Towards a proof theory for Basic Logic

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Abstract. Proof systems based on rules with the property that all formulas contained in the assumptions are contained as subformulas in the conclusion as well, are particularly suitable for automated proof search. Systems of this kind were found for several well-known fuzzy logics. However, for BL (Basic Logic), the logic of continuous t-norms and their residua, the situation is less satisfactory. We consider two proof systems for BL which fulfill the desired property in quite contrasting ways.

1 Introduction

Fuzzy logics are usually presented in the Hilbertian style; propositions are proved from a rather comprehensive set of axioms by means of a usually small number of rules, among which there is modus ponens and often not more. The logics which we have in mind are those whose propositions are interpreted by values from the real unit interval and whose language contains a conjunction interpreted by a left-continuous t-norm, an implication interpreted by the corresponding residuum, and a falsity constant interpreted by the real value 0. As a rich source of information about these logic, we recommend P. Hájek's monograph [8].

For automated proof search, Hilbert-style systems are clearly inappropriate. The problem is that modus ponens does not have the subformula property; the formula which disappears when using this rule cannot be reconstructed from the conclusion. In recent years, proof systems for various fuzzy logics were presented consisting exclusively of rules which do have the subformula property [2–4, 6, 10, 11]. They are called analytic, as they offer the possibility to decompose a proposition whose provability is to be checked step by step into its atomic constituents. In these systems, the question of provability of a proposition is reducible to the much easier tractable question about the validity of statements which do not involve any logical connective.

Analytic proof systems were, for instance, found for Łukasiewicz logic [10], product logic [11], Gödel logic [4], and the logic MTL [2]. The tool on which these calculi are based are generalizations of Gentzen's sequents. In particular, hypersequents are used, which are multisets of sequents. Hypersequents were further generalized to r-hypersequents in [6], and proof systems based on r-hypersequents were found for all the three standard extensions of BL, in a way that the logical rules coincide [6].

Each of the mentioned calculi features a set logical rules of the minimal possible size and moreover an easily apprehensible set of structural rules. It is an open question if a similarly elegant proof system exists also for Basic Logic, the logic of continuous t-norms and their respective residua. In this paper, we discuss recent work on this problem.

There are two proof systems for BL which, at least formally, come relatively close to the desired type. The first one is the calculus RHBL from [5], which we will shortly review (Section 3). The second one is our calculus rHMŁ [15]. We will give an introduction to rHMŁ and compile its basic properties (Section 4) and in particular explain its proof search capabilities (Section 5).

2 Basic Logic - the usual Hilbert-style formulation

Basic (Fuzzy) Logic, or BL for short, was introduced by P. Hájek [8]. We summarize the basic facts.

The propositional version of BL uses the language $\odot, \rightarrow, 0$. An evaluation of BL is a structure-preserving map from the algebra of propositions to an algebra $([0,1]; \odot, \rightarrow, 0)$, where $[0,1]$ is the real unit interval, $\odot$ is a continuous t-norm, and $\odot, \rightarrow$ form an adjoint pair. The valid propositions are those being assigned 1 by all evaluations.

The notion of a proof in BL is as follows. Using the axiom schemes

$$
\begin{aligned}
&\text{(A1)}\ \ [(\alpha \rightarrow \beta) \odot (\beta \rightarrow \gamma)] \rightarrow (\alpha \rightarrow \gamma),\\
&\text{(A2)}\ \ \alpha \odot \beta \rightarrow \alpha,\\
&\text{(A3)}\ \ \alpha \odot \beta \rightarrow \beta \odot \alpha,\\
&\text{(A4)}\ \ [\alpha \odot (\alpha \rightarrow \beta)] \rightarrow [\beta \odot (\beta \rightarrow \alpha)],\\
&\text{(A5a)}\ \ [(\alpha \odot \beta) \rightarrow \gamma] \rightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)],\\
&\text{(A5b)}\ \ [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \odot \beta) \rightarrow \gamma],\\
&\text{(A6)}\ \ 0 \rightarrow \alpha,\\
&\text{(A7)}\ \ [((\alpha \rightarrow \beta) \rightarrow \gamma) \odot ((\beta \rightarrow \alpha) \rightarrow \gamma)] \rightarrow \gamma,
\end{aligned}
$$

we prove propositions by means of the modus ponens, i.e. the rule

$$\frac{\alpha \quad \alpha \rightarrow \beta}{\beta}.$$

We have (weak) standard completeness: A proposition is provable in BL exactly if it is valid in BL.

