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A way to interpret Łukasiewicz Logic and Basic Logic

Abstract. Fuzzy logics are in most cases based on an ad-hoc decision about the interpretation of the conjunction. If they are useful or not can typically be found out only by testing them with example data. Why we should use a specific fuzzy logic can in general not be made plausible. Since the difficulties arise from the use of additional, unmotivated structure with which the set of truth values is endowed, the only way to base fuzzy logics on firm ground is the development of alternative semantics to all of whose components we can associate a meaning.

In this paper, we present one possible approach to justify ex post Łukasiewicz Logic as well as Basic Logic. The notion of ambiguity is central. Our framework consists of a Boolean or a Heyting algebra, respectively, endowed with an equivalence relation expressing ambiguity. The quotient set bears naturally the structure of an MV- or a BL-algebra, respectively, and thus can be used to interpret propositions of the mentioned logics.

Keywords: Łukasiewicz Logic, Basic Logic, Boolean algebra, Heyting algebra, a-equivalence relation, MV-algebra, BL-algebra

1. Introduction

To study fuzzy logics can be motivated in at least two different ways. Both imply fundamental conceptual difficulties. In view of the origins of the field, the intention is to find out appropriate ways to reason about vague statements. The aim is to develop a framework for dealing with statements which are not necessarily fully true or fully false, but which hold in general to a certain degree. A natural approach is to modify the classical propositional logic accordingly. The simple idea is to associate with propositions generalized truth values; rather than restricting to the values 0 and 1, elements from the whole real unit interval are permitted. Whereas this initial step is natural, the need to redefine the connectives brings arbitrariness into the concept; there is in particular no criterion telling us in which way the conjunction should be interpreted for values strictly between 0 and 1. A canonical choice not being available, either an ad-hoc decision is made, or the connectives

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are required to fulfill certain properties believed to be natural and validity is defined with regard to all these allowed choices. However we decide, as long as the truth values of compound statements depend on the truth values of the constituents in some specific way, we may develop a corresponding formal calculus, to be called a fuzzy logic [COM, Haj, Got].

We may also come upon the field of fuzzy logics from the syntactical side. Namely, certain proof systems allow a very clear view on which kind of reasoning is allowed for a particular logic and which not. We have in particular those systems in mind which are based on Gentzen sequents or on one of their generalizations. The question is then natural what is still derivable when one or more rules are dropped. In particular, when discarding substructural rules from Gentzen's sequent system for classical or intuitionistic propositional logic, we arrive at logics closely related to fuzzy logics. For instance, Höhle's Monoidal Logic [Hoe], whose extension by the prelinearity axiom scheme gives the logic MTL, is obtained by dropping the contraction rule from Gentzen's calculus LJ [Tak].

Choosing whatever approach, we are faced with the problem that it is not clear what we actually reason about. This is a natural consequence for the second approach and may in that case not be seen as a problem. But the conception of the first approach clearly refers to our intuition; however, the drawn picture is not specific enough to justify a specific logic. Indeed, only in the exceptional case, it is possible to associate with one of the known fuzzy-logical calculi a clearly defined situation in which reasoning according to this logic is appropriate.

It is our aim to contribute to a solution of this dilemma by proposing for the most common fuzzy logics alternative semantics. In this paper, we concentrate on two logics. We first study a logic whose logical connectives were once chosen ad hoc: the Łukasiewicz logic [COM]. Second, we consider a logic which arose out of the undecidedness about a specific way to interpret the logical "and": the Basic Logic, the logic of continuous t-norms and their associated residua [Haj].

Several approaches to endow fuzzy logics with alternative semantics have been elaborated in the past. For a survey of ways how to interpret t-norm based fuzzy logics, see [Par]. We note, however, that we demand here not only an appealing interpretation of the conjunction; this problem seen isolated is comparably easy to solve. What we want is an interpretation of all what is needed to base the semantics of fuzzy logics on it – the implication included. Actually not really many approaches can be found in the literature fulfilling this requirement. Let us mention one particularly convincing way to interpret reasoning according to fuzzy logics: the dialog games. The game-theoretic approach was originally developed by R. Giles for Łukasiewicz logic [Gil] and was generalized for further fuzzy logics by C. Fermüller and others; see e.g. [Fer].

The concept on which this paper is based goes in a different direction. Recall that the semantics of a typical fuzzy logic may be based on any collection of algebras generating the same variety as the algebras used for the canonical semantics. We search for appropriate representations of the corresponding class of residuated lattices; in the first case, these are the MV-algebras, in the second case BL-algebras. We do not intend to specify a concrete situation; as an approach which is more specific about the referred situation and which concerns the same pair of logics as we consider here, we mention the Ulam-game based semantics [Mun, CiMu]. In the present work, we rather stay on the abstract level; what we propose is a mathematical framework which might offer a basis to develop a new way to interpret the fuzzy logics taken into account here. It is actually our hope that our framework will serve as a starting point for an interpretation which is more specific, but also more convincing than what can be associated with the standard semantics.

