Linear orthosets and orthogeometries

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Abstract

Anisotropic Hermitian spaces can be characterised as anisotropic orthogeometries, that is, as projective spaces that are additionally endowed with a suitable orthogonality relation. But linear dependence is uniquely determined by the orthogonality relation and hence it makes sense to investigate solely the latter. It turns out that by means of orthosets, which are structures based on a symmetric, irreflexive binary relation, we can achieve a quite compact description of the inner-product spaces under consideration. In particular, Pasch's axiom, or any of its variants, is no longer needed.

Having established the correspondence between anisotropic Hermitian spaces on the one hand and so-called linear orthosets on the other hand, we moreover consider the respective symmetries. We present a version of Wigner's Theorem adapted to the present context.

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1 Introduction

The characterisation of linear spaces by means of the linear dependence relation is a topic whose roots go far into the past. As the first milestone, we might consider von Staudt's contributions to what he called "Geometrie der Lage", published in 1856–1857 [Sta]. In 1908, Veblen and Young laid the basis for the coordinatisation theorem of projective geometry as we know it today [VeYo]. Presumably much later inner-product spaces were studied in this context. In addition to the ternary linear dependence relation, the binary relation of orthogonality was considered. It is in the seminal paper [BiNe] of Birkhoff and von Neumann where we find the key fact needed to understand the mutual relationship between orthogonality and an inner product.

In the present paper we are interested in linear spaces endowed with a sesquilinear form that is symmetric and anisotropic. That is, we focus on anisotropic Hermitian spaces, in the sequel simply referred to as Hermitian spaces. By an *orthogeometry*, we mean a projective space endowed with an orthogonality relation $\perp$ such that $\perp$ is symmetric and irreflexive, any point orthogonal to two points is orthogonal to the whole line spanned by them, and any point on a line is orthogonal to another point on that line. It turns out that orthogeometries coincide, up to isomorphism, with those arising from Hermitian spaces [FaFr].

To formulate this result, however, it seems to be unnecessary to make use of the linear dependence relation. Indeed, the linear span $\langle u, v \rangle$ of two vectors u and v of a Hermitian space coincides with $\{u, v\}^{\perp\perp}$ and is thus expressible by the orthogonality relation. Here, for any set S of vectors, $S^{\perp\perp}$ is the subspace consisting of those vectors that are orthogonal to all vectors orthogonal to every member of S. At least in principle, it is thus immediate that Hermitian spaces can be characterised solely by means of the orthogonality relation. It is less evident that this can be done in such a way that the conditions on the linear dependence relation are largely omitted. In the language of projective geometry, we may say that our conditions do not involve planes but only lines. In particular, we can dispense with Pasch's axiom, which we consider as the condition hardest to motivate among the axioms of projective geometry.

To provide a concise characterisation of the orthogonality relation $\perp$ on a Hermitian space is the primary purpose of this paper. Our starting point is the notion of an *orthogonality space*, or an *orthoset* for short: a set X endowed with a symmetric and irreflexive binary relation $\perp$. Our focus is on the following combinatorial principle. We call an orthoset *linear* if, for any distinct elements $e, f \in X$, there is a third one $g \in X$ such that the orthogonal complements of $\{e, f\}$ and $\{e, g\}$ coincide and exactly one of f and g is orthogonal to e. This is all we need; we will establish the mutual correspondence between linear orthosets and Hermitian spaces. As usual we have to exclude the case of low ranks; the theory works only under the assumption that the orthosets contain at least four mutually orthogonal elements.

The idea to consider orthosets in the present context goes back to David Foulis and his collaborators; see, e.g., [Dac, Wlc]. The collection of one-dimensional subspaces of a Hilbert space has the natural structure of an orthoset and in principle, this is for us the guiding example as well. The question how to characterise (generalised or classical) Hilbert spaces as orthosets was dealt with, e.g., in [Mac, HePu, Bru, Rum, Vet1, Vet2]. Here, we address a more general context. Indeed, the present work might be understood as a "local" study of inner-product spaces. Roughly speaking, we focus on properties that pairs of vectors may have and neglect properties that involve subspaces of possibly infinite dimensions. In this context it is unlikely, say, to ensure the orthomodularity of the subspace lattice, which would give us the chance to apply, e.g., Wilbur's theory [Wlb] or Solèr's Theorem [Sol].

We already established in [Vet1] that linear orthosets of finite rank are representable by Hermitian spaces. To explain the difficulties that we had to handle in the case of infinite rank we have to refer to the lattice-theoretic results that we relied on. Any orthoset gives rise in a canonical manner to a complete ortholattice; under a mild assumption, the latter contains the former as its atom space. A lattice-theoretic characterisation of Hermitian spaces is in turn well-known; provided that the dimension is at least 4, the set of orthoclosed subspaces can be described as a complete, irreducible, atomistic ortholattice with the covering property [MaMa, Section 34]. It is not difficult to verify that the linearity of an orthoset implies the covering property of the corresponding ortholattice if the rank is finite [Vet1]. In case of an infinite rank, however, the argument is no longer applicable. In [Vet2], we provided an alternative version of the representation theorem for Hermitian spaces, with the effect that the conditions imposed on a complete atomistic ortholattice involve finite elements only. However, one of the conditions from [Vet2] turned out to be redundant and as a consequence, a formulation is possible involving elements of height 2 only. Consequently, we are in the position to describe Hermitian spaces without restriction of the dimension by orthosets in the mentioned simple way.

It would certainly be desirable to extend the correspondence between linear orthosets and Hermitian spaces to include structure-preserving maps. For a suitable categorical framework, we refer the reader to [PaVe1, PaVe2]. Here, we restrict to the "less involved" part of this theory: we discuss the correlation between the automorphisms of both kinds of structures. Numerous results on the symmetries of projective Hermitian spaces can be found in the literature and they are commonly referred to as "Wigner's Theorem". In the present context, two versions are relevant. Piron has dealt with the symmetries of the ortholattice of closed subspaces of an orthomodular space [Pir]. Uhlhorn has dealt with the symmetries of the orthoset of one-dimensional subspaces of a classical Hilbert space [Uhl]. Dealing with the one-dimensional subspaces of a Hermitian space, we establish the natural generalisation of both variants. Although no new methods of proof are needed, we feel that we fill a gap and we provide the necessary arguments in a concise form.

