Residuated lattices arising from equivalence relations on Boolean and Browerian algebras

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Abstract

Logics designed to deal with vague statements typically allow algebraic semantics such that propositions are interpreted by elements of residuated lattices. The structure of these algebras is in general still unknown, and in the cases that a detailed description is available, to understand its significance for logics can be difficult. So the question seems interesting under which circumstances residuated lattices arise from simpler algebras in some natural way. A possible construction is described in this paper.

Namely, we consider pairs consisting of a Browerian algebra (i.e. a dual Heyting algebra) and an equivalence relation. The latter is assumed to be in a certain sense compatible with the partial order, with the formation of differences, and with the formation of suprema of pseudoorthogonal elements; we then call it an s-equivalence relation. We consider operations which, under a suitable additional assumption, naturally arise on the quotient set. The result is that the quotient set bears the structure of a residuated lattice. Further postulates lead to dual BL-algebras. In the case that we begin with Boolean algebras instead, we arrive at dual MV-algebras.

Keywords: Browerian algebra, Heyting algebra, Boolean algebra, s-equivalence relation, divisible residuated lattice, BL-algebra, MV-algebra

1 Introduction

In recent years, fuzzy logics and their algebraic aspects have been a field of intensive research. Among the currently available monographs we may mention [COM], [Haj], and [Got]. A characteristic property of commonly known logics dealing with vagueness is that they possess an alternative algebraic semantics; in the typical case, a variety containing bounded, integral commutative residuated lattices can be used.

Actually, residuated lattices are well suitable to interpret vague propositions. Namely, we expect that a set of propositions is lattice-ordered and endowed with a monotonous monoidal operation; the order reflects the strength of propositions, and the monoidal operation plays the role of a conjunction. Furthermore, in order to base a logic on ℓ-monoids, a further condition should hold: for any two propositions α and β, there is a third one, namely $\alpha \to \beta$, which is the weakest proposition γ such that α together with γ implies β; and this is exactly what the residuation property expresses.

It would be of great interest to know what the structure of an arbitrary residuated lattice is like. For the known facts, consult for instance [JiTs] and the references given there. At the time being, only certain subclasses are well understood, like the BL-algebras [AgMo]. In these case, however, it is still not evident how the discovered structural properties are related to the assumption that we have to do with a model of vague propositions.

These considerations motivate us to study the question if residuated lattices can be represented by means of other, better known structures. Several approaches to this question can be found in the literature already; most importantly, we may mention the representations by means of algebras of relations. For instance, see [BuWo] and the references given there for the representation of residuated posets. Furthermore, representations of quantales are developed in [Val]. Finally, Jipsen defined residuated lattices of binary relations and showed that this class contains, up to isomorphism, the BL-algebras [Jip].

Our own construction is based alternatively on Boolean algebras or Browerian algebras, by which in this paper we mean the duals of Heyting algebras. These algebras are assumed to be endowed with a so-called s-equivalence relation, and the quotient set is endowed with monoidal operations which arise from the pointwise connection of two equivalence classes. Starting with Boolean algebras, we arrive at MV-algebras. Otherwise, we are led to residuated lattices, and further conditions ensure that divisibility and prelinearity hold, that is, that we get BL-algebras.

The present paper continues certain previous work. We could say that, compared to what we proposed earlier, the present approach rises the level of abstraction. The paper [Vet2] was based on Boolean algebras endowed with a "symmetry group" modelling ambiguity. In the present case we refer to an equivalence relation which is possibly, but not by assumption, induced by such a group. In the older work [Vet1], we assumed a Boolean algebra to be endowed with a measure. Interestingly, in the present framework, the existence of a measure is implied as well. But again, we work here directly with an equivalence relation, whose properties are not necessarily derivable from the structure of the measure. Note in addition that the inclusion of Browerian algebras is new in the present work.

The paper is organized as follows. After preliminary remarks (Section 2), we are first concerned with Boolean algebras. On a Boolean algebra, we define what we call an s-equivalence relation (Section 3). We then wonder which kind of operations we may naturally define on the quotient set; more specifically, we wonder which equivalence class is naturally associated to the set arising from pointwise connecting two equivalence classes by a binary Boolean operation. Under an appropriate assumption called residuability, the quotient set is lattice-ordered and has actually the structure of a residuated lattice. In fact, what we get is a dual MV-algebra, and every totally ordered dual MV-algebra can be the result of the construction (Section 4). We finally treat the question of sufficient conditions for residuability (Section 5).

