Partial quantum logics revisited

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Abstract

Partial Boolean algebras (PBA's) were introduced by Kochen and Specker as an algebraic model reflecting the mutual relationships among quantum-physical yes-no tests. The fact that not all pairs of tests are compatible was taken into special account.

In this paper, we review PBA's from two sides. First, we generalize the concept, taking into account also those yes-no tests which are based on unsharp measurements. Namely, we introduce partial MV-algebras, and we define a corresponding logic.

Second, we turn to the representation theory of PBA's. In analogy to the case of orthomodular lattices, we give conditions for a PBA to be isomorphic to the PBA of closed subspaces of a complex Hilbert space. Hereby, we do not restrict to purely algebraic statements; we rather give preference to conditions involving automorphisms of a PBA.

We conclude outlining a critical view on the logico-algebraic approach to the foundational problem of quantum physics.

Keywords: Partial Boolean algebras, partial MV-algebras, quantum logic, Hilbert space

1 Introduction

With this note, we intend to contribute to a research line around which it has become calm during the last years. During many years, it was the aim of an ambitious program to justify the basic model of quantum physics on base of a few basic and easily comprehensible principles. The Hilbert space, which otherwise seems to be chosen ad hoc, was to be reconstructed in an algebraic or even logical framework. Understandably, the research focused on the set of two-valued measurements associated with some quantum-physical arrangement. A Hilbert space model of the experiment being given, the yes-no tests are alternatively modelled by the closed subspaces, or more generally by the effects, that is, by the positive operators below the identity.

The closed subspaces form an orthomodular lattice (OML), and this fact led to an extensive research about this, admittedly fascinating, kind of algebra, about which, by the way, numerous important problems are still open. However, as apparent from the work of [6], the weak point about the OML-based approach is the fact that to apply lattice-theoretical operations in the algebra means to combine possibly incompatible experiments. Already in 1965, Kochen and Specker made an alternative proposal; they endowed the set of closed subspaces of a Hilbert space with the structure of a partial Boolean algebra. The idea is to allow lattice-theoretical operations only in case that the two subspaces commute. Moreover, they defined a logic in which exactly those propositions are derivable which are valid in all partial Boolean algebras.

Compared to the approach based on OML's, the price to pay is certainly high; to deal with partial algebras is in general quite difficult. This might have been the reason why the Kochen-Specker approach did by far not achieve the same amount of attention as later the OML-based one. However, the arguments are convincing, and the unpleasant aspects of the partiality of operations just reflect a basic principle in quantum physics.

We review in this paper partial Boolean algebras, and we extend the theory slightly into two directions. First, rather than studying the set of closed subspaces of a Hilbert space, we may, according to the more general approach, consider the set of effects. An effect is a self-adjoint operator E of a Hilbert space such that $0 \leq E \leq I$, where I is the identity operator; E models a generalized yes-no measurement. The set of effects might be consider the "fuzzified" counterpart of the set of closed subspaces.

The set of effects was given an algebraic structure according to an approach of [5], who introduced effect algebras. In this case, it became unavoidable to work with partial algebras; effect algebras are based on an addition which is not defined for all pairs of elements. The standard effect algebra is the set of Hilbert space effects endowed with a partial addition defined as the usual sum of operators whenever the result is an effect again. Dalla Chiara and Giuntini introduced in [3] the propositional logic UPaQL, which is based on effect algebras; namely, with respect to these algebras, the calculus was shown to be sound and complete.

Now, the critics due to Kochen and Specker's work concerning the set of Hilbert space subspaces, apply for the generalized approach as well. Namely, two effects may be connected - namely, their sum may be formed - even if these effects represent incompatible experiments. Indeed, any two effects, if they commute or not, may each be multiplied by $\frac{1}{2}$ to become summable.

We propose here to treat the set of effects in a way analogous to the way proposed in [6]. We introduce operations in analogy to MV-algebras, but defined only in the case that two effects commute. We are then led to the notion of a partial MV-algebra. We may furthermore formulate a logic for these algebras and show a completeness theorem.

The second part of our note on partial Boolean algebras concerns the old question how to get the Hilbert space structure from natural postulates. The aim to identify the standard Hilbert space, that is, the $\aleph_0$-dimensional complex Hilbert space, with an appropriate logic, has failed. However, the aim to characterize the standard Hilbert space by pure algebraic means was achieved, aside from the excusable restriction that an infinitary condition must be allowed; the conditions are formulated for OML's. Certain of these conditions are quite cumbersome; however, there is an alternative way, not formulated in an algebraic language, but to be considered at least as natural as the lattice-theoretical conditions: the existence of certain automorphisms. This was, in particular, pointed out in [9, 8].

We give a representation theorem along these lines for partial Boolean algebras. To this end, we work with a condition which has been called transitivity in [1], meaning to assume that our partial algebra is partially ordered. This is not a critical point, however, since it is not the partial order, but the lattice structure which is sensitive to interpretational questions. In particular, we do not assume the existence of infima and suprema; this is rather a consequence of the conditions on the existence of automorphisms.

We conclude with a general evaluation of the quantum logical, or quantum structural, approach to the foundations of quantum theory.

2 Partial Boolean algebras

Let $\mathcal{H}$ be a complex Hilbert space; let $C(\mathcal{H})$ be the set of closed subspaces of $\mathcal{H}$. We can endow $C(\mathcal{H})$ with an algebraic structure as follows. The relation $\diamond$ is the compatibility relation on $C(\mathcal{H})$; two subspaces $A, B \in C(\mathcal{H})$ are called compatible if there are mutually orthogonal elements $A_0, B_0, C \in C(\mathcal{H})$ such that $A = [A_0 \cup C]$ and $B = [B_0 \cup C]$; by $[X]$, we denote the smallest closed subspace containing $X \subseteq \mathcal{H}$. Moreover, for two compatible subspaces $A, B \in C(\mathcal{H})$, define $A \cap_\diamond B = A \cap B$ and leave $A \cap_\diamond B$ undefined otherwise. Finally, put $A^\perp = \{a \in \mathcal{H} \colon\ a \perp v \text{ for all } v \in A\}$. Then, the structure $(C(\mathcal{H}); \diamond, \cap_\diamond, ^\perp, \{0\}, \mathcal{H})$ is the prototypical example of the following notion, introduced in [6].

We note that here and in the sequel, our definitions might be slightly modified compared to the original ones, but never in an essential way.

Definition 2.1 The structure $(L; \diamond, \wedge, \neg, 0, 1)$ is called a *partial Boolean algebra*, or *PBA* for short, if the following conditions are fulfilled:

(PB1) $\diamond$ is a symmetric and reflexive binary relation. Elements a and b such that $a \diamond b$ are called *compatible*.

(PB2) $\wedge$ is a partial binary operations, and for $a, b \in L$, $a \wedge b$ is defined if and only if $a \diamond b$. Moreover, $\neg$ is a total unary operation.

(PB3) Let B be a finite subset of L such that $a \diamond b$ for any $a, b \in B$. Then any term formed from elements of B and the constants $0, 1$ by means of the operations $\wedge$ and $\neg$ is defined. Let $\bar{B}$ be the set containing all these elements; then $a \diamond b$ for all $a, b \in \bar{B}$, and $(\bar{B}; \wedge, \neg, 0, 1)$, is a Boolean algebra.

In the sequel, when we say that an equation containing partial operations holds, we mean that the partial operations are defined and the equality holds.

Note that we use only the infimum, and not the supremum, as an own operation of a PBA. We will rather treat expressions $a \vee b$ as defined by $\neg(\neg a \wedge \neg b)$. Note that then, $a \vee b$ is defined exactly if $a \wedge b$ is.

