Regular left-continuous t-norms

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May 2008

– The original publication is available at www.springerlink.com. –

Abstract

A left-continuous (l.-c.) t-norm $\odot$ is called regular if there is an $n < \omega$ such that the map $x \mapsto x \odot a$ has, for any $a \in [0,1]$, at most n discontinuity points, and if the function mapping every $a \in [0,1]$ to the set $\{x \in [0,1] : \lim_{y \searrow x} y \odot a = x\}$ behaves in a specifically simple way. The t-norm algebras based on regular l.-c. t-norms generate the variety of MTL-algebras.

With each regular l.-c. t-norm, we associate certain characteristic data, which in particular specifies a finite number of constituents, each of which belongs to one out of six different types. The characteristic data determines the t-norm to a high extent; we focus on those t-norms which are actually completely determined by it. Most of the commonly known l.-c. t-norms are included in the discussion.

Our main tool of analysis is the translation semigroup of the totally ordered monoid $([0,1]; \leq, \odot, 0, 1)$, which consists of commuting functions from the real unit interval to itself.

1 Classifying left-continuous t-norms - a hopeless case?

In fuzzy logics, the real unit interval is frequently used as a model of statements involving vagueness. The question then arises how to interpret the conjunction. It is believed to be natural to interpret the conjunction in fuzzy logics by a binary function $\odot \colon [0,1]^2 \to [0,1]$ with the following minimal properties: commutativity, associativity, monotonicity in both arguments, and neutrality of the truth value 1. Then, $\odot$ is called a t-norm. It is furthermore common to interpret a fuzzy logic's implication $\to$ in a way that the conjunction and the implication form an adjoint pair. This is reasonable: for two propositions α and β, $\alpha \to \beta$ should be understood as the weakest proposition γ such that α together with γ implies β. In order to ensure the existence of a residuum, the t-norm must be assumed left-continuous.

To justify any further property of the function interpreting the conjunction in fuzzy logics is difficult. So it might be reasonable not to go beyond the mentioned conditions; there is actually no need to make a specific choice. Esteva and Godo's logic MTL [EsGo], for instance, is based on the whole set of left-continuous t-norms together with their respective residua.

However, we might not feel comfortable with a notion which is purely based on abstract axiomatics. We should wonder with what kind of functions we actually work when applying MTL or a related logic. Unfortunately, the insight gained from many years of research on left-continuous t-norms, is generally not considered as satisfactory. At least, there are well-understood subclasses. We have in particular a fully satisfactory theory for continuous t-norms; see [MoSh] or e.g. [KMP]. As an example of a further condition which turned out to be strong enough to allow a structure theory, we mention cancellativity; see [Hor].

How to describe left-continuous t-norms in general, remains mysterious though. Generally applicable tools for an analysis being unknown, various ways of their construction were proposed; see [Jen4] for an overview. Now, out of the work summarized in [Jen4], quite a large collection of different t-norms arose. As we are not able to deal with the general case, we decided to review this collection of well-known left-continuous t-norms and tried to detect common properties. Our paper is motivated by the aim to systematize and to characterize a class of as many left-continuous t-norms as possible; at the very least, the well-known ones should be included.

By now, there have been just a few efforts to bring the known constructions methods into a single line. One such approach is contained in the author's work [Vet1], in which the observation is made that, apart from few known cases, left-continuous t-norms can be represented by means of totally ordered Abelian groups.

The approach on which the present paper is based, is not related to the ideas used in [Vet1]. Rather than working with the abstract notion of a totally ordered group, we refer to a common source of inspiration about the properties of t-norms: a t-norm may be visualized by means of its graph, which is a three-dimensional geometrical object. However, we slightly modify this approach; we work with the – two-dimensional – vertical cuts through the three-dimensional graph.

Namely, a t-norm is a binary operation $\odot$ on the real unit interval making $([0,1]; \leq, \odot, 0, 1)$ a totally ordered commutative monoid. We may associate with it its semigroup Λ of (inner right) translations. Λ then actually possesses the structure of a totally ordered monoid as well and, as such, Λ is isomorphic to $([0,1]; \leq, \odot, 0, 1)$. So rather than studying $\odot$, we may equally well study Λ, and this is what we propose here. Now, the translation by an element $a \in [0,1]$ is the function $[0,1] \to [0,1], \quad x \mapsto x \odot a$; this is nothing but the vertical cut at the point a. So from the geometrical point of view, Λ may be considered as the set containing all the vertical cuts of $\odot$.

We introduce in this paper so-called regular left-continuous t-norms. From the universal-algebraic point of view, regularity might be considered as a not too serious restriction: we will show that the t-norm algebras based on a regular left-continuous t-norm and the corresponding residuum, generate the whole variety of MTL-algebras.

Now, if a left-continuous t-norm is regular, we may partition the real unit interval into certain subintervals, called basic intervals, and we may consider the set of translations restricted in an appropriate way to one of them. It turns out that the resulting restricted translation semigroup, which is actually again a totally ordered monoid, belongs to one out of six different isomorphism types. The number of basic intervals and their respective isomorphism types is contained in what we call the characteristic data of a regular left-continuous t-norm. The characteristic data determines the t-norm, up to isomorphism, to a high extent, and we consider here the case that the t-norm is actually fully determined by it. We cover in this way most of the left-continuous t-norms explicitly defined somewhere in the literature, as far as known to us.

The paper is organized as follows. We start with the one-to-one correspondence between t-norm monoids and the totally ordered monoids of their translations, and we characterize the latter (Section 2). We then introduce our basic notion – regularity (Section 3), and we discuss the implications with regard to the variety of MTL-algebras. We next see how to bring structure into the translation semigroup by means of regularity (Section 4). The next part is preparatory; we discuss certain systems of continuous and idempotent functions (Section 5). We then describe the basic constituents of regular left-continuous t-norms (Section 6) and how the t-norm is built up from these constituents (Section 7). We conclude with a list of examples, to illustrate how specific t-norms are characterized within our framework (Section 8). We also add the example of a set of non-regular left-continuous t-norms, closed under the pointwise formation of the arithmetic mean; we answer in this way a question proposed in [AFS] positively (Section 9).

2 Left-continuous t-norms as functional algebras

We study in this paper the totally ordered monoids which are based on left-continuous t-norms.

Definition 2.1 Let $[0,1]$ be the real unit interval, endowed with its natural order. An operation

$\odot\colon [0,1]^2 \to [0,1]$ is called a *t-norm* if, for all $a,b,c \in [0,1]$, (i) $(a \odot b) \odot c = a \odot (b \odot c)$, (ii) $a \odot b = b \odot a$, (iii) $a \odot 1 = a$, and (iv) $a \le b$ implies $a \odot c \le b \odot c$. A t-norm $\odot$ is called *left-continuous*, or *l.-c.* for short, if for every $a \in [0,1]$, the function $(0,1] \to [0,1]$, $x \mapsto x \odot a$ is left-continuous.

Let $\odot$ be a l.-c. t-norm. Then we call $([0,1]; \le, \odot, 0, 1)$ the *t-norm monoid* based on $\odot$. Moreover, let $\to\colon [0,1]^2 \to [0,1]$, $(a,b) \mapsto \max\{x\colon a \odot x \le b\}$ be the *residuum* of $\odot$; then we call $([0,1]; \wedge, \vee, \odot, \to, 0, 1)$ the *t-norm algebra* based on $\odot$.

In other words, a t-norm monoid is a structure $([0,1]; \le, \odot, 0, 1)$, where $[0,1]$ is the naturally ordered real unit interval, $([0,1]; \le, \odot, 1)$ is a totally ordered commutative monoid, and $\cdot \odot a$ preserves, for any a, arbitrary suprema. In the sequel, a totally ordered commutative monoid $(L; \le, \odot, 1)$ will be called a tomonoid, and a structure $(L; \le, \odot, 0, 1)$ such that $(L; \le, \odot, 1)$ is a tomonoid with the bottom element 0 and the top element 1, will be called a 0, 1-tomonoid. Using these notions, a t-norm monoid is a 0, 1-tomonoid such that $\cdot \odot a$ preserves, for any a, arbitrary suprema.

Alternatively, we may view a t-norm monoid as a strictly two-sided commutative quantale whose base set is $[0,1]$ endowed with the natural order.

For an overview of results on totally ordered semigroups, see [HoLa] and the references given herein. For tomonoids in particular, see the comprehensive article [EKMMW]. As a rich source of information about quantales, we refer to [Ros]. For an account focused directly on t-norms, see [KMP].

We have been originally interested in a classification of l.-c. t-norm algebras, since it is them to play the important role for the semantics of fuzzy logics. However, apart from our considerations about the question how comprehensive the class of regular l.-c. t-norms is within the class of all l.-c. t-norms, our analysis will concern exclusively the monoidal operation $\odot$; this is why we will deal mostly with t-norm monoids rather than t-norm algebras. For our general aim to classify l.-c. t-norm algebras, this is certainly not a restriction as $\to$ is definable from the monoidal operation and the order.

The main idea of this paper is to examine the set of inner right translations of the semigroup $([0,1]; \odot)$ rather than the t-norm itself. This is a set of pairwise commuting functions from $[0,1]$ to $[0,1]$. We introduce and discuss the relevant notions step by step.

Definition 2.2 Let $\odot$ be a l.-c. t-norm. We denote by

$$\lambda_a^\odot\colon [0,1] \to [0,1], \quad x \mapsto x \odot a$$

the *translation* by $a \in [0,1]$ based on $\odot$. The set of all translations based on $\odot$ will be denoted by

$$\Lambda^\odot = \{\lambda_a^\odot\colon a \in [0,1]\}.$$

Because we have to do with a commutative semigroup with an identity, the attributes "right" and "inner" are superfluous in the present context; this is why we speak simply about "translations". Furthermore, in the sequel, we will drop the superscript $^\odot$ and write λ_a and Λ rather than $\lambda_a^\odot$ and $\Lambda^\odot$, respectively, whenever the reference to a specific t-norm is clear.

Geometrically, Λ consists of the vertical cuts through the graph of the t-norm. Namely, cutting the three-dimensional graph $\{(x, y, x \odot y)\colon x, y \in [0,1]\}$ of a t-norm $\odot$ by a vertical plane parallel to the x-axis at the point $a \in [0,1]$ on the y-axis, yields the graph of $\lambda_a^\odot$, the translation by a.

However, the most convenient way to visualize the set Λ of all translations based on a t-norm makes use of just two dimensions. Namely, it is most instructive to visualize λ_a, where a ranges over $[0,1]$, as a continuum of functions; we start with the largest function, λ_1, which is always the identity function of $[0,1]$, proceed successively to smaller ones, and end up finally with λ_0, which is always the constant 0 function. Throughout this paper, we implicitly rely on this way to imagine a t-norm as a little "movie". Corresponding graphical illustrations for several l.c. t-norms can be found in the paper [Vet2].

We further note that, apart from the theory of totally ordered semigroups and the theory of quantales, the theory of functional equations is involved here, although we will make relatively limited use of it. For facts originating from this field, we refer to [KCG].

We next establish the exact properties of the set of translations based on a l.-c. t-norm. On these properties, the results of this paper fully rely.

In what follows, we will frequently deal with functions which preserve or reverse the order or the strict order from one poset to another one, and we will use the following simplified terminology. We call a function $\varphi\colon P \to Q$ between the posets $(P; \leq)$ and $(Q; \leq)$ increasing, strictly increasing, decreasing, or strictly decreasing if, for $a, b \in P$, $a < b$ implies $\varphi(a) \leq \varphi(b)$, $\varphi(a) < \varphi(b)$, $\varphi(a) \geq \varphi(b)$, or $\varphi(a) > \varphi(b)$, respectively. Increasing or decreasing bijections will also be called order isomorphisms or order antiisomorphisms, respectively, and in the case of a coincidence of the domain and the range, order automorphisms or order antiautomorphisms, respectively.

Moreover, a function $f \in [0,1] \to [0,1]$ is called left-continuous if f is left-continuous on $(0,1]$ and continuous at 0. Note that if $f(x) \leq x$ for all $x \in [0,1]$, then continuity at 0 it automatic. We also recall at this point that each increasing left-continuous function $f\colon [0,1] \to [0,1]$ is regulated, that is, all right-side limits exist. We write

$$f^+(a) = \lim_{x \searrow a} f(x), \qquad a \in [0,1)$$

and, in addition, we set $f^+(1) = f(1)$. – Similar remarks apply for functions from any other closed real interval to itself.

Theorem 2.3 *Let Λ be the set of translations based on a l.-c. t-norm $\odot$. Then Λ contains functions from $[0,1]$ to $[0,1]$ with the following properties:*

(T1) *Every $f \in \Lambda$ is increasing.*

(T2) *Every two functions in Λ commute, that is, $f \circ g = g \circ f$ for any $f, g \in \Lambda$.*

(T3) *For every $t \in [0,1]$, there is exactly one $f \in \Lambda$ such that $f(1) = t$.*

(T4) *Every $f \in \Lambda$ is left-continuous.*

Conversely, let Λ be a set of functions from $[0,1]$ to $[0,1]$ fulfilling (T1)–(T4)*. Then there is a l.-c. t-norm $\odot$ such that Λ is the set of translations based on $\odot$. The t-norm $\odot$ is uniquely determined by*

$$a \odot b = f(a), \text{ where } f \in \Lambda \text{ is such that } f(1) = b \tag{1}$$

for $a, b \in [0,1]$.

Proof. It is not difficult to see that the set of translations based on a l.-c. t-norm fulfils the conditions (T1)–(T4).

Conversely, assume that Λ is a set of mappings from $[0,1]$ to $[0,1]$ fulfilling (T1)–(T4). By (T3), we may associate with every $t \in [0,1]$ a unique $\lambda_t \in \Lambda$ such that $\lambda_t(1) = t$. It follows that $\odot\colon [0,1]^2 \to [0,1]$ is well-defined by (1).

We then have $a \odot b = \lambda_b(a) = \lambda_b(\lambda_a(1)) = \lambda_a(\lambda_b(1)) = \lambda_a(b) = b \odot a$ by (T2); so $\odot$ is commutative. Furthermore, $(a \odot b) \odot c = \lambda_c(a \odot b) = \lambda_c(\lambda_b(a)) = \lambda_b(\lambda_c(a)) = (a \odot c) \odot b$ again by (T2); so in view of the commutativity of $\odot$, the associativity follows as well. Moreover, $a \odot 1 = 1 \odot a = \lambda_a(1) = a$. And $a \leq b$ implies $a \odot c = \lambda_c(a) \leq \lambda_c(b) = b \odot c$ by (T1). Finally, $\odot$ is l.-c. because, by (T4), every $f \in \Lambda$ is.

So $\odot$ is a l.-c. t-norm, and obviously, Λ contains exactly the translations based on $\odot$. □

In other words, there is a one-to-one correspondence between the l.-c. t-norms and the functions from $[0,1]$ to $[0,1]$ fulfilling (T1)–(T4). Note that this correspondence can be extended to all t-norms, by dropping the condition (T4).

We now proceed to endow a l.-c. t-norm's set of translations with the usual algebraic structure. Λ is a semigroup under composition, and there is natural partial order. So with each pair $f, g\colon [0,1] \to [0,1]$, we associate their composition $f \circ g$. Furthermore, for pairs of functions $f, g\colon [0,1] \to [0,1]$, we denote the pointwise order by $\leq$; $f < g$ will mean $f \leq g$ and $f(x) < g(x)$ for at least one x.

Definition 2.4 Let $\odot$ be a l.-c. t-norm, and let Λ be the set of translations based on $\odot$. Endow Λ with the pointwise order $\leq$, with the composition of functions $\circ$, and with the constants *0* and *id*. Then we call $(\Lambda; \leq, \circ, \mathit{0}, \mathit{id})$ the *translation tomonoid* of $\odot$.

The basic properties of the order $\leq$ and of the operation $\circ$ on Λ are stated next.

Lemma 2.5 *Let $\odot$ be a l.-c. t-norm. Then Λ, the set of translations based on $\odot$, has the following properties:*

(T5) *$(\Lambda; \leq)$, where $\leq$ is the pointwise order, is isomorphic to $([0,1]; \leq)$, where $\leq$ is the natural order. The bottom element of Λ is*

$$\mathit{0}\colon [0,1] \to [0,1], \quad x \mapsto 0,$$

and the top element of Λ is

$$\mathit{id}\colon [0,1] \to [0,1], \quad x \mapsto x.$$

Moreover, suprema in Λ are calculated pointwise.

(T6) *Λ is closed under composition, that is, $f \circ g \in \Lambda$ for any $f, g \in \Lambda$.*

Proof. These properties are not difficult to check from the fact that Λ consists of the translations based on $\odot$. □

Alternative proof. We find it instructive to see how (T5)–(T6) derive from the axioms (T1)–(T4); the following procedure may be seen as an illustration. So let us assume that Λ is a set of functions from $[0,1]$ to $[0,1]$ such that (T1)–(T4) hold.

Again, in view of (T3), let us denote the unique $f \in \Lambda$ such that $f(1) = t$ by λ_t, where $t \in [0,1]$. Let $\lambda_t, \lambda_u \in \Lambda$ be such that $t \leq u$. We then have $\lambda_t(x) = \lambda_t(\lambda_x(1)) = \lambda_x(\lambda_t(1)) = \lambda_x(t) \leq \lambda_x(u) = \lambda_u(x)$ for all x by (T2) and (T1); so the order of Λ is total. By (T3), $(\Lambda; \leq)$ is isomorphic to $([0,1]; \leq)$.

Furthermore, λ_0 is constant 0; so $\mathit{0} \in \Lambda$. And for every $x \in [0,1]$ we have $\lambda_1(x) = \lambda_x(1) = x$; so $\lambda_1 = \mathit{id} \in \Lambda$. Clearly, Λ is bounded by $\mathit{0}$ and id.

Finally, let t_ι, $\iota \in I$, be a subset of $[0,1]$, and let $t = \bigvee_\iota t_\iota$. Then $\bigvee_\iota \lambda_{t_\iota} = \lambda_t$. Using (T4), we have $\lambda_t(x) = \lambda_x(t) = \lambda_x(\bigvee_\iota t_\iota) = \bigvee_\iota \lambda_x(t_\iota) = \bigvee_\iota \lambda_{t_\iota}(x)$ for every $x \in (0,1]$. So suprema are calculated pointwise. This finishes the proof of (T5).

