Normal orthogonality spaces

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Abstract

An orthogonality space is a set X together with a symmetric and irreflexive binary relation $\perp$, called the orthogonality relation. A block partition of X is a partition of a maximal set of mutually orthogonal elements of X, and a decomposition of X is a collection of subsets of X each of which is the orthogonal complement of the union of the others. $(X, \perp)$ is called normal if any block partition gives rise to a unique decomposition of the space. The set of one-dimensional subspaces of a Hilbert space equipped with the usual orthogonality relation provides the motivating example.

Together with the maps that are, in a natural sense, compatible with the formation of decompositions from block partitions, the normal orthogonality spaces form a category, denoted by $\mathcal{NOS}$. The objective of the present paper is to characterise both the objects and the morphisms of $\mathcal{NOS}$ from various perspectives as well as to compile basic categorical properties of $\mathcal{NOS}$.

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1 Introduction

The motivation underlying the present work is to contribute to a characterisation of Hilbert spaces, the basic models underlying quantum physics. There are various possibilities of associating with a Hilbert space H an algebraic or relational structure. Following the traditional approach, for instance, we may consider the collection $\mathcal{C}(H)$ of closed subspaces of H. Partially ordered by set-theoretic inclusion, $\mathcal{C}(H)$

is a complete lattice and equipped with the orthogonality relation, $\mathcal{C}(H)$ becomes an ortholattice, in fact an orthomodular lattice [BrHa]. Alternatively, let $P(H)$ be the collection of one-dimensional subspaces of H. Endowed with the triple relation of linear dependence, $P(H)$ is a projective geometry and taking into account also the orthogonality relation, we are led to an orthogeometry [FaFr]. An enormous amount of work has been devoted to the investigation of these as well as numerous related structures, see, e.g., [EGL1, EGL2].

It is an easy, yet interesting observation that the relation of linear dependence in $P(H)$ and hence also the lattice structure of $\mathcal{C}(H)$ can be derived from the orthogonality relation alone. Accordingly, a further option is to start from structures that were once proposed by D. Foulis and his collaborators: a so-called orthogonality space (sometimes also referred to as an orthoset) is solely based on a binary relation, assumed to be symmetric and irreflexive. This idea is exploited in the present paper. Our guiding example is $(P(H), \perp)$, where $\perp$ is the usual orthogonality relation.

The first systematic account of orthogonality spaces from the point of view of the foundations of quantum physics is due to J. R. Dacey [Dac]. In subsequent years, research focused to a large extent on the more general concept of a test space [Wlc]. Since recently, orthogonality spaces have been considered again and the question was raised whether they allow a reasonable characterisation of inner-product spaces. It has turned out that, under a quite simple combinatorial condition called linearity, orthogonality spaces of finite rank are exactly those of the form $(P(H), \perp)$, where H is a Hermitian space [Vet3]. Further work focused on the characterisation of real or complex Hilbert spaces [Bru, Vet1, Rum1, Vet2, Vet3]. In this context, actually a variety of further issues is touched. Rump's paper [Rum1], e.g., is also motivated by properties of the structure group of a set-theoretic solution of the Yang-Baxter equation; cf. also [Rum2]. We may furthermore observe that the structures considered in this paper occur in more general frameworks. For example, the collection of pure states on any C*-algebra carries naturally the structure of an orthogonality space, see, e.g., [DiNg].

The question seems natural whether there exists a reasonable categorical framework for orthogonality spaces. A first discussion of this issue is contained in our preceding paper [PaVe]. The category of all orthogonality spaces and orthogonality-preserving maps has turned out to be unsuitable. Indeed, in this case an aspect that is central in the quantum-physical formalism is left out of account: the possibility of decomposing a Hilbert space into the direct sum of closed subspaces. As a consequence, we restrict our attention to those orthogonality spaces in which decompositions are determined analogously to the case of Hilbert spaces. More specifically, a block partition is a collection $(E_\alpha)_{\alpha<\kappa}$ of pairwise disjoint non-empty subsets of X whose union is a maximal set of orthogonal elements. A decomposition of X is a collection $(A_\alpha)_{\alpha<\kappa}$ of non-empty subsets of X each of which is the orthogonal complement of the union of the others. We call an orthogonality space normal if for any block partition (E_1, E_2), there is a unique decomposition $(\bar{E}_1, \bar{E}_2)$ of the space such that $E_1 \subseteq \bar{E}_1$ and $E_2 \subseteq \bar{E}_2$. The number of cells of the block partition plays no role here; normality implies that, for any block partition $(E_\alpha)_{\alpha<\kappa}$, there is a unique decomposition $(\bar{E}_\alpha)_{\alpha<\kappa}$ of the space such that $E_\alpha \subseteq \bar{E}_\alpha$ for all $\alpha < \kappa$. In order to define a category of normal orthogonality spaces, we choose a notion of morphism that takes the formation of decompositions from block partitions in a natural way into account. We say that an orthogonality-preserving map $\varphi\colon X \to Y$ between orthogonality spaces is normal if the following holds true: whenever the image of each cell of a block partition of X is a cell of a block partition of Y, then φ maps, in the obvious sense, the associated decomposition of X into the associated decomposition of Y. Together with these maps, normal orthogonality spaces become a category, denoted by $\mathcal{NOS}$.

A detailed discussion of $\mathcal{NOS}$, both from the conceptual and the categorical viewpoint, is the objective of the present paper. We proceed as follows. We start in Section 2 with an introduction to the notion of normality of orthogonality spaces, providing a comprehensive list of equivalent formulations. For instance, an orthogonality space is normal if and only if any maximal set of mutually orthogonal elements gives rise to a Boolean subalgebra of the associated ortholattice. We moreover draw the connection to test spaces and the corresponding notion of algebraicity. Sets of the form $E^{\perp\perp}$, where E consists of mutually orthogonal elements of some orthogonality space $(X, \perp)$, are called propositions and we consider the collection $\Pi(X, \perp)$ of all propositions, which is the analogue of the logic of a test space. It turns out that $(X, \perp)$ is normal if and only if $\Pi(X, \perp)$ has the structure of an orthomodular poset. We also characterise Dacey spaces among the normal orthogonality spaces, that is, those that give rise to an orthomodular lattice.

The subsequent Section 3 is devoted to the structure-preserving maps between normal orthogonality spaces, that is, to the normal homomorphisms. Also for maps, normality can be characterised from various different viewpoints. For instance, normality ensures a property reminding of the preservation of linear dependence by maps between linear spaces. Indeed, the double complementation ${}^{\perp\perp}$ makes any orthogonality space into a closure space and an orthogonality-preserving map φ between normal orthogonality spaces is normal if and only if, for any set A of mutually orthogonal elements, we have $\varphi(A^{\perp\perp}) \subseteq \varphi(A)^{\perp\perp}$.

Section 4 contains the categorical part. We characterise the mono- and epimorphisms and we show that any morphism factorises into a quasi-surjective map followed by an embedding. We define horizontal sums and direct products in $\mathcal{NOS}$. These constructions also make negative results apparent; horizontal sums or direct products are not a categorical sum. We finally deal with the decomposition of normal orthogonality spaces into what we call its irreducible subspaces.

2 Normal orthogonality spaces

We deal in this paper with the following relational structures [Dac, Wlc].

Definition 2.1. An *orthogonality space* is a non-empty set X equipped with a symmetric, irreflexive binary relation $\perp$, called the *orthogonality relation*.

A subset of X consisting of mutually orthogonal elements is called a $\perp$-*set*. The supremum of the cardinalities of $\perp$-sets is called the *rank* of $(X, \perp)$.

The examples that we have in mind are associated with Hilbert spaces. Equipped with the usual orthogonality relation, the set $P(H)$ of one-dimensional subspaces of a Hilbert space H is obviously an orthogonality space. The rank of $(P(H), \perp)$ coincides with the (Hilbert) dimension of H.

Orthogonality spaces are essentially the same as undirected graphs and hence must be considered as very general when considered in the context of the foundations of quantum physics. We will restrict our discussion to orthogonality spaces of a specific type, requiring a property that corresponds to a basic feature of the quantum-physical model. We follow the lines of our previous work [PaVe]. However, to make ideas as transparent as possible, we will introduce our main notions in a different way. The equivalence with previous definitions will be apparent from subsequent lemmas.

Our considerations circle around ways of decomposing an orthogonality space into constituents. For the definition of what this means, let us start with the case of decomposing the space just into two parts. For a subset A of an orthogonality space X, the *(orthogonal) complement* of A is

$$A^{\perp} \;=\; \{x \in X : x \perp a \text{ for all } a \in A\}.$$

By a *(two-fold) decomposition* of X, we mean a pair (A_1, A_2) of non-empty subsets of X such that $A_1 = A_2^{\perp}$ and $A_2 = A_1^{\perp}$. The sets A_1, A_2 are called the components of the decomposition. Note that for the two components of a decomposition we have $A_1 \perp A_2$ and $(A_1 \cup A_2)^{\perp} = \varnothing$.

An essential feature of the quantum-physical formalism is the property of Hilbert spaces to decompose into direct sums. In particular, let B be an orthonormal basis of a Hilbert space H and let (B_1, B_2) be a partition of B into two non-empty subsets. Then H is the direct sum of closed subspaces one of which contains B_1 and the other B_2. Clearly, the decomposition is uniquely determined: the subspaces are the closure of the linear span of B_1 and B_2, respectively.

We will require orthogonality spaces to have the analogous property. For an orthogonality space $(X, \perp)$, a maximal $\perp$-set of X is a *block*, and by a *block partition*, we mean a partition of a block of X into non-empty subsets, called its cells. Given a block partition (E_1, E_2), we can always decompose X in such a way that each cell of the partition is contained in exactly one component of the decomposition: simply consider $(E_1^{\perp\perp}, E_1^{\perp})$. But also $(E_2^{\perp}, E_2^{\perp\perp})$ is a decomposition with this property and can differ from the former. What we are interested in is the case that there is not more than one possibility.