As opposed to many approaches to the interpretation of logics, we make the notion of ambiguity central. We will start from a Boolean algebra or, alternatively, from a Heyting algebra. Both kinds of algebras are adequate to model systems of crisp properties. We will endow these algebras with a certain kind of equivalence relation; this relation is meant to express ambiguity of properties. Propositions of fuzzy logic will be interpreted in the corresponding quotient set. The point is that, when choosing the relevant notions in the right way, the quotient set naturally bears the structure of a residuated lattice; indeed, what comes out is an MV- or BL-algebra.

The paper is organized as follows. We first review the foundational issue of fuzzy logics and shortly elaborate our opinion on this delicate subject (Section 2). The subsequent mathematical part is based on results of our paper [Vet3], adapted to the present context. We deal with Łukasiewicz and Basic Logic (Section 3). We are first concerned with Boolean algebras endowed with what we call an a-equivalence relation (Section 4). Endowing a Boolean algebra with a measure and identifying elements whose complements are of equal size, provides an example of an a-equivalence relation. Under an appropriate assumption called residuability, the quotient set is a lattice and has the structure of a residuated monoid; in fact, what we get is an MV-algebra, and every totally ordered MV-algebra can be the result of this construction. In the following part of the paper, we modify our definitions and results, aiming at constructing a larger class of residuated lattices (Section 5). Starting with Heyting algebras rather than Boolean algebras, we are led to BL-algebras. So in the first case, we provide alternative semantics for Łukasiewicz logic, in the second case for Hájek's Basic Logic (Section 6).

2. The foundational problem of fuzzy logic

The interpretation of fuzzy logics has been an intensively discussed subject. For a survey of contributions from the mathematical and applicational point of view, we may refer for instance to [Urq]. Fuzzy logic is furthermore a particularly prominent topic within the philosophy of vagueness. To get an idea about the approaches to vagueness propagated by various philosophical schools, see for instance [KeSm, Kee, Wil].

In this section, we like to summarize our point of view on the nature of vagueness and on the difficulties in modelling this phenomenon by means of formal logical calculi.

In our opinion, to approach the phenomenon of vagueness we should analyze the role of statements in formal or natural languages from an empiristic point of view. We think that the contents of statements expressing observations of what is around us, should not be thought of as referring to something existing in some absolute sense. To assume that observations themselves are something absolute might be useful; but to assume that there is something absolute behind it, is not useful.

Certainly, most of the time, we all adopt the realists' attitude, assuming that the objects in the world would exist even if we did not observe them and that physical laws are independent of our means of observation. Apparently, even most physicists think this way, although this clearly contradicts the basic principles of quantum theories and has caused an endless debate on the proper interpretation of quantum phenomena. For the question how to understand vagueness, the discussion seems to follow certain similar and similarly hopeless lines. By tendency, vagueness is searched outside of us, not as part of the very conditions under which we observe something. The dilemma then is that things modelled in the realistic way, that is, by classical physics, behave in an exact way, and it is difficult to bring this in line with vague phenomena.

We find it useful for the present context to adopt the opposite way of dealing with reality: to reduce the absolute to the sole fact that we observe something. In fact, the easiest way to deal with the absurdities of quantum physics is to understand this theory in a non-traditional way: the theory does not explain how particles move around the world, but how probabilities are associated to specific sequences of observations.

This point of view simplifies not only modelling quantum phenomena, but it helps considerably to understand mathematical modelling in general. When designing a mathematical theory, we should not assume that we will be able to describe what is going on outside of us, but just to bring structure into our observations and to process it afterwards in our mind in a well-defined manner. For example, the two most basic structures are the natural numbers and the real numbers. There is simply no need to claim that in reality we find objects properly characterized by means of these structures; it is more appropriate to say that these structures are extremely useful to describe an observed picture and to figure out the probable further development.

On a more basic level, consider the kind of propositions we deal with so often in everyday life: the two-valued ones, assumed to be true or not true. It is hard to justify that a proposition asserted about anything observed is true in some absolute sense, that is, true independently from the observer. We prefer to argue that a yes-no proposition serves to divide the set of possibilities how a specific situation can be described, into two disjoint and exhaustive subsets. These subsets are assumed to be disjoint and exhaustive as long as we process the information without further reference to the involved objects. But if we deal with these objects repeatedly, for instance in a more precise manner, we might have to revise the classification. All in all, classical logic provides us a thinking model, not more, and it need not be considered as a copy of a part of reality.

In particular, the significance of mathematical modelling should not be overestimated. It should not be understood as a tool to describe the reality properly, but to sort our thoughts. The procedure to design a mathematical theory is the following one. We begin with a type of situation which we would like to describe, involving a constellation of objects. No situation is ever described completely, but with respect to selected properties. The crucial step is to endow a class of possible constellations of the involved objects with algebraic structure, describing the mutual relationships. Formal statements have to refer to this structure.