For $f, g \in \Lambda$, let $k \in \Lambda$ be the unique function such that $k(1) = f(g(1))$. Then, for any $x \in [0,1]$, we have $k(x) = k(\lambda_x(1)) = \lambda_x(k(1)) = \lambda_x(f(g(1))) = f(g(\lambda_x(1))) = f(g(x))$, so $k = f \circ g$, and (T6) is shown. □

The correspondence between a l.-c. t-norm and the set of translations based on it, may easily be shown to preserve the respective algebraic structure; cf. [ClPr, Chapter 1.2].

Theorem 2.6 *Let $\odot$ be a l.-c. t-norm. Then the t-norm monoid based on $\odot$, $([0,1]; \leq, \odot, 0, 1)$, and the translation tomonoid of $\odot$, $(\Lambda; \leq, \circ, \mathit{0}, \mathit{id})$, are isomorphic. The isomorphism is given by $[0,1] \to \Lambda,\ \ t \mapsto \lambda_t$.*

Proof. It is a well-known fact that $([0,1]; \odot)$, a commutative semigroup with identity, is isomorphic to its (inner right) translation semigroup $(\Lambda; \circ)$. Also the remaining facts are straightforward to see. □

We finally endow the translation tomonoid of a l.-c. t-norm with a topology. Namely, we will assume in the sequel that the translation tomonoid is endowed with the supremum metric. Uniform convergence refers to this metric.

The following lemma originates from [Fra, Lemma 1.12].

Lemma 2.7 *Let $f: [0,1] \to [0,1]$ be continuous, and let $g_i: [0,1] \to [0,1]$, $i < \omega$, be increasing left-continuous functions such that $g_0 \leq g_1 \leq \ldots$ and, for each $x \in [0,1]$, $f(x) = \bigvee_i g_i(x)$. Then the sequence $(g_i)_i$ converges to f uniformly.*

Lemma 2.8 *Let $(\Lambda; \leq, \circ, \mathit{0}, \mathit{id})$ be the translation tomonoid of a l.-c. t-norm, and let $f \in \Lambda$ be continuous and > 0. Then f is the uniform limit of a sequence of functions strictly below f.*

Proof. Let $a \in (0,1]$ such that $f = \lambda_a$, and let $b_0 \leq b_1 \leq \ldots < a$ be such that $\bigvee_i b_i = a$. Then by (T3), $f = \bigvee_i \lambda_{b_i}$ in Λ. By (T5), $(\lambda_{b_i})_i$ converges to f pointwise; so $(\lambda_{b_i})_i$ converges to f uniformly by Lemma 2.7. □

In what follows, we will characterize t-norms only up to isomorphism. We recall that two t-norms $\odot_1$ and $\odot_2$ are called isomorphic if there is an order automorphism $\varphi : [0,1] \to [0,1]$ such that $a \odot_2 b = \varphi^{-1}(\varphi(a) \odot_1 \varphi(b)),\ \ a, b \in [0,1]$. This is obviously the same as to say the t-norm monoids $([0,1]; \leq, \odot_1, 0, 1)$ and $([0,1]; \leq, \odot_2, 0, 1)$ are isomorphic. The corresponding notion for the translation tomonoids is conjugacy; cf. [KCG, Chapter 8].

Definition 2.9 Let F and G consist of functions from $[0,1]$ to $[0,1]$. Then F and G are called *conjugate* if there is an order automorphism $\varphi\colon [0,1] \to [0,1]$ such that $G = \{\varphi^{-1} \circ f \circ \varphi\colon f \in F\}$.

Lemma 2.10 *Let $\odot_1$ and $\odot_2$ be l.-c. t-norms with associated sets of translations $\Lambda^{\odot_1}$ and $\Lambda^{\odot_2}$. Then $\odot_1$ are $\odot_2$ are isomorphic if and only if $\Lambda^{\odot_1}$ and $\Lambda^{\odot_2}$ are conjugate.*

Proof. Let φ be an order automorphism φ such that $\varphi(a) \odot_1 \varphi(b) = \varphi(a \odot_2 b)$ for all $a, b \in [0,1]$. Then $\lambda_b^{\odot_2}(a) = a \odot_2 b = \varphi^{-1}(\varphi(a) \odot_1 \varphi(b)) = \varphi^{-1}(\lambda_{\varphi(b)}^{\odot_1}(\varphi(a)))$ for $a \in [0,1]$. It follows that $\Lambda^{\odot_1}$ and $\Lambda^{\odot_2}$ are isomorphic.

Conversely, let $\Lambda^{\odot_1}$ and $\Lambda^{\odot_2}$ be conjugate, φ being the order automorphism. Let $a, b \in [0,1]$. Then $\varphi^{-1} \circ \lambda_{\varphi(b)}^{\odot_1} \circ \varphi \in \Lambda^{\odot_2}$ and $\varphi^{-1}(\lambda_{\varphi(b)}^{\odot_1}(\varphi(1))) = b$; hence $\lambda_b^{\odot_2} = \varphi^{-1} \circ \lambda_{\varphi(b)}^{\odot_1} \circ \varphi$. So $\varphi(a) \odot_1 \varphi(b) = \lambda_{\varphi(b)}^{\odot_1}(\varphi(a)) = \varphi(\lambda_b^{\odot_2}(a)) = \varphi(a \odot_2 b)$. □

3 Regular left-continuous t-norms

In this paper, we study l.-c. t-norms subject to two conditions. The first one is as follows.

Definition 3.1 Let $\odot$ be a l.-c. t-norm. We say that $\odot$ has *few discontinuity points* if there is an $n < \omega$ such that each translation based on $\odot$ has at most n points of discontinuity.

In other words, a l.-c. t-norm has few discontinuity points if the number of points at which the maps $\lambda_a\colon\ [0,1] \to [0,1],\ \ x \mapsto x \odot a$, where $a \in [0,1]$, are not continuous, is globally bounded by a finite number.

Let us ask how strong this condition is, first in the informal way. As a rule of thumb, we may say that l.-c. t-norms defined in a finitary way, that is, by distinction of finitely many cases which are distinguished by means of algebraic inequalities involving finite expressions and to each of which a finite algebraic expression is assigned, all have few discontinuity points.

So the t-norms found in the literature with an explicit definition, typically have this property. On the other hand, we have:

- Hájek's t-norm defined in [Haj2, proof of Theorem 2] has translations with infinitely many discontinuity points.
- Hliněná's t-norm defined in [Smu, Proposition 1] has translations whose discontinuity points are even dense in $[0,1]$.

More t-norms of this type can be derived using the construction method presented in [JeMo2, Section 5]. More specifically, H-transformations as defined in [Mes] lead to t-norms similar to Hájek's.

We now turn to a serious way to decide if a condition for left-continuous t-norms is against our aim to develop a general structure theory. The following might be a reasonable criterion. Recall that the variety generated by all t-norm algebras $([0,1]; \wedge, \vee, \odot, \to, 0, 1)$, where $\odot$ is a l.-c. t-norm, consists of the MTL-algebras. So we may ask: Is the variety generated by those t-norm algebras which are based on l.-c. t-norms fulfilling the given condition, the whole variety of MTL-algebras? We recall the definition of MTL-algebras; for further basic information on them as well as the logic MTL, see [EsGo].

Definition 3.2 An algebra $(L; \wedge, \vee, \odot, \to, 0, 1)$ is an *MTL-algebra* if (i) $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, (ii) $(L; \odot, 1)$ is a commutative monoid, (iii) $a \odot b \leq c$ if and only if $a \leq b \to c$ for all $a, b, c \in L$, and (iv) $(a \to b) \vee (b \to a) = 1$ for all $a, b \in L$.

For l.-c. t-norms with few discontinuity points, the question can be answered affirmatively. The following construction is due to Jenei and Montagna [JeMo1].

Example 3.3 Let $(L; \leq, \odot, 0, 1)$ be a finite $0,1$-tomonoid. We may add the residuum $\to$ of $\odot$, given by

$$a \to b \;=\; \max\,\{x\colon\, a \odot x \leq b\} \quad \text{for } a, b \in L, \tag{2}$$

and replace the order relation by the infimum and supremum operations; then $(L; \wedge, \vee, \odot, \to, 0, 1)$ is an MTL-algebra.

We are going to enlarge the set L as follows; let

$$\hat{L} \;=\; \{(a, r)\colon\, a \in L \setminus \{0\},\;\; r \in (0, 1]\} \;\cup\; \{(0, 1)\}, \tag{3}$$

endowed with the lexicographical order. Note that then $\hat{L}$ is isomorphic to the real unit interval. For $(a, r), (b, s) \in \hat{L}$ define

$$(a, r) \odot (b, s) \;=\; (a, r) \wedge (b, s) \wedge (a \odot b, 1). \tag{4}$$

We have constructed the structure $(\hat{L}; \leq, \odot, (0, 1), (1, 1))$, about which we can say the following [JeMo1]:

Lemma 3.4 *$(\hat{L}; \leq, \odot, (0, 1), (1, 1))$ from Example 3.3 is again a $0,1$-tomonoid. Moreover, the residuum $\to$ of $\odot$ exists, and $(\hat{L}; \wedge, \vee, \odot, \to, (0, 1), (1, 1))$ is an MTL-algebra. The mapping $\vartheta : L \to \hat{L},\;\; a \mapsto (a, 1)$ is an isomorphic embedding of MTL-algebras.*

We draw the obvious conclusions.

Lemma 3.5 *Let $(L; \wedge, \vee, \odot, \to, 0, 1)$ be a finite totally ordered MTL-algebra. Then L may be isomorphically embedded into a t-norm algebra $([0, 1]; \wedge, \vee, \odot, \to, 0, 1)$, where $\odot$ is a l.-c. t-norm with few discontinuity points.*

Proof. Let $\hat{L}$ be the MTL-algebra constructed from L as shown in Example 3.3. Clearly, $\hat{L}$ is isomorphic to a l.-c. t-norm algebra. It is not difficult to check that each translation based on the t-norm has less discontinuity points than there are elements in L; so the t-norm has few discontinuity points. □

Theorem 3.6 *The t-norm algebras based on l.-c. t-norms with few discontinuity points generate the variety of MTL-algebras.*

Proof. By Lemma 3.5, any finite totally ordered MTL-algebra is a subalgebra of a t-norm algebra based on a l.-c. t-norm with few discontinuity points. Moreover, the variety MTL is generated by its finite totally ordered members [CMM]. The assertion follows. □

We now turn to our second condition; the restriction of the number of discontinuity points is not yet enough to make an easily comprehensible structure theory for l.-c. t-norms possible. We need some preparations.

Definition 3.7 For any increasing left-continuous function $f\colon\, [0, 1] \to [0, 1]$ such that $f \leq id$, we define

$$Q(f) \;=\; \{x \in [0, 1]\colon\, f^+(x) = x\}.$$

Moreover, let $\odot$ be a l.-c. t-norm. Then we put

$$q^{\odot}(a) \;=\; Q(\lambda_a^{\odot})$$

for each $a \in [0,1]$. Finally, let $\mathcal{C}([0,1])$ be the set of closed subsets of $[0,1]$, and partially order $\mathcal{C}([0,1])$ by the set-theoretic inclusion.

Again, whenever the reference to a specific t-norm $\odot$ is clear, we will write q instead of $q^{\odot}$.

When visualizing an element f of a translation tomonoid Λ by its graph, $Q(f)$ represents the intersection of the closure of the graph of f and the graph of the identity function. Moreover, we have $q(f(1)) = Q(f)$; so given Λ and some $a \in [0,1]$, we determine $q(a)$ by first selecting among the graphs of the functions in Λ the unique one whose right-most point is $(1, f(1)) = (1, a)$, and second associating with it the intersection of its closure with the identity line.

In the next lemma, we call the element x of a subset C of $[0,1]$ a *right-limit point* of C if $x = 0$ or, for any $0 < \varepsilon \leq x$, $(x - \varepsilon, x)$ has a non-empty intersection with C.

Lemma 3.8 *Let $f\colon [0,1] \to [0,1]$ be increasing, left-continuous and below the identity. Then $Q(f)$ is a closed subset of $[0,1]$. If x is a right-limit point of $Q(f)$, in particular if x is in the interior of $Q(f)$, then $f(x) = x$. Conversely, if $f(x) = x$, then $x \in Q(f)$. Finally, we have $0 \in Q(f)$.*

Proof. Elementary. □

Concerning q, we have the following obvious property:

Lemma 3.9 *Let $\odot$ be a l.-c. t-norm. Then q is an increasing mapping from $[0,1]$ to $\mathcal{C}([0,1])$ such that $q(0) = \{0\}$ and $q(1) = [0,1]$.*

Apart from monotonicity, there is little more to say about q in general. Let

$$q^-(a) = \bigvee_{x<a} q(x) = \overline{\bigcup_{x<a} q(x)} \quad \text{for } a \in (0,1], \tag{5}$$

$$q^+(a) = \bigcap_{x>a} q(x) \quad \text{for } a \in [0,1). \tag{6}$$

Then we have $q(a)^- \subseteq q(a) \subseteq q(a)^+$ for $a \in (0,1)$, and the inclusions can be proper: q preserves from $[0,1]$ to $\mathcal{C}([0,1])$ in general neither suprema nor infima. Indeed, q may not preserve suprema even if $\odot$ is continuous; for instance, in case that $\odot$ is the product t-norm, $q(a) = \{0\}$ for $a < 1$, but $q(1) = [0,1]$.

We will require that q has intervalwise a simple structure; our condition is as follows. Here, for a subinterval K of $[0,1]$, we denote by $\mathcal{C}K$ the set of subsets of K which are relatively closed in K.

Definition 3.10 Let $\odot$ be a l.-c. t-norm. We say that q is *simple* if there is a finite partition $(0,1] = \dot{\bigcup}_{1 \leq i \leq k} (b_i, c_i]$ such that, for each i, the function

$$q_i\colon [0,1] \to \mathcal{C}(b_i, c_i), \quad x \mapsto q(x) \cap (b_i, c_i) \tag{7}$$

fulfills the following conditions: (i) For each $x \in [0,1]$, $q_i(x)$ is of the form $[d, c_i)$ for $b_i < d \leq c_i$, or $(b_i, d]$ for $b_i \leq d < c_i$, or equal to (b_i, c_i). (ii) For some $u, v \in (0,1]$ such that $u \leq v$, $q_i(t) = \emptyset$ for $t < u$, $q_i(t) = (b_i, c_i)$ for $t \geq v$, and, provided that $u < v$, $q_i|_{[u,v]}$ is an order isomorphism between $[u,v]$ and a maximal chain in $\mathcal{C}(b_i, c_i)$.

Let us call a l.-c. t-norm $\odot$ *regular* if $\odot$ has few discontinuity points and q is simple.

Note that even simplicity of q does not imply that q preserves suprema or infima. However, it preserves infima up to finitely many single points. For a closed set $C \subseteq [0,1]$, let us call $x \in C$ a *singleton* of C if $x > 0$ and x is an isolated point of C. We put

$$\mathrm{regcl}(C) = \{x \in C\colon\, x \text{ is not a singleton of } C\}.$$

Lemma 3.11 *Let q be simple. Then, for every $a \in [0,1]$, $q(a)$ is the union of finitely many closed subintervals of $[0,1]$. Moreover, provided that $a < 1$, we have*

$$\mathrm{regcl}(q^+(a)) \subseteq q(a) \subseteq q^+(a).$$

Proof. The first assertion is clear by condition (i) of Definition 3.10 and the fact that $q(a)$ is closed.

Let $a \in [0,1)$. By the monotonicity of q, we have $q(a) \subseteq q^+(a)$.

Let furthermore $(0,1] = \dot{\bigcup}_{1 \leq i \leq k} (a_i, b_i]$ be given according to Definition 3.10. Then $q(a) \cap (a_i, b_i) = q^+(a) \cap (a_i, b_i)$ for each i; so $q(a)$ differs from $q^+(a)$ by a subset of the boundary points $a_1, b_1, ..., a_k, b_k$. Since $q(a)$ is closed, the points in $q^+(a) \setminus q(a)$ are isolated points of $q^+(a)$; it follows $\mathrm{regcl}(q^+(a)) \subseteq q(a)$. □

To find l.-c. t-norms in the literature which have few continuity points, but which are not regular, is not very easy. We have the following situation.

- All continuous t-norms are regular.
- A finite ordinal sum of regular l.-c. t-norms is again regular.
- All l.-c. t-norms constructed from regular l.-c. t-norms by means of annihilation, rotation, or rotation-annihilation are again regular. See [Jen4] for an overview on these constructions.

Our counterexample is somewhat artificial in nature. Namely, consider $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$, the four-element totally ordered MV-algebra. Embed then L into a l.-c. t-norm algebra, as explained in Example 3.3. The l.-c. t-norm $\odot$ arising in this way has few discontinuity points; but q is not simple. In particular, $\odot$ is not regular. In Section 9, an explicit definition of $\odot$ and similar l.-c. t-norms can be found.

We devote the remainder of this section to the proof that the t-norm algebras based on regular l.-c. t-norms generate the variety of MTL-algebras. To this end, we will modify the construction of Example 3.3.

Example 3.12 Let $(L; \leq, \odot, 0, 1)$ be a finite $0,1$-tomonoid. We will use the following auxiliary notation. For any $a > 0$, we denote by Pa the immediate predecessor of a w.r.t. the total order. Furthermore, we write $a \Lleftarrow b$ if b is a non-zero idempotent, $a \odot b = a$ and $a \odot Pb < a$.

Let $\hat{L}$ be defined by (3), endowed with the lexicographical order. Define $\odot \colon \hat{L}^2 \to \hat{L}$ as follows: for $(a,r), (b,s) \in \hat{L}$ such that $(a,r) \leq (b,s)$ let

$$\begin{aligned}(a,r) \odot (b,s) &= (b,s) \odot (a,r) \\ &= \begin{cases} (a, rs) & \text{if } a \Lleftarrow b \text{ and } a = b, \\ (a, r^{\frac{1}{s}}) & \text{if } a \Lleftarrow b \text{ and } a < b, \\ (a,r) \wedge (a \odot b, 1) & \text{else.} \end{cases}\end{aligned} \tag{8}$$

For the structure $(\hat{L}; \leq, \odot, (0,1), (1,1))$ defined this way, we have:

Lemma 3.13 *Let $(L; \leq, \odot, 0, 1)$ be a finite $0,1$-tomonoid. Then $(\hat{L}; \leq, \odot, (0,1), (1,1))$, as defined in Example 3.12, is again a $0,1$-tomonoid.*

Proof. Clearly, $(\hat{L}; \leq)$ is a total order with bounds $(0,1)$ and $(1,1)$. By construction, $\odot$ is commutative, and $(1,1)$ is easily seen to be a neutral element.