Our intention is to replace this Hilbert-style proof system by an alternative one. As a preparatory step, we will in the remainder of this section comment on the semantics on which BL is based, and subsequently we will recall known approaches for proof systems of fuzzy logics.

Validity in BL refers canonically to the set of all continuous t-norms. As exhibited e.g. in [13], it is possible to restrict this set without enlarging the set of valid propositions. Namely, the variety of BL-algebra, which is generated by all continuous t-norm algebras, is already generated by all those t-norm algebras which are based on finite ordinal sums of Łukasiewicz t-norms. Indeed, for any non-valid proposition φ, there is an evaluation v into a t-norm algebra such that $v(\varphi) < 1$; obviously, the range of v may be assumed the ordinal sum of finitely many Łukasiewicz and product algebras; moreover, by cutting off unused parts of the negative cones constituting the product algebras, we may replace the image of v by a t-norm algebra composed from Łukasiewicz algebras only.

So let for $k \geq 1$

$$\begin{aligned} S_k \; &= \{(n,r) \in \mathbb{Z} \times \mathbb{R} \colon \; n = 0, \;\; -1 \leq r \leq 0 \\ &\quad \text{or } \; -(k-1) \leq n \leq -1, \;\; -1 \leq r < 0\}; \end{aligned}$$

endow S_k with the lexicographical order; define

$$(m,r) \odot (n,s) \;=\; \begin{cases} (m, (r+s) \vee -1) & \text{if } m = n, \\ (m,r) \wedge (n,s) & \text{else,} \end{cases} \tag{1}$$

$$(m,r) \to (n,s) \;=\; \begin{cases} (n,s) & \text{if } m > n, \\ (m, s-r) & \text{if } m = n \text{ and } r > s, \\ (0,0) & \text{if } (m,r) \leq (n,s) \end{cases} \tag{2}$$

for any $(m,r), (n,s) \in S_k$; and let $e = (0,0)$ and $z_k = (-(k-1), -1)$. (Here, $r \wedge s$ and $r \vee s$ denote the smaller and the larger of two reals r and s, respectively.) Then, in accordance with our previous remarks, a proposition is valid in BL if and only if it is assigned e by all evaluations into the algebra $(S_k; \leq, \odot, \to, z_k, e)$.

Rather than basing the validity of BL on the countably many algebras $S_1, S_2, \ldots$, we may also use a single one. A possibility is to take

$$S_\infty \;= \{(n,r) \in \mathbb{N} \times \mathbb{R} \colon \; -1 \leq r < 0\} \cup \{\infty\}, \tag{3}$$

where ∞ is a new element added on top. The operations $\odot$ and $\to$ may be defined similarly to (1) and (2), respectively, but some special care is needed for the element ∞. The zero element is $(-1, 0)$, the one element is ∞.

Note that the necessity in this case to add a single isolated element is caused by the presence of the constant 0. For Hoop Logic, the logic based on basic hoops [7], the situation would be different; we could simply take the union of the S_k, $k < \omega$, and use the unmodified definitions (1) and (2).

Next, let us address the topic of alternative proof systems for fuzzy logics. We do not refer especially to BL here. The concept which we are going to explain is an advancement of Gentzen's sequent calculus and has been elaborated in numerous papers in recent years. The so-called hypersequents are due to Avron [1] and Pottinger [14]; in [3], sequents-of-relations were introduced; a common generalization both these concepts are the r-hypersequents, introduced in [6]. It is the latter notion with which we will deal here.