In the second half of the paper, we present our results concerning Browerian algebras, which take over the role of Boolean algebras. This theory is more involved. We are led to residuated lattices, and under two additional assumptions to dual BL-algebras (Sections 6, 7, 8).

2 Residuated lattices and fuzzy logic

We are going to develop in this paper a particular way to construct bounded, integral, and commutative residuated lattices. Among the represented algebras, there are all the totally ordered BL-algebras, the latter being the counterpart of Hájek's Basic Logic. In particular, there are all totally ordered MV-algebras, the latter being the counterpart of Łukasiewicz logic.

As a motivational background, we recall in this section where these algebras originate and what we know about their structure. Apart from that, however, the implications of our results for many-valued logics will not be elaborated here. We restrict to the algebraic aspects and might treat the interpretational issue in a further paper.

Both mentioned types of algebras are associated with specific fuzzy logics: the Łukasiewicz Logic ŁL [COM, Haj] and the Basic Fuzzy Logic BL [Haj]. Their common language is $\{\odot, \to, 0\}$, where $\odot$ and $\to$ are binary connectives and 0 is a constant. Their canonical semantics is based on t-norms, that is, binary operations on the real unit interval $[0,1]$ making $([0,1]; \wedge, \vee, \odot, 1)$ an ℓ-monoid w.r.t. the natural order of the reals.

More specifically, let $(\mathcal{F}; \odot, \to, 0)$ be the algebra of propositions of ŁL and BL. Then ŁL is the logic proving those propositions to which all structure-preserving mappings from $(\mathcal{F}; \odot, \to, 0)$ to $([0,1]; \odot_L, \to_L, 0)$ assign 1, where $\odot_L \colon [0,1]^2 \to$

$[0,1]$, $(a,b) \mapsto (a+b-1) \vee 0$ and $\to_L$: $[0,1]^2 \to [0,1]$, $(a,b) \mapsto (1-a+b) \wedge 1$. Moreover, BL is the logic proving those propositions to which all structure-preserving mappings from $(\mathcal{F}; \odot, \to, 0)$ to $([0,1]; \bar{\odot}, \bar{\to}, 0)$ assign 1, where $\bar{\odot}$ is any continuous t-norm and $\bar{\to}$ the residuum corresponding to $\bar{\odot}$. Hilbert-style proof systems for these logics can be found e.g. in [COM] and [Haj].

The equivalent algebraic semantics for ŁL is based on MV-algebras, the variety generated by the Łukasiewicz algebra $([0,1]; \wedge, \vee, \odot_L, \to_L, 0, 1)$. The algebraic semantics for BL is based on the BL-algebras, the variety generated by all continuous t-norm algebras $([0,1]; \wedge, \vee, \bar{\odot}, \bar{\to}, 0, 1)$.

At this point, we have to stress that we are going to work with the dual algebras. The main reason to reverse the order is that the argumentation of this paper is in this way considerably easier to comprehend. So what we will construct are actually the algebras in the variety generated by the Łukasiewicz conorm-algebra or by the continuous t-conorm algebras, respectively.

The order in propositional logics is a matter of agreement, and we do not think that the reversed order is necessarily counterintuitive. Recall that, following the usual viewpoint, propositions are associated with the set of possible situations in which they hold true; accordingly, the weaker propositions are considered the larger ones. However, we may just as well adapt the opposite viewpoint, namely that asserting a proposition excludes certain possibilities. Since the weaker propositions exclude less possibilities than stronger propositions, it is not unreasonable to consider the former as the smaller ones.

So our main notion are bounded, integral, and commutative dual residuated lattices. However, we will omit the attributes bounded, integral, and commutative.

Definition 2.1 An algebra $(L; \wedge, \vee, \oplus, \ominus, 0, 1)$ is called a *dual residuated lattice* if $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, $(L; \oplus, 0)$ is a commutative monoid, and $a \leq b \oplus c$ iff $a \ominus b \leq c$ for all $a, b, c \in L$. Moreover, L is called *divisible* if $a \vee b = (a \ominus b) \oplus b$; L is called *prelinear* if $(a \ominus b) \wedge (b \ominus a) = 0$; L is called *involutive* if $1 \ominus (1 \ominus a) = a$.