Note next that for any $(a,r), (b,s) \in \hat{L}$,

$$(a,r) \odot (b,s) \;=\; (a \odot b, t) \tag{9}$$

for some t. Furthermore, if either $r = 1$ or $s = 1$, we have

$$(a,r) \odot (b,s) \;=\; (a,r) \wedge (b,s) \wedge (a \odot b, 1). \tag{10}$$

We now show that $\odot$ is in both variables increasing. Note first that $(0,1)$ is an annihilator in $\hat{L}$. Let now $(a,r),(b,s) \in \hat{L}$ be such that $(0,1) < (a,r) \leq (b,s)$, and let $A = (a,r) \odot (b,s)$. It is easily checked that A depends monotonously on r and on s. So it suffices to show that $B = (Pa,1) \odot (b,s) \leq A$ and $C = (a,r) \odot (Pb,1) \leq A$.

Assume first that $a \Lleftarrow b$ and $a = b$. Then $A = (a, rs)$, and a is a non-zero idempotent. We conclude $B, C < A$ from (9).

Assume next that $a \Lleftarrow b$ and $a < b$; then $A = (a, r^{\frac{1}{s}})$. Since $Pa \odot b < a$ and $a \odot Pb < a$, we again use (9) to conclude $B, C < A$.

Assume that $a \not\Lleftarrow b$ and $a \odot b = a$; then $A = (a,r)$. We have $B < A$ by (9), and $C \leq A$ by (10).

Finally, assume that $a \not\Lleftarrow b$ and $a \odot b < a$; then $A = (a \odot b, 1)$. We have $B, C \leq A$ by (9).

We next show the associativity of $\odot$. Assume $(a,r) \leq (b,s) \leq (c,t)$. We have to check that $A = [(a,r) \odot (b,s)] \odot (c,t)$ and $B = (a,r) \odot [(b,s) \odot (c,t)]$ and $C = [(a,r) \odot (c,t)] \odot (b,s)$ all coincide.

Assume that the first case in (8) applies at least to one pair among a, b, c; this means there are two equal idempotents among a, b, c. If $a = b = c$, we have $A = B = C = (a, rst)$. If $a = b < c$, all three values equal (a, rs). Finally, let $a < b = c$. If then $a \Lleftarrow b$; then we get $(a, r^{\frac{1}{st}})$. If $a \not\Lleftarrow b$ and $a \odot b = a$, we have (a,r). If $a \not\Lleftarrow b$ and $a \odot b < a$, we get $(a \odot b, 1)$.

Assume now that the first case of (8) does not apply for any pair among a, b, c, but that the second case of (8) does apply at least to one pair. Assume first that $a < b < c$ and $a \Lleftarrow b$. Note that then $a \odot b \odot c = a,\ b \odot c = b,\ a \not\Lleftarrow c$, and $b \not\Lleftarrow c$. We get $A = B = C = (a, r^{\frac{1}{s}})$.

Assume next that $a \leq b < c$ and both $a \Lleftarrow c$ and $b \Lleftarrow c$. Note that then $a \odot b \odot c = a \odot b < a$ and $a \not\Lleftarrow b$. We get $(a \odot b, 1)$ in all three cases.

Assume now that $a \leq b < c$, $a \Lleftarrow c$ and $b \not\Lleftarrow c$. Note again that $a \odot b \odot c = a \odot b < a$ and $a \not\Lleftarrow b$ and furthermore $a \not\Lleftarrow b \odot c$. We again get $(a \odot b, 1)$.

Finally, assume that $a \leq b < c$, $a \not\Lleftarrow c$, and $b \Lleftarrow c$. Then $a \not\Lleftarrow b$. If $a \odot b = a$, we get (a,r); if $a \odot b < a$, we get $(a \odot b, 1)$.

Let us now assume that for all pairs a, b and a, c and b, c the third possibility in (8) applies. Note that then, (10) holds for any $(a,r),(b,s) \in \hat{L}$. In case $a \odot b = a$, we calculate $A = C = (a,r)$, and since $a \odot b \odot c = a$ and $a \leq b \odot c$, we get also $B = (a,r)$. In case $a \odot b \odot c = a \odot b < a$, we have $A = B = C = (a \odot b, 1)$. Finally, in case $a \odot b \odot c < a \odot b < a$, we get $A = B = C = (a \odot b \odot c, 1)$. The proof is complete. □

Lemma 3.14 *Let* $(L; \leq, \odot, 0, 1)$ *be a finite* $0,1$*-tomonoid, and let* $(\hat{L}; \leq, \odot, (0,1), (1,1))$ *be the* $0,1$*-tomonoid defined in Example* 3.12*. Then the residuum* $\to$ *of* $\odot$ *exists in L and in* $\hat{L}$*, and* $(L; \wedge, \vee, \odot, \to, 0, 1)$ *as well as* $(\hat{L}; \wedge, \vee, \odot, \to, (0,1), (1,1))$ *are MTL-algebras. The mapping* $\vartheta\colon L \to \hat{L},\ \ a \mapsto (a,1)$ *is an isomorphic embedding of MTL-algebras.*

Proof. Defining $\to$ by (2), we get an MV-algebra $(L; \wedge, \vee, \odot, \to, 0, 1)$, as was already stated above. To see that also $\hat{L}$ can be expanded to an MV-algebra, we have to show that in $\hat{L}$, we can define $\to$ by (2). Let $(a,r),(b,s) \in \hat{L}$. Note that, for any $(x,t) \in \hat{L}$, the second component of $(a,r) \odot (x,t)$ depends on t continuously, and t takes values in $(0,1]$. It follows that the maximum of $\{(x,t)\colon\ (a,r) \odot (x,t) \leq (b,s)\}$ always exists.

Clearly, ϑ preserves the order, and ϑ is injective. Moreover, let $a, b \in L$. We see from (10) that $(a,1) \odot (b,1) = (a \odot b, 1)$. Again by (10), $(a,1) \to (b,1) = \max\,\{(x,t)\colon\ (a,1) \odot (x,t) \leq (b,1)\} = \max\,\{(x,t)\colon\ (x,t) \wedge (a \odot x, 1) \leq (b,1)\} = \max\,\{(x,t)\colon\ x \leq b \text{ or } a \odot c \leq b\} = \max\,\{(x,t)\colon\ a \odot x \leq b\} = (a \to b, 1)$. So we have shown that $\vartheta\colon L \to \hat{L},\ \ a \mapsto (a,1)$ is a homomorphism of MTL-algebras. □

Lemma 3.15 *Let* $(L; \leq, \odot, 0, 1)$ *be a finite totally ordered MTL-algebra. Then L may be isomorphically embedded into a t-norm algebra* $([0,1]; \wedge, \vee, \odot, \to, 0, 1)$*, where* $\odot$ *is a regular l.-c. t-norm.*

Proof. Let $(\hat{L}; \leq, \odot, (0,1), (1,1))$ be the $0,1$-tomonoid constructed from L as in Example 3.12. From Lemma 3.14 and the fact that $\hat{L}$ is order-isomorphic to the real unit interval, we may conclude that $(\hat{L}; \wedge, \vee, \odot, \rightarrow, (0,1), (1,1))$ is isomorphic to a l.-c. t-norm algebra. Let $\iota : \hat{L} \to [0,1]$ be the isomorphism.

We have to show that the t-norm $\odot$ is regular. Discontinuities of translations associated $\odot$ can arise only at the points $\iota(a,1)$, where $a \in L$; so $\odot$ has few discontinuity points.

It remains to prove that q is simple. Consider any of the intervals $I_a = \{\iota(a,r) : r \in (0,1]\}$, where $a \in L \setminus \{0\}$. Let b be the smallest idempotent $\geq a$ such that $a \odot b = a$.

Assume first that a is idempotent. Then $b = a$, and $a \Lleftarrow b$. So for all $r \in (0,1]$, we have $(a,r) \odot (b,1) = (a,r)$, and $(a,r) \odot (b,s) = (a,rs) < (a,r)$ if $s < 1$.

Assume next that a is not idempotent; then $a < b$. We claim that $a \odot Pb < a$ and consequently $a \Lleftarrow b$. Indeed, assume $a \odot Pb = a$, and let $k \geq 1$ be large enough such that $c = (Pb)^k$ is idempotent; then $a \odot c = a < c < b$ in contradiction to the minimality of b. So $(a,r) \odot (b,s) = (a, r^{\frac{1}{s}})$, where $r, s \in (0,1]$. In particular, $(a,r) \odot (b,1) = (a,r)$ and for $r \in (0,1)$, we have $(a,r) \odot (b,s) < (a,r)$ if $s < 1$.

So in both cases, $q(t)$ contains the whole interval I_a for all $t \geq \iota^{-1}(b,1)$, and $q(t)$ has empty intersection with the interior of I_a for all $t < \iota^{-1}(b,1)$. □

So we arrive at the announced result:

Theorem 3.16 *The t-norm algebras based on regular l.-c. t-norms generate the variety of MTL-algebras.*

Proof like for Theorem 3.6. □

4 The basic intervals and their associated algebras

In this section, we start examining the regular l.-c. t-norms in a systematic manner. The structural characteristics which we will compile, refer to the associated translation tomonoids. Namely, using the simplicity of the function q from Definition 3.7, we will partition the unit interval into certain subintervals and restrict the translations to them in a specific way. As we will see later, each of the resulting restricted translation tomonoids belongs, up to isomorphism, to one out of six different types.

Throughout this section, let $\odot$ be a regular l.-c. t-norm.

Lemma 4.1 *There is a finite chain $0 = a_0 < a_1 < \ldots < a_k = 1$ such that, for any $i \in \{1, ..., k\}$, exactly one of the following possibilities hold:*

(Cont) *For a unique $v_i \in (0,1]$, we have: (α) $[a_{i-1}, a_i] \subseteq q(t)$ for $t \geq v_i$; (β) $(a_{i-1}, a_i) \cap q(t) = \emptyset$ for $t < v_i$; and (γ) $[a_{i-1}, a_i]$ is a maximal closed interval with the properties (α) and (β).*

(Idp-l) *For a uniquely determined pair $u_i, v_i \in [0,1]$ such that $u_i < v_i$, there is an order antiisomorphism $m\colon (u_i, v_i] \to [a_{i-1}, a_i)$ such that $q(t) \cap (a_{i-1}, a_i) = [m(t), a_i)$ for $t \in (u_i, v_i)$.*

(Idp-r) *For a uniquely determined pair $u_i, v_i \in [0,1]$ such that $u_i < v_i$, there is an order isomorphism $m\colon (u_i, v_i] \to (a_{i-1}, a_i]$ such that $q(t) \cap (a_{i-1}, a_i) = (a_{i-1}, m(t)]$ for $t \in (u_i, v_i)$.*

Proof. Choose $0 = a_0 < a_1 < \ldots < a_k = 1$ such that $(0,1] = \dot{\bigcup}_{1 \leq i \leq k} (a_{i-1}, a_i]$ is a partition as specified in Definition 3.10. Assume that this partition is a coarsest w.r.t. the conditions of Definition 3.10.

Let $1 \leq i \leq k$. There are three possibilities:

Case (i): There is a $v_i \in (0,1]$ such that $(a_{i-1}, a_i) \subseteq q_i(t)$ if $t \geq v_i$, and $(a_{i-1}, a_i) \cap q_i(t) = \emptyset$ if $t < v_i$.

Case (ii): There is a pair $u_i, v_i \in [0,1]$ such that $u_i < v_i$ and an order isomorphism $[u_i, v_i] \to \{[d, a_i) \setminus \{a_{i-1}\} \colon a_{i-1} \le d \le a_i\}, \quad t \mapsto q(t) \cap (a_{i-1}, a_i)$.

Case (iii): There is a pair $u_i, v_i \in [0,1]$ such that $u_i < v_i$ and an order isomorphism $[u_i, v_i] \to \{(a_{i-1}, d] \setminus \{a_i\} \colon a_{i-1} \le d \le a_i\}, \quad t \mapsto q(t) \cap (a_{i-1}, a_i)$.

Evidently, these possibilities are mutually exclusive. In case (i), conditions (α) and (β) of (Cond) are obviously fulfilled. In view of our assumption that the partition is coarsest, also the maximality condition (γ) is assured. Finally, in case (ii), (Idp-l) is fulfilled; in case (iii), (Idp-r) is fulfilled. □

Accordingly, we will associate with the t-norm a number of specific points.

Definition 4.2 Let $0 = a_0 < a_1 < \ldots < a_k = 1$ be such that for each $i \in \{1, ..., k\}$ (Cont) or (Idp-l) or (Idp-r) holds, in accordance with Lemma 4.1. Then $(a_{i-1}, a_i]$, $1 \le i \le k$, are called *basic intervals* of $\odot$.

Let $i \in \{1, ..., k\}$ be such that (Cont) holds, and let v_i be as specified in (Cont). Then $(a_{i-1}, a_i]$ is called a *continuity interval*; and v_i is called the *opening point* of $(a_{i-1}, a_i]$. Furthermore, let $u_i \le v_i$ be minimal such that $\lambda_t(a_i) \in (a_{i-1}, a_i]$ for all $t \in (u_i, v_i]$. Then $(u_i, v_i]$ is called the *parameter set* associated to $(a_{i-1}, a_i]$.

Similarly, let i be such that (Idp-l) or (Idp-r) holds, and let u_i, v_i be as specified by (Idp-l) or (Idp-r), respectively. Then $(a_{i-1}, a_i]$ is called an *idempotency interval* or, more specifically, a *left-* or *right-idempotency interval*, respectively; and v_i is called the *opening point* of $(a_{i-1}, a_i]$. Moreover, $(u_i, v_i]$ is called the *parameter set* associated to $(a_{i-1}, a_i]$.

Assume finally that any two parameter sets associated to idempotency intervals either coincide or are disjoint. Then we call $(a_0, a_1, \ldots, a_k)$ a *frame* for $\odot$.

Note that the continuity intervals are assigned to the l.-c. t-norm $\odot$ in a unique way. So the union of all idempotency intervals is uniquely determined as well. Indeed, a point $x > 0$ is in some continuity interval iff x it is not in an idempotency interval iff there is a $y < x$ such that the set $\{q(t) \cap (y, x) \colon t \in [0,1]\}$ is only two-element. The choice of the idempotency intervals themselves is not unique, though; for instance, a left-idempotency interval may be divided into two intervals of this type. Uniqueness might be achieved by a maximality condition; we do not make such a requirement, not to complicate matters unnecessarily.

We rather assume now that a fixed choice of basic intervals has been made. We will actually assume that we are given a frame $(a_0, \ldots, a_k)$; this is easily seen to be possible.

Lemma 4.3 *There is a frame for* $\odot$.

Proof. Let $0 = a_0 < a_1 < \ldots < a_k = 1$ be such that for each $i \in \{1, ..., k\}$ (Cont) or (Idp-l) or (Idp-r) holds, in accordance with Lemma 4.1. Let there be two parameter sets $(u_i, v_i]$ and $(u_j, v_j]$ associated to the distinct idempotency intervals $(a_{i-1}, a_i]$ and $(a_{j-1}, a_j]$, and assume that $u_i < u_j < v_i < v_j$.

We may then replace the basic intervals $(a_{i-1}, a_i]$ and $(a_{j-1}, a_j]$ by four intervals. Namely, we choose $m_1 \in (a_{i-1}, a_i]$ and $m_2 \in (a_{j-1}, a_j]$ such that the new basic intervals $(a_{i-1}, m_1]$, $(m_1, a_i]$, $(a_{j-1}, m_2]$, $(m_2, a_j]$ have the parameter sets $(u_i, u_j]$, $(u_j, v_i]$, and $(v_i, v_j]$. Note that $(u_j, v_i]$ is the parameter set associated to two of the four basic intervals.

Proceeding in the same way with any pair of distinct parameter sets with non-empty intersection, we obtain a frame for $\odot$. □

We next collect the basic facts about the two types of basic intervals.

Definition 4.4 A function $e \colon [0,1] \to [0,1]$ is called *idempotent* if $e \circ e = e$. An idempotent e is called *proper* if $0 < e < id$.

Lemma 4.5 *Let* $e \colon [0,1] \to [0,1]$ *be left-continuous, increasing, and below the identity. Then* e *is idempotent if and only if* (i) $Q(e)$ *has no singletons and* (ii) e *is determined by* $Q(e)$ *in the following* *way:*

$$e(x) = \begin{cases} x & \text{if } x \text{ is a right-limit point of } Q(e), \\ \text{the largest element of } Q(e) \text{ strictly below } x & \text{else} \end{cases} \tag{11}$$

for $x \in [0, 1]$.

Proof. Let e be an idempotent. For $x \in [0, 1]$, if x is a right-limit point of $Q(e)$, then $e(x) = x$ by Lemma 3.8. Let $x \notin Q(e)$. Then $e(x) \leq e^+(x) < x$. Because $e(e(x)) = e(x)$ and e is increasing, e is constant $e(x)$ on $[e(x), x]$. Moreover, $e(x) \in Q(e)$ by Lemma 3.8, and $e(x) < y \leq x$ implies $y \notin Q(e)$; so $e(x)$ is the largest element below x of $Q(e)$. Finally, let $x \in Q(e) \setminus \{0\}$ such that x is a right-limit point of the complement of $Q(e)$. By left-continuity, $e(x)$ is the largest element of $Q(e)$ strictly below x. So (11) is shown.

Assume furthermore that $x > 0$ is a singleton of $Q(e)$. If then $x < 1$, there is an $y > x$ such that, by (11), $e(y) = e(x) \leq x < y$. If $x = 1$, then, by (11), e is constant $e(1) < 1$ on $(e(1), 1)$, so even on $[e(1), 1]$ by idempotency and left-continuity; but this contradicts $x \in Q(e)$. So there can be no singleton in $Q(e)$.

Conversely, it is clear that any e fulfilling (i) and (ii) is idempotent. □

Lemma 4.6 *For any* $f \in \Lambda$, *we have* $\text{regcl}(Q(f \circ f)) = \text{regcl}(Q(f))$.