(PB3) could certainly be formulated more scarcely; since the defining equations of Boolean algebras involve maximally three elements, (PB3) could be replaced by the requirement that any three mutually compatible elements fulfil the equations valid for Boolean algebras and that Boolean combinations preserve compatibility. We preferred the more detailed version, which more directly expresses the intention: any finite subset of mutually compatible elements generates a Boolean algebra.

For a PBA L, it is in general hardly possible to derive any global property. In particular, for two arbitrary elements $a, b \in L$, $a \wedge b$ and $a \vee b$ are not defined in general, so L need not be a lattice. Even worse, L need not even be partially orderable in some natural way. Only in the second half of this paper, we will use a modification of the notion of a PBA according to [1], such that a partial order will be guaranteed.

Clearly, $(C(\mathcal{H}); \diamond, \cap_\diamond, {}^\perp, \{0\}, \mathcal{H})$ is a PBA.

We may now define a logic for PBA's, to be called **LPB**. We follow [6] as closely as possible. We first define the language of **LPB**.

Definition 2.2 An expression built up from a set $\varphi_0, \varphi_1, \ldots$ of *propositional variables* and the *constants* 0 and 1 by means of the binary operation $\wedge$ and the unary operation $\neg$ is called a *proper proposition* of **LPB**. We denote the set of all proper propositions by $\mathcal{F}_p$. Moreover, an expression of the form $\alpha \diamond \beta$, where $\alpha, \beta \in \mathcal{F}_p$, is called a *compatibility proposition* of **LPB**. We denote the set of compatibility propositions by $\mathcal{F}_c$. We define $\mathcal{F} = \mathcal{F}_p \cup \mathcal{F}_c$ as the set of *propositions* of **LPB**.

We will use the usual abbreviations in propositions of **LPB**. Namely, if $\alpha, \beta \in \mathcal{F}_p$, $\alpha \vee \beta$ stands for $\neg(\neg\alpha \wedge \neg\beta)$; $\alpha \rightarrow \beta$ is $\neg\alpha \vee \beta$; and $\alpha \leftrightarrow \beta$ is $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. Furthermore, for $\alpha_1, \ldots, \alpha_k$, where $k \geq 0$, the expression $\diamond\, (\alpha_1, \ldots, \alpha_k)$ replaces the set $\alpha_i \diamond \alpha_j,\ \ 1 \leq i \leq j \leq k$, of compatibility propositions. In particular, if $k = 0$, this is the empty sequence; if $k = 1$, we drop the brackets and write $\diamond\, \alpha_1$, meaning $\alpha_1 \diamond \alpha_1$; $\diamond\, (\alpha_1, \alpha_2)$ is $\alpha_1 \diamond \alpha_1,\ \alpha_1 \diamond \alpha_2,\ \alpha_2 \diamond \alpha_2$; and so on.

We now define which formulas are valid in all PBA's. Validity is in case of a partial algebra not a straightforward notion; the definition goes as follows.

Definition 2.3 Let $(L; \diamond, \wedge, \neg, 0, 1)$ be a PBA. Then a partial mapping $v\colon\ \mathcal{F}_p \rightarrow L$ is called an *evaluation* for **LPB** if, for any $\gamma \in \mathcal{F}_p$, the following conditions are fulfilled:

(1) If γ is atomic, $v(\gamma)$ is defined. Moreover, $v(0) = 0$ and $v(1) = 1$.

(2) If γ is of the form $\alpha \wedge \beta$, then $v(\gamma)$ is defined if and only if $v(\alpha)$ is defined and $v(\beta)$ is defined and $v(\alpha) \diamond v(\beta)$ holds. In this case, $v(\gamma) = v(\alpha) \wedge v(\beta)$.

(3) If γ is of the form $\neg\alpha$, then $v(\gamma)$ is defined if and only if $v(\alpha)$ is defined. In this case, $v(\gamma) = \neg v(\alpha)$.

Let $\alpha \in \mathcal{F}_p$. An evaluation v being given, α is said to be *satisfied* by v if $v(\alpha)$ is defined and equals 1. α is called *valid* in **LPB** if α is satisfied by any evaluation v such that $v(\alpha)$ is defined; in this case, we write $\models \alpha$.

Moreover, let $\alpha, \beta \in \mathcal{F}_p$. An evaluation v for **LPB** being given, $\alpha \diamond \beta$ is said to be satisfied by v if $v(\alpha)$ and $v(\beta)$ are defined and $v(\alpha) \diamond v(\beta)$ holds.

Note the role of propositions of the form $\diamond\, \alpha$: It is satisfied by an evaluation v iff $v(\alpha)$ is defined. Clearly, $\diamond\, \alpha$ is valid for any α.

We next define proofs of the logic **LPB**. By a Boolean tautology, we mean a proper proposition provable in classical propositional logic; we assume a proof system to derive the Boolean tautologies to be given.

Definition 2.4 For any proper propositions $\alpha, \alpha_1, \ldots, \beta$, the following are the rules of **LPB**:

$$\text{(R1)}\ \frac{\alpha \diamond \beta}{\beta \diamond \alpha}, \quad \text{(R2)}\ \frac{\alpha \diamond \beta \quad \beta \leftrightarrow \gamma}{\alpha \diamond \gamma}, \quad \text{(R3)}\ \frac{\alpha \diamond \beta}{\alpha \diamond \neg\beta}, \quad \text{(R4)}\ \frac{\diamond\,(\alpha, \beta, \gamma)}{\alpha \diamond \beta \wedge \gamma},$$

$$\text{(R5)}\ \frac{\diamond\,(\alpha_1, \ldots, \alpha_k)}{\varphi(\alpha_1, \ldots, \alpha_k)}, \text{ where } k \geq 0 \text{ and } \varphi \text{ is any Boolean tautology},$$

$$\text{(R6)}\ \frac{\alpha \quad \alpha \rightarrow \beta}{\beta}.$$

A finite sequence $(\varphi_1, \ldots, \varphi_k)$ of propositions is called an α*-admissible proof* in **LPB** if, for any $i = 1, \ldots, k$, either φ_i is of the form $\diamond\,\beta$ and β is a subformula of α, or φ_i is of the form $\beta \diamond \gamma$ and $\beta \wedge \gamma$ is a subformula of α, or φ_i is derived by means of one of the rules (R1)–(R6) from propositions among $\varphi_1, \ldots, \varphi_{i-1}$.

A proper proposition α is called *provable* in **LPB** if there is an α-admissible proof whose last element is α; in this case, we write $\vdash \alpha$.

The main result of [6] is the following. We will reproduce in a rough way the proof from [6], just detailed enough to make additional explanations in case of the more general logic discussed below unnecessary.

Theorem 2.5 *The logic* **LPB** *is sound and complete: Any proper proposition α is provable if and only if it is valid.*

Proof: Assume $\vdash \alpha$, and let $v : \mathcal{F}_p \rightarrow L$ be an evaluation such that $v(\alpha)$ is defined. We have to show that $v(\alpha) = 1$. By assumption, v is defined for all subformulas of α, and for any pair of subformulas β and γ of α connected by $\wedge$, we have $v(\beta) \diamond v(\gamma)$. It follows that all compatibility propositions in a proof of α are satisfied by v. Moreover, every rule obviously preserves satisfiability by v. The claim follows.

