Logic of approximate entailment in quasimetric and in metric spaces

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Abstract

It is known that a quasimetric space can be represented by means of a metric space; the points of the former space become closed subsets of the latter one, and the role of the quasimetric is assumed by the Hausdorff quasidistance. In this paper, we show that, in a slightly more special context, a sharpened version of this representation theorem holds. Namely, we assume a quasimetric to fulfil separability in the original sense due to Wilson. Then any quasimetric space can be represented by means of a metric space such that distinct points are assigned disjoint closed subsets.

This result is tailored to the solution of an open problem from the area of approximate reasoning. Following the lines of E. Ruspini's work, the Logic of Approximate Entailment (LAE) is based on a graded version of the classical entailment relation. We present a proof calculus for LAE and show its completeness with regard to finite theories.

1 Introduction

A quasimetric is defined similarly to a metric; the assumption of symmetry, however, is dropped. This generalisation of the concept of a distance naturally occurs in many real-world situations. An often cited example is the time that a walker needs to get from one place to another one within a mountainous area. Quasimetric spaces are moreover closely related to weighted directed graphs. The latter play a significant role in computer science, in particular for the formulation of network flow problems [AMO, HPS].

It is certainly also true that the notion of a quasimetric is by far less common in mathematics than its symmetric counterpart. Remarkably, however, the metric spaces themselves give rise to a non-symmetric distance function. The *Hausdorff quasidistance* quantifies the difference between subsets of a metric space, rather than between its points. It shares with a metric certain characteristic properties like the triangle inequality, but symmetry does in general not hold. Hyperspaces consisting of subsets of a metric space, endowed with the Hausdorff quasidistance, are used in a number of contexts. An example is the area of point-free geometry [DiGe, Dic]. Measuring the degree of distinctness between subsets of a metric space is moreover an issue in fuzzy set theory; see, e.g., [Ger].

It is reasonable to ask whether distance functions that violate symmetry but otherwise resemble a metric always arise from a metric space in the above way mentioned. We find the affirmative answer in P. Vitolo's paper [Vit]. Vitolo has studied spaces (W, q), where $q \colon W \times W \to \mathbb{R}^+$ fulfils the triangle inequality as well as the following version of the separation axiom: $q(a,b) = q(b,a) = 0$ if and only if $a = b$. He proved that any such space can be embedded into the hyperspace of non-empty closed sets of a metric space, the role of the quasimetric being taken by the Hausdorff quasidistance. Also among the results in [Ger], Vitolo's representation theorem can be found.

In the present paper, we study distance functions of a more special type. In this paper, a quasimetric is understood to be a mapping $q \colon W \times W \to \bar{\mathbb{R}}^+$ such that, for $a, b, c \in W$, $q(a,c) \leqslant q(a,b) + q(b,c)$, and $q(a,b) = 0$ if and only if $a = b$. This definition is in accordance with the early work of W. A. Wilson on the topic [Wil]; however, it is evidently stronger than the one used in [Vit, Ger]. We prove a representation theorem similar to Vitolo's, but with certain additional features.

The motivation underlying Vitolo's work was to characterise the quasimetrisability of a topological space. Gerla's interest originated from the connection between quasimetrics and fuzzy orders. The results of the present paper are developed yet for another purpose: we are interested in logics for approximate reasoning, in line with the framework proposed in E. Ruspini's seminal paper [Rus].

Recall that in classical propositional logic, properties are modelled by subsets and the logical entailment corresponds to the subsethood relation. Following [Rus], a graded version of the subsethood relation may be used in order to express the approximate entailment of properties. To this end, we endow the underlying space with a similarity function. A similarity function is a flexible tool that allows the quantification of the distinctness of points. One possibility is to take (the dual of) a metric and we presuppose this choice here. Then the Hausdorff quasidistance associated with the metric provides a means of expressing that one subset is contained in another one to a certain degree.

To capture this idea on the basis of a logical calculus turns out to be difficult. We focus here on the so-called Logic of Approximate Entailment, LAE for short, which was introduced by R. Rodríguez in his PhD thesis [Rod]. The completeness of LAE in a finitary setting has long been known; see [Rod], cf. also [EGRV]. But an axiomatisation of LAE in a more general setting has remained an open problem; cf. [Rod, Section 8.3].

In our previous paper [Vet], we have dealt with a closely related setting. Namely, we have considered a modification of LAE, the logic LAE^q, which is based on quasimetric spaces instead of metric spaces. We have shown that a certain calculus, denoted by $\mathbf{LAE}$ and consisting of only six rules [Vet], is sound and complete for LAE^q. The question has been open if this result could be useful also for the logic LAE. To this end, a means of representing a quasimetric space within a metric space is required. Indeed, what we need is a representation theorem similar to the one given in [Vit]. Vitolo's result itself is, unfortunately, not applicable. The crucial problem is that points are assigned subsets of a metric space and subsets assigned to distinct points are in general not disjoint. The present paper offers a solution for this situation.

Our core result is the following. For any quasimetric space (X, q), there is a metric space (Y, d) and a mapping $\iota \colon X \to \mathcal{P}(Y)$ such that the following holds: ι maps distinct points to disjoint closed subsets of Y and under this embedding the quasimetric on X coincides with the Hausdorff quasidistance on Y. We in fact even show that the natural extension of ι to the power set of X preserves the Hausdorff quasidistance. We apply this result in order to show that the calculus $\mathbf{LAE}$, which is known to be complete for the logic LAE^q based on quasimetric spaces, is actually complete for LAE, which is based on metric spaces. We may consequently say that we are able to define a logical calculus that is well in line with the framework for approximate reasoning in its original form due to Ruspini.

The paper is structured as follows. In the following Section 2, we introduce the main notions under consideration. Section 3 contains the representation theorem for quasimetric spaces. We apply this result in Section 4 to the Logic of Approximate Entailment LAE, showing that the calculus $\mathbf{LAE}$ from [Vet] is sound and complete for LAE. An outlook on possible further work is contained in the concluding Section 5.