The paper is structured as follows. Notation, terminology, and basic facts are compiled in the subsequent Section 2. Section 3 is devoted to the improved version of a lattice-theoretic characterisation of Hermitian spaces. Applying this result, we are led in Section 4 to a correspondence between Hermitian spaces on the one hand and linear orthosets on the other hand. Section 5 contains the version of Wigner's Theorem relevant in the present context.

2 Hermitian spaces, orthosets, and orthogeometries

We start by fixing the basic terminology and providing some background information.

A $\star$-*sfield* is meant be an sfield (i.e., a division ring) equipped with an involutory antiautomorphism. Let H be a (left) linear space over the $\star$-sfield F with the antiautomorphism *. A *Hermitian form* on H is a symmetric sesquilinear form, that is, a map $(\cdot,\cdot) : H \times H \to F$ such that, for any $u, v, w \in H$ and $\alpha, \beta \in F$,

$$\begin{aligned} (\alpha u + \beta v, w) &= \alpha\,(u, w) + \beta\,(v, w)\,, \\ (w, \alpha u + \beta v) &= (w, u)\,\alpha^\star + (w, v)\,\beta^\star, \\ (u, v) &= (v, u)^\star\,. \end{aligned}$$

We additionally assume in this paper Hermitian forms to be *anisotropic*, meaning that $(u, u) = 0$ holds only in case $u = 0$. Endowed with an (anisotropic) Hermitian form, H becomes a *Hermitian space.*

For a linear space V, we put $V^{\bullet} = V \setminus \{0\}$. For vectors $u_1, \ldots, u_k \in V^{\bullet}$, we denote by $\langle u_1, \ldots, u_k\rangle$ the subspace spanned by $u_1, \ldots, u_k$. We define $P(V)$ to be the collection of one-dimensional subspaces of V, that is,

$$P(V) = \{\langle u\rangle : u \in V^{\bullet}\}.$$

Our primary issue is the reduction of a Hermitian space H to an algebraic or a relational structure. In order to do so, it seems to make sense to proceed in two steps: first to consider the linear dependence relation, then the inner product. We can deal with both aspects by endowing $P(H)$ with a suitable structure. Let us review the basic facts.

Definition 2.1 *An* (irreducible) projective space *is a non-empty set P together with a map $\star \colon P \times P \to \mathcal{P}(P)$ such that, for any $e, f, g, h \in P$, the following conditions hold.*

(P1) *$e, f \in e \star f$, and $e \star f$ contains an element distinct from e and f if and only if $e \neq f$.*
(P2) *If $g, h \in e \star f$ and $g \neq h$, then $g \star h = e \star f$.*
(P3) *Let $e \neq g$ and $f \neq h$. Then $e \star f \cap g \star h \neq \varnothing$ implies $e \star g \cap f \star h \neq \varnothing$.*

For a subset B of a projective space P, we denote by $\langle B\rangle$ the smallest subset of P containing B and such that $e, f \in \langle B\rangle$ implies $e \star f \subseteq \langle B\rangle$. The rank *of P is the minimal cardinality of the subsets $B \subseteq P$ such that $\langle B\rangle = P$.*

The elements of a projective space are commonly called *points*; $\langle e, f\rangle = e \star f$ is called the *line* spanned by the distinct points e and f; and $\langle e, f, g\rangle$ is called a *plane*, provided that the points e, f, g are not contained in a line. By (P1), each line contains at least three points, and by (P2), any two distinct points belong to exactly one line. (P3), also called Pasch's axiom, might be found geometrically appealing – "two lines in a plane have a non-empty intersection" –, but has otherwise no straightforward interpretation. We note, however, that we may extend $\star$ elementwise to an operation on subsets and replace (P3) with the condition that $\star$ is associative.

A subset B of a projective space P is called *independent* if $e \notin \langle B \setminus \{e\}\rangle$ for any $e \in B$. A *basis* of P is an independent set $B \subseteq P$ such that $\langle B\rangle = P$. If P is of finite rank n, then any basis consists of n points.

Provided that we deal with dimensions $\geqslant 4$, projective spaces provide the possibility of describing linear spaces. We cite the classical coordinatisation theorem; see, e.g., [FaFr, Theorem 9.2.6].

Theorem 2.2 *Let V be a linear space. Then $P(V)$, together with the map $\star$ given by*

$$\langle u\rangle \star \langle v\rangle \;=\; \{\langle w\rangle \colon w \in \langle u, v\rangle^{\bullet}\},$$

is a projective space. Up to isomorphism, any projective space of rank $\geqslant 4$ arises in this way.

Next, let H be a Hermitian space. We call vectors $u, v \in H^{\bullet}$ *orthogonal* if $(u, v) = 0$; we write $u \perp v$ in this case. The binary relation $\perp$ is compatible with the transition from H to $P(H)$ and we arrive at the following abstraction from H.

Definition 2.3 *An* orthoset *(or* orthogonality space*) is a non-empty set X equipped with a symmetric, irreflexive binary relation $\perp$, called the* orthogonality relation*.*

A subset of an orthoset X consisting of mutually orthogonal elements is called a $\perp$-set*. The supremum of the cardinalities of $\perp$-sets is called the* rank *of $(X, \perp)$.*

By the symmetry and anisotropy of the Hermitian form, it is immediate that any Hermitian space H gives rise to the orthoset $(P(H), \perp)$. Further properties of the orthogonality relation come along with the notion of an orthogeometry, which is discussed in Faure and Frölicher's monograph [FaFr, Section 14.1].

Definition 2.4 *Let P be a projective space and let $\perp$ be a binary relation making P into an orthoset. Then we call $(P, \perp)$ an* orthogeometry *provided that the following conditions hold:*

(O1) *If* $e \perp f, g$, *then* $e \perp h$ *for any* $h \in f \star g$.
(O2) *If* $e \neq f$, *there is a* $g \in e \star f$ *such that* $g \perp e$.

We note that our definition differs from the one given in [FaFr]; here, we assume orthogeometries to be anisotropic. The subsequent lemma shows that an orthogeometry understood according to Definition 2.4 is actually the same as an anisotropic orthogeometry in the sense of Faure and Frölicher.