Definition 2.2. We call an orthogonality space $(X, \perp)$ *normal* if, for any block partition (E_1, E_2), there is a unique decomposition $(\bar{E}_1, \bar{E}_2)$ such that $E_1 \subseteq \bar{E}_1$ and $E_2 \subseteq \bar{E}_2$.

As already indicated, our prototypical example has this property.

Example 2.3. *Let H be a Hilbert space. Let S_1 and S_2 be closed subspaces of H such that $S_1 \perp S_2$ and $H = S_1 + S_2$. Writing $P(S)$ for the set of one-dimensional subspaces of a subspace S of H, we have that $(P(S_1), P(S_2))$ is a decomposition of the orthogonality space $(P(H), \perp)$. Moreover, any two-fold decomposition is of this form.*

Let E be a block of $(P(H), \perp)$. Then $E = \{\text{span}(b) \colon b \in B\}$ for some orthogonal basis B of H. Let (E_1, E_2) be any partition of E into two non-empty subsets. Then $E_1 = \{\text{span}(b) \colon b \in B_1\}$ and $E_2 = \{\text{span}(b) \colon b \in B_2\}$ for a partition (B_1, B_2) of B. Let $\text{cl}(\text{span}(B_1))$ and $\text{cl}(\text{span}(B_2))$ be the closed linear span of B_1 and B_2, respectively. Then $\big(P\big(\text{cl}(\text{span}(B_1))\big), P\big(\text{cl}(\text{span}(B_2))\big)\big)$ is the unique decomposition of $(P(H), \perp)$ whose components contain E_1 and E_2, respectively. Hence $(P(H), \perp)$ is normal.

For later use we include a further, particularly simple example.

Example 2.4. *For $n \in \mathbb{N} \setminus \{0\}$, let $\mathbf{n}$ denote an n-element set. Then $(\mathbf{n}, \neq)$ is an orthogonality space and the two-fold decompositions are in one-to-one correspondence with partitions of $\mathbf{n}$ into two subsets. Obviously, $(\mathbf{n}, \neq)$ is normal.*

In this section, we will reformulate the property of an orthogonality space to be normal in a number of alternative ways.

We recall that any orthogonality space $(X, \perp)$ can be regarded in a natural way as a closure space; cf. [PaVe]. Indeed, the map $\mathcal{P}(X) \to \mathcal{P}(X),\ A \mapsto A^{\perp\perp}$ is a closure operator [Ern]. A set $A \subseteq X$ closed w.r.t. ${}^{\perp\perp}$ is called *orthoclosed*. Thus A is orthoclosed if and only if $A = A^{\perp\perp}$ if and only if there is a $B \subseteq X$ such that $A = B^{\perp}$.

Note that the decompositions of X are exactly the pairs $(A, A^{\perp})$, where A is orthoclosed. Indeed, for any orthoclosed set $A \subseteq X$, $(A, A^{\perp})$ is clearly a decomposition. Conversely, if (A_1, A_2) is a decomposition of X, then $A_1 = A_2^{\perp}$ is orthoclosed and $A_2 = A_1^{\perp}$.

Proposition 2.5. *Let $(X, \perp)$ be an orthogonality space. Then the following conditions are equivalent.*

(1) *$(X, \perp)$ is normal.*

(2) *For any block partition (E_1, E_2), there is a decomposition $(\bar{E}_1, \bar{E}_2)$ of X such that $E_2^{\perp} \subseteq \bar{E}_1$ and $E_1^{\perp} \subseteq \bar{E}_2$.*

(3) *For any block partition (E_1, E_2), $(E_2^{\perp}, E_1^{\perp})$ is a decomposition of X.*

(4) *For any block partition (E_1, E_2), $(E_1^{\perp\perp}, E_2^{\perp\perp})$ is a decomposition of X.*

(5) *For any block partition (E_1, E_2), we have $E_1^{\perp} \perp E_2^{\perp}$.*

Proof. (1) $\Rightarrow$ (3): Let X be normal and let (E_1, E_2) be a block partition. That is, E_1 and E_2 are disjoint and $E_1 \cup E_2$ is a block. Then $(E_1^{\perp\perp}, E_1^{\perp})$ is a decomposition such that $E_1 \subseteq E_1^{\perp\perp}$ and $E_2 \subseteq E_1^{\perp}$. Likewise, $(E_2^{\perp}, E_2^{\perp\perp})$ is a decomposition such that $E_1 \subseteq E_2^{\perp}$ and $E_2 \subseteq E_2^{\perp\perp}$. By uniqueness, we conclude $E_2^{\perp} = E_1^{\perp\perp}$ and $E_1^{\perp} = E_2^{\perp\perp}$. In particular, $(E_2^{\perp}, E_1^{\perp})$ is a decomposition.

(3) $\Rightarrow$ (4): Let (E_1, E_2) be a block partition and assume that $(E_2^{\perp}, E_1^{\perp})$ is a decomposition. Then $E_2^{\perp\perp} = E_1^{\perp}$ and $E_1^{\perp\perp} = E_2^{\perp}$.

(4) ⇒ (1): Let (E_1, E_2) be a block partition and assume that $(E_1^{\perp\perp}, E_2^{\perp\perp})$ is a decomposition. Let $(\bar{E}_1, \bar{E}_2)$ be a further decomposition such that $E_1 \subseteq \bar{E}_1$ and $E_2 \subseteq \bar{E}_2$. Then $\bar{E}_1$ is orthoclosed and hence $E_1^{\perp\perp} \subseteq \bar{E}_1$. Moreover, $\bar{E}_1 = \bar{E}_2{}^{\perp} \subseteq E_2^{\perp} = E_1^{\perp\perp}$ and we conclude $\bar{E}_1 = E_1^{\perp\perp}$. Hence we also have $\bar{E}_2 = E_2^{\perp\perp}$. Normality follows.

(4) ⇒ (5): Let (E_1, E_2) be a block partition and assume that $(E_1^{\perp\perp}, E_2^{\perp\perp})$ is a decomposition. Then $E_1^{\perp} = E_1^{\perp\perp\perp} = E_2^{\perp\perp}$ and hence $E_1^{\perp} \perp E_2^{\perp}$.

(5) ⇒ (4): Let (E_1, E_2) be a block partition and assume that $E_1^{\perp} \perp E_2^{\perp}$. Then $E_2 \subseteq E_1^{\perp} \subseteq E_2^{\perp\perp}$ implies that $E_1^{\perp} = E_2^{\perp\perp}$. It follows $E_2^{\perp} = E_1^{\perp\perp}$, hence $(E_1^{\perp\perp}, E_2^{\perp\perp})$ is a decomposition.

Finally, it is obvious that (4) implies (2) and that (2) implies (5). □

We note that criterion (5) in Proposition 2.5 is the one that we chose to define normality in [PaVe].

For any non-zero cardinal κ, a *(κ-fold) decomposition* of X is meant to be a collection $(A_\alpha)_{\alpha<\kappa}$ of non-empty subsets of X such that, for any $\alpha < \kappa$, we have $A_\alpha = (\bigcup_{\beta<\kappa,\, \beta\neq\alpha} A_\beta)^{\perp}$. We again refer to the sets A_α, $\alpha < \kappa$, as the components of the decomposition. Clearly, for a κ-fold decomposition to exist a necessary condition is that κ is at most the rank of X.

Note again that a κ-fold decomposition $(A_\alpha)_{\alpha<\kappa}$ consists of orthoclosed subsets. Moreover, we have $A_\alpha \perp A_\beta$ for any $\alpha \neq \beta$, as well as $(\bigcup_{\alpha<\kappa} A_\alpha)^{\perp} = \varnothing$.

It turns out that the notion of normality would not change if we based its definition on κ-fold decompositions for arbitrary cardinals κ. Proposition 2.5 could also be modified accordingly; we do so, as an example, for criterion (4).

Proposition 2.6. *Let $(X, \perp)$ be an orthogonality space. The following conditions are equivalent.*

(1) *$(X, \perp)$ is normal.*

(2) *Let $(E_\alpha)_{\alpha<\kappa}$ be a block partition. Then there is a unique decomposition $(\bar{E}_\alpha)_{\alpha<\kappa}$ of X such that $E_\alpha \subseteq \bar{E}_\alpha$, $\alpha < \kappa$.*

(3) *Let $(E_\alpha)_{\alpha<\kappa}$ be a block partition. Then $(E_\alpha{}^{\perp\perp})_{\alpha<\kappa}$ is a decomposition of X.*

Proof. (1) ⇒ (3): Assume that X is normal and that $(E_\alpha)_{\alpha<\kappa}$ is a block partition. By Proposition 2.5, $(E_\alpha^{\perp\perp}, (\bigcup_{\beta<\kappa,\, \beta\neq\alpha} E_\beta)^{\perp\perp})$ is a decomposition for any $\alpha < \kappa$, hence $(\bigcup_{\beta<\kappa,\, \beta\neq\alpha} E_\beta^{\perp\perp})^{\perp} = (\bigcup_{\beta<\kappa,\, \beta\neq\alpha} E_\beta)^{\perp} = E_\alpha^{\perp\perp}$ for all $\alpha < \kappa$. We conclude that $(E_\alpha{}^{\perp\perp})_{\alpha<\kappa}$ is a decomposition of X.

(3) ⇒ (2): Let $(E_\alpha)_{\alpha<\kappa}$ be a block partition and assume that $(E_\alpha{}^{\perp\perp})_{\alpha<\kappa}$ is a decomposition of X. Let $(\bar{E}_\alpha)_{\alpha<\kappa}$ be a further κ-fold decomposition of X such that $E_\alpha \subseteq \bar{E}_\alpha$, $\alpha < \kappa$. For all $\alpha < \kappa$, we then have $E_\alpha^{\perp\perp} \subseteq \bar{E}_\alpha = (\bigcup_{\beta<\kappa,\, \beta\neq\alpha} \bar{E}_\beta)^{\perp} \subseteq (\bigcup_{\beta<\kappa,\, \beta\neq\alpha} E_\beta)^{\perp} = (\bigcup_{\beta<\kappa,\, \beta\neq\alpha} E_\beta^{\perp\perp})^{\perp} = E_\alpha^{\perp\perp}$, that is, $E_\alpha^{\perp\perp} = \bar{E}_\alpha$. This shows (2).

