)$, which moreover generates $C(L)$ and consequently $G(L)$. ☐

3.5 The representation of d.p.r. tomonoids

So far, we have associated with a d.p.r. tomonoid an R-chain; we have represented this partial algebra as an ordinal sum of naturally ordered R-chains; and we have shown that each naturally ordered R-chain embeds into a totally ordered Abelian group. Summarising these results, we may now formulate a representation theorem for d.p.r. tomonoids.

For convenience, let us introduce simple notions for our basic constituents.

DEFINITION 3.5.1. *Let $\langle G, \leq, +, 0 \rangle$ be a totally ordered group. Then we call the tomonoid $\langle G^+, \leq, +, 0 \rangle$ a* group cone.

Moreover, let $u \in G$ such that $u > 0$. Let $[0, u] = \{g \in G\colon 0 \leq g \leq u\}$, and define

$$a \oplus b \;=\; (a + b) \wedge u, \quad a, b \in [0, u].$$

Then we call the tomonoid $\langle [0, u], \leq, \oplus, 0 \rangle$ a group interval.

Note that group cones and group intervals are actually d.p.r. tomonoids.

LEMMA 3.5.2. *Let $\langle L, \leq, \oplus, 0 \rangle$ be a d.p.r. tomonoid such that its associated partial algebra is a naturally ordered R-chain. Then L is either a group cone or a group interval.*

Proof. Let $\iota\colon L \to G$ the embedding of L into a totally ordered group according to Theorem 3.4.8. We distinguish two cases:

Case 1. The addition of L is total. As G^+ is generated by $\iota(L)$, it follows $\iota(L) = G^+$, that is, $\langle L, \leq, \oplus, 0 \rangle$ is a group cone in this case.

Case 2. There are $a, b \in L$ such that $a + b$ is not defined. Let then $u = a \oplus b$; we claim that u is the top element of L and hence $\iota(L) = \{g \in G^+\colon g \leq \iota(u)\}$.

Assume to the contrary that there is a $v \in L$ such that $u < v$. Let $a' \leq a$ and $b' \leq b$ such that $a' + b' = u$. Then either $a' < a$ or $b' < b$; we assume $b' < b$ and we can proceed similarly in the case $a' < a$. As L, as an R-chain, is naturally ordered, there is a $d > 0$ such that $v = u + d$. This means that the sum $a' + b' + d$ is defined; then $b'' = (b' + d) \wedge b > b'$ and $a' \oplus b'' = a \oplus b = u$. However, the sum $a' + b''$ is defined and strictly greater than $a' + b' = a \oplus b = u$. Our claim is proved.

If now $a, b \in L$ such that $a + b$ exists, we have $\iota(a \oplus b) = \iota(a + b) = \iota(a) + \iota(b) \in L$, that is, $\iota(a) + \iota(b) \leq \iota(u)$. If $a + b$ does not exist, $\iota(a) + \iota(b) \notin \iota(L)$, that is $\iota(a) + \iota(b) > \iota(u)$. In this case, $a \oplus b = u$, hence $\iota(a \oplus b) = \iota(u)$. We conclude

$$\iota(a \oplus b) \;=\; (\iota(a) + \iota(b)) \wedge u;$$

hence L is a group interval in this case. □

We next define ordinal sums of d.p.r. tomonoids.

DEFINITION 3.5.3. *Let* $\langle I, \leq \rangle$ *be a chain, and for each* $i \in I$*, let* $\langle L_i, \leq_i, \oplus_i, 0_i \rangle$ *be a d.p.r. tomonoid. Put* $L = \dot{\bigcup}_{i \in I} (L_i \backslash \{0_i\}) \cup \{0\}$*, where* 0 *is a new element and* $\dot{\cup}$ *denotes the disjoint union. For* $a, b \in L$*, let* $a \leq b$ *if either* $a = 0$*, or* $a \in L_i$ *and* $b \in L_j$ *such that* $i < j$*, or* $a, b \in L_i$ *for some* i *and* $a \leq b$ *holds in* L_i*. Moreover, for* $a, b \in L$*, define* $a \oplus 0 = 0 \oplus a = a$*;* $a \oplus b = a \oplus_i b$ *if* $a, b \in L_i$ *for some* $i \in I$*; and* $a \oplus b = b \oplus a = a$ *if* $a \in L_i$ *and* $b \in L_j$ *such that* $i < j$*. Then* $\langle L, \leq, \oplus, 0 \rangle$ *is called the* ordinal sum *of the d.p.r. tomonoids* L_i *w.r.t.* $\langle I, \leq \rangle$*.*

We obviously have:

LEMMA 3.5.4. *The ordinal sum of d.p.r. tomonoids is again a d.p.r. tomonoid.*

We can finally state our main result.

THEOREM 3.5.5. *Each d.p.r. tomonoid is the ordinal sum of d.p.r. tomonoids* L_i*,* $i \in I$*, such that each* L_i *is either a group cone or a group interval.*

Proof. Let $\langle L, \leq, \oplus, 1 \rangle$ be a d.p.r. tomonoid. By Theorem 3.2.9, its associated partial algebra $\langle L, \leq, +, 0 \rangle$ is an R-chain, and $\oplus$ is determined by $+$ according to (3). By Theorem 3.3.4, the R-chain L is the ordinal sum of naturally ordered R-chains $\langle L_i, \leq, +, 0 \rangle$, where $i \in I$ and I is a chain.

From (3) we conclude that L_i is closed under $\oplus$. Consequently, $\langle L_i, \leq, \oplus, 0 \rangle$ is a tomonoid, in fact a d.p.r. tomonoid. It is furthermore easily seen that $\langle L_i, \leq, +, 0 \rangle$ is its associated R-chain. As the latter is naturally ordered, $\langle L_i, \leq, \oplus, 0 \rangle$ is by Lemma 3.5.2 a group cone or a group interval. The assertion follows. □

In the dual picture, Theorem 3.5.5 provides a representation of divisible, integral residuated chains, or totally ordered pseudohoops [8]. Adding the assumption that there is a bottom element, we arrive at a representation of totally ordered pseudo-BL algebras. Finally, adding commutativity, we get the well-known representation of totally ordered BL algebras [1]; cf. Chapter V of Volume 1.

4 Coextensions of totally ordered monoids

4.1 The idea

The second approach that we are going to present in this chapter follows similar aims than the first one; our concern is a better understanding of the structure of residuated chains. However, what now follows could hardly be more different in style from what we have discussed so far.

Our starting point is a simple observation. Recall that the quotients of an integral residuated chain are in a one-to-one correspondence with its filters, and the set of filters is itself totally ordered w.r.t. set-theoretical inclusion. With any integral residuated chain we may hence associate the chain of their quotients. The bottom element of this chain is the trivial algebra, consisting of a single element; and the top element is the algebra under consideration. The intermediate elements may be seen as leading us stepwise from the trivial algebra to more and more fine-grained structures up to the algebra under consideration.

This intuitive picture is certainly easily overridden by the real situation: although the set of quotients cannot be ordered in a completely arbitrary manner, this chain can be very complicated. An example is the Cantor set endowed with its natural order; in such cases we can hardly speak about a stepwise construction process.

Nevertheless it seems to make sense to explore neighbouring elements in the chain, provided that there are any. If two quotients directly follow one another the filter inducing the congruence is Archimedean and accordingly we speak about Archimedean coextensions then. The construction of Archimedean coextensions is again intractable in general, but there is a condition that reduces possibilities drastically, namely, the condition that the congruence classes are order-isomorphic to real intervals.

The fact that we deal with the real line might reveal our original motivation underlying the present study: our ultimate aim has been a classification of left-continuous t-norms. Given the tomonoid based on a t-norm, the detection of filters and their induced quotients may already imply the entanglement of a possibly complicated structure. By this step alone, seemingly exotic cases can often be easily categorised. Moreover, the regular representation of monoids is a convenient geometric tool that accompanies our analysis with a clear intuition.

With regards to t-norms, our main results implies the following. Consider the tomonoid arising from a left-continuous t-norm and assume that it possesses an Archimedean filter. Then the t-norm can be described in terms of the quotient induced by this filter and the only essential information needed is the order type of the congruence classes.

We proceed as follows. The subsequent Section 4.2 introduces the class of totally ordered monoids that we consider this time. The property of divisibility will no longer play a role; but we will deal with the commutative case only and we will assume an order-theoretic completeness condition. In the subsequent Section 4.3, we turn to the chain of quotients of the tomonoids induced by filters.

As a preparation for what follows, and the same time as a visualisation tool, we discuss in Section 4.4 the regular representation of tomonoids, which we call Cayley tomonoids. Section 4.5 contains our main result: a method of constructing from a given tomonoid an Archimedean coextension.

4.2 Q.n.c. tomonoids and their quotients

In this second part of the present chapter, totally ordered monoids, or tomonoids for short, serve again as our algebraic framework. This time, however, we will use the multiplicative notation.

Our tomonoids will be assumed to be negative, and we deal with the commutative case only. Moreover, we assume that the tomonoids are almost complete. Here, a poset is called *almost complete* if arbitrary non-empty suprema exist. Finally, we will assume that the multiplication distributes over arbitrary joins. We combine the latter two conditions to one notion named "quantic" because quantales are in fact defined similarly [34].

DEFINITION 4.2.1. *A tomonoid* $\boldsymbol{L} = \langle L, \le, \odot, 1\rangle$ *is called* quantic *if* (i) $\boldsymbol{L}$ *is almost complete and* (ii) *for any elements* a, b_ι, $\iota \in I$, *of* L *we have*

$$a \odot \bigvee_\iota b_\iota \;=\; \bigvee_\iota (a \odot b_\iota) \quad \textit{and} \quad (\bigvee_\iota b_\iota) \odot a \;=\; \bigvee_\iota (b_\iota \odot a).$$

The tomonoids that we consider here are quantic, negative, and commutative. We abbreviate these three properties with "q.n.c.". Note that q.n.c. tomonoids are residuated. In fact, q.n.c. tomonoids are in a one-to-one correspondence with almost complete, integral, commutative residuated chains, or almost complete totally ordered basic semihoops.

The motivating examples arise from left-continuous triangular norms.

EXAMPLE 4.2.2. *Let* $[0, 1]$ *be the real unit interval. Let* $\star\colon [0, 1]^2 \to [0, 1]$ *be a left-continuous t-norm; let* $\le$ *be the natural order on* $[0, 1]$. *Then* $\langle [0, 1], \le, \star, 1\rangle$ *is a q.n.c. tomonoid.*

We will use in the sequel occasionally the residual of a q.n.c. tomonoid; we denote it by $\to$.

A q.n.c. tomonoid does not necessarily possess a bottom element. If not, we can add an additional element with this role in the usual way.

DEFINITION 4.2.3. *Let* $\boldsymbol{L}$ *be a q.n.c. tomonoid. Let* $\boldsymbol{L}^0 = \boldsymbol{L}$ *if* L *has a bottom element. Otherwise, let* $\boldsymbol{L}^0 = \langle L^0, \le, \odot, 1\rangle$ *arise from* $\boldsymbol{L}$ *by adding a new element* 0*; in this case, we extend the total order to* L^0 *such that* 0 *is the bottom element, and we extend the monoidal operation to* L^0 *such that* 0 *is absorbing.*

Obviously, for any q.n.c. tomonoid $\boldsymbol{L}$, $\boldsymbol{L}^0$ is again a q.n.c. tomonoid, whose total order is complete and which hence can be seen as a quantale.

In the context of almost complete chains, it makes sense to speak about intervals analogously to the case of reals. An *interval* of a q.n.c. tomonoid $\boldsymbol{L}$ will be a non-empty convex subset of L. An interval J of $\boldsymbol{L}$ possesses in $\boldsymbol{L}^0$ an infimum u and a supremum v, and we will refer to J by (u, v), $(u, v]$, $[u, v)$, or $[u, v]$, depending on whether or not u and v belong to J.

We now turn to quotients of tomonoids. We note that the following definition could be simplified if we included the infimum or supremum to the signature instead of the total order relation.

DEFINITION 4.2.4. *Let* $\boldsymbol{L} = \langle L, \le, \odot, 1\rangle$ *be a q.n.c. tomonoid. An equivalence relation* $\sim$ *on* L *is called a* tomonoid congruence *if* (i) $\sim$ *is a congruence of* $\boldsymbol{L}$ *as a monoid and* (ii) *the* $\sim$*-classes are convex. We endow then the quotient* $[\boldsymbol{L}]_\sim$ *with the total order given by*

$$[a]_\sim \le [b]_\sim \textit{ if } a' \le b' \textit{ for some } a' \sim a \textit{ and } b' \sim b$$

for $a, b \in L$, *with the induced operation* $\odot$, *and with the constant* $[1]_\sim$. *The resulting structure* $\langle [\boldsymbol{L}]_\sim, \le, \odot, [1]_\sim\rangle$ *is called a* tomonoid quotient *of* $\boldsymbol{L}$.