In particular, for any $t \in [0, 1]$, $q(t) \setminus q(t \odot t)$ *has an empty interior.*

Proof. Because $f \circ f \leq f$, we have $Q(f \circ f) \subseteq Q(f)$. Moreover, if x is in the interior of $Q(f)$, then $f(x) = x$ by Lemma 3.8; so $f(f(x)) = x$ and again by Lemma 3.8, $x \in Q(f \circ f)$. So the interior of $Q(f)$ equals the interior of $Q(f \circ f)$, and the first assertion follows.

Furthermore, $Q(f) \setminus Q(f \circ f)$ contains only singletons of $Q(f)$. The second part follows as well. □

Lemma 4.7 *Let* $t \in (0, 1]$.

(i) *Let* t *be in the parameter set of an idempotency interval. Then* λ_t *is idempotent.*

(ii) *Let* t *be an opening point of any basic interval. Then* λ_t *is idempotent.*

Proof. Let t be either contained in the parameter interval of an idempotency interval, or let t be an opening point of a continuity interval. In view of Lemma 4.1, we conclude that $q(t) \setminus q(s)$ has a non-empty interior for any $s < t$.

Assume now that λ_t is not idempotent. Then $\lambda_t \circ \lambda_t$ is different from λ_t, and this means by (T3) that $t \odot t = \lambda_t(\lambda_t(1)) < \lambda_t(1) = t$. So it follows from Lemma 4.6 that $q(t) \setminus q(t \odot t)$ has an empty interior. The assertions are proved. □

Lemma 4.8 *Let* $(a, b]$ *be a continuity interval with parameter set* $(u, v]$. *Then we have:*

(i) $\lambda_v|_{(a,b]} = id|_{(a,b]}$, *and for any* $x \in (a, b)$ *and* $t < v$, *we have* $\lambda_t(x) \leq \lambda_t^+(x) < x$.

(ii) *For any* $t \in (u, v)$, λ_t *is not idempotent.*

Proof. (i) Because $q(v) \supseteq [a, b]$, $\lambda_v(x) = x$ for $x \in (a, b]$ by Lemma 3.8. For $x \in (a, b)$ and $t < v$, we have $q(t) \cap (a, b) = \emptyset$, so $\lambda_t^+(x) < t$. The inequality $\lambda_t(x) \leq \lambda_t^+(x)$ holds in general.

(ii) Let $x \in (a, b)$ be such that $a < \lambda_t(x)$. So $\lambda_t(x) \in (a, b)$, and it follows $\lambda_t(\lambda_t(x)) < \lambda_t(x)$ by part (i). So λ_t is not idempotent. □

Lemma 4.9 *Let* $(t, u]$ *and* $(v, w]$ *be the parameter sets of two distinct basic intervals. Then either both* $(t, u]$ *and* $(v, w]$ *are associated to idempotency intervals and* $(t, u] = (v, w]$, *or both are associated to continuity intervals and* $u = w$, *or* $(t, u]$ *and* $(v, w]$ *are disjoint.*

Proof. Assume that $(t, u]$ and $(v, w]$ are not disjoint. If then $(t, u]$ is associated to an idempotency interval, $(v, w]$ cannot be associated to a continuity interval by Lemmas 4.7(i) and 4.8(ii). So $(v, w]$ is associated to an idempotency interval as well, and by our general assumption to work with a frame, the two intervals must coincide. Similarly, if $(t, u]$ is associated to a continuity interval, then so is $(v, w]$. Since $q(u)$ and $q(w)$ are idempotent by 4.7(ii), we conclude $u = w$ by Lemma 4.8(ii). □

We are now ready to describe in more detail how the parameter sets are related to the basic intervals.

Lemma 4.10 *Let d be the opening point of some basic interval. Then there is a $c < d$ such that $(c, d]$ is a basic interval with the parameter set $(c, d]$. For each $t \in (c, d]$, λ_t is constant t on $[d, 1]$.*

Assume that $(c, d]$ is an idempotency interval. Let $(a, b]$ be a further basic interval with opening point d; then $b \le c$. In this case, also $(a, b]$ is an idempotency interval, and $(c, d]$ is the parameter set of $(a, b]$.

Assume that $(c, d]$ is a continuity interval. Let $(a, b]$ be a further basic interval whose opening point is d; then $b \le c$. In this case, also $(a, b]$ is a continuity interval.

Finally, the interval $(c, d]$ does not contain any further opening point apart from d.

Proof. λ_d is idempotent by Lemma 4.7, and $\lambda_d(1) = d$. So, by Lemma 4.5, there is a $w < d$ such that $\lambda_d(x) = x$ for $x \in (w, d]$, and $\lambda_d(x) = d$ for $x \in [d, 1]$.

It follows that $q(d) \cap (d, 1] = \emptyset$, $q(d) \supseteq [w, d]$, and for any $t < d$, $q(t) \cap [w, d]$ is a proper subset of $[w, d]$ and in particular $d \notin q(t)$. Furthermore, any basic interval with opening point d is contained in $q(d)$, which in turn is contained in $[0, d]$.

Let now $(c, d']$ be the basic interval containing d, and let $(u, v]$ be the associated parameter set. By Lemma 4.9, the opening point d cannot be in the interior of $(u, v]$; so either $d \le u$ or $v < d$ or $v = d$. If $d \le u$, then $[w, d] \subseteq q(d) \subseteq q(u)$, in contradiction to $q(u) \cap (c, d'] = \emptyset$. If $v < d$, then $d \in (c, d'] \subseteq q(v)$ contradicts the fact that $d \notin q(t)$ for any $t < d$. So $d = v$, and $(u, d]$ is the parameter set associated to the basic interval $(c, d']$. But then $(c, d'] \subseteq [0, d]$ and $d' \ge d$; so actually, $d = d'$, and $(u, d]$ is the parameter set associated to $(c, d]$.

From $q(d) \supseteq (c, d]$, we have $\lambda_d(x) = x$ for $x \in (c, d]$. So for $t \in (c, d]$ and $x \in [d, 1]$, we conclude $\lambda_t(x) = \lambda_x(t) = \lambda_d(t) = t$, that is, λ_t is constant t on $[d, 1]$ for all $t \in (c, d]$.

It moreover follows that for any $t \in [0, 1]$, $\lambda_t(d) > c$ if and only if $t > c$; so $u = c$, that is, the parameter set of $(c, d]$ is actually $(c, d]$ itself.

The second, third and forth paragraph follow from Lemma 4.9. □

So we see that a point d may be an opening point of a finite number of basic intervals, all of which are either idempotency of continuity intervals. Moreover, one of these intervals is distinguished by the fact that its right border is d. So it is of the form $(c, d]$ for some $c < d$, and $(c, d]$ is then also its parameter set. The remaining basic intervals with opening point d, if there are any, are located on the left side of c.

Definition 4.11 A basic interval $(a, b]$ is called *primary* if its opening point is b, otherwise *secondary*.

We now come to a construction which is simple, but fundamental for this paper. Namely, for each basic interval $[a, b]$, we may restrict the action of any translation $f \in \Lambda$ to $[a, b]$ as follows.

Definition 4.12 Let $(a, b]$ be some basic interval. For any $f \in \Lambda$, let

$$f_{(a,b]} \colon [a, b] \to [a, b], \quad x \mapsto f(x) \vee a \tag{12}$$

and put $\Lambda_{(a,b]} = \{f_{(a,b]} \colon f \in \Lambda\}$. We endow $\Lambda_{(a,b]}$ with the pointwise order $\le$; with the composition of functions $\circ$; with the constant $0_{(a,b]} \colon [a, b] \to [a, b]$, $x \mapsto a$; and with the constant $id_{(a,b]} \colon [a, b] \to [a, b]$, $x \mapsto x$. Then $(\Lambda_{(a,b]}; \le, \circ, 0_{(a,b]}, id_{(a,b]})$ is called the *basic tomonoid* associated with $(a, b]$.

For any $f \in \Lambda$, we call $f_{(a,b]}$ the *projection* of f into $\Lambda_{(a,b]}$. Moreover, we say that $f \in \Lambda$ *crosses* $\Lambda_{(a,b]}$ if $0_{(a,b]} < f_{(a,b]} < id_{(a,b]}$.

So given a translation $f \in \Lambda$ and a basic interval $(a,b]$, we have that $f_{(a,b]}(x) = f(x)$ if $x > a$ and $f(x) > a$, and else $f_{(a,b]}(x) = a$. Geometrically, we may visualize the transition from f to $f_{(a,b]}$ as follows: We consider the graph of $f|_{[a,b]}$, that is, $\{(x, f(x)) : x \in [a,b]\}$, and we project any point $(x, f(x))$ such that $x \in [a,b]$ and $f(x) < a$, to (x, a); the points which are then located within the triangle with vertices (a,a), (b,a), and (b,b), form the graph of $f_{(a,b]}$.

Note that $f_{(a,b]}$ coincides with f on $\{x \in (a,b] : f(x) > a\}$. So there is a $d \in [a,b]$ such that $f_{(a,b]}(x) = a$ for $x \in [a,d]$, and $f_{(a,b]}(x) = f(x) > a$ for $x \in (d,b]$. Furthermore, we have $f_{(a,b]} = 0_{(a,b]}$ iff $f(x) \leq a$ for any $x \in (a,b]$ iff $f_{(a,b]}$ maps the whole interval $[a,b]$ to a. Similarly, $f_{(a,b]} = id_{(a,b]}$ iff $f(x) = x$ for any $x \in (a,b]$ iff $f_{(a,b]}$ is the identity on $[a,b]$. In particular, f crosses $\Lambda_{(a,b]}$ iff $f(b) > a$ and, for some $x \in (a,b]$, $f(x) < x$.

To determine a basic tomonoid, it is enough to consider the translations by the elements of the associated parameter set:

Lemma 4.13 *Let $(a,b]$ be a basic interval with the associated parameter set $(u,v]$. Then*

$$\Lambda_{(a,b]} = \{\lambda_{t\,(a,b]} : t \in [u,v]\}.$$

Proof. We have $\lambda_{u\,(a,b]} = 0_{(a,b]}$ and $\lambda_{t\,(a,b]} > 0_{(a,b]}$ if $t > u$; similarly, $\lambda_{v\,(a,b]} = id_{(a,b]}$ and $\lambda_{t\,(a,b]} < id_{(a,b]}$ if $t < v$. □

We next see how the a translation tomonoid and one of the basic tomonoids are related, and in particular that the latter, endowed with the pointwise order and the operation $\circ$, is indeed a tomonoid.

Lemma 4.14 *Let $(a,b]$ be a basic interval.*

(i) *$(\Lambda_{(a,b]}; \leq)$ is isomorphic to $([0,1]; \leq)$. The bottom element is $0_{(a,b]}$ and the top element is $id_{(a,b]}$. Moreover, the suprema are calculated pointwise.*

(ii) *$\Lambda_{(a,b]}$ is closed under $\circ$, and the mapping*

$$\iota_{(a,b]} : \Lambda \to \Lambda_{(a,b]}, \quad f \mapsto f_{(a,b]}.$$

is a homomorphism of the tomonoids $(\Lambda; \leq, \circ, 0, id)$ and $(\Lambda_{(a,b]}; \leq, \circ, 0_{(a,b]}, id_{(a,b]})$. Moreover, $\iota_{(a,b]}$ preserves arbitrary suprema.

Proof. We clearly have $0_{(a,b]} \leq f_{(a,b]} \leq id_{(a,b]}$ for all $f \in \Lambda$. From the definition of $f_{(a,b]}$, it is obvious that the order of $\Lambda_{(a,b)}$ is total and that ι is monotone. We furthermore easily see that, for any set $f_\iota \in \Lambda$, $\iota \in I$, we have $\bigvee_\iota f_{\iota\,(a,b]} = (\bigvee_\iota f_\iota)_{(a,b]}$. In particular, the order of $\Lambda_{(a,b]}$ is complete. It also follows that suprema are calculated pointwise. Finally, we easily check that, for $f, g \in \Lambda$, we have $f_{(a,b]} \circ g_{(a,b]} = (f \circ g)_{(a,b]}$.

The proof of part (ii) is done; for part (i), it remains to show that $(\Lambda_{(a,b]}; \leq)$ is isomorphic to $([0,1]; \leq)$.

To this end, assume first that $(a,b]$ is an idempotency interval with the associated parameter set $(u,v]$. Then λ_t is idempotent for each $t \in (u,v]$, and consequently $\lambda_{t\,(a,b]}$ is uniquely determined by $q(t) \cap (a,b)$. So by Lemma 4.13, the map $[u,v] \to \Lambda_{(a,b]}$, $t \mapsto \lambda_{t\,(a,b]}$ is a bijection, which is moreover an order isomorphism. In particular, $\Lambda_{(a,b]}$ is order-isomorphic to $[0,1]$.

Assume now that $(a,b]$ is a continuity interval with the associated parameter set $(u,v]$. Then, by Lemma 4.8(i), $\lambda_t(x) < x$ for all $t < v$ and $x \in (a,b)$. Let $f, g \in \Lambda$ be such that $f_{(a,b]} < g_{(a,b]}$. Then $f(x) \vee a < g(x)$ for some $x \in (a,b)$; note that $g(x) \in (a,b)$. Now, $id_{(a,b]} = \lambda_{v\,(a,b]}$ is the pointwise supremum of all $\lambda_{t\,(a,b]}$, where $t < v$; so for some t, we have $f(x) \vee a < \lambda_t(g(x)) < g(x)$, that is, $f_{(a,b]} < (\lambda_t \circ g)_{(a,b]} < g_{(a,b]}$. It follows that the order of $\Lambda_{(a,b]}$ is dense.

Moreover, $\{\lambda_{r\,(a,b]} : r \in [0,1] \cap \mathbb{Q}\}$ is dense in $\Lambda_{(a,b]}$. So $(\Lambda_{(a,b]}; \leq)$ is a complete totally ordered set which is moreover dense and separable; consequently, $\Lambda_{(a,b]}$ is order-isomorphic to $[0,1]$. □

A basic tomonoid $\Lambda_{(a,b]}$ inherits most of the properties of the translation tomonoid Λ. In the sequel, by one of the conditions (Ti), $i = 1, ..., 6$, to hold for $\Lambda_{(a,b]}$, we understand that in (Ti), each appearance of Λ, 0, or id is replaced by $\Lambda_{(a,b]}$, $0_{(a,b]}$, or $id_{(a,b]}$, respectively.

Lemma 4.15 *Let $(a, b]$ be a basic interval. Then the basic tomonoid $(\Lambda_{(a,b]}; \leq, \circ, 0_{(a,b]}, id_{(a,b]})$ fulfils* (T1)*,* (T2)*,* (T4)*,* (T5)*, and* (T6)*.*

Proof. (T1) and (T4) are evident. (T2) and (T6) hold by Lemma 4.14(ii). (T5) holds by Lemma 4.14(i). □

5 Algebras of idempotent functions and algebras of continuous functions

In this section, we will characterize six specific tomonoids of left-continuous functions. The results will be used in the sequel to describe the basic tomonoids associated to the basic intervals of a regular l.-c. t-norm. Apart from an insertion concerning continuous t-norms, the results of this section do not explicitly refer to the theory of t-norms.

We first deal with the case of tomonoids which contain exclusively idempotent functions.

Definition 5.1 Let F be a set of left-continuous functions from $[0, 1]$ to $[0, 1]$, defined as follows.

(i) Let F contain the functions

$$f_t\colon [0,1] \to [0,1], \quad x \mapsto \begin{cases} 0 & \text{for } x \leq t, \\ x & \text{for } x > t, \end{cases} \tag{13}$$

where $t \in [0, 1]$. Then $(F; \leq, \circ, 0, id)$ is called the *left-idempotency monoid.*

(ii) Let F contain the functions

$$f_t\colon [0,1] \to [0,1], \quad x \mapsto \begin{cases} x & \text{for } x \leq t, \\ a & \text{for } x > t, \end{cases} \tag{14}$$

where $t \in [0, 1]$. Then $(F; \leq, \circ, 0, id)$ is called the *right-idempotency monoid.*

The following Theorem is not essential in the sequel, but we like to include it for the sake of completeness.

Theorem 5.2 *Let F be a set of functions from $[0, 1]$ to $[0, 1]$ such that* (T1)*,* (T2)*,* (T4)*,* (T5)*, and the following conditions are fulfilled:*

(I1) *Every $f \in F$ is idempotent.*

(I2) *For every $f \in F$, $Q(f)$ is connected and contains either* 0 *or* 1*.*

(I3) *For every non-empty open interval $(c, d) \subseteq [0, 1]$, there is an $f \in F$ such that $\emptyset \subsetneqq \{f(x)\colon x \in (c, d)\} \cap (c, d) \subsetneqq (c, d)$.*

Then $(F; \leq, \circ, id)$ is either the left- or to the right-idempotency monoid.

Proof. By (I2) and Lemma 4.5, which we may apply due to (T1), (T4), (T5), and (I1), every $f \in F$ is either of the form (13) or (14). By (T5), $0, id \in F$, and there is a proper idempotent in F; furthermore, by (T5), every two elements of F are comparable. It follows that either all $f \in F$ are of the form (13), or all $f \in F$ are of the form (14); so $F = \{f_t\colon t \in S\}$, where $\{0, 1\} \subset S \subseteq [0, 1]$ and f_t is defined by (13) or (14), respectively.

By (S5), $(S; \leq)$ is a dense and complete totally ordered set. Moreover, by (I3), there is no non-empty open interval $(c, d) \subseteq [0, 1]$ such that $(c, d) \cap S = \emptyset$. It follows that $S = [0, 1]$, and the proof is finished. □

We insert a note on the parametrization of the algebras considered in Theorem 5.2.

Lemma 5.3 *Let $u < v$, and let $F = \{g_s : [0,1] \to [0,1] : s \in [u,v]\}$ be such that* (T1), (T2), (T4), (T5) *and* (I1)–(I3) *are fulfilled and such that $g_s \leq g_t$ iff $s \leq t$ for any $s, t \in [u,v]$. Then there is an order automorphism $\varphi : [0,1] \to [0,1]$ such that, w.r.t. the notation of Definition* 5.1, *$\varphi^{-1} \circ g_{(1-t)u+tv} \circ \varphi = f_t$ for all $t \in [0,1]$.*

Proof. By Theorem 5.2, $F = \{f_t : t \in [0,1]\}$, where f_t is defined by (13) or (14). So for some order automorphism $\varphi\colon [0,1] \to [0,1]$, we have $g_{(1-t)u+tv} = f_{\varphi(t)}$. We easily check that $\varphi \circ f_t \circ \varphi^{-1} = f_{\varphi(t)}$, $t \in [0,1]$, and the assertion follows. □

We now turn to the more difficult case that we are given tomonoids containing only continuous functions.