2 Quasimetric spaces and metric spaces

We start by fixing notation and terminology used in this paper.

We let $\mathbb{R}^+ = \{r \in \mathbb{R} \colon r \geqslant 0\}$ be the set of positive reals and we denote by $\bar{\mathbb{R}}^+$ the extended set of positive reals, that is, $\bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$. Here, ∞ is a new element such that $r < \infty$ for any $r \in \mathbb{R}^+$, and we define $r + \infty = \infty + r = \infty$ for any $r \in \bar{\mathbb{R}}^+$.

Definition 2.1. Let X be a non-empty set. A *quasimetric* on X is a mapping $q \colon X \times X \to \bar{\mathbb{R}}^+$ such that the following conditions hold:

(M1) For any $a \in X$, we have $q(a, a) = 0$.

(M2) For any $a, b \in X$, $q(a, b) = 0$ implies $a = b$.

(M3) For any $a, b, c \in X$, we have $q(a, c) \leqslant q(a, b) + q(b, c)$.

In this case we refer to (X, q) as a *quasimetric space*.

Moreover, we call q a *metric* and (X, q) a *metric space* if, in addition, the following condition holds:

(M4) For any $a, b \in X$, we have $q(a, b) = q(b, a)$.

As mentioned in the introduction, there is a close relationship between quasimetric spaces on the one hand and weighted directed graphs on the other hand. The details are as follows.

Remark 2.2. *Consider a finite directed graph, consisting of the vertices V and edges $E \subseteq V \times V$, together with a weight function $w \colon E \to \mathbb{R}^+$ such that, for any $(v_1, v_2) \in E$, $w(v_1, v_2) = 0$ if and only if $v_1 = v_2$. Define the length of a directed path as the sum of the weights of its edges. Then, for any pair of vertices $v_1, v_2 \in V$, let $q(v_1, v_2) = 0$ if $v_1 = v_2$, let $q(v_1, v_2)$ be the minimal length of a path from v_1 to v_2 if $v_1 \neq v_2$ and there is at least one path from v_1 to v_2, and let $q(v_1, v_2) = \infty$ if $v_1 \neq v_2$ and there is no path from v_1 to v_2. We easily verify that q makes V into a quasimetric space.*

Assuming that the length of the paths between each two distinct vertices is lower bounded by a strictly positive value, we may generalise the observation to the infinite case.

Conversely, we may associate with each quasimetric space (X, q) the directed graph whose set of vertices is X and whose set of edges is $\{(v_1, v_2) \in X \times X \colon q(v_1, v_2) < \infty\}$. From this graph (X, q) arises as indicated above.

In contrast to the case of a metric, the definition of a quasimetric is in the literature not uniform. Most remarkably, instead of the separability axiom (M2), the following one is often used:

(M2') For any $a, b \in X,\ q(a,b) = q(b,a) = 0$ implies $a = b$.

Assuming (M2'), we evidently could not conclude from $q(a,b) = 0$ or $q(b,a) = 0$ alone that a and b coincide. W. A. Wilson [Wil], who is reportedly the first to study quasimetric spaces on an axiomatic basis, uses condition (M2) as well. In [Vit] and [Ger], however, (M2') is used.

A further difference is that a quasimetric is often defined to take values in $\mathbb{R}^+$ only. Although this deviation might sound less significant, we should note that the procedure presented in the subsequent section relies on it in an essential way.

We endow a quasimetric space (X, q) with a topology in the usual way: its basis consists of the sets of the form $\{b \in X \colon q(a,b) < \varepsilon\}$, where $a \in X$ and $\varepsilon > 0$. By (M2), X is a T_1-space, hence all singletons are closed. Note that (M2') would imply not more than T_0.

Definition 2.3. Let (X, q) be a quasimetric space. For a point $a \in X$ and a set $B \subseteq X$ we put

$$p_q(a, B) \ = \ \inf_{b \in B} \ q(a,b),$$

and we define the *Hausdorff quasidistance* between subsets A and B of X by

$$e_q(A, B) \ = \ \sup_{a \in A} \ p_q(a, B).$$

We compile a few straightforward facts on the Hausdorff quasidistance.

Lemma 2.4. *Let (X, q) be a quasimetric space. Then we have, for any $A, A_\iota, B, C \subseteq X$, $\iota \in I$:*

(i) $e_q(A, B) = 0$ *if $A \subseteq B$. If B is closed, also the converse holds.*

(ii) $e_q(\bigcup_\iota A_\iota, B) \ = \ \sup_\iota \ e_q(A_\iota, B)$.

(iii) $e_q(A, B) \geqslant e_q(A, C)$ *if $B \subseteq C$.*

(iv) $e_q(A, C) \leqslant e_q(A, B) + e_q(B, C)$.

(v) $e_q(\varnothing, B) = 0$. *Moreover, if A is non-empty, $e_q(A, \varnothing) = \infty$.*

Proof. The first part of (i) as well as (ii), (iii), (v) are clear.

For the second part of (i), assume that $e_q(A, B) = 0$ and B is closed. Then $p_q(a, B) = 0$ for any $a \in A$ and since B is closed, it follows that $a \in B$ for any $a \in A$, that is, $A \subseteq B$.

To verify (iv), assume that A, B, C are non-empty; otherwise, the inequality is clear from (v). We calculate

$$\begin{aligned} e_q(A, B) + e_q(B, C) &= \sup_{a \in A} \inf_{b \in B} q(a, b) + \sup_{b \in B} \inf_{c \in C} q(b, c) \\ &= \sup_{a \in A} \inf_{b \in B} (q(a, b) + \sup_{b' \in B} \inf_{c \in C} q(b', c)) \\ &\geqslant \sup_{a \in A} \inf_{b \in B} (q(a, b) + \inf_{c \in C} q(b, c)) \\ &= \sup_{a \in A} \inf_{b \in B} \inf_{c \in C} (q(a, b) + q(b, c)) \\ &\geqslant \sup_{a \in A} \inf_{c \in C} q(a, c) \;=\; e_q(A, C), \end{aligned}$$

using (M3). □

By Lemma 2.4, the Hausdorff quasidistance e_q, defined on the hyperspace of closed sets of a quasimetric space X, fulfils conditions (M1), (M2'), and (M3). This is actually the guiding example in [Vit]. From Lemma 2.4(i), however, it is also clear that condition (M2) is in general violated.