Obviously, the congruence classes of a tomonoid quotient are intervals and we have $[a]_\sim < [b]_\sim$ if and only if $a' < b'$ for all $a' \sim a$ and $b' \sim b$.

To describe the totality of quotients of q.n.c. tomonoids is in general difficult. Here, we are interested in only one way of forming quotients: by means of filters.

DEFINITION 4.2.5. *Let $\boldsymbol{L}$ be a q.n.c. tomonoid. Then a* filter *of $\boldsymbol{L}$ is a subtomonoid $\boldsymbol{F} = \langle F, \leq, \odot, 1\rangle$ of $\boldsymbol{L}$ such that $f \in F$ and $g \geq f$ imply $g \in F$.*

By the *trivial* tomonoid, we mean the one-element tomonoid, consisting of 1 alone. Each q.n.c. tomonoid $\boldsymbol{L}$ possesses the following filters: $\{1\}$, the *trivial* filter, and $\boldsymbol{L}$, the *improper* filter. Thus each non-trivial q.n.c. tomonoid has at least two filters.

As we easily check, a filter of a q.n.c. tomonoid is again a q.n.c. tomonoid. A filter may or may not possess a bottom element; this is actually the reason for which we defined quanticity by requiring an almost complete rather than a complete order.

DEFINITION 4.2.6. *Let $\boldsymbol{F}$ be a filter of a q.n.c. tomonoid $\boldsymbol{L}$. Let d be the infimum of F in $\boldsymbol{L}^0$; then we call d the* boundary *of $\boldsymbol{F}$. If d belongs to F, we write $F = d^{\leq}$; if d does not belong to F, we write $F = d^{<}$.*

Thus each filter of a q.n.c. tomonoid $\boldsymbol{L}$ is of the form $d^{<} = (d, 1]$ for some $d \in L^0 \backslash \{1\}$, or $d^{\leq} = [d, 1]$ for some $d \in L$. Each filter $\boldsymbol{F}$ is uniquely determined by its boundary d together with the information whether or not d belongs to F. We note that, for some $d \in L$, it is possible that both $d^{<}$ and $d^{\leq}$ are filters.

LEMMA 4.2.7. *Let $\boldsymbol{L}$ be a q.n.c. tomonoid, and let $d \in L$.*

(i) *$d^{\leq}$ is a filter if and only if d is idempotent.*

(ii) *$d^{<}$ is a filter if and only if $d \neq 1$, $d = \bigwedge_{a>d} a$, and $d < a \odot b$ for all $a, b > d$.*

Proof. (i) $[d, 1]$ is a filter if and only if $[d, 1]$ is closed under multiplication if and only if $d \odot d = d$, that is, if d is idempotent.

(ii) Let $d^{<}$ be a filter. Then $d < 1$ because each filter contains 1; $d = \inf d^{<} = \inf (d, 1] = \bigwedge_{a>d} a$; and $(d, 1]$ is closed under multiplication, that is, $a \odot b > d$ for each $a, b > d$.

Conversely, let $d \neq 1$ such that $d = \bigwedge_{a>d} a$ and $d < a \odot b$ for any $a, b > d$. Then $\{a \in L \colon a > d\}$ is a filter whose infimum is d, that is, which equals $d^{<}$. □

In the broader context of residuated lattices, the relevant substructures are convex normal subalgebras; each of the latter induces a quotient and every quotient of a residuated lattice arises in this way [24]. Here we consider a special case of this situation: filters of q.n.c. tomonoids lead to tomonoid quotients. We should, however, be aware of the fact that not all quotients of totally ordered monoids are induced by filters.

DEFINITION 4.2.8. *Let $\boldsymbol{F}$ be a filter of a q.n.c. tomonoid $\boldsymbol{L}$. For $a, b \in L$, let*

$$a \sim_F b \quad \text{if there is an } f \in F \text{ such that } b \odot f \leq a \text{ and } a \odot f \leq b.$$

Then we call $\sim_F$ the congruence induced by $\boldsymbol{F}$.

Equivalence relations of this type do not only preserve the tomonoid structure, but also all the three properties that we generally assume here.

LEMMA 4.2.9. *Let $\boldsymbol{L}$ be a q.n.c. tomonoid, and let $\boldsymbol{F}$ be a filter of $\boldsymbol{L}$. Then the congruence induced by $\boldsymbol{F}$ is a tomonoid congruence, and the tomonoid quotient is again quantic, negative, and commutative.*

Proof. It is easily checked that $\sim_F$ is compatible with $\odot$ and that the equivalence classes are convex. Clearly, negativity and commutativity are preserved.

Our next aim is to prove that the tomonoid quotient $[\boldsymbol{L}]_{\sim_F}$ is almost complete. For simplification, equivalence classes w.r.t. $\sim_F$ will be denoted by $[\cdot]$. We will prove the following statement, which obviously implies almost completeness:

($\star$) Let $a_\iota \in L$, $\iota \in I$, be such that among $[a_\iota]$, $\iota \in I$, there is no largest element; then

$$\bigvee_\iota [a_\iota] \;=\; [\bigvee_\iota a_\iota]. \tag{7}$$

To see ($\star$), let $a_\iota \in L$, $\iota \in I$, and assume that the $[a_\iota]$ do not possess a largest element. Let $a = \bigvee_\iota a_\iota$. Then $[a] \geq [a_\iota]$ for all ι. Moreover, let $b \in L$ be such that $[b] \geq [a_\iota]$ for all ι. Then b is not equivalent to any a_ι, hence $[b] > [a_\iota]$; consequently $b > a_\iota$ for all ι, so that $b \geq a$ and $[b] \geq [a]$. Thus (7) follows.

It remains to show that $\odot$ distributes over suprema in $[\boldsymbol{L}]_{\sim_F}$. Let $b_\iota \in L$, $\iota \in I$, and $a \in L$. Assume first that the elements $[a \odot b_\iota]$, $\iota \in I$, do not possess a maximal element. Then also the $[b_\iota]$ do not possess a maximal element, and ($\star$) implies

$$[a] \odot \bigvee_\iota [b_\iota] \;=\; \bigvee_\iota ([a] \odot [b_\iota]). \tag{8}$$

Assume second that the $[a \odot b_\iota]$ possess the maximal element $[a \odot b_\kappa]$, but that the $[b_\iota]$ do not possess a maximal element. Let $\iota \in I$ such that $[b_\iota] > [b_\kappa]$. Then $a \odot b_\iota \sim a \odot b_\kappa$, and we have $a \odot b_\kappa \leq a \odot \inf[b_\iota] = a \odot \bigwedge_{f \in F} (b_\iota \odot f) \leq \bigwedge_{f \in F} (a \odot b_\iota \odot f) = \inf[a \odot b_\iota] = \inf[a \odot b_\kappa] \leq a \odot b_\kappa$. We conclude that $a \odot b_\iota = a \odot b_\kappa$ for any $\iota \in I$ such that $b_\iota > b_\kappa$. Thus $a \odot \bigvee_\iota b_\iota = a \odot b_\kappa$. By ($\star$), $[a] \odot \bigvee_\iota [b_\iota] = [a \odot \bigvee_\iota b_\iota] = [a \odot b_\kappa] = \bigvee_\iota ([a] \odot [b_\iota])$, and (8) is proved.

Assume third that the $[b_\iota]$, $\iota \in I$, possess the maximal element $[b_\kappa]$. Then $[a \odot b_\kappa]$ is maximal among the $[a \odot b_\iota]$. Then obviously, (8) holds as well. $\square$

As we will deal in the sequel exclusively with congruences induced by filter, we simplify our notation as follows.

DEFINITION 4.2.10. *Let $\boldsymbol{L}$ be a q.n.c. tomonoid, and let $\boldsymbol{F}$ be a filter of $\boldsymbol{L}$. Let $\sim_F$ be the congruence induced by $\boldsymbol{F}$. We will refer to the $\sim_F$-classes as* $\boldsymbol{F}$-classes *and we denote them by $[\cdot]_F$. Similarly, let P be the quotient of $\boldsymbol{L}$ by $\sim_F$. Then we refer to P as the* quotient of $\boldsymbol{L}$ by $\boldsymbol{F}$ *and we denote it by $[\boldsymbol{L}]_F$.*

We furthermore call in this case $\boldsymbol{L}$ an coextension of P by $\boldsymbol{F}$*, and we refer to $\boldsymbol{F}$ as the* extending *tomonoid.*

4.3 The chain of quotients of a q.n.c. tomonoid

Each filter of a q.n.c. tomonoid induces a quotient. Let us now consider the collection of such quotients as a whole.

The most basic observation is that, for any two filters, one is included in the other one: the set of all filters is totally ordered by set-theoretical inclusion.

DEFINITION 4.3.1. *Let $\boldsymbol{L}$ be a q.n.c. tomonoid. We denote the set of all filters of $\boldsymbol{L}$ by $\mathbb{F}$, and we endow $\mathbb{F}$ with the set-theoretical inclusion $\subseteq$ as a total order.*

Again, if we included in our signature the lattice operations instead of the total order, the proof of the next lemma could be kept shorter, as it would follow from the Second Isomorphism Theorem of Universal Algebra [5].

LEMMA 4.3.2. *Let $\boldsymbol{L}$ be a q.n.c. tomonoid, and let $\boldsymbol{F}$ and $\boldsymbol{G}$ be filters of $\boldsymbol{L}$ such that $\boldsymbol{F} \subseteq \boldsymbol{G}$. Then $[\boldsymbol{G}]_F$ is a filter of $[\boldsymbol{L}]_F$, and $[\boldsymbol{L}]_G$ is isomorphic to the quotient of $[\boldsymbol{L}]_F$ by $[\boldsymbol{G}]_F$.*

Proof. We claim that, for $a, b \in L$, $a \sim_G b$ if and only if $[a]_F \sim_{[G]_F} [b]_F$. Indeed, assume $a \leq b$; then $a \sim_G b$ if and only if there a $g \in G$ such that $b \odot g \leq a$. Since $\boldsymbol{F}$ is a filter contained in $\boldsymbol{G}$, the latter holds if and only if there are a $g \in G$ and an $f \in F$ such that $b \odot g \odot f \leq a$ if and only if $[b \odot g]_F \leq [a]_F$ for some $g \in G$ if and only if $[b]_F \odot [g]_F \leq [a]_F$ for some $[g]_F \in [\boldsymbol{G}]_F$ if and only if $[a]_F \sim_{[G]_F} [b]_F$.

It follows that we can define

$$\varphi\colon [\boldsymbol{L}]_G \to [[\boldsymbol{L}]_F]_{[G]_F}, \quad [a]_G \mapsto [[a]_F]_{[G]_F}$$

and that φ is a bijection. Moreover, φ preserves $\odot$ and is an order-isomorphism. The lemma follows. □

Lemma 4.3.2 is the basis of our loose statement that a q.n.c. tomonoid is the result of a linear construction process. In general, this process does not proceed in a stepwise fashion. But we can speak about a single step if there is a pair of successive filters; the following definition addresses this case.

For an element a of a tomonoid and $n \geq 1$, we write a^n for $a \odot \ldots \odot a$ (n factors).

DEFINITION 4.3.3. *A q.n.c. tomonoid $\boldsymbol{L}$ is called* Archimedean *if, for each $a, b \in L$ such that $a < b < 1$, we have $b^n \leq a$ for some $n \geq 1$.*

A coextension of a q.n.c. tomonoid by an Archimedean tomonoid is called Archimedean.

For two filters $\boldsymbol{F}, \boldsymbol{G} \in \mathbb{F}$, we will write $\boldsymbol{F} \subset\!\!\cdot\; \boldsymbol{G}$ to express that $\boldsymbol{G}$ is the immediate successor of $\boldsymbol{F}$ in $\mathbb{F}$, that is, $\boldsymbol{G}$ is the next smallest filter to $\boldsymbol{F}$.