Lemma 2.5 *Let* $(P, \perp)$ *be an orthogeometry. Then, for any* e, f, h *such that* $e \neq f$, *there is a* $g \in e \star f$ *such that* $h \perp g$.

Proof Let e and f be distinct points and h any further point. If $h \in e \star f$, the assertion is clear from (P2) and (O2). Assume that $h \notin e \star f$. By (O2), there is an $e' \perp h$ such that $h \star e = h \star e'$ and, similarly, there is a $f' \perp h$ such that $h \star f = h \star f'$. Then $e \star f$ and $e' \star f'$ are lines contained in the plane $\langle e, f, h \rangle$ and hence possess a common point g. By (O1), g fulfils the requirements. $\square$

By the rank of an orthogeometry $(P, \perp)$ we mean the rank of $(P, \perp)$ as an orthoset. If finite, this value coincides with the rank of P as a projective space.

Lemma 2.6 *Let* $(P, \perp)$ *be an orthogeometry. Then the rank of (the orthoset)* $(P, \perp)$ *is finite if and only if so is the rank of* P *as a projective space, and in this case the two values coincide.*

Proof Let m be the rank of the orthoset and let n be the rank of the projective space.

Assume first that m is finite. Then there is a maximal $\perp$-set $B = \{e_1, \ldots, e_m\}$. We claim that B is a basis of P, implying that $m = n$. Indeed, by (O1), B is independent. Assume that there is an $f \notin \langle B \rangle$. By (O2), there is an $f_1 \in e_1 \star f$ such that $f_1 \perp e_1$, and by (P2) we have $e_1 \star f = e_1 \star f_1$. Moreover, $f_1 \neq e_2$ because otherwise $f \in e_1 \star e_2$ in contradiction to our assumption. Hence we may argue again that there is an $f_2 \perp e_2$ such that $e_2 \star f_1 = e_2 \star f_2$. Note that then $\langle e_1, e_2, f \rangle = \langle e_1, e_2, f_2 \rangle$ and $f_2 \perp e_1, e_2$. Applying the same argument repeatedly, we eventually conclude that there is an $f_m \perp B$, contradicting the maximality of the $\perp$-set B. It follows $\langle B \rangle = P$, that is, B is a basis of P and consequently $m = n$.

Assume second that m is infinite. Then P contains an infinite independent set, hence n is infinite as well. $\square$

Orthogeometries characterise Hermitian spaces of dimension $\geqslant 4$. In the finite-dimensional case, a related statement was shown in [BiNe], cf. also [MaMa, (34.3)]. For the general case, we refer to [FaFr, Section 14.1].

Theorem 2.7 *Let* H *be a Hermitian space. Then* $(P(H), \perp)$ *is an orthogeometry. Up to isomorphism, any orthogeometry of rank* $\geqslant 4$ *arises in this way.*

Proof $P(H)$ is a projective space by Theorem 2.2, and (O1) obviously holds. Moreover, for linearly independent vectors $u, v \in H^{\cdot}$, we have $v - (v,u)\,(u,u)^{-1}u \perp u$ and we conclude that also (O2) is fulfilled.

For the converse assertion, let $(P, \perp)$ be an orthogeometry of rank $\geqslant 4$. Then, by Theorem 2.2, there is a linear space H such that the projective spaces P and $P(H)$ are isomorphic. By Lemma 2.6, the projective space $P(H)$ has rank $\geqslant 4$ and consequently H has dimension $\geqslant 4$. By [FaFr, Theorem 14.1.8], there is a Hermitian form on H inducing the orthogonality relation on $P(H)$ imparted by P. □

3 Lattice-theoretic description of Hermitian spaces

The aim of this section is to establish a lattice-theoretic characterisation of Hermitian spaces. Provided that the dimension is at least 4, Hermitian spaces correspond to the complete, irreducible atomistic ortholattices with the covering property [MaMa, Theorems (34.2), (34.5)]. Our aim is to replace the lattice-theoretic conditions by weaker ones. More specifically, the covering property involves all elements of the lattice including the non-finite ones and we wish to replace it with a condition that involves finite elements only. We have shown in [Vet2] that this is possible and we will present here an improved version. Our proof will moreover be comparatively short, this time not following the lines of [MaMa] but being based on Theorem 2.7.

We recall that an *ortholattice* is a bounded lattice equipped with an order-reversing involution $^{\perp}$ such that, for any a, $a^{\perp}$ is a complement of a. Elements a and b of an ortholattice are called *orthogonal* if $a \leqslant b^{\perp}$; we write $a \perp b$ in this case.

Furthermore, a lattice with 0 is *atomistic* if every element is a join of atoms, and the joins of finitely many atoms (including 0) are called the *finite* elements.

For an atomistic ortholattice, we shall consider the following condition:

(L) For any distinct atoms p and q, there is an atom $r \perp p$ such that $p \vee q = p \vee r$.

Let us compile the consequences of condition (L).

Lemma 3.1 *Let L be an atomistic ortholattice fulfilling* (L)*. Let a be a finite element of L and let p be an atom of L not below a. Then there is an atom $q \perp a$ such that $a \vee p = a \vee q$.*

Proof The assertion follows from the following fact, cf. [Vet2]:

($\star$) Let $r_1, \ldots, r_k$, where $k \geqslant 1$, be pairwise orthogonal atoms and let p be an atom not below $r_1 \vee \ldots \vee r_k$. Then there is an atom $q \perp r_1, \ldots, r_k$ such that $r_1 \vee \ldots \vee r_k \vee p = r_1 \vee \ldots \vee r_k \vee q$.

To see ($\star$), we construct successively a sequence $q_0, \ldots, q_k$ of atoms such that, for any $i \geqslant 1$, $q_i \perp r_1, \ldots, r_i$ and $r_1 \vee \ldots \vee r_i \vee q_i = r_1 \vee \ldots \vee r_i \vee p$. Namely, set $q_0 = p$. Given q_{i-1}, we may by (L) choose $q_i \perp r_i$ such that $r_i \vee q_{i-1} = r_i \vee q_i$. □

Lemma 3.2 *Let L be an atomistic ortholattice fulfilling* (L). *Let a be finite and let p, q be atoms orthogonal to a. Then $a \vee p \leqslant a \vee q$ implies $p = q$.*

Proof Assume to the contrary that $a \vee p \leqslant a \vee q$ and $p \neq q$. By (L), there is an atom $p' \perp q$ such that $p \vee q = p' \vee q$. Then, on the one hand, $p' \leqslant p \vee q \perp a$, hence $p' \perp a \vee q$. On the other hand, $p' \leqslant a \vee p' \vee q = a \vee p \vee q = a \vee q$, a contradiction. □

Let L be an atomistic lattice. We write $a \lessdot b$ if a is covered by b. We recall that L is said to fulfil the *covering property* if, for any $a \in L$ and any atom $p \in L$ such that $p \nleqslant a$, we have $a \lessdot a \vee p$. If this is the case whenever a is finite, L is said to fulfil the *finite covering property*.