The situation is different for hyperspaces consisting of mutually disjoint subsets of a metric space. Our own guiding example of a quasimetric space is the following one, which is given in [Wil] right at the beginning.

Lemma 2.5. *Let (X, d) be a metric space, let $\mathcal{Y}$ be a partition of X into non-empty closed subsets, and let e_d be the Hausdorff quasidistance restricted to $\mathcal{Y}$. Then $(\mathcal{Y}, e_d)$ is a quasimetric space.*

Proof. Let $A, B \in \mathcal{Y}$. Clearly, $e_d(A, A) = 0$, that is, (M1) holds. Furthermore, $e_d(A, B) = 0$ implies $A \subseteq B$ by Lemma 2.4(i). By assumption, A and B are non-empty subsets of X that are either equal or disjoint. Hence $A = B$ and (M2) follows. (M3) holds by Lemma 2.4(iv). □

We will see that any quasimetric space arises in this way from a metric.

3 Representation of quasimetric spaces

In this section, we show how quasimetric spaces can be led back to metric spaces. Intuitively speaking, we "expand" each point of a quasimetric space to a set of possibly infinite cardinality and we endow the (disjoint) union of all these sets with a metric. This is done such that the quasimetric between two points of the original space becomes the Hausdorff quasidistance between the two associated subsets of the new space. We shall furthermore see that our representation even preserves the Hausdorff quasidistance on the original space.

We start with an auxiliary lemma. By a *pseudometric*, we mean a mapping $q\colon X \times X \to \bar{\mathbb{R}}^+$ fulfilling (M1), (M3), and (M4).

Lemma 3.1. *Let X be a non-empty set and let $D \subseteq X \times X$ be such that $(a,a) \in D$ for all $a \in X$, and $(a,b) \in D$ implies $(b,a) \in D$ for any $a,b \in X$. Let $\hat{d}\colon D \to \bar{\mathbb{R}}^+$ be such that the following conditions hold:*

(α) *Let $a,b \in X$. Then $(a,b) \in D$ and $\hat{d}(a,b) = 0$ if and only if $a = b$.*

(β) *Let $(a,b) \in D$. Then $\hat{d}(a,b) = \hat{d}(b,a)$.*

(γ) *Let $a_0,\ldots,a_n \in X$ be pairwise distinct. If $(a_0,a_1),\ldots,(a_{n-1},a_n)$, where $n \geqslant 2$, as well as (a_0,a_n) are in D, then $\hat{d}(a_0,a_n) \leqslant \hat{d}(a_0,a_1) + \ldots + \hat{d}(a_{n-1},a_n)$.*

Moreover, let $d\colon X \times X \to \bar{\mathbb{R}}^+$ be defined as follows. For $a,b \in X$, let $d(a,b) = 0$ if $a = b$ and otherwise

$$d(a,b) \;=\; \inf\,\{\hat{d}(c_0,c_1) + \ldots + \hat{d}(c_{n-1},c_n)\colon \text{ there are pairwise distinct elements } c_0,\ldots,c_n\ (n \geqslant 1) \text{ in } X \text{ such that } a = c_0,\ b = c_n, \text{ and } (c_0,c_1),\ldots,(c_{n-1},c_n) \in D\}. \quad (1)$$

Then d is the largest pseudometric on X extending $\hat{d}$.

Proof. We claim that $d'\colon X \times X \to \bar{\mathbb{R}}^+$ defined by

$$d'(a,b) \;=\; \inf\,\{\hat{d}(c_0,c_1) + \ldots + \hat{d}(c_{n-1},c_n)\colon\ c_0,\ldots,c_n \in X \text{ such that } a = c_0,\ b = c_n, \text{ and } (c_0,c_1),\ldots,(c_{n-1},c_n) \in D\}$$

coincides with d. In fact, let $a,b \in X$. If $a = b$, we have $d'(a,b) = 0 = d(a,b)$ by (α). Otherwise consider a sequence $a = c_0,\ldots,c_n = b$ such that all pairs of successive elements are in D. Then we can determine a subsequence as follows:

if $c_i = c_j$ for some $0 \leqslant i < j \leqslant n$, we remove the elements $c_{i+1}, \ldots, c_j$; and we repeat doing so until no element appears twice. Let $c'_0, \ldots, c'_m$ be the resulting sequence. Then the first and last element is the same as before, that is, $c'_0 = c_0 = a$ and $c'_m = c_n = b$. Furthermore, pairs of successive elements are still in D. As $\hat{d}(c'_0, c'_1) + \ldots + \hat{d}(c'_{m-1}, c'_m) \leqslant \hat{d}(c_0, c_1) + \ldots + \hat{d}(c_{n-1}, c_n)$, we conclude $d = d'$.

It is readily checked that d' and hence d is a pseudometric. Furthermore, by (γ), d extends $\hat{d}$. By construction, d is the largest pseudometric extending $\hat{d}$. □

We now turn to the core result of the present paper. We will denote by $A^{<\omega}$ the set of all finite sequences of elements of a set A. A finite sequence $(a_1, \ldots, a_n)$ is said to *extend* another one $(b_1, \ldots, b_m)$ if $m \leqslant n$ and $a_1 = b_1, \ldots, b_m = a_m$. Furthermore, we write $\varnothing$ for the sequence of length 0; and for $\sigma \in A^{<\omega}$ and $a \in A$, we denote by $\sigma^\frown a$ the sequence arising from σ by adding a as the new last element.