Assume that $\not\vdash \alpha$. We have to construct a PBA L and an evaluation $v : \mathcal{F}_p \rightarrow L$ such that $v(\alpha)$ is defined, but not equal to 1. Call any $\varphi \in \mathcal{F}_p$ α-provable if there is an α-admissible proof whose last element is φ. Note that, by assumption, α is not α-provable. Let Ω be the set of all proper propositions φ such that $\diamond\,\varphi$ is

α-provable. Then Ω contains all subformulas of α. Furthermore, we easily check that $0, 1 \in \Omega$, and that $\alpha \in \Omega$ implies $\neg\alpha \in \Omega$.

For $\varphi, \psi \in \Omega$, let $\varphi \sim \psi$ if $\varphi \leftrightarrow \psi$ is α-provable. Then $\sim$ is an equivalence relation on Ω. Indeed, from $\diamond\, \varphi$, we derive $\varphi \leftrightarrow \varphi$ by (R5). Furthermore, $\varphi \sim \psi$ implies $\diamond\,(\varphi, \psi)$ by (R2). So from $\varphi \sim \psi$, we may derive $\psi \sim \varphi$ by (R5) and (R6). Similarly, $\varphi \sim \psi$ and $\psi \sim \xi$ imply $\diamond\,(\varphi, \psi, \xi)$, whence $\varphi \sim \xi$ follows.

$\sim$ is compatible with $\neg$. Indeed, let $\varphi, \psi \in \Omega$; from $\diamond\,(\varphi, \psi)$ we derive $\diamond\,(\varphi, \neg\varphi, \psi, \neg\psi)$, and consequently $\varphi \sim \psi$ implies $\neg\varphi \sim \neg\psi$. Similarly, $\sim$ is compatible with $\wedge$: for $\varphi, \psi, \xi \in \Omega$ such that $\varphi \diamond \psi$ and $\psi \sim \xi$, we have $\varphi \diamond \xi$ and $\varphi \wedge \psi \sim \varphi \wedge \xi$. To see this, note that $\diamond\,(\varphi, \psi, \xi)$ in this case, whence $\varphi \wedge \psi \leftrightarrow \varphi \wedge \xi$ is derivable by (R5) and (R6). Finally, $\sim$ is compatible with $\diamond$. Indeed, if $\varphi \diamond \psi$ and $\psi \sim \xi$, we have $\varphi \diamond \xi$ by (R2).

Let $([Q]; \diamond, \wedge, \neg, [0], [1])$ be the partial algebra induced by $\sim$. Clearly, $[Q]$ is a partial Boolean algebra. $v : \mathcal{F}_p \to [Q], \ \ \varphi \mapsto [\varphi]$ is an evaluation such that $v(\alpha) = [\alpha]$ is defined, but not equal to $[1]$. We conclude $\not\models \alpha$. □

What makes Kochen and Specker's completeness proof somewhat unusual is the fact that the PBA constructed to show that a non-provable proposition α is not valid, depends on α. In fact, there seems to be no way to construct a reasonable analog of the Lindenbaum-Tarski algebra, just like in the case of common total logics, without any special compatibility assumptions.

3 Partial MV-algebras

Again, let $\mathcal{H}$ be a complex Hilbert space; in this section, we shall be concerned with the set of effects of $\mathcal{H}$:

$$\mathcal{E}(\mathcal{H}) \ = \ \{E \in B_{\mathrm{sa}}(\mathcal{H}) \colon \, 0 \leq E \leq I\},$$

where $B_{\mathrm{sa}}(\mathcal{H})$ is the set of bounded self-adjoint operators of $\mathcal{H}$, 0 is the zero operator and I is the identity operator. According to the standard approach, $\mathcal{E}(\mathcal{H})$ is endowed with a partial binary operation as follows: For two effects E and F, $E + F$ is defined as the usual sum of operators if the result is an effect again, otherwise $E + F$ remains undefined. We then get the standard effect algebra $(\mathcal{E}(\mathcal{H}); +, 0, I)$, a partial algebra intensively discussed in the literature; see e.g. [4].

Here, we proceed differently. First of all, we wish to understand effects as fuzzy sets. Namely, let $E \in \mathcal{E}(\mathcal{H})$; then there is a compact, second countable Hausdorff space X endowed with a Radon integral, and there is an isomorphism U :

$L^2(X) \to \mathcal{H}$ such that, for an $e \in \mathcal{L}^\infty$, we have $E = UM_eU^{-1}$, where

$$M_e \colon L^2(X) \to L^2(X), \quad v \mapsto v\,e,$$

that is, M_e is the pointwise multiplication operator. (See e.g. [11].) Since the spectrum of E is in $[0,1]$, e is actually (up to a set of measure zero) a fuzzy set on X: e maps from X to $[0,1]$.

Clearly, for arbitrary two effects, a joint representation of this kind is impossible. However, what we have in mind is to connect pairs of effects only in case they are compatible, that is, if they commute. Let $\mathcal{A}$ be a set of effects such that any two of them commute; then we have a representation as before. Namely, there is a compact, second countable Hausdorff space X endowed with a Radon integral, an isomorphism $U \colon L^2(X) \to \mathcal{H}$, and for each $E \in \mathcal{A}$ there is an $e(E) \in \mathcal{L}^\infty$ such that $E = UM_{e(E)}U^{-1}$.

This representation being given, we may introduce an algebraic structure on the set of fuzzy sets $\{e(E) : E \in \mathcal{A}\}$. To this end, assume that $\mathcal{A}$ is a maximal set of pairwise commuting effects. We choose a standard conjunction from fuzzy logic, the Łukasiewicz t-norm, and the standard negation:

$$\odot \colon [0,1]^2 \to [0,1], \quad (s,t) \mapsto (s+t-1) \vee 0, \tag{1}$$

$$\neg \colon [0,1] \to [0,1], \quad t \mapsto 1-t. \tag{2}$$

These operations apply pointwise to fuzzy sets; we will use the same notation in this case. So for a pair $E, F \in \mathcal{A}$, we may consider their representations $M_{e(E)}$ and $M_{e(F)}$ in $L^2(X)$, and we may associate to them the operator $M_{e(E)\odot e(F)}$; the corresponding operator in $\mathcal{H}$ will be denoted by $E \odot F$. Similarly, we may associate to $M_{e(E)}$ the operator $M_{\neg e(E)}$; the corresponding operator in $\mathcal{H}$ will be denoted by $\neg E$. Note that $E \odot F$ and $\neg E$ are effects again. Furthermore, the definition of $\odot$ and $\neg$ on the set $\mathcal{A}$ does not depend on the representation U of $\mathcal{H}$.

Similarly like in the case of the closed subspaces, we endow $\mathcal{E}(\mathcal{H})$ with a compatibility relation; we define

$$E \diamond F \quad \text{if } E \text{ and } F \text{ commute.}$$

For any $E, F \in (\mathcal{E}(\mathcal{H})$, we define $E \odot F$ as above if $E \diamond F$, and we let $E \odot F$ undefined otherwise. Similarly, for any $E \in (\mathcal{E}(\mathcal{H})$, we define $\neg E$ as above. The resulting partial algebra $(\mathcal{E}(\mathcal{H}); \diamond, \odot, \neg, 0, I)$ shall be called the standard partial MV-algebra.

We recall next the notion of an MV-algebra. A structure $(L; \wedge, \vee, \odot, \neg, 0, 1)$ is an MV-algebra if $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, $(L; \wedge, \vee, \odot, 1)$ is an ℓ-monoid,

$\neg$ is an involutive order-reversing unary operation, and $a \wedge b = a \odot \neg(a \odot \neg b)$ for any $a, b \in L$. For a detailed exposition on MV-algebras, see e.g. [2].