Note that if L has the finite covering property, it also has the *finite exchange property*. Indeed, if in this case a is a finite element of L, p and q are atoms, and $p \nleqslant a$ but $p \leqslant a \vee q$, then $a < a \vee p \leqslant a \vee q$ and thus $a \vee p = a \vee q$.

Lemma 3.3 *Let L be an atomistic ortholattice fulfilling* (L). *Then L fulfils the finite covering property.*

Proof Let $a \in L$ be a finite element and let p be an atom of L such that $p \nleqslant a$. Let $b \in L$ be such that $a < b \leqslant a \vee p$. Then there is an atom $q \leqslant b$ such that $q \nleqslant a$ and hence $a < a \vee q \leqslant a \vee p$.

By Lemma 3.1, there are $p', q' \perp a$ be such that $a \vee p = a \vee p'$ and $a \vee q = a \vee q'$. Moreover, by Lemma 3.2, $a \vee q' \leqslant a \vee p'$ implies $p' = q'$. It follows $a \vee p = a \vee q \leqslant b \leqslant a \vee p$, that is, $b = a \vee p$. □

We insert the following lattice-theoretic result, cf. [MaMa, (27.4)].

Lemma 3.4 *Let L be an atomistic ortholattice with the covering property. Then for any $a, b \in L$, $a \lessdot a \vee b$ implies $a \wedge b \lessdot b$.*

Proof Let $a, b \in L$ be such that $a \lessdot a \vee b$. Then $a^\perp \wedge b^\perp \lessdot a^\perp$ and hence there is an atom $p \perp a$ but $p \not\perp b$. We have $(a^\perp \wedge b^\perp) \vee p \leqslant a^\perp$ and by the covering property it in fact follows $(a^\perp \wedge b^\perp) \vee p = a^\perp$. Hence $a^\perp \vee b^\perp = b^\perp \vee p$. Again by the covering property, we conclude $b^\perp \lessdot a^\perp \vee b^\perp$, that is, $a \wedge b \lessdot b$. □

For an element d of an ortholattice L, let us consider the sublattice $\downarrow d = \{a \in L \colon a \leqslant d\}$. We endow $\downarrow d$ with the operation

$$^{\perp_d} \colon \downarrow d \to \downarrow d,\ a \mapsto a^\perp \wedge d.$$

Equipped with $^{\perp_d}$, $\downarrow d$ becomes an ortholattice if and only if $^{\perp_d}$ is involutory.

Lemma 3.5 *Let L be an atomistic ortholattice fulfilling* (L) *and let $d \in L$ be finite. Then the $\downarrow d$ is an atomistic ortholattice with the covering property.*

Proof Clearly, $\downarrow d$ is an atomistic lattice. By Lemma 3.3, $\downarrow d$ has the finite covering property. Consequently, by [MaMa, (8.8), (8.17)], $\downarrow d$ consists of finite elements only. In particular, $\downarrow d$ has the covering property.

To see that $\downarrow d$ is an ortholattice, we have to verify that ${}^{\perp_d}$ is involutory. For any $a \leqslant d$, we clearly have $a \leqslant a^{\perp_d \perp_d}$; let us assume $a < a^{\perp_d \perp_d}$. By Lemma 3.1, there is an atom $r \perp a$ such that $r \leqslant a^{\perp_d \perp_d} = (a \vee d^\perp) \wedge d$. But then $r \leqslant a^\perp \wedge d$ as well as $r \leqslant a \vee d^\perp$, a contradiction. □

An element z of an ortholattice L is called *central* if $x = (x \wedge z) \vee (x \wedge z^\perp)$ for any $x \in L$. In this case,

$$L \to \downarrow z \times \downarrow z^\perp, \; x \mapsto (x \wedge z, \, x \wedge z^\perp)$$

defines an isomorphism between L and the direct product of the ortholattices $\downarrow z$ and $\downarrow z^\perp$. We call L *irreducible* if L does not contain any central elements apart from 0 and 1.

Moreover, we denote by $\mathcal{A}(L)$ the collection of atoms of L, called the *atom space* of L.

We shall show that condition (L) implies a complete, atomistic ortholattice to be decomposable into irreducible components. The following easy observation will be useful.

Lemma 3.6 *Assume that, for any pair p, q of orthogonal atoms of an ortholattice, there is a third atom $r \leqslant p \vee q$. Then L is irreducible.*

Proof Let z be a central element distinct from 0 and 1. Let p, q be atoms such that $p \leqslant z$ and $q \leqslant z^\perp$. Assume that r is an atom such that $r \leqslant p \vee q$. Then $r = (r \wedge z) \vee (r \wedge z^\perp)$ and it follows that either $r \leqslant z$ or $r \leqslant z^\perp$. In the first case, we have $r = r \wedge z \leqslant (p \vee q) \wedge z = p$, that is, $r = p$; in the second case, we similarly see that $r = q$. Hence there is no third atom below $p \vee q$. □

Theorem 3.7 *Let L be an complete, atomistic ortholattice fulfilling* (L).

(i) *There are mutually orthogonal central elements z_ι, $\iota \in I$, such that, for each $\iota \in I$, $\downarrow z_\iota$ is an irreducible ortholattice and $L \to \Pi_{\iota \in I} \downarrow z_\iota$, $x \mapsto (x \wedge z_\iota)_\iota$ defines an isomorphism between L and the direct product of $\downarrow z_\iota$, $\iota \in I$.*
(ii) *L is irreducible if and only if, for any orthogonal atoms p and q, there is a third atom $r \leqslant p \vee q$.*

Proof For atoms $p, q \in L$, we define $p \sim q$ to hold if $p = q$ or otherwise there is a third atom $r \leqslant p \vee q$. We claim that $\sim$ is an equivalence relation on $\mathcal{A}(L)$. Clearly, $\sim$ is reflexive and symmetric. Let p, q, r be pairwise distinct atoms such that $p \sim q$ and $q \sim r$. We need to show that $p \sim r$. Recall that, by Lemma 3.5, $\downarrow(p \vee q \vee r)$ is an ortholattice with the covering property.