It is at the point of transition from an unstructured picture to a model which involves some of its characteristic properties, that vagueness comes into play. By a model, we mean either the well-defined structure on which a formal argumentation relies, or a structure underlying implicitly a statement of natural language. In any case, the creation of a model is nothing unambiguous, nothing perfect, and often only something temporary. It is based on certain observations, and afterwards more specific or new observations may make the model obsolete. The lack of the possibility to make a fixed picture in mind coincide with the object it refers to, is a source of vagueness.

Let us now turn to the peculiarities of nowadays' fuzzy logic. The aim of fuzzy logic is to take into account the mentioned inaccuracy of models. There is nothing special about this; for the design of propositional fuzzy logics, just the usual steps mentioned for the more general case are followed: We extract structure from the picture which we have in mind, and make a mathematical theory from it.

Even before starting, however we must be aware of the fact that we will have to get along with the incompleteness of our theory in the same way as in the case of any other mathematical theory. We can describe certain objects in certain ways, but if we add new aspects, we cannot expect that our theory can treat also these. For instance, when modelling ways to think about one version of the sorites paradox using an appropriate logic, we cannot expect that this logic is adequate also for all other versions, or even more absurdly, for all other cases where vagueness appears.

There are many possibilities how to start; let us choose the picture that we are given a region R of space and collection of subsets of R whose borders are unsharp. The common procedure is to take functions from R to a structure expressing gradedness. The latter is a bounded and dense linear order; so the rational unit interval would be the natural choice, but since in mathematics we are more familiar with the reals, it is the real unit interval which is practically always used. So we deal with fuzzy sets on R, understood as functions $u\colon\ R \to [0,1]$.

Next, we need to endow the set of fuzzy sets with an algebraic structure. For fuzzy propositional logic, the basic decision made is: fuzzy sets are to be connected pointwise. So what we have to consider in the end is just the real unit interval. Recall why we chose it: because it is a bounded dense linear order. So this is the structure associated with it.

In fuzzy logic, however, more structure is used. But any structure with which our set of truth values is endowed is to be derived from relations appearing in the situations which we are going to model. The dilemma becomes manifest: The respective situations do in general not offer more structure than the mentioned order-theoretical one. As opposed to that, the structure which is actually used originates from elsewhere, namely from the fact that the real numbers form a field.

If we want more structure, then it should arise from somewhere. We provide in this paper possible ways of motivation; we shall try to justify two specific fuzzy logics ex post.

3. Łukasiewicz Logic and Basic Logic

We consider in this paper two kinds of fuzzy logics: the Łukasiewicz Logic ŁL [COM, Haj] and the Basic Logic BL [Haj]. Sometimes, to avoid confusion with another, equally denoted logic, the latter is also called Basic Fuzzy Logic. We restrict to the propositional versions.

We shortly review the basic formal facts. Propositions of ŁL and BL are built up from atomic propositions and the constant 0 by means of the binary connectives $\odot$ an $\rightarrow$. Let $(\mathcal{F}; \odot, \rightarrow, 0)$ be the algebra of propositions of ŁL and BL. A continuous t-norm algebra is a structure $([0,1]; \wedge, \vee, \bar{\odot}, \bar{\rightarrow}, 0)$, where $[0,1]$ is the real unit interval, $\wedge$ and $\vee$ denote, respectively, the infimum and supremum of the reals w.r.t. their natural order, $\bar{\odot}$ is any continuous t-norm and $\bar{\rightarrow}$ is the residuum corresponding to $\bar{\odot}$. The Łukasiewicz algebra is the continuous t-norm algebra based on the Łukasiewicz t-norm $\odot_L\colon\ [0,1]^2 \rightarrow [0,1],\ \ (a,b) \mapsto (a+b-1) \vee 0$ and its residuum $\rightarrow_L\colon\ [0,1]^2 \rightarrow [0,1],\ \ (a,b) \mapsto (1-a+b) \wedge 1$.

In ŁL, a proposition φ is called valid if every structure-preserving mapping from $(\mathcal{F}; \odot, \rightarrow, 0)$ to $([0,1]; \odot_L, \rightarrow_L, 0)$ assigns 1 to φ. In BL, a proposition φ is called valid if every structure-preserving mapping from $(\mathcal{F}; \odot, \rightarrow, 0)$ to the reduct $([0,1]; \bar{\odot}, \bar{\rightarrow}, 0)$ of a continuous t-norm algebra assigns 1 to φ. For axiomatizations of ŁL and BL, see [COM, Haj].

In this paper, we will propose an alternative way to interpret propositions of ŁL and BL; the real unit interval will no longer play a role then.

The set of canonical models, which in case of ŁL contains just the Łukasiewicz t-norm algebra and which in case of BL contains all continuous t-norm algebras, may be enlarged without changing the notion of validity. Namely, we may take the generated varieties: the Łukasiewicz t-norm algebra generates the variety of MV-algebras, and the continuous t-norm algebras generate the variety of BL-algebras.