In MV-algebras, the lattice-theoretic operations $\wedge$ and $\vee$ are both term-definable from the remaining operations. This is clear in case of the infimum, and the supremum is definable from $\wedge$ and $\neg$. From now on, we will not consider $\wedge$ and $\vee$ as own operations; an MV-algebra will be a structure of the form $(L; \odot, \neg, 0, 1)$.

The standard examples of MV-algebras are sets of fuzzy sets. Let some non-empty set X be given, and L consist of fuzzy sets on X, that is, functions $u\colon X \to [0, 1]$. Assume furthermore that L is closed under the operations $\odot$ and $\neg$ and that L contains the constant zero fuzzy set $\bar{0}$ and the constant one fuzzy set $\bar{1}$. It is easily checked that $(L; \odot, \neg, \bar{0}, \bar{1})$ is an MV-algebra. By applying the mentioned formulas for the discarded operations $\wedge$ and $\vee$, it may also be seen that the infimum is in L simply the pointwise minimum, and the supremum is the pointwise maximum.

Just like Boolean algebras generalize to partial Boolean algebras, MV-algebras generalize to partial MV-algebras.

Definition 3.1 The structure $(L; \Diamond, \odot, \neg, 0, 1)$ is called a *partial MV-algebra* if the following conditions are fulfilled:

(PM1) $\Diamond$ is a symmetric and reflexive binary relation. Elements a and b such that $a \Diamond b$ are called *compatible*.

(PM2) $\odot$ is a partial binary operations, and for $a, b \in L$, $a \odot b$ is defined if and only if $a \Diamond b$. Moreover, $\neg$ is a total unary operation.

(PM3) Let M be a finite subset of L such that $a \Diamond b$ for any $a, b \in M$. Then any term formed from elements of M and the constants $0, 1$ by means of the operations $\odot$ and $\neg$ is defined. Let $\bar{M}$ be the set containing all these elements; then $a \Diamond b$ for all $a, b \in \bar{M}$, and $(\bar{M}; \odot, \neg, 0, 1)$, is an MV-algebra.

It is obvious from the definition that exactly in case that the relation $\Diamond$ is total, a partial MV-algebra is an MV-algebra.

Again, we may note that (PM3) could be replaced by a requirement involving only three mutually compatible elements.

It should be clear from the above discussion that $(\mathcal{E}(\mathcal{H}); \Diamond, \odot, \neg, 0, I)$ is a partial MV-algebra.

As in the case of partial Boolean algebras, there is in general no reasonable way to endow a partial MV-algebra with a partial order. Namely, we may tentatively put

$a \preceq b$ if a and b are compatible and in the MV-algebra generated by a and b, we have $a \leq b$. However, $\preceq$ is not necessarily transitive. Consider $\mathcal{E}(\mathcal{H})$; for effects E, F, G, we may have $E \leq F$ and $E \diamond F$, furthermore $F \leq G$ and $F \diamond G$, so also $E \leq G$, but we cannot conclude that $E \diamond G$.

We now define a logic for partial MV-algebras in complete analogy to the logic **LPB**; we will call it **LPM**. We avoid repetitions where possible.

Definition 3.2 The set $\mathcal{F}_p$ of *proper propositions* and the set $\mathcal{F}_c$ of *compatibility propositions* of **LPM** are defined as in Definition 2.2, but using the binary connective $\odot$ instead of $\wedge$. We let $\mathcal{F} = \mathcal{F}_p \cup \mathcal{F}_c$ be the set of *propositions* of **LPM**.

The abbreviations needed are the following. For $\alpha, \beta \in \mathcal{F}_p$, $\alpha \rightarrow \beta$ stands for $\neg(\alpha \odot \neg\beta)$, and $\alpha \leftrightarrow \beta$ is $(\alpha \rightarrow \beta) \odot (\beta \rightarrow \alpha)$. For $\alpha_1, \ldots, \alpha_k$, where $k \geq 0$, we define $\diamond(\alpha_1, \ldots, \alpha_k)$ as above. Additional abbreviations are $\alpha \wedge \beta$ for $\alpha \odot (\alpha \rightarrow \beta)$ and $\alpha \vee \beta$ for $(\alpha \rightarrow \beta) \rightarrow \beta$.

We now define validity w.r.t. partial MV-algebras.

Definition 3.3 Let $(L; \diamond, \odot, \neg, 0, 1)$ be a partial MV-algebra. Then a partial mapping $v\colon \mathcal{F}_p \rightarrow L$ is called an *evaluation* for **LPM** if, for any $\gamma \in \mathcal{F}_p$, the following conditions are fulfilled:

(1) If γ is atomic, $v(\gamma)$ is defined. Moreover, $v(0) = 0$ and $v(1) = 1$.

(2) If γ is of the form $\alpha \odot \beta$, then $v(\gamma)$ is defined if and only if $v(\alpha)$ is defined and $v(\beta)$ is defined and $v(\alpha) \diamond v(\beta)$ holds. In this case, $v(\gamma) = v(\alpha) \odot v(\beta)$.

(3) If γ is of the form $\neg\alpha$, then $v(\gamma)$ is defined if and only if $v(\alpha)$ is defined. In this case, $v(\gamma) = \neg v(\alpha)$.

Satisfaction by some evaluation and validity for propositions is defined in analogy to Definition 2.3 above.

We next define proofs in the logic **LPM**. We note first that the propositions which are valid in all MV-algebras are exactly the tautologies of Łukasiewicz logic, or Łukasiewicz tautologies for short. We assume that a proof system for this logic has been defined. We refer to [2] for a Hilbert-style proof system, and to [10] for a r-hypersequent-based analytic proof system.

Definition 3.4 For any proper propositions $\alpha, \alpha_1, \ldots, \beta$, the following are the axioms and rules of **LPM**:

$$\text{(R1)}\ \frac{\alpha \diamond \beta}{\beta \diamond \alpha}, \quad \text{(R2)}\ \frac{\alpha \diamond \beta \quad \beta \leftrightarrow \gamma}{\alpha \diamond \gamma}, \quad \text{(R3)}\ \frac{\alpha \diamond \beta}{\alpha \diamond \neg\beta}, \quad \text{(R4)}\ \frac{\diamond\,(\alpha, \beta, \gamma)}{\alpha \diamond \beta \odot \gamma},$$

$$\text{(R5)}\ \frac{\diamond\,(\alpha_1, \ldots, \alpha_k)}{\varphi(\alpha_1, \ldots, \alpha_k)}, \text{ where } k \geq 0 \text{ and } \varphi \text{ is a Łukasiewicz tautology},$$

$$\text{(R6)}\ \frac{\alpha \quad \alpha \to \beta}{\beta}.$$

A finite sequence $(\varphi_1, \ldots, \varphi_k)$ of propositions is an α-*admissible proof* if, for any $i = 1, \ldots, k$, either φ_i is of the form $\diamond\,\beta$ and β is a subformula of α, or φ_i is of the form $\beta \diamond \gamma$ and $\beta \odot \gamma$ is a subformula of α, or φ_i is derived by means of one of the rules (R1)–(R6) from propositions among $\varphi_1, \ldots, \varphi_{i-1}$.

A proper proposition α is called *provable* in **LPM** if there is an α-admissible proof whose last element is α. We write $\vdash \alpha$ in this case.

