Orthogonality spaces of finite rank and the complex Hilbert spaces

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Abstract

An orthogonality space is a set endowed with a symmetric, irreflexive binary relation. By means of the usual orthogonality relation, each anisotropic quadratic space gives rise to such a structure. We investigate in this paper the question to which extent this strong abstraction suffices to characterise complex Hilbert spaces, which play a central role in quantum physics. To this end, we consider postulates concerning the nature and existence of symmetries. Together with a further postulate excluding the existence of non-trivial quotients, we establish a representation theorem for finite-dimensional orthomodular spaces over a dense subfield of $\mathbb{C}$.

Keywords: Characterisation of the complex Hilbert space; orthogonality space; orthomodular lattice; orthomodular space over an ordered $\star$-field

MSC: 81P10; 06C15; 46C05

1 Introduction

Quantum physics uses the complex Hilbert space as its basic model. The reason why this particular structure has proven suitable is not obvious and has in fact been the topic of much debate. The probably oldest approach towards a better understanding of the quantum physical formalism investigates the possibility of a structural reduction. The question is whether the Hilbert space can be recovered from a simpler structure. In the field that has become known under the name "quantum logic", several algebraic approaches have been proposed to this end and have led to a wide range of results; see, e.g., [EGL2, EGL3]. There are, for instance, ways to characterise the infinite-dimensional complex Hilbert space by means of its orthomodular lattice of closed subspaces [Wlb].

The present paper is a further contribution along these lines. However, in contrast to certain previous works including [Wlb], it is not our objective to find a characterisation of purely algebraic nature, that is, to restrict to what a first-order language associated with some class of algebras has to offer. Our guidelines are the following. First of all, we choose a structure that we think is a particularly modest one. If a quantum physical model is to be based on any principle at all, it is probably the principle of distinguishability: an observation of a certain kind may lead to one out of several mutually exclusive descriptions of a quantum system. The concept of orthogonality can be found, in one form or another, in all algebraic frameworks that have been proposed in the present context, most obviously in cases like orthomodular lattices, partial Boolean algebras, or orthoalgebras. The notion of an orthogonality space, which is due to D. J. Foulis and his collaborators, is solely based on this idea. An orthogonality space is a set endowed with a symmetric, irreflexive binary relation, the prototype consisting of the one-dimensional subspaces of a Hilbert space together with the usual orthogonality relation. We remark that test spaces, which have been proposed at a later time by D. J. Foulis and C. H. Randall, are structures following a closely related concept; see [Wlc].

In spite of its simplicity, the notion of an orthogonality space is amazingly powerful. The binary relation gives rise to a closure operator and the closed sets form a complete ortholattice. However, without a further extension of the concept, we can hardly derive any specific structure; we are certainly in a quite general context when compared to the prototype.

A further aspect that we consider as central is associated with the notion of symmetry. Indeed, any quantum physical model should be able to describe a change of perspective and it is natural to assume that this is done by means of symmetries of the model. We are thus led to the idea of requiring an orthogonality space to possess a suitable amount of automorphisms.

In the lattice-theoretic context, postulates involving automorphisms have been proposed by several authors. We may for instance mention the work of Holland [Hol3], Mayet [May], Aerts and Van Steirteghem [AeSt], Engesser, Gabbay, and Lehmann [EGL1], and Cassinelli and Lahti [CaLa1, CaLa2]. Mayet discusses in [May] the subspace lattices of the classical Hilbert spaces and he showed a way of singling out the field of complex numbers among the three classical division rings. In the present work, we will investigate the effect of such postulates in a more general context.

Let us give an idea of what we have in mind. Let X be an orthogonality space. We may postulate that there is an automorphism changing the relationship of certain elements of X with regard to other elements in a specific way. Most simply, we may require that, for any $e, f \in X$ there is an automorphism mapping e to f; cf. [Hol3, AeSt]. Similarly, we may require that, for any distinct $e, f \in X$ there is an automorphism mapping e to an element orthogonal to f. In both cases, we may add the condition that all elements not involved, that is, the elements orthogonal to e and f, remain fixed. A further concern may be the existence of "roots" of automorphisms: we may require for an automorphism φ and any $k \geqslant 2$ a further automorphism ψ to exist such that $\psi^k = \varphi$ and ψ leaves an element of X fixed if so does φ.

We will propose four postulates along these lines and show them to be sufficient to characterise the orthogonality spaces arising from an orthomodular space over an ordered $\star$-field K. If Archimedean, K is by a theorem of Holland [Hol1] embeddable into one of the classical fields, $\mathbb{R}$ or $\mathbb{C}$. We further show that otherwise the orthogonality space possesses a quotient that is representable by a space over an Archimedean $\star$-field. The formation of the quotient has the peculiar property that any automorphism is compatible with it, provided that it is induced by a unitary operator. By excluding such a situation, we are able to conclude that K is a subfield of $\mathbb{C}$.

As indicated in the title, the present work depends largely on a finiteness assumption. Indeed, for the issue of characterising the complex Hilbert space it makes a big difference whether we deal with the case of finite or infinite dimensions. In the present paper, we deal exclusively with the former. This is to say that our orthogonality spaces will be assumed to contain only finitely many mutually orthogonal elements and the representing spaces will accordingly be finite-dimensional. In the infinite case, additional tools might greatly simplify the axiomatics. In particular, Solèr's Theorem would be available and we might be lead to a procedure that differs not just in some details. We will turn to this case in a separate paper. Presently, we may refer to [Vet], where we have proposed a representation theorem for partial Boolean algebras.

We proceed as follows. We establish in Section 2 that an orthogonality space, assumed to be of finite rank and to possess automorphisms of a certain type, gives rise to an irreducible atomistic modular ortholattice. Before applying this result to the algebraic theory of linear spaces, we establish in Section 3 the necessary facts on ordered $\star$-sfields (skew fields with an involutorial antiautomorphism) and orthomodular spaces over them. The subsequent Section 4 is then devoted to the representation of the orthogonality space by an orthomodular space. Next we address the problem that we cannot yet delimit the $\star$-sfield in a reasonable way; the Archimedean property may fail. In Section 5 we show that in the non-Archimedean case the orthogonality space possesses a quotient with the desired properties. We conclude in Section 6 that a postulate on the non-existence of such quotients leads us to an orthomodular space over a subfield of $\mathbb{C}$. Finally, Section 7 contains some concluding remarks.

2 Orthogonality spaces

The objective of this paper is to characterise complex Hilbert spaces by means of the orthogonality relation between its one-dimensional subspaces. The following structure represents the corresponding abstraction [Dac].

Definition 2.1. An *orthogonality space* consists of a non-empty set X and a symmetric, irreflexive binary relation $\perp$, called the *orthogonality relation*. The supremum of the number of mutually orthogonal elements of X is called the *rank* of $(X, \perp)$.

Example 2.2. *Let $P(H)$ be the set of one-dimensional subspaces of a Hilbert space H and let $\perp$ be the usual orthogonality relation. Then $(P(H), \perp)$ is an orthogonality space and the dimension of H coincides with the rank of $(P(H), \perp)$.*

Given a complex Hilbert space H, our concern is to investigate the characteristic properties of $(P(H), \perp)$. Our focus is on the finite-dimensional case. We will in fact discuss exclusively orthogonality spaces of finite rank.

Let us consider a further example, which is of a completely different kind.

Example 2.3. *Let B be a Boolean algebra with at least two elements and let $B^\star$ consist of its non-zero elements. For $a, b \in B^\star$, put $a \perp b$ if $a \wedge b = 0$. Then $(B^\star, \perp)$ is an orthogonality space. If B is finite, then the rank of $(B^\star, \perp)$ is the number of atoms of B.*

A *homomorphism* between orthogonality spaces $(X, \perp)$ and $(Y, \perp)$ is a map $\varphi \colon X \to Y$ preserving the orthogonality relation. An injective homomorphism $\varphi \colon X \to Y$ is called *full* if, for any $x_1, x_2 \in X$, we have $x_1 \perp x_2$ if and only if $\varphi(x_1) \perp \varphi(x_2)$. Moreover, a surjective homomorphism $\varphi \colon X \to Y$ is called *faithful* if, for any $y_1, y_2 \in Y$ such that $y_1 \perp y_2$, there are $x_1, x_2 \in X$ such that $x_1 \perp x_2$, $\varphi(x_1) = y_1$, and $\varphi(x_2) = y_2$. An *automorphism* of an orthogonality space is a bijection φ such that both φ and φ^{-1} are homomorphisms. Clearly, an automorphism is full and faithful.

Studying orthogonality spaces is facilitated by the fact that we are directly led to lattice theory. We may associate with any orthogonality space a complete ortholattice, in a way that automorphisms of the former structure become automorphisms of the latter.

For the remainder of this section, let us fix an orthogonality space $(X, \perp)$. For $A \subseteq X$, we put $A^\perp = \{x \in X : x \perp a \text{ for all } a \in A\}$. Then the map $\mathcal{P}(X) \to \mathcal{P}(X), \;\; A \mapsto A^{\perp\perp}$ is a closure operator. The sets A such that $A^{\perp\perp} = A$ are called *orthoclosed* and the set of all orthoclosed subsets is denoted by $\mathcal{C}(X, \perp)$.

For the lattice-theoretic facts of which we make use in this paper, we may refer, e.g., to [MaMa].

Lemma 2.4. *$\mathcal{C}(X, \perp)$, partially ordered by set-theoretical inclusion and endowed with the unary operation $^\perp$, is a complete ortholattice. The bottom element is $\varnothing$, the top element is X.*

Furthermore, let $\varphi \colon X \to X$ be an automorphism of $(X, \perp)$. Then the mapping $\bar{\varphi} \colon \mathcal{C}(X, \perp) \to \mathcal{C}(X, \perp), \;\; A \mapsto \{\varphi(x) : x \in A\}$ is an automorphism of the ortholattice $\mathcal{C}(X, \perp)$.