Definition 3.1. An algebra $(L; \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a *residuated lattice* if $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, $(L; \odot, 1)$ is a commutative monoid, and $a \odot b \leq c$ iff $a \leq b \rightarrow c$ for all $a, b, c \in L$. Moreover, L is called *divisible* if $a \wedge b = a \odot (a \rightarrow b)$ holds for any $a, b \in L$; L is called *prelinear* if $(a \rightarrow b) \vee (b \rightarrow a) = 1$ holds for any $a, b \in L$; L is called *involutive* if $(a \rightarrow 0) \rightarrow 0 = a$ holds for every $a \in L$.

A divisible, prelinear residuated lattice is called a *BL-algebra.* An involutive BL-algebra is called an *MV-algebra.*

4. a-Equivalence relations on Boolean algebras

We shall prepare in this section a framework for the interpretation of the propositions and the connectives of ŁL, the Łukasiewicz logic. The framework will consist of a Boolean algebra and a certain type of equivalence relation on it, to be called an a-equivalence relation, where the "a" stands for "ambiguity". We will show that, under a suitable condition, the set of equivalence classes bears the structure of an MV-algebra.

The results in this and the next section are based on our work [Vet3], where complete proofs and many further facts may be found. Note, however, that in [Vet3], the results are not put into the context of logics; as a consequence, the terminology is chosen slightly different, and in particular, the results are formulated in the dual order.

A *Boolean algebra* is a structure $(L; \wedge, \vee, \neg, 0, 1)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is a complementation function. We denote the relative complement of a w.r.t. b by $a \rightarrow b = \neg a \vee b$. Furthermore, we say that two elements a, b of a Boolean algebra are *non-implicative* and we write $a \top b$ if $a \rightarrow b = b$. Note that $a \top b$ if and only if $a \vee b = 1$; in particular, the relation $\top$ is symmetric.

Any Boolean algebra can be represented by a system $\mathcal{B}$ of subsets of some fixed set X such that $\mathcal{B}$ is closed under intersection, union, complementation and $\mathcal{B}$ contains the empty set and X. This is why we may say that any Boolean algebra models a system of crisp properties.

The idea is now to model the case that we cannot refer to a property, represented by an element of a Boolean algebra, unambiguously. We are going to reflect the ambiguity by an equivalence relation. There are certainly many ways to choose an equivalence relation for this objective. The basic requirement is in our case that it should preserve the strength of the property. Namely, if we introduce a measure on the Boolean algebra assumed to indicate the strength of the modelled property, then identifying elements of equal measure provides the typical example of the notion which we will introduce now.

Definition 4.1. Let $(L; \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. Let $\sim$ be an equivalence relation $\sim$ on L, and write $a \precsim b$ if $a' \leq b'$ for some $a' \sim a$ and $b' \sim b$. $\sim$ is called an *a-equivalence relation* on L if the following conditions hold:

(B1) If $a \precsim b$, then there is an $a' \sim a$ such that $a' \leq b$.

(B2) Let $a \sim a'$, $b \sim b'$ and $a \leq b$, $a' \leq b'$. Then $b \rightarrow a \sim b' \rightarrow a'$.

(B3) Let $a \sim a'$, $b \sim b'$, and $a \top b$, $a' \top b'$. Then $a \wedge b \sim a' \wedge b'$.

Given an a-equivalence relation $\sim$ on an algebra L, we will in the sequel denote the $\sim$-equivalence class of some element a by $[a]$, and $[L] = \{[a]\colon a \in L\}$ will be the quotient set w.r.t. $\sim$.

Example 4.2. Let $(L; \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. Moreover, let $\mu\colon L \to [0,1]$ be a measure on L in the real unit interval $[0,1]$; this means (i) $\mu(a \vee b) = \mu(a) + \mu(b)$ for $a, b \in L$ such that $a \wedge b = 0$, and (ii) $\mu(1) = 1$. Call the measure μ *homogeneous* if (i) $a > 0$ implies $\mu(a) > 0$ and (ii) for any $a, b \in L$ such that $\mu(a) \leq \mu(b)$, there is an $a' \leq b$ such that $\mu(a') = \mu(a)$. Informally, as mentioned, we should think of each element of the Boolean algebra as some crisp property, and its measure expresses its strength in the sense that elements of smaller measure are stronger than those of larger measure.

Now, for $a, b \in L$, define $a \sim b$ if $\mu(a) = \mu(b)$. It is evident that $\sim$ is an a-equivalence relation on L if μ is homogeneous. Note that $\sim$ has the additional property that no a is equivalent to any element strictly below a. The equivalence classes are in a one-to-one correspondence with the range of μ; in particular, the induced order is total.

We should underline that, as seen from this example, an a-equivalence relation $\sim$ on a Boolean algebra L is in general not compatible with the lattice operations. However, by (B2), $\sim$ is compatible with the complementation. Furthermore, $\sim$ is a congruence of L seen as a bounded poset.