In fuzzy logics, an r-sequent $\Gamma \lesseqqgtr \Delta$ consists of two finite multisets of propositions $\Gamma = \gamma_1, ..., \gamma_m$ and $\Delta = \delta_1, ..., \delta_n$ and moreover a relation symbol $\lesseqqgtr$, which is either $\leq$ or $<$. The validity of r-sequents is defined individually for each logic; in the general case, we may say the following. For a given evaluation v of the logic's propositions into a set of truth values S, a sequent is called valid under v if $\bigotimes_{i=1}^{m} v(\gamma_i) \lesseqqgtr \bigotimes_{i=1}^{n} v(\delta_i)$, where $\otimes$ is a binary isotone associative function operating on some upper bounded totally ordered set $T \supseteq S$, and $\lesseqqgtr$ refers to the order or strict order of T, respectively. For instance, in case $S = S_1 = [-1, 0]$, $\otimes$ can be the addition of reals and $T = \mathbb{R}^-$; in general, however, $\otimes$ and T can be determined rather arbitrarily.

Furthermore, an r-hypersequent $\Gamma_1 \lesseqqgtr \Delta_1 | \ldots | \Gamma_n \lesseqqgtr \Delta_n$ is a multiset of r-sequents. A hypersequent $\mathcal{H}$ is called valid under some evaluation v if one of the sequents contained in $\mathcal{H}$ is valid under v. $\mathcal{H}$ is called valid if $\mathcal{H}$ is valid under all evaluations.

Now, a rule is a pair of a finite set of r-hypersequents – the assumptions –, and a single r-hypersequent – the conclusion. A calculus is a collection of rules. Proofs are finite trees of instances of rules, such that every assumption of a rule is the conclusion of an immediately preceding rule. A proposition φ is defined to be provable if the r-hypersequent $\emptyset \leq \varphi$ is the conclusion at root of some proof.

3 The Bova-Montagna calculus RHBL

It seems that for the logic BL, the concept of an r-hypersequent-based proof system as explained in the previous section and applied successfully to a large number of fuzzy logics, must be modified in some way if analyticity is the property we aim at. One possible way to generalize the involved notions was recently proposed by S. Bova and F. Montagna [5].

Let us roughly summarize this approach. First of all, validity in BL is considered with respect to the single algebra whose base set is S_∞ given by (3). The main idea in [5] is to generalize the concept of an r-hypersequent. Namely, in view of the special structure of S_∞, further binary relations apart from $\leq$ and $<$ can easily be defined. The calculus RHBL is based on r-hypersequents, where the relation symbol may be one of $\ll$ or $\preccurlyeq$ or $\preccurlyeq_z$, where $z \in \mathbb{Z}$. For instance, $\ll$ applies to pairs of multisets of length ≤ 1 and refers to strict inequality of the first components of elements from S_∞.

The calculus RHBL consists of eight logical rules. The rules fulfill a property even more desirable than the subformula property: they are invertible. There is a rule for each connective and each side, separately for $\ll$-hypersequents and for the general case. The idea when introducing e.g. the formula $\alpha \odot \beta$, is that there are different proof branches according to the different possibilities for the mutual relationship of α and β. For instance, in the premises of the rule introducing $\alpha \odot \beta$, five different possibilities are distinguished how α is related to β.

By means of the calculus RHBL, it is possible to decompose a proposition step by step into atomic r-hypersequents. Proving the latter's' validity is not possible within RHBL; valid r-hypersequents are axioms. However, a method is described in [5] how the validity of an atomic r-hypersequent is checked effectively by means of a linear program.

4 The calculus rHMŁ for a conservative extension of BL

A proof system for BL based on a different idea was proposed in [15]. This system actually does not refer to BL directly, but to MŁ, the so-called Logic of Multiples of the Łukasiewicz t-Norm.