We note that it is in general not possible to reconstruct the orthogonality space $(X, \perp)$ from the ortholattice $\mathcal{C}(X, \perp)$. When $(X, \perp)$ is based on a Boolean algebra as in Example 2.3, then $\mathcal{C}(X, \perp)$ is its Dedekind-MacNeille completion and thus in a sense a finer structure, from which we cannot in general deduce the original one. A converse situation arises when X contains distinct elements x and y such that $\{x\}^\perp = \{y\}^\perp$. The distinguishability of x and y being lost, we might then view $\mathcal{C}(X, \perp)$ as a coarser structure, from which we cannot recover the original one either. The latter case implies that there is in general also no one-to-one correspondence between the automorphisms of $(X, \perp)$ and of $\mathcal{C}(X, \perp)$. We will get back to these issues below.

In order to characterise the canonical orthogonality space, our strategy consists of finding natural conditions regarding the existence of automorphisms of $(X, \perp)$. In particular, we will require that, for any two elements e and f, there is an automorphism mapping e to an element e' that is compatible with f. Here, compatibility of e' with f means that e' either coincides with f or is orthogonal to f. A guiding principle will furthermore be that the automorphisms are required to act "locally", that is, to leave any element orthogonal to the involved elements unchanged.

We say that an automorphism φ of $(X, \perp)$ *fixes* an $x \in X$ if $\varphi(x) = x$. If φ fixes every $x \in A$ for some $A \subseteq X$, we say that φ *is the identity on* A.

We define the following properties of $(X, \perp)$.

(F1) Let $e, f \in X$ be distinct. Then there is an automorphism $\varphi\colon X \to X$ such that

- (i) $\varphi(e) \perp f$,
- (ii) φ fixes any $x \in X$ such that $x \perp e, f$ or $x \perp \varphi(e), f$.

(F2) Let $e, f \in X$ be distinct. Then there is an automorphism $\varphi\colon X \to X$ such that

- (i) $\varphi(e) = f$,
- (ii) φ fixes any $x \in X$ such that $x \perp e, f$.

We will see that condition (F1) has amazingly strong consequences. To begin with, Example 2.3 shows that the lattice $\mathcal{C}(X, \perp)$ is in general not atomistic. It follows from (F1) that $\mathcal{C}(X, \perp)$ does have this property.

Lemma 2.5. *Let* $(X, \perp)$ *fulfil* (F1). *Then* $\mathcal{C}(X, \perp)$ *is atomistic, the atoms being* $\{e\}^{\perp\perp}$, $e \in X$.

Proof. Let $e \in X$. We will show that $\{e\}^{\perp\perp}$ is an atom of $\mathcal{C}(X, \perp)$; the assertion will then follow.

Assume that $\{e\}^{\perp\perp}$ is not an atom. Then there is an $f \in X$ such that $\{f\}^{\perp\perp} \subsetneqq \{e\}^{\perp\perp}$. In particular $e \neq f$ and hence, by (F1), there is an automorphism φ of X such that $\varphi(f) \perp e$ and φ fixes any $x \in X$ such that $x \perp e, f$. Then $\varphi(x) = x$ for any $x \in \{e\}^\perp$, because $x \perp e$ implies $x \perp f$. Hence $f \perp \{e\}^\perp$ implies $\varphi(f) \perp \{e\}^\perp$. But we also have $\varphi(f) \in \{e\}^\perp$, a contradiction. $\square$

An orthogonality space $(X, \perp)$ is called *irredundant* if, for any $x, y \in X$, $\{x\}^\perp = \{y\}^\perp$ implies $x = y$. Condition (F1) happens to imply this property.

Lemma 2.6. *Let* $(X, \perp)$ *fulfil* (F1). *Then* $(X, \perp)$ *is irredundant.*

Proof. Let $x, y \in X$ be such that $\{x\}^\perp = \{y\}^\perp$ and assume that x and y are distinct. By (F1), there is an automorphism φ such that $\varphi(x) \perp y$ and $\varphi(z) = z$ for any $z \perp x, y$. Then $\varphi(z) = z$ for any $z \perp y$, in particular $\varphi(\varphi(x)) = \varphi(x)$ and hence $\varphi(x) = x$. But this means $x \perp y$, a contradiction. $\square$

Proposition 2.7. *Let* $(X, \perp)$ *fulfil* (F1). *Then the set of atoms of* $\mathcal{C}(X, \perp)$ *is the set of singletons* $\{\{x\}\colon x \in X\}$ *and, endowed with the orthogonality relation inherited from* $\mathcal{C}(X, \perp)$*, is isomorphic with* $(X, \perp)$.

Proof. We have to show that the singletons are orthoclosed, that is, $\{x\}^{\perp\perp} = \{x\}$ for any $x \in X$. Let $y \in \{x\}^{\perp\perp}$. Then $\{y\}^{\perp\perp} \subseteq \{x\}^{\perp\perp}$, and from Lemma 2.5 it follows that $\{y\}^{\perp\perp} = \{x\}^{\perp\perp}$. Hence $\{y\}^{\perp} = \{x\}^{\perp}$. Since X is by Lemma 2.6 irredundant, we conclude $y = x$.

By Lemma 2.5, $\{\{x\} \colon x \in X\}$ is the set of atoms of $\mathcal{C}(X, \perp)$. The lemma follows. □

Proposition 2.8. *Let* $(X, \perp)$ *fulfil* (F1). *Then the automorphisms of* $(X, \perp)$ *and those of the ortholattice* $\mathcal{C}(X, \perp)$ *are in a one-to-one correspondence.*

Proof. Recall from Lemma 2.4 that any automorphism φ of X induces an automorphism $\bar{\varphi}$ of $\mathcal{C}(X, \perp)$.

Conversely, let $\bar{\varphi}$ be an automorphism of $\mathcal{C}(X, \perp)$. In view of Proposition 2.7, we may define $\varphi \colon X \to X$ to be the map such that, for any $x \in X$, $\varphi(x)$ is the single element of $\bar{\varphi}(\{x\})$. Clearly, φ is a bijection. Let $x, y \in X$. Then $x \perp y$ in X iff $\{x\} \perp \{y\}$ in $\mathcal{C}(X, \perp)$ iff $\bar{\varphi}(\{x\}) \perp \bar{\varphi}(\{y\})$ in $\mathcal{C}(X, \perp)$ iff $\varphi(x) \perp \varphi(y)$ in X. We conclude that φ is an automorphism of $(X, \perp)$.

Clearly, the assignment $\varphi \mapsto \bar{\varphi}$ defines a one-to-one correspondence. □

Our next aim is to show the modularity of $\mathcal{C}(X, \perp)$, provided that X is of finite rank. To this end, we cite a lemma of J. R. Dacey containing a criterion for $\mathcal{C}(X, \perp)$ to be orthomodular; see [Dac] or [Wlc, Theorem 35]. For an overview of orthomodular lattices, see, e.g., [BrHa].

In what follows, a subset D of X is called *orthogonal* if so are any two distinct elements of D.

Lemma 2.9. *Let* $(X, \perp)$ *be an orthogonality space. Then* $\mathcal{C}(X, \perp)$ *is orthomodular if and only if, for any* $A \in \mathcal{C}(X, \perp)$ *and any maximal orthogonal subset D of A, we have* $A = D^{\perp\perp}$.

Lemma 2.10. *Let* $(X, \perp)$ *have finite rank and fulfil* (F1). *Let* $D \subseteq X$ *be orthogonal and let* $e \in X$ *be such that* $e \notin D^{\perp\perp}$. *Then there is an* $f \perp D$ *such that* $(D \cup \{e\})^{\perp\perp} = (D \cup \{f\})^{\perp\perp}$.

Proof. Since X has finite rank, D is finite; let $D = \{d_1, \ldots, d_k\}$.

If $e \perp D$, we let $f = e$ and we are done. Assume $e \not\perp D$. Then there is a smallest $j \in \{1, \ldots, k\}$ such that $e \perp d_1, \ldots, d_{j-1}$ but $e \not\perp d_j$. By (F1), there is an automorphism φ such that $e' = \varphi(e) \perp d_j$ and $\varphi(x) = x$ if $x \perp d_j, e$ or $x \perp d_j, e'$. Since $d_1, \ldots, d_{j-1} \perp d_j, e$, we have that $d_1, \ldots, d_{j-1}$ are fixed by φ and we conclude $e' \perp d_1, \ldots, d_j$. We furthermore have that $x \perp D, e$ if and only if $x \perp D, e'$. This means $(D \cup \{e\})^{\perp\perp} = (D \cup \{e'\})^{\perp\perp}$.

Thus we have determined an $e' \notin D^{\perp\perp}$ such that $e' \perp d_1, \ldots, d_j$ and $(D \cup \{e\})^{\perp\perp} = (D \cup \{e'\})^{\perp\perp}$. Repeating the argument, we eventually get an element f as desired. □

We will abbreviate "modular ortholattice" by "MOL".

Lemma 2.11. *Let* $(X, \perp)$ *be of finite rank* n *and fulfil* (F1). *Then* $\mathcal{C}(X, \perp)$ *is an atomistic MOL of length* n.

Proof. By Lemma 2.5, the ortholattice $\mathcal{C}(X, \perp)$ is atomistic.

Furthermore, from Lemma 2.10 and Dacey's criterion (Lemma 2.9), it follows that $\mathcal{C}(X, \perp)$ is orthomodular. Since the maximal number of orthogonal atoms is n, it also follows that $\mathcal{C}(X, \perp)$ has length n.

We next show that $\mathcal{C}(X, \perp)$ fulfils the covering property. As an atomistic ortholattice of finite length with this property is modular [MaMa, Lemma 30.3], the assertion will then follow.

Let $A \in \mathcal{C}(X, \perp)$ and let $e \in X$ be such that $e \notin A$. According to Dacey's criterion, let D be an orthogonal set such that $A = D^{\perp\perp}$. By Lemma 2.10, there is an $f \perp D$ such that $A \vee \{e\} = (D \cup \{e\})^{\perp\perp} = (D \cup \{f\})^{\perp\perp} = A \vee \{f\}$. Note that $\{f\}$ is an atom orthogonal to A. Hence it follows by the orthomodularity of $\mathcal{C}(X, \perp)$ that $A \vee \{e\}$ covers A. □

We continue by introducing our third condition on $(X, \perp)$. This time the idea is that certain automorphisms should possess "roots" of all orders.