We begin considering the case that the translation tomonoid of a l.-c. t-norm consists of continuous functions.

Lemma 5.4 *Let $e : [0,1] \to [0,1]$ be continuous, increasing, and below the identity. Then e is idempotent if and only if there is an $a \in [0,1]$ such that, for $x \in [0,1]$,*

$$e(x) = \begin{cases} x & \text{for } x \leq a, \\ a & \text{for } x \geq a. \end{cases} \tag{15}$$

Proof by Lemma 4.5. □

Theorem 5.5 *Let Λ be the translation tomonoid of the l.-c. t-norm $\odot$. Then $\odot$ is continuous if and only if Λ consists of continuous functions only.*

Moreover, let Λ not contain any proper idempotent. Then $\odot$ is isomorphic either to the Łukasiewicz t-norm or to the product t-norm.

Proof. The first part is immediate. For the second, note that, by Theorem 2.6, Λ contains a proper idempotent if and only if $\odot$ has an idempotent different from 0 and 1. A continuous t-norm whose only idempotents are 0 and 1, is isomorphic either to the Łukasiewicz t-norm or to the product t-norm. This well-known fact can be found in [MoSh]; otherwise see e.g. [KMP]. □

For the functional algebras which now follow, we have in mind any closed subinterval $[a,b]$ of the real unit interval as their domain and range. For simplicity, however, we will use in this section the whole interval $[0,1]$. Isomorphisms between sets of functions on real intervals are defined in the obvious way:

Definition 5.6 Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$ be such that $a_1 < b_1$ and $a_2 < b_2$. Let F consist of functions from $[a_1, b_1]$ to $[a_1, b_1]$, and let G consist of functions from $[a_2, b_2]$ to $[a_2, b_2]$. Then F and G are called *isomorphic* if there is an order-preserving bijection $\varphi\colon [a_1, b_1] \to [a_2, b_2]$ such that $G = \{\varphi \circ f \circ \varphi^{-1} : f \in F\}$.

In case that we can choose φ linear, we say that G is *scaled* from F.

The following are the structures which we will discuss.

Definition 5.7 Let F be a set of continuous functions from $[0,1]$ to $[0,1]$, defined as follows.

(i) Let F contain the functions $f_t\colon [0,1] \to [0,1]$, $x \mapsto t \cdot x$, where $t \in [0,1]$ and $\cdot$ is the multiplication of reals. Then $(F; \leq, \circ, 0, id)$ is called the *product monoid.*

(ii) Let F contain the functions $f_t : [0,1] \to [0,1]$, $x \mapsto (t + x - 1) \vee 0$, where $t \in [0,1]$. Then $(F; \leq, \circ, 0, id)$ is called the *Łukasiewicz monoid.*

(iii) Let F contain the functions $f_t\colon [0,1] \to [0,1]$, $x \mapsto \frac{(t+x-1)\vee 0}{t}$, where $t \in (0,1]$, and $f_0 = 0$. Then $(F; \leq, \circ, 0, id)$ is called the *reversed product monoid.*

(iv) Let F contain the functions $f_t\colon [0,1] \to [0,1]$, $x \mapsto x^{\frac{1}{t}}$, where $t \in (0,1]$, and $f_0 = 0$. Then $(F; \leq, \circ, 0, id)$ is called the *power monoid.*

We remark that the Łukasiewicz monoid has also been called the nil interval. Note that it arises from the product monoid as the Rees quotient w.r.t. the ideal $[0, \frac{1}{2}]$.

In the next lemma, for some bijection f from a set to itself, we define f^k for each $k \in \mathbb{Z}$, by putting $f^0 = f$; $f^k = \underbrace{f \circ \ldots \circ f}_{k \text{ times}}$ for $k \geq 1$; and $f^k = \underbrace{f^{-1} \circ \ldots \circ f^{-1}}_{-k \text{ times}}$ for $k \leq -1$.

Lemma 5.8 *Let $g_k\colon [0,1] \to [0,1]$, $0 \leq k < \omega$, be strictly increasing continuous functions such that, for each k,* (i) *$g_k(0) = 0$, $g_k(x) < x$ for $x \in (0,1)$, and $f(1) = 1$,* (ii) *$g_{k+1}^2 = g_k$, and* (iii) *the functions g_k converge uniformly to id. Then there is an order automorphism $\varphi\colon [0,1] \to [0,1]$ such that, for each k, $g_k(\varphi(x)) = \varphi(x^{2^{(\frac{1}{2})^k}})$, $x \in [0,1]$.*

Proof. We will identify g_k, $k < \omega$, and φ with their restrictions to the open unit interval; the problem then reads as follows and is obviously equivalent to the original one: We are given for each k an order automorphism $g_k\colon (0,1) \to (0,1)$ such that $g_k(x) < x$ for all $x \in (0,1)$; we have $g_k = g_{k+1}^2$; and we have to determine an order automorphism $\varphi\colon (0,1) \to (0,1)$ such that $g_k(\varphi(x)) = \varphi(x^{2^{(\frac{1}{2})^k}})$ holds for $x \in (0,1)$. We proceed as explained in [PoMa].

We first set $y = \ln x \in (-\infty, 0)$ and $\psi\colon (-\infty,0) \to (0,1)$, $y \mapsto \varphi(e^y)$. Then $\varphi(x) = \psi(y)$, and the problem is to solve $g_k(\psi(y)) = \psi(2^{(\frac{1}{2})^k} y)$ for $y \in (-\infty, 0)$, where ψ must be an order isomorphism. We next set $z = \ln(-y) \in \mathbb{R}$ and $\chi\colon \mathbb{R} \to (0,1)$, $z \mapsto \psi(-e^z)$. Then $\psi(y) = \chi(z)$, and the problem is to find an order antiisomorphism $\chi\colon \mathbb{R} \to (0,1)$ such that

$$g_k(\chi(z)) = \chi(z + \tfrac{\ln 2}{2^k}), \quad z \in \mathbb{R}, \tag{16}$$

for each $k \geq 0$. We set $\chi(0) = \frac{1}{2}$, and for any $k \geq 0$ and $n \in \mathbb{Z}$, let $\chi(\frac{n}{2^k} \ln 2) = g_k^n(\frac{1}{2})$; we readily check that this defines χ unambiguously on the set $R = \{\frac{n}{2^k} \ln 2\colon k \geq 0,\ n \in \mathbb{Z}\}$. For $z \in R$, (16) is then fulfilled. Moreover, because $g_k(x) < x$ for each k and each $x \in (0,1)$, (16) implies that χ, seen as a function on R, is strictly decreasing.

We next show that χ can be continuously extended to the whole real line. Let $r \in \mathbb{R}$, and let $(n_k)_k$ be the sequence of natural numbers such that $r \in [\frac{n_k}{2^k} \ln 2, \frac{n_k+1}{2^k} \ln 2)$ for every k. We have to prove that the length of the interval $[\chi(\frac{n_k+1}{2^k} \ln 2), \chi(\frac{n_k}{2^k} \ln 2)]$ converges to 0 for $k \to \infty$. But this is the case because $\chi(\frac{n_k+1}{2^k} \ln 2) = g_k(\chi(\frac{n_k}{2^k} \ln 2))$ and $(g_k)_k$ converges uniformly to the identity. Note that the function $\chi\colon \mathbb{R} \to [0,1]$ is decreasing and fulfils (16), because χ and g_k are continuous.

It remains to show is that χ is surjective. Recall that $g_0(x) < x$ for all $x \in (0,1)$, let $u = \bigwedge_n g_0^n(\frac{1}{2})$, and assume $u > 0$. Then, by the continuity of g_0, we have $g_0(u) = u$, a contradiction; so $u = 0$. Similarly, we conclude $\bigvee_n g_0^{-n}(\frac{1}{2}) = 1$. So the image of χ covers the whole interval $(0,1)$. □

We next insert an auxiliary lemma, to be used also at later times, concerning the local generation of algebras of, not necessarily continuous, functions.

Definition 5.9 Let G be a set of functions from $[0,1]$ to $[0,1]$ such that $g \leq id$ for each $g \in F$. For an $f \in G$, the set

$$\{g \in F\colon\ g \geq f^k \text{ for some } k \geq 1\}$$

is called the *filter generated by f* in G.

In what follows, we define for a left-continuous function $f\colon [0,1] \to [0,1]$ the support of f by

$$\text{supp } f \ = \ \{x \in [0,1]\colon\ f(x) > 0\};$$

so if f is increasing and left-continuous, we have supp $f = (d, 1]$ for some $d \in [0,1]$. By a property to hold for f on its support, we mean that the property holds for f restricted to supp f.

Lemma 5.10 *Let G be a set of increasing left-continuous functions from $[0,1]$ to $[0,1]$ such that for every $f \in G$ the following holds: (α) $f = id$ or else $f^+(x) < x$ for every $x \in (0,1)$, (β) $f \leq g$ or $g \leq f$ for all $g \in G$, (γ) $f \circ g = g \circ f \in G$ for all $g \in G$, (δ) f has only finitely many discontinuity points. Then the following holds:*

(i) *Let $f \in G$ be such that $f < id$. If some $g \in G$ is not contained in the filter generated by f, then $g = 0$. In particular, $G \setminus \{0\}$ is contained in the filter generated by f.*

(ii) *If there are $h_i < id$, $i < \omega$, in G such that the sequence $(h_i)_i$ converges pointwise to id, then every $f \in G$ is on its support strictly increasing.*

Proof. (i) Let $[a,b] \subseteq (0,1)$. By (α) and (δ), there is an $\varepsilon > 0$ such that $f(x) < x - \varepsilon$ for all $x \in [a,b]$. It follows, for $x \in [a,b]$, that if $y \leq f^k(x)$ for all k, then $y \leq a$. Since $[a,b]$ was arbitrarily chosen within $(0,1)$, we conclude $\bigwedge_k f^k(x) = 0$ for all $x \in (0,1)$.

Let g be not in the filter generated by f. By (β), $g \leq f^k$ for all k; that is, $g(x) = 0$ for $x \in (0,1)$. By monotonicity and left-continuity, we have $g = 0$.

(ii) Let $f \in G$, let $0 < a < b < 1$, and assume that $f(a) > 0$ and $f_{[a,b]}$ is constant $z = f(a)$. Note that $z \in (0,1)$. Let $h \in G$ be such that $h(x) < x$ for all $x \in (0,1)$ and $a < h(b) < b$. It follows $(f \circ h)(b) = z$ and $(h \circ f)(b) < z$, in contradiction to (γ). □

Theorem 5.11 *Let F be a set of functions from $[0,1]$ to $[0,1]$ such that* (T1), (T2), (T5), (T6)*, and the following conditions are fulfilled:*

(C1) *Every $f \in F$ is continuous.*

(C2) *$f = id$, or else $f(x) < x$ for all $x \in (0,1)$.*

Then $(F; \leq, \circ, id)$ is isomorphic either to the product, Łukasiewicz, reversed product, or power monoid.

Proof. Let $f \in F$ be such that $0 < f < id$. If $f(1) < 1$, then $g(1) < 1$ for all $g \leq f$; and if $f(1) = 1$, then $f^k(1) = 1$ for all $k \geq 1$. Since by Lemma 5.10(i) the filter generated by f contains the whole F expect possibly 0, we conclude that either $g(1) = 1$ for all $g \in F \setminus \{0\}$ or $g = id$ is the only element of $F \setminus \{0\}$ with this property.

Similarly, if $f(x) = 0$ for some $x \in (0,1]$, then $g(x) = 0$ for all $g \leq f$; and if $f(x) > 0$ for all $x \in (0,1]$, then $f^k(x) > 0$ for all $k \geq 1$ and $x \in (0,1]$. So again by Lemma 5.10(i), $g(x) = 0$ implies $x = 0$ either for all $g \in F \setminus \{0\}$, or for id as the only element of $F \setminus \{0\}$.

Accordingly, we will distinguish four cases; still, let $0 < f < id$: (i) $f(x) > 0$ for $x \in (0,1]$, and $f(1) < 1$, (ii) $f(x) = 0$ for some $x \in (0,1]$, and $f(1) < 1$, (iii) $f(x) = 0$ for some $x \in (0,1]$, and $f(1) = 1$, (iv) $f(x) > 0$ for $x \in (0,1]$, and $f(1) = 1$.

Note next that by (T5), id is the pointwise supremum of all $h \in F$ such that $h < id$. Since F contains only continuous functions and $[0,1]$ is compact, id is even the uniform limit of a sequence of functions $< id$ in F.

Consequently, by Lemma 5.10(ii), every $f \in F$ is on its support strictly increasing. In particular, there are no proper idempotents in F.

It furthermore follows that if $f(v) = g(v) > 0$ for some $v \in (0,1)$, then f equals g. Indeed, assume $0 < f(v') < g(v')$ for some further $v' \in (0,1)$; then $h \circ g$ and f are not comparable for an $h < id$ sufficiently close to id. So $f(x) = g(x)$ for $x \in \operatorname{supp} f \cap \operatorname{supp} g$, and by continuity and monotonicity, it follows $f = g$. In the cases (i) and (ii), the argument obviously works for the case $v = 1$ as well.

We next show that for any $v \in (0,1)$, or $v \in (0,1]$ in the cases (i) and (ii), we have $F(v) = \{f(v) : f \in F\} = [0,v]$. Clearly, 0 is the smallest element and 1 is the largest element in $F(v)$. We claim that $F(v)$ is closed under infima and suprema, calculated in $[0,1]$. $F(v)$ is closed under suprema, because by (T5), all suprema exist in F and are calculated pointwise. Furthermore, let $g_\iota \in F$, $\iota \in I$, and assume that $a = \bigwedge_\iota g_\iota(v) \in (0,1)$. For any $\varepsilon > 0$, we may choose $h < id$ sufficiently close to id such that $a - \varepsilon < h(a) < a$; then $a - \varepsilon < h(g_\iota(v)) < a$ for some ι by the continuity of h. So since $F(v)$

is closed under suprema, $a \in F(v)$, whence $F(v)$ is closed under infima as well. A similar argument shows that $F(v)$ is densely ordered. It follows that $F(v) = [0, v]$.

A corollary of these results is that the infimum of any subset of F is calculated pointwise.

Case (i). We may identify every $f \in F$ with $f(1)$; this means that F fulfills (T3). So F is a translation tomonoid of a l.-c. t-norm $\odot$. By Theorem 5.5, $\odot$ is continuous, and since there are no proper idempotents in F, $\odot$ is actually isomorphic to the Łukasiewicz or product t-norm. However, for any $f \in F$, $f(x) = 0$ only if $x = 0$, so $\odot$ is isomorphic to the product t-norm. It follows by Lemma 2.10 that F is isomorphic to the algebra $\{f\colon [0,1] \to [0,1],\ \ x \mapsto x \cdot a\colon a \in [0,1]\}$, the product monoid.

Case (ii). We may argue similarly to case (i). Namely, F fulfills (T3) and is the translation tomonoid of a continuous t-norm without proper idempotents. For $f \in F$ such that $0 < f < id$, we have, because $f(x) = 0$ for some $x > 0$ and also $f(1) < 1$, $f^k = 0$ for some k. So this t-norm is nilpotent and hence is isomorphic to the Łukasiewicz t-norm. F is isomorphic to the Łukasiewicz monoid.

Case (iii). Let $f \in F$ be such that $0 < f < id$. Then f is by assumption zero on an interval $[0, z]$, where $0 < z < 1$. Furthermore, f is strictly increasing on $[z, 1]$, and $f(1) = 1$.

Let F' contain 0 as well as the continuous functions $f' : [0,1] \to [0,1]$, where $f \in F \setminus \{0\}$ and $f'(x) = 1 - f^{-1}(1-x)$ for $x \in [0,1)$. Then every $f' \in F$ is increasing; so F' fulfils (T1). Moreover, let $f, g \in F \setminus \{0\}$ and $x \in (0,1]$; then $f^{-1}(x), g^{-1}(x)$ exist and are in $(0,1]$, whence $f^{-1}(g^{-1}(x)) = (g \circ f)^{-1}(x)$ for $x \in (0,1]$, which implies $f' \circ g' = (g \circ f)'$. It follows that (T2) and (T6) holds for F'.

Note next that, for $f, g \in F \setminus \{0\}$, we have $f \leq g$ iff $f' \leq g'$. Moreover, $0, id \in F'$, and we conclude that the order of F' coincides with the order of F, which proves the first part of (T5). For the second part of (T5), let $f \in F \setminus \{0\}$, $x \in [0,1)$, and $0 < \varepsilon < f'(x)$; then there is a $g \in F$ such that $1 - x \leq g(y + \varepsilon) < f(y + \varepsilon)$, where $y = f^{-1}(1-x)$. Then $f'(x) - \varepsilon \leq g'(x) < f'(x)$, and it follows that suprema are calculated in F' pointwise. Finally, F' fulfils (C1), and it is easily checked that F' fulfils also (C2).

So F' is a set of functions fulfilling all assumptions of this theorem. F' belongs to case (i), hence F' is isomorphic to the product monoid. Consequently, F itself is isomorphic to the reversed product monoid.

Case (iv). Fix some $v \in (0,1)$. We may identify any $f \in F$ with $f(v)$, and $F(v) = [0, v]$.

We claim that for any $f \in F$, there is a g such that $g^2 = f$. Let $g = \bigwedge\{h : h^2 \geq f\}$. Because infima calculate pointwise, $g(v) = \bigwedge\{h(v) : h^2 \geq f\}$. Since F consists of continuous and pairwise commuting functions, we get $g(g(v)) = \bigwedge\{h(k(v) : h^2, k^2 \geq f\}$. Since any two functions in F are comparable, we further conclude $g(g(v)) = \bigwedge\{h^2(v) : h^2 \geq f\} \geq f(v)$. Assume $g(g(v)) > f(v)$; then we may choose $h < id$ is sufficiently close to id such that $(g \circ h)^2(v) > f(v)$ as well, which means $(g \circ h)^2 > f$, although $g \circ h < g$. It follows $g(g(v)) = f(v)$, so $g^2 = f$.