Theorem 3.5 *Let α be a proper proposition. Then α is provable if and only if α is valid.*

Proof: The proof works like in case of Theorem 2.5. □

It remains to demonstrate that this logic is new; it is actually not straightforward to see that the set of tautologies of a partial logic defined in the way we did, differs from the set of tautologies of the corresponding total logic. Considering, for instance, the law of distributivity of $\odot$ over $\vee$,

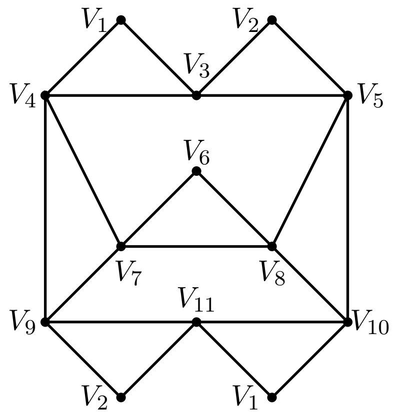
$$\alpha \odot (\beta \vee \gamma) \ \leftrightarrow\ (\alpha \odot \beta) \vee (\alpha \odot \gamma),$$

does not help to clarify the picture; it is valid not only in all MV-algebras, but also in all partial MV-algebras. The reason is the rather restrictive notion of validity; evaluations deciding about the validity of a proposition are only those which have the proposition in question in their domain. In the present case, α, β, and γ are requested to be interpreted pairwise compatibly.

The example given in [6] for the case of PBA's does not work here; the proposition

$$((\alpha \leftrightarrow \beta) \leftrightarrow (\gamma \leftrightarrow \delta)) \leftrightarrow ((\alpha \leftrightarrow \delta) \leftrightarrow (\beta \leftrightarrow \gamma))$$

is not a tautology of Łukasiewicz logic. However, another example referring to what is called a "partial algebra" in [6, Section 4], does help after the necessary modifications. Consider the following graph:



(Note that V_1 and V_2 are depicted twice.) Let F be the set of functions whose domain consists of three distinct points V_i, V_j, V_k, $i, j, k \in \{1, ..., 11\}$, such that V_i, V_j, and V_k are pairwise connected, and whose range is the real unit interval $[0, 1]$. Let $\neg$ be the Łukasiewicz negation, and let $\odot$ be the Łukasiewicz conjunction. For $f \in F$, let $\neg f \in F$ have the same domain like f, and put $(\neg f)(A) = \neg(f(A))$ for $A \in \text{dom } f$. For $f, g \in F$, let $f \diamond g$ if dom $f =$ dom g; in this case, we let $f \odot g$ have the same domain like f and g, and put $(f \odot g)(A) = f(A) \odot g(A)$ for $A \in \text{dom } f$.

Define now the equivalence relation $\approx$ on F as follows. Let $f, g \in F$ have the domains $\{A, B, C\}$ and $\{A', B', C'\}$, respectively. Let $f \approx g$ if either $f = g$, or the domains of f and g have exactly one point, say $A = A'$, in common, $f(A) = g(A')$, and $f(B) = f(C) = g(B') = g(C')$, or the domains of f and g are disjoint and $f(A) = f(B) = f(C) = g(A') = g(B') = g(C')$. Let $\bar{F} = \{\bar{f} : f \in F\}$ be the set of equivalence classes $\bar{f}$ of the functions $f \in F$. Furthermore, let $\diamond$, $\odot$, and $\neg$ be the operations on $\bar{F}$ induced by the equally named relation and operations on F. Finally, let $\bar{0}$ and $\bar{1}$ the equivalence class of some constant 0 and constant 1 function, respectively.

Note that finitely many elements $\bar{f}_1, \ldots, \bar{f}_k$ of $\bar{F}$ fulfil pairwise the $\diamond$-relation exactly if there are $f_1' \approx f_1, \ldots, f_k' \approx f_k$ with coinciding domains. It is then straightforward to check that $(\bar{F}; \diamond, \odot, \neg, \bar{0}, \bar{1})$ is a partial MV-algebra.

Let us next consider the proposition

$$(\alpha \odot \beta) \odot (\gamma \odot \delta) \leftrightarrow (\alpha \odot \delta) \odot (\beta \odot \gamma). \tag{3}$$

Clearly, (3) is valid in all MV-algebras, that is, a tautology of Łukasiewicz logic. However, (3) is not valid in all partial MV-algebras, that is, it is not a tautology of **LPM**. Consider an evaluation v with range $\bar{F}$, where $v(\alpha) = \bar{a}$, $v(\beta) = \bar{b}$, $v(\gamma) = \bar{c}$, $v(\delta) = \bar{d}$, and where $a, b, c, d \in F$ are defined as follows (we identify the functions with their graphs):

$$\begin{aligned}
a &= \{(V_1, 0.7), (V_3, 1), (V_4, 1)\} \approx \{(V_1, 0.7), (V_{10}, 1), (V_{11}, 1)\}, \\
b &= \{(V_1, 1), (V_3, 0.7), (V_4, 1)\} \approx \{(V_2, 1), (V_3, 0.7), (V_5, 1)\}, \\
c &= \{(V_2, 0.7), (V_3, 1), (V_5, 1)\} \approx \{(V_2, 0.7), (V_9, 1), (V_{11}, 1)\}, \\
d &= \{(V_2, 1), (V_9, 1), (V_{11}, 0.7)\} \approx \{(V_1, 1), (V_{10}, 1), (V_{11}, 0.7)\}.
\end{aligned}$$

Then $(\bar{a} \odot \bar{b}) \odot (\bar{c} \odot \bar{d})$ contains $\{(V_4, 0.7), (V_7, 0.4), (V_9, 0.7)\}$, hence also $\{(V_6, 0.7), (V_7, 0.4), (V_8, 0.7)\}$. On the other hand, $(\bar{a} \odot \bar{d}) \odot (\bar{b} \odot \bar{c})$ contains $\{(V_6, 0.7), (V_8, 0.4), (V_{10}, 0.7)\}$, hence also $\{(V_6, 0.7), (V_7, 0.7), (V_8, 0.4)\}$.

4 Representation of partial Boolean algebras

We now turn to the representation problem for partial Boolean algebras. We are going to characterize the standard PBA, namely, the partial algebra $(C(\mathcal{H}); \diamond, \cap_\diamond, {}^\perp, \{0\}, \mathcal{H})$, where $\mathcal{H}$ is the complex Hilbert space of countably infinite dimension, $C(\mathcal{H})$ is the set of closed subspaces of $\mathcal{H}$, $\diamond$ is the compatibility relation between subspaces, $\cap_\diamond$ denotes the intersection restricted to pairs of compatible subspaces, and $^\perp$ is the complementation function.

We begin recalling the lattice-theoretical characterization of the system of closed subspaces of a Hilbert space. As in the previous sections, we will treat $\vee$, the supremum, always as a defined operation, defined by the respective infimum and complementation operations.

Definition 4.1 Let K be a division ring endowed with the antiautomorphism $^\star : K \to K$; let $\mathcal{H}$ be a linear space over K; and let $(\cdot,\cdot)\colon \mathcal{H} \times \mathcal{H} \to K$ a semilinear form such that, for $a, b \in E$, $(a, b)^\star = (b, a)$, and $(a, a) = 0$ implies $a = 0$. Then $(\mathcal{H}, (\cdot,\cdot))$ is called a *hermitean space*.

In a hermitean space $\mathcal{H}$, let $C(\mathcal{H})$ be the system of subspaces A such that $A^{\perp\perp} = A$, where, for a subspace B, $B^\perp = \{x \in \mathcal{H} \colon (x, y) = 0 \text{ for all } y \in \mathcal{H}\}$. $(\mathcal{H}, (\cdot,\cdot))$ is called an *orthomodular space* if $\mathcal{H} = A + A^\perp$ for all $A \in C(\mathcal{H})$.