For an automorphism φ of X and $k \geqslant 1$, we write φ^k for $\varphi \circ \ldots \circ \varphi$ (k times). Let us consider the following property of $(X, \perp)$.

(F3) Let $e, f \in X$ be distinct and let $\varphi\colon X \to X$ be an automorphism such that φ fixes any $x \perp e, f$. Then, for any $k \geqslant 1$, there is an automorphism $\psi\colon X \to X$ such that

  (i) $\psi^k = \varphi$,

  (ii) ψ fixes any $x \in X$ fixed by φ.

An immediate consequence of (F3), together with (F2), is as follows. We will say that the orthogonality space $(X, \perp)$ is *reducible* if X is the union of two non-empty disjoint subsets A and B such that $x \perp y$ for any $x \in A$ and $y \in B$. Otherwise, we will call $(X, \perp)$ *irreducible*.

Proposition 2.12. *Let* $(X, \perp)$ *fulfil* (F2) *and* (F3). *Then* $(X, \perp)$ *is irreducible.*

Proof. We shall show that for any orthogonal $e, f \in X$ there is a $g \in X$ such that $g \not\perp e, f$. The assertion will obviously then follow.

Let $e \perp f$. By (F2) and (F3), there is an automorphism φ such that $\varphi^2(e) = f$ and $\varphi(x) = x$ whenever $x \perp e, f$. Let $g = \varphi(e)$ and assume that $g \perp e$. Then $g = \varphi(e) \perp \varphi(g) = f$. Hence φ fixes g, that is, $e = \varphi^{-1}(g) = g = \varphi(g) = f$, a contradiction. Hence we have $g \not\perp e$. Similarly, assume $g \perp f$. Then $g = \varphi(e) = \varphi^{-1}(f) \perp \varphi^{-1}(g) = e$, again a contradiction. We conclude that we also have $g \not\perp f$. □

We call a bounded lattice L *reducible* if L is isomorphic to the direct product of bounded lattices with at least two elements, and otherwise *irreducible*. Note that it makes sense to apply these properties also to ortholattices. Indeed, let L be an ortholattice and assume that L, viewed as a bounded lattice, is reducible. Then any decomposition $L \cong L_1 \times L_2$, where L_1 and L_2 are bounded lattices, is a direct decomposition of L as an ortholattice as well. Note that in such a situation we have $(a, 0) \perp (0, b)$ for any $a \in L_1$ and $b \in L_2$.

We arrive at the main result of this section.

Theorem 2.13. *Let $(X, \perp)$ be of finite rank n and fulfil* (F1)–(F3). *Then $\mathcal{C}(X, \perp)$ is an irreducible atomistic MOL of length n.*

Proof. In view of Lemma 2.11, only the irreducibility remains to be shown.

Assume that $\mathcal{C}(X, \perp)$ is reducible. Then the collection of atoms of $\mathcal{C}(X, \perp)$ can be partitioned into two non-empty subsets such that any atom contained in the first set is orthogonal to any atom contained in the second set. Hence also X can be partitioned in this way. But by Proposition 2.12, $(X, \perp)$ is irreducible, a contradiction. ☐

3 Orthomodular spaces over ordered $\star$-fields

Atomistic modular ortholattices are representable by means of inner-product spaces, provided that their length is at least 4. In this section we compile some basic facts on such spaces. Further information can be found, e.g., in [Bae, Piz, Hol2].

We start by considering the scalars. Recall that a *$\star$-ring* is a (unital) ring K equipped with an involutorial antiautomorphism *. If K is a skew field (i.e., a division ring) or a field, we will refer to K as a *$\star$-sfield* or a *$\star$-field*, respectively.

For a $\star$-sfield K, we define $S_K = \{\alpha \in K : \alpha^\star = \alpha\}$ to be the set of *symmetric* elements. Note that S_K is an additive subgroup of K, but S_K is a sub-sfield only if K is commutative.

Following Baer [Bae], an order on K actually means an order on S_K.

Definition 3.1. An *ordered $\star$-sfield* is a $\star$-sfield K together with a subset $S_K^+ \subseteq S_K$, called the *domain of positivity*, such that

(1) $S_K^+ + S_K^+ \subseteq S_K^+$,

(2) $1 \in S_K^+$,

(3) if $\alpha \in S_K^+$, then $\beta\alpha\beta^\star \in S_K^+$ for any $\beta \in K$,

(4) $S_K^+ \cap -S_K^+ = \{0\}$,

(5) $S_K^+ \cup -S_K^+ = S_K$.

For instance, the classical $\star$-sfields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are canonically endowed with the set of positive reals as their domain of positivity.

By properties (1) and (4) of Definition 3.1, we may partially order the set of symmetric elements of an ordered $\star$-field K by defining

$$\alpha \leqslant \beta \quad \text{if} \quad \beta - \alpha \in S_K^+,$$

where $\alpha, \beta \in S_K$. In the following lemma, we list some elementary properties of S_K, which are (in case that K is a field) due to A. Prestel [Pre1]. Here, $\mathbb{Q}$ is seen as a subset of S_K and, w.r.t. to the natural order of the rationals, we put $\mathbb{Q}^+ = \{\varrho \in \mathbb{Q} \colon \varrho \geqslant 0\}$.

Lemma 3.2. *Let K be an ordered $\star$-field. Then $(S_K; \leqslant, +, 0)$, the additive group of the symmetric elements endowed with $\leqslant$, is a totally ordered abelian group. We moreover have:*

(i) *For any $\alpha, \beta \in S_K$ and $\gamma \in K \setminus \{0\}$, $\alpha < \beta$ if and only if $\gamma\alpha\gamma^\star < \gamma\beta\gamma^\star$.*

(ii) *For any $\alpha, \beta \in S_K$ and $\varrho \in \mathbb{Q}^+ \setminus \{0\}$, $\alpha < \beta$ if and only if $\varrho\alpha < \varrho\beta$.*

(iii) *$\mathbb{Q}$ is ordered in the natural way.*

(iv) *For $\alpha, \beta \in S_K$, $0 < \alpha < \beta$ implies $\alpha\beta^{-1}\alpha < \alpha$ and $\beta < \beta\alpha^{-1}\beta$.*

(v) *For $\alpha, \beta \in S_K$, $0 < \alpha < \beta$ if and only if $0 < \beta^{-1} < \alpha^{-1}$.*

(vi) *Let $\alpha \in S_K^+$, $\varrho \in \mathbb{Q}^+$, and $k \geqslant 1$. Then $\alpha < \varrho$ iff $\alpha^k < \varrho^k$, and $\varrho < \alpha$ iff $\varrho^k < \alpha^k$.*

Proof (sketched). $\leqslant$ makes S_K into a totally ordered group by (1), (4), and (5). Furthermore, (3) implies (i).

Let $\alpha, \beta \in S_K$ such that $\alpha < \beta$ and let $n \in \mathbb{N} \setminus \{0\}$. Then $n\alpha < n\beta$ by (1). Furthermore, $\frac{1}{n}\beta \leqslant \frac{1}{n}\alpha$ would imply $\beta \leqslant \alpha$, hence $\frac{1}{n}\alpha < \frac{1}{n}\beta$. Both facts together show (ii). Setting $\alpha = 0$ and $\beta = 1$ in (ii) and using (2), we also see (iii).

We note next that, for any $\alpha > 0$, we have $\alpha^{-1} \in S_K$ and hence by (i) $\alpha^{-1} = \alpha^{-1}\alpha\alpha^{-1} > 0$.

Assume now $0 < \alpha < \beta$. Then $0 < ((\beta-\alpha)^{-1} + \alpha^{-1})^{-1} = (\alpha(\beta-\alpha)^{-1} + 1)^{-1}\alpha = (\beta-\alpha)(\alpha+\beta-\alpha)^{-1}\alpha = \alpha - \alpha\beta^{-1}\alpha$ and hence $\alpha\beta^{-1}\alpha < \alpha$. By (i), it further follows $\beta^{-1} < \alpha^{-1}\alpha\alpha^{-1} = \alpha^{-1}$. Once again by (i), we conclude $\beta = \beta\beta^{-1}\beta < \beta\alpha^{-1}\beta$. (iv) and (v) are shown.

To see (vi), let $\alpha \in S_K^+$ and $\varrho \in \mathbb{Q}^+$. Assume $\alpha < \varrho$. We intend to show that $\alpha^k < \varrho^k$ holds for all $k \geqslant 1$. If $\alpha = 0$, this is clear by (iii). Let $\alpha \neq 0$. Then $\varrho^{-1}\alpha^2 < \alpha < \varrho$ by (iv) and hence $\alpha^2 < \varrho^2$ by (ii). By (i) and (ii), it follows $\alpha^4 < \varrho^2\alpha^2 < \varrho^4$ and similarly $\alpha^k < \varrho^k$ for all even k. Furthermore, $\alpha < \varrho$ and $\alpha^2 < \varrho^2$ imply $\alpha^3 < \varrho\alpha^2 < \varrho^3$ by (i) and (ii), thus we get $\alpha^k < \varrho^k$ also for all uneven k.