Theorem 3.2. *Let (X, q) be a quasimetric space. Then there is a metric space (Y, d) and a mapping $\iota \colon X \to \mathcal{P}(Y)$ such that $\{\iota(a) \colon a \in X\}$ is a partition of Y and*

$$q(a, b) \;=\; e_d(\iota(a), \iota(b)) \tag{2}$$

holds for any $a, b \in X$.

Proof. The assertion is trivial if q is actually a metric. Assume that this is not the case.

Let

$$U \;=\; \{(a, b) \in X \times X \colon q(a, b) < q(b, a)\}$$

and put

$$Y \;=\; X \times U^{<\omega}.$$

Let D be the subset of $Y \times Y$ consisting of all pairs of the following form:

$$\begin{aligned} &((a, \sigma), (b, \sigma)), \text{ where } a, b \in X \text{ and } \sigma \in U^{<\omega}, \\ \text{or }\ &((a, \sigma), (b, \sigma^\frown(a, b))), \text{ where } \sigma \in U^{<\omega} \text{ and } (a, b) \in U, \\ \text{or }\ &((b, \sigma^\frown(a, b)), (a, \sigma)), \text{ where } \sigma \in U^{<\omega} \text{ and } (a, b) \in U. \end{aligned}$$

Let us define a map $\hat{d} \colon D \to \bar{\mathbb{R}}^+$ as follows. For $a, b \in X$ and $\sigma \in U^{<\omega}$, let

$$\hat{d}((a, \sigma), (b, \sigma)) \;=\; \max\{q(a, b),\, q(b, a)\};$$

and for $\sigma \in U^{<\omega}$ and $(a, b) \in U$, let

$$\hat{d}((a, \sigma), (b, \sigma^\frown(a, b))) \;=\; \hat{d}((b, \sigma^\frown(a, b)), (a, \sigma)) \;=\; q(a, b).$$

Our aim is to show that Lemma 3.1 is applicable to $\hat{d}$. It is clear that the domain $D \subseteq Y \times Y$ of $\hat{d}$ fulfils the requirements. Moreover, we easily check that $\hat{d}$ fulfils conditions (α) and (β).

Before turning to condition (γ), we show two auxiliary facts.

(A) Let $(x_1, \sigma_1), \ldots, (x_n, \sigma_n)$, where $n \geqslant 2$, be a finite sequence of pairwise distinct elements of Y. Assume that $\hat{d}$ is defined for each pair of successive elements and that σ_n extends σ_1. Then, for each $i = 1, \ldots, n-1$, σ_{i+1} extends σ_i.

Indeed, assume the contrary. We have that, for each $i = 1, \ldots, n-1$, either $\sigma_i = \sigma_{i+1}$, or σ_{i+1} extends σ_i by one element, or σ_i extends σ_{i+1} by one element. Since σ_n extends σ_1, we conclude that the sequence contains two pairs of successive elements such that one is of the form $(x, \sigma^\frown(a,b)), (y, \sigma)$ and the other one $(x', \sigma), (y', \sigma^\frown(a,b))$. Since these two pairs are in D, we conclude $x = y' = b$ and $y = x' = a$, in contradiction to our assumption that the elements of the sequence are pairwise distinct.

(B) Let $(x_0, \sigma_0), \ldots, (x_n, \sigma_n)$, where $n \geqslant 1$, be a finite sequence of pairwise distinct elements of Y. Assume that $\hat{d}$ is defined for each pair of successive elements and that σ_n extends σ_0. Then

$$q(x_0, x_n) \leqslant \hat{d}((x_0, \sigma_0), (x_1, \sigma_1)) + \ldots + \hat{d}((x_{n-1}, \sigma_{n-1}), (x_n, \sigma_n)).$$

Indeed, by (A), σ_{i+1} extends σ_i for each $i = 0, \ldots, n-1$. By the definition of $\hat{d}$ it follows that $q(x_i, x_{i+1}) \leqslant \hat{d}((x_i, \sigma_i), (x_{i+1}, \sigma_{i+1}))$ for each $i = 0, \ldots, n-1$. Hence

$$\begin{aligned} q(x_0, x_n) &\leqslant q(x_0, x_1) + \ldots + q(x_{n-1}, x_n) \\ &\leqslant \hat{d}((x_0, \sigma_0), (x_1, \sigma_1)) + \ldots + \hat{d}((x_{n-1}, \sigma_{n-1}), (x_n, \sigma_n)), \end{aligned}$$

as asserted.

To see (γ), we have to show

$$\hat{d}((x_0, \sigma_0), (x_n, \sigma_n)) \leqslant \hat{d}((x_0, \sigma_0), (x_1, \sigma_1)) + \ldots + \hat{d}((x_{n-1}, \sigma_{n-1}), (x_n, \sigma_n)), \tag{3}$$

provided that $(x_0, \sigma_0), \ldots, (x_n, \sigma_n)$, where $n \geqslant 2$, are pairwise distinct and that $\hat{d}$ is defined in each indicated case.

We have that either σ_n extends σ_0 or vice versa. Converting the order of the sequence if necessary, we may assume that the former possibility applies, that is, σ_n extends σ_0. Consequently, by (B), we have

$$q(x_0, x_n) \leqslant \hat{d}((x_0, \sigma_0), (x_1, \sigma_1)) + \ldots + \hat{d}((x_{n-1}, \sigma_{n-1}), (x_n, \sigma_n)). \tag{4}$$

To see (3), we distinguish two cases.

Case 1. σ_0 and σ_n coincide. By (A), we then have $\sigma_0 = \sigma_1 = \ldots = \sigma_n$. Moreover, by (B), (4) holds with "$q(x_0, x_n)$" being replaced by "$q(x_n, x_0)$". As $\hat{d}((x_0, \sigma_0), (x_n, \sigma_n)) = \max\{q(x_0, x_n),\ q(x_n, x_0)\}$, we conclude (3).

Case 2. σ_n extends σ_0 by one element. Then $\hat{d}((x_0, \sigma_0), (x_n, \sigma_n)) = q(x_0, x_n)$ and (3) holds again by (4).