Lemma 4.3. *Let $\sim$ be an a-equivalence relation on a Boolean algebra L. Then $\sim$ is compatible with the partial order; that is, setting*

$$[a] \leq [b] \;\; \textit{if} \;\; a \precsim b \;\; \textit{for } a, b \in L$$

makes $([L]; \leq, [0], [1])$ a bounded poset, and $\iota\colon L \to [L]$, $a \mapsto [a]$ is a surjective homomorphism of bounded posets.

We wonder about the internal structure of the set $[L]$ of $\sim$-equivalence classes. We will compile a set of operations definable on $[L]$ under a further assumption. Namely, given $a, b \in L$, we may pointwise connect the equivalence classes $[a]$ and $[b]$. This gives in particular the set of meets $\{a' \vee b'\colon a' \sim a,\ b' \sim b\}$ and the set of joins $\{a' \wedge b'\colon a' \sim a,\ b' \sim b\}$. In both cases, we may, if possible, select an element which is up to $\sim$-equivalence minimal or maximal with respect to the preorder $\precsim$.

Definition 4.4. Let $\sim$ be an a-equivalence relation on a Boolean algebra $(L; \wedge, \vee, \neg, 0, 1)$. Consider the bounded poset $([L]; \leq, [0], [1])$. If, for every $a, b \in L$, the set $\{[a' \wedge b'] \colon \ a' \sim a, \ b' \sim b\}$ contains a maximal element, we say that $[L]$ is *residuable.*

In other words, we require for every pair of elements a, b of the Boolean algebra that there are $\bar{a} \sim a$ and $\bar{b} \sim b$ such that $\bar{a} \wedge \bar{b} \gtrsim a' \wedge b'$ for any $a' \sim a$ and $b' \sim b$. Note that in this case the elements $\bar{a}$ and $\bar{b}$ are in general by a and b not uniquely determined, but only the equivalence class of the infimum $\bar{a} \wedge \bar{b}$.

Lemma 4.5. *Let $\sim$ be an a-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Let $a, b \in L$. Then*

$$\begin{aligned}
&\max \{[a' \wedge b'] \colon \ a' \sim a, \ b' \sim b\}, \\
&\min \{[a' \wedge b'] \colon \ a' \sim a, \ b' \sim b\}, \\
&\max \{[a' \vee b'] \colon \ a' \sim a, \ b' \sim b\}, \\
&\min \{[a' \vee b'] \colon \ a' \sim a, \ b' \sim b\}
\end{aligned}$$

all exist. Moreover, let $\bar{a} \sim a$ and $\bar{b} \sim b$ represent the first listed element, that is, assume $\bar{a} \wedge \bar{b} \gtrsim a' \wedge b'$ for all $a' \sim a$ and $b' \sim b$. Then $[\bar{a} \wedge \bar{b}] = [a] \wedge [b]$ and $[\bar{a} \vee \bar{b}] = [a] \vee [b]$. In particular, $[L]$ is lattice-ordered.

Furthermore, rather than connecting pairs of equivalence classes pointwise by the lattice operations, we may use the implication as well.

Lemma 4.6. *Let $\sim$ be an a-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Let $a, b \in L$, and let $\bar{a} \sim a$ and $\bar{b} \sim b$ be such that $[\bar{a} \wedge \bar{b}] = [a] \wedge [b]$. Then*

$$[\bar{a} \rightarrow \bar{b}] = \max \{[a' \rightarrow b'] \colon \ a' \sim a, \ b' \sim b\}.$$

We are led to the possibility to endow $[L]$ with the structure of a residuated lattice.

Definition 4.7. Let $\sim$ be an a-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Then, for $a, b \in L$, let

$$\begin{aligned}
[a] \odot [b] &= \min \{[a' \wedge b'] \colon \ a' \sim a, \ b' \sim b\}, \\
[a] \rightarrow [b] &= \max \{[a' \rightarrow b'] \colon \ a' \sim a, \ b' \sim b\},
\end{aligned}$$

and let $\mathit{0} = [0]$ and $\mathit{1} = [1]$. We call the structure $([L]; \wedge, \vee, \odot, \rightarrow, \mathit{0}, \mathit{1})$ the *ambiguity algebra* associated to the pair $(L, \sim)$.

EXAMPLE 4.8. Let L be the Boolean algebra generated by the intervals $(u, v]$ of the real unit interval, where $0 \leq u \leq v \leq 1$, by means of intersection, union, and complementation. For $a, b \in L$, define $a \sim b$ if the Borel measures of a and b coincide. Then $[L]$ can be identified with $[0, 1]$, ordered in the natural way. Moreover, $\odot$ is the Łukasiewicz t-norm, $\rightarrow$ is the Łukasiewicz implication. So the $([L]; \wedge, \vee, \odot, \rightarrow, \mathit{0}, \mathit{1})$ is the Łukasiewicz t-norm algebra.