Assume now $\varrho < \alpha$. We shall show that $\varrho^k < \alpha^k$ for all $k \geqslant 1$. Consider first the case $\varrho = 0$. We have $0 < \alpha^2$ by (2) and (3), thus $0 < \alpha^k$ follows for any k by (i). Let now $\varrho \neq 0$. Then $\alpha < \varrho^{-1}\alpha^2$ by (iv), hence $\varrho^2 < \varrho\alpha < \alpha^2$ by (ii). We argue similarly as in the previous paragraph to see that $\varrho^k < \alpha^k$ holds for all k. Part (vi) follows. □

We follow now the lines of Holland' work [Hol2]. For an ordered $\star$-field K, we define the sets

$$\begin{aligned} I_K &= \{\alpha \in K \colon \alpha\alpha^\star < \tfrac{1}{n} \text{ for all } n \in \mathbb{N} \setminus \{0\}\}, \\ F_K &= \{\alpha \in K \colon \alpha\alpha^\star \leqslant n \text{ for some } n \in \mathbb{N}\} \end{aligned}$$

of *infinitesimal* and *finite* elements of K, respectively. We note that, by Lemma 3.2(vi), an element $\alpha \in S_K^+$ is infinitesimal if and only if $\alpha < \frac{1}{n}$ for all n, and $\alpha \in S_K^+$ is finite if and only if $\alpha \leqslant n$ for some n. We moreover have that $I_K + I_K = I_K$, $F_K + F_K = F_K$, $I_K \cdot F_K = F_K \cdot I_K = I_K$, and $F_K \cdot F_K = F_K$, and both I_K and F_K are closed under *.

We define in addition

$$M_K \;=\; F_K \setminus I_K$$

to be the set of *medial* elements. From Lemma 3.2(v) we see that M_K is a subgroup of the multiplicative group $K \setminus \{0\}$.

We call K *Archimedean* if $I_K = \{0\}$. By Lemma 3.2(v), K is Archimedean if and only if $F_K = K$. This property has strong consequences [Hol1, Theorem 2].

Theorem 3.3. *An Archimedean ordered $\star$-sfield is order-isomorphic to an ordered sub-$\star$-sfield of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

In the general case, we may form the quotient of the finite elements modulo the infinitesimals and we are led to an Archimedean ordered $\star$-sfield [Hol2, Theorem 4.3]

Theorem 3.4. *Let K be an ordered $\star$-field. Then F_K is a sub-$\star$-ring of K and I_K is the unique maximal left (right) ideal of F_K. Hence $\hat{K} = F_K / I_K$ is a skew field. Moreover, F_K and I_K are closed under * and by defining*

$$(\alpha / I_K)^\star = \alpha^\star / I_K, \quad \alpha \in K,$$

we make $\hat{K}$ into a $\star$-sfield. The set of symmetric elements of $\hat{K}$ is $S_{\hat{K}} = (S_K \cap F_K)/I_K$.

Finally, letting $S_{\hat{K}}^+ = (S_K^+ \cap F_K)/I_K$ the domain of positivity, we make $\hat{K}$ into an ordered $\star$-sfield, which is Archimedean.

This finishes our discussion of ordered $\star$-sfields and we now turn to linear spaces equipped with an inner product.

Definition 3.5. Let H be a linear space over a $\star$-sfield K. A map $(\cdot,\cdot) : H \times H \to K$ is called a *hermitian form* if, for any $x, y, z \in H$ and $\alpha, \beta \in K$, we have:

$$\begin{aligned} (\alpha x + \beta y, z) &= \alpha\,(x,z) + \beta\,(y,z)\,, \\ (z, \alpha x + \beta y) &= (z,x)\,\alpha^\star + (z,y)\,\beta^\star, \\ (x,y) &= (y,x)^\star\,. \end{aligned}$$

Moreover, the form is called *anisotropic* if $(x,x) = 0$ holds only in case $x = 0$.

Let H be a linear space equipped with an anisotropic hermitian form. For $x, y \in H$, we write $x \perp y$ if $(x, y) = 0$, and for a subspace E of H, we define $E^\perp = \{x \in H : x \perp y \text{ for all } y \in E\}$. Note that we have $E \cap E^\perp = \{0\}$. In case that also $H = E + E^\perp$ holds, E is called *splitting*. Moreover, E is *closed* if $E = E^{\perp\perp}$ and we denote by $\mathcal{C}(H)$ the set of closed subspaces of H. Any splitting subspace is closed but the converse does not in general hold. We partially order $\mathcal{C}(H)$ by set-theoretic inclusion and we equip $\mathcal{C}(H)$ with the orthocomplementation $^\perp$. Then $\mathcal{C}(H)$ is an ortholattice, which is orthomodular if and only if any closed subspace is splitting.

Definition 3.6. A linear space H over a $\star$-sfield together with an anisotropic hermitian form is called an *orthomodular space* if all closed subspaces are splitting.

Here, we are primarily interested in the case of finite dimensions. If H is a finite-dimensional space equipped with an anisotropic hermitian form, then all subspaces are splitting and hence H is an orthomodular space. Moreover, $\mathcal{C}(H)$ consists simply of all subspaces of H and is a modular ortholattice.

We write $[x_1, \ldots, x_k]$ for the subspace spanned by $x_1, \ldots, x_k \in H$. For a subspace E of H we put $E^\bullet = E \setminus \{0\}$. We let $P(H) = \{[x] : x \in H^\bullet\}$ and we define $[x] \perp [y]$ if $x \perp y$, where $x, y \in H^\bullet$. Then $(P(H); \perp)$ is an orthogonality space, giving rise to the ortholattice $\mathcal{C}(P(H), \perp)$.

Obviously, the orthoclosed subsets of $P(H)$ correspond one-to-one to the closed subspaces of H, that is, the ortholattices $\mathcal{C}(P(H), \perp)$ and $\mathcal{C}(H)$ can be identified. In the sequel, it will usually be more convenient to refer to the latter.

Definition 3.7. Let H be an orthomodular space over the $\star$-sfield K. A bijective map $U : H \to H$ is called *semiunitary* if there is an automorphism $^\sigma$ of K (as a skew field) and an element $\lambda \in K$ such that, for any $x, y \in H$ and $\alpha \in K$, we have:

$$\begin{aligned} U(x + y) &= U(x) + U(y); \\ U(\alpha x) &= \alpha^\sigma\, U(x); \\ (U(x), U(y)) &= (x, y)^\sigma\; \lambda. \end{aligned}$$

Moreover, U is called *unitary* if $^\sigma$ is the identity and $\lambda = 1$.

The following version of Wigner's Theorem is due to Piron [Pir, Theorem 3.28]; see also [May, Lemma 1].

Theorem 3.8. *Let H be an orthomodular space of dimension $\geqslant 3$ and let φ be an automorphism of the ortholattice $\mathcal{C}(H)$. Then there is a semiunitary map U inducing φ, that is, such that*

$$\varphi(E) = \{U(x) : x \in E\}, \quad E \in \mathcal{C}(H).$$

Moreover, if there is an at least two-dimensional subspace F such that $\varphi|_{[0,F]}$ is the identity, then U can uniquely be chosen as a unitary operator such that $U|_F$ is the identity.

We have seen that we may form the quotient $\hat{K}$ of the subring F_K of finite elements of an ordered $\star$-sfield K modulo the infinitesimal elements. Holland has established that we may proceed analogously for linear spaces equipped with a hermitian form [Hol2].

For a finite-dimensional orthomodular space H over the ordered $\star$-sfield K, we define the sets

$$\begin{aligned} I_H &= \{x \in H\colon\ (x,x) \in I_K\}, \\ F_H &= \{x \in H\colon\ (x,x) \in F_K\} \end{aligned}$$

of *infinitesimal* and *finite* vectors in H, respectively. We also define

$$M_H \;=\; F_H \setminus I_H \;=\; \{x \in H\colon\ (x,x) \in M_K\}$$

to be the set of *medial* vectors, which that are finite but not infinitesimal.

We recall that the form of H is called *positive definite* if $(x,x) > 0$ for any $x \neq 0$. Note that the existence of a unit vector in each one-dimensional subspace implies positive definiteness. A vector x is in this case infinitesimal if and only if $(x,x) < \frac{1}{n}$ for all n, and x is finite if and only if $(x,x) < n$ for some n.

The finite vectors of H modulo the infinitesimal ones give rise to a linear space over the Archimedean $\star$-sfield $\hat{K}$ [Hol2, Theorem 5.4].

Theorem 3.9. *Let H be a finite-dimensional orthomodular space over the ordered $\star$-sfield K and assume that each one-dimensional subspace contains a unit vector. Then F_H is a F_K-module and I_H is a F_K-submodule; moreover, $\alpha x \in I_H$ if $\alpha \in I_K$ and $x \in F_H$. Let $\hat{H} = F_H/I_H$ be the quotient F_K-module and let $\hat{K} = F_K/I_K$ be the $\star$-sfield as specified in Theorem 3.4. Defining*

$$\alpha/I_K \cdot x/I_H = (\alpha x)/I_H, \quad \alpha \in F_K,\ x \in F_H,$$

we make $\hat{H}$ into a linear space over $\hat{K}$.

Furthermore, $(x,y) \in F_K$ for any $x, y \in F_H$ and $(x,y) \in I_K$ for any $x, y \in F_H$ such that $x \in I_H$ or $y \in I_H$. Defining

$$(x/I_H, y/I_H) = (x,y)\,/I_K, \quad x, y \in F_H,$$

we make $\hat{H}$ into an orthomodular space such that each one-dimensional subspace contains a unit vector. Moreover, the dimensions of H and $\hat{H}$ coincide.

4 Orthomodular spaces arising from orthogonality spaces

We discuss in this section the representation of orthogonality spaces by orthomodular spaces.

Orthomodular space can be characterised lattice-theoretically as follows [MaMa, Theorems 34.2 and 34.5]:

Theorem 4.1. *Let H be a finite-dimensional orthomodular space. Then $\mathcal{C}(H)$ is an irreducible atomistic MOL of finite length.*

Conversely, let L be an irreducible atomistic MOL of finite length $\geqslant 4$. Then there is an orthomodular space H such that L is isomorphic to $\mathcal{C}(H)$.

Our first representation theorem is now immediate.

Theorem 4.2. *Let $(X, \perp)$ be an orthogonality space of finite rank $\geqslant 4$ and fulfilling* (F1)–(F3). *Then $(X, \perp)$ is isomorphic to $(P(H), \perp)$, where H is an orthomodular space such that each subspace contains a unit vector.*

Proof. By Theorems 2.13 and 4.1, there is an orthomodular space H over a $\star$-sfield K such that $\mathcal{C}(X, \perp)$ is isomorphic to $\mathcal{C}(H)$. It follows that $(X, \perp)$ is isomorphic to $(P(H), \perp)$.