Clearly, (2) implies (1). □

We conclude that any block partition of a normal orthogonality space gives rise to a unique decomposition such that each cell of the partition is contained in exactly one component. In this case, each component is the closure of the cell that it contains. Given a block partition $\mathbf{E} = (E_\alpha)_{\alpha<\kappa}$, we shall write accordingly

$$\bar{\mathbf{E}} \;=\; ({E_\alpha}^{\perp\perp})_{\alpha<\kappa}. \tag{D}$$

We call a decomposition of this form *propositional*. It should be noted that in general a decomposition need not be of this type.

We shall next see what the condition of normality means when orthogonality spaces are considered as test spaces. We recall that an (irredundant) *test space* is a collection of subsets of a non-empty set such that none of these subsets is properly contained in another one. For a comprehensive overview of test spaces, we refer the reader to the A. Wilce's handbook chapter [Wlc].

For any orthogonality space $(X, \perp)$, the collection of blocks of X is an example of a test space. For a test space of this form, we recall the common notions. A $\perp$-set E is called a *complement* of another $\perp$-set F if (E, F) is a block partition. Moreover, the test space is called *algebraic* if any two $\perp$-sets E and F that possess a common complement possess the same set of complements.

The following lemma shows that an orthogonality space is normal exactly if the associated test space is algebraic. In its proof as well as at some places in the sequel, we denote the union of sets A and B by $A \,\dot\cup\, B$ in order to express that A and B are disjoint.

Proposition 2.7. *Let $(X, \perp)$ be an orthogonality space. Then the following conditions are equivalent.*

(1) *$(X, \perp)$ is normal.*

(2) *If (E_1, E_2), (F_1, E_2), and (E_1, F_2) are block partitions, then so is (F_1, F_2).*

Proof. (1) ⇒ (2): Let X be normal and let (E_1, E_2), (F_1, E_2), (E_1, F_2) be block partitions. By criterion (4) of Proposition 2.5, we have that $F_1^{\perp\perp} = E_2^{\perp} = E_1^{\perp\perp}$ and hence $(F_1^{\perp\perp}, F_2^{\perp\perp}) = (E_1^{\perp\perp}, F_2^{\perp\perp})$ is a decomposition of X. It follows that (F_1, F_2) is a block partition as well.

(2) ⇒ (1): Assume that condition (2) holds and that (E_1, E_2) is a block partition. Let $f_2 \perp E_1$ and $f_1 \perp E_2$. Then we extend $E_1 \cup \{f_2\}$ to a block $E_1 \,\dot\cup\, F_2$, and we choose similarly F_1. Then (E_1, E_2), (F_1, E_2), and (E_1, F_2) are block partitions and by assumption it follows $F_1 \perp F_2$. In particular we have $f_1 \perp f_2$ and by criterion (5) of Proposition 2.5, the normality follows. □

We next turn to a lattice-theoretic characterisation of normality. We consider to this end the collection of all orthoclosed subsets of X, denoted by $\mathcal{C}(X, \perp)$. W.r.t. the set-theoretical inclusion, $\mathcal{C}(X, \perp)$ is a complete lattice. Together with the orthocomplementation, $\mathcal{C}(X, \perp)$ is in fact a complete ortholattice. The meet of closed subsets A_α, $\alpha < \kappa$, is given by $\bigcap_{\alpha<\kappa} A_\alpha$ and the join is $(\bigcup_{\alpha<\kappa} A_\alpha)^{\perp\perp}$. Furthermore, the orthocomplement of an orthoclosed subset A is $A^\perp$, and the bottom and top elements are $\varnothing$ and X, respectively.

With reference to the ortholattice $(\mathcal{C}(X,\perp); \cap, \vee, ^\perp, \varnothing, X)$, we may express normality as follows. A subalgebra $\mathcal{B}$ of an ortholattice is called *Boolean* if $\mathcal{B}$ is distributive, that is, a Boolean algebra. Moreover, we call a Boolean subalgebra $\mathcal{B}$ of $\mathcal{C}(X,\perp)$ *complete* if $\mathcal{B}$ is closed under arbitrary meets and joins. This means that arbitrary meets and joins exist in $\mathcal{B}$ and coincide with those in $\mathcal{C}(X,\perp)$. We note that, in contrast, the atomicity of a Boolean subalgebra $\mathcal{B}$ will be meant to refer to $\mathcal{B}$ only. Thus $\mathcal{B}$ is atomic if below any of its elements there is an atom of $\mathcal{B}$, but this atom is not required to be an atom of $\mathcal{C}(X,\perp)$.

Proposition 2.8. *Let $(X,\perp)$ be an orthogonality space. Then X is normal if and only if, for any block E of X, $\mathcal{B}_E = \{A^{\perp\perp} \colon A \subseteq E\}$ is a Boolean subalgebra of $\mathcal{C}(X,\perp)$. In this case, $\mathcal{B}_E$ is in fact a complete atomic Boolean subalgebra, and $\gamma \colon \mathcal{P}(E) \to \mathcal{B}_E$, $A \mapsto A^{\perp\perp}$ is an isomorphism between the Boolean algebra of subsets of E and $\mathcal{B}_E$.*

Proof. Assume that X is normal and let $E \subseteq X$ be a block. Then $\mathcal{B}_E \subseteq \mathcal{C}(X,\perp)$. Obviously, we have $\varnothing, X \in \mathcal{B}_E$.

For subsets A_α, $\alpha < \kappa$, of E we have

$$\bigvee_{\alpha<\kappa} A_\alpha^{\perp\perp} = \Big(\bigcup_{\alpha<\kappa} A_\alpha^{\perp\perp}\Big)^{\perp\perp} = \Big(\bigcup_{\alpha<\kappa} A_\alpha\Big)^{\perp\perp} \in \mathcal{B}_E,$$

hence $\mathcal{B}_E$ is closed under arbitrary joins. Furthermore, $\mathcal{B}_E$ is closed under the orthocomplement. Indeed, for any $A \subseteq E$, $((E \setminus A)^\perp, A^\perp)$ is, by Proposition 2.5, a decomposition and hence

$$(A^{\perp\perp})^\perp = A^\perp = (E \setminus A)^{\perp\perp} \in \mathcal{B}_E.$$

We conclude that $\mathcal{B}_E$ is closed under joins, meets, and the orthocomplementation. In particular, $\mathcal{B}_E$ is a subalgebra of $\mathcal{C}(X,\perp)$ and γ defines an isomorphism between the Boolean algebra of subsets of E and $\mathcal{B}_E$. Hence $\mathcal{B}_E$ is a complete atomic Boolean algebra.

For the converse direction, assume that (E_1, E_2) is a block partition and that $\mathcal{B}_E = \{A^{\perp\perp} \colon A \subseteq E\}$ is a Boolean subalgebra of $\mathcal{C}(X,\perp)$, where $E = E_1 \cup E_2$. Then $E_1^\perp = (E_1^{\perp\perp})^\perp = F^{\perp\perp}$ for some $F \subseteq E$. We have on the one hand $F \cap E_1 = \varnothing$ and hence $F \subseteq E_2$. On the other hand, $E \setminus F \subseteq F^\perp = E_1^{\perp\perp} \perp E_2$ and thus $E_2 \cap (E \setminus F) = \varnothing$, that is, $E_2 \subseteq F$. We conclude $F = E_2$, that is, $E_1^\perp = E_2^{\perp\perp}$ and $E_2^\perp = E_1^{\perp\perp}$. But this means that $(E_1^{\perp\perp}, E_2^{\perp\perp})$ is a decomposition of X. By Proposition 2.5, it follows that $(X,\perp)$ is normal. □

Let E be a block in a normal orthogonality space X. Then $\{e\}^{\perp\perp}$, $e \in E$, generate by Proposition 2.8 a complete Boolean subalgebra of $\mathcal{C}(X,\perp)$, which we will, as in Proposition 2.8, denote by $\mathcal{B}_E$. More generally, for any $\perp$-set $A \subseteq X$, we denote by $\mathcal{B}_A$ the complete Boolean algebra consisting of the joins of $\{e\}^{\perp\perp}$, $e \in A$, and whose top element is $A^{\perp\perp}$.

The orthoclosed subsets occurring in propositional decompositions are of a particular form and we next focus our attention on these. We call an orthoclosed subset A of an orthogonality space a *proposition* if there is a $\perp$-set E such that $A = E^{\perp\perp}$. We denote by $\Pi(X, \perp)$ the set of all propositions of $\mathcal{C}(X, \perp)$. In analogy to the case of test spaces, we could refer to $\Pi(X, \perp)$ as the *logic* of $(X, \perp)$.

As shown in [FGR, Theorem 6.5], a coherent algebraic test space gives rise to an orthomodular poset. In the present, more special context, we may observe that the occurrence of an orthomodular structure is characteristic for normality.

Proposition 2.9. *Let $(X, \perp)$ be an orthogonality space. Then X is normal if and only if $\Pi(X, \perp)$ is closed under orthocomplementation and, partially ordered by set-theoretic inclusion and equipped with the orthocomplementation, is an orthomodular poset.*

Proof. Assume that X is normal. For any $A \in \Pi(X, \perp)$, there is a $\perp$-set E such that $A = E^{\perp\perp}$. Let $F \subseteq X$ be such that (E, F) is a block partition. Then $A^{\perp} = E^{\perp} = F^{\perp\perp}$ by Proposition 2.5, hence $\Pi(X, \perp)$ is closed under orthocomplementation. Furthermore, let $A, B \in \Pi(X, \perp)$ be such that $A \subseteq B$. Then we may choose $\perp$-sets $E, F \subseteq X$ such that $E^{\perp\perp} = A$ and $F^{\perp\perp} = B^{\perp}$, and we may extend $E \cup F$ to a block $(E \cup F) \dot{\cup} G$. Then $B = B^{\perp\perp} = F^{\perp} = (E \cup G)^{\perp\perp} = (A \cup G^{\perp\perp})^{\perp\perp}$. In $\mathcal{C}(X, \perp)$, and consequently also in $\Pi(X, \perp)$, we hence have $B = A \vee G^{\perp\perp}$, where $G^{\perp\perp} \perp A$. Hence $\Pi(X, \perp)$ is an orthomodular poset.