Let $s \neq p, q$ be such that $s \leqslant p \vee q$, and let $t \neq q, r$ be such that $t \leqslant q \vee r$. Assume first $q \leqslant s \vee t$. Then $s \neq t$ and we conclude $s \leqslant q \vee t$. Hence $p \leqslant s \vee q \leqslant q \vee t = q \vee r$ and thus $q \leqslant p \vee r$, showing that $p \sim r$. Assume second $q \nleqslant s \vee t$. Then $s \vee t \lessdot s \vee t \vee q =$

$(s \vee q) \vee (t \vee q) = (s \vee p) \vee (t \vee r) = (s \vee t) \vee (p \vee r)$. By Lemma 3.4 we conclude that $(s \vee t) \wedge (p \vee r) \lessdot p \vee r$. This implies that there is an atom u below both $s \vee t$ and $p \vee r$. Then $u \neq p$ because otherwise $q \leqslant p \vee q \vee t = p \vee s \vee t = s \vee t$, in contradiction to our assumption; and similarly we see that $u \neq r$. We conclude that $p \sim r$.

Let now $A_\iota \subseteq \mathcal{A}(L)$, $\iota \in I$, be the equivalence classes w.r.t. $\sim$, and put $z_\iota = \bigvee A_\iota$. By (L), $p \sim q$ holds for any distinct non-orthogonal atoms p and q. Hence the elements z_ι, $\iota \in I$, are pairwise orthogonal. Let $a \in L$. Then $a \wedge z_\iota = \bigvee\{p \in \mathcal{A}(L) \colon p \leqslant a, p \in A_\iota\}$ for each $\iota \in I$ and hence $a = \bigvee_\iota (a \wedge z_\iota)$. For each $\iota \in I$, we moreover have $z_\iota^\perp = \bigvee_{\kappa \neq \iota} z_\kappa$ and hence $a = (a \wedge z_\iota) \vee (a \wedge z_\iota^\perp)$. Thus z_ι is central and, by Lemma 3.6, $\downarrow z_\iota$ is irreducible. The assertions follow. □

We arrive at the main theorem of this section. We equip the atom space $\mathcal{A}(L)$ of an atomistic lattice L with the operation $\star$, defined for $p, q \in \mathcal{A}(L)$ by

$$p \star q \;=\; \{r \in \mathcal{A}(L) \colon r \leqslant p \vee q\}.$$

We furthermore equip $\mathcal{A}(L)$ with the orthogonality relation inherited from L.

Theorem 3.8 *Let L be a complete, irreducible, atomistic ortholattice fulfilling* (L). *Then $(\mathcal{A}(L), \perp)$ is an orthogeometry.*

Proof We have to verify that $\star$ makes $\mathcal{A}(L)$ into a projective space.

Ad (P1): Clearly, $p \in \mathcal{A}(L)$ is the only atom below $p \vee p = p$. Moreover, if $p, q \in \mathcal{A}(L)$ are distinct, then, by (L) and by Theorem 3.7(ii), there is an atom $r \neq p, q$ such that $r \leqslant p \vee q$.

Ad (P2): Let $p, q, r, s \in \mathcal{A}(L)$ such that $r, s \leqslant p \vee q$ and $r \neq s$. Then one of r or s, say r, is distinct from p. By Lemma 3.3, L has the finite exchange property and hence $p \vee q = p \vee r$. Similarly, it follows that $p \vee r = r \vee s$. That is, we have $r \vee s = p \vee q$.

Ad (P3): Let $p, q, r, s \in \mathcal{A}(L)$ such that $p \neq r$, $q \neq s$, and $t = (p \vee q) \wedge (r \vee s) \neq 0$. We have to show that then $(p \vee r) \wedge (q \vee s) \neq 0$.

Case 1. Assume that t is not an atom. Then there are two distinct atoms u and v such that $u, v \leqslant p \vee q, r \vee s$. By property (P2) that we just proved, it follows $u \vee v = p \vee q = r \vee s$. A further application of (P2) implies $u \vee v = p \vee r$ and similarly $u \vee v = q \vee s$. The claim follows.

Case 2. Assume that t is an atom. If $q \leqslant p \vee r$, the claim is clear. Assume $q \nleqslant p \vee r$. Then $t \neq r$, because otherwise $r \leqslant p \vee q$ and hence $q \leqslant p \vee r$ by the finite exchange property. We have $t \leqslant r \vee s$ and by the finite exchange property $s \leqslant t \vee r \leqslant p \vee q \vee r$. By the finite covering property, we have $p \vee r \lessdot p \vee r \vee q = (p \vee r) \vee (q \vee s)$. By Lemma 3.5, $\downarrow(p \vee r \vee q)$ is an atomistic ortholattice with the covering property. Hence, by Lemma 3.4, we conclude that $(p \vee r) \wedge (q \vee s) \lessdot q \vee s$. But this means that $(p \vee r) \wedge (q \vee s)$ cannot be 0.

It remains to show that $(\mathcal{A}(L), \perp)$ is actually an orthogeometry. Clearly, $(\mathcal{A}(L), \perp)$ is an orthoset. Moreover, condition (O1) is obviously fulfilled, and (O2) holds by (L). □

We shall apply this result to the description of Hermitian spaces.

The *orthocomplement* of a subset S of a Hermitian space H is the subspace $S^\perp = \{u \in H : u \perp v \text{ for all } v \in S\}$. The subspaces S of H such that $S = S^{\perp\perp}$ are called *orthoclosed*. The collection of all orthoclosed subspaces, ordered by set-theoretic inclusion and equipped with the orthocomplementation, is an ortholattice, which we denote by $\mathcal{C}(H)$.

Lemma 3.9 *Let H be a Hermitian space. For any orthoclosed subspace S of H and any vector $u \in H^\bullet$, also $S + \langle u \rangle$ is orthoclosed.*

In particular, any finite-dimensional subspace of H is orthoclosed.