We are ready to formulate the main theorem of this section.

THEOREM 4.9. *Let $\sim$ be an a-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Then the ambiguity algebra $([L]; \wedge, \vee, \odot, \rightarrow, \mathit{0}, \mathit{1})$ associated to $(L, \sim)$ is an MV-algebra.*

Up to isomorphism, all totally ordered MV-algebras arise in this way.

We close this section with one more example.

EXAMPLE 4.10. Let $X = \{(0, r)\colon\ r \in \mathbb{Z},\ r \geq 0\} \cup \{(1, r)\colon\ r \in \mathbb{Z},\ r \leq 0\}$ and endow X with the lexicographical order. Then X can be seen as the unit interval of the unital ℓ-group $\mathbb{Z} \times \mathbb{Z}$, where the addition is defined componentwise, the order is the lexicographical one, and the unit element is $(1, 0)$. The MV-algebra to which this ℓ-group interval gives rise is the Chang algebra; cf. e.g. [COM].

Let us see how the Chang algebra is the result of our construction. Let L be the set of all finite unions of intervals $(u, v] = \{x \in X\colon\ u < x \leq v\}$, where $u \leq v$. Then L, endowed with the intersection, union, complementation, and the constants $\varnothing$ and X, is a Boolean algebra. Given some $a \in L$, let $a = \dot{\cup}_i (u_i, v_i]$ be represented by disjoint intervals, and let $\mu(a) = \Sigma_i (u_i - v_i)$; for any $a, b \in L$, define $a \sim b$ if $\mu(a) = \mu(b)$. Then it is easily checked that the ambiguity algebra $[L]$ associated to $(L, \sim)$ is isomorphic to the Chang algebra.

The difficult problem in the present context is to find reasonable sufficient conditions for residuability. For this topic, we refer to [Vet3].

5. a-Equivalence relations on Heyting algebras

In this section, we generalize the framework introduced in Section 4. Rather than Boolean algebras, we will use Heyting algebras. The aim is to develop a representation theory not only for MV-algebras, but also for the more general BL-algebras, the algebraic counterpart of BL.

A *Heyting algebra* is a bounded lattice $(L; \wedge, \vee, 0, 1)$ such that, for all $a, b \in L$, the relative pseudocomplement of a w.r.t. b, given by

$$a \rightarrow b = \max\{x\colon\ b \wedge x \leq a\},$$

exists. Heyting algebras are distributive lattices such that $\wedge$ and $\rightarrow$ form an adjoint pair: $a \wedge b \leq c$ if and only if $a \leq b \rightarrow c$ for all a, b, c. For more information, see for instance [RaSi].

For elements a, b of a Heyting algebra, we again say that a and b are *non-implicative* and write $a \top b$ if $a \rightarrow b = b$. Note, however, that $\top$, in contrast to Boolean algebras, is in general not symmetric.

By distributivity, Heyting algebras can be represented by a lattice $\mathcal{B}$ of subsets of a set X. Then $\mathcal{B}$ is closed under intersection and union and $\mathcal{B}$ contains $\varnothing$ and X; in contrast to the case of Boolean algebras, however, $\mathcal{B}$ is not necessarily closed under complementation. Instead, there is for any two sets $A, B \in \mathcal{B}$ a largest one $C \in \mathcal{B}$ such that $A \cap C \subseteq B$.

Consequently, we may interpret a Heyting algebra as a model of a set of crisp properties such that the following holds: the conjunction and the disjunction of any two properties exists, and for any two properties there is a third one which, together with the first one, implies the second one.

We next present the adapted version of Definition 4.1 for the case of Heyting algebras.

Definition 5.1. Let $(L; \wedge, \vee, 0, 1)$ be a Heyting algebra. Let $\sim$ be an equivalence relation $\sim$ on L, and write $a \precsim b$ if $a' \leq b'$ for some $a' \sim a$ and $b' \sim b$. $\sim$ is called an *a-equivalence relation* on L if the following conditions hold:

(H0) $a \sim 1$ implies $a = 1$.

(H1) If $a \precsim b \precsim c$ and $a \leq c$, then there is a $b' \sim b$ such that $a \leq b' \leq c$.

(H2) Let $a \sim a'$, $b \sim b'$. Then $a \vee b \sim a' \vee b'$ if and only if $a \rightarrow b \sim a' \rightarrow b'$.

(H3) Let $a \sim a$, $b \sim b'$ and $a \top b$, $a' \top b'$. Then $a \wedge b \sim a' \wedge b'$.

As above, we denote by $[L]$ the set of equivalence classes of an a-equivalence relation $\sim$. By (H1), $\precsim$ induces a partial order on $[L]$, which will also be denoted by $\leq$, and $\iota\colon L \rightarrow [L]$, $a \mapsto [a]$ is a surjective homomorphism of bounded posets.

Note the new condition (H0), which has the following immediate consequence.