Conversely, assume that $\Pi(X, \perp)$ is closed under orthocomplementation and is in fact an orthomodular poset. Let (E_1, E_2) be a block partition. In $\mathcal{C}(X, \perp)$ and hence in $\Pi(X, \perp)$, we then have $E_1^{\perp\perp} \vee E_2^{\perp\perp} = X$ and $E_1^{\perp\perp} \perp E_2^{\perp\perp}$. By orthomodularity, it follows $E_1^{\perp} = E_2^{\perp\perp}$ and $E_2^{\perp} = E_1^{\perp\perp}$, that is, $(E_1^{\perp\perp}, E_2^{\perp\perp})$ is a decomposition. By Proposition 2.5, we conclude that X is normal. □

Dacey spaces

Normality of an orthogonality space means that any block partition gives rise to a unique decomposition of the space. We consider now the condition that all decompositions arise in this way.

Definition 2.10. An orthogonality space $(X, \perp)$ is called *Dacey* if X is normal and any two-fold decomposition of X is propositional.

In other words, an orthogonality space X is Dacey if any block partition with two cells gives rise to a unique decomposition of X and any two-fold decomposition of X arises in this way. As usual we refer to X in this case as a “Dacey space” rather than a “Dacey orthogonality space”. For an overview of results on Dacey spaces, we refer again to [Wlc].

The following proposition contains some alternative characterisations of Dacey spaces. We see in particular that also the property of being Dacey would not change if its definition was based on partitions and decompositions with more than two constituents.

Proposition 2.11. *Let $(X, \perp)$ be a normal orthogonality space. The following are equivalent:*

(1) *X is Dacey.*

(2) *Any orthoclosed subset of X is a proposition, that is, $\Pi(X, \perp) = \mathcal{C}(X, \perp)$.*

(3) *Any κ-fold decomposition, where κ is a non-zero cardinal, is propositional.*

(4) *For every atomic complete Boolean subalgebra $\mathcal{B}$ of $\mathcal{C}(X, \perp)$ there is a block E such that $\mathcal{B}$ is a subalgebra of $\{A^{\perp\perp} : A \subseteq E\}$.*

Proof. (1) $\Rightarrow$ (2): Let $A \in \mathcal{C}(X, \perp)$. Then $(A, A^\perp)$ is a decomposition. If X is Dacey, there is a $\perp$-set E such that $A = E^{\perp\perp}$.

(2) $\Rightarrow$ (3): Assume that every orthoclosed set is a proposition and let $(A_\alpha)_{\alpha<\kappa}$ be a decomposition of X. For each $\alpha < \kappa$, choose a $\perp$-set $E_\alpha \subseteq X$ such that $A_\alpha = E_\alpha^{\perp\perp}$. Then $\bigcup_{\alpha<\kappa} E_\alpha$ is a block, that is, $(E_\alpha)_{\alpha<\kappa}$ is a block partition. Hence the decomposition is propositional.

(3) $\Rightarrow$ (4): Let $\mathcal{B}$ be an atomic complete Boolean subalgebra of $\mathcal{C}(X, \perp)$ and let $(A_\alpha)_{\alpha<\kappa}$ be the atoms of $\mathcal{B}$. Then $(A_\alpha)_{\alpha<\kappa}$ is a decomposition of X. Assume now that there is a block partition $(E_\alpha)_{\alpha<\kappa}$ such that $A_\alpha = E_\alpha^{\perp\perp}$ for all $\alpha < \kappa$. By normality, we have from Proposition 2.8 that $\mathcal{B}_E = \{A^{\perp\perp} : A \subseteq E\}$, where $E = \bigcup_{\alpha<\kappa}$, is a complete Boolean subalgebra of $\mathcal{C}(X, \perp)$, and $\mathcal{B}$ is a subalgebra of $\mathcal{B}_E$.

(4) $\Rightarrow$ (1): Let (A_1, A_2) be a decomposition of X. Then $\{X, A_1, A_2, \varnothing\}$ is an atomic Boolean subalgebra of $\mathcal{C}(X, \perp)$. Assume (4). Then there is block partition (E_1, E_2) such that $A_1 = E_1^{\perp\perp}$ and $A_2 = E_2^{\perp\perp}$. We conclude that X is Dacey. $\square$

Dacey spaces are usually defined without reference to normality. We next see that the common definition coincides with ours. The equivalence of condition (3) and (4) in the following lemma is due to Dacey [Dac].

Proposition 2.12. *Let $(X, \perp)$ be an orthogonality space. Then the following are equivalent:*

(1) *X is Dacey.*

(2) *Every two orthogonal elements of $\mathcal{C}(X, \perp)$ are contained in a Boolean subalgebra of $\mathcal{C}(X, \perp)$.*

(3) *$\mathcal{C}(X, \perp)$ is an orthomodular lattice.*

(4) *For any $A \in \mathcal{C}(X, \perp)$ and any maximal $\perp$-set E contained in A, we have $A = E^{\perp\perp}$.*

Proof. (1) $\Rightarrow$ (2): Assume that X is Dacey and let $A, B \in \mathcal{C}(X, \perp)$ be such that $A \perp B$. By Proposition 2.11, there are $\perp$-sets $E, F \subseteq X$ such that $A = E^{\perp\perp}$ and $B = F^{\perp\perp}$. Choose $G \subseteq X$ to extend $E \cup F$ to a block $(E \cup F) \mathbin{\dot\cup} G$. By Proposition 2.8, A and B belong to a Boolean subalgebra of $\mathcal{C}(X, \perp)$.

(2) $\Rightarrow$ (3): Let $A, B \in \mathcal{C}(X, \perp)$ such that $A \subseteq B$. Then $A \perp B^\perp$. Assume that A and $B^\perp$ are contained in a Boolean subalgebra. This means that the subalgebra of $\mathcal{C}(X, \perp)$ generated by A and B is Boolean. It follows $B = A \vee (A^\perp \cap B)$ and we conclude that $\mathcal{C}(X, \perp)$ is orthomodular.

(3) $\Rightarrow$ (4): Assume that $\mathcal{C}(X, \perp)$ is orthomodular. Let $A \in \mathcal{C}(X, \perp)$ and let E be a maximal $\perp$-set contained in A. If $E^{\perp\perp}$ was strictly contained in A, there would be an $e \in A$ such that $e \perp E$, in contradiction to the maximality of E.

(4) $\Rightarrow$ (1): Assume that (4) holds. Then any orthoclosed subset of X is a proposition. Hence, by Proposition 2.11, it suffices to prove that X is normal. Let (E_1, E_2) be a block partition. Then E_1 is a maximal $\perp$-set contained in $E_2^\perp$ and by assumption it follows $E_1^{\perp\perp} = E_2^\perp$. Hence $(E_2^\perp, E_1^\perp)$ is a decomposition and we conclude by Proposition 2.5 that X is normal. □

3 Normal homomorphisms

We now turn to the question which maps between orthogonality spaces we should consider as structure-preserving. The most fundamental aspect is certainly the preservation of the orthogonality relation.

Definition 3.1. A map $\varphi\colon X \to Y$ between the orthogonality space X and Y is called a *homomorphism* if, for any $e, f \in X$, $e \perp f$ implies $\varphi(e) \perp \varphi(f)$.

Homomorphisms can be considered as the structure-preserving maps between orthogonality spaces in general. For a discussion of categories of orthogonality spaces, see, e.g., [Faw]. Here, our focus is on normal orthogonality spaces and a more special choice of morphisms seems to be natural.

Let $\varphi\colon X \to Y$ be a homomorphism between orthogonality spaces. Let $\mathbf{E}$ be a block partition of X and let $\mathbf{F}$ be a block partition of Y. We say that φ maps $\mathbf{E}$ *into* $\mathbf{F}$ if, for any cell E of $\mathbf{E}$, $\varphi(E)$ is a cell of $\mathbf{F}$. Likewise, we say that φ maps the decomposition $\mathbf{A} = (A_\alpha)_{\alpha<\kappa}$ of X *into* the decomposition $\mathbf{B}$ of Y if there are pairwise distinct components $B_\alpha \in \mathbf{B}$, $\alpha < \kappa$, such that $\varphi(A_\alpha) \subseteq B_\alpha$, $\alpha < \kappa$.

By Proposition 2.6, a characteristic property of the normality of orthogonality spaces is the following: each block partition $\mathbf{E}$ gives rise to the decomposition $\bar{\mathbf{E}}$ consisting of the closures of the cells of $\mathbf{E}$, cf. (D). This fact motivates the following definition of normality for homomorphisms.

Definition 3.2. Let $(X, \perp)$ and $(Y, \perp)$ be normal orthogonality spaces. We call a homomorphism $\varphi\colon X \to Y$ *normal* if the following condition holds:

(N) Let φ map the block partition $\mathbf{E}$ of X into the block partition $\mathbf{F}$ of Y. Then φ maps the decomposition $\bar{\mathbf{E}}$ into the decomposition $\bar{\mathbf{F}}$.

Also the normality of homomorphisms allows a number of different formulations. Probably most important, a normal homomorphism fulfils for $\perp$-sets the continuity condition w.r.t. the closure operator ${}^{\perp\perp}$ [Ern].