THEOREM 4.3.4. *Let $\boldsymbol{L}$ be a q.n.c. tomonoid. Then we have:*

(i) *The largest and smallest elements of $\mathbb{F}$ are $\boldsymbol{L}$ and $\{1\}$, respectively. Moreover, $[\boldsymbol{L}]_L$ is the trivial tomonoid, and $[\boldsymbol{L}]_{\{1\}}$ is isomorphic to $\boldsymbol{L}$.*

(ii) *For each $\boldsymbol{F} \in \mathbb{F}\backslash\{\boldsymbol{L}\}$ such that $\boldsymbol{F}$ is not an immediate predecessor, $\sim_F = \bigcap_{\boldsymbol{G} \supset \boldsymbol{F}} \sim_G$.*

(iii) *For each $\boldsymbol{F} \in \mathbb{F}\backslash\{\{1\}\}$ such that $\boldsymbol{F}$ is not an immediate successor, $\sim_F = \bigcup_{\boldsymbol{G} \subset \boldsymbol{F}} \sim_G$.*

(iv) *For each $\boldsymbol{F}, \boldsymbol{G} \in \mathbb{F}$ such that $\boldsymbol{F} \subset\!\!\cdot\; \boldsymbol{G}$, $[\boldsymbol{L}]_F$ is an Archimedean coextension of $[\boldsymbol{L}]_G$.*

Proof. (i) The largest filter is $\boldsymbol{L}$, and the quotient $[\boldsymbol{L}]_L$ is one-element, that is, trivial. The smallest filter is $\{1\}$, and the quotient $[\boldsymbol{L}]_{\{1\}}$ has singleton classes only, that is, coincides with $\boldsymbol{L}$.

(ii) Let $\boldsymbol{F} \in \mathbb{F}$ such that $\boldsymbol{F}$ is neither $\boldsymbol{L}$ nor the predecessor of another filter. As $\mathbb{F}$ is closed under arbitrary intersections, we then have $F = \bigcap_{\boldsymbol{G} \supset \boldsymbol{F}} G$. Let $a, b \in L$ such that $a \leq b$. We have to show that $a \sim_F b$ if and only if, for each $\boldsymbol{G} \supset \boldsymbol{F}$, $a \sim_G b$. Clearly, $a \sim_F b$ implies $a \sim_G b$ for each $\boldsymbol{G} \supset \boldsymbol{F}$. Conversely, assume $a \sim_G b$ for each $\boldsymbol{G} \supset \boldsymbol{F}$. Then for each $\boldsymbol{G} \supset \boldsymbol{F}$ there is a $g_G \in G$ such that $b \odot g_G \leq a$. It follows $b \odot f \leq a$, where $f = \bigvee_{\boldsymbol{G} \supset \boldsymbol{F}} g_G \in F$, hence $a \sim_F b$.

(iii) Let $\boldsymbol{F} \in \mathbb{F}$ such that $\boldsymbol{F}$ is neither $\{1\}$ nor the successor of another filter. As $\mathbb{F}$ is closed under arbitrary unions, $F = \bigcup_{\boldsymbol{G} \subset \boldsymbol{F}} G$ then. For $a \leq b$, we have $a \sim_F b$ if and only if $b \odot f \leq a$ for some $f \in F$ if and only if $b \odot f \leq a$ for some $f \in G$ such that $\boldsymbol{G} \subset \boldsymbol{F}$ if and only if $a \sim_G b$ for some $\boldsymbol{G} \subset \boldsymbol{F}$.

(iv) Let $\boldsymbol{F}, \boldsymbol{G} \in \mathbb{F}$ such that $\boldsymbol{F} \subset\!\!\cdot\; \boldsymbol{G}$. By Lemma 4.3.2, $[\boldsymbol{L}]_G$ is then isomorphic to the quotient of $[\boldsymbol{L}]_F$ by the filter $[\boldsymbol{G}]_F$.

Assume that $[\boldsymbol{G}]_F$ is not Archimedean. Then there is a filter $\boldsymbol{H}$ of $[\boldsymbol{G}]_F$ such that $\{[1]_F\} \subset H \subset [G]_F$. But then $\bigcup H$ is a filter of $\boldsymbol{L}$ such that $F \subset \bigcup H \subset G$, a contradiction. □

4.4 The Cayley tomonoid

A monoid can be identified with a monoid under composition of mappings, namely, with the set of mappings acting on the monoid by left (or right) multiplication. This is the regular representation [7], which is due to A. Cayley for the case of groups. If the monoid is commutative, any two of the mappings commute. Moreover, the presence of a compatible total order on the monoid means that the mappings are order-preserving.

Representations of partially ordered monoids by order-preserving mappings have been studied in a more general context under the name S-posets [15]. An adaptation of our terminology might be a future issue; for the results presented in this chapter, however, such a step would most likely not improve clarity.

The reason to consider the regular representation of tomonoids is twofold. Most important, it gives us a means to specify tomonoid coextensions in a "modular" way. We will see below that the coextension of a tomonoid splits up into constituents each of which we may specify separately. Second, there is an informal aspect that, in the context of structures that are as theoretical as residuated chains, should not be neglected. The regular representation provides a geometric view on tomonoids that is not to be mixed up with the traditionally used three-dimensional graphs of t-norms. It is rather well in line with the algebraic orientation of this study, visualising quotients in a clear way and, in addition, getting along with two dimensions.

DEFINITION 4.4.1. *Let* $\langle R, \leq \rangle$ *be a chain, and let* Φ *be a set of order-preserving mappings from* R *to* R. *We denote by* $\leq$ *the pointwise order on* Φ, *by* $\circ$ *the functional composition, and by* id_R *the identity mapping on* R. *Assume that* (i) $\leq$ *is a total order on* Φ, (ii) Φ *is closed under* $\circ$, *and* (iii) $\mathrm{id}_R \in \Phi$. *Then we call* $\boldsymbol{\Phi} = \langle \Phi, \leq, \circ, \mathrm{id}_R \rangle$ *a* composition tomonoid *on* R.

In order to characterise the composition tomonoids associated with q.n.c. tomonoids, we introduce the following properties of a composition tomonoid $\mathbf{\Phi}$ on a chain R:

(C1) $\circ$ is commutative.

(C2) id_R is the top element.

(C3) Every $\lambda \in \Phi$ is sup-preserving.

(C4) Pointwise calculated suprema of non-empty subsets of Φ exist and are in Φ.

(C5) R has a top element 1, and for each $a \in R$ there is a unique $\lambda \in \Phi$ such that $\lambda(1) = a$.

PROPOSITION 4.4.2. *Let $\mathbf{\Phi}$ be a composition tomonoid over a chain R. Then $\mathbf{\Phi}$ is a tomonoid. Furthermore, we have:*

(i) *$\mathbf{\Phi}$ is commutative if and only if $\mathbf{\Phi}$ fulfils* (C1)*.*

(ii) *$\mathbf{\Phi}$ is negative if and only if $\mathbf{\Phi}$ fulfils* (C2)*.*

(iii) *If $\mathbf{\Phi}$ fulfils* (C3) *and* (C4)*, $\mathbf{\Phi}$ is quantic.*

Proof. The fact that $\mathbf{\Phi}$ is a tomonoid is easily checked, and so are parts (i) and (ii).

Assume (C3) and (C4). Then any non-empty subset of Φ possesses by (C4) w.r.t. the pointwise order a supremum; that is, $\mathbf{\Phi}$ is almost complete. Furthermore, let $\lambda_\iota, \mu \in \Phi$, $\iota \in I$. Then we have by (C4) for any $r \in R$

$$(\bigvee_\iota \lambda_\iota \circ \mu)(r) \;=\; (\bigvee_\iota \lambda_\iota)(\mu(r)) \;=\; \bigvee_\iota \lambda_\iota(\mu(r)) \;=\; \bigvee_\iota (\lambda_\iota \circ \mu)(r) \;=\; (\bigvee_\iota (\lambda_\iota \circ \mu))(r).$$

Moreover, we have by (C3) and (C4) for any $r \in R$

$$(\mu \circ \bigvee_\iota \lambda_\iota)(r) \;=\; \mu(\bigvee_\iota \lambda_\iota(r)) \;=\; \bigvee_\iota \mu(\lambda_\iota(r)) \;=\; \bigvee_\iota (\mu \circ \lambda_\iota)(r) \;=\; (\bigvee_\iota (\mu \circ \lambda_\iota))(r).$$

We conclude that $\mathbf{\Phi}$ is quantic. □

By Proposition 4.4.2, each composition tomonoid is a tomonoid. We next recall that, conversely, each tomonoid can be viewed as a composition tomonoid.

PROPOSITION 4.4.3. *Let $\langle L, \leq, \odot, 1\rangle$ be a q.n.c. tomonoid. For each $a \in L$, put*

$$\lambda_a \colon L \to L, \quad x \mapsto a \odot x, \tag{9}$$

and let $\Lambda = \{\lambda_a \colon a \in L\}$. Then $\langle \Lambda, \leq, \circ, id_L\rangle$ is a composition tomonoid on L fulfilling (C1)–(C5)*. Moreover,*

$$\pi \colon L \to \Lambda, \quad a \mapsto \lambda_a \tag{10}$$

is an isomorphism of the tomonoids $\langle L, \leq, \odot, 1\rangle$ and $\langle \Lambda, \leq, \circ, id_L\rangle$.

DEFINITION 4.4.4. *Let $\boldsymbol{L} = \langle L, \leq, \odot, 1 \rangle$ be a tomonoid. For each $a \in L$, the mapping λ_a defined by* (9) *is called the* (left) translation *by a. Furthermore, the composition tomonoid $\langle \Lambda, \leq, \circ, \mathrm{id}_L \rangle$ assigned to $\boldsymbol{L}$ according to Proposition 4.4.3 is called the* Cayley tomonoid *associated with $\boldsymbol{L}$.*

Let us state what Proposition 4.4.3 means for t-norms: a left-continuous t-norm corresponds to a monoid under composition of pairwise commuting, order-preserving, and left-continuous mappings from $[0, 1]$ to $[0, 1]$ such that for any $a \in [0, 1]$ exactly one of them maps 1 to a.

EXAMPLE 4.4.5. *The Cayley tomonoids associated with the three standard t-norms are shown in Figure 1. A selection of translations are indicated in a schematic way.*

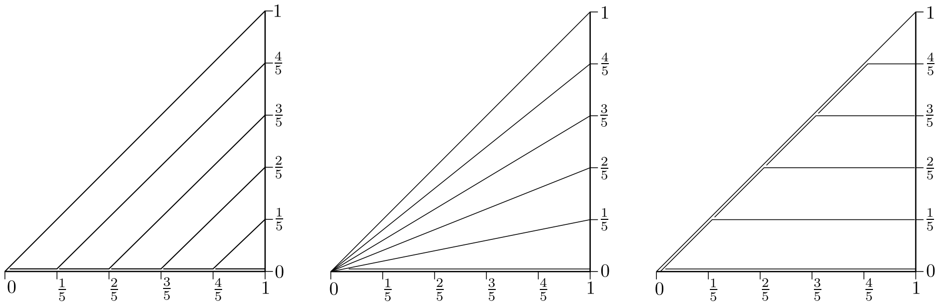


Figure 1. The Cayley tomonoids associated with the tomonoids based on the Łukasiewicz, product, and Gödel t-norm.

Quotients and Cayley tomonoids

Our next aim is to see how quotients of a q.n.c. tomonoid are reflected by its associated Cayley tomonoid.

We will use the following notation and conventions. Let $\boldsymbol{L}$ be a q.n.c. tomonoid and let $\boldsymbol{P}$ be the quotient of $\boldsymbol{L}$ by the filter $\boldsymbol{F}$. Then any $R \in P$ will be considered as a subset of L, namely as a class of the congruence on $\boldsymbol{L}$ that yields $\boldsymbol{P}$.

For any $f \in F$, λ_f maps R to itself. We write $\lambda_f^R \colon R \to R$ for λ_f with its domain and range being restricted to R, and we put $\Lambda^R = \{\lambda_f^R \colon f \in F\}$. Note that Λ^F is the Cayley tomonoid associated with $\boldsymbol{F}$.

Moreover, let $R \in P$ and $T \in P \backslash \{F\}$, and let $S = R \odot T$. Then for any $t \in T$, λ_t maps R to S. We write $\lambda_t^{R,S} \colon R \to S$ for λ_t with its domain restricted to R and its range restricted to S, and we put $\Lambda^{R,S} = \{\lambda_t^{R,S} \colon t \in T\}$.

Finally, we denote a function that maps all values of a set A to the single value b by $c^{A,b}$.

The following lemma describes the sets Λ^R, where R is an $\boldsymbol{F}$-class; cf. Figure 2.