Lemma 5.2. *Let $\sim$ be an a-equivalence relation on a Heyting algebra L. Let $a, b \in L$. Then $a \sim b$ and $a \leq b$ imply $a = b$.*

If $\sim$ is an a-equivalence relation on a Boolean algebra L, then the relation $\sim$ on L, viewed as a Heyting algebra, fulfills (H1)–(H3), but not necessarily the new condition (H0). We did not require (H0) above because it was not needed. So if we were pedantic, we would have chosen a new notion for Definition 5.1; we did not do so to keep the notation simple.

In the case of Boolean algebras, we suggested to imagine $\sim$ as a relation identifying elements of equal extent. In a more general form, this picture can be applied to the present context as well.

EXAMPLE 5.3. Let $(I; \leq)$ be a totally ordered set. For every $\iota \in I$, let $(L_\iota; \wedge, \vee, 0, 1)$ be a Boolean algebra. Mimicking the ordinal sum construction, we combine these algebras to a new one as follows. Let L consist of those $(a_\iota)_{\iota \in I} \in \prod_{\iota \in I} L_\iota$ such that, for $\kappa \in I$, $a_\kappa < 1$ implies $a_\iota = 0$ for all $\iota > \kappa$. Endow L with the pointwise order. Then L is obviously a lattice with smallest the element $(0)_\iota$ and the largest element $(1)_\iota$. It is straightforward that L is actually a Heyting algebra.

Assume now in addition that, for every ι, $\mu_\iota \colon L_\iota \to [0,1]$ is a homogeneous measure on L_ι. Define $(a_\iota)_\iota \sim (b_\iota)_\iota$ if $\mu_\iota(a_\iota) = \mu_\iota(b_\iota)$ for all ι. We may check that $\sim$ is an a-equivalence relation on L. The induced order is total.

We will now endow the quotient of a Heyting algebra with respect to an a-equivalence relation with the structure of a residuated lattice, analogously to the procedure in Section 4. As to be expected, we will have to require here stronger conditions to obtain similar results. We are then again led to a class of residuated lattices, which is larger than before.

DEFINITION 5.4. Let $\sim$ be an a-equivalence relation on a Heyting algebra $(L; \wedge, \vee, \neg, 0, 1)$. Consider the bounded poset $([L]; \leq, \mathit{0}, \mathit{1})$. If, for every $a, b \in L$, the set $\{[a' \wedge b'] \colon a' \sim a, b' \sim b\}$ contains a minimal and a maximal element, we say that $[L]$ is *residuable*.

Mutatis mutandis, Lemma 4.5 holds for Heyting algebras as well. So we can define the operations $\odot$ and $\to$ as above.

DEFINITION 5.5. Let $\sim$ be an a-equivalence relation on a Heyting algebra L such that $[L]$ is residuable. Then we define for $a, b \in L$

$$\begin{aligned} [a] \odot [b] &= \min \{[a' \wedge b'] \colon a' \sim a, b' \sim b\}, \\ [a] \to [b] &= \max \{[a' \to b'] \colon a' \sim a, b' \sim b\}, \end{aligned}$$

and we let $\mathit{0} = [0]$ and $\mathit{1} = [1]$. We call the structure $([L]; \wedge, \vee, \odot, \to, \mathit{0}, \mathit{1})$ the *ambiguity algebra* associated to the pair $(L, \sim)$.

THEOREM 5.6. *Let $\sim$ be an a-equivalence relation on a Heyting algebra L such that $[L]$ is residuable. Then the ambiguity algebra $([L]; \wedge, \vee, \odot, \to, \mathit{0}, \mathit{1})$ associated to $(L, \sim)$ is a divisible residuated lattice.*

It is open if an ambiguity algebra associated to a Heyting algebra fulfills prelinearity. We will introduce an additional condition to ensure this property.

DEFINITION 5.7. Let L be a Heyting algebra L, and let $\sim$ be an a-equivalence relation on L. Then we say that $[L]$ is *prelinear* if the following condition holds:

(PL) Let $a, b \in L$ be such that $a \wedge b \gtrsim a' \wedge b'$ for all $a' \sim a$ and $b' \sim b$. Then $c_1 \sim c_2$, $c_1 \geq a \rightarrow b$ and $c_2 \geq b \rightarrow a$ imply $c_1 = c_2 = 1$.

In other words, $[L]$ is prelinear if $[a \wedge b] = [a] \wedge [b]$ implies $[a \rightarrow b] \vee [b \rightarrow a] = \mathit{1}$.

We arrive at this section's main theorem.

THEOREM 5.8. *Let $\sim$ be an a-equivalence relation on a Heyting algebra L such that $[L]$ is residuable and prelinear. Then the ambiguity algebra $([L]; \wedge, \vee, \odot, \rightarrow, \mathit{0}, \mathit{1})$ associated to $(L, \sim)$ is a BL-algebra.*

Moreover, all totally ordered BL-algebras arise in this way.

Again, we add examples to illustrate the meaning of this theorem.