Proposition 3.3. *Let $\varphi\colon X \to Y$ be a homomorphism between normal orthogonality spaces. Then the following are equivalent.*

(1) *φ is normal.*

(2) *For any $\perp$-set $A \subseteq X$, we have $\varphi(A^{\perp\perp}) \subseteq \varphi(A)^{\perp\perp}$.*

(3) *For any block $E \subseteq X$, we have $\varphi(X)^{\perp\perp} = \varphi(E)^{\perp\perp}$.*

Proof. (1) ⇒ (2): Let φ be normal and let A be a $\perp$-set. Let $B \subseteq X$ be such that (A, B) is a block partition of X, and let $C \subseteq Y$ be such that $(\varphi(A), \varphi(B), C)$ is a block partition of Y. Then, by normality, φ maps the decomposition $(A^{\perp\perp}, B^{\perp\perp})$ of X into the decomposition $(\varphi(A)^{\perp\perp}, \varphi(B)^{\perp\perp}, C^{\perp\perp})$ of Y. It follows that $\varphi(A^{\perp\perp}) \subseteq \varphi(A)^{\perp\perp}$.

(2) ⇒ (3): Let E be a block of X. Then (2) implies that $\varphi(X) = \varphi(E^{\perp\perp}) \subseteq \varphi(E)^{\perp\perp} \subseteq \varphi(X)^{\perp\perp}$ and we conclude $\varphi(X)^{\perp\perp} = \varphi(E)^{\perp\perp}$.

(3) ⇒ (2): Let A be a $\perp$-set of X. Extend A to a block E and let $C \subseteq Y$ be such that $\varphi(E) \,\dot\cup\, C$ is a block of Y. Then $x \in A^{\perp\perp}$ implies $x \perp E \setminus A$ and hence $\varphi(x) \perp \varphi(E \setminus A)$. Assume now that $\varphi(X)^{\perp\perp} = \varphi(E)^{\perp\perp}$. Then we also have that $\varphi(x) \perp C$. By normality, it follows $\varphi(x) \in \varphi(A)^{\perp\perp}$. We have shown that $\varphi(A^{\perp\perp}) \subseteq \varphi(A)^{\perp\perp}$.

(2) ⇒ (1): This is obvious. □

An automorphism of an orthogonality space is a bijection $\varphi\colon X \to X$ such that both φ and φ^{-1} are homomorphisms. We remark that automorphisms are always normal.

Proposition 3.4. *Let X be a normal orthogonality space and let $\varphi\colon X \to X$ be an automorphism. Then φ is normal.*

Proof. Let E be a block. Then also $\varphi(E)$ is a block and hence $\varphi(E)^{\perp\perp} = X = \varphi(X) = \varphi(X)^{\perp\perp}$. By criterion (3) of Proposition 3.3, φ is normal. □

We shall next see that normality implies the preservation of a preorder with which any orthogonality space is naturally equipped.

Definition 3.5. Let $(X, \perp)$ be an orthogonality space. We define the *specialisation order* $\preccurlyeq$ on X as follows: For $e, f \in X$, we let $e \preccurlyeq f$ if, for any $x \in X$, $x \perp f$ implies $x \perp e$.

Our choice of terminology should remind of the analogous situation in closure spaces [Ern].

Note that the specialisation order is reflexive and transitive but in general not a partial order. An orthogonality space whose specialisation order is antisymmetric, and hence a partial order, is called *irredundant* [Vet3]. In case of our guiding examples, orthogonality spaces associated to Hermitian spaces as in Example 2.3, the specialisation order is in fact trivial, that is, for any e, f, $e \preccurlyeq f$ implies $e = f$. We call an orthogonality space with this property *strongly irredundant* [PaVe]. Unlike in our work [PaVe], our present discussion is not restricted from the outset to irredundant or even strongly irredundant spaces.

For elements e and f of an orthogonality space, we have $e \preccurlyeq f$ if and only if $\{f\}^\perp \subseteq \{e\}^\perp$ if and only if $\{e\}^{\perp\perp} \subseteq \{f\}^{\perp\perp}$ if and only if $e \in \{f\}^{\perp\perp}$. A homomorphism does not in general preserve the specialisation order. It does so, for instance, under the particular situation described in the next lemma.

Lemma 3.6. *Let $\varphi \in X \to Y$ be a surjective map between the orthogonality spaces X and Y such that, for any $e, f \in X$, $e \perp f$ if and only if $\varphi(e) \perp \varphi(f)$. Then φ is a homomorphism that preserves the specialisation order.*

Proof. Let $e, f \in X$ be such that $e \preccurlyeq f$. Let $z \perp \varphi(f)$. Then there is an $x \in X$ such that $z = \varphi(x)$. It follows $x \perp f$, hence $x \perp e$ and $z = \varphi(x) \perp \varphi(e)$, that is $\varphi(e) \preccurlyeq \varphi(f)$. □

The normality of homomorphisms can be understood as a sharpening of the condition to preserve the specialisation order.

Proposition 3.7. *Let $\varphi\colon X \to Y$ be a homomorphism between normal orthogonality spaces. Then the following are equivalent.*

(1) *φ is normal.*

(2) *Let $e \in X$ and let $A \subseteq X$ be a $\perp$-set. If, for any $x \in X$, $x \perp A$ implies $x \perp e$, then, for any $y \in Y$, $y \perp \varphi(A)$ implies $y \perp \varphi(e)$.*

In this case, φ preserves the specialisation order.

Proof. For any $e \in X$ and any $\perp$-set $A \subseteq X$, we have that $A^\perp \subseteq \{e\}^\perp$ implies $\varphi(A)^\perp \subseteq \{\varphi(e)\}^\perp$ if and only if $\{e\}^{\perp\perp} \subseteq A^{\perp\perp}$ implies $\{\varphi(e)\}^{\perp\perp} \subseteq \varphi(A)^{\perp\perp}$ if and only if $e \in A^{\perp\perp}$ implies $\varphi(e) \in \varphi(A)^{\perp\perp}$ if and only if $\varphi(A^{\perp\perp}) \subseteq \varphi(A)^{\perp\perp}$. Hence the asserted equivalence follows from Proposition 3.3.

For the final assertion, apply condition (2) to a singleton. □

We next consider the normality of homomorphisms in relation to the partial order among decompositions. As is common in case of partitions, we say that a decomposition $\mathbf{A}$ is *finer* than a decomposition $\mathbf{B}$ if any component of $\mathbf{A}$ is contained in a component of $\mathbf{B}$. In this case, we also call $\mathbf{A}$ a *refinement* of $\mathbf{B}$ and $\mathbf{B}$ a *coarsening* of $\mathbf{A}$.

The normality of a homomorphism between normal orthogonality spaces means, in the sense of the following proposition, its compatibility with the partial order among propositional decompositions.

Proposition 3.8. *Let $\varphi\colon X \to Y$ be a homomorphism between normal orthogonality spaces. Then the following are equivalent.*

(1) *φ is normal.*

(2) *φ preserves the specialisation order and the following condition holds: If φ maps the propositional decomposition $\mathbf{A}$ of X into the propositional decomposition $\mathbf{B}$ of Y, then for any coarsening $\mathbf{B}'$ of $\mathbf{B}$ there is a coarsening $\mathbf{A}'$ of $\mathbf{A}$ such that φ maps $\mathbf{A}'$ into $\mathbf{B}'$.*

Proof. (1) $\Rightarrow$ (2): This follows from Proposition 3.7 and criterion (2) of Proposition 3.3.

(2) $\Rightarrow$ (1): Assume that (2) holds. Let $E \subseteq X$ be a block and let $F \subseteq Y$ be such that $\varphi(E) \dot{\cup} F$ is a block. As φ preserves the specialisation order, φ maps the decomposition $(\{e\}^{\perp\perp})_{e\in E}$ into the decomposition of Y consisting of $\{\varphi(e)\}^{\perp\perp}$, $e \in E$, and $F^{\perp\perp}$. Moreover, $(\varphi(E)^{\perp\perp}, F^{\perp\perp})$ is a coarsening of the latter decomposition. The only coarsening of $(\{e\}^{\perp\perp})_{e\in E}$ that is mapped by φ into $(\varphi(E)^{\perp\perp}, F^{\perp\perp})$ is the decomposition consisting of a single component and we conclude that $\varphi(X) \subseteq \varphi(E)^{\perp\perp}$. Hence φ is normal by criterion (3) of Proposition 3.3. □

Finally, with regard to the associated ortholattices, the normality of homomorphism reads as follows. It is this characterisation that we have used in [PaVe].

Any homomorphism $\varphi\colon X \to Y$ between orthogonality spaces induces a map $\bar{\varphi}$ between the associated ortholattices:

$$\bar{\varphi}\colon \mathcal{C}(X,\perp) \to \mathcal{C}(Y,\perp),\ A \mapsto \varphi(A)^{\perp\perp}.$$

In general, all we can say about this map is that it preserves the order and the orthogonality relation.

Proposition 3.9. *Let $(X,\perp)$ and $(Y,\perp)$ be normal orthogonality spaces. A map $\varphi\colon X \to Y$ is a normal homomorphism if and only if, for any block E of X, $\bar{\varphi}$ maps $\mathcal{B}_E$ isomorphically to $\mathcal{B}_{\varphi(E)}$.*

Proof. Assume first that $\varphi\colon X \to Y$ is a normal homomorphism. Let E be a block of X and extend $\varphi(E)$ to a block F of Y. Then, for $A \subseteq E$, we have by Proposition 3.3 that $\bar{\varphi}(A^{\perp\perp}) = \varphi(A^{\perp\perp})^{\perp\perp} \subseteq \varphi(A)^{\perp\perp} \subseteq \varphi(A^{\perp\perp})^{\perp\perp} = \bar{\varphi}(A^{\perp\perp})$, that is, $\bar{\varphi}(A^{\perp\perp}) = \varphi(A)^{\perp\perp}$. By Proposition 2.8, $\mathcal{B}_E = \{A^{\perp\perp}\colon A \subseteq E\}$ and $\mathcal{B}_{\varphi(E)} = \{B^{\perp\perp}\colon B \subseteq \varphi(E)\}$ are Boolean algebras, isomorphic to the Boolean algebras of subsets of E and $\varphi(E)$, respectively. We conclude that $\bar{\varphi}$ establishes an isomorphism between $\mathcal{B}(E)$ and $\mathcal{B}(\varphi(E))$.