LEMMA 4.4.6. *Let $\boldsymbol{L} = \langle L, \leq, \odot, 1 \rangle$ be a q.n.c. tomonoid that possesses the non-trivial filter $\boldsymbol{F}$. Let $\boldsymbol{P}$ be the quotient of $\boldsymbol{L}$ induced by $\boldsymbol{F}$.*

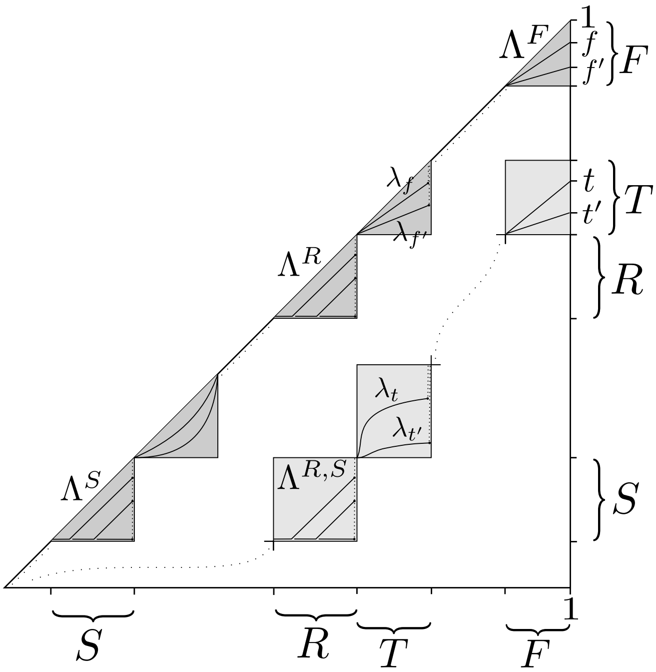


Figure 2. The Cayley tomonoid associated with a q.n.c. tomonoid $\boldsymbol{L}$, which possesses a filter $\boldsymbol{F}$. The translations by the elements $t', t, f', f, 1$ are depicted schematically. It is furthermore indicated how the Cayley tomonoid associated with the quotient of $\boldsymbol{L}$ by $\boldsymbol{F}$ arises. The translation by an element T is shown in light grey, the translation by F, which is the identity mapping, is shown in dark grey.

(i) *The top element of $\boldsymbol{P}$ is F. Let $u = \inf F \in L^0$; then $u < 1$, and F is one of $(u, 1]$ or $[u, 1]$. Moreover, $\langle \Lambda^F, \leq, \circ, \text{id}_F \rangle$ is the Cayley tomonoid of $\boldsymbol{F}$. We have:*

(a) *Let $f \in F$. If $F = [u, 1]$, $\lambda^F_f(u) = u$; if $F = (u, 1]$, $\bigwedge_{g \in F} \lambda^F_f(g) = u$. Moreover, $\lambda^F_f(1) = f$.*

(b) *If $F = [u, 1]$, Λ^F has the bottom element $\text{c}^{F,u}$.*

Finally,

$$\pi\colon F \to \Lambda^F, \quad f \mapsto \lambda^F_f$$

is an isomorphism between $\langle F, \leq, \odot, 1 \rangle$ and $\langle \Lambda^F, \leq, \circ, \text{id}_F \rangle$.

(ii) *Let $R \in P \backslash \{F\}$. Let $u = \inf R$ and $v = \sup R$. If $u = v$, then $R = \{u\}$ and $\lambda^R_f(u) = u$ for any $f \in F$.*

Assume now $u < v$. Then R is one of (u, v), $[u, v)$, $(u, v]$, or $[u, v]$. Moreover, $\langle \Lambda^R, \leq, \circ, \text{id}_R \rangle$ is a composition tomonoid on R fulfilling (C1)–(C4) *as well as* *the following properties:*

(c) *Let $f \in F$. If $u \in R$, $\lambda_f^R(u) = u$; if $u \notin R$, $\bigwedge_{r \in R} \lambda_f^R(r) = u$. Moreover, if $v \notin R$, $\bigvee_{r \in R} \lambda_f^R(r) = \lambda_f(v) = v$.*

(d) *If $R = [u, v]$, Λ^R has the bottom element $\mathsf{c}^{R,u}$. If $R = [u, v)$, then $\mathsf{c}^{R,u} \notin \Lambda^R$.*

Finally,

$$\varrho \colon F \to \Lambda^R, \quad f \mapsto \lambda_f^R \tag{11}$$

is a surjective sup-preserving homomorphism from $\langle F, \leq, \odot, 1\rangle$ to $\langle \Lambda^R, \leq, \circ, \mathbf{id}_R\rangle$.

Proof. (i) Here, Proposition 4.4.3 is applied to the q.n.c. tomonoid $\boldsymbol{F}$.

(a) Let $f \in F$. If $u \in F$, clearly $\lambda_f^F(u) = u$. If $u \notin F$, we have $u \leq \bigwedge_{g \in F} \lambda_f^F(g) \leq \bigwedge_{g \in F} g = u$, that is, $\bigwedge_{g \in F} \lambda_f^R(g) = u$. Clearly, $\lambda_f(1) = f \odot 1 = f$.

(b) If $u \in F$, $\lambda_u^F = \mathsf{c}^{F,u}$ is the bottom element of Λ^F.

(ii) The case that R is a singleton is trivial. Assume $u < v$.

The fact that Λ^R is a composition tomonoid fulfilling (C1)–(C4) and that ϱ, defined by (11), is a sup-preserving and surjective homomorphism follows from Proposition 4.4.3.

(c) Let $f \in F$. We see like in the proof of (a) that $\lambda_f^R(u) = u$ if $u \in R$, and $\bigwedge_{r \in R} \lambda_f^R(r) = u$ otherwise. Moreover, if $v \notin R$, then $\lambda_f(v) \notin R$ and consequently $r < \lambda_f(v) \leq v$ for any $r \in R$, that is, $\lambda_f(v) = v$.

(d) Assume $u \in R$. If R has a largest element v as well, $v \odot z = u$ for some $z \in F$, and hence $\mathsf{c}^{R,u} = \lambda_z^R \in \Lambda^R$. If R does not contain its supremum v, then by (c), $\bigvee_{r \in R} \lambda_f^R(r) = v$ for any $f \in F$, and it follows $\mathsf{c}^{R,u} \notin \Lambda^R$. □

We next turn to the set $\Lambda^{R,S}$, where R and S are two $\boldsymbol{F}$-classes; cf. again Figure 2.

In what follows, we call a pair A, B of elements of the q.n.c. tomonoid P *$\odot$-maximal* if $A = B \to A \odot B$ and $B = A \to A \odot B$.

LEMMA 4.4.7. *Let $\boldsymbol{L} = \langle L, \leq, \odot, 1\rangle$ be a q.n.c. tomonoid that possesses the non-trivial filter $\boldsymbol{F}$. Let $\boldsymbol{P}$ be the quotient of $\boldsymbol{L}$ induced by $\boldsymbol{F}$.*

Let $R, T \in P$ such that $T < F$, and let $S = R \odot T$.

(i) *Let R, T be $\odot$-maximal. Then $S < R$. Let $u = \inf R$, $v = \sup R$, $u' = \inf S$, and $v' = \sup S$. If $u = v$, then $R = \{u\}$, $u' \in S$, and $\lambda_t^{R,S}(u) = u'$ for all $t \in T$. If $u' = v'$, then $S = \{u'\}$ and $\lambda_t^{R,S} = \mathsf{c}^{R,u'}$ for all $t \in T$.*

Assume now $u < v$ and $u' < v'$. If then $u \in R$, we have $u' \in S$. Moreover, $\Lambda^{R,S} = \{\lambda_t^{R,S} \colon t \in T\}$ is a set of mappings from R to S with the following properties:

(a) *R and S are conditionally complete, and for any $t \in T$, $\lambda_t^{R,S}$ is sup-preserving.*

(b) *Let $t \in T$. If $u \in R$, $\lambda_t^{R,S}(u) = u'$; if $u \notin R$, $\bigwedge_{r \in R} \lambda_t^{R,S}(r) = u'$.*

(c) *Under the pointwise order, $\Lambda^{R,S}$ is totally ordered.*

(d) *Let* $K \subseteq \Lambda^{R,S}$ *such that* $\bigvee_{\lambda \in K} \lambda(r) \in S$ *for all* $r \in R$. *Then the pointwise calculated supremum of* K *is in* $\Lambda^{R,S}$.

(e) *If* $u' \in S$ *and* $v \in R$, $\Lambda^{R,S}$ *has the bottom element* $c^{R,u'}$. *If* $u' \in S$ *and* $v \notin R$, *then either* $\Lambda^{R,S} = \{c^{R,u'}\}$ *or* $c^{R,u'} \notin \Lambda^{R,S}$. *If* $v \notin R$ *and* $v' \in S$, *then* $u' \in S$ *and* $\Lambda^{R,S} = \{c^{R,u'}\}$.

(f) *For any* $t \in T$ *and* $f \in F$, $\lambda_f^S \circ \lambda_t^{R,S}$ *and* $\lambda_t^{R,S} \circ \lambda_f^R$ *are in* $\Lambda^{R,S}$ *and coincide.*

Finally,

$$\tau \colon T \to \Lambda^{R,S}, \quad t \mapsto \lambda_t^{R,S} \tag{12}$$

is a sup-preserving mapping from T *to* $\Lambda^{R,S}$ *such that, for any* $f \in F$ *and* $t \in T$,

$$\tau(\lambda_f^T(t)) = \lambda_f^S \circ \tau(t) = \tau(t) \circ \lambda_f^R. \tag{13}$$

(ii) *Let* R, T *not be* $\odot$*-maximal. Then* S *contains a smallest element* u', *and* $\lambda_t^{R,S} = c^{R,u'}$ *for all* $t \in T$.

Proof. (i) We clearly have $S \leq R$. If $S = R$, the maximal element Y such that $R \odot Y = R \odot T$ would be F, in contradiction to the assumptions that $T < F$ and R, T is a $\odot$-maximal pair. Thus $S < R$.

We consider first the case that R is a singleton, that is, $R = \{u\}$. Then $u \odot f = u$ for all $f \in F$. Let $t \in T$; then $\lambda_t^{R,S}(u) \odot f = u \odot t \odot f = u \odot t = \lambda_t^{R,S}(u)$ for any $f \in F$; hence $u' \in S$ and $\lambda_t^{R,S}(u) = u'$.

The case that S is a singleton is trivial.

Assume now $u < v$ and $u' < v'$. Let $u \in R$. Then $\lambda_t^{R,S}(u) = u' \in S$ for any $t \in T$. Indeed, we again have $u \odot f = u$ and consequently $\lambda_t^{R,S}(u) \odot f = \lambda_t^{R,S}(u)$ for any $f \in F$.

(a), (c), (d), and the fact that τ, defined by (12), is sup-preserving follow from Proposition 4.4.3.

(b) Let $t \in T$. If $u \in R$, we have seen above that $\lambda_t^{R,S}(u) = u'$. If $u \notin R$, choose some $\tilde{r} \in R$; then $\bigwedge_{r \in R} \lambda_t^{R,S}(r) = \bigwedge_{f \in F} \lambda_t^{R,S}(\tilde{r} \odot f) = \bigwedge_{f \in F} (\lambda_t^{R,S}(\tilde{r}) \odot f) = \inf S = u'$.

(e) Let $u' \in S$ and $v \in R$. Then, for an arbitrary $\tilde{t} \in T$, $\lambda_{\tilde{t}}(v)$ and u' are both in the congruence class S, whose smallest element is u'. Thus, for some $f \in F$, we have $\lambda_{\tilde{t}}(v) \odot f = u'$, and consequently $\lambda_t^{R,S} = c^{R,u'}$, where $t = \tilde{t} \odot f \in T$.

Next, let $u' \in S$ and $v \notin R$. For any $t, t' \in T$ such that $t \sim_F t'$, we have $\lambda_t(v) \sim_F \lambda_{t'}(v)$. Consequently, either $\lambda_t(v) \in S$ for all $t \in T$, or $\lambda_t(v) \notin S$ for all $t \in T$. Furthermore, from $v \notin R$ it follows $v \odot f = v$ and thus $\lambda_t(v) \odot f = v \odot t \odot f = v \odot t = \lambda_t(v)$ for all $t \in T$ and $f \in F$. We conclude that, in the former case, $\lambda_t(v) = u'$ for any $t \in T$, that is, $\Lambda^{R,S} = \{c^{R,u'}\}$. In the latter case, $v' \leq \lambda_t(v) = \bigvee_{r \in R} \lambda_t^{R,S}(r) \leq v'$, that is, $\lambda_t(v) = v'$ for all $t \in T$, and $c^{R,u'} \notin \Lambda^{R,S}$.

Finally, let $v \notin R$ and $v' \in S$. Let $t \in T$. Then $\lambda_t(v) = \bigvee_{r \in R} \lambda_t^{R,S}(r) \in S$ and $\lambda_t(v) \odot f = v \odot t \odot f = v \odot t = \lambda_t(v)$ for any $f \in F$; thus $\lambda_t(v) = u' \in S$, that is, $\lambda_t^{R,S} = c^{R,u'}$, and we conclude again $\Lambda^{R,S} = \{c^{R,u'}\}$.

(f) Let $t \in T,\ f \in F$, and $r \in R$. We have $(\lambda^S_f \circ \lambda^{R,S}_t)(r) = r \odot t \odot f = \lambda^{R,S}_{t\odot f}(r) = \lambda^{R,S}_{f\odot t}(r) = r \odot f \odot t = (\lambda^{R,S}_t \circ \lambda^R_f)(r)$.