The language of MŁ contains like BL the connectives $\odot, \to$ and the constant 0; in addition, however, there is a unary connective $\triangledown$. Propositions of MŁ are interpreted in t-norm algebras which are ordinal sums of Łukasiewicz algebras; and $\triangledown$ is interpreted by the function mapping a truth value t to the greatest idempotent below t. So in particular, a proposition of MŁ not containing the new connective $\triangledown$ is valid if and only if it is valid in BL; that is, MŁ is a conservative extension of BL. So a calculus suitable for MŁ may be used to check provability in BL equally well as provability in MŁ.

An extension of BL similar to our MŁ was proposed earlier by Hájek [9] and others. However, in contrast e.g. to the logic defined in [9], MŁ is not based on all continuous t-norm algebras, but only on ordinal sums of Łukasiewicz algebras; we use $\triangledown$ to rule out those ordinal sums in which the product algebra appears.

Hájek's ideas were further elaborated by Montagna in [12], where a so-called storage operator was introduced for any appropriate class of MTL-algebras. In fact, our connective $\triangledown$ is Montagna's storage operator applied to ordinal multiples of Łukasiewicz algebras.

Although we think that MŁ is interesting in itself, we admit that the primary reason to consider this logic is that we may define an analytic proof system for MŁ apparently much easier than for BL. However, concerning the question if it makes sense to consider a logic like MŁ, we should mention Montagna's paper a second time; the introduction of [12] contains interesting hints concerning the interpretation of a storage operator, so in particular of our $\triangledown$.

The calculus for MŁ which we are going to discuss here, is called rHMŁ. We list its key features for easy reference:

(i) rHMŁ is based on r-hypersequents as introduced in [6], that is, with relation symbols $\leq$ and $<$ only.

(ii) The interpretation of r-hypersequents generalizes the one of rHŁ in [6].

(iii) There is a rule for each binary connective and each side, separately for the case that $\triangledown$ is the outermost connective of the introduced formula or not. For instance, a proposition $\alpha \odot \beta$ is composed from α and β, whereas $\triangledown(\alpha \odot \beta)$ is composed from $\triangledown\alpha$ and $\triangledown\beta$.

(iv) The rules introducing the binary logical connectives are invertible.

(v) The structural rules treat r-hypersequents which contain literals only, where a literal is an atom or an atom to which the connective $\triangledown$ is applied.

(vi) rHMŁ has elementary axioms, that is, the axioms are r-sequents with at most one literal in the antecedent and succedent.

(vii) Among the structural rules, there is a rule distinguishing the cases $\alpha \leq \beta$ and $\beta < \alpha$, where α, β are literals. We are referring to the rule (Cut≤/>) below, which may be called an analytic cut.

We will now explain the logic MŁ and its calculus in detail.

As in the case of BL, it is more convenient to evaluate propositions of MŁ in the algebras S_k rather than in t-norm algebras. So a proposition φ of MŁ is valid if for every evaluation v into $(S_k; \odot, \rightarrow, \triangledown, z_k, e)$, we have $v(\varphi) = e$, where $\triangledown\colon S_k \rightarrow S_k$ is defined by

$$\triangledown(n, r) \;=\; \begin{cases} (n, -1) & \text{if } r < 0; \\ (0, 0) & \text{if } (n, r) = (0, 0) \end{cases}$$

for $(n, r) \in S_k$.

The validity of r-hypersequents will be based on algebras different from the S_k; we use the same "trick" as in the case of the calculus rHŁ for Łukasiewicz logic defined in [6]. Let for every $k \geq 1$

$$\begin{aligned} T_k \;=\; \{(n, r) \in \mathbb{Z} \times \mathbb{R}\colon\; & n = 0,\;\; r \leq 0 \\ \text{or } & -(k-1) \leq n \leq -1,\;\; r < 0\}; \end{aligned}$$