Proof See [Piz2, Proposition (2.1.5)]. □

Theorem 3.10 *Let H be a Hermitian space. Then $\mathcal{C}(H)$ is a complete, irreducible, atomistic ortholattice fulfilling* (L)*.*

Conversely, let L be a complete, irreducible, atomistic ortholattice of length $\geqslant 4$ fulfilling (L)*. Then there is a Hermitian space H such that L is isomorphic to the ortholattice $\mathcal{C}(H)$.*

Proof Clearly, $\mathcal{C}(H)$ is a complete, atomistic ortholattice, and the irreducibility of $\mathcal{C}(H)$ follows from Lemma 3.6. By Lemma 3.9, any two-dimensional subspace of H is orthoclosed. Hence (L) follows from property (O2) of the orthogeometry $(P(H), \perp)$, see Theorem 2.7.

To prove the second part, let L be an ortholattice as indicated. By Theorem 3.8, $(\mathcal{A}(L), \perp)$ is an orthogeometry. As L is of length $\geqslant 4$, we conclude from Lemma 3.1 that L contains 4 mutually orthogonal atoms, that is, the rank of $(\mathcal{A}(L), \perp)$ is at least 4.

By Theorem 2.7, there is a Hermitian space H and an isomorphism $\varphi\colon \mathcal{A}(L) \to P(H)$ of orthogeometries. In particular, φ is an automorphism of orthosets: atoms p and q of L are orthogonal if and only if so are the one-dimensional subspaces $\varphi(p)$ and $\varphi(q)$ of H.

For $a \in L$, let

$$\tau(a) \;=\; \bigcup\{\varphi(p) \colon p \in \mathcal{A}(L) \text{ such that } p \leqslant a\},$$

that is, the union of all one-dimensional subspaces $\varphi(p)$, where p is an atom below a. We claim that τ establishes an isomorphism between the ortholattices L and $\mathcal{C}(H)$. Indeed, for $a \in L$ we have

$$\begin{aligned} \tau(a^\perp) \;&=\; \bigcup\{\varphi(p) \colon p \perp a\} \\ &= \bigcup\{\varphi(p) \colon p \perp q \text{ for all atoms } q \leqslant a\} \\ &= \bigcup\{\varphi(p) \colon \varphi(p) \perp \varphi(q) \text{ for all atoms } q \leqslant a\} \\ &= \{u \in H \colon u \perp \varphi(q) \text{ for all atoms } q \leqslant a\} \\ &= \tau(a)^\perp. \end{aligned} \tag{1}$$

In particular, $\tau(a) = \tau(a)^{\perp\perp}$ and hence $\tau(a) \in \mathcal{C}(H)$ for each $a \in L$.

It is clear that $\tau\colon L \to \mathcal{C}(H)$ preserves the order and is injective. By (1), τ preserves orthocomplements. To see that τ is also surjective, consider the orthoclosed subspace $S^\perp$, where $S \subseteq H$. Let $a = \bigvee\{\varphi^{-1}(\langle u \rangle) \colon u \in S^\bullet\}$. Then $\tau(a^\perp) =$

$\bigcup\{\varphi(p) \colon p \perp a\} = \bigcup\{\varphi(p) \colon p \perp \varphi^{-1}(\langle u \rangle) \text{ for all } u \in S^{\bullet}\} = \bigcup\{\varphi(p) \colon \varphi(p) \perp u$ for all $u \in S\} = S^{\perp}$. □

We note the following lattice-theoretic consequence of our results.

Corollary 3.11 *Let L be a complete, irreducible, atomistic ortholattice. Then L fulfils* (L) *if and only if L has the covering property.*

Proof (⇒): Let L fulfil (L). If L is of finite length, L has the covering property by Lemma 3.3. Otherwise, L is, by Theorem 3.10, isomorphic to $\mathcal{C}(H)$ for some Hermitian space H. Then L has the covering property by Lemma 3.9.

(⇐): Let L have the covering property. For distinct atoms $p, q \in L$, $p^{\perp} \lessdot 1 = p^{\perp} \vee (p \vee q)$ implies $r = p^{\perp} \wedge (p \vee q) \lessdot p \vee q$ by Lemma 3.4. Hence r is an atom orthogonal to p, and $p \vee r = p \vee q$, that is, (L) holds. □

In view of Corollary 3.11, one might wonder whether Theorem 3.10 still holds when condition (L) is replaced with the finite covering property. The following example shows that this is not the case.

Example 3.12 *Let H be an infinite-dimensional Hilbert space and $z \in H^{\bullet}$. Let $L = \mathcal{C}(H) \setminus \{\langle z \rangle, \langle z \rangle^{\perp}\}$, partially ordered by set-theoretic inclusion. Then L is a complete lattice. Indeed, for any collection $A_\iota \in L$, $\iota \in I$, we have that $\bigwedge_\iota A_\iota = \bigcap_\iota A_\iota$ if $\bigcap_\iota A_\iota \neq \langle z \rangle$ and $\bigwedge_\iota A_\iota = \{0\}$ otherwise.*

Equipped with the orthocomplementation $^{\perp}$, L is furthermore an ortholattice. Indeed, $^{\perp}$ is an order-reversing involution and for any $A \in L$, $A^{\perp} \cap A = \{0\}$ implies that $A^{\perp} \wedge A = \{0\}$. We readily check that L is atomistic. Finally, for any vectors $u, v \in H^{\bullet}$ there is some $w \in \langle u, v \rangle^{\bullet}$ such that $w \notin \langle u \rangle, \langle v \rangle, \langle z \rangle$. We conclude from Lemma 3.6 that L is irreducible.

The lattice $\mathcal{F}(L)$ of finite elements of L consists of all finite-dimensional subspaces of H except for $\langle z \rangle$. The atoms are the one-dimensional subspaces distinct from $\langle z \rangle$. The supremum in $\mathcal{F}(L)$ is the algebraic sum, that is, for finite-dimensional subspaces $A, B \in L$ we have $A \vee B = A + B$. As this is the case in $\mathcal{C}(H)$ as well, L has the finite covering property. However, let $y \in H^{\bullet}$ be such that $y \perp z$. Then there is no atom below $\langle y, z \rangle$ that is orthogonal to $\langle y \rangle$. Hence L does not fulfil (L).