The proof is complete that $\hat{d}$ fulfils property (γ) of Lemma 3.1. We conclude that there is a largest pseudometric d extending $\hat{d}$ to the whole $Y \times Y$, given by (1).

We claim that d is in fact a metric. Let $(a, \sigma), (b, \sigma') \in Y$ be distinct. We have to show $d((a, \sigma), (b, \sigma')) > 0$. If $\sigma = \sigma'$, the pair is in D and the assertion is clear by (M2). Assume that σ and σ' are distinct. W.l.o.g., we can then assume that there is a $(c, d) \in U$ that occurs in σ' but not in σ. Let $(a, \sigma) = (x_0, \sigma_0), \ldots, (x_n, \sigma_n) = (b, \sigma')$ be such that $\hat{d}$ is defined for each pair of successive elements. Then there is a $0 \leqslant i \leqslant n{-}1$ such that $\sigma_{i+1} = \sigma_i{}^\frown(c, d)$ and this means $\hat{d}((x_i, \sigma_i), (x_{i+1}, \sigma_{i+1})) = q(c, d)$. It follows that $d((a, \sigma), (b, \sigma') \geqslant q(c, d)$ and since $c \neq d$ the assertion follows from (M2).

We next note:

(C) For any $(a, \sigma), (b, \sigma') \in Y$ such that σ' extends σ, we have

$$q(a, b) \leqslant d((a, \sigma), (b, \sigma')).$$

Indeed, this is immediate from (B).

We now define

$$\iota \colon X \to \mathcal{P}(Y), \quad a \mapsto \{(a, \sigma) \colon \sigma \in U^{<\omega}\}.$$

Let $a, b \in X$; we have to show (2). By (C), we have for any $\sigma \in U^{<\omega}$

$$d((a, \varnothing), (b, \sigma)) \geqslant q(a, b),$$

and it follows that $e_d(\iota(a), \iota(b)) \geqslant q(a, b)$. In order to prove the converse inequality, let us first assume that $q(a, b) < q(b, a)$. Then $(a, b) \in U$ and, for any $\sigma \in U^{<\omega}$,

$$d((a, \sigma), (b, \sigma^\frown(a, b))) \ = \ q(a, b),$$

hence $e_d(\iota(a), \iota(b)) \leqslant q(a, b)$. Assume second that $q(a, b) \geqslant q(b, a)$. Since, for any $\sigma \in U^{<\omega}$,

$$d((a, \sigma), (b, \sigma)) \ = \ \max\{q(a, b),\ q(b, a)\} \ = \ q(a, b),$$

we again have $e_d(\iota(a), \iota(b)) \leqslant q(a, b)$. The proof of (2) is complete. □

The remainder of this section is devoted to several refinements of Theorem 3.2. We could have integrated the following lemmas directly into Theorem 3.2; we did not do so in order to avoid the lengthy proof to be even further extended.

We first consider the topologies on the represented and representing spaces.

Lemma 3.3. *Let* (X, q) *be a quasimetric space. Let* $\iota\colon X \to \mathcal{P}(Y)$ *be the representation of* (X, q) *by means of the metric space* (Y, d) *according to Theorem* 3.2. *Then, for each* $a \in X$, $\iota(a)$ *is a closed subset of* Y.

Proof. With reference to the proof of Theorem 3.2, let $a \in X$ and let $(b, \sigma) \in Y$ such that $a \neq b$. To show that $\iota(a) = \{(a, \sigma')\colon \sigma' \in U^{<\omega}\}$ is closed, we will determine an $\varepsilon > 0$ such that $d((a, \sigma'), (b, \sigma)) > \varepsilon$ for any $\sigma' \in U^{<\omega}$.

For each $\sigma' \in U^{<\omega}$, one of the following alternatives applies:

Case 1. σ' extends σ. Then $d((a, \sigma'), (b, \sigma)) \geqslant q(b, a) > 0$ by (C).

Case 2. There is a first element $(c, d) \in U$ in σ where σ differs from σ'. Let then $(a, \sigma') = (x_0, \sigma_0), \ldots, (x_n, \sigma_n) = (b, \sigma)$ be such that $\hat{d}$ is defined for each pair of successive elements. Then there is a $0 \leqslant i \leqslant n-1$ such that $\sigma_{i+1} = \sigma_i{}^\frown(c, d)$ and this means $\hat{d}((x_i, \sigma_i), (x_{i+1}, \sigma_{i+1})) = q(c, d)$. It follows that $d((a, \sigma'), (b, \sigma)) \geqslant q(c, d) > 0$.

We conclude that $d((a, \sigma'), (b, \sigma))$ is bounded from below by finitely many non-zero values, and the assertion follows. □

We conclude that any quasimetric space is of the form indicated in Lemma 2.5.

For the following lemma, recall that a topology is called discrete if any singleton and consequently any subset is open.

Lemma 3.4. *Let* (X, q) *be a quasimetric space. Let* $\iota\colon X \to \mathcal{P}(Y)$ *be the representation of* (X, q) *by means of the metric space* (Y, d) *according to Theorem* 3.2. *If the topology of* X *is discrete, then so is the topology of* Y.

Proof. We refer again to the proof of Theorem 3.2. Assume that X has the discrete topology; then $\{a\}$ is open and hence there is an $\varepsilon > 0$ such that $q(a, b) \geqslant \varepsilon$ for any $b \neq a$. Let $\sigma \in U^{<\omega}$. We have to show that $\{(a, \sigma)\}$ is open in Y as well.

If σ is of length $\geqslant 1$, let $(c, d) \in U$ be the last element of σ and choose $0 < \delta < \varepsilon \wedge q(c, d)$. If $\sigma = \varnothing$, choose $0 < \delta < \varepsilon$. We claim that the δ-neighbourhood of (a, σ) is a singleton.