endow T_k with the lexicographical order; for $(m, r), (n, s) \in T_k$, define

$$(m, r) \cdot (n, s) \;=\; \begin{cases} (m, r + s) & \text{if } m = n, \\ (m, r) \wedge (n, s) & \text{else,} \end{cases}$$

$$(m, r) \rightarrow (n, s) \;=\; \begin{cases} (n, s) & \text{if } m > n; \\ (m, s - r) & \text{if } m = n \text{ and } r > s, \\ (0, 0) & \text{if } (m, r) \leq (n, s); \end{cases}$$

and let $e = (0, 0)$. Then an r-sequent $\alpha_1, \ldots, \alpha_m \lesseqqgtr \beta_1, \ldots, \beta_n$, where $m, n \geq 0$, is valid under the evaluation v with range S_k if

$$v(\alpha_1) \cdot \ldots \cdot v(\alpha_m) \;\lesseqqgtr\; v(\beta_1) \cdot \ldots \cdot v(\beta_n);$$

here, S_k is considered a subset of T_k and the product $\cdot$ refers to T_k; moreover, the product of the empty set is understood to be e. – The validity of r-hypersequents is defined accordingly.

We next define the calculus itself. In each rule, the three dots at the beginning of each r-hypersequent replace an arbitrary finite multiset of r-sequents, for each rule uniformly. Furthermore, the symbol $\lesseqqgtr$ is to be replaced by $\leq$ or $<$, for each rule uniformly.

In addition, an r-hypersequent will be called quasiatomic if every proposition contained in it is of the form α or $\triangledown\alpha$ for an atom α. Finally, for a multiset Γ and an atom α, $\Gamma \setminus \alpha$ denotes the multiset Γ with all occurrences of α and $\triangledown\alpha$ removed.

Definition 1. *The logical rules of the calculus rHMŁ are the following:*

$$(\odot\mathrm{l})\ \frac{\ldots \mid \Gamma,\alpha,\beta \lesseqgtr \Delta \qquad \ldots \mid \Gamma,\nabla\alpha \lesseqgtr \Delta \mid \Gamma,\nabla\beta \lesseqgtr \Delta}{\ldots \mid \Gamma,\alpha\odot\beta \lesseqgtr \Delta}$$

$$(\nabla\odot\mathrm{l})\ \frac{\ldots \mid \Gamma,\nabla\alpha \lesseqgtr \Delta \mid \Gamma,\nabla\beta \lesseqgtr \Delta}{\ldots \mid \Gamma,\nabla(\alpha\odot\beta) \lesseqgtr \Delta}$$

$$(\odot\mathrm{r})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta,\alpha,\beta \mid \Gamma \lesseqgtr \Delta,\nabla\alpha \qquad \ldots \mid \Gamma \lesseqgtr \Delta,\alpha,\beta \mid \Gamma \lesseqgtr \Delta,\nabla\beta}{\ldots \mid \Gamma \lesseqgtr \Delta,\alpha\odot\beta}$$

$$(\nabla\odot\mathrm{r})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta,\nabla\alpha \qquad \ldots \mid \Gamma \lesseqgtr \Delta,\nabla\beta}{\ldots \mid \Gamma \lesseqgtr \Delta,\nabla(\alpha\odot\beta)}$$

$$(\rightarrow\mathrm{l})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta \mid \Gamma,\beta \lesseqgtr \Delta,\alpha \qquad \ldots \mid \Gamma \lesseqgtr \Delta \mid \beta < \alpha}{\ldots \mid \Gamma,\alpha\rightarrow\beta \lesseqgtr \Delta}$$

$$(\nabla\rightarrow\mathrm{l})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta \mid \beta < \alpha \qquad \ldots \mid \Gamma,\nabla\beta \lesseqgtr \Delta \mid \alpha \leq \beta}{\ldots \mid \Gamma,\nabla(\alpha\rightarrow\beta) \lesseqgtr \Delta}$$

$$(\rightarrow\mathrm{r})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta \qquad \ldots \mid \Gamma,\alpha \lesseqgtr \Delta,\beta \mid \alpha \leq \beta}{\ldots \mid \Gamma \lesseqgtr \Delta,\alpha\rightarrow\beta}$$