Furthermore, $\tau(\lambda^T_f(t))(r) = \lambda^{R,S}_{t\odot f}(r) = r \odot t \odot f$, and also (13) follows. The proof of part (i) is complete.

(ii) Consider first the case that there is an $R' > R$ such that $R' \odot T = S$. Let $r \in R$, $t \in T$, and $r' \in R'$. Then $r < r' \odot f$ for any $f \in F$, and consequently $r \odot t \leq r' \odot f \odot t$ for any $f \in F$. As $r' \odot t \in S$, we conclude that $r \odot t$ is the smallest element of S, that is, $\lambda^{R,S}_t(r) = r \odot t = u'$, where $u' = \inf S \in S$.

Similarly, we argue in the case that there is a $T' > T$ such that $R \odot T' = S$. Let $r \in R,\ \ t \in T$, and $t' \in T'$. Then $t < t' \odot f$ for any $f \in F$, and consequently $r \odot t \leq r \odot t' \odot f$ for any $f \in F$. We conclude again that $u' = \inf S \in S$ and $\lambda^{R,S}_t(r) = r \odot t = u'$. □

Again, let $\boldsymbol{L}$ be a q.n.c. tomonoid, $\boldsymbol{F}$ a filter of $\boldsymbol{L}$, and $\boldsymbol{P}$ the quotient of $\boldsymbol{L}$ by $\boldsymbol{F}$. From an intuitive point of view, we may say with reference to Figure 2 that the Cayley tomonoid associated with $\boldsymbol{L}$ is composed from triangular and rectangular sections, one for each $R \in P$ and for each pair of elements $R, S \in P$, respectively. Lemma 4.4.6(i) deals with top congruence class, the filter F, whose associated Cayley tomonoid Λ^F is located in the uppermost triangle. Lemma 4.4.6(ii) describes the set Λ^R for some $R \in P \backslash \{F\}$, located in one of the remaining triangles. Finally, let $S = R \odot T < R$, where $R, S, T \in P$. Then Lemma 4.4.7 deals with $\Lambda^{R,S}$, located in the rectangular section associated with R and S. If R, T is not $\odot$-maximal, $\Lambda^{R,S}$ is trivial by part (ii).

We will provide in the sequel some examples of q.n.c. tomonoids based on left-continuous t-norms. Definitions of t-norms are often involved; to keep them as short as possible, we will in general not provide full specifications, but assume commutativity to be used to cover all cases.

EXAMPLE 4.4.8. *Let us consider the following t-norm:*

$$a \star_H b \;=\; \begin{cases} 4ab - 3a - 3b + 3 & \text{if } a, b > \frac{3}{4}, \\ 4ab - 3a - 2b + 2 & \text{if } \frac{1}{2} < a \leq \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a - b + 1 & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a & \text{if } a \leq \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ 2ab - a - b + \frac{3}{4} & \text{if } \frac{1}{2} < a, b \leq \frac{3}{4}, \\ ab - \frac{1}{2}a - \frac{1}{4}b + \frac{1}{8} & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \\ 0 & \text{if } a \leq \frac{1}{4} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \text{ or } a, b \leq \frac{1}{2}. \end{cases} \tag{14}$$

$\star_H$ *is a modification of a t-norm defined by Hájek in* [21]. *The tomonoid* $\langle [0,1], \leq, \star_H, 1 \rangle$ *possesses the filter* $F = (\frac{3}{4}, 1]$ *and the* $\boldsymbol{F}$*-classes are* $\{0\}$, $(0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, *and* $(\frac{3}{4}, 1]$. *The quotient by* $\boldsymbol{F}$ *is isomorphic to* $\boldsymbol{L}_5$, *the five-element Łukasiewicz chain. An illustration, showing the sets* Λ^R *and* $\Lambda^{R,S}$ *for all the congruence classes* R *and* S, *can be found in Figure 3.*

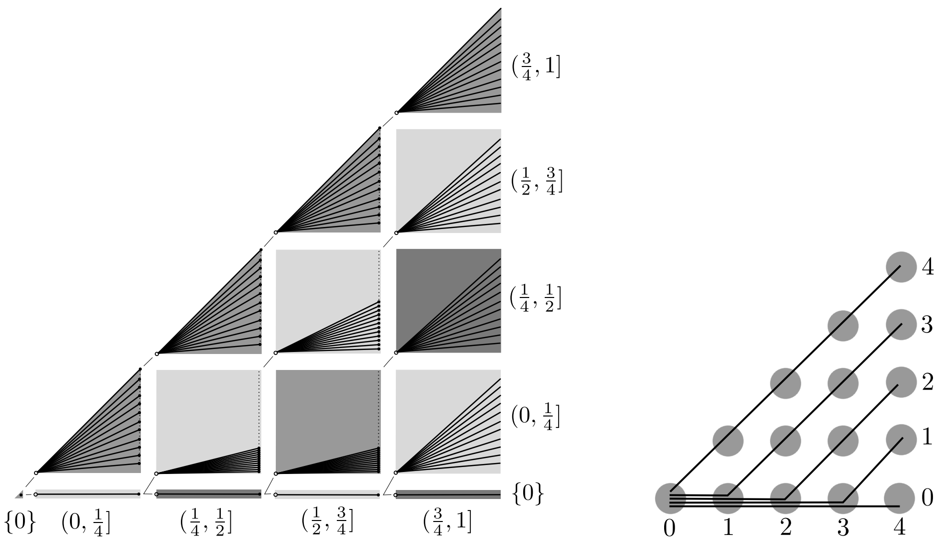


Figure 3. Left: The tomonoid $\langle [0,1], \leq, \star_{\mathrm{H}}, 1 \rangle$. To increase clarity, we have separated the congruence classes by margins. Right: The five-element quotient $\boldsymbol{L}_5$ by the filter $(\frac{3}{4}, 1]$.

4.5 Real Archimedean coextensions

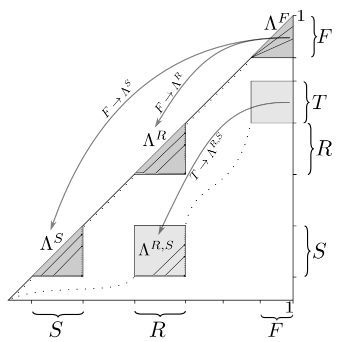
As we have seen in Subsection 4.3, we may associate with a q.n.c. tomonoid the chain of quotients induced by its filters. We shall now have a closer look at the case of two successive elements of this chain. By part (iv) of Theorem 4.3.4, the corresponding coextension is in this case Archimedean.

In what follows, a *real interval* is meant to be a one-element set or one of (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ for $a, b \in \mathsf{R}$ such that $a < b$.

DEFINITION 4.5.1. *Let $\boldsymbol{P}$ be the quotient of the q.n.c. tomonoid $\boldsymbol{L}$ by an Archimedean filter such that each congruence class is order-isomorphic to a real interval. Then we call $\boldsymbol{P}$ a* real Archimedean quotient *of $\boldsymbol{L}$, and we call $\boldsymbol{L}$ a* real Archimedean coextension *of $\boldsymbol{P}$.*

Given a q.n.c. tomonoid $\boldsymbol{P}$, our aim is to describe the real Archimedean coextensions $\boldsymbol{L}$ of $\boldsymbol{P}$. To this end, we will specify, following the lines of Lemmas 4.4.6 and 4.4.7, sectionwise the Cayley tomonoid Λ of $\boldsymbol{L}$. That is, for each pair R and S of congruence classes, we specify the translations restricted in domain to R and in range to S. The illustration below may serve as a guide through this section, indicating which section of the Cayley tomonoid is described in which proposition or theorem.

We will use a few auxiliary notions. A *non-minimal* element of a chain A is any $a \in A$ such that a is not the smallest element of A. Furthermore, let χ be an order-preserving mapping from A to another chain B. Then we call $\{x \in A \colon \chi(x) \text{ is non-minimal in } B\}$



- The Cayley tomonoid Λ^F of the extending filter $\boldsymbol{F}$: Proposition 4.5.3.
- For each $R \in P$ such that $R < F$,
  - the composition tomonoid Λ^R: Theorem 4.5.6;
  - the homomorphism $F \to \Lambda^R$, $f \mapsto \lambda^R_f$: Proposition 4.5.7.
- For each pair $R, S \in P$ such that $S = R \odot T$ for some $T \in P \backslash \{F\}$,
  - the set of mappings $\Lambda^{R,S}$: Proposition 4.5.8;
  - the mapping $T \to \Lambda^{R,S}$, $t \mapsto \lambda^{R,S}_t$: Proposition 4.5.9.

Figure 4. The way we specify a real Archimedean coextension of a q.n.c. tomonoid.

the *support* of χ. Obviously, the support of χ is the whole set A if B does not possess a smallest element; and the support of χ is empty if and only if B possesses a smallest element u and $\chi = c^{A,u}$.

A composition tomonoid Φ on a chain R will be called *c-isomorphic* to another composition tomonoid Ψ on a chain S if there is an order isomorphism $\iota \colon R \to S$ such that $\Psi = \{\iota \circ \lambda \circ \iota^{-1} \colon \lambda \in \Phi\}$. Note that c-isomorphic composition tomonoids are also isomorphic (as tomonoids); the converse, however, does not in general hold.

We will first be concerned with the sets Λ^R, where R is an element of the quotient of the tomonoid $\boldsymbol{L}$ that we are going to construct, identifiable with an element of the given tomonoid $\boldsymbol{P}$.

DEFINITION 4.5.2. (i) *Let* Φ *consist of the functions* $\lambda_t \colon [0,1] \to [0,1],\;\; x \mapsto (x+t-1) \vee 0$ *for each* $t \in [0,1]$. *Then* $\langle \Phi, \leq, \circ, id_{[0,1]} \rangle$ *is called the* Łukasiewicz composition tomonoid.

(ii) *Let* Φ *consist of the functions* $\lambda_t \colon (0,1] \to (0,1],\;\; x \mapsto t \cdot x$ *for each* $t \in (0,1]$. *Then* $\langle \Phi, \leq, \circ, id_{(0,1]} \rangle$ *is called the* product composition tomonoid.

(iii) *Let* Φ *consist of the functions* $\lambda_t \colon [0,1) \to [0,1),\;\; x \mapsto \frac{(x+t-1) \vee 0}{t}$ *for each* $t \in (0,1]$. *Then* $\langle \Phi, \leq, \circ, id_{[0,1)} \rangle$ *is called the* reversed product composition tomonoid.

(iv) *Let* Φ *consist of the functions* $\lambda_t \colon (0,1) \to (0,1),\;\; x \mapsto x^{\frac{1}{t}}$ *for each* $t \in (0,1]$. *Then* $\langle \Phi, \leq, \circ, id_{(0,1)} \rangle$ *is called the* power composition tomonoid.

A composition tomonoid on a chain R *that is c-isomorphic to one of these four will be called a* standard composition tomonoid.

The four standard composition tomonoids are schematically shown in Figure 5. Note that the key property in which they differ is their base set: the real unit interval with, without the left, right margin.

We start by specifying the uppermost composition tomonoid.

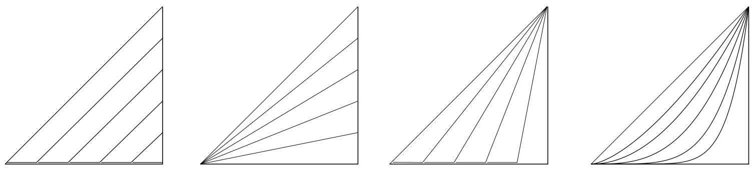


Figure 5. The standard composition tomonoids.

PROPOSITION 4.5.3. *Let $\boldsymbol{P}$ be a real Archimedean quotient of the q.n.c. tomonoid $\boldsymbol{L}$ by the filter $\boldsymbol{F}$. Then $\langle \Lambda^F, \leq, \circ, \mathrm{id}_F \rangle$ is c-isomorphic to the Łukasiewicz composition tomonoid or to the product composition tomonoid.*

Proof. By assumption, F is order-isomorphic to a real interval, which is right-closed and either left-closed or left-open.

Assume first that F possesses a smallest element. Then $\langle F, \leq, \odot, 1 \rangle$ is a q.n.c. tomonoid such that F is order-isomorphic to the real unit interval; that is, $\boldsymbol{F}$ is isomorphic to a tomonoid based on a t-norm. Let $\odot$ be this t-norm. By assumption, $\boldsymbol{F}$ is Archimedean. By [27, Proposition 2.16], $\odot$ is continuous, and due to the Archimedean property, $\odot$ is in fact isomorphic to the Łukasiewicz t-norm. Consequently, Λ^F is c-isomorphic to the Łukasiewicz composition tomonoid.