F consists of strictly increasing continuous functions f such that $f(0) = 0$ and $f(1) = 1$ and furthermore, if $f < id$, $f(x) < x$ for all $x \in (0,1)$. Fix an arbitrary $g_0 < id$. For $k \geq 1$, let g_k be such that $g_k^{2^k} = g_0$. Then $g_0 < g_1 < \ldots < id$. Moreover, $(g_k)_k$ converges uniformly to id. Indeed, let $g = \bigvee_k g_k$; then $g_0 \leq g^k$ for every k, and $g < id$ would imply $g_0 = 0$ by Lemma 5.10(i); so $(g_k)_k$ converges to id pointwise and consequently uniformly.

By Lemma 5.8, there is an order automorphism $\varphi\colon [0,1] \to [0,1]$ such that $\tilde{F} = \{\varphi^{-1} \circ f \circ \varphi\colon f \in F\}$ contains $\tilde{g}_k : [0,1] \to [0,1],\ \ x \mapsto x^{2^{(\frac{1}{2})^k}}$ for each $0 \leq k < \omega$. It follows that the functions $x \mapsto x^r$, where $r = 2^{\frac{m}{2^n}}$ for $m, n \in \mathbb{N}$, are dense in $\tilde{F}$. So we conclude that $\tilde{F}$ consist of the functions $x \mapsto x^r$, where $r \in \{s \in \mathbb{R}\colon s \geq 1\}$; that is, $\tilde{F}$ is the power monoid. □

We conclude this section with a note on a possible freedom in the choice of the order automorphism whose existence is claimed in Theorem 5.11.

Definition 5.12 Let $F = \{f_t \in t \in [0,1]\}$ be the product, Łukasiewicz, reversed product, or power monoid, where f_t, $t \in [0,1]$, is given by Definition 5.7(i), (ii), (iii), or (iv), respectively. Then we call $f_{\frac{1}{2}}$ the *middle element* of F.

Lemma 5.13 *Let F be a set of functions from $[0,1]$ to $[0,1]$ such that conditions* (T1)*,* (T2)*,* (T5)*,* (T6)*,* (C1)*, and* (C2) *are fulfilled and moreover $f^k > 0$ holds for any $f > 0$ in F and any $k \geq 1$. Let $h \in F$ be such that $0 < h < id$. Then there is an order automorphism $\varphi\colon [0,1] \to [0,1]$, such that $\tilde{F} = \{\varphi^{-1} \circ f \circ \varphi\colon f \in F\}$ is the product, reversed product, or power monoid and $\varphi^{-1} \circ h \circ \varphi$ is the middle element of $\tilde{F}$.*

Proof. In view of Theorem 5.11, we may w.l.o.g. assume that F is the product, reversed product, or power monoid.

Assume first that F is the product monoid, and in accordance with the notation of Definition 5.7(i), let $b \in (0,1)$ be such that $h = f_b$. We have to show that there is an order automorphism $\psi\colon [0,1] \to [0,1]$ such that $G = \{\psi^{-1} \circ f \circ \psi\colon f \in F\}$ coincides with F and $\psi^{-1} \circ h \circ \psi = f_{\frac{1}{2}}$.

Let $c = -\frac{\ln b}{\ln 2}$; then $c > 0$. Let $\psi\colon [0,1] \to [0,1]$, $x \mapsto x^c$; this is an order automorphism of $[0,1]$, and we have $G = F$. Finally, $\psi^{-1}(h(\psi(x))) = b^{\frac{1}{c}} x = \frac{1}{2}x$ for $x \in [0,1]$.

So the assertion is proved if F is the product monoid, and the case that F is the reversed product monoid follows as well.

Finally, let F be the power monoid. In this case, let us reconsider the proof of Theorem 5.11. There, we have to choose an arbitrary function $g_0 \in F$ such that $0 < g_0 < id$; this may be $g_0 = h$. It was shown that, for the automorphism $\psi\colon [0,1] \to [0,1]$ converting the given algebra into the power monoid, $\psi^{-1} \circ g_0 \circ \psi = f_{\frac{1}{2}}$. This completes the proof. □

6 The basic tomonoids

In this section, we show that each basic tomonoid associated to a basic interval of a regular l.-c. t-norm is isomorphic to one of the six tomonoids introduced in Section 5.

We assume $\odot$ to be a regular l.-c. t-norm, and we fix a frame $(a_0, ..., a_k)$ for $\odot$.

The case of an idempotency interval is easy.

Theorem 6.1 *Let $(a,b]$ be a left- or right-idempotency interval. Then $(\Lambda_{(a,b]}; \leq, \circ, 0_{(a,b]}, id_{(a,b]})$ is a scaled left- or right-idempotency monoid, respectively.*

Proof. Let $(a,b]$ be a left-idempotency interval; the other case works analogously. Let $(u,v]$ be the parameter set of $(a,b]$. By Lemma 4.7(i), every $\lambda_t \in \Lambda$ such that $t \in (u,v]$ is idempotent. From (Idp-l) and Lemma 4.5 it follows that $\Lambda_{(a,b]}$ consists, up to scaling, exactly of the functions (13) in Definition 5.1(i). □

We now consider the case of continuity intervals. Some preparations are needed.

Lemma 6.2 *Let $\varepsilon > 0$, and let $h\colon [0,1] \to [0,1]$ be a left-continuous function such that (α) $h^+(x) < x$ for all $x \in (0,1)$, (β) h is on its support strictly increasing, and (γ) for all $x \in [0,1]$, $x - h(x) < \varepsilon$. If, for some $k \geq 1$, $k\varepsilon < 1 - k\varepsilon$ and h is discontinuous at some point in $(k\varepsilon, 1 - k\varepsilon)$, then h^k has at least k points of discontinuity.*

Proof. Assume $k \geq 2$, and let h be discontinuous at $x_1 \in (k\varepsilon, 1 - k\varepsilon)$. By assumption, $0 < h(x_1) < h^+(x_1) < x_1$. Let $x_2 = \max\{x : h(x) \leq x_1\}$; then $x_1 < x_2$. From $h(x_2) \leq x_1$, we conclude $x_2 < x_1 + \varepsilon < 1 - (k-1)\varepsilon < 1$. Note further that $h(x_2) \leq x_1 \leq h^+(x_2)$. Consider now $h^2 = h \circ h$. We have $h^2(x_2) > x_2 - 2\varepsilon > (k-2)\varepsilon \geq 0$. Furthermore, $h^2(x_2) \leq h(x_1)$; $(h^2)^+(x_2) \geq h^+(x_1) > h(x_1)$; and, because $h^+(x_2) < x_2$, also $(h^2)^+(x_2) < x_1$.

So, h^2 is discontinuous at $x_2 \in (k\varepsilon, 1 - (k-1)\varepsilon)$, and we have $0 < h^2(x_2) < (h^2)^+(x_2) < x_1$. If $k \geq 3$, we continue in the same way. Namely, we define $x_3 = \max\{x : h^2(x) \leq x_1\}$; then $x_2 < x_3 < 1 - (k-2)\varepsilon < 1$. We have $h^2(x_3) \leq x_1 \leq (h^2)^+(x_3)$. Concerning h^3, we see the following. We have $h^3(x_2) > 0$, and $h^3(x_3) \leq h(x_1)$; $(h^3)^+(x_3) \geq h^+(x_1) > h(x_1)$; and, because $h^+(x_3) < x_3$, also $(h^3)^+(x_3) < x_1$.

So, h^3 is discontinuous at $x_3 \in (k\varepsilon, 1-(k-2)\varepsilon)$, and we have $0 < h^3(x_3) < (h^3)^+(x_3) < x_1$. Defining similarly $x_4, \ldots, x_k \in (k\varepsilon, 1)$, we see, for $i = 1, \ldots, k$, that h^i is discontinuous at x_i.

Because $h^k(x_1) > 0$, $x_1, \ldots, x_k \in \text{supp}\, h^k$. Moreover, $h, \ldots, h^k$ are strictly increasing on $[k\varepsilon, 1]$. It follows that h^k is discontinuous at $x_1, \ldots, x_k$. □

Lemma 6.3 *Let G be a set of increasing left-continuous functions from $[0,1]$ to $[0,1]$ such that for every $f \in G$ the following holds: (α) $f = id$ or else $f^+(x) < x$ for every $x \in (0,1)$, (β) $f \leq g$ or $g \leq f$ for all $g \in G$, (γ) $f \circ g = g \circ f \in G$ for all $g \in G$, (δ) f has only finitely many discontinuity points. Assume furthermore that the following condition holds: if $[a,b] \subseteq (0,1)$ and $\varepsilon > 0$, then there is an $h \in G$ such that h is continuous on $[a,b]$ and $x - \varepsilon < h(x) < x$ for $x \in [a,b]$. Then every $f \in G$ is continuous.*

Proof. Let $g \in G$ be such that $0 < g < id$. Let $[a,b] \subseteq (0,1)$ be such that $g(b) > a$, and let $\varepsilon > 0$. By assumption, there is an $h \in G$ such that $g \leq h < id$, $h|_{[a,b]}$ is continuous, and $x - \varepsilon < h(x) < x$ for $x \in [a,b]$.

Let now $G_{(a,b]} = \{f_{(a,b]} : f \in G\}$, where $f_{(a,b]} : [a,b] \to [a,b],\ x \mapsto x \vee a$, cf. (12). W.r.t. the domain $[a,b]$ instead of $[0,1]$, we see that $G_{(a,b]}$ has all the properties (α)–(δ). So by Lemma 5.10(i), applied to $G_{(a,b]}$, there is a natural number $k \geq 1$ such that $h^{k+1}_{(a,b]} \leq g_{(a,b]} \leq h^k_{(a,b]}$. We have $h^k_{(a,b]}(x) - g_{(a,b]}(x) \leq h^k_{(a,b]}(x) - h^{k+1}_{(a,b]}(x) < \varepsilon$ for $x \in [a,b]$.

So $g_{(a,b]}$ is the uniform limit of continuous functions and hence continuous as well. It follows that also g itself is continuous. □

Lemma 6.4 *Let $(a,b]$ be a continuity interval. Then for any $f \in \Lambda$, we have:*

(i) *$f_{(a,b]}$ is continuous.*

(ii) *Either $f_{(a,b]} = id_{(a,b]}$, or else $f_{(a,b]}(x) < x$ for all $x \in (a,b)$.*

Proof. For any $f \in \Lambda$ such that $f_{(a,b]} < id_{(a,b]}$, we have by Lemma 4.8(i) that $f^+_{(a,b]}(x) < x$ for all $x \in (a,b)$. This proves part (ii).

For part (i), note first that by Lemma 4.15, $(\Lambda_{(a,b]}; \leq, \circ, 0_{(a,b]}, id_{(a,b]})$ fulfills (T2), (T5), and (T6). So the conditions (α)–(δ) of Lemma 5.10 are all fulfilled. Furthermore, making use of (T5), we may apply Lemma 2.7 to $id_{(a,b]}$ to conclude that there is for every $\varepsilon > 0$ an $h \in \Lambda$ such that $x - \varepsilon < h_{(a,b]}(x) < x$ for $x \in (a,b)$. Finally, by Lemma 5.10(ii), every $f_{(a,b]} \in \Lambda_{(a,b]}$ is on its support strictly increasing.

Let now a', b' be such that $a < a' < b' < b$. By Lemma 6.2 and the fact that the number of discontinuity points of the functions in Λ is globally bounded, there is an $\varepsilon > 0$ such that, for all $g_{(a,b]} \in \Lambda_{(a,b]}$, if $x - g_{(a,b]}(x) < \varepsilon$ for all $x \in [a,b]$, then $g_{(a,b]}$ is continuous on $[a', b']$.

So it follows from Lemma 6.3 that $\Lambda_{(a,b]}$ consists of continuous functions. □

We arrive at the key result of this section.

Theorem 6.5 *Let $(a,b]$ be a continuity interval. Then $(\Lambda_{(a,b]}; \leq, \circ, 0_{(a,b]}, id_{(a,b]})$ is isomorphic to the product or Łukasiewicz or reversed product or power monoid.*

Proof. This follows by Lemmas 4.15 and 6.4 from Theorem 5.11. □

The parametrization of a basic tomonoid leaves some freedom, which to restrict is the purpose of the next definition.

Definition 6.6 Let $(a,b]$ be a continuity interval with parameter set $(u,v]$. We say that $\Lambda_{(a,b]}$ is *normally parametrized* if $\lambda_{\frac{u+v}{2}}$ projects to the middle element of $\Lambda_{(a,b]}$.

Moreover, let $(a, b]$ be an idempotency interval with parameter set $(u, v]$. We say that $\Lambda_{(a,b]}$ is *normally parametrized* if, for any $p \in [0, 1]$, the length of the interval $q((1-p)\,u + p\,v) \cap [a, b]$ is $p\,(b-a)$.

The following definition suggests itself.

Definition 6.7 We say that the regular l.-c. t-norm $\odot$ is in *normal form* if, for some $k \geq 1$, (i) $(0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1)$ is the frame for $\odot$, (ii) each basic tomonoid $\Lambda_{(\frac{i-1}{k}, \frac{i}{k}]}$, $i = 1, ..., k$, is a scaled product or Łukasiewicz or reversed product or power or left-idempotency or right-idempotency monoid, and (iii) each basic tomonoid which is a scaled product or reversed product or power monoid or a scaled left- or right-idempotency monoid, is normally parametrized.

We associate to $\odot$ in this case the following *characteristic data*: (CD1) the number k; (CD2) to each $i = 1, \ldots, k$, the isomorphism type of $\Lambda_{(\frac{i-1}{k}, \frac{i}{k}]}$, which is one of product, Łukasiewicz, reversed product, power, left-idempotency, or right-idempotency; (CD3) to each $i = 1, \ldots, k$ the parameter set associated to $(\frac{i-1}{k}, \frac{i}{k}]$, which equals $(\frac{i-1}{k}, \frac{i}{k}]$ or else $(u, \frac{j}{k}]$ for some $j > i$ and $u \in (\frac{i-1}{k}, \frac{j}{k})$.

Theorem 6.8 *Every regular l.-c. t-norm is isomorphic to a regular l.-c. t-norm in normal form.*

Proof. Clearly, we may find an isomorphism such that the basic intervals obtain all the same length; then condition (i) in Definition 6.7 is satisfied.

By Lemma 4.10, every basic interval is either a continuity or an idempotency interval. So condition (ii) is satisfiable by Theorems 6.1 and 6.5.

Finally, condition (iii) can be fulfilled by Lemmas 5.3 and 5.13. □

In the sequel, we will not in general assume that our given t-norm $\odot$ is in normal form; however, we will refer to its characteristic data, applicable to the isomorphic t-norm in normal form. Furthermore, we will say, for instance, that a basic tomonoid $\Lambda_{(a,b]}$ is *of type Łukasiewicz* to express that the algebra $(\Lambda_{(a,b]}; \leq, \circ, 0_{(a,b]}, id_{(a,b]})$ is isomorphic to the Łukasiewicz monoid.

We may wonder if we can sharpen condition (iii) for the characteristic data. In particular, if $(a, b]$ and $(c, d]$ are basic intervals with the common opening point d, then we know that the parameter set associated to $(c, d]$ is $(c, d]$, and that the parameter set of $(a, b]$ is of the form $(c', d]$ for some $c' < d$; but can we say anything about c' relative to c? Unfortunately, we cannot; apart from $c = c'$, both $c' > c$ and $c' < c$ is possible. Examples $\odot_6$ and $\odot_8$ in Section 8 will confirm this inconvenience.

Our question is now to which degree the characteristic data of a regular l.-c. t-norm determines the t-norm up to isomorphism. In the remainder of the present section, we show how to reconstruct from the characteristic data those pieces of the translations which either coincide with the identity function or are specified by their projection into the basic tomonoids. The problem how to reconstruct the remaining pieces of the translations, will be discussed in the subsequent section.

The first concern is to derive from the characteristic data the function q.

Lemma 6.9 *Let the characteristic data of the regular l.-c. t-norm $\odot$ be given, and assume that $\odot$ is in normal form. Then the function q is uniquely determined.*

Proof. We are given a frame for $\odot$; we know about each basic tomonoid if it is a continuity or left-idempotency or right-idempotency monoid; we know in each case the associated parameter set; and we know that the parametrization of each idempotency interval is normal. Clearly, this information determines q. □

The characteristic data being given, we want to find out next how to associate to a projection of a translation f into a basic tomonoid the value $f(1)$. In particular, we need to tell when projections into different basic tomonoids belong to the same translation.

Lemma 6.10 *Let $(a, b]$ be a primary basic interval. Then $\Lambda_{(a,b]}$ is of type product or Łukasiewicz or right-idempotency. Moreover, any $f \in \Lambda$ crossing $\Lambda_{(a,b]}$ is constant on $[b, 1]$; in particular, $f(1) = f(b) = f_{(a,b]}(b)$.*

Proof. By Lemma 4.10, the parameter set of $(a, b]$ is $(a, b]$, and for every $t \in (a, b]$, λ_t is constant t on $[b, 1]$. We have $\lambda_t(b) = t < b$ for $t \in (a, b]$; so $\Lambda_{(a,b]}$ cannot be of type reversed product or power or left-idempotency. □

Lemma 6.11 *Let $(a, b]$ be a primary basic interval, and let $f, g \in \Lambda$ cross $\Lambda_{(a,b]}$. Then $f_{(a,b]} = g_{(a,b]}$ implies $f = g$.*

Proof. By Lemma 6.10, $f_{(a,b]} = g_{(a,b]}$ implies $f(1) = g(1)$, so $f = g$. □

Unfortunately, we cannot generalize Lemma 6.11 to the case of an arbitrary basic interval; see the t-norm $\odot_8$ in Section 8. However, the following weaker assertion holds.