We proceed as, e.g., in [Hol3] to show that we may modify the inner product of H with the effect that there exists a unit vector. Let $u \in H^{\bullet}$. We define a new inner product by $(x, y)' = (x, y)\,(u, u)^{-1}$ and a new antiautomorphism by $\alpha^{\#} = (u, u)\,\alpha^{\star}\,(u, u)^{-1}$. In this way we get an orthomodular space H', whose orthogonality relation is the same as in H. Hence $(X, \perp)$ is isomorphic to $(P(H'), \perp)$. Moreover, u is in H' a unit vector.

Let $z \in H'$ be linearly independent from u. By (F2), there is an automorphism φ of $(P(H'), \perp)$ such that $\varphi([u]) = [z]$ and $\varphi([x]) = [x]$ for any $x \perp u, z$. By Lemma 2.4, φ extends to an automorphism of $\mathcal{C}(P(H'), \perp)$, that is, of $\mathcal{C}(H')$. By Theorem 3.8, φ is induced by a unitary map U. This means that $U(u)$ is a unit vector in $[z]$. We conclude that there is a unit vector in every one-dimensional subspace of H'. □

We see that, having the usual lattice-theoretic machinery available, we may quite easily associate with an orthogonality space a suitable inner-product space over some $\star$-sfield K. It is, however, less easy to be more specific about K. We will in fact need to add further assumptions.

We introduce the following condition on an orthogonality space $(X, \perp)$, which is inspired by Baer's "Second Fundamental Theorem of Projective Geometry" [Bae, Sec. III.3].

(F4) Let $e, f \in X$ be such that $e \not\perp f$. Then any automorphism that fixes e, f, and any $x \perp e, f$ is the identity on $\{e, f\}^{\perp\perp}$.

For the remainder of this section, let us fix an orthogonality space $(X, \perp)$ that is of finite rank $\geqslant 4$ and fulfils (F1)–(F4). Moreover, let H be the orthomodular space over the $\star$-sfield K that represents $(X, \perp)$ according to Theorem 4.2.

We start with a simple observation; cf. [Jon].

Lemma 4.3. *K has characteristic* 0.

Proof. By assumption, there is an $x_1 \in H$ such that $(x_1, x_1) = 1$. Clearly, $x_1 \neq 0$. Let now $n \geqslant 1$ and assume that there is an $x_n \in H^{\bullet}$ such that $(x_n, x_n) = n$. Let

$y \in H$ be such that $y \perp x_n$ and $(y, y) = 1$. Putting $x_{n+1} = x_n + y$, we have $x_{n+1} \neq 0$ and $(x_{n+1}, x_{n+1}) = n + 1$. By induction, we conclude that for any $n \geqslant 1$, there is an $x_n \in H^\bullet$ such that $(x_n, x_n) = n$. By anisotropy, the assertion follows. $\square$

Let $T_K = \{\varepsilon \in K : \varepsilon\varepsilon^\star = 1\}$ be the multiplicative group of unit elements of K.

Lemma 4.4. *For any $\varepsilon \in T_K$ and $k \geqslant 1$, there is an $\zeta \in T_K$ such that $\zeta^k = \varepsilon$.*

Proof. Let $\varepsilon \in T_K$ and $k \geqslant 1$. H possesses an orthogonal basis, and as we assume every subspace to contain a unit vector, H actually possesses an orthonormal basis $\{e_1, \ldots, e_n\}$. Let $U : H \to H$ be the linear map such that

$$U(e_1) = e_1, \ \ldots, \ U(e_{n-1}) = e_{n-1}, \text{ and } U(e_n) = \varepsilon\, e_n.$$

Then U is a unitary map, inducing an automorphism φ of $(P(H), \perp)$. By (F3), there is an automorphism ψ of $(P(H), \perp)$, and hence of $\mathcal{C}(H)$, such that $\psi^k = \varphi$ and $\psi([x]) = [x]$ whenever $\varphi([x]) = [x]$, where $x \in H^\bullet$. Then $\psi|_{[0,[e_1,\ldots,e_{n-1}]]}$ is the identity and hence, by Theorem 3.8, ψ is induced by a unitary map V such that $V|_{[e_1,\ldots,e_{n-1}]}$ is the identity. We have $V(e_1) = e_1, \ldots, V(e_{n-1}) = e_{n-1}$ as well as $V([e_n]) = [e_n]$, that is, $V(e_n) = \zeta\, e_n$ for some $\zeta \in T_K$. The maps V^k and U induce the same automorphism of $\mathcal{C}(H)$ and the uniqueness statement in Theorem 3.8 implies $V^k = U$. We conclude $\zeta^k = \varepsilon$. $\square$

Our next aim is to show that K is commutative. Let $Z(K)$ be the centre of K. Note that $Z(K)$ is a sub-$\star$-sfield of K, which is in fact a $\star$-field.

Lemma 4.5. *K is a $\star$-field. Moreover, $K = S_K(i)$, where $i \in K$ is such that $i^2 = -1$ and $i^\star = -i$.*

Proof. Using the argument of [Bae, Sec. III.3], we first show that $T_K \subseteq Z(K)$. Let $\varepsilon \in T_K$, and let $e_1, \ldots, e_n$ be again an orthonormal basis. Requiring $U(e_1) = \varepsilon\, e_1$, $U(e_2) = \varepsilon\, e_2$, and $U(e_j) = e_j$, $j = 3, \ldots, n$, we can define a unitary map U of H. Let φ be the automorphism of $(P(H), \perp)$ induced by U. Note that $U(e_1 + e_2) = \varepsilon\,(e_1 + e_2)$. Hence φ fixes $[e_1]$ and $[e_1 + e_2]$ and moreover any $[x]$ such that $x \perp e_1, e_1 + e_2$. By (F4), φ is the identity on $\{[e_1], [e_1 + e_2]\}^{\perp\perp} = \{[e_1], [e_2]\}^{\perp\perp}$. Thus $[U(x)] = [x]$ for every $x \in [e_1, e_2]$, so that we have for any $\alpha \in K$

$$[e_1 + \alpha e_2] \;=\; [U(e_1 + \alpha e_2)] \;=\; [\varepsilon\, e_1 + \alpha\, \varepsilon\, e_2] \;=\; [e_1 + \varepsilon^\star\, \alpha\, \varepsilon\, e_2],$$

and we conclude that $\alpha = \varepsilon^\star\, \alpha\, \varepsilon$, or $\varepsilon\, \alpha = \alpha\, \varepsilon$. Hence ε is in the centre of K.

We have shown that the $\star$-field $Z(K)$ contains T_K. By Lemma 4.4, there is an $i \in T_K$ such that $i^2 = -1$. Since $i^{\star 2} = (i^2)^\star = (-1)^\star = -1$, either $i^\star = i$ or $i^\star = -i$. Assume $i^\star = i$ and let $u, v \in H$ be orthogonal unit vectors; then $(u + iv, u + iv) = 0$ and thus $u = -iv$, a contradiction. We conclude that $i^\star = -i$.

We claim that $S_K \subseteq Z(K)$. Let $\alpha \in S_K$. Note that $\alpha \neq -i$. Let $\varepsilon = (\alpha - i)(\alpha + i)^{-1} = (\alpha + i)^{-1}(\alpha - i)$. Then $\varepsilon \in T_K$ and hence $\varepsilon \in Z(K)$. Note that $\varepsilon \neq 1$. From

$(\alpha+i)\,\varepsilon = \alpha - i$, we conclude $\alpha\,(1-\varepsilon) = i\,(1+\varepsilon)$ and thus $\alpha = i\,(1+\varepsilon)\,(1-\varepsilon)^{-1} \in Z(K)$ as asserted.

Finally, let $\alpha \in K$. We may by Lemma 4.3 define

$$\operatorname{Re}\alpha = \tfrac{1}{2}(\alpha + \alpha^\star), \quad \operatorname{Im}\alpha = \tfrac{1}{2}i(\alpha^\star - \alpha). \tag{1}$$

Then $\operatorname{Re}\alpha, \operatorname{Im}\alpha \in S_K$, and we have $\alpha = \operatorname{Re}\alpha + i\operatorname{Im}\alpha \in Z(K)$. We conclude that K is commutative. In particular, S_K is a subfield of K, and $K = S_K(i)$. □

Our final lemma shows that K can be made into an ordered $\star$-field.

Lemma 4.6. *K is orderable.*

Proof. We have to specify a domain of positivity $P \subseteq S_K$. Let $P_0 = \{(x,x) : x \in H\}$. We shall show that P_0 fulfils the properties (1)–(4) of Definition 3.1. Obviously, $P_0 \subseteq S_K$. Moreover, it follows from (F1) or (F2) that for any $x, y \in H^{\bullet}$, there is a unitary map U such that $U(x) \perp y$ and hence $(x,x)+(y,y) = (U(x),U(x))+(y,y) = (U(x)+y, U(x)+y)$; so $P_0 + P_0 \subseteq P_0$. As H contains a unit vector, we have $1 \in P_0$. For any $x \in H$ and $\beta \in K$, we have $\beta\,(x,x)\,\beta^\star = (\beta x, \beta x)$; this means that $\alpha \in P_0$ implies $\beta\alpha\beta^\star \in P_0$. Finally, assume that $P_0 \cap -P_0$ contains a non-zero element, that is, $(x,x) = -\,(y,y)$ for some $x, y \in H^{\bullet}$. We choose again a unitary map U such that $U(x) \perp y$ and we conclude $(U(x)+y, U(x)+y) = 0$, so that $U(x)+y = 0$ and hence $x = 0$, a contradiction.

As any vector in H is a multiple of a unit vector, we have $P_0 = \{\alpha\alpha^\star \colon \alpha \in K\}$. It follows that $\beta \in P_0 \setminus \{0\}$ implies $\frac{1}{\beta} \in P_0$.