$$(\nabla\rightarrow\mathrm{r})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta \mid \beta < \alpha \qquad \ldots \mid \Gamma \lesseqgtr \Delta,\nabla\beta \mid \alpha \leq \beta}{\ldots \mid \Gamma \lesseqgtr \Delta,\nabla(\alpha\rightarrow\beta)}$$

$$(\nabla\nabla\mathrm{l})\ \frac{\ldots \mid \Gamma,\nabla\alpha \lesseqgtr \Delta}{\ldots \mid \Gamma,\nabla\nabla\alpha \lesseqgtr \Delta} \qquad (\nabla\nabla\mathrm{r})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta,\nabla\alpha}{\ldots \mid \Gamma \lesseqgtr \Delta,\nabla\nabla\alpha}$$

The following structural rules of rHMŁ refer to quasiatomic r-hypersequents. Any expression $\nabla\alpha$ in a rule's conclusion, where $\alpha = \nabla\beta$ for some atom β, is meant to be $\nabla\beta$.

$$(\mathrm{A1})\ \varnothing \leq \varnothing \qquad (\mathrm{A2})\ \alpha \leq \alpha \qquad (\mathrm{A3})\ 0 \leq \alpha \qquad (\mathrm{A4})\ 0 < \varnothing$$

$$(\mathrm{EW})\ \frac{\ldots}{\ldots \mid \Gamma \lesseqgtr \Delta} \qquad (\mathrm{EC})\ \frac{\ldots \mid \Gamma \lesseqgtr \Delta \mid \Gamma \lesseqgtr \Delta}{\ldots \mid \Gamma \lesseqgtr \Delta}$$

$$(\mathrm{Cut}{\leq}/{>})\ \frac{\ldots \mid \Gamma \leq \Delta \qquad \ldots \mid \Delta < \Gamma}{\ldots},$$

where Δ and Γ contain at most one literal,
and any variable in $\Delta \cup \Gamma$ appears in the side r-hypersequent

$$(\mathrm{O})\ \frac{\ldots \mid \Gamma \setminus \alpha \lesseqgtr \Delta \setminus \alpha}{\ldots \mid \Gamma \lesseqgtr \Delta \mid \nabla\alpha \leq \nabla\beta},$$

where α and β are atoms such that α or $\nabla\alpha$ is in $\Gamma \cup \Delta$, and β or $\nabla\beta$ is in $\Gamma \cup \Delta$

$$(\nabla\text{l})\ \frac{\ldots \mid \Gamma, \alpha \overset{<}{\leq} \Delta}{\ldots \mid \Gamma, \nabla\alpha \overset{<}{\leq} \Delta} \qquad (\nabla\text{lr})\ \frac{\ldots \mid \beta \overset{<}{\leq} \alpha}{\ldots \mid \nabla\beta \leq \nabla\alpha} \qquad (\nabla\text{r})\ \frac{\ldots \mid \emptyset \leq \alpha}{\ldots \mid \emptyset \leq \nabla\alpha}$$

$$(\text{wl})\ \frac{\ldots \mid \Gamma \overset{<}{\leq} \Delta}{\ldots \mid \Gamma, \alpha \overset{<}{\leq} \Delta} \qquad (\text{w0l})\ \frac{\ldots \mid \Gamma \leq \Delta}{\ldots \mid \Gamma, 0 < \Delta}$$

$$(\text{w}\nabla\text{l})\ \frac{\ldots \mid \alpha_1, \ldots, \alpha_n \leq \Delta}{\ldots \mid \alpha_1, \ldots, \alpha_n, \nabla\beta < \Delta \mid \nabla\alpha_1 < \nabla\beta \mid \ldots \mid \nabla\alpha_n < \nabla\beta \mid \emptyset \leq \nabla\beta},$$