Consequently, L does not have the covering property. Indeed, $\langle y, z \rangle^{\perp} \vee \langle y \rangle = 1$ but 1 does not cover $\langle y, z \rangle^{\perp}$.

4 Hermitian spaces as linear orthosets

On the basis of Theorem 3.10, we establish in this section that Hermitian spaces correspond to orthosets subject to a simple combinatorial axiom. In the case of finite rank, our result is already known [Vet1].

We define the *orthocomplement* of a subset A of an orthoset $(X, \perp)$ to be $A^{\perp} = \{f \in X \colon f \perp e \text{ for all } e \in A\}$.

Definition 4.1 *An orthoset* $(X, \perp)$ *is called* linear *if, for any distinct element* $e, f \in X$*, there is a* $g \in X$ *such that* $\{e, f\}^\perp = \{e, g\}^\perp$ *and exactly one of* f *and* g *is orthogonal to* e*.*

Lemma 4.2 *For any linearly independent vectors* u*,* v *of a Hermitian space, we have* $\{\langle u\rangle, \langle v\rangle\}^{\perp\perp} = \langle u\rangle \star \langle v\rangle$*.*

Proof We have $\{\langle u\rangle, \langle v\rangle\}^{\perp\perp} = \{\langle w\rangle : \langle w\rangle \perp \langle x\rangle$ for all $x \in H^{\bullet}$ such that $\langle x\rangle \perp \langle u\rangle, \langle v\rangle\} = \{\langle w\rangle : w \perp x$ for all $x \perp u, v\} = \{\langle w\rangle : w \in \{u, v\}^{\perp\perp}\}$. Moreover, by Lemma 3.9, $\{u, v\}^{\perp\perp} = \langle u, v\rangle$. The assertion follows. □

Example 4.3 *The orthoset* $(P(H), \perp)$ *associated with any Hilbert space* H *is linear. In fact, for any Hermitian space* H*,* $(P(H), \perp)$ *is linear. This is easily seen from Lemma 4.2.*

Any orthoset gives rise to an ortholattice in a natural way. Namely, the map sending any $A \subseteq X$ to $A^{\perp\perp} = (A^\perp)^\perp$ is a closure operation and the sets in its image are called *orthoclosed*. The lattice of orthoclosed subsets of X is denoted by $\mathcal{C}(X, \perp)$ and the orthocomplementation $^\perp$ makes $\mathcal{C}(X, \perp)$ into an ortholattice.

Lemma 4.4 *Let* $(X, \perp)$ *be a linear orthoset. Then* $\mathcal{C}(X, \perp)$ *is a complete, irreducible, atomistic ortholattice fulfilling* (L)*. Moreover,* $(X, \perp)$ *is isomorphic to* $(\mathcal{A}(\mathcal{C}(X, \perp)), \perp)$*.*

Proof As in case of any closure space, $\mathcal{C}(X, \perp)$ is a complete lattice and we have already noted that, equipped with $^\perp$, $\mathcal{C}(X, \perp)$ is an ortholattice.

Moreover, we have $\{e\}^{\perp\perp} = \{e\}$ for any $e \in X$. Indeed, if $\{e\}^{\perp\perp}$ contained an element f distinct from e, then, by linearity, there would be a $g \in \{e\}^\perp$ such that $g \in \{e, g\}^{\perp\perp} = \{e, f\}^{\perp\perp} = \{e\}^{\perp\perp}$, a contradiction. We conclude that the atoms of $\mathcal{C}(X, \perp)$ are the singletons and in particular, $\mathcal{C}(X, \perp)$ is atomistic. In particular, the map $X \to \mathcal{A}(\mathcal{C}(X, \perp))$, $e \mapsto \{e\}$ defines an isomorphism of orthosets. Moreover, the linearity of $(X, \perp)$ implies condition (L) and, in view of Lemma 3.6, also the irreducibility. □

We now arrive at the above-mentioned correspondence between linear orthosets and Hermitian spaces.

Theorem 4.5 *Let* $(X, \perp)$ *be an orthoset of rank* $\geqslant 4$*. Then* $(X, \perp)$ *is linear if and only if there is a Hermitian space* H *such that* $(X, \perp)$ *is isomorphic to* $(P(H), \perp)$*.*

Proof The "only if" part is the consequence of Lemma 4.4 and Theorem 3.10. The "if" part is clear from Example 4.3. □

We conclude the section by an informal comparison of the axioms defining orthogeometries with the conditions proposed here to characterise Hermitian spaces. Let us reformulate to this end Theorem 4.5 in a way that comes closer to the style of projective geometry. For elements e and f of an orthoset, we put $e \star f = \{e, f\}^{\perp\perp}$.

Theorem 4.6 *Let $(X, \perp)$ be an orthoset of rank $\geqslant 4$. Then there is a Hermitian space H such that $(P(H), \perp)$ is isomorphic to $(X, \perp)$ if and only if the following holds:*

(M1) *For any distinct elements $e, f \in X$, there is a $g \in e \star f$ such that $g \perp e$ and $e \star f = e \star g$.*

(M2) *For any orthogonal elements $e, f \in X$, $e \star f$ contains an element distinct from e and f.*

Proof To see the "if" part, we argue similarly as in case of Lemma 4.4 to see that (M1) and (M2) imply $\mathcal{C}(X, \perp)$ to be an complete, irreducible, atomistic ortholattice fulfilling (L). So the assertion follows again from Theorem 3.10. The "only if" part is clear. □

We observe that all conditions of Theorem 4.6 can in some form or the other be found among (P1)–(P3), (O1)–(O2). Indeed, the requirement that $(P(H), \perp)$ is an orthoset is part of the definition of an orthogeometry, (M1) corresponds to (O2) provided that (P2) holds, and (M2) is part of (P1).

Conversely, we may check for analogues of (P1)–(P3), (O1)–(O2). The requirement of (P1) that the line $e \star f$ contains e and f holds by construction and so does (O1). Moreover, the requirement of (P1) according to which $e \star e$ is a singleton does not occur, and a particular case of (P2) as well as (O2) is covered by (M1). Finally, there is no analogue of Pasch's axiom (P3).

5 The correspondence of automorphisms

It would be desirable to extend the correspondence between linear orthosets and Hermitian spaces to a categorical level. For this issue, we refer the reader to [PaVe1, PaVe2]. Here, we shall deal with the close relationship between the respective automorphism groups.