Lemma 6.12 *Let $(a, b]$ and $(c, d]$ be distinct continuity intervals with the same opening point. Let $f, g \in \Lambda$ cross both $\Lambda_{(a,b]}$ and $\Lambda_{(c,d]}$. Then $f_{(a,b]} = g_{(a,b]}$ if and only if $f_{(c,d]} = g_{(c,d]}$.*

Proof. Assume that $f_{(a,b]} = g_{(a,b]}$, but $f_{(c,d]} < g_{(c,d]}$. Choose $h \in \Lambda$ such that $h_{(c,d]} < id_{(c,d]}$ is large enough such that $f_{(c,d]} < (h \circ g)_{(c,d]}$. Since $(a, b]$ and $(c, d]$ have the same opening point, $h_{(a,b]} < id_{(a,b]}$ holds as well, and consequently $(h \circ g)_{(a,b]} = (h \circ f)_{(a,b]} < f_{(a,b]}$. The assertion follows. □

We are now going to show how to reconstruct from the characteristic data the functions $t \mapsto \lambda_{t(a,b]}$, where $(a, b]$ is a basic interval. We have to make a restriction; the case that a parameter set exceeds the basic interval with which it shares the upper bound, shall be excluded.

Lemma 6.13 *Let the characteristic data of the regular l.-c. t-norm $\odot$ be given, and assume that $\odot$ is in normal form. Assume that the parameter set of any basic interval is a subset of a basic interval. Then the functions $t \mapsto \lambda_{t(a,b]}$, where $(a, b]$ are the basic intervals, are uniquely determined.*

Proof. Let $(a, b]$ be a basic interval. For any projection $f_{(a,b]}$ of a translation $f \in \Lambda$ crossing $\Lambda_{(a,b]}$, we shall show that f is by $f_{(a,b]}$ uniquely determined and we shall determine $f(1)$. Clearly, the assertion will be proved then.

Let $(a, b]$ be an idempotency interval, and let f cross $\Lambda_{(a,b]}$. Then $f_{(a,b]}$ is clearly the projection of a unique translation, and we may derive $f(1)$ from the function q; recall that q is uniquely determined by Lemma 6.9.

Let $(a, b]$ be a primary continuity interval, and let f cross $\Lambda_{(a,b]}$. Then $f_{(a,b]}$ is clearly the projection of a unique translation by Lemma 6.11, and we have $f(1) = f_{(a,b)}(b)$ by Lemma 6.10.

Let $(a, b]$ be a secondary continuity interval. Let $(c', d]$ be its parameter set, contained in the primary basic interval $(c, d]$. Note that any $f \in \Lambda$ crossing $\Lambda_{(a,b]}$ also crosses $\Lambda_{(c,d]}$; so by Lemma 6.12, if $f, g \in \Lambda$ cross $\Lambda_{(a,b)}$, then $f_{(a,b]} = g_{(a,b]}$ iff $f_{(c,d]} = g_{(c,d]}$ iff $f = g$. In particular, any projection of a translation crossing $\Lambda_{(a,b]}$ belongs to a unique translation.

Let now $h_{(a,b]}$ be the middle element of $\Lambda_{(a,b]}$. We are going to determine $h_{(c,d]}$ or, equivalently, $h(1) = h_{(c,d]}(d)$. Then, we will be done; indeed, for any $k \geq 2$, there is exactly one $f \in \Lambda$ such that ${f_{(a,b]}}^k = h_{(a,b]}$ and ${f_{(c,d]}}^k = h_{(c,d]}$, so that $f(1) = f_{(c,d]}(d)$ will be determined as well.

If $\Lambda_{(a,b]}$ is not of type Łukasiewicz, $\Lambda_{(a,b]}$ is, by assumption, normally parametrized. It follows $h(1) = \frac{c'+d}{2}$.

Let $\Lambda_{(a,b]}$ be of type Łukasiewicz. By Lemma 6.10, $\Lambda_{(c,d]}$ is of type product or Łukasiewicz. If $c = c'$, $\Lambda_{(c,d]}$ cannot be of type product, so it must be of type Łukasiewicz. Furthermore, in this case, $f \in \Lambda$ crosses $\Lambda_{(a,b)}$ iff f crosses $\Lambda_{(c,d)}$, and it follows that $h(1) = \frac{c+d}{2}$. If $c < c'$, f crosses $\Lambda_{(a,b)}$ iff $f_{(c,d]}(d) \in (c', d)$, and it follows that $h_{(c,d]}$ is the unique element of $\Lambda_{(c,d]}$ such that ${h_{(c,d]}}^2 = \lambda_{c'(c,d]}$. □

7 Building the t-norm from the basic tomonoids

The basic tomonoids and their parameter sets are specified by what we call the characteristic data of a l.-c. t-norm. In this section, we will continue exploring the question to what extent the knowledge of the characteristic data is sufficient to determine the whole translation tomonoid and thus the t-norm, up to isomorphism.

We again assume throughout the section that $\odot$ is a regular l.-c. t-norm, and we fix a frame $(a_0, ..., a_k)$ for $\odot$. The characteristic data of $\odot$ given, we have already established how to determine the function q and projections of the translations f into the basic tomonoids in dependence from $f(1)$. Now, we are going to examine what can be said about the remaining part of a translation $f \in \Lambda$.

Lemma 7.1 *Let $(b, c]$ be a continuity interval with the opening point v, and let f cross $\Lambda_{(b,c]}$. Let $a < b$ and $d > c$.*

(i) *Assume that λ_v is constant c on $[c, d]$. Then $f_{(b,c]}$ is constant on $[c, d]$.*

(ii) *Assume that λ_v is constant a on $[a, b]$. Then, for all $x \in (b, c]$ such that $f_{(b,c]}(x) = b$, $f(x) \leq a$.*

Proof. (i) We have $\lambda_v(x) = x$ for $x \in (b, c]$ and $\lambda_v(x) = c$ for $x \in [c, d]$. Let $x \in [c, d]$. Since $f(c) \in (b, c]$ and $f \leq \lambda_v$, we have $f(x) \in (b, c]$, and it follows $\lambda_v(f(x)) = f(x)$. On the other hand, $f(\lambda_v(x)) = f(c)$. The assertion is proved.

(ii) We have $\lambda_v(x) = a$ for $x \in [a, b]$ and $\lambda_v(x) = x$ for $x \in (b, c]$. Let $x \in (b, c]$ be such that $f_{(b,c]}(x) = b$, that is, $f(x) \leq b$. Assume $f(x) > a$. Then $f(x) = f(\lambda_v(x)) = \lambda_v(f(x)) = a$. So $f(x) \leq a$. □

Lemma 7.2 *Let $(a, b]$ and $(c, d]$ be continuity intervals such that $b < c$, and let v be their common opening point. Let $\Lambda_{(a,b]}$ be of type product or Łukasiewicz, let $\Lambda_{(c,d]}$ be of type Łukasiewicz or reversed product, and let λ_v be constant b on $[b, c]$. Then, for any $f \in \Lambda$ crossing both $\Lambda_{(a,b]}$ and $\Lambda_{(c,d]}$, there is an $e \in (c, d]$ such that the following holds:* (i) *f is constant $f(b)$ on $[b, c]$;* (ii) *f is on $[c, e]$ uniquely determined by the basic tomonoids $\Lambda_{(a,b]}$ and $\Lambda_{(c,d]}$ and the projections $f_{(a,b]}$ and $f_{(c,d]}$;* (iii) *$f(e) = b$ and $f^+(e) = c$.*

Moreover, let $f \in \Lambda$ be such that $f(d) > a$. Then $f(d) \in (a, b] \cup (c, d]$.

Proof. Let f cross $\Lambda_{(a,b]}$ and $\Lambda_{(c,d]}$. By Lemma 7.1(i), f is constant on $[b, c]$; so (i) is clear.

By Lemma 7.1(ii), $f(x) > c$ or $f(x) \leq b$ for $x \in (c, d]$. Let e be the unique element in $(c, d]$ be such that $f^+(e) = c$ and $f(x) > c$ if $x > e$. We have $f(e) \in [f(b), b]$; we claim that $f(e) = b$. Indeed, let $f(e) < b$. Then we may choose a $g \in \Lambda$ such that $f_{(a,b]} < g_{(a,b]} < id_{(a,b]}$ and $g(f(e)) < f(c)$. But by Lemma 6.12, it follows $f_{(c,d]} < g_{(c,d]} < id_{(c,d]}$, whence $g(e) > c$ and consequently $f(g(e)) \geq f(c)$. So (iii) follows.

To show (ii), assume that we are given $f_{(a,b]}$, $f_{(c,d]}$, and the algebras $\Lambda_{(a,b]}$ and $\Lambda_{(c,d]}$. This means in particular that we know that $f_{(a,b]}$ and $f_{(c,d]}$ belong to the same translation f; by Lemma 6.12, we may consequently identify all pairs $g_{(a,b]}$ and $g_{(c,d]}$, where g is any translation crossing $\Lambda_{(a,b]}$ and $\Lambda_{(c,d]}$. Let $x \in (c, e)$. Choose $g_{(c,d]} > f_{(c,d]}$ such that $g(e) = g_{(c,d]}(e) = x$; then $f(x) = f(g(e)) = g(f(e)) = g(b) = g_{(a,b]}(b)$.

Assume next that, for some $f \in \Lambda$, $f(d) \in (b, c]$. By left-continuity, there is also a $d' \in (c, d)$ such that $f(d') \in (b, c]$. But then, for an $h < \lambda_v$ such that $h_{(c,d]}$ is sufficiently close to $id_{(c,d]}$, we have $f(h(d')) > b$ and $h(f(d') \leq b$. So also the second part is shown. □

We are ready to show that any translation crossing a primary basic tomonoid, is completely determined by the information derivable from the characteristic data on base of Lemmas 6.9 and 6.13.

Lemma 7.3 *Let $f \in \Lambda$ cross $\Lambda_{(c,d]}$, where $(c, d]$ is a primary basic interval. Then f is uniquely determined by the projection $f_{(c,d]}$, the function q and the mappings $t \mapsto \lambda_{t\,(a,b]}$, where $(a, b]$ are the basic intervals.*

Proof. By Lemma 6.11, $f_{(c,d]}$ is the projection of a unique translation f, and $f(1) = f_{(c,d]}(d)$.

If $(c,d]$ is an idempotency interval, f is idempotent and thus the whole function f is determined from $Q(f) = q(f(1))$ by 4.5.

Let us assume that $(c,d]$ is a continuity interval. Comparing $q(d)$ with $q^-(d)$, we may determine all continuity intervals whose opening point is d as well; namely, up to the boundary points, these are the connected components of $q(d) \setminus q^-(d)$.

Let now $g \in \Lambda$ be such that $f_{(c,d]} \leq g_{(c,d]} < id_{(c,d]}$ and $f_{(c,d]} = {g_{(c,d]}}^k$ for some $k \geq 1$ and g crosses all the basic tomonoids with opening point d. Again, we have $g(1)$, and so the projections of g into the basic tomonoids belongs to the given information. We are going to determine from this information the whole function g; then we have determined f as well, because $f_{(c,d]} = g^k_{(c,d]}$ by Lemma 4.14 and hence $f = g^k$ by Lemma 6.11.

Let $(a,b]$ be any of the continuity intervals with opening point d. Let $l \leq a$ be the largest element such that either $l \in q^-(d)$ or, for some $k < l$, $(k,l]$ is a continuity interval with the opening point d as well. Given $g_{(a,b]}$, we shall specify g on the interval $(l,b]$.

Note that $g(x) \geq l$ for $x > l$. If $l = a$, then $l \in q^-(d)$, and $g|_{(l,b]}$ is clearly uniquely determined. Let $l < a$ and $l \in q^-(d)$. Then λ_d is constant l on $(l,a]$, and $g_{(l,b]}$ is specified by Lemma 7.1(ii). If $l < a$ and $l \notin q^-(d)$, then there is a continuity interval $(k,l]$ with opening point d, and we conclude that again λ_d is constant l on $(l,a]$. In this case, $g_{(l,a]}$ is determined by Lemmas 7.1 or 7.2.

Similarly, let u be the smallest element in $q^-(d)$ such that $u \geq b$, and assume that $u > b$ and $(b,u]$ does not contain any further continuity interval with the opening point d. We want to specify $g_{(b,u]}$. We have that λ_d is constant b on $(b,u]$; so $g_{(b,u]}$ is determined by Lemma 7.1(i).

Finally, if $x \in q^-(d)$, we clearly have $g(x) = x$. This finishes the proof that g can be reconstructed from the information assumed. □

We get the following about translations crossing a primary basic tomonoid.

Theorem 7.4 *Let the characteristic data of the regular l.-c. t-norm $\odot$ be given, and assume that $\odot$ is in normal form. Assume that the parameter set of any basic interval is a subset of a basic interval. Then for any t contained in a primary basic interval, λ_t is uniquely determined.*

Proof. By Lemma 6.9, q is uniquely determined, and by Lemma 6.13, functions $t \mapsto \lambda_{t\,(a,b]}$, where $(a,b]$ are the basic intervals, are uniquely determined. Given $t \in (c,d]$, where $(c,d]$ is a basic interval with opening point d, we get $f_{(c,d]}$ from the characteristic data. So may reconstruct the whole translation λ_t by Lemma 7.3. □

In general, however, the characteristic data does not determine the whole translation tomonoid. In particular, those translations which do not cross any primary basic tomonoid, can be undetermined. We now turn to cases that the characteristic data is sufficient to specify the t-norm completely.

Definition 7.5 We say that the regular l.-c. t-norm $\odot$ is *locally determined* if every regular t-norm with the same characteristic data is isomorphic to $\odot$.

The following criterion, sufficient for a t-norm to be locally determined, is very easy to check and furthermore applicable for any of those examples from Section 8 which are taken from the literature.

Theorem 7.6 *Assume that for each primary basic interval $(c,d]$ such that $c > 0$, λ_c restricted to $[0,c]$ is idempotent. Then $\odot$ is locally determined.*

Proof. We assume to be given the characteristic data and that $\odot$ is in normal form. Note that the parameter set of a basic interval cannot exceed the basic interval with which it shares the upper bound; so Lemma 6.13 applies and we can derive q as well as the mappings $t \mapsto \lambda_{t\,(a,b]}$ for all basic intervals $(a,b]$.

Let d be any opening point, and assume that we have already determined all translations λ_s for $s > d$. Let d' be the next smaller opening point, or 0 if there is none; we will show that the translations λ_s, where $d' < s \leq d$, are determined as well. Because 1 is always an opening point, the assertion will then follow by induction.

By Lemma 4.10, there is a primary basic interval $(c, d]$. The idempotent translation λ_d is determined by $Q(\lambda_d)) = q(d)$, and the λ_t, $t \in (c, d)$, are determined according to Lemma 7.3.

Assume that $c > d'$, that is, $c > 0$ and c is not an opening point. By assumption, $\lambda_c|_{[0,c]}$ is idempotent. In particular, $\lambda_c|_{[0,c]}$ is determined by $q(c)$ according to Lemma 4.5. Let $b \leq c$ be such that λ_c is constant b on $[b, c]$. Then $b < c$. Indeed, if $b = c$, it follows that λ_c is constant c on $[c, 1]$ and hence that c is an opening point.

From $\lambda_c(\lambda_c(1)) = \lambda_c(c) = b = \lambda_b(1)$, we have $\lambda_c^2 = \lambda_b$. It follows $\lambda_b|_{[0,c]} = \lambda_c^2|_{[0,c]} = \lambda_c|_{[0,c]}$. So for $t \in (b, c]$, the translations λ_t depend as follows from $\lambda_c|_{[0,c]}$ and the translations λ_s, $s > c$. We have $\lambda_t(x) = \lambda_c(x)$ for $x \in [0, c]$, and $\lambda_t(x) = \lambda_x(t)$ for $x \in (c, 1]$.

Since $\lambda_b^2(1) = \lambda_b(b) = \lambda_c(b) = b = \lambda_b(1)$, the translation λ_b is idempotent, and it follows that λ_b is the largest idempotent below λ_c. So either $\lambda_b = \mathit{0}$, or b is an opening point, that is, $b = d'$. □

We conclude the section with a further theorem containing conditions which guarantee that the t-norm is locally determined. These conditions refer to the characteristic data directly, and the theorem is based on the previous Theorem 7.6.

Theorem 7.7 *For the regular l.-c. t-norm $\odot$, assume that the following conditions hold:*

(LD1) *Let $(a, b]$ be a basic interval with the opening point v. If there is a basic interval $(b, c]$ with the opening point $> v$, then $\Lambda_{(a,b]}$ is of type product or Łukasiewicz or right-idempotency.*

(LD2) *Let $(b, c]$ be a basic interval with the opening point v. If there is a basic interval $(a, b]$ with the opening point $> v$, then $\Lambda_{(c,d]}$ is of type reversed product or Łukasiewicz or left-idempotency.*

(LD3) *Let $(b, c]$ be a basic interval with the opening point v. and let $(a, b]$ and $(c, d]$ be basic intervals with opening points $> v$. Then $v = c$.*

(LD4) *Let $(a, b]$ and $(b, c]$ be basic intervals with the common opening point v. Then $v = c$.*

(LD5) *Let $(a, b]$ and $(c, d]$, where $b < c$, be basic intervals with the common opening point v. Then there is a basic interval contained in $(b, c]$ with the opening point $< v$.*

Then $\odot$ is locally determined.

Proof. Let $(c, d]$ be a primary basic interval such that $c > 0$. We shall prove that $\lambda_c|_{[0,c]}$ is idempotent; the assertion then follows by Theorem 7.6.

The basic interval located immediately left from $(c, d]$ may have an opening point $< d$, $= d$, or $> d$. Accordingly, we have to consider the following cases.

Case (i). There is a $b < c$ such that $(b, c]$ is a basic interval with opening point $< d$. Then $\lambda_c|_{(b,c]} = id|_{(b,c]}$.

Case (ii). There is a $b < c$ such that $(b, c]$ is a basic interval whose opening point is d as well. If then $b = 0$, we have $\lambda_c|_{[0,c]} = \mathit{0}|_{[0,c]}$. Let $b > 0$. If then there is a basic interval $(a, b]$ with opening point $< d$, we have that $\lambda_c|_{(a,b]} = id|_{(a,b]}$ and $\lambda_c|_{(b,c]}$ is constant b. Else, as a consequence of (LD4), there an $a < b$ such that λ_d is constant a on $(a, b]$. In this case, by (LD2), $(b, c]$ is of type reversed product or Łukasiewicz or left-idempotency. If $a = 0$, we have $\lambda_c|_{[0,c]} = \mathit{0}|_{[0,c]}$. Otherwise, by (LD5), there is a basic interval $(z, a]$ with opening point $< d$, and we have $\lambda_c|_{(z,a]} = id|_{(z,a]}$ and $\lambda_c|_{[a,c]}$ is constant a.