The remaining part of the proof follows the lines of [Pre2, §1]. Let $P \supseteq P_0$ be a maximal subset of S_K fulfilling the properties (1)–(4) of Definition 3.1. Our aim is to show that P also fulfils property (5). Assume that there is a $\gamma \in S_K$ such that $\gamma, -\gamma \notin P$ and put $P' = P - \gamma P_0$. Then P' fulfils (1)–(3) and because P' properly contains P, it follows that P' does not fulfil (4). Hence there are $\alpha_1, \alpha_2 \in P$ and $\beta_1, \beta_2 \in P_0$ such that $\alpha_1 - \gamma\beta_1 = -(\alpha_2 - \gamma\beta_2) \neq 0$, that is, $\alpha_1 + \alpha_2 = \gamma(\beta_1 + \beta_2)$. If $\beta_1 + \beta_2 \neq 0$, we would have $\gamma = (\alpha_1+\alpha_2)(\beta_1+\beta_2)^{-1} \in P$, because $(\beta_1+\beta_2)^{-1} \in P_0$ and $P \cdot P_0 \subseteq P$. But $\gamma \notin P$, hence $\beta_1 + \beta_2 = 0$ and this implies $\beta_1 = \beta_2 = 0$, because $P_0 \cap -P_0 = \{0\}$. It further follows that $\alpha_1 + \alpha_2 = 0$ and hence $\alpha_1 = \alpha_2 = 0$ for the same reason. We conclude that one of γ or $-\gamma$ is in P, that is, $S_K = P \cup -P$. □

We may summarise the results of this section as follows.

Theorem 4.7. *Let $(X, \perp)$ be an orthogonality space of finite rank $\geqslant 4$ and fulfilling* (F1)–(F4). *Then $(X, \perp)$ is isomorphic to $(P(H), \perp)$, where H is an orthomodular space H over an ordered $\star$-field K such that each subspace contains a unit vector. Moreover, $K = S_K(i)$, where $i^2 = -1$ and $i^\star = -i$.*

5 Quotients of orthomodular spaces

In this section we discuss in some more detail Holland's construction described in Theorem 3.9. With respect to the notation of that theorem, it is our aim to investigate how the orthogonality space $(P(H), \perp)$ and the subspace lattice $\mathcal{C}(H)$ are related to the corresponding structures in $\hat{H}$.

Our considerations restrict to the case of a $\star$-sfield that has the properties according to the conclusion of Theorem 4.7. Namely, in this section K is an ordered $\star$-field of the form $S_K(i)$, where S_K is what is called the fixed field and $i^2 = -1, \quad i^\star = -i$. Moreover, H will be an orthomodular space of finite dimension $\geqslant 4$ over K, containing a unit vector in every one-dimensional subspace.

We recall that H is the disjoint union of three types of vectors. Considering (x, x) as the length of an $x \in H$, the infinitesimal vectors are those whose length is below $\frac{1}{n}$ for all $n \in \mathbb{N} \setminus \{0\}$; the medial vectors have a length between $\frac{1}{n}$ and n for some $n \in \mathbb{N}$; and there are finally those vectors whose length larger than n for all $n \in \mathbb{N}$.

By assumption, every one-dimensional subspace of H contains a medial vector, so that $P(H) = \{[x] \colon x \in M_H\}$. The next lemma shows that two medial vectors contained in a one-dimensional subspace of H differ by a factor that is medial as well.

Lemma 5.1. *Let $x \in M_H$. Then $y \in [x] \cap M_H$ if and only if $y = \alpha x$ for some $\alpha \in M_K$.*

Proof. Let $\alpha \in K$ and assume that $y = \alpha x \in M_H$. This means $(y, y) = \alpha\alpha^\star\,(x, x) \in M_K$. It follows that $\alpha\alpha^\star = (y, y)\,(x, x)^{-1} \in M_K \cap S_K^+$ and consequently $\alpha \in M_K$.

Conversely, if $y = \alpha x$ for some $\alpha \in M_K$, then $(y, y) = \alpha\alpha^\star\,(x, x) \in M_K$, that is, $y \in M_H$. $\square$

For $x \in M_H$, let $[\![x]\!] = [x/I_H]$ be the subspace of $\hat{H}$ spanned by x/I_H. Note that if x is infinitesimal, x/I_H is the null vector of $\hat{H}$. Consequently, every one-dimensional subspace of $\hat{H}$ is arises from a medial vector as indicated, that is, $P(\hat{H}) = \{[\![x]\!] \colon x \in M_H\}$.

For $x, y \in M_H$, we define $[x] \approx [y]$ if there are medial vectors $x' \in [x]$ and $y' \in [y]$ such that $x' - y' \in I_H$.

Lemma 5.2. *Let $x, y \in M_H$. Then $[x] \approx [y]$ if and only if $[\![x]\!] = [\![y]\!]$ if and only if $x = \alpha y + s$ for some $\alpha \in M_K$ and $s \in I_H$.*

In particular, $\approx$ is an equivalence relation on $P(H)$ and we may identify $P(H)/{\approx}$ with $P(\hat{H})$.

Proof. Assume $[x]{\approx}[y]$. By Lemma 5.1, there are $\alpha, \beta \in M_K$ such that $\alpha x - \beta y \in I_H$. It follows $[\![x]\!] = [x/I_H] = [(\alpha x)/I_H] = [(\beta y)/I_H] = [y/I_H] = [\![y]\!]$.

Assume $[\![x]\!] = [\![y]\!]$. Then $x/I_H = \alpha/I_K \cdot y/I_H = (\alpha y)/I_H$ for some $\alpha \in M_K$, thus $x = \alpha y + s$ for some $s \in I_H$.

If $x = \alpha y + s$ for some $\alpha \in M_K$ and $s \in I_H$, then $\alpha y \in M_H$ and hence $[x] \approx [y]$. $\square$

We add a further useful criterion for the equivalence w.r.t. $\approx$ in $P(H)$. Here, the more special assumptions on K come into play.

Lemma 5.3. *Let $u, v \in H$ be unit vectors. Then $0 \leqslant (u,v)\,(u,v)^\star \leqslant 1$. Moreover, $[u] \approx [v]$ if and only if $1-(u,v)\,(u,v)^\star \in I_K$.*

Proof. The first assertion follows from the Cauchy-Schwarz inequality [Hol2, Sec. 5.1]. Alternatively, it is seen from

$$0 \leqslant (u-(u,v)\,v,\; u-(u,v)\,v) = 1-(u,v)\,(u,v)^\star . \tag{2}$$

Assume now that $[u] \approx [v]$. By Lemma 5.2 there are an $\alpha \in M_K$ and $s \in I_H$ such that $v = \alpha u + s$. From $1 = (v,v) = (\alpha u + s, \alpha u + s)$ we conclude, on the one hand, that $1-\alpha\alpha^\star \in I_K$. It follows, on the other hand, that $(s,s) = (\alpha u - v, \alpha u - v) = 1+\alpha\alpha^\star - 2\,\mathrm{Re}\,(\alpha u, v) = 2(1-\mathrm{Re}\,(\alpha u,v)) - (1-\alpha\alpha^\star) \in I_K$, where we used the notation as in (1). We conclude $1-\mathrm{Re}\,(\alpha u,v) \in I_K$ and moreover

$$\begin{aligned} 0 \leqslant \alpha\alpha^\star(1-(u,v)\,(u,v)^\star) \;&=\; \alpha\alpha^\star - (\alpha u,v)\,(\alpha u,v)^\star \\ &= \alpha\alpha^\star - (\mathrm{Re}\,(\alpha u,v))^2 - (\mathrm{Im}\,(\alpha u,v))^2 \leqslant \alpha\alpha^\star - (\mathrm{Re}\,(\alpha u,v))^2 \\ &= (1-(\mathrm{Re}\,(\alpha u,v))^2) - (1-\alpha\alpha^\star) \\ &= (1-\mathrm{Re}\,(\alpha u,v))(1+\mathrm{Re}\,(\alpha u,v)) - (1-\alpha\alpha^\star) \in I_K, \end{aligned}$$

because $1+\mathrm{Re}\,(\alpha u,v) \in F_K$. It follows that $\alpha\alpha^\star(1-(u,v)\,(u,v)^\star) \in I_K$ and hence $1-(u,v)\,(u,v)^\star \in I_K$.

Conversely, if $1-(u,v)\,(u,v)^\star \in I_K$, then we may observe from (2) that $[u] \approx [v]$. □

We next turn to the orthogonality relation in $P(\hat{H})$.

Lemma 5.4. *Let $x, y \in M_H$. Then $[\![x]\!] \perp [\![y]\!]$ if and only if $[x'] \perp [y']$ for some $x', y' \in M_H$ such that $[x'] \approx [x]$ and $[y'] \approx [y]$ if and only if $[x'] \perp [y]$ for some $x' \in M_H$ such that $[x'] \approx [x]$.*

Proof. Assume $[\![x]\!] \perp [\![y]\!]$. This means $(x/I_H, y/I_H) = 0/I_K$, that is, $(x,y) \in I_K$. Putting $x' = x - (x,y)\,(y,y)^{-1}\,y \in M_H$, we get $x - x' \in I_H$ and hence $[x'] \approx [x]$, and we have $[x'] \perp [y]$.

Let now $x', y' \in M_H$ be such that $[x'] \approx [x]$, $[y'] \approx [y]$, and $[x'] \perp [y']$. Then $x - \alpha x'$, $y - \beta y' \in I_H$ for some $\alpha, \beta \in M_K$. It follows $(x,y) \in I_K$, that is, $[\![x]\!] \perp [\![y]\!]$. □

We wish to describe the natural quotient map from $P(H)$ to $P(\hat{H})$. A key observation is the following.

Lemma 5.5. *Let $x, y, z \in M_H$. If $[\![x]\!] \subseteq [\![y]\!] \vee [\![z]\!]$, then there is an $x' \in M_H$ such that $[x'] \approx [x]$ and $x' \in [y,z]$. If $[y] \not\approx [z]$, then also the converse is true.*

Proof. Assume that $[\![x]\!] \subseteq [\![y]\!] \vee [\![z]\!]$. This means $x/I_H \in [y/I_H, z/I_H]$, that is, there are $\alpha, \beta \in F_K$ such that $\alpha y + \beta z - x \in I_H$. Then $x' = \alpha y + \beta z$ fulfils the indicated requirements.