Conversely, assume that for any block E of X, $\bar{\varphi}$ maps $\mathcal{B}_E$ isomorphically to $\mathcal{B}_{\varphi(E)}$. Then, any pair of orthogonal elements $e, f \in X$ is contained in a block and $\{e\}^{\perp\perp} \perp \{f\}^{\perp\perp}$ implies $\bar{\varphi}(\{e\}^{\perp\perp}) \perp \bar{\varphi}(\{f\}^{\perp\perp})$. Thus we have $\varphi(e) \perp \varphi(f)$, that is, φ is a homomorphism. Moreover, for any block E of X, $\bar{\varphi}$ maps the top element of $\mathcal{B}_E$ to the top element of $\mathcal{B}_{\varphi(E)}$, that is, $\varphi(X)^{\perp\perp} = \bar{\varphi}(E^{\perp\perp}) = \varphi(E)^{\perp\perp}$. From criterion (3) of Proposition 3.3 we conclude that φ is normal. □

4 The category $\mathcal{NOS}$ of normal orthogonality spaces

The composition of normal homomorphisms between normal orthogonality spaces is a normal homomorphism again. This is most directly seen from criterion (2) of

Proposition 3.3. By $\mathcal{NOS}$, we denote the category of normal orthogonality spaces and normal homomorphisms. We compile in this section a number of properties of $\mathcal{NOS}$.

We start by characterising the monomorphisms and epimorphisms. We consider to this end a doubling point construction, which is explained in the following lemma.

To increase clarity, we will occasionally use subscripts for the denotation of orthogonality relations and the complements in the associated ortholattices.

Lemma 4.1. *Let $(X, \perp_X)$ be a normal orthogonality space and $x \in X$. Let Z arise from X by replacing x with two new elements x_1 and x_2. We define the orthogonality relation $\perp_Z$ on Z as follows: For $e, f \in X \setminus \{x_1, x_2\}$ such that $e \perp_X f$, we let $e \perp_Z f$; and for $e \in X$ such that $e \perp_X x$, we let $x_1, x_2 \perp_Z e$ and $e \perp_Z x_1, x_2$. Then $(Z, \perp_Z)$ is a normal orthogonality space.*

Moreover, we define $f_1, f_2 \colon X \to Z$ as follows: $f_1(z) = f_2(z) = z$ if $z \neq x$; $f_1(x) = x_1$; and $f_2(x) = x_2$. Then f_1, f_2 are morphisms in $\mathcal{NOS}$.

Proof. To prove that $(Z, \perp_Z)$ is normal, we use criterion (5) of Proposition 2.5. Let (E_1, E_2) be a block partition of Z and assume $e \perp_Z E_2$ and $f \perp_Z E_1$. There are the following two cases.

Case 1. $(E_1 \cup E_2) \cap \{x_1, x_2\} \neq \varnothing$. We assume that $x_1 \in E_1$; the other cases work similarly. Then $x_2 \notin E_1$ and $x_1, x_2 \notin E_2$. We have $e \perp_X E_2$ or $e \in \{x_1, x_2\}$; moreover, $f \perp_X (E_1 \setminus \{x_1\}) \cup \{x\}$ and $f \perp_Z x_1, x_2$. As $((E_1 \setminus \{x_1\}) \cup \{x\}, E_2)$ is a block partition of X, we conclude that $e \perp_Z f$.

Case 2. $(E_1 \cup E_2) \cap \{x_1, x_2\} = \varnothing$. Then $e \perp_X E_2$, or $e \in \{x_1, x_2\}$ and $x \perp_X E_2$. Similarly for f. As (E_1, E_2) is a block partition of X, we conclude that $e \perp_Z f$.

We now show that f_1 is a morphism of $\mathcal{NOS}$; the case of f_2 is similar. Clearly, f_1 is a homomorphism. We use Proposition 3.3 to show that f_1 is normal. Let E be a block of X. Then $f_1(E)$ is a block of Z and hence $f_1(E)^{\perp_Z \perp_Z} = Z$. Furthermore, we have $f_1(X)^{\perp_Z \perp_Z} = \varnothing^{\perp_Z} = Z$. We conclude that f_1 is normal. □

Proposition 4.2. *Let $\varphi \colon X \to Y$ be a morphism in $\mathcal{NOS}$. Then we have:*

(i) *φ is a monomorphism in $\mathcal{NOS}$ if and only if φ is injective.*

(ii) *φ is an epimorphism in $\mathcal{NOS}$ if and only if φ is surjective.*

Proof. Ad (i): Assume that φ is a monomorphism in $\mathcal{NOS}$. Let $x_1, x_2 \in X$ be such that $\varphi(x_1) = \varphi(x_2)$. Let $(\mathbf{1}, \neq) = (\{p\}, \varnothing)$ be the one-element orthogonality space, cf. Example 2.4. Clearly, $(\{p\}, \varnothing)$ is normal. Then the maps $\widehat{x_1}, \widehat{x_2} \colon \{p\} \to X$, given by $\widehat{x_1}(p) = x_1$ and $\widehat{x_2}(p) = x_2$ are morphisms in $\mathcal{NOS}$. It follows $\varphi \circ \widehat{x_1} = \varphi \circ \widehat{x_2}$ and hence $\widehat{x_1} = \widehat{x_2}$. We conclude $x_1 = x_2$, that is, φ is injective.

The reverse direction is evident.

Ad (ii): Assume that φ is an epimorphism in $\mathcal{NOS}$ that is not surjective. Let $y \in Y$ be such that $y \notin \varphi(X)$. Let $Z = (Y \setminus \{y\}) \cup \{y_1, y_2\}$, where y_1, y_2 are new elements, and let $\perp_Z$ be defined as in Lemma 4.1, such that $(Z, \perp_Z)$ becomes a normal orthogonality space. Likewise, let $f_1, f_2 \colon Y \to Z$ be such that $f_1(z) = f_2(z) = z$ if $z \neq y$, $f_1(y) = y_1$, and $f_2(y) = y_2$. By Lemma 4.1, f_1 and f_2 are morphisms in $\mathcal{NOS}$. But from $f_1 \circ \varphi = f_2 \circ \varphi$ we conclude $f_1 = f_2$, a contradiction.

The other direction is again evident. □

Let $\varphi \colon X \to Y$ be a morphism in $\mathcal{NOS}$. We call φ *quasi-surjective* if $Y = \varphi(X)^{\perp\perp}$. Clearly, if φ is surjective, φ is also quasi-surjective. Moreover, we call φ *full* if, for any $x_1, x_2 \in X$ such that $\varphi(x_1) \perp \varphi(x_2)$, there are $x_1', x_2' \in X$ such that $x_1' \perp x_2'$ and $\varphi(x_1) = \varphi(x_1')$ and $\varphi(x_2) = \varphi(x_2')$. Finally, we call φ an *embedding* if φ is injective and full.

Let A be an orthoclosed subset of an orthogonality space $(X, \perp)$. The restriction of the orthogonality relation to A, which we will denote by $\perp$ as well, makes A into an orthogonality space as well. We call $(A, \perp)$ a *subspace* of X. We should, however, note that a subspace of a normal orthogonality space is not in general normal; cf. [PaVe].

We may factorise a morphism in $\mathcal{NOS}$ as follows.

Theorem 4.3. *Let $\varphi \colon X \to Y$ be a morphism in $\mathcal{NOS}$. Then there are morphisms $\alpha \colon X \to Z$ and $\beta \colon Z \to Y$ such that $\varphi = \beta \circ \alpha$, where α is quasi-surjective and β is an embedding.*

Proof. In this proof, we mark the ortholattice complement on $\mathcal{C}(Z, \perp)$ by a subscript Z, whereas the unmarked ones refer to $\mathcal{C}(X, \perp)$ or $\mathcal{C}(Y, \perp)$.

We claim that the subspace $Z = \varphi(X)^{\perp\perp}$ of Y is normal. Let E be a block of X and let $\varphi(E) \,\dot{\cup}\, F$ be a block of Y. As φ is normal, we have by Proposition 3.3 that $Z = \varphi(E)^{\perp\perp}$. From the normality of Y, it furthermore follows that $Z = F^{\perp}$. Let now G be a block of Z. We readily see that then $G \cup F$ is a block of Y. By the normality of Y, we have $Z = F^{\perp} = G^{\perp\perp}$.

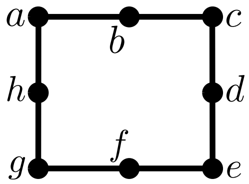
Let (G_1, G_2) be a partition of G. By Proposition 2.8, $G \cup F$ generates a Boolean subalgebra of $\mathcal{C}(Y)$. We conclude that $G_1^{\perp_Z} = G_1^{\perp} \cap Z = G_1^{\perp} \cap F^{\perp} = (G_1 \cup F)^{\perp\perp} = G_2^{\perp\perp}$ and similarly $G_2^{\perp_Z} = G_2^{\perp} \cap Z = G_1^{\perp\perp}$, hence $G_1^{\perp_Z} \perp G_2^{\perp_Z}$ and consequently $G_1^{\perp_Z} \perp_Z G_2^{\perp_Z}$. By criterion (5) of Proposition 2.5, $(Z, \perp_Z)$ is normal.

Let $\alpha \colon X \to Z,\ x \mapsto \varphi(x)$ and let $\beta \colon Z \to Y$ be the inclusion map. Clearly, α and β preserve the orthogonality, and $\varphi = \beta \circ \alpha$. To see that α is normal, let again E be a block of X and let F be as above. Then $\alpha(X)^{\perp_Z\perp_Z} = \varphi(X)^{\perp_Z\perp_Z} = (\varphi(X)^{\perp} \cap Z)^{\perp} \cap Z = (Z^{\perp} \cap Z)^{\perp} \cap Z = Z$ and $\alpha(E)^{\perp_Z\perp_Z} = \varphi(E)^{\perp_Z\perp_Z} = (\varphi(E)^{\perp} \cap Z)^{\perp} \cap Z = (Z^{\perp} \cap Z)^{\perp} \cap Z = Z$, hence the assertion holds by Proposition 3.3. It also follows that α is quasi-surjective.