Case (iii). The idempotent translation λ_d is constant b on $(b, c]$ for some $b \leq c$. By (LD2), $(c, d]$ is of type reversed product or Łukasiewicz or left-idempotency. If $b = 0$, $\lambda_c|_{[0,c]} = \mathit{0}|_{[0,c]}$. Otherwise, let $a < b$ be such that $(a, b]$ is a basic interval. By (LD5), $(a, b]$ has an opening point $< d$, and by (LD2), $(c, d]$ is of type Łukasiewicz. It follows that $\lambda_c|_{(a,b]} = id|_{(a,b]}$ and $\lambda_c|_{(b,d]}$ is constant b.

We consider next an arbitrary further basic interval $(a, b]$ with opening point d.

Case (i). For some $b' > b$, $(b, b']$ is a basic interval with opening point d. Then, by (LD4), $(b, b'] = (c, d]$, which is Case (ii) above.

Note that there cannot be a basic interval with opening point d located on the left side of $(a, b]$; this would contradict (LD4).

Case (ii). The idempotent translation λ_d is constant z on $(z, a]$ for some $z < a$. By (LD3) and (LD5), $z = 0$ or there is a basic interval $(y, z]$ with opening point $< d$, and furthermore there is a basic interval $(b, c]$ with opening points $< d$. By (LD1), $(a, b]$ is of type reversed product or Łukasiewicz or left-idempotency, and it follows that λ_c is the identity on $(y, z]$, respectively, and $(b, c]$ and is constant z on $(z, a]$.

Case (iii). The idempotent translation λ_d is constant b on $(b, c]$ for some $c > b$. Then we argue similarly to Case (ii).

In view of Lemma 4.5, we conclude that λ_c, restricted to $[0, c]$, is idempotent. □

We will give in a subsequent section examples of regular l.c. t-norms which, by fulfilling the conditions (LD1)–(LD5), are locally determined. All t-norms explicitly defined in [Jen4] are included.

The description of a general regular l.c. t-norm requires more information than what is contained in its characteristic data. We will not further discuss this problem here. However, we may say that even in the general case, the basic tomonoids determine the rest of the translation tomonoid to a high extent. Namely, for a pair of two basic intervals $(a, b]$ and $(c, d]$, we may consider the set $H_{(a,b]}^{(c,d]} \colon (a, b] \to (c, d], \;\; x \mapsto (f(x) \vee c) \wedge d$, in analogy to Definition 4.12. In dependence of the basic tomonoids $\Lambda_{(a,b]}$ and $\Lambda_{(c,d]}$, there is a set $\bar{H}$ of functions from $(a, b]$ to $(c, d]$, totally ordered w.r.t. the pointwise ordering, such that H is always a subset of $\bar{H}$; cf. the proof of Lemma 7.2.

8 Examples of regular l.-c. t-norms

We shall give several examples of regular l.-c. t-norms, together with their characteristic data.

First to mention, the Łukasiewicz, the product, and the Gödel t-norm are clearly regular. Each of these t-norms has one basic interval, namely $(0, 1]$, with parameter set $(0, 1]$; and the basic tomonoid belonging to it, is the full translation tomonoid.

The following definitions are formally incomplete; they are to be completed by using the commutativity.

The rotated product t-norm [Jen2] is given according to

$$a \odot_1 b = \begin{cases} \frac{a+b-1}{2b-1} & \text{if } a \leq \frac{1}{2}, \;\; b > \frac{1}{2}, \text{ and } a + b > 1, \\ 2ab - a - b + 1 & \text{if } a, b > \frac{1}{2}, \\ 0 & \text{if } a + b \leq 1 \end{cases}$$

for $a, b \in [0, 1]$. There are the two basic intervals $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$; $\Lambda_{(0,\frac{1}{2}]}$ is of type reversed product monoid, and $\Lambda_{(\frac{1}{2},1]}$ is of type product monoid; the common parameter set is $(\frac{1}{2}, 1]$.

The nilpotent minimum t-norm [Fod] is defined by

$$a \odot_2 b = \begin{cases} a \wedge b & \text{if } a + b > 1, \\ 0 & \text{else} \end{cases}$$

for $a, b \in [0, 1]$. There are again two basic intervals, $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$; $\Lambda_{(0,\frac{1}{2}]}$ is of type left-idempotency, and $\Lambda_{(\frac{1}{2},1]}$ is of type right-idempotency; the common parameter set is $(\frac{1}{2}, 1]$.

We proceed with the t-norms having three basic intervals. The t-norm J [Jen1] is given by

$$a \odot_3 b = \begin{cases} a \wedge b & \text{if } a + b > 1, \;\text{ and }\; a \leq \frac{1}{3} \text{ or } a > \frac{2}{3}, \\ a + b - \frac{2}{3} & \text{if } a + b > 1 \;\text{ and }\; \frac{1}{3} < a, b \leq \frac{2}{3}, \\ 0 & \text{if } a + b \leq 1 \end{cases}$$

for $a, b \in [0, 1]$. We have the basic intervals $(0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3}]$; and $(\frac{2}{3}, 1]$; $\Lambda_{(0,\frac{1}{3}]}$ is of type left-idempotency monoid, $\Lambda_{(\frac{1}{3},\frac{2}{3}]}$ is of type Łukasiewicz monoid, and $\Lambda_{(\frac{2}{3},1]}$ is of type right-idempotency monoid; the common parameter set of the idempotency intervals is $(\frac{2}{3}, 1]$, and the middle one is parametrized by $(\frac{1}{3}, \frac{2}{3}]$.

The rotation-annihilation of two Łukasiewicz t-norms [Jen3] is defined according to

$$a \odot_4 b = \begin{cases} a+b-1 & \text{if } a,b > \frac{2}{3} \text{ and } a+b > \frac{5}{3}, \\ & \text{or } a \leq \frac{1}{3},\ b > \frac{2}{3} \text{ and } a+b > 1, \\ \frac{2}{3} & \text{if } a,b > \frac{2}{3} \text{ and } a+b \leq \frac{5}{3}, \\ a+b-\frac{2}{3} & \text{if } \frac{1}{3} < a,b \leq \frac{2}{3} \text{ and } a+b > 1, \\ a & \text{if } \frac{1}{3} < a \leq \frac{2}{3} \text{ and } b > \frac{2}{3}, \\ 0 & \text{if } a+b \leq 1. \end{cases}$$

for $a, b \in [0, 1]$. We have the basic intervals $(0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3}]$; and $(\frac{2}{3}, 1]$; all three basic tomonoids are of type Łukasiewicz; the common parameter set of the marginal intervals is $(\frac{2}{3}, 1]$, and for the middle one it is $(\frac{1}{3}, \frac{2}{3}]$.

Let us now consider the following t-norm; let

$$a \odot_5 b = \begin{cases} 2ab-a-b+1 & \text{if } a,b > \frac{1}{2}, \\ \frac{1}{2}(2a)^{\frac{1}{2b-1}} & \text{if } a \leq \frac{1}{2} \text{ and } b > \frac{1}{2}, \\ 0 & \text{if } a,b \leq \frac{1}{2}. \end{cases}$$

Then there are two basic intervals $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$; $\Lambda_{(0,\frac{1}{2}]}$ is of type power monoid, $\Lambda_{(\frac{1}{2},1]}$ is of type product monoid; the common parameter set is $(\frac{1}{2}, 1]$.

The examples so far show that all six different types of basic tomonoids may actually arise.

Next, we we consider a t-norm similar to $\odot_4$; however, when imagining the translations in decreasing order, we observe that the "speed" with which the lower Łukasiewicz-type basic tomonoid "grows" is doubled.

$$a \odot_6 b = \begin{cases} a+2b-2 & \text{if } a \leq \frac{1}{3},\ b > \frac{5}{6}, \text{ and } a+2b > 2 \\ 0 & \text{if } a \leq \frac{1}{3},\ b > \frac{5}{6}, \text{ and } a+2b \leq 2 \\ 0 & \text{if } a \leq \frac{1}{3} \text{ and } b \leq \frac{5}{6} \\ a & \text{if } \frac{1}{3} < a \leq \frac{2}{3} \text{ and } b > \frac{2}{3}, \\ a+b-\frac{2}{3} & \text{if } \frac{1}{3} < a,b \leq \frac{2}{3} \text{ and } a+b > 1, \\ 0 & \text{if } \frac{1}{3} < a,b \leq \frac{2}{3} \text{ and } a+b \leq 1, \\ a+b-1 & \text{if } a,b > \frac{2}{3} \text{ and } a+b > \frac{5}{3}, \\ \frac{2}{3} & \text{if } a,b > \frac{2}{3} \text{ and } a+b \leq \frac{5}{3} \end{cases}$$

for $a, b \in [0, 1]$. We have the same characteristic data like for $\odot_4$; however, the parameter set of the lower Łukasiewicz interval is $(\frac{5}{6}, 1]$ rather than $(\frac{2}{3}, 1]$.

The t-norm $\odot_6$ shows that, for a regular l.c. t-norm, if d is the opening point of the basic intervals $(a, b]$ and $(c, d]$, then the parameter set of $(a, b]$ may be properly contained in $(c, d]$, which is the parameter set of $(c, d]$.

Finally, we give two examples of regular l.-c. t-norms which are not covered by Theorem 7.7. The following variation of Hájek's t-norm [Haj2] does not fulfil the condition (LD4) of Theorem 7.7. Let

$$a \odot_7 b = \begin{cases} a(3b-2) & \text{if } a \leq \frac{1}{3} \text{ and } b > \frac{2}{3}, \\ 3ab-2a-b+1 & \text{if } \frac{1}{3} < a \leq \frac{2}{3} \text{ and } b > \frac{2}{3}, \\ 3ab-2a-2b+2 & \text{if } a,b > \frac{2}{3}, \\ 0 & \text{if } a \leq \frac{1}{3} \text{ and } b \leq \frac{2}{3}, \\ 3ab-a-b+\frac{1}{3} & \text{if } \frac{1}{3} < a,b \leq \frac{2}{3} \end{cases}$$

for $a, b \in [0, 1]$. We have here three basic intervals, each of which is of type product and has the parameter set $(\frac{2}{3}, 1]$.

Our last t-norm exemplifies the situation described in Lemma 7.2. It does not fulfil condition (LD5). Let

$$a \odot_8 b = \begin{cases} 2ab & \text{if } a, b \le \frac{1}{4}, \\ \frac{a}{2} & \text{if } a \le \frac{1}{4} \text{ and } \frac{1}{4} < b \le \frac{1}{2}, \\ 2^{4b-3}a & \text{if } a \le \frac{1}{4} \text{ and } \frac{1}{2} < b \le \frac{3}{4}, \\ a & \text{if } a \le \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ \frac{1}{8} & \text{if } \frac{1}{4} < a, b \le \frac{1}{2}, \\ 2^{4b-5} & \text{if } \frac{1}{4} < a \le \frac{1}{2} \text{ and } \frac{1}{2} < b \le \frac{3}{4}, \\ 4ab - 3a - b + 1 & \text{if } \frac{1}{4} < a \le \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ 2^{4(a+b)-7} & \text{if } \frac{1}{2} < a, b \le \frac{3}{4} \text{ and } a + b \le \frac{5}{4}, \\ a + b - \frac{3}{4} & \text{if } \frac{1}{2} < a, b \le \frac{3}{4} \text{ and } a + b > \frac{5}{4}, \\ a & \text{if } \frac{1}{2} < a \le \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a - 3b + 3 & \text{if } a, b > \frac{3}{4} \end{cases}$$

for $a, b \in [0, 1]$. Here, we have the four basic intervals $(0, \frac{1}{4}], (\frac{1}{4}, \frac{1}{2}], (\frac{1}{2}, \frac{3}{4}], (\frac{3}{4}, 1]$. The common parameter set of the intervals $(\frac{1}{4}, \frac{1}{2}]$ and $(\frac{3}{4}, 1]$ is $(\frac{3}{4}, 1]$; both $\Lambda_{(\frac{1}{4}, \frac{1}{2}]}$ and $\Lambda_{(\frac{3}{4}, 1]}$ are of type product. Furthermore, the parameter set of $(\frac{1}{2}, \frac{3}{4}]$ is $(\frac{1}{2}, \frac{3}{4}]$, and $\Lambda_{(\frac{1}{2}, \frac{3}{4}]}$ is of type Łukasiewicz. Finally, $\Lambda_{(0, \frac{1}{4}]}$ is of type product; and the parameter set is $(0, \frac{3}{4}]$. Note the remarkable fact that λ_t crosses $\Lambda_{(0, \frac{1}{4}]}$ iff $0 < t < \frac{3}{4}$; but $\lambda_t|_{(0, \frac{1}{4}]} = \lambda_{\frac{1}{4}}|_{(0, \frac{1}{4}]}$ for any $t \in [\frac{1}{4}, \frac{1}{2}]$.

So the t-norm $\odot_8$ reveals inconvenient facts about regular l.c. t-norms. Namely, if d is the opening point of the basic intervals $(a, b]$ and $(c, d]$, where $b \le c$, then the parameter set of $(a, b]$ may properly contain $(c, d]$, the parameter set of $(c, d]$. Furthermore, for a continuity interval $(a, b]$ with the parameter set $(u, v]$, the mapping $(u, v] \to \Lambda_{(a,b]}, \quad t \mapsto \lambda_{t(a,b]}$ may be not injective.

9 Examples of non-regular l.-c. t-norms, and the arithmetic mean of t-norms

Let us finally reconsider the case that a l.-c. t-norm is not regular, but has few continuity points.

For some $p \in (0, \frac{1}{3}]$, let

$$a \odot_{(p)} b = \begin{cases} a \wedge b & \text{if } a > \frac{2}{3} \text{ or } b > \frac{2}{3}, \\ p & \text{if } \frac{1}{3} < a, b \le \frac{2}{3}, \\ 0 & \text{if } a, b \le \frac{2}{3}, \text{ and } a \le \frac{1}{3} \text{ or } b \le \frac{1}{3} \end{cases}$$

for $a, b \in [0, 1]$. Then for any p, $\odot_{(p)}$ is a l.-c. t-norm, and $\odot_{(\frac{1}{3})}$ is the t-norm mentioned in Section 3. We have $q(t) \supseteq [0, \frac{2}{3}]$ for each $t > \frac{2}{3}$, but $q(t) = \{0\}$ for $t \le \frac{2}{3}$; in particular, $\odot_{(p)}$ is not regular. It is moreover evident that the discontinuity points of any translation based on $\odot_{(p)}$ are among $\{\frac{1}{3}, \frac{2}{3}\}$.

We can derive a statement concerning a problem proposed by Alsina, Frank, and Schweizer [AFS, Problem 5]:

Problem. Are there two distinct t-norms whose pointwise calculated arithmetic mean is a t-norm again?

Certain versions of this problem have been studied in [Jen5] and in [MeMe].

Now, w.t.r. the above example, given $r, s \in (0, \frac{1}{3}]$, we have

$$a \odot_{(\frac{r+s}{2})} b \;=\; \frac{1}{2}(a \odot_{(r)} b \;\;+\;\; a \odot_{(s)} b)$$

for $a, b \in [0, 1]$. So $\odot_{(\frac{r+s}{2})}$ is the arithmetic mean of $\odot_{(r)}$ and $\odot_{(s)}$. We have shown: The arithmetic mean of two distinct l.-c. t-norms can be a l.-c. t-norm again.

10 Conclusion

We defined a special kind of left-continuous t-norms, called regular l.-c. t-norms. We showed that the t-norm algebras based on a t-norm in this class, generate the variety of MTL-algebras.

We studied the totally ordered monoids $([0,1]; \leq, \odot, 0, 1)$ based on a regular l.-c. t-norm. To this end, we analyzed their translation semigroup, which, geometrically, contains the t-norm's vertical cuts. We have shown that we may partition the real unit interval into a finite number of half-open subintervals such that the translations restricted appropriately to one of these subintervals, form a tomonoid which belongs to one out of six different isomorphism types. We associated to each regular l.-c. t-norm its characteristic data, which contains the mentioned isomorphism types of all constituents together with the order in which the constituents are traversed when proceeding from larger to smaller cuts. The characteristic data determines the t-norm to a high extent, and under an easily checked condition, which is met in case of the well-known t-norms, even completely.

Our work may be continued along various different lines. The most natural issue is to find a kind of converse for Theorem 6.8. The characteristic data of a regular l.-c. t-norm cannot be arbitrary, but must obey certain rules; the question is how these rules could look like. Next, the part of the translation tomonoid which is not determined by the characteristic data is not arbitrary; but to find the exact ways allowed for the construction of a t-norm is not an easy problem. Another project is to check how known constructions, like rotation and annihilation, translate into rules within the present framework.

Furthermore, it is likely to be possible that our approach can be generalized to cover more l.-c. t-norms. Consider the examples of t-norms which do not have few discontinuity points, given in Chapter 3. We conjecture that it is not too difficult to generalize our method to include also Hájek's t-norm. Hliněná's t-norm seems to be a difficult case though – in accordance with Hliněná's own decision to call her article "On a peculiar t-norm". Note however that Hliněná's t-norm is po-group representable [Vet1] and hence might be covered in the framework of a more algebraically oriented approach.

In general, however, some care might be in order when weakening the notion of regularity; a guideline could be that the l.-c. t-norms similar to our strange example from Section 9 must remain excluded. Note that in case of this example, you may change the translation tomonoid in a certain area practically arbitrarily, and the result is still the translation tomonoid of a l.-c. t-norm; this is the situation to be avoided.

Finally, the most ambitious project would be to develop the corresponding theory on purely algebraic ground: a theory of "regular MTL-algebras". Given the difficulties which had to be overcome when the theory of continuous t-norm algebras was generalized to a structure theory for BL-algebras, this might be considered the most difficult problem.

The list could easily be enlarged. All in all, the described formalism offers a kind of playground for the construction of l.-c. t-norms; it is up to the creativity of anybody interested, to find new and interesting functions suitable to interpret the conjunction in fuzzy logics.

Acknowledgements

This research was partially supported by the Austrian Science Foundation (FWF) Grant P18563-N12.

Furthermore, I would like to thank the anonymous referee whose suggestions led to a substantial improvement of this paper.

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