Let $(b, \sigma') \in Y$ be distinct from (a, σ) and such that $\hat{d}((a, \sigma), (b, \sigma'))$ is defined. There are the following alternatives:

Case 1. $\sigma = \sigma'$. Then $a \neq b$ and hence $\hat{d}((a,\sigma),(b,\sigma')) = \max\{q(a,b),\, q(b,a)\} \geqslant q(a,b) \geqslant \varepsilon > \delta$.

Case 2. $\sigma' = \sigma^\frown(a,b)$. Then $\hat{d}((a,\sigma),(b,\sigma')) = q(a,b) \geqslant \varepsilon > \delta$.

Case 3. $\sigma = \sigma'^\frown(c,d)$. Note that then $(c,d) = (b,a)$. We have $\hat{d}((a,\sigma),(b,\sigma')) = q(b,a) = q(c,d) > \delta$.

The assertion follows. □

We finally show that the Hausdorff quasidistance between subsets of the represented quasimetric space is preserved.

Lemma 3.5. *Let (X,q) be a quasimetric space. Let $\iota\colon X \to \mathcal{P}(Y)$ be the representation of (X,q) by means of the metric space (Y,d) according to Theorem 3.2. Let us denote the extension of ι to subsets of X again by ι. Then*

$$e_q(A,B) \;=\; e_d(\iota(A),\iota(B)) \tag{5}$$

holds for any $A,B \subseteq X$.

Proof. The assertion is clear from Lemma 2.4(v) if A or B is empty. Let us assume that A and B are non-empty.

Let $a \in X$ and $B \subseteq X$. We will show

$$p_q(a,B) \;=\; e_d(\iota(a),\iota(B)); \tag{6}$$

then (5) will follow by Lemma 2.4(ii).

By definition,

$$p_q(a,B) \;=\; \inf_{b\in B} q(a,b)$$

and

$$e_d(\iota(a),\iota(B)) \;=\; \sup_{x\in\iota(a)} \inf_{y\in\iota(B)} d(x,y).$$

To see that these two values coincides, we make use of the particular definition of the metric d from the proof of Theorem 3.2. Let $a,b \in X$ and $\sigma \in U^{<\omega}$. If $q(a,b) < q(b,a)$, we have $(a,b) \in U$ and $d((a,\sigma),(b,\sigma^\frown(a,b))) = q(a,b)$. If $q(b,a) \leqslant q(a,b)$, we have $d((a,\sigma),(b,\sigma)) = \max\{q(a,b),\, q(b,a)\} = q(a,b)$. We conclude $\inf_{y\in\iota(b)} d(x,y) \leqslant q(a,b)$ for any $x \in \iota(a)$ and hence

$$\inf_{y\in\iota(B)} d(x,y) = \inf_{b\in B} \inf_{y\in\iota(b)} d(x,y) \leqslant \inf_{b\in B} q(a,b) = p_q(a,B).$$

We conclude $e_d(\iota(a),\iota(B)) \leqslant p_q(a,B)$.

To see the converse inequality, let again $a, b \in X$. We have $d((a, \varnothing), (b, \sigma)) \geqslant q(a, b)$ for any $\sigma \in U^{<\omega}$. Moreover, if $q(a, b) < q(b, a)$, then $(a, b) \in U$, and $d((a, \varnothing), (b, (a, b))) = q(a, b)$. If $q(b, a) \leqslant q(a, b)$, then $d((a, \varnothing), (b, \varnothing)) = \max\{q(a, b),\ q(b, a)\} = q(a, b)$. We conclude that $\inf_{y \in \iota(b)} d((a, \varnothing), y) = q(a, b)$. Hence

$$\begin{aligned} e_d(\iota(a), \iota(B)) &\geqslant \inf_{y \in \iota(B)} d((a, \varnothing), y) \\ &= \inf_{b \in B} \inf_{y \in \iota(b)} d((a, \varnothing), y) \;=\; \inf_{b \in B} q(a, b) = p_q(a, B), \end{aligned}$$

and (6) is shown. □

Our results are compiled in the following theorem.

Theorem 3.6. *Let (X, q) be a quasimetric space. Then there is a metric space (Y, d) and a mapping $\iota \colon X \to \mathcal{P}(Y)$ such that $\{\iota(a) \colon a \in X\}$ is a partition of Y into closed subsets and $e_q(A, B) = e_d(\iota(A), \iota(B))$ holds for any $A, B \subseteq X$.*

4 The Logic of Approximate Entailment

We now turn to a context in which Theorem 3.6 has an immediate consequence: to logics that allow reasoning in an approximate manner.

Following the lines of Ruspini in [Rus], we deal with the implication between properties up to a certain degree of tolerance. To this end, properties are modelled by subsets of a metric space (X, d), and a property modelled by $A \subseteq X$ is considered to imply a property $B \subseteq X$ to the degree d if $A \subseteq U_d(B)$, where $U_d(\cdot)$ denotes the d-neighbourhood.

We note that the framework proposed in [Rus] allows some additional flexibility. Instead of a metric, the central notion is a $\odot$-similarity function, where $\odot$ is a fixed t-norm [KMP]. A $\odot$-similarity function is a mapping $s \colon X \times X \to [0, 1]$ such that, for any $a, b, c \in X$, (i) $s(a, b) = 1$ iff $a = b$, (ii) $s(a, c) \leqslant s(a, b) \odot s(b, c)$, and (iii) $s(a, b) = s(b, a)$. Under the correspondence $[0, 1] \to \bar{\mathbb{R}}^+, \quad s \mapsto \begin{cases} -\ln s & \text{if } s > 0, \\ \infty & \text{if } s = 0 \end{cases}$ we see that a metric can be regarded as the same as a $\odot$-similarity if $\odot$ is the product t-norm.

In order to realise Ruspini's concept, a number of logics have been proposed in the literature [DPEGG, EGGR, Rod, GoRo, EGRV]. For instance, logics endowed with a set of modal operators were studied, indexed by the elements of the real unit interval and interpreted by the neighbourhood in similarity spaces. Furthermore, a formalism using a more restricted language and getting along without modalities was introduced in [Rod]. It is based on a graded version of the implication and it is the calculus to be considered in the present paper.