Orthomodular spaces are lattice-theoretically characterized in the following way.

Definition 4.2 An ortholattice $(L; \wedge, \neg, 0, 1)$ is called a *Hilbert lattice* if (i) L is orthomodular, i.e. for any a, b such that $a \leq b$ there is a unique $c \perp a$ such that $a \vee c = b$; (ii) L is an AC-lattice, i.e. L is atomistic and fulfils the covering property; (iii) L is irreducible; (iv) L is complete.

Theorem 4.3 *For any Hilbert lattice $(L; \wedge, \neg, 0, 1)$ of length ≥ 4, there is a uniquely determined orthomodular space $\mathcal{H}$ such that $(L; \wedge, \neg, 0, 1)$ is isomorphic to $(C(\mathcal{H}); \cap, {}^{\perp}, \{0\}, \mathcal{H})$.*

By the celebrated theorem of [12], an orthomodular space containing an infinite sequence of pairwise orthogonal vectors of the same non-zero length, is a Hilbert space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We use this result in the form presented in [8].

We need the following auxiliary notion. For some automorphism φ of a Hilbert lattice, we call an element $a \in L$ *fixed by* φ if $\varphi(x) = x$ for all $x \leq a$. Now, for some Hilbert lattice $(L; \wedge, {}^{\perp}, 0, 1)$, we consider the following conditions:

(HL1) There is a sequence $e_1, e_2, \ldots$ of pairwise orthogonal atoms.

(HL2) For any two orthogonal atoms e and f, there is an automorphism φ of L such that $\varphi(e) = f$ and $\varphi(x) = x$ for all $x \perp e, f$.

(HL3) For any $n \geq 2$ and for any automorphism φ of L such that there is an element of length ≥ 2 fixed by φ, there is an automorphism ψ such that (α) $\psi^n = \varphi$ and (β) for any atom e, $\psi(e) = e$ whenever $\varphi(e) = e$.

Lemma 4.4 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Then $(C(\mathcal{H}); \cap, {}^{\perp}, \{0\}, \mathcal{H})$, the ortholattice of closed subspaces, is a Hilbert lattice with the properties* (HL1), (HL2), *and* (HL3).

Proof: It is well-known that $C(\mathcal{H})$ is a Hilbert lattice, and it is evident that conditions (HL1) and (HL2) are fulfilled.

To see that (HL3) holds, let φ be an automorphism of $C(\mathcal{H})$ such that $\varphi(B) = B$ for all $B \leq A$, where $A \in C(\mathcal{H})$ is at least two-dimensional. By Wigner's theorem, φ is induced by a semiunitary operator U_φ. Since φ is the identity on a subspace of dimension ≥ 2, we may choose U_φ unitary [8, Lemma 1].

We may furthermore assume that $\mathcal{H} = L^2(X)$ for some compact Hausdorff space X endowed with a Radon integral, and that, for some $u \in \mathcal{L}^\infty(X)$ with values in $[0, 2\pi)$, $U(v) = e^{iu(\cdot)}v(\cdot)$, $v \in \mathcal{H}$. Then, for any non-zero $v \in \mathcal{H}$, v and $U(v)$ span the same one-dimensional subspace iff there is a $\lambda \in [0, 2\pi)$ such that $v = e^{-i\lambda}U(v)$ iff, for some $\lambda \in [0, 2\pi)$, $v(x) = e^{i(u(x)-\lambda)}v(x)$ for almost all x

iff, for some $\lambda \in [0, 2\pi)$, either $v(x) = 0$ or $u(x) = \lambda$ for almost all x. So given $n \geq 2$, we may put $V(v) = e^{i\frac{u(\cdot)}{n}} v(\cdot)$, $v \in \mathcal{H}$; then $V^n = U$, and $[U(v)] = [v]$ implies $[V(v)] = [v]$. □

Theorem 4.5 *Let* $(L; \wedge, \neg, 0, 1)$ *be a Hilbert lattice with the properties* (HL1), (HL2), *and* (HL3). *Then L is isomorphic to the ortholattice of closed subspaces of an infinite-dimensional complex Hilbert space.*

Proof: (sketched; for further details, see [8]). Let $(\mathcal{H}, (\cdot, \cdot))$ be the orthomodular space such that $(C(\mathcal{H}); \cap, ^\perp, \{0\}, \mathcal{H})$ is isomorphic to $(L; \wedge, \neg, 0, 1)$. By (HL1), there is an infinite sequence $(v_i)_{i<\omega}$ of pairwise orthogonal vectors. By (HL2), there is an automorphism φ of $C(\mathcal{H})$ mapping $[v_1]$ to $[v_2]$ and mapping the elements below $[v_i, v_j]^\perp$ to itself. Then φ is induced by a semiunitary operator U_φ of $\mathcal{H}$, which, according to the same argumentation as above, can in fact be chosen unitary. In particular, v_1 and $U_\varphi(v_1) \in [v_2]$ are of the same length, and repeating the argument, we conclude that $\mathcal{H}$ possesses an orthonormal basis. By Solèr's theorem, $\mathcal{H}$ is a Hilbert space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Let now $A \in C(\mathcal{H})$ such that A and $A^\perp$ are both at least two-dimensional. Let G be the set of those automorphisms φ of $C(\mathcal{H})$ such that $\varphi(B) = B$ for all $B \in C(\mathcal{H})$ contained in A or $A^\perp$. Let C_K be the centre of K and $E_K = \{k \in C_K : \, kk^\star = 1\}$. Then each $g \in G$ is induced by a unitary operator of the form U_k for some $k \in E_K$, where $U_k(u + v) = ku + v$ if $u \in A$ and $v \in A^\perp$. By (HL3), for each $\varphi \in G$, there is $\psi \in G$ such that $\psi^2 = \varphi$. If now $K = \mathbb{R}$ or $K = \mathbb{H}$, then $E_K = \{-1, 1\}$, so (HL3) cannot be fulfilled. Consequently, $K = \mathbb{C}$, in which case $E_K = SO(2)$. □

We note that the condition (HL3) for Hilbert lattices, which refers to automorphisms φ such that an element of dimension ≥ 2 is fixed by φ, might be replaceable by the requirement that, the appropriate topology on the set of automorphisms being given, φ belongs to the identity component. However, here we will not to work with topological notions.

We are now going to reformulate the above representation theorem for partial Boolean algebras. As the first step, the notion of a PBA will be slightly modified; the following notion of a tPBA is due to [1]. However, we use axioms which differ from [1] and which moreover do not extend those in Definition 2.1; we rather prefer to provide an independent picture, which is as appropriate as possible in the present context.

Definition 4.6 The structure $(L; \leq, \diamond, 0, 1)$ is called a *transitive partial Boolean* *algebra*, or *tPBA* for short, if the following conditions hold:

(tPB1) $(P; \diamond, \leq, 0, 1)$ is a bounded poset.

(tPB2) $\diamond$ is a symmetric and reflexive binary relation. Elements a and b such that $a \diamond b$ are called *compatible*.

(tPB3) (α) Let $a, b \in P$ such that $a \leq b$. Then $a \diamond b$, and for exactly one element d compatible with a we have $a \wedge d = 0$ and $a \vee d = b$. d is then compatible with all those elements which are compatible with a and b.

(β) Any set A of pairwise compatible elements possesses an infimum $\bigwedge A$ and a supremum $\bigvee A$. $\bigwedge A$ and $\bigvee A$ are compatible with all those elements which are compatible with every $a \in A$.