where $n \geq 0$; in case $n = 0$ the r-sequents "$\ldots < \nabla\beta$" are omitted

$$(\text{M})\ \frac{\ldots \mid \Gamma_1 \leq \Delta_1 \qquad \ldots \mid \Gamma_2 \leq \Delta_2}{\ldots \mid \Gamma_1, \Gamma_2 \leq \Delta_1, \Delta_2} \qquad (\text{S})\ \frac{\ldots \mid \Gamma_1, \Gamma_2 \overset{<}{\leq} \Delta_1, \Delta_2}{\ldots \mid \Gamma_1 \overset{<}{\leq} \Delta_1 \mid \Gamma_2 \overset{<}{\leq} \Delta_2}$$

$$(\text{S}<)\ \frac{\ldots \mid \Gamma, \alpha_1, \ldots, \alpha_n \leq \Delta_1, \Delta_2}{\ldots \mid \Gamma \leq \Delta_1 \mid \alpha_1, \ldots, \alpha_n < \Delta_2 \mid \nabla\alpha_1 < \nabla\beta \mid \ldots \mid \nabla\alpha_n < \nabla\beta},$$

where (i) $n \geq 1$, and (ii) $\beta \in \Gamma \cup \Delta_1$

It is tedious, but not really difficult to check that rHMŁ is sound and that the logical rules are all invertible, w.r.t. the above defined validity. In particular, all propositions provable in rHMŁ are valid in MŁ.

To see that rHMŁ is actually complete, we rely on the fact that, by backwards application of the logical rules, we may decompose a proposition step by step until we arrive at a (possibly quite large) number of quasiatomic r-hypersequents. It is, furthermore, not so trivial to see that the quasiatomic r-hypersequents are provable by means of the structural rules of rHMŁ. For this proof, we refer to [15]. Note that among the structural rules – in contrast to RHBL not among the logical rules –, there is the rule (Cut≤/>) which distinguishes the relative order of two atoms. Taken all mentioned facts together, we get weak standard completeness for rHMŁ:

Theorem 1. *The calculus rHMŁ is sound and complete for MŁ: A proposition* α *is valid in MŁ if and only if* α *is provable in rHMŁ.*

5 Proof search with the calculus rHMŁ

By backwards application of the proof rules, we may use the calculus rHMŁ to check if a proposition of MŁ is valid or not; in particular, we may check the validity of a proposition of BL. We will outline the method.

First Fact. (i) All logical rules of rHMŁ are invertible. (ii) Applying the logical rules successively upwards terminates after finitely many steps with quasiatomic r-hypersequents.

This procedure is in particular applicable to the r-hypersequent $\emptyset \leq \alpha$ for any given proposition α. It follows that the question if α is valid in MŁ is reducible to the question if certain quasiatomic r-hypersequents are valid.

In what follows, we call an r-sequent $\Gamma \lesseqgtr \Delta$ basic if Γ is either empty or contains one proposition of the form $\nabla\alpha$ for an atom α, and similarly for Δ.

Second Fact. The following rules are admissible in rHMŁ and furthermore invertible:

(ExtCut≤/>)

$$\frac{\ldots \mid \Gamma \setminus \alpha \lesseqgtr \Delta \setminus \alpha \mid \nabla\alpha \leq \nabla\beta \qquad \ldots \mid \Gamma \setminus \beta \lesseqgtr \Delta \setminus \beta \mid \nabla\beta \leq \nabla\alpha \qquad \ldots \mid \Gamma \lesseqgtr \Delta \mid \nabla\alpha < \nabla\beta \mid \nabla\beta < \nabla\alpha}{\ldots \mid \Gamma \lesseqgtr \Delta},$$

where Γ and Δ are quasiatomic, and
both atoms α and β are subformulas in $\Gamma \cup \Delta$;

(ExtCut≤/>0) $$\frac{\ldots \mid \Gamma \setminus \alpha \lesseqgtr \Delta \setminus \alpha \mid \nabla\alpha < \emptyset \qquad \ldots \mid \Gamma \lesseqgtr \Delta \mid \emptyset \leq \nabla\alpha}{\ldots \mid \Gamma \lesseqgtr \Delta},$$

where Γ and Δ are quasiatomic
and the atom α is a subformula in $\Gamma \cup \Delta$.