We define the *Logic of Approximate Entailment*, or LAE for short, as follows. The language of LAE includes *variables* $\varphi_0, \varphi_1, \ldots$ and the two constants $\bot$ (false), and $\top$ (true). A *Boolean formula* is built up from the variables and the constants by means of the operations $\wedge$ (and), $\vee$ (or), and $\neg$ (not). Furthermore, a *graded implication*, or an *implication* for short, is a triple consisting of two Boolean formulas α and β as well as a number $d \in \mathbb{R}^+$, denoted

$$\alpha \stackrel{d}{\rightarrow} \beta.$$

A *model* for LAE is a metric space (X, d). An *evaluation* in a model X is a mapping v from the Boolean formulas to $\mathcal{P}(X)$ such that $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta),\ v(\alpha \vee \beta) = v(\alpha) \cup v(\beta),\ v(\neg\alpha) = \complement v(\alpha),\ v(\bot) = \varnothing$, and $v(\top) = X$. We then say that v *satisfies* an implication $\alpha \stackrel{d}{\rightarrow} \beta$ if

$$e_d(v(\alpha), v(\beta)) \leqslant d.$$

A theory is a set of implications. We say that a theory $\mathcal{T}$ *semantically entails* an implication Φ in LAE if any evaluation v in some model X that satisfies all elements of $\mathcal{T}$ also satisfies Φ.

We note that also modified versions of LAE have been considered. The present definition has been chosen in best possible accordance with the framework originally proposed in [Rus].

An axiomatisation of LAE is a non-trivial issue. To some extent, the mentioned modifications are a consequence of these difficulties. In [Rod], for instance, the language is assumed to contain a fixed finite set of variables. This restriction was discarded in [Vet]; however, the assumption of symmetry of the distance function was dropped in turn.

Let us recall the calculus proposed in [Vet]. This calculus is quite similar to the axiomatic system that has been shown to be sound and complete for the finitary version of LAE [Rod, GoRo, EGRV]. A difference, however, is that the syntactical features relying on a fixed finite number of variables are not available here.

Definition 4.1. **LAE** consists of the following axiom and rules, where α, β, γ are any Boolean formulas and $c, d \in \mathbb{R}^+$:

(R1) $\alpha \stackrel{0}{\rightarrow} \beta$ if $\alpha \rightarrow \beta$ is a tautology of CPL (R2) $\dfrac{\alpha \stackrel{0}{\rightarrow} \beta}{\alpha \wedge \gamma \stackrel{0}{\rightarrow} \beta \wedge \gamma}$

$$\text{(R3)} \quad \frac{\alpha \stackrel{c}{\to} \beta}{\alpha \stackrel{d}{\to} \beta} \text{ where } d \geqslant c \qquad \text{(R4)} \quad \frac{\alpha \stackrel{c}{\to} \bot}{\alpha \stackrel{0}{\to} \bot}$$

$$\text{(R5)} \quad \frac{\alpha \stackrel{c}{\to} \gamma \quad \beta \stackrel{c}{\to} \gamma}{\alpha \vee \beta \stackrel{c}{\to} \gamma} \qquad \text{(R6)} \quad \frac{\alpha \stackrel{c}{\to} \beta \quad \beta \stackrel{d}{\to} \gamma}{\alpha \stackrel{c+d}{\to} \gamma}$$

Let $\mathcal{T}$ be a theory and Φ be an implication. A *proof* of Φ from $\mathcal{T}$ in $\mathbf{LAE}$ is defined in the expected way.

We have shown in [Vet] that $\mathbf{LAE}$ is sound and complete for an entailment relation that is closely related to LAE. Namely, we have considered in [Vet] the logic LAE^q. This logic is defined similarly to LAE. The syntax of LAE^q is actually defined in the same way as for LAE. The semantic entailment relation, however, is modified as follows. A model for LAE^q is any *quasi*metric space (X, q) and an evaluation v satisfies an implication $\alpha \stackrel{d}{\to} \beta$ if $e_q(v(\alpha), v(\beta)) \leqslant d$.

Theorem 4.2. *Let $\mathcal{T}$ be a finite theory and Φ an implication. Then $\mathcal{T}$ proves Φ in $\mathbf{LAE}$ if and only if $\mathcal{T}$ semantically entails Φ in* LAE^q.

We make now use of the results of Section 3 to show that $\mathbf{LAE}$ is in fact also complete with respect to metric spaces.

Theorem 4.3. *Let $\mathcal{T}$ be a finite theory and Φ an implication. Then $\mathcal{T}$ proves Φ in $\mathbf{LAE}$ if and only if $\mathcal{T}$ semantically entails Φ in* LAE.

Proof. The soundness part holds by Theorem 4.2. To see completeness, assume that $\mathcal{T}$ does not prove Φ in $\mathbf{LAE}$. By Theorem 4.2, there exists a quasimetric space (X, q) and an evaluation e in (X, q) such that e satisfies all elements of $\mathcal{T}$ but not Φ.

Let $\iota\colon X \to \mathcal{P}(Y)$ be the representation of (X, q) by means of a metric space (Y, d) according to Theorem 3.6. Then the natural extension of ι to $\mathcal{P}(X)$ preserves the set-theoretic operations as well as the Hausdorff quasidistance. The assertion follows. $\square$

We conclude this section with some additional notes. We first remark that the metric spaces used in the present context are of a particular nature. This observation can be seen as a consequence of the fact that Theorem 4.3 is restricted to finite theories.

Remark 4.4. *We have shown in* [Vet] *the following stronger version of the completeness Theorem* 4.2*: Assume that the finite theory $\mathcal{T}$ does not prove the implication Φ in $\mathbf{LAE}$. Then there exists an evaluation in a quasimetric space (X, q)*

that satisfies all elements of $\mathcal{T}$ but not Φ. Moreover, the range of $q\colon X \times X \to \bar{\mathbb{R}}^+$ contains a smallest non-zero element.