(γ) For any pairwise compatible elements $a, b, c \in P$, we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Let $\mathcal{H}$ be the complex Hilbert space of countably infinite dimension; then $(C(\mathcal{H}); \diamond, \subseteq, \{0\}, \mathcal{H})$ is a tPBA, called the *standard tPBA*.

As should be expected, tPBA's may be understood as special PBA's, as shown in the next two lemmas.

Lemma 4.7 *Let $(L; \leq, \diamond, 0, 1)$ be a tPBA. For any $a, b \in L$, in accordance with* (tPB3)(β)*, put $a \wedge' b = a \wedge b$ if $a \diamond b$, and leave $\wedge'$ undefined otherwise. Moreover, for any $a \in L$, in accordance with* (tPB3)(α)*, let $\neg a$ be the unique complement of a compatible with a. Then $(L; \diamond, \wedge', \neg, 0, 1)$ is a PBA fulfilling the following conditions:*

(PB4) *If, for some $a, b, c \in L$, $a = a \wedge' b$ and $b = b \wedge' c$, then $a = a \wedge' c$.*

(PB5) *Let B be a maximal subset of L consisting of pairwise compatible elements. Then $0, 1 \in B$, and $(B; \wedge', \neg, 0, 1)$ is a complete Boolean algebra.*

Moreover, let $A \subseteq B$, and let B' be a further maximal set of pairwise compatible elements such that $A \subseteq B'$. Then the infimum of A in B and the infimum of A in B' coincide.

Proof: (PB1) holds by (tPB2). (PB2) as well as (PB4) hold by construction.

Let A be a non-empty set of pairwise compatible elements. (tPB3)(α) implies that we may join 0 and 1 to A, and still all elements are pairwise compatible. Again by (tPB3)(α), when joining, for any $a \in A$, the element $\neg a$ to A, the resulting set still consists of pairwise compatible elements. Similarly, by (tPB3)(β), the infimum and supremum of an arbitrary subset of A exists and can be joined to A with the same effect. Denote by $\bar{A}$ the closure of $A \cup \{0, 1\}$ under $\neg$ as well as arbitrary infima and suprema. Then $(\bar{A}; \leq)$ is a complete lattice, $0, 1 \in \bar{A}$, and every element a possesses $\neg a$ as the unique complement in $\bar{A}$. Finally, by (tPB3)(γ), $\bar{A}$ is distributive, hence a Boolean algebra. (PB3) and the first part of (PB5) follow. Furthermore, infima in $\bar{A}$ are infima w.r.t. $\leq$; so also the second part of (PBA5) follows. □

Lemma 4.8 *Let* $(L; \diamond, \wedge', \neg, 0, 1)$ *be a PBA fulfilling conditions* (PB4) *and* (PB5). *For* $a, b \in L$, *let* $a \leq b$ *if* $a = a \wedge' b$. *Then* $(L; \leq, \diamond, 0, 1)$ *is a tPBA.*

Proof: By (PB4), $\leq$ is transitive, and (tPB1) follows. Note next that, by construction, $a \leq b$ implies $a \diamond b$. Moreover, for $a, b \in L$ such that $a \diamond b$, $a \wedge' b$ is the infimum w.r.t. $\leq$; indeed, $a \wedge' b \leq a, b$, and if $x \leq a, b$, then $x, a, b, a \wedge' b$ are pairwise compatible, and it follows $x \leq a \wedge' b$. So if $a \diamond b$, then $a \wedge' b = a \wedge b$ and, similarly, $a \vee' b = a \vee b$.

(tPB2) holds by (PB1). To prove (tPB3)(α), let $a \leq b$. Then $a \diamond b$; so a and b generate a Boolean subalgebra, and we conclude that, putting $d = \neg a \wedge b$, d is compatible with a, furthermore $a \vee' d = b$, whence $a \vee d = b$, and $a \wedge' d = 0$, whence $a \wedge d = 0$. d is the unique element with these three properties; indeed, if $d' \diamond a$, $a \vee d' = b$ and $a \wedge d' = 0$, then a, b, d' are pairwise compatible, so $a \vee' d' = b$ and $a \wedge' d' = 0$ and consequently $d' = \neg a \wedge' b = d$. Finally, $x \diamond a, b$ implies $x \diamond d$ by (PB3). The proof of (tPB3)(α) is complete.

Let now A be a set of pairwise compatible elements, and let $c \diamond a$ for all $a \in A$. Let B be a maximal set of pairwise compatible elements which contains $A \cup \{c\}$. By (PB5), $(B; \wedge', \neg, 0, 1)$ is a complete Boolean algebra; let b be the infimum of A in B. Certainly, $c \diamond b$. We claim that $b = \bigwedge A$, where the infimum refers to the partial order $\leq$ of L. Clearly, $b \leq a$ for all $a \in A$. Let $x \in L$ such that $x \leq a$ for all $a \in A$. Extend $A \cup \{x\}$ to a maximal set B' of pairwise compatible elements. Then $x \leq b'$, where b' is the infimum of A taken in B'. But by (PB5), $b' = b$ and hence $x \leq b$. So the part of (tPB3)(β) concerning infima is proved, and by the self-duality of Boolean algebras, the other half follows as well.

(tPB3)(γ) holds by (PB3). □

We next restrict tPBA's with respect to cardinality. We wish that any Boolean subalgebra does not contain more than countably many independent elements; and it shall possibly contain, on the other hand, any finite number of independent elements.

A further requirement is that there is no element different from 0 and 1 which may be joined to any given Boolean subalgebra.

Definition 4.9 *Let $(L; \leq, \Diamond, 0, 1)$ be a tPBA. We call L of countably infinite size if:*

(i) *L, as a poset, is atomistic.*

(ii) *Any set of atoms every pair of which has infimum zero, is countable.*

(iii) *There is a countably infinite set of atoms every pair of which has infimum zero.*

Furthermore, we call L irreducible if for any a, there is a b incompatible with a.

The crucial condition on tPBA's comes next: We postulate the existence of sufficiently many automorphisms. We define an element to be fixed by an automorphism in analogy to the case of Hilbert lattices.

Definition 4.10 *Let $(L; \leq, \Diamond, 0, 1)$ be a tPBA. Put $a \perp b$ if $a \Diamond b$ and $a \wedge b = 0$ We call L flexible if:*

(i) *For any $a \in L$ and any atom $e \in L$ not below a, there is an automorphism φ such that (α) $\varphi(e) \perp a$ and (β) $\varphi(x) = x$ whenever $x \perp a, e$ or $x \perp a, \varphi(e)$.*

(ii) *For any $n \geq 2$ and any automorphism φ such that some element of length ≥ 2 is fixed by φ, there is an automorphism ψ such that (α) $\psi^n = \varphi$ and (β) $\psi(e) = e$ whenever $\varphi(e) = e$ for any atom e.*

We next check that the standard tPBA has the properties just defined.

Lemma 4.11 *Let $\mathcal{H}$ be a complex Hilbert space of countably infinite dimension. Then $(C(\mathcal{H}); \subseteq, \Diamond, \{0\}, \mathcal{H})$ is a tPBA which is of countably infinite size, flexible, and irreducible.*

Proof: We have already noticed that $C(\mathcal{H})$ is a tPBA. Clearly, $C(\mathcal{H})$ is of countably infinite size and irreducible. Moreover, let U be a unitary operator of $\mathcal{H}$, and let φ_U be the map induced by U on $C(\mathcal{H})$. Then evidently, φ_U is an automorphism of the tPBA $C(\mathcal{H})$. This fact given, part (i) of the definition of flexibility is easily checked, and part (ii) follows from Lemma 4.4. $\square$

Lemma 4.12 *Let $(L; \leq, \diamond, 0, 1)$ be a flexible tPBA of countably infinite size. Then, L is lattice-ordered. Furthermore, for $a \in L$, let $\neg a$ be the unique element such that $\neg a \diamond a$, $a \wedge \neg a = 0$, and $a \vee \neg a = 1$. Then, $(L; \wedge, \neg, 0, 1)$ is a complete orthomodular AC-lattice.*

Moreover, if L is irreducible as a tPBA, then so is L as an ortholattice.