To see that β is normal, let again G be a block of Z. We have seen above that then $Z = G^{\perp\perp}$. Hence $\beta(Z)^{\perp\perp} = Z^{\perp\perp} = Z = G^{\perp\perp} = \beta(G)^{\perp\perp}$, hence the normality follows from Proposition 3.3. The fact that β is an embedding is obvious. □

The next propositions deal with equalisers as well as with two ways of constructing orthogonality spaces in $\mathcal{NOS}$ from given ones. We need the following example from [PaVe].

Example 4.4. *The following orthogonality space $(X, \perp)$ is normal but not a Dacey space:*



Here, two elements are orthogonal if they lie both on a straight line. For instance, a, b, and c are mutually orthogonal.

Proposition 4.5. *The category $\mathcal{NOS}$ does not have equalisers.*

Proof. Let us consider the normal orthogonality space $(X, \perp_X)$ from Example 4.4. We define $\varphi\colon X \to X,\ a \mapsto a,\ b \mapsto h,\ c \mapsto g,\ d \mapsto f,\ e \mapsto e,\ f \mapsto d,\ g \mapsto c,\ h \mapsto b$. Then φ is an automorphism of X and hence, by Proposition 3.4, a morphism of $\mathcal{NOS}$.

Let us assume that the pair of arrows $X \overset{\varphi}{\underset{\mathrm{id}_X}{\rightrightarrows}} X$ in $\mathcal{NOS}$ possesses an equaliser $\psi\colon Y \to X$. Since the diagram $Y \xrightarrow{\ \psi\ } X \overset{\varphi}{\underset{\mathrm{id}_X}{\rightrightarrows}} X$ commutes, the image of ψ must be contained in $\{a, e\}$. We consider two cases.

Case 1. Assume that ψ is a constant map, that is, $\psi(Y) = \{a\}$ or $\psi(Y) = \{e\}$. We assume that $\psi(Y) = \{a\}$; the other case is similar. Let again $(\mathbf{1}, \neq) = (\{p\}, \varnothing)$ be the normal orthogonality space consisting of a single element and consider the morphism $\widehat{e}\colon \{p\} \to X,\ p \mapsto e$. Then $\varphi \circ \widehat{e} = \mathrm{id}_X \circ \widehat{e}$. But there is no map $k\colon \{p\} \to Y$ such that $\widehat{e} = \psi \circ k$.

Case 2. Assume that $\psi(Y) = \{a, e\}$. Let $y \in Y$ be such that $\psi(y) = a$. Because $a \not\perp e$, we have that $\{y\}$ is a block of Y. Moreover, $\psi(Y)^{\perp_X\perp_X} = \{a, e\}^{\perp_X\perp_X} = \{a, e\} \neq \{a\} = \{a\}^{\perp_X\perp_X} = \{\psi(y)\}^{\perp_X\perp_X}$, in contradiction to the normality of ψ.

We conclude that the pair φ, id_X does not possess an equaliser. □

Let $(X_i, \perp_i)$, $i \in I$, be normal orthogonality spaces. In the category $\mathcal{NOS}$, we call an object $(X, \perp_X)$ together with morphisms $\mathrm{inj}_i\colon X_i \to X$, $i \in I$, a *horizontal sum* if the following holds: for any morphisms $\varphi_i\colon X_i \to Y$, $i \in I$, such that $\varphi_i(X_i)^{\perp_Y\perp_Y} = \varphi_j(X_j)^{\perp_Y\perp_Y}$ for all $i, j \in I$, there is a unique morphism $\varphi\colon X \to Y$ such that $\varphi_i = \varphi \circ \mathrm{inj}_i$ for every $i \in I$.

Proposition 4.6. *The category $\mathcal{NOS}$ has horizontal sums.*

Proof. Let $(X_i, \perp_i)$, $i \in I$, be normal orthogonality spaces. We assume that the sets X_i, $i \in I$, are mutually disjoint. Let $X = \bigcup_{i \in I} X_i$ and for $e, f \in X$, let $e \perp f$ if there is an $i \in I$ such that $e, f \in X_i$ and $e \perp_i f$.

Clearly, $(X, \perp)$ is an orthogonality space. We claim that $(X, \perp)$ is normal. Let (E_1, E_2) be a block partition of X. Then (E_1, E_2) is a block partition of X_i for some $i \in I$. As X_i is normal, we have $E_1^{\perp_i} \perp_i E_2^{\perp_i}$ and consequently $E_1^{\perp} \perp E_2^{\perp}$. The normality of X holds by criterion (5) of Proposition 2.5.

For every $i \in I$, let $\mathrm{inj}_i \colon X_i \to X$ be the inclusion map. We claim that inj_i is a morphism. By construction, inj_i is orthogonality-preserving. Let E be a block of X_i. Then E is also a block of X. Hence $\mathrm{inj}_i(E)^{\perp\perp} = E^{\perp\perp} = \varnothing^{\perp} = X^{\perp\perp} = X_i^{\perp\perp} = \mathrm{inj}_i(X_i)^{\perp\perp}$ and the normality follows from Proposition 3.3.

Let now $(Y, \perp_Y)$ be a further normal orthogonality space and let $\varphi_i \colon X_i \to Y$, $i \in I$, be morphisms such that $\varphi_i(X_i)^{\perp_Y\perp_Y} = \varphi_j(X_j)^{\perp_Y\perp_Y}$ for all $i, j \in I$. We have to show that there exists a unique morphism $\varphi \colon X \to Y$ such that $\varphi_i = \varphi \circ \mathrm{inj}_i$ for every $i \in I$. The only map φ fulfilling the latter requirement is defined as follows: for $x \in X$, we let $\varphi(x) = \varphi_i(x)$ for the unique $i \in I$ such that $x \in X_i$. Clearly, φ is a homomorphism. To see that φ is normal, let E be a block of X. Then E is a block of X_j for some $j \in I$. Applying Proposition 3.3 to φ_j, we get

$$\begin{aligned} \varphi(E)^{\perp_Y\perp_Y} &= \varphi_j(E)^{\perp_Y\perp_Y} = \varphi_j(X_j)^{\perp_Y\perp_Y} = \bigvee_{i\in I} \varphi_i(X_i)^{\perp_Y\perp_Y} \\ &= \left(\bigcup_{i\in I} \varphi_i(X_i)^{\perp_Y\perp_Y}\right)^{\perp_Y\perp_Y} = \left(\bigcup_{i\in I} \varphi_i(X_i)\right)^{\perp_Y\perp_Y} = \varphi(X)^{\perp_Y\perp_Y}, \end{aligned}$$

and again from Proposition 3.3 the normality of φ follows. □

Let again $(X_i, \perp_i)$, $i \in I$, be normal orthogonality spaces. In $\mathcal{NOS}$, we call an object $(X, \perp)$ together with morphisms $\mathrm{in}_i \colon X_i \to X$, $i \in I$, a *direct product* if the following holds: for any morphisms $\varphi_i \colon X_i \to Y, i \in I$, such that $\varphi_i(X_i) \perp_Y \varphi_j(X_j)$ for all $i \neq j$, there is a unique morphism $\varphi \colon X \to Y$ such that $\varphi_i = \varphi \circ \mathrm{in}_i$ for all $i \in I$.

Proposition 4.7. *The category $\mathcal{NOS}$ has direct products.*

Proof. Let $(X_i, \perp_i)$, $i \in I$ be normal orthogonality spaces. We assume again that the sets X_i, $i \in I$, are mutually disjoint. Let $X = \bigcup_{i\in I} X_i$ and for $e, f \in X$, let $e \perp f$ if either there is an $i \in I$ such that $e, f \in X_i$ and $e \perp_i f$, or there are $i, j \in I$, $i \neq j$ such that $e \in X_i$ and $f \in X_j$.

Obviously, $(X, \perp)$ is an orthogonality space. We claim that $(X, \perp)$ is normal. Let (E, F) be a block partition and assume $e \perp F$ and $f \perp E$. Let $e \in X_i$ and $f \in X_j$. If $i \neq j$, we have $e \perp f$ by construction. Otherwise, $e \perp F \cap X_i$ and $f \perp E \cap X_i$. Since $(E \cap X_i, F \cap X_i)$ is a block partition of $(X_i, \perp_i)$, the normality of X_i implies $e \perp_i f$, that is, we have $e \perp f$ also in this case. The assertion follows now from criterion (5) of Proposition 2.5.

For every $i \in I$, let $\mathrm{in}_i \colon X_i \to X$ be the inclusion map. We claim that in_i is a morphism in $\mathcal{NOS}$. Evidently, in_i preserves the orthogonality. Moreover, let E_i be a block of X_i. Then $E_i^{\perp} = \bigcup_{j\in I, j\neq i} X_j$ and hence $E_i^{\perp\perp} = X_i$. We conclude $\mathrm{in}_i(E_i)^{\perp\perp} = E_i^{\perp\perp} = X_i = X_i^{\perp\perp} = \mathrm{in}_i(X_i)^{\perp\perp}$ and by Proposition 3.3, in_i is normal.

Let now $(Y, \perp_Y)$ be a further normal orthogonality space and let $\varphi_i \colon X_i \to Y$, $i \in I$, be morphisms in $\mathcal{NOS}$ such that $\varphi_i(X_i) \perp_Y \varphi_j(X_j)$ for all $i \neq j$. We have to show that there exists a unique morphism $\varphi \colon X \to Y$ such that $\varphi_i = \varphi \circ \mathrm{in}_i$ for every

$i \in I$. Again, there is only one map φ fulfilling the latter requirement: for $x \in X$, we let $\varphi(x) = \varphi_i(x)$ for the unique $i \in I$ such that $x \in X_i$. We readily observe that φ preserves the orthogonality. It remains to show that φ is normal. Let E be a block of X. Then $E_i = E \cap X_i$ is a block of X_i for every $i \in I$. By Proposition 3.3 applied to φ_i, $i \in I$, we get

$$\begin{aligned}\varphi(E)^{\perp_Y\perp_Y} &= \left(\bigcup_{i\in I}\varphi(E_i)\right)^{\perp_Y\perp_Y} = \left(\bigcup_{i\in I}\varphi(E_i)^{\perp_Y\perp_Y}\right)^{\perp_Y\perp_Y} \\ &= \left(\bigcup_{i\in I}\varphi_i(X_i)^{\perp_Y\perp_Y}\right)^{\perp_Y\perp_Y} = \left(\bigcup_{i\in I}\varphi_i(X_i)\right)^{\perp_Y\perp_Y} = \varphi(X)^{\perp_Y\perp_Y},\end{aligned}$$

and the normality of φ follows once more from Proposition 3.3. □

We remark that, if $(X, \perp_X)$ is the horizontal sum (respectively, the direct product) of normal orthogonality spaces $(X_i, \perp_i)$, $i \in I$, then $\mathcal{C}(X, \perp)$ is the horizontal sum (respectively, the direct product) of the ortholattices $\mathcal{C}(X_i, \perp_i)$, $i \in I$.