As a consequence of the last mentioned fact, the topology on (X, q) is discrete. By Lemma 3.4*, so is the topology of the metric space (Y, d) representing (X, q). In other words, Theorem* 4.3 *remains valid when restricting the models to metric spaces whose induced topology is discrete.*

We finally raise the question how our completeness theorem is related to those presented in previous works.

Remark 4.5. *As mentioned above, several completeness results have been established for a finitary version of* LAE [Rod, GoRo, EGRV]. *In that case, the language is assumed to contain a fixed finite set of variables and certain axioms rely on maximal elementary conjunctions, or m.e.c. for short. An m.e.c. is a conjunction of literals such that all variables occur exactly once.*

We may say that $\mathbf{LAE}$ *arises from the axiomatic system proposed, e.g., in* [EGRV] *essentially by dropping the axioms containing m.e.c.s. Hence it is natural to ask whether, in some sense, the latter are provable in* $\mathbf{LAE}$*. This is, however, not the case.*

For instance, the axiom (A5) *from* [EGRV, Def. 4.2] *expresses the symmetry of the distance between the subsets interpreting two m.e.c.s. The axiom is sound because the m.e.c.s are assumed to be interpreted by sets containing at most one element. In the present context, in contrast, the conjunction of finitely many literals can be interpreted by a set of an arbitrary size; hence symmetry cannot be expected here.*

It follows in particular that the completeness theorems obtained in the mentioned previous works are not special cases of the results presented here.

5 Conclusion

Quasimetrics have often be considered from the topological angle: under which conditions is a topology quasimetrisable? Assuming the more general version of the separation axiom, P. Vitolo has shown that this is the case if and only if the space is embeddable into a hyperspace of closed sets endowed with the Hausdorff quasidistance [Vit]. Quasimetrics moreover play a role for logical calculi generalising classical propositional logic, for instance in fuzzy logic [Ger] or in approximate reasoning [Rus, Rod]. The present paper is devoted to the last mentioned aspect.

We have presented a representation theorem for quasimetric spaces that is, in a sense, analogous to Vitolo's. However, we have dealt with quasimetric spaces in a narrower sense and could then enhance the representation by additional features. In this way, an open issue concerning the Logic of Approximate Entailment could be solved. For this logic, defined closely to its original formulation by Ruspini, a sound and complete axiomatisation had not been known. Based on our result on quasimetric spaces, we were able to fill this gap.

A further progress in the field of approximate reasoning remains certainly desirable in several respects. For instance, our completeness theorem is for finite theories only; the question is open if the finiteness condition could be dropped. To this end, the complexity of the procedure of proving completeness, contained in our previous paper [Vet] as well as in the present one, might be worthy of consideration. A non-trivial amount of partly technically demanding auxiliary facts were required and it is open if there is an alternative, more direct way. Furthermore, we have considered by now Rodríguez's Logic of Approximate Entailment only. It might be interesting to apply the same methods to related logics. Fuzzy logics or logics aiming at the formalisation of vague statements might well be considered in a framework similar to the one considered in this paper and might offer new perspectives both in formal and informal respects.

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References

[AMO] R. K. Ahuja, Th. L. Magnanti, J. B. Orlin, "Network Flows: Theory, Algorithms, and Applications", Prentice Hall, Upper Saddle River 1993.

[Dic] A. Di Concilio, Point-free geometries: proximities and quasi-metrics, *Math. Comput. Sci.* **7** (2013), 31 - 42.

[DiGe] A. Di Concilio, G. Gerla, Quasi-metric spaces and point-free geometry, *Math. Struct. Comput. Sci.* **16** (2006), 115 - 137.

[DPEGG] D. Dubois, H. Prade, F. Esteva, P. Garcia, L. Godo, A logical approach to interpolation based on similarity relations, *Int. J. Approx. Reasoning* **17** (1997), 1 - 36.

[HPS] G. T. Heineman, Gary Pollice, Stanley Selkow, "Algorithms in a Nutshell", Chapter 8: Network Flow Algorithms, O'Reilly, Sebastopol 2009.

[KMP] E. P. Klement, R. Mesiar, E. Pap, "Triangular Norms", Kluwer Academic Publishers, Dordrecht 2000

[EGGR] F. Esteva, P. Garcia, L. Godo, R. O. Rodríguez, A modal account of similarity-based reasoning, *Int. J. Approx. Reasoning* **16** (1997), 235 - 260.

[EGRV] F. Esteva, L. Godo, R. O. Rodríguez, T. Vetterlein, Logics for approximate and strong entailment, *Fuzzy Sets Syst.* **197** (2012), 59 - 70.

[Ger] G. Gerla, Representation theorems for fuzzy orders and quasi-metrics, *Soft Comput.* **8** (2004), 571 - 580.

[GoRo] L. Godo, R. O. Rodríguez, Logical approaches to fuzzy similarity-based reasoning: an overview, in: G. Della Riccia, D. Dubois, H.-J. Lenz, R. Kruse (Eds.), "Preferences and similarities", Springer-Verlag, Berlin 2008; pp. 75 - 128.

[Rod] R. O. Rodríguez, "Aspectos formales en el Razonamiento basado en Relaciones de Similitud Borrosas", PhD thesis, Technical University of Catalonia (UPC), 2002.

[Rus] E. H. Ruspini, On the semantics of fuzzy logic, *Int. J. Approx. Reasoning* **5** (1991), 45 - 88.

[Vet] T. Vetterlein, Logic of approximate entailment in quasimetric spaces, *Int. J. Approx. Reasoning* **64** (2015), 39 - 53.

[Vit] P. Vitolo, A representation theorem for quasi-metric spaces, *Topology Appl.* **65** (1995), 101 - 104.

[Wil] W. A. Wilson, On quasi-metric spaces, *Am. J. Math.* **53** (1931), 675 - 684.