Proof: Note first that the definition of $\neg$ is possible due to (tPB3)(α). As is furthermore easily seen, $\neg$ is a complementation function. Note also that $a \perp b$ iff $a \leq \neg b$.

We show next that for some $a \in L$ and an atom $e \in L$, the supremum $a \vee e$ exists in L. By flexibility, there is an automorphism φ such that $\varphi(e) \perp a$ and $\varphi(x) = x$ if $x \perp a$ and either $x \perp e$ or $x \perp \varphi(e)$. Then $a \vee \varphi(e)$ exists, and we claim that this is the supremum of a and e. Clearly, $a \leq a \vee \varphi(e)$. Furthermore, from $\neg(a \vee \varphi(e)) \perp a, \varphi(e)$, we conclude that $e \leq \varphi^{-1}(a \vee \varphi(e)) = a \vee \varphi(e)$. Let $a, e \leq c$. Then $\neg c \perp a, e$, so $\varphi(e) \leq \varphi(c) = c$ and $a \vee \varphi(e) \leq c$.

To see that L is a complete lattice, it is, by the atomicity of L, sufficient to prove that an arbitrary set e_ι, $\iota \in I$, of atoms possesses a supremum. Let $J \subseteq I$ such that $\bigvee_{\iota \in J} e_\iota$ exists. For any $\kappa \in I \setminus J$, then by the preceding paragraph, also $\bigvee_{\iota \in J \cup \{\kappa\}} e_\iota$ exists. Furthermore, any chain in L consists of pairwise compatible elements and thus possesses a supremum. By Zorn's Lemma, the supremum $\bigvee_{\iota \in I} e_\iota$ exists.

So $(L; \wedge, \neg, 0, 1)$ is shown to be a complete ortholattice. By (tPB3)(α), it is clear that $(L; \wedge, \neg, 0, 1)$ is actually an orthomodular lattice. Furthermore, from the second paragraph, it is easily seen that L fulfils the covering property. So L is an AC-lattice.

Finally, let L, as an ortholattice, be reducible. Then there is an $a \in L$ such that $0 < a < 1$ and for any $b \in L$, we have $b = (b \wedge a) \vee (b \wedge \neg a)$. From $b \wedge a \leq a$, $b \wedge \neg a \leq \neg a$, $b \wedge a \leq \neg(b \wedge \neg a)$, we have that $a, b \wedge a, b \wedge \neg a$ are pairwise compatible; in particular, a is compatible with b. So L is not irreducible as a tPBA. The proof is complete. $\square$

Lemma 4.13 *Let $(L; \leq, \diamond, 0, 1)$ be a flexible tPBA of countably infinite size, and let φ be an automorphism of L. Then φ is also an automorphism of L as an ortholattice.*

Proof: φ preserves the partial order as well as $\perp$. $\square$

Theorem 4.14 *Let* $(L; \leq, \diamond, 0, 1)$ *be a tPBA which is of countably infinite size, flexible, and irreducible. Then* L *is isomorphic to the standard tPBA.*

Proof: We may define $\neg$ as in Lemma 4.12; then, $\wedge$ being the infimum operation, $(L; \wedge, \neg, 0, 1)$ is a Hilbert lattice.

We have to verify the conditions (HL1)–(HL3) for Hilbert lattices. (HL1) holds because L is countably infinite size. In view of Lemma 4.13, we may identify automorphisms of L with automorphisms of L viewed as an ortholattice. Let e, f be orthogonal atoms; by irreducibility, there is a third atom g distinct from e, f such that $g \leq e \vee f$. By flexibility, there are automorphisms φ_1, φ_2 such that $\varphi_1(g) = f,\ \varphi_2(g) = e$, and $\varphi_1(x) = \varphi_2(x) = x$ if $x \perp e \vee f$. So taking $\varphi_1 \circ \varphi_2^{-1}$, we see that (HL2) holds. (HL3) clearly follows from flexibility as well.

By Theorem 4.5, $(L; \wedge, \neg, 0, 1)$ is isomorphic to $(C(\mathcal{H}); \cap, {}^{\perp}, \{0\}, \mathcal{H})$ for some $\aleph_0$-dimensional complex Hilbert space $\mathcal{H}$. We claim that the compatibility relation is in both cases the same. Indeed, in the tPBA L, if $a \diamond b$, then a and b generate a Boolean subalgebra, so evidently $a = a_0 \vee c$ and $b = b_0 \vee c$ for mutally orthogonal elements a_0, b_0, c. Conversely, if a and b possess this representation, then a_0, b_0, c are pairwise compatible, so also $a \diamond b$. The proof of the theorem is complete. $\square$

5 Conclusion

We have studied two different aspects of the theory of partial Boolean algebras (PBA's). In the first part, we re-considered a logic based on PBA's, introduced in [6] and called **LPB** in this paper. The calculus **LPB** reflects the possible logical considerations with respect to testable yes-no statements about a quantum-physical system, and the strict rule is followed that logical interrelations are applicable only for compatible statements. To include unsharp statements as well, we have defined the logic **LPM**; **LPM** may be regarded as a fuzzy version of **LPB** and is based on partial MV-algebras. The second aspect which we studied concerned the representation of PBA's. Namely, we characterized the standard PBA, the partial algebra of closed subspaces of an $\aleph_0$-dimensional complex Hilbert space.

Both parts of this note follow the logico-algebraic approach to the foundation of quantum physics. The idea of this approach is to justify the basic model used in quantum physics – the complex Hilbert space. However, we would like to underline that, whereas the work as presented in this paper might be considered interesting from a mathematical point of view, we guess that its value for the foundational debate on quantum physics is limited.

Let us explain our viewpoint, first with regard to logics. We are convinced that the means provided by some appropriate logical calculus are not sufficient to cope with the fundamental interpretational difficulties connected to quantum physics. The most important limitation is reflected in the choice of the present paper's subject: logical interrelations can hold among propositions which can be considered the same time, but it does not make sense to treat incompatible propositions by logical methods. All we can do in the framework of logics is to exhibit just this pure fact: not all propositions may be considered the same time. However, the crucial requirement in physics is to model the change from one observational framework to another one, that is, the change from one set of jointly testable propositions to another such set. In this respect, logics are most likely not of help.

Second, there are related problems with respect to the research on quantum structures. This line is possibly more fruitful, simply because there are more possibilities available to specify a structure. However, a problem analogous to the one already mentioned, exists in this case as well: the restriction to a language of algebra. We followed this line here as well, although we included conditions which else seem to be avoided: those concerning automorphisms.

To make a real progress in quantum structures with respect to the original aims of the field, we rather think that it is the characterization of the automorphisms which is of fundamental importance. What we propose for the future is to work towards a justification of this viewpoint. Quite a lot of work has been done to understand those entities of Hilbert space which model yes-no tests; quite a few work was done for the question how to characterize its automorphism group. To characterize the unitary group - again by algebraic means - seems to be extremely difficult, and this is probably also the case for any kind of linear group; cf. e.g. [7]. At least to the author, only very few results are known, and those which are known are probably not (yet) well usable for the aim of better understanding the quantum-mechanical formalism.

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