Proposition 4.8. *The horizontal sum as well as the direct product of Dacey spaces is again a Dacey space.*

Proof. Let $(X_i, \perp_i)$, $i \in I$, be Dacey spaces and let $(Z, \perp_Z)$ be their horizontal sum. We shall see that $(Z, \perp_Z)$ fulfils criterion (4) of Proposition 2.12. Let $A \in \mathcal{C}(Z, \perp_Z)$ and let D be a maximal $\perp$-set contained in A. We have to show that $D^{\perp\perp} = A$. This is clear if $A = \varnothing$ or $A = Z$. Otherwise, there is an $i \in I$ such that A is a non-empty, proper subset of X_i. We have that $A^{\perp_i\perp_i} = A^{\perp_Z\perp_Z} = A$, that is, $A \in \mathcal{C}(X_i, \perp_i)$. Furthermore, D is a maximal subset of A consisting of mutually orthogonal elements of X_i. As X_i is Dacey, it follows $A = D^{\perp_i\perp_i} = D^{\perp_Z\perp_Z}$.

Moreover, let $(X, \perp)$ be the direct product of the Dacey spaces $(X_i, \perp_i)$, $i \in I$. Then $X_i^{\perp\perp} = X_i$, that is, $X_i \in \mathcal{C}(X, \perp)$. Let $A \in \mathcal{C}(X, \perp)$ and let D be a maximal $\perp$-set contained in A. Let $A_i = A \cap X_i$ and $D_i = D \cap X_i$, $i \in I$. Then D_i is a maximal subset of A_i consisting of mutually orthogonal elements of X_i. As X_i is Dacey, it follows $A = D^{\perp_i\perp_i} = D^{\perp_Z\perp_Z}$. Hence

$$A = \bigcup_{i\in I} A_i = \bigcup_{i\in I} D_i^{\perp\perp} \subseteq \left(\bigcup_{i\in I} D_i\right)^{\perp\perp} = D^{\perp\perp} \subseteq A.$$

We conclude that $(X, \perp)$ is Dacey. □

We may observe that the two preceding constructions do not ensure the existence of categorical sums in $\mathcal{NOS}$. The following example shows that neither the horizontal sum nor the direct product, together with the respective injection mappings, is a categorical sum in $\mathcal{NOS}$.

Example 4.9. *Let us consider the orthogonality spaces $(\{a, b\}, \neq)$ and $(\{c, d\}, \neq)$, which by Example 2.4 are normal.*

Let $(X, \perp_X) = (\{a,b,c,d\}, \neq)$ *be their direct product. Let* $\varphi_1\colon \{a,b\} \to \{a,b\}$ *be the identity map and let* $\varphi_2\colon \{c,d\} \to \{a,b\}$, $c \mapsto a$, $d \mapsto b$. *Then* φ_1 *and* φ_2 *are normal homomorphisms. But there is obviously no orthogonality-preserving map from* $(\{a,b,c,d\}, \neq)$ *to* $(\{a,b\}, \neq)$. *We conclude that* $(\{a,b,c,d\}, \neq)$ *is not a categorical sum of* $(\{a,b\}, \neq)$ *and* $(\{c,d\}, \neq)$.

Let now $(Z, \perp_Z)$ *be the horizontal sum of* $(\{a,b\}, \neq)$ *and* $(\{c,d\}, \neq)$. *Then we have* $Z = \{a,b,c,d\}$ *and* $\perp_Z = \{(a,b),(b,a),(c,d),(d,c)\}$. *The inclusion maps* $\iota_1\colon \{a,b\} \to X$ *and* $\iota_2\colon \{c,d\} \to X$ *are normal homomorphisms. If* $\psi\colon Z \to X$ *is such that* $\psi \circ \mathrm{inj}_1 = \iota_1$ *and* $\psi \circ \mathrm{inj}_2 = \iota_2$, *then* ψ *is the identity map. But* ψ *is not normal. Indeed,* $\{a,b\}$ *is a block of* Z *and we have* $\psi(\{a,b\})^{\perp_X\perp_X} = \{a,b\}$ *but* $\psi(Z)^{\perp_X\perp_X} = X$. *Hence also* $(Z, \perp_Z)$ *is not the categorical sum of* $(\{a,b\}, \neq)$ *and* $(\{c,d\}, \neq)$.

The direct product construction may be used to describe the decomposition of an orthogonality space $(X, \perp)$ into its irreducible components. We say that $(X, \perp)$ is *reducible* if there is a decomposition (A, B) of X such that $X = A \cup B$, otherwise X is called *irreducible*; cf. [Vet3].

Following the idea of Rump who has studied the connectedness of L-algebras [Rum1], we make the following definitions. For an orthogonality space $(X, \perp)$, let $\perp^c = (X \times X) \setminus \perp$. Then $\perp^c$ is a symmetric, reflexive relation. Endowed with $\perp^c$, X can be considered as a graph with loops. We say that two elements x and y of X are *dually connected* if x and y are connected in this graph, that is, if there is a finite sequence $x_0, \ldots, x_n$ in X such that $x = x_0 \perp^c x_1 \perp^c \cdots \perp^c x_n = y$. We write $x \,\mathrm{dc}\, y$ in this case. Note that dc is an equivalence relation on X and the dc-classes are pairwise orthogonal.

Lemma 4.10. *Let* $(X, \perp)$ *be an orthogonality space. Then the following conditions are equivalent.*

(1) $(X, \perp)$ *is irreducible.*

(2) *Any two elements of* X *are dually connected, that is,* $\mathrm{dc} = X \times X$.

Proof. (1) ⇒ (2): Assume that dc possesses more than one equivalence class. Then there are disjoint non-empty subsets $A, B \subseteq X$ such that $X = A \cup B$ and $e \perp f$ for any $e \in A$ and $f \in B$. Then $A = B^\perp$ and $B = A^\perp$, that is, (A, B) is a decomposition.

(2) ⇒ (1): Assume that there is a decomposition (A, B) such that $X = A \cup B$. Then $e \perp f$ for any $e \in A$ and $f \in B$. Hence the dc-class of any $e \in A$ is contained in A, whereas the dc-class of any $f \in B$ is contained in B. In particular, there are at least two dc-classes. □

For any orthogonality space $(X, \perp)$, X is the union of its dc-classes X_i, $i \in I$. Clearly, X_i is orthoclosed for each i, that is, $(X_i, \perp)$ is a subspace of X. Hence the X_i are exactly the maximal irreducible subspaces of X and we call them the *irreducible components* of $(X, \perp)$.

Proposition 4.11. *Every normal orthogonality space is the direct product of its irreducible components.*

Proof. Let $(X, \perp)$ be a normal orthogonality space and let $(X_i, \perp)$ be its irreducible components. That is, let X_i be the dc-classes endowed with the restriction of the orthogonality relation.

We have to show that $(X_i, \perp)$ is normal, then the assertion will be clear. We apply once more criterion (5) of Proposition 2.5. Let (E, F) be a block partition of X_i and let $e, f \in X_i$ be such that $e \perp F$ and $f \perp E$. Let $G \subseteq X$ be such that $(E \cup F) \dot{\cup} G$ is a block of X. Then $G \subseteq \bigcup_{j \neq i} X_j$ and hence $e \perp F \cup G$. By the normality of X, we conclude $e \perp f$. □

Corollary 4.12. *The irreducible components of a Dacey space are Dacey as well. Hence every Dacey space is the direct product of irreducible Dacey subspaces.*

Proof. A subspace of a Dacey space is again Dacey. Hence the assertion follows from Proposition 4.11. □

We conclude mentioning a further negative result. Let $(X_i, \perp_i)$, $i \in I$, be orthogonality spaces. On $\prod_{i \in I} X_i$, we define the orthogonality relation componentwise, that is, we let $(e_i)_{i \in I} \perp (f_i)_{i \in I}$ if $e_i \perp f_i$ for all $i \in I$. Then, obviously, $(\prod_{i \in I} X_i, \perp)$ is an orthogonality space.

However, as the following example shows, normality is in general not preserved under this construction.

Example 4.13. *Consider $(P(H_1), \perp)$ and $(P(H_2), \perp)$, where H_2 is a 4-dimensional Hilbert space, H_1 is a 2-dimensional subspace of H_2, and $\perp$ denotes the usual orthogonality relations. Then $(P(H_1) \times P(H_2), \perp)$ is not normal.*

Indeed, let $w, x \in H_2$ be an orthogonal basis of H_2, and let w, x, y, z be an orthogonal basis of H_4. Then $\{(\text{span}(w), \text{span}(w)), (\text{span}(x), \text{span}(x))\}$ is a block of $(P(H_1) \times P(H_2), \perp)$. Moreover, we have that

$$(\text{span}(x), \text{span}(y)) \perp (\text{span}(w), \text{span}(w)),$$
$$(\text{span}(w), \text{span}((y + z))) \perp (\text{span}(x), \text{span}(x))$$

but $(\text{span}(x), \text{span}(y)) \not\perp (\text{span}(w), \text{span}((y + z)))$. This is in contradiction to criterion (5) for normality in Proposition 2.5.

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