Assume second that F does not have a smallest element. Then $\langle F^0, \leq, \odot, 1 \rangle$ is a q.n.c. tomonoid such that F^0 is order-isomorphic to the real unit interval; that is, $\boldsymbol{F}^0$ is again isomorphic to a tomonoid based on a t-norm. Let $\odot$ be this t-norm. Then $(0, 1]$ together with the restriction of $\odot$ to $(0, 1]$ is an Archimedean tomonoid. Thus we conclude as before from [27, Proposition 2.16] that $\odot$ is continuous, but this time isomorphic to the product t-norm. Consequently, Λ^F is in this case c-isomorphic to the product composition tomonoid. □

Our next aim is to characterise the composition tomonoids associated with some of the remaining congruence classes. Several preparations are needed.

For chains A and B that are order-isomorphic to real intervals, *continuity* of a mapping from A to B will be understood in the obvious way.

LEMMA 4.5.4. *Let $\boldsymbol{P}$ be a real Archimedean quotient of the q.n.c. tomonoid $\boldsymbol{L}$. Let $R \in P$, and assume that R is not a singleton. Then $\langle \Lambda^R, \leq, \circ, \mathrm{id}_R \rangle$ is a composition tomonoid on R fulfilling* (C1)–(C4). *Moreover, the following holds:*

(C6) *Any $\lambda \in \Lambda^R$ is continuous.*

(C7) *For any $\lambda \in \Lambda^R \backslash \{\mathrm{id}_R\}$ and any non-minimal element r of R, $\lambda(r) < r$.*

Proof. As $\boldsymbol{P}$ is assumed to be a real Archimedean quotient, R is order-isomorphic to a real interval. Furthermore, by Proposition 4.5.3, the extending filter $\boldsymbol{F}$ is isomorphic to $\langle [0, 1], \star, \leq, 1 \rangle$, where $\star$ is the Łukasiewicz t-norm, or to $\langle (0, 1], \cdot, \leq, 1 \rangle$, where $\cdot$ is the product t-norm.

By Lemma 4.4.6, Λ^R fulfils (C1)–(C4). Thus we only have to prove (C6) and (C7). We will first show (C7) as well as a strengthened form of (C7) and then (C6).

(C7) Let $f \in F \backslash \{1\}$ and let $r \in R$. Assume that $\lambda_f^R(r) = r \odot f = r$. Then $r \odot f^n = r$ for any $n \geq 1$, and since $\boldsymbol{F}$ is Archimedean, it follows that $r \odot g = r$ for all $g \in F$; thus r is the smallest element of the congruence class R. We conclude that if r is not the smallest element of R, then $\lambda_f^R(r) < r$.

We next prove:

($\star$) For any $\lambda \in \Lambda^R \backslash \{id_R\}$ and any $r \in R$ that is neither the smallest nor the largest element of R, $\bigwedge_{x>r} \lambda(x) < r$.

Let $f \in F \backslash \{1\}$ and let $r \in R$ be neither the smallest nor the largest element of R. Let then $g \in F$ be such that $f \leq g^2 < g < 1$. Assume that $\lambda_g^R(x) = x \odot g > r$ for all $x \in R$ such that $x > r$; then $x \odot g^n > r$ for any $n \geq 1$, and since $\boldsymbol{F}$ is Archimedean, it further follows $x \odot h > r$ for all $h \in F$, in contradiction to the fact that x and r are in the same congruence class R. Hence there is an $x \in R$ such that $x > r$ and $\lambda_g^R(x) = x \odot g \leq r$. As r is non-minimal and λ_g is not the identity, we conclude by (C7) that $\lambda_f^R(x) = x \odot f \leq x \odot g \odot g \leq r \odot g < r$. The proof of ($\star$) is complete.

(C6) Let $f \in F$ and assume that λ_f^R is discontinuous at $r \in R$. Note that then $f < 1$ and r is neither the smallest nor the largest element of R. Let $p = \lambda_f^R(r)$ and $q = \bigwedge_{x>r} \lambda_f^R(x)$; then $p < q < r$ by ($\star$). By (C4) and (C7), we may choose a $\lambda \in \Lambda^R$ such that $p < \lambda(q) < q$ and $q < \lambda(r) < r$. By ($\star$), there is an $x > r$ such that $\lambda(x) \leq r$. Then $\lambda_f^R(\lambda(x)) \leq \lambda_f^R(r) = p$ and $\lambda(\lambda_f^R(x)) \geq \lambda(q) > p$, a contradiction. □

The proof of the following technical lemma proceeds according to [30].

LEMMA 4.5.5. *For each $k \in \mathsf{N}$, let g_k be an order-automorphism of the open real unit interval $(0,1)$. Assume that, for each k,* (i) *$g_k(x) < x$ for $x \in (0,1)$;* (ii) *$g_{k+1}^2 = g_k$; and* (iii) *the functions g_k converge uniformly to $\mathrm{id}_{(0,1)}$. Then there is an order-automorphism φ of $(0,1)$ such that $g_k(\varphi(x)) = \varphi(x^{2^{(\frac{1}{2})^k}})$ for each $k \in \mathsf{N}$ and $x \in (0,1)$.*

Proof. We have to determine an order-automorphism $\varphi\colon (0,1) \to (0,1)$ such that $(\varphi^{-1} \circ g_k \circ \varphi)(x) = x^{2^{(\frac{1}{2})^k}}$ for each $x \in (0,1)$ and $k \geq 0$. We shall reformulate this problem twice. First, let $y = \ln x \in (-\infty, 0)$ and $\psi\colon (-\infty,0) \to (0,1),\ y \mapsto \varphi(e^y)$. Then $\varphi(x) = \psi(y)$, and our problem is to determine an order isomorphism $\psi\colon (-\infty,0) \to (0,1)$ such that $(\psi^{-1} \circ g_k \circ \psi)(y) = 2^{(\frac{1}{2})^k} y$ for each $y \in (-\infty,0)$ and $k \geq 0$. We next set $z = \ln(-y) \in \mathsf{R}$ and $\chi\colon \mathsf{R} \to (0,1),\ z \mapsto \psi(-e^z)$. Then $\psi(y) = \chi(z)$, and our problem is finally to find an order antiisomorphism $\chi\colon \mathsf{R} \to (0,1)$ such that

$$(\chi^{-1} \circ g_k \circ \chi)(z) = z + \tfrac{\ln 2}{2^k}, \quad z \in \mathsf{R},\ k \geq 0. \tag{15}$$

We set $\chi(0) = \frac{1}{2}$, and for any $k \geq 0$ and $n \in \mathsf{Z}$, let $\chi(\frac{n}{2^k} \ln 2) = g_k^n(\frac{1}{2})$. We readily check that this defines χ unambiguously on the set $R = \{\frac{n}{2^k} \ln 2\colon k \geq 0,\ n \in \mathsf{Z}\}$. For $z \in R$, (15) is then fulfilled. Moreover, because $g_k(x) < x$ for each k and $x \in (0,1)$, (15) implies that χ is strictly decreasing.

We next show that χ can be continuously extended to the whole real line. Let $r \in \mathsf{R}$, and let $(n_k)_k$ be the sequence of natural numbers such that $r \in [\frac{n_k}{2^k} \ln 2, \frac{n_k+1}{2^k} \ln 2)$ for every k. We have to prove that the length of the interval $[\chi(\frac{n_k+1}{2^k}\ln 2), \chi(\frac{n_k}{2^k}\ln 2)]$ converges to 0 for $k \to \infty$. But this is the case because $\chi(\frac{n_k+1}{2^k}\ln 2) = g_k(\chi(\frac{n_k}{2^k}\ln 2))$ and $(g_k)_k$ converges uniformly to the identity. Note that the function $\chi\colon \mathsf{R} \to [0,1]$ is decreasing and fulfils (15) because χ and the g_k are continuous.

It remains to show is that χ is surjective. Recall that $g_0(x) < x$ for all $x \in (0,1)$, let $u = \bigwedge_n g_0^n(\frac{1}{2})$, and assume $u > 0$. Then, by the continuity of g_0, we have $g_0(u) = u$, a contradiction; so $u = 0$. Similarly, we conclude $\bigvee_n {g_0^{-}}^n(\frac{1}{2}) = 1$. So the image of χ covers the whole interval $(0,1)$. □

We are now ready to characterise the composition tomonoids Λ^R, where R is any congruence class.

THEOREM 4.5.6. *Let $\boldsymbol{P}$ be a real Archimedean quotient of the q.n.c. tomonoid $\boldsymbol{L}$. Let $R \in P$, and assume that R is not a singleton. Then $\langle \Lambda^R, \leq, \circ, id_R \rangle$ is a standard composition tomonoid.*

In fact, if then R has a smallest and a largest element, Λ^R is c-isomorphic to the Łukasiewicz composition tomonoid. If R has a largest but no smallest element, Λ^R is c-isomorphic to the product composition tomonoid. If R has a smallest but no largest element, Λ^R is c-isomorphic to the reversed product composition tomonoid. If R has no smallest and no largest element, Λ^R is c-isomorphic to the power composition tomonoid.

Proof. We can assume that R is a real interval with the boundaries 0 and 1.

By Lemma 4.5.4, $\langle \Lambda^R, \leq, \circ, id_R \rangle$ is a composition tomonoid fulfilling (C1)–(C4) and (C6)–(C7). Before beginning the actual proof, we will state some auxiliary facts.

(a) By (C6), each $\lambda \in \Lambda^R$ is continuous and, by Lemma 4.4.6(ii)(c), if $0 \notin R$ the right limit of λ at 0 is 0, and if $1 \notin R$ the left limit of λ at 1 is 1. Moreover, id_R is the uniform limit of (any increasing sequence in) $\Lambda^R \backslash \{id_R\}$. In fact, by (C4), id_R is the pointwise supremum of these mappings, and if 0 or 1 are not in R, the continuous extension of any $\lambda \in \Lambda^R$ maps 0 to 0 and 1 to 1; thus the claim follows from the compactness of $[0,1]$.

(b) (C7) and (a) imply that each $\lambda \in \Lambda^R$, restricted to its support, is strictly increasing.

(c) If $\lambda(r) = \lambda'(r) > 0$ for some $\lambda, \lambda' \in \Lambda^R$ and $r \in R$, then $\lambda = \lambda'$. Indeed, if then $0 < \lambda(s) < \lambda'(s)$ for some $s \in R$, the pair $\kappa \circ \lambda'$ and λ is not comparable for a sufficiently large $\kappa \in \Lambda^R \backslash \{id_R\}$. Thus λ and λ' coincide on the meet of their supports, and by continuity and monotonicity, we conclude $\lambda = \lambda'$.

(d) Infima of subsets Λ^R that possess some lower bound exist and are calculated pointwise. Indeed, let $\lambda_\iota \in \Lambda^R$, $\iota \in I$, be lower bounded. Let $r \in R$ be such that $s = \bigwedge_\iota \lambda_\iota(r) > 0$. Then, for any $\varepsilon > 0$, there is a $\kappa \in \Lambda^R$ such that $s - \varepsilon < \kappa(a) < s$; as κ is continuous, there is a ι such that $s - \varepsilon < (\kappa \circ \lambda_\iota)(r) < s$. We conclude that the supremum of the lower bounds of λ_ι, $\iota \in I$, is their pointwise infimum.

(e) For any $r \in R$, $\{\lambda(r) \colon \lambda \in \Lambda^R\} = \{a \in R \colon a \leq r\}$. Indeed, this set is closed under suprema by (C4) and under infima by (d). Moreover, we conclude from (a) and (c) that the set is dense. In view of (c), we see in particular that $1 \in R$ implies that Λ^R fulfils (C5).

We distinguish four cases.

Case (i). Let $R = [0,1]$. By (e), Λ^R fulfils (C1)–(C5) and is thus the Cayley tomonoid associated with a q.n.c. tomonoid. As in the proof of Proposition 4.5.3, we conclude that Λ^R is c-isomorphic to the Łukasiewicz composition tomonoid.

Case (ii). Let $R = (0,1]$. We proceed as for Case (i) to see that Λ^R is c-isomorphic to the product composition tomonoid.

Case (iii). Let $R = [0,1)$. For $\lambda \in \Lambda^R$, let $z_\lambda = \max\,\{r \in R\colon \lambda(r) = 0\}$; note that this definition is possible because $\lambda(0) = 0$ and $\lim_{r \nearrow 1} \lambda(r) = 1$. We claim:

(f) For each $z \in R$, there is exactly one $\lambda \in \Lambda^R$ such that $z = z_\lambda$. Indeed, by (C7) and (a), $\lambda \neq \lambda'$ implies $z_\lambda \neq z_{\lambda'}$. Furthermore, Λ^R is closed under suprema and infima, hence $\{z_\lambda\colon \lambda \in \Lambda^R\}$ is a closed subset of R, which by (a) is dense and does not possess a largest element and thus equals R.