For the converse direction, we may, w.l.o.g., assume that x, y, and z are unit vectors. Let $[y] \not\approx [z]$ and let $x' \in M_H$ be such that $[x'] \approx [x]$ and $x' \in [y, z]$. By replacing x' with a scalar multiple if necessary, we can assume that $x' - x \in I_H$. We have $x' = \alpha y + \beta z$ for some $\alpha, \beta \in K$ and we will show that $\alpha, \beta \in F_K$; this will imply that $[\![x]\!] \subseteq [\![y]\!] \vee [\![z]\!]$.

Since $[y] \not\approx [z]$, we have $1 - (y, z)\,(y, z)^\star \in M_K$ by Lemma 5.3. Putting $r = y - (y, z)\,z$ and $\beta' = \beta + \alpha\,(y, z)$, we have $r \perp z$ and $x' = \alpha y + \beta z = \alpha r + \beta' z$. Then $0 \leqslant \alpha\alpha^\star\,(r, r) \leqslant \alpha\alpha^\star\,(r, r) + \beta'\beta'^\star = (x', x') \in M_K$ and hence $\alpha\alpha^\star\,(r, r) \in F_K$. Since $(r, r) = 1 - (y, z)\,(y, z)^\star \in M_K$, we have $\alpha\alpha^\star \in F_K$ and hence $\alpha \in F_K$. Similarly, we see that also $\beta' \in F_K$ and thus $\beta = \beta' - \alpha\,(y, z) \in F_K$. □

It is noticeable what Lemma 5.5 entails and what not. Given $x, y, z \in M_H$ such that x in the linear hull of y and z, this linear dependence is preserved in $\hat{H}$, provided that $[\![y]\!]$ and $[\![z]\!]$ are distinct subspaces, that is, provided that there is no medial scalar γ such that γy differs from z by an infinitesimal vector only.

If the latter condition is not fulfilled, however, no conclusion is possible. Indeed, let there be a non-zero infinitesimal $\varepsilon \in K$. Let $x, y \in M_H$ be any vectors such that $[x] \not\approx [y]$. Then, putting $z = y + \varepsilon x$, we have that x is in the linear hull of y and z, but $[\![x]\!] \nsubseteq [\![y]\!] = [\![z]\!] = [\![y]\!] \vee [\![z]\!]$.

Theorem 5.6. *The map*

$$q\colon P(H) \to P(\hat{H}), \quad [x] \mapsto [\![x]\!] \quad (x \in M_H) \tag{3}$$

is a faithful surjective homomorphism of orthogonality spaces. Moreover, for $x, y, z \in M_H$ such that $[y] \not\approx [z]$, we have that $[x] \subseteq [y] \vee [z]$ implies $q([x]) \subseteq q([y]) \vee q([z])$.

Proof. By Lemma 5.4, q preserves the orthogonality relation. Clearly, q is surjective and q is faithful by Lemmas 5.4 and 5.2. The first assertion follows. The second one holds by Lemma 5.5. □

It is natural to ask whether the quotient map q can be extended from $P(H)$ to the ortholattice $\mathcal{C}(H)$.

Theorem 5.7. *The map q given by* (3) *induces the surjective map*

$$\bar{q}\colon \mathcal{C}(H) \to \mathcal{C}(\hat{H}), \quad E \mapsto \bigvee_{x \in E^\bullet} q([x]). \tag{4}$$

We have:

(i) *$\bar{q}$ is order-preserving;*

(ii) *$\bar{q}$ preserves dimensions;*

(iii) $\bar{q}$ *preserves the orthocomplement.*

Proof. We first observe that we have

$$\bar{q}(E) = \{x/I_H : x \in E \cap F_H\}, \quad E \in \mathcal{C}(H).$$

Indeed, $\bar{q}(E)$ is assumed to contain $q([x])$ for all $x \in E^{\bullet}$, or equivalently, for all $x \in E \cap M_H$. It follows that $\bar{q}(E) \supseteq \{x/I_H : x \in E \cap F_H\}$, and since the latter set is a linear subspace of $\hat{H}$, the assertion follows.

We next claim that, for any orthogonal unit vectors $e_1, \ldots, e_k \in H$, we have

$$\bar{q}([e_1, \ldots, e_k]) \;=\; [e_1/I_H, \ldots, e_k/I_H]. \tag{5}$$

Indeed, let $x \in [e_1, \ldots, e_k] \cap F_H$. Then $x = \alpha_1 e_1 + \ldots + \alpha_k e_k$, where we have $\alpha_i = (x, e_i) \in F_H$. We conclude that $x/I_H \in [e_1/I_H, \ldots, e_k/I_H]$. This shows one inclusion in (5); the converse inclusion is obvious.

To show that $\bar{q}$ is surjective, let $B \in \mathcal{C}(\hat{H})$. Recall that $\hat{H}$ has the same dimension as H, which is finite. Hence also B is finite-dimensional and we conclude that B possesses an orthogonal basis $e_1/I_H, \ldots, e_k/I_H$ such that $e_1, \ldots, e_k$ are unit vectors. By (5), we then have $\bar{q}([e_1, \ldots, e_k]) = B$.

(i) is clear by construction, and (ii) holds by (5). To see (iii), let $E \in \mathcal{C}(H)$ and choose an orthonormal basis $e_1, \ldots, e_n$ of H such that $E = [e_1, \ldots, e_k]$, $0 \leqslant k \leqslant n$. By (5), $\bar{q}(E) = [e_1/I_H, \ldots, e_k/I_H]$ and $\bar{q}(E^\perp) = [e_{k+1}/I_H, \ldots, e_n/I_H]$. Again by (5), $e_1/I_H, \ldots, e_n/I_H$ is an orthonormal basis of $\hat{H}$. We conclude $\bar{q}(E^\perp) = \bar{q}(E)^\perp$. □

We conclude the section showing the fact of which we will actually make use in the present paper: the transition from $P(H)$ to $P(\hat{H})$ is compatible with the automorphisms of $P(H)$ induced by unitary maps.

Theorem 5.8. *Let φ be an automorphism of $(P(H), \perp)$ such that, for some linearly independent vectors $x, y \in H$, φ is the identity on $\{[x], [y]\}^{\perp\perp}$. Then, for any $x, y \in H^{\bullet}$, we have $[x] \approx [y]$ if and only if $\varphi([x]) \approx \varphi([y])$. Setting*

$$\hat{\varphi}(q([x])) \;=\; q(\varphi([x])), \quad x \in M_H,$$

we may moreover define an automorphism $\hat{\varphi}$ of $P(\hat{H})$, where q is given by (3).

Proof. By Lemma 2.4, φ extends to an automorphism of the ortholattice $\mathcal{C}(H)$. By Theorem 3.8, there is a unitary map $U : H \to H$ such that $\varphi([x]) = [U(x)]$, $x \in H^{\bullet}$.

We have $z \in I_H$ if and only if $U(z) \in I_H$. Hence we may define $\hat{U}(x/I_H) = U(x)/I_H$, where $x \in F_H$. Then $\hat{U}$ is an endomorphism of $\hat{H}$ preserving the inner product, hence a unitary map. Thus $\hat{U}$ induces an automorphism $\hat{\varphi}$ as asserted.

Let now $x, y \in H^{\bullet}$. Then $[x] \approx [y]$ if and only if $x' - y' \in I_H$ for some $x' \in [x] \cap M_H$ and $y' \in [y] \cap M_H$. Similarly, $\varphi([x]) \approx \varphi([y])$ if and only if $[U(x)] \approx [U(y)]$ if and only if $x'' - y'' \in I_H$ for some $x'' \in [U(x)] \cap M_H$ and $y'' \in [U(y)] \cap M_H$. Since $x' - y' \in I_H$ is equivalent to $Ux' - Uy' \in I_H$, and M_H is invariant under U, the two statements are equivalent. □

We may formulate an analogous result with respect to the subspace ortholattices.

Theorem 5.9. *Let φ be an automorphism of $\mathcal{C}(H)$ such that, for some two-dimensional subspace F of H, $\varphi|_{[0,F]}$ is the identity. Setting $\hat{\varphi}(\bar{q}(E)) = \bar{q}(\varphi(E))$, $E \in \mathcal{C}(H)$, we may then define an automorphism $\hat{\varphi}$ of $\mathcal{C}(\hat{H})$, where $\bar{q}$ is given according to* (4).

Proof. As in the proof of Theorem 5.8, there is a unitary map U of H such that $\varphi([x]) = [U(x)]$ for $x \in H^{\bullet}$, and a unitary map $\hat{U}$ on $\hat{H}$ such that $\hat{U}(x/I_H) = U(x)/I_H$ for $x \in F_H$.

Putting $\hat{\varphi}(B) = \{\hat{U}(y) \colon y \in B\}$, where $B \in \mathcal{C}(\hat{H})$, we define an automorphism of $\mathcal{C}(\hat{H})$. Then, for any $E \in \mathcal{C}(H)$, we have $\hat{\varphi}(\bar{q}(E)) = \hat{\varphi}(\{x/I_H \colon x \in E \cap F_H\}) = \{\hat{U}(x/I_H) \colon x \in E \cap F_H\} = \{U(x)/I_H \colon x \in E \cap F_H\}$ and $\bar{q}(\varphi(E)) = \bar{q}\{U(x) \colon x \in E\} = \{U(x)/I_H \colon x \in E,\ U(x) \in F_H\} = \{U(x)/I_H \colon x \in E \cap F_H\}$. The assertion follows. $\square$

6 Orthogonality spaces arising from complex Hilbert spaces

We turn in this section to our primary concern: the description of those orthogonality spaces that arise from finite-dimensional complex Hilbert spaces. We will not provide an exact characterisation of this canonical example of an orthogonality space. However, our main result establishes the representation by an orthomodular space over a subfield of $\mathbb{C}$, and consequently an embedding into a space of the desired type.