This means that a proof search for a quasiatomic r-hypersequent $\mathcal{H}$ goes as follows. If $\mathcal{H}$ contains a non-basic r-sequent $\Gamma \lesseqgtr \Delta$ such that the distinct atoms α and β appear in it and such that $\mathcal{H}$ does not contain both $\nabla\alpha < \nabla\beta$ and $\nabla\beta < \nabla\alpha$, apply (ExtCut≤/>) backwards. Alternatively, if $\mathcal{H}$ contains a non-basic r-sequent $\Gamma \lesseqgtr \Delta$ such that the variable α appears in $\Gamma \cup \Delta$ and such that $\mathcal{H}$ does not contain $\emptyset \leq \nabla\alpha$, apply (ExtCut≤/>0). Proceed then in the same way as long as this is possible.

Consider next successively all the r-hypersequents at the leaves. Let $\mathcal{L}$ be one of them; and let $\mathcal{L}_b$ be the r-hypersequent arising from $\mathcal{L}$ by deleting all r-sequents which are not basic. Note that the validity of $\mathcal{L}_b$ can be translated to a statement about bounded totally ordered sets and thus be checked effectively. We proceed in dependence of the result:

(i) Let $\mathcal{L}_b$ be valid. Then we may prove $\mathcal{L}_b$ by means of the structural rules; see [15, Lemma 3.7]. Subsequently, $\mathcal{L}$ is proved from $\mathcal{L}_b$ by external weakening, (EW).

(ii) Let $\mathcal{L}_b$ be not valid. Then we may write $\mathcal{L} = \mathcal{L}_1 | \ldots | \mathcal{L}_k | \mathcal{R}$ such that (i) $\mathcal{L}_i$ and $\mathcal{L}_j$, $i \neq j$, do not have any atom in common; (ii) for any i and any distinct atoms α and β in $\mathcal{L}_i$, we have that $\nabla\alpha < \nabla\beta$ and $\nabla\beta < \nabla\alpha$ is in $\mathcal{L}_i$; similarly, for any variable α, we have that $\emptyset \leq \nabla\alpha$ is in $\mathcal{L}_i$; (iii) we may discard $\mathcal{R}$ without affecting the validity of $\mathcal{L}$.

We conclude that $\mathcal{L}$ is valid iff at least one of the $\mathcal{L}_i$ is valid. For a given i, however, $\mathcal{L}_i$ is valid if $\mathcal{L}'_i$ is valid in rHŁ [6], where $\mathcal{L}'_i$ arises from $\mathcal{L}_i$ by replacing all r-sequents $\emptyset \leq \triangledown\alpha$ by $\emptyset \leq \alpha$ and then all remaining expressions $\triangledown\alpha$ by 0.

Third Fact. The validity of atomic r-hypersequents of rHŁ can be checked effectively.

Indeed, this problem is equivalent to the problem if an associated system of linear inequalities is inconsistent, and can be solved by linear programming methods. This is a result of [6].

Adding more details would clearly go beyond the scope of this paper, but the material on which the present argumentation is based, can be found in [15].

6 Conclusion

It is a peculiar fact that for one of the best known fuzzy logics, the logic BL of continuous t-norms, the conception of an analytic proof system based on r-hypersequents causes serious problems. We considered two alternatives to Hilbert-style systems available at the moment – Bova's and Montagna's system RHBL, which adapts the notion of an r-hypersequent to a specific model of the propositions of BL, and our system rHML, which is a calculus for a conservative extension of BL. In both cases, the validity of a proposition w.r.t. BL can be effectively decided, the decision procedure being exponential. Further simplifications of the methods are desirable.

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