For each $\lambda \in \Lambda^R$, we have that $\lambda(r) = 0$ for $r \in [0, z_\lambda]$, strictly increasing on $[z, 1)$, and $\lim_{r \nearrow 1} \lambda(1) = 1$. Thus we can define

$$\tilde{\lambda}\colon (0,1] \to (0,1], \quad x \mapsto \begin{cases} 1 - \lambda^{-1}(1-x) & \text{if } x < 1, \\ 1 - z_\lambda & \text{if } x = 1. \end{cases}$$

It is somewhat tedious, but not difficult to check that $\tilde{\Lambda}^R = \{\tilde{\lambda}\colon \lambda \in \Lambda^R\}$ is a composition tomonoid fulfilling (C1)–(C4). By (f), $\tilde{\Lambda}^R$ fulfils also (C5). We conclude as in Case (ii) that $\tilde{\Lambda}^R$ is c-isomorphic to the product composition tomonoid. Hence Λ^R itself is c-isomorphic to the reversed product composition tomonoid.

Case (iv). Let $R = (0,1)$. Then Λ^R consists of order-automorphisms of $(0,1)$.

Let $\lambda \in \Lambda^R$ and put $\kappa = \bigwedge\{\mu \in \Lambda^R\colon \mu^2 \geq \lambda\}$. We claim that then $\kappa^2 = \lambda$. Indeed, by (d), $\kappa(r) = \bigwedge\{\mu(r)\colon \mu^2 \geq \lambda\}$ for any $r \in R$. By continuity, we calculate $\kappa^2(r) = \bigwedge\{\mu(\mu'(r))\colon \mu^2, {\mu'}^2 \geq \lambda\} = \bigwedge\{\mu^2(r)\colon \mu^2 \geq \lambda\} \geq \lambda(r)$. If this inequality was strict, there would be a $\nu \in \Lambda^R\backslash\{id_R\}$ such that $\nu^2(\kappa^2(r)) > \lambda(r)$; but then $(\kappa \circ \nu)^2(r) > \lambda(r)$ although $\kappa \circ \nu < \kappa$, a contradiction. Thus $\kappa^2(r) = \lambda(r)$, that is, $\kappa^2 = \lambda$.

Let now $\lambda_0 \in \Lambda^R\backslash\{id_R\}$, and let λ_k be the unique mapping such that $\lambda_k^{2^k} = \lambda_0$.

Then $\lambda_0 < \lambda_1 < \ldots < id$. Moreover, $(\lambda_k)_k$ converges uniformly to id. Indeed, let $\lambda = \bigvee_k \lambda_k$; then $\lambda_0 \leq \lambda^k$ for every k and it follows $\lambda = id$. Thus $(\lambda_k)_k$ converges to id pointwise and consequently uniformly.

By Lemma 4.5.5, Λ^R is c-isomorphic to a composition tomonoid $\tilde{\Lambda}^R$ containing the mappings $\tilde{\lambda}_k\colon (0,1) \to (0,1), \quad r \mapsto r^{2^{(\frac{1}{2})^k}}$ for each $k \in \mathsf{N}$. It follows that the functions $r \mapsto r^q$, where $q = 2^{\frac{m}{2^n}}$ for $m, n \in \mathsf{N}$, are dense in $\tilde{\Lambda}^R$. We conclude that $\tilde{\Lambda}^R$ consist of the mappings $x \mapsto x^q$, where $q \in \{s \in \mathsf{R}\colon s \geq 1\}$; that is, $\tilde{\Lambda}^R$ is the power composition tomonoid. □

Theorem 4.5.6 describes each composition tomonoid Λ^R separately. Each element of Λ^R is the restriction of a translation λ_f, where f is an element of the extending filter $\boldsymbol{F}$, to R. It remains to determine which mapping in Λ^R belongs to which element of F. According to our next proposition, the homomorphism $\varrho\colon F \to \Lambda^R, \quad f \mapsto \lambda_f^R$ is already uniquely determined by one non-trivial assignment.

PROPOSITION 4.5.7. *Let* $\mathbf{\Phi}$ *be a standard composition tomonoid on a chain* R*; let* $\langle F, \leq, \odot, 1\rangle$ *be either the product or the Łukasiewicz tomonoid; let* $\tilde{f} \in F\backslash\{1\}$ *be non-minimal, and let* $\tilde{\lambda} \in \Phi\backslash\{\mathrm{id}_R\}$ *have a non-empty support. Then there is at most one surjective sup-preserving homomorphism* $\varrho\colon F \to \Phi$ *such that* $\varrho(\tilde{f}) = \tilde{\lambda}$.

Proof. Let $n \geq 1$. As $\boldsymbol{F}$ is the product or the Łukasiewicz tomonoid, and $\tilde{f}$ is a non-minimal element of it, there is a unique $f_n \in F$ such that $f_n^n = \tilde{f}$. Similarly, $\mathbf{\Phi}$ is a standard composition tomonoid, and $\tilde{\lambda}$ is a non-minimal element of it; it is readily checked that in each of the four possible cases there is a unique $\lambda_n \in \Phi$ such that $\lambda_n^n = \tilde{\lambda}$.

It follows that any homomorphism mapping $\tilde{f}$ to $\tilde{\lambda}$ must map f_n to λ_n. As ϱ is supposed to be a sup-preserving homomorphism, the claim follows. □

We now turn to the sets $\Lambda^{R,S}$. We will see that Λ^R and Λ^S largely determine which mappings can be contained in $\Lambda^{R,S}$. Figure 6 gives an impression of the situation.

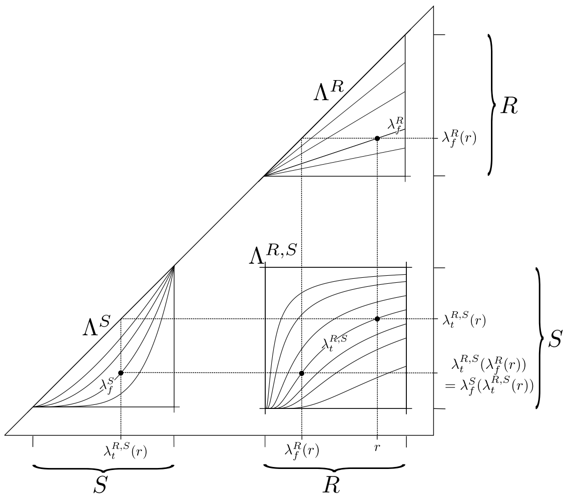


Figure 6. Each mapping contained in $\Lambda^{R,S}$ is uniquely determined by its value at a single point. The figure shows how the value of $\lambda_t^{R,S}$ at r determines its value at $\lambda_f^S(r)$.

PROPOSITION 4.5.8. *Let* $\mathbf{\Phi}$ *be a standard composition tomonoid on the chain* R *and let* $\mathbf{\Psi}$ *be a standard composition tomonoid on the chain* S*. Furthermore, let* $\langle F, \leq, \odot, 1\rangle$ *be either the product or the Łukasiewicz tomonoid, and assume that there are surjective homomorphisms* $F \to \Phi,\ \ f \mapsto \Phi_f$ *and* $F \to \Psi,\ \ f \mapsto \psi_f$*. Let*

$$\Xi \;=\; \{\xi\colon R \to S :\ \text{ for all } f \in F,\ \xi \circ \varphi_f = \psi_f \circ \xi\}. \tag{16}$$

Moreover, let Ξ' *be a set of mappings from* R *to* S *such that* (1) *for any* $\xi \in \Xi'$ *and* $f \in F$, $\psi_f \circ \xi$ *and* $\xi \circ \varphi_f$ *coincide and are in* Ξ', (2) *if the pointwise calculated* *supremum of a subset of* Ξ' *exists, it is in* Ξ'*, and* (3) *if* S *has a smallest element* u'*,* $c^{R,u'} \in \Xi'$*. Then either* $\Xi' = \Xi$ *or there is a* $\zeta \in \Xi$ *such that* $\Xi' = \{\xi \in \Xi \colon \xi \leq \zeta\}$*.*

Proof. If $u' \notin S$ and Ξ is empty, or $u' \in S$ and Ξ contains only $c^{R,u'}$, the assertion is trivial. Let us assume that neither of these possibilities apply.

We first show two auxiliary lemmas about Ξ.

(a) For any $r \in R$ and any non-minimal $s \in S$, there is at most one $\xi \in \Xi$ such that $\xi(r) = s$.

Indeed, let $\xi, \upsilon \in \Xi$ be such that $\xi(r) = \upsilon(r) = s$. For any $r' < r$, there is an $f \in F$ such that $\varphi_f(r) = r'$, because φ is a standard composition tomonoid. Thus $\xi(r') = \xi(\varphi_f(r)) = \psi_f(\xi(r)) = \psi_f(\upsilon(r)) = \upsilon(\varphi_f(r)) = \upsilon(r')$. For any $r' > r$, there an $f \in F$ such that $\varphi_f(r') = r$, thus $\psi_f(\xi(r')) = \xi(\varphi_f(r')) = \xi(r) = \upsilon(r) = \upsilon(\varphi_f(r')) = \psi_f(\upsilon(r'))$, and since Ψ is a standard composition tomonoid and s is non-minimal, it follows $\xi(r') = \upsilon(r')$ again. We conclude $\xi = \upsilon$ and (a) is proved.

(b) Let $\xi, \upsilon \in \Xi$ have a non-empty support. Then there is an $f \in F$ such that either $\xi = \upsilon \circ \varphi_f$ or $\upsilon = \xi \circ \varphi_f$. In particular, ξ and υ are comparable.

Indeed, let $r \in R$ be in the support of both ξ and υ. Assume $\xi(r) \leq \upsilon(r)$, and let $f \in F$ be such that $(\upsilon \circ \varphi_f)(r) = \psi_f(\upsilon(r)) = \xi(r)$. Note that $\upsilon \circ \varphi_f \in \Xi$. Then it follows by (a) that $\xi = \upsilon \circ \varphi_f$. Similarly, $\upsilon(r) \leq \xi(r)$ implies that there is an $f \in F$ such that $\upsilon = \xi \circ \varphi_f$. The proof of (b) is complete.

Let Ξ' be a set of functions from R to S such that properties (1)–(3) hold. By (1), $\Xi' \subseteq \Xi$. Assume that $\xi \in \Xi'$, $\upsilon \in \Xi$, and $\upsilon \leq \xi$; we claim that then $\upsilon \in \Xi'$. Indeed, either S has the smallest element u' and $\upsilon = c^{R,u'}$; thus $\upsilon \in \Xi'$ by (3). Or υ has a non-empty support; by (b), then $\upsilon = \xi \circ \varphi_f$ for some $f \in F$; thus $\upsilon \in \Xi'$ by (1).

Assume now that Ξ' is a proper subset of Ξ. Because Ξ is totally ordered, any element of $\Xi \backslash \Xi'$ is an upper bound of Ξ'; hence the pointwise supremum ζ of Ξ' exists and is, by (2), in Ξ'. Hence also $\zeta \in \Xi$, and we conclude $\Xi' = \{\xi \in \Xi \colon \xi \leq \zeta\}$. □

Again, Proposition 4.5.8 describes the sets $\Lambda^{R,S}$ separately and it remains to determine which mapping in $\Lambda^{R,S}$ belongs to which translation. Similarly as in case of Proposition 4.5.7, the mapping $\tau \colon T \to \Lambda^{R,S},\ \ t \mapsto \lambda_t^{R,S}$ is uniquely determined by a single assignment.

PROPOSITION 4.5.9. *Let* R*,* $\mathbf{\Phi}$*,* S*,* $\mathbf{\Psi}$*,* $\boldsymbol{F}$*, as well as the mappings* $f \mapsto \varphi_f$ *and* $f \mapsto \psi_f$ *be as in Proposition 4.5.8, and let* Ξ *be defined by* (16)*. Let* $\boldsymbol{X}$ *be a further standard composition tomonoid on the chain* T*, and let* $F \to X,\ \ f \mapsto \chi_f$ *be a surjective sup-preserving homomorphism. Let* $\tilde{t} \in T$ *be non-minimal, and let* $\tilde{\xi} \in \Xi$ *have a non-empty support. Then there is at most one mapping* $\tau \colon T \to \Xi$ *such that* $\tau(\chi_f(t)) = \psi_f \circ \tau(t)$ *for any* $t \in T$ *and* $\tau(\tilde{t}) = \tilde{\xi}$*.*

Proof. Assume that the mappings $\tau_1, \tau_2 \colon T \to \Xi$ are as indicated.