To begin with, we show that for complex Hilbert spaces the conditions that we have considered so far are fulfilled.

Lemma 6.1. *Let H be a Hilbert space of finite dimension $\geqslant 4$ over $\mathbb{C}$. Then $(P(H), \perp)$ fulfils* (F1)–(F4).

Proof. Ad (F1) and (F2): Let $u, v \in H^{\bullet}$ be linearly independent. Then there is a unitary map U such that $U(u) \in [v]^{\perp} \cap [u, v]$ or $U(u) \in [v]$, respectively, and such that $U|_{[u,v]^{\perp}}$ is the identity.

Ad (F3): Let E be a two-dimensional subspace of H and let U be a unitary map such that $U|_{E^{\perp}}$ is the identity. Let u, v be an orthonormal basis of E such that $Uu = e^{i\alpha}u$ and $Uv = e^{i\beta}v$, where $0 \leqslant \alpha, \beta < 2\pi$. Let V be the unitary map such that $Vu = e^{\frac{i\alpha}{k}}u$ and $Vv = e^{\frac{i\beta}{k}}v$ and $V|_{E^{\perp}}$ is the identity. Then $V^k = U$ and we moreover readily check that, for any $x \in H$, $Ux \in [x]$ implies $Vx \in [x]$.

Ad (F4): Let $u, v \in H$ be orthogonal unit vectors and let $w = \alpha u + \beta v$, where $\alpha, \beta \neq 0$. Assume that U is a unitary map such that $Uu \in [u]$ and $Uw \in [w]$. Then there are $\gamma_1, \gamma_2 \in \mathbb{C}$ such that $Uu = \gamma_1 u$ and $Uw = \gamma_2 w$. Hence $\alpha\gamma_1 u + \beta Uv = Uw = \gamma_2\alpha u + \gamma_2\beta v$ and since $(Uv, u) = 0$ we conclude $\alpha\gamma_1 = \gamma_2\alpha$. Thus $\gamma_1 = \gamma_2$ and it follows that $U(x) \in [x]$ for any $x \in [u, v]$. $\square$

In order to establish the announced representation theorem, we need to consider, in addition to (F1)–(F4), still one more condition.

An equivalence relation $\sim$ on an orthogonality space $(X, \perp)$ is called a *congruence* if there is a faithful surjective homomorphism φ to another orthogonality space such that, for $x, y \in X$, we have $x \sim y$ iff $\varphi(x) = \varphi(y)$. Furthermore, we say that an automorphism φ of $(X, \perp)$ is *compatible* with some equivalence relation $\sim$ on X if $x \sim y$ is equivalent to $\varphi(x) \sim \varphi(y)$ for any $x, y \in X$.

(F5) Let $\approx$ be a congruence on $(X, \perp)$ and assume that any automorphism $\varphi\colon X \to X$ that is the identity on $\{e, f\}^{\perp\perp}$ for some distinct $e, f \in X$ is compatible with $\approx$. Then $\approx$ is the diagonal.

Lemma 6.2. *Let H be a Hilbert space of finite dimension $\geqslant 3$ over $\mathbb{C}$. Then $(P(H), \perp)$ fulfils* (F5).

Proof. Let $\approx$ be an equivalence relation on $(P(H), \perp)$ as specified in condition (F5). Then $[x] \approx [y]$ implies $[Ux] \approx [Uy]$ for all those unitary operators U on H that are the identity when restricted to a two-dimensional subspace. As these operators generate the whole unitary group, the implication holds in fact for any unitary operator U.

Let $u, v \in H$ be linearly independent unit vectors such that $[u] \approx [v]$. Let $x, y \in H$ be arbitrary unit vectors. We shall show that $[x] \approx [y]$; from this contradiction the assertion will follow.

Let $c = |(u, v)|$. Then $0 \leqslant c < 1$. We distinguish several cases.

Case 1. Let $|(x, y)| = 1$. Then $[x] = [y]$ and hence $[x] \approx [y]$.

Case 2. Let $|(x, y)| = c$. Then there is a unitary map U such that $U([u]) = [x]$ and $U([v]) = [y]$, hence $[x] \approx [y]$.

Case 3. Let $c < |(x, y)| < 1$. Let $\varepsilon \in \mathbb{C}$ be such that $|\varepsilon| = 1$ and $(x, \varepsilon y) = |(x, y)|$ and let $\gamma = \frac{1}{\sqrt{2(1+|(x,y)|)}}$. Then $w = \gamma x + \gamma \varepsilon y$ is a unit vector and $1 \geqslant |(x, w)| = |(y, w)| \geqslant |(x, y)| \geqslant c$. Let z be a unit vector orthogonal to x and y. Then we can find $\alpha, \beta \in \mathbb{R}$ such that $t = \alpha w + \beta z$ is a unit vector and $|(x, t)| = |(y, t)| = c$. It follows from Case 2 that $[x] \approx [y]$.

Case 4. Let $|(x, y)| < c$. Then we can find a finite sequence of unit vectors, beginning with x and ending with y, such that the absolute value of the scalar product of each successive pair is $> c$. From Case 3, we conclude $[x] \approx [y]$. □

We are ready to formulate our main result: an orthogonality space fulfilling conditions (F1)–(F5) arises from an orthomodular space over a subfield of the complex numbers.

We will call a subfield K of $\mathbb{C}$ *dense* if K is a dense subset of $\mathbb{C}$ endowed with the standard topology.

Theorem 6.3. *Let $(X, \perp)$ be an orthogonality space of finite rank $n \geqslant 4$ fulfilling* (F1)–(F5). *Then $(X, \perp)$ is isomorphic to $(P(H), \perp)$, where H is an n-dimensional orthomodular space H over a dense sub-$\star$-field K of $\mathbb{C}$ such that each one-dimensional subspace contains a unit vector.*

Proof. By Theorem 4.7, there is an orthomodular space H over an ordered $\star$-field K such that each one-dimensional subspace contains a unit vector and $(X, \perp)$ is isomorphic to $P(H)$.

According to Theorem 3.9, let $\hat{H}$ be the orthomodular space over $\hat{K}$ arising as the quotient of the finite elements of H modulo the infinitesimals. By Theorem 5.6, Lemma 5.2, and Theorem 5.8, there is then a faithful surjective homomorphism from $(P(H), \perp)$ to $(P(\hat{H}), \perp)$ such that any automorphism of $(P(H), \perp)$ that is the identity on a two-dimensional subspace is compatible with the associated congruence on $(P(H), \perp)$. By (F5), the congruence is the diagonal.

It follows that $I_K = \{0\}$, that is, K is Archimedean. Indeed, let $u, v \in M_H$ be linearly independent. If there was a non-zero infinitesimal $\alpha \in K$, then $[u]$ and $[u + \alpha v]$ would be distinct elements of $P(H)$ such that $[u] \approx [u + \alpha v]$.

By Holland's Theorem 3.3, K is isomorphic to a sub-$\star$-field of $\mathbb{R}$ or $\mathbb{C}$. By (F3), only the latter possibility can apply.

Identifying K with a sub-$\star$-field of $\mathbb{C}$, the fixed field S_K is a subfield of $\mathbb{R}$, and $i \in K$ can w.l.o.g. be assumed to be the imaginary unit of $\mathbb{C}$. As $\mathbb{Q} \subseteq S_K$ and $K = S_K(i)$, we conclude that K is dense in $\mathbb{C}$. $\square$

We conclude with the following, possibly more intuitive, version of Theorem 6.3. By an embedding of orthogonality spaces, we mean a full injective homomorphism.

Theorem 6.4. *Let $(X, \perp)$ be an orthogonality space of finite rank $n \geqslant 4$ fulfilling* (F1)–(F5). *Then $(X, \perp)$ can be embedded into $(P(\mathbb{C}^n), \perp)$, where $\mathbb{C}^n$ is endowed with the standard hermitian form.*

Proof. Let H be the n-dimensional orthomodular space over a sub-$\star$-field K of $\mathbb{C}$, representing $(X, \perp)$ according to Theorem 6.3. As H possesses an orthonormal basis, H is isomorphic to K^n equipped with the standard hermitian form.

We can consider K^n as a subset of $\mathbb{C}^n$. Then each one-dimensional subspace of K^n is contained in a unique one-dimensional subspace of $\mathbb{C}^n$. Moreover, the natural map from $P(K^n)$ to $P(\mathbb{C}^n)$ is a full injective homomorphism. The assertion follows. $\square$

7 Conclusion

To characterise the complex Hilbert space by algebraic means is a demanding endeavour, which at least in the infinite-dimensional case has led to a good success [Wlb, Sol, May]. The conditions that are used, however, are often complex and seem in some cases fairly arbitrary. Moreover, in the finite-dimensional case, no comparable results seem to be available.

The present paper is an attempt to revive the issue from a somewhat modified point of view. Choosing a structure that is even "lighter" than ortholattices, we have put the emphasis on symmetries. Namely, we have dealt with orthogonality spaces, which were introduced in the 1960s by David Foulis, and we investigated the effect of conditions concerning the existence of automorphisms. We have moreover focused exclusively on the finite-dimensional case. The result is not as perfect as it would be desirable, but we succeeded to delimit the representing inner-product spaces to those over subfields of $\mathbb{C}$.

Several issues call for further elaboration. First of all, a comparable approach could well be tried also in the infinite-dimensional case. Some ideas to this end are already contained our paper [Vet]. As mentioned already above, the procedure would be quite different, the main tool being Solèr's Theorem.

A further progress might require conceptual modifications. One idea is the following. We have worked with automorphisms of the orthogonality space consisting of the one-dimensional subspaces of a Hilbert space. Each of these one-dimensional subspaces, however, rise to automorphisms itself, in fact to a group homomorphism from the unit circle to the unitary group. Understanding the Hilbert space in terms of these homomorphisms, which have played no major role in the present work, seems to be worthwhile.

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