EXAMPLE 5.9. Let $X = \mathbb{R}^+ \cup \{\infty\}$, that is, the positive cone of the totally ordered group of reals, enriched with one additional element ∞ on top. Let L consist of the following subsets of X: (i) the empty set $\varnothing$ and (ii) the unions a of finitely many intervals $(u, v] = \{x \in X : \ u < x \leq v\}$, where $u \leq v$, such that $\infty \in a$. Then L, endowed with the intersection, union, and the constants $\varnothing$ and X, is a Heyting algebra of subsets of X.

Given $a, b \in L$, let $a \sim b$ if neither a nor b is the empty set and if the Borel measure of the complement of a and b coincides. Then we may check that the ambiguity algebra $[L]$ associated to $(L, \sim)$ is isomorphic to the product algebra.

EXAMPLE 5.10. To construct the third standard BL-algebra, let $X = [0, 1]$ be the real unit interval, and let L consist of the subintervals $[0, a]$, where $a \in [0, 1]$. Then L is a Heyting algebra of subsets of X. Let $\sim$ be the equivalence relation on X which does not identify any pair of elements. Then the ambiguity algebra $[L]$ associated to $(L, \sim)$ is isomorphic to the Gödel algebra.

6. Alternative semantics for fuzzy logics

In this last section, we shall make explicit how our results apply to fuzzy logics.

Recall that the varieties of MV- and BL-algebras are generated by their totally ordered members. So in view of Theorems 4.9 and 5.8, we may base the semantics of ŁL and of BL on ambiguity algebras arising from Boolean or Heyting algebras, respectively.

DEFINITION 6.1. Let $\sim$ be an a-equivalence relation on a Boolean algebra L such that $[L]$ is residuable, and let $([L]; \wedge, \vee, \odot, \rightarrow, \mathit{0}, \mathit{1})$ be the ambiguity algebra associated to $(L, \sim)$. An *evaluation* of the propositions $\mathcal{F}$ of ŁL in $[L]$ is a mapping $v\colon\ \mathcal{F} \rightarrow [L]$ such that, for $\alpha, \beta \in \mathcal{F}$,

$$\begin{aligned} v(\alpha \odot \beta) &= v(\alpha) \odot v(\beta), \\ v(\alpha \rightarrow \beta) &= v(\alpha) \rightarrow v(\beta), \\ v(0) &= \mathit{0}. \end{aligned}$$

A proposition φ is called *valid* in $[L]$ if $v(\varphi) = \mathit{1}$ for all evaluations in $[L]$.

Similarly, let $\sim$ be an a-equivalence relation on a Heyting algebra L such that $[L]$ is residuable and prelinear. Let $([L]; \wedge, \vee, \odot, \rightarrow, \mathit{0}, \mathit{1})$ be the ambiguity algebra associated to $(L, \sim)$. Define an *evaluation* of the propositions of BL, and define a proposition of BL to be *valid*, analogously to the case of ŁL.

THEOREM 6.2. *A proposition φ of ŁL is valid in ŁL if and only if for all pairs of a Boolean algebra L and an a-equivalence relation $\sim$ on L such that $[L]$ is residuable, φ is valid in the ambiguity algebra $[L]$ associated to $(L, \sim)$.*

Similarly, a proposition φ of BL is valid in BL if and only if for all pairs of a Heyting algebra L and an a-equivalence relation $\sim$ on L such that $[L]$ is residuable and prelinear, φ is valid in the ambiguity algebra $[L]$ associated to $(L, \sim)$.

7. Conclusion

To justify the use of a specific t-norm based many-valued logic, its semantical framework must be more specific, compared to the common case that the real unit interval endowed with a t-norm is used. Working towards this aim, we have proposed alternative semantics for Łukasiewicz and Basic Logic. The idea was to derive the corresponding algebraic structures, MV- and BL-algebras, in some natural way. The result is as follows: We use pairs consisting of a Boolean or Heyting algebra and an equivalence relation subject to conditions which are chosen in accordance with the case that elements of equal strength are identified; under a natural assumption, the set of equivalence classes gives rise to an MV- or a BL-algebra, respectively.

It would be desirable to have similar results for further fuzzy logics. Note however that, like e.g. in case of the logic MTL, a general structure theory for the algebras of the corresponding varieties is typically not available; this makes approaches like the present one difficult. Furthermore, for many fuzzy logics, first-order versions were defined, and to develop an approach like the present one also for these logics would again be certainly desirable. However, here the problems are located even on a more basic level. Namely, we find that to interpret first-order fuzzy logics properly is even much more involved than in case of the propositional logics; at least we have to admit that we have not yet found a satisfying approach.

On the positive side, however, we are convinced that apart from the approach chosen here, there are many more possibilities how to approach the aim of making the content of fuzzy logics like ŁL and BL more clear. Further research can continue along as many lines as specific situations can be imagined in which a certain fuzzy logic is applicable.

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