There are many versions of Wigner's theorem. The formulations that come closest to our context are Piron's lattice-theoretic version [Pir, Theorem (3.28)] and Uhlhorn's orthogonality-based version [Uhl, Theorem 4.2]. The former is given for orthomodular spaces, the latter for complex Hilbert spaces. Dealing with Hermitian spaces, we may consider our version as a common generalisation of both.

Let V be a linear space over a sfield F. A map $A\colon V \to V$ is called *semilinear* if A is a homomorphism of the additive group and, for some automorphism $^\sigma$ of F, we have $A(\alpha u) = \alpha^\sigma A(u)$ for all $u \in V$. If, in addition, A is bijective, then also A^{-1} is semilinear and we speak of a *semilinear automorphism*.

Let us consider the projective space $P(V)$ associated with V. An automorphism of $P(V)$ is a bijection $\varphi\colon P(V) \to P(V)$ such that, for any

$e, f \in P(V)$, $e \in f \star g$ if and only if $\varphi(e) \in \varphi(f) \star \varphi(g)$. For any semilinear automorphism $A \colon V \to V$, we have that

$$P(A) \colon P(V) \to P(V),\ \langle u \rangle \mapsto \langle A(u) \rangle$$

is an automorphism of $P(V)$, called the automorphism *induced by* A.

Theorem 5.1 *Let V be a linear space of dimension $\geqslant 3$ over the sfield F. Let φ be an automorphism of $P(V)$. Then φ is induced by a semilinear automorphism of V.*

Assume moreover that S is a two-dimensional subspace of V and that φ, restricted to $P(S)$, is the identity. Then φ is induced by a unique linear automorphism which, restricted to S, is the identity.

Proof The first part holds according to the Fundamental Theorem of Projective Geometry; see, e.g., [Bae, Section III.1].

To see the second part, let A be a semilinear automorphism such that $\varphi = P(A)$ and let the subspace S have the indicated properties. For linear independent $u, v \in S$, we have $A(u) = \alpha u$, $A(v) = \beta v$, and $A(u+v) = \gamma(u+v)$ for some $\alpha, \beta, \gamma \in F \setminus \{0\}$. Then $\gamma u + \gamma v = A(u+v) = A(u) + A(v) = \alpha u + \beta v$ and hence $\alpha = \beta = \gamma$. We conclude that there is an $\alpha \in F \setminus \{0\}$ such that $A(u) = \alpha u$ for all $u \in S$.

Furthermore, $B = \alpha^{-1} A$ is likewise a semilinear automorphism such that $\varphi = P(B)$. Denote by ${}^\tau$ the automorphism of F belonging to B. We have $B(u) = u$ for any $u \in S$. Moreover, for any $\alpha \in F$ and $u \in S^\bullet$, $\alpha^\tau u = \alpha^\tau B(u) = B(\alpha u) = \alpha u$ and hence $\alpha = \alpha^\tau$. So ${}^\tau = \mathrm{id}$, that is, B is actually a linear automorphism.

If $\tilde{B}$ is a further linear automorphism inducing φ, then $\tilde{B}^{-1} B$ induces the identity and we conclude again that $\tilde{B}^{-1} B$ is a multiple of the identity. If $\tilde{B}|_S$ is the identity, it follows that $B = \tilde{B}$. □

Theorem 5.2 *Let H be a linear space of dimension $\geqslant 2$ over the sfield F. Let $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ be Hermitian forms w.r.t. involutory antiautomorphisms * and ${}^{\star'}$, respectively. Assume that $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ induce the same orthogonality relation on $P(H)$. Then there is a $\lambda \in F \setminus \{0\}$ such that, for all $u, v \in H$, $(u,v)' = (u,v)\,\lambda$ and, for all $\alpha \in F$, $\alpha^{\star'} = \lambda^{-1} \alpha^\star \lambda$.*

Proof Apply [Piz1, Theorem (2.2)] to the identity map on H. □

An automorphism of an orthoset $(X, \perp)$ is a bijection $\varphi \colon X \to X$ such that, for any $e, f \in X$, we have $e \perp f$ if and only if $\varphi(e) \perp \varphi(f)$.

Lemma 5.3 *Let H be a Hermitian space and let φ be an automorphism of $(P(H), \perp)$. Then φ is likewise an automorphism of $P(H)$ as a projective space.*

Proof For $e, f, g \in P(H)$, we have that $e \in \{f, g\}^{\perp\perp}$ if and only if $\varphi(e) \in \{\varphi(f), \varphi(g)\}^{\perp\perp}$. By Lemma 4.2, $\{f, g\}^{\perp\perp} = f \star g$ and $\{\varphi(f), \varphi(g)\}^{\perp\perp} = \varphi(f) \star \varphi(g)$. The assertion follows. □

We arrive at the announced variant of Wigner's Theorem. A linear automorphism U of a Hermitian space is called *unitary* if U preserves the Hermitian form.

Theorem 5.4 *Let H be a Hermitian space over the $\star$-sfield F of dimension $\geqslant 3$. Any unitary operator induces an automorphism of $(P(H), \perp)$.*

Conversely, let φ be an automorphism of $(P(H), \perp)$ and assume that there is an at least two-dimensional subspace S of H such that $\varphi(e) = e$ for any $e \in P(S)$. Then φ is induced by a unique unitary operator U which, restricted to S, is the identity.

Proof The first part is obvious.

To see the second part, let φ and S be as indicated. By Lemma 5.3 and Theorem 5.1, φ is induced by a unique linear automorphism U of H such that $U|_S$ is the identity. For $u, v \in H$, let $(u, v)' = (U(u), U(v))$. Then $(\cdot, \cdot)'$ is a sesquilinear form inducing the same orthogonality relation on $P(V)$ as $(\cdot, \cdot)$. By Theorem 5.2, there is a $\lambda \in F$ such that $(u, v)' = (u, v)\, \lambda$ for all $u, v \in H$.

In particular, for any $u \in S^{\bullet}$, we have $(u, u) = (U(u), U(u)) = (u, u)' = (u, u)\, \lambda$ and hence $\lambda = 1$. Thus $(U(u), U(v)) = (u, v)$ for all $u, v \in H$, that is, U is unitary. □

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