Let $t > \tilde{t}$ and put $\xi_1 = \tau_1(t),\ \ \xi_2 = \tau_2(t)$. As $\boldsymbol{X}$ is a standard composition tomonoid, there is an $f \in F$ such that $\chi_f(t) = \tilde{t}$. We have $\psi_f \circ \xi_1 = \psi_f \circ \tau_1(t) = \tau_1(\chi_f(t)) = \tau_1(\tilde{t}) = \tilde{\xi}$ and similarly $\psi_f \circ \xi_2 = \tilde{\xi}$. Let r be in the support of $\tilde{\xi}$; then $\psi_f(\xi_1(r)) = \psi_f(\xi_2(r))$ is non-minimal, and we conclude $\xi_1(r) = \xi_2(r)$. We proceed as in the proof of Proposition 4.5.8 to conclude that $\xi_1 = \xi_2$, that is, $\tau_1(t) = \tau_2(t)$.

Let now $t < \tilde{t}$. Then there is an $f \in F$ such that $\chi_f(\tilde{t}) = t$ and we have $\tau_1(t) = \tau_1(\chi_f(\tilde{t})) = \psi_f \circ \tau_1(\tilde{t}) = \psi_f \circ \tau_2(\tilde{t}) = \tau_2(\chi_f(\tilde{t})) = \tau_2(t)$ also in this case.

Hence $\tau_1 = \tau_2$ and the assertion follows. □

This concludes our specification of real Archimedean coextensions. We now demonstrate on the basis of some examples how Proposition 4.5.3, Theorem 4.5.6 and Propositions 4.5.7, 4.5.8, 4.5.9 can be used to determine the real Archimedean coextensions of a given tomonoid.

EXAMPLE 4.5.10. *As a first example of how our theory works, let us review Example 4.4.8. Again, let $\boldsymbol{L}_5$ be the five-element Łukasiewicz chain. We are going to determine the real Archimedean coextensions of $\boldsymbol{L}_5$ such that the bottom element is left unaltered and the remaining four elements are expanded to left-open right-closed real intervals.*

By Proposition 4.5.3, Λ^F, where $\boldsymbol{F}$ is the extending tomonoid, is the product or Łukasiewicz composition tomonoid. As F does not possess a smallest element, the former possibility applies.

Furthermore, by Theorem 4.5.6, $\Lambda^{(0,\frac{1}{4}]}$, $\Lambda^{(\frac{1}{4},\frac{1}{2}]}$, $\Lambda^{(\frac{1}{2},\frac{3}{4}]}$ are c-isomorphic to the product composition tomonoid as well.

To determine the translations λ_t, $\frac{3}{4} < t < 1$, it is by Proposition 4.5.7 sufficient to specify one of them. To this end, we choose one element distinct from the identity from each composition tomonoid $\Lambda^{(0,\frac{1}{4}]}$, $\Lambda^{(\frac{1}{4},\frac{1}{2}]}$, $\Lambda^{(\frac{1}{2},\frac{3}{4}]}$, and $\Lambda^{(\frac{3}{4},1]}$, and we require that these mappings arise from the same translation.

It remains to determine the sets $\Lambda^{R,S}$, where R and S are among $\{0\}$, $(0,\frac{1}{4}]$, $(\frac{1}{4},\frac{1}{2}]$, $(\frac{1}{2},\frac{3}{4}]$, $(\frac{3}{4},1]$. The case that the singleton $\{0\}$ is involved is trivial and covered by Lemma 4.4.7(ii). Let both R and S be distinct from $\{0\}$; then Proposition 4.5.8 applies. It is straightforward to calculate Ξ according to (16) *from Λ^R and Λ^S, which are both product composition tomonoids. The actual set $\Lambda^{R,S}$ results from Ξ by determining a largest element ζ. Note that only in this respect, our construction allows an essential choice.*

Still given R and S, it remains to determine the mapping $T \to \Lambda^{R,S},\ \ t \mapsto \lambda_t^{R,S}$, where $T = R \to S$. By Proposition 4.5.9 it is sufficient to make a single assignment. But one assignment is already clear, namely, $\lambda_t^{R,S} = \zeta$, where t is the maximal element of T.

The t-norm $\star_H$ is a possible result of this construction; cf. Figure 3 and (14).

EXAMPLE 4.5.11. *Next, we construct the real Archimedean coextensions of the four-element drastic tomonoid, that is, the tomonoid specified in Figure 7 (left). We assign to the four elements the real intervals $[0,\frac{1}{3}]$, $(\frac{1}{3},\frac{2}{3})$, $\{\frac{2}{3}\}$, and $(\frac{2}{3},1]$, respectively.*

The universe of the extending tomonoid is the left-open interval $(\frac{2}{3},1]$ and $\Lambda^{(\frac{2}{3},1]}$ is consequently again the product composition tomonoid.

Moreover, the composition tomonoids $\Lambda^{[0,\frac{1}{3}]}$, and $\Lambda^{(\frac{1}{3},\frac{2}{3})}$ are, according to Theorem 4.5.6, c-isomorphic to the Łukasiewicz and the power composition tomonoid, respectively. By Lemma 4.4.7(ii), $\Lambda^{\{\frac{2}{3}\}}$ consists of the mapping assigning $\frac{2}{3}$ to itself.

We next define an arbitrary translation λ_t, where $\frac{2}{3} < t < 1$, by selecting from each of the three non-trivial composition tomonoids one mapping different from the identity. Then the translations λ_t are uniquely determined also for all remaining $t \in (\frac{2}{3},1]$.

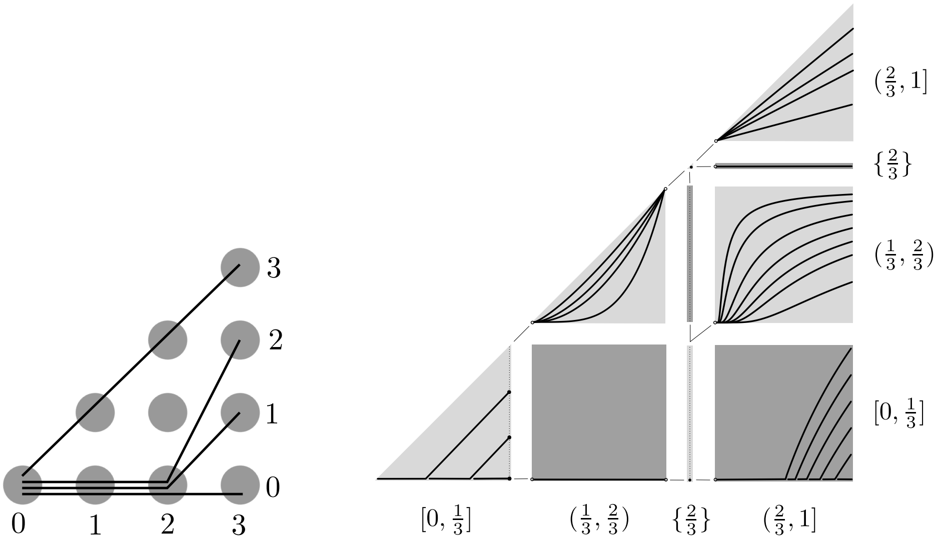


Figure 7. A coextension of a four-element tomonoid.

We proceed by constructing the set $\Lambda^{(\frac{2}{3},1],(\frac{1}{3},\frac{2}{3})}$ *on the basis of* (16)*. The whole set* (16) *is needed in this case because, by condition* (C5)*, for each* $t \in (\frac{1}{3}, \frac{2}{3})$ *there must be a translation mapping* 1 *to* t*. The situation is similar in the case of* $\Lambda^{(\frac{2}{3},1],[0,\frac{1}{3}]}$.

Finally, Lemma 4.4.7(i)(e) implies that $\Lambda^{(\frac{1}{3},\frac{2}{3}),[0,\frac{1}{3}]}$ *contains the constant* 0 *mapping only. Again by Lemma 4.4.7(i),* $\Lambda^{\{\frac{2}{3}\},[0,\frac{1}{3})}$ *consists of the single mapping assigning* $\frac{2}{3}$ *to* 0*. Also* $\Lambda^{(\frac{2}{3},1],\{\frac{2}{3}\}}$ *is trivial, consisting of the constant* $\frac{2}{3}$ *mapping.*

The Cayley tomonoid is thus completely determined. The result is a left-continuous t-norm, for instance the following one:

$$a \star b \;=\; \begin{cases} 3ab - 2a - 2b + 2 & \text{if } a, b > \frac{2}{3}, \\ \frac{1}{3}((3a-1)^{\frac{1}{3b-2}} + 1) & \text{if } \frac{1}{3} < a \leq \frac{2}{3} \text{ and } b > \frac{2}{3}, \\ (a + \frac{1}{3}\log_2(3b-2)) \vee 0 & \text{if } a \leq \frac{1}{3} \text{ and } b > \frac{2}{3}, \\ 0 & \text{if } a, b \leq \frac{2}{3}. \end{cases}$$

5 Historical remarks and further reading

5.1 Partial algebras

The idea of using partial algebras for the representation of residuated lattices originates from a research field that is led by quite different concerns from those of many-valued logics. Quantum structures were originally meant to be partially ordered algebras occurring in the context of quantum mechanics; an example is the orthomodular lattice of closed subspaces of a complex Hilbert space [31]. Orthomodular lattices have later turned out to be an interesting research object in their own right, and the same applies for their generalisations [9]. Viewing closed subspaces as a model of "sharp" measurements—like the position of some object within a certain interval—, effects, which are the positive operators below the identity, are considered as a model of "unsharp" measurements—like the position of some object within a fuzzy set over the reals [6]. The set of effects possesses the internal structure of an effect algebra and the latter structure has turned out to be a rewarding research object [17].

In particular, K. Ravindran studied the relationship between effect algebras and partially ordered groups. He has shown that an effect algebra fulfilling a certain Riesz decomposition property is representable as an interval of a partially ordered group [32]. We note that the underlying technique was the first time employed by R. Baer in a more general context [2]. Among the effect algebras to which the technique is applicable, we do not find the standard effect algebra known from physics. But we do find here the so-called MV-effect algebras, a class of partial algebras that are in a one-to-one correspondence with MV-algebras [16].

We have explored this connection between algebras originating from fuzzy logic and techniques developed for quantum structures in a series of papers [10–13, 35, 36, 38, 39]. The structures under consideration were chosen more and more general. For instance, commutativity and boundedness turned out to be dispensable conditions.

The present exposition treats the most general case with which we have coped so far. However, we consider here the case of a total order only; this restriction makes a remarkable optimisation possible. The reader interested in a more general framework is referred to [39].

To summarise, the literature on effect algebras and their various generalisations is rich. Residuated structures viewed as partial algebras, however, have most likely been mainly examined in the afore-mentioned papers. The interested reader can find a more detailed and a more general account than the one given here in particular in [39]. We note that the latter paper is the only one in which the notation of residuated structures is adapted to what is common in logics; a partial multiplication instead of a partial addition is used.

5.2 Coextensions of tomonoids

The second part of this chapter addresses various topics and we are not able to provide a comprehensive overview over the related activities. We restrict to several short remarks.

Monoids endowed with a compatible partial order or, more generally, semigroups with a compatible preorder, have been considered in a number of different contexts. As an example, we may mention the paper [3] on compatible preorders and associated decompositions of semigroups. Several particular topics have furthermore been studied; see, e.g., [20], [25], or [26].

The more special totally ordered semigroups appear in the literature less frequently. An early survey is [19]; a comprehensive paper from more recent times is [14]. A property of tomonoids that has recently drawn attention is formal integrality; see, e.g., [23].

A central part of our discussion was devoted to representations of tomonoids by monoids of mappings under composition. We have mostly considered the simplest such representation, the regular representation of tomonoids. Such representations of partially ordered semigroups have been considered in a general context in a series of papers; they are known under the name S-posets. The initial paper was [15]; among the newer contributions, we may mention, e.g., [4]. Another viewpoint on the same topic can be found, e.g., in [33].

Quotients of partially ordered monoids in general have apparently not really been considered as a rewarding topic; they are indeed not easy to characterise. In contrast, congruences of residuated tomonoids that preserve the residuals as well are well understood, being identifiable with normal convex subalgebras. See [24] for the general case and [28] also for more special cases like MTL-algebras.

The last topic to be mentioned is a field to which we have not only given a good amount of space in this chapter but which is also very present in the literature: triangular norms. Many results on t-norms are compiled in [27], which is endowed also with a comprehensive bibliography. As a general way of classifying t-norms does not exist, several methods of their construction have been compiled over the years, not all of which are in line with the algebraic structure as exposed in this chapter. For a review of construction methods from a more algebraic perspective, see, e.g., [29]. For an account of MTL-algebras, we refer to [28].

Finally, how tomonoids can be analysed by means of their quotients has been described in our papers [37] and [40]; the former adopts a more geometrical, the latter a more algebraic point of view. In these papers a considerable amount of information can be found in addition to what we have presented here.

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