A divisible, prelinear dual residuated lattice is called a *dual BL-algebra*. An involutive dual BL-algebra is called a *dual MV-algebra*.

Concerning the structure of the two algebras mentioned last, the first basic fact is that every dual MV- or BL-algebra is the subdirect product of totally ordered algebras of the respective type. Second, the totally ordered algebras are in both cases conveniently described by means of Abelian groups.

Theorem 2.2 *Let $(G; \leq, +, 0)$ be a totally ordered Abelian group, and let $u \geq 0$. Let $L = \{g \in G\colon\ 0 \leq g \leq u\}$. For $g, h \in L$, let*

$$g \oplus h = (g+h) \wedge u, \qquad g \ominus h = (g-h) \vee 0.$$

Then $(L; \wedge, \vee, \oplus, \ominus, 0, u)$ is a totally ordered dual MV-algebra. Up to isomorphism, every totally ordered dual MV-algebra arises in this way.

We note that we can describe dual MV-algebras in the same way even in the general case, by admitting lattice-ordered groups instead of totally ordered ones [Mun, DvPu].

For a proof of the following theorem, see e.g. [AgMo].

Theorem 2.3 *Let $(I; \leq)$ be a totally ordered set. For every $\iota \in I$, let $(G_\iota; \leq, +_\iota, 0_\iota)$ be a totally ordered Abelian group, and define the algebra $(L_\iota; \oplus_\iota, \ominus_\iota, 0_\iota)$ as follows: Either let $L_\iota = \{g \in G_\iota \colon\ 0 \leq g \leq u_\iota\}$ for some $u_\iota \geq 0$ and define, for $g, h \in L_\iota$, $g \oplus_\iota h = (g +_\iota h) \wedge u_\iota$ and $g \ominus_\iota h = (g -_\iota h) \vee 0_\iota$. Or let $L_\iota = G_\iota^+$ and define, for $g, h \in L_\iota$, $g \oplus_\iota h = g +_\iota h$ and $g \ominus_\iota h = (g -_\iota h) \vee 0_\iota$.*

Let then $L = \dot{\bigcup}(L_\iota \setminus \{0_\iota\}) \mathbin{\dot\cup} \{0\}$, where 0 is a new element. Extend the total order from the L_ι to a total order on L as follows: if $g \in L_\iota$ and $h \in L_\kappa$ for distinct indices ι and κ, then $g \leq h$ if $\iota < \kappa$; and 0 is the bottom element of L. Define, for $g, h \in L$

$$g \oplus h = \begin{cases} g \oplus_\iota h & \text{if } g, h \in L_\iota, \\ g \vee h & \text{else;} \end{cases} \qquad g \ominus h = \begin{cases} 0 & \text{if } g \leq h \\ g \ominus_\iota h & \text{if } g > h \text{ and } g, h \in L_\iota, \\ g & \text{else.} \end{cases}$$

Assume now that I has the top element λ and that L_λ has the top element u_λ. Then $(L; \wedge, \vee, \oplus, \ominus, 0, u_\lambda)$ is a totally ordered dual BL-algebra. Up to isomorphism, all totally ordered dual BL-algebras arise in this way.

3 s-Equivalence relations on Boolean algebras

We describe in this and the subsequent section a framework suitable for the representation of dual MV-algebras. We will be concerned with a pair consisting of a Boolean algebra and what we will call an s-equivalence relation.

A Boolean algebra is a structure $(L; \wedge, \vee, \neg, 0, 1)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is a complementation function. We say that two elements a, b of a Boolean algebra are orthogonal if $a \wedge b = 0$, and in this case we write $a \perp b$. Furthermore, we denote the difference of a and b by $a \setminus b = a \wedge \neg b$.

Definition 3.1 Let $(L; \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. Let $\sim$ be an equivalence relation $\sim$ on L, and write $a \precsim b$ if $a' \leq b'$ for some $a' \sim a$ and $b' \sim b$. $\sim$ is called a *s-equivalence relation* on L if the following conditions hold:

(B1) If $a \precsim b$, then there is a $a' \sim a$ such that $a' \leq b$.

(B2) Let $a \sim a'$, $b \sim b'$ and $a \leq b$, $a' \leq b'$. Then $b \setminus a \sim b' \setminus a'$.

(B3) Let $a \sim a'$, $b \sim b'$ and $a \perp b$, $a' \perp b'$. Then $a \vee b \sim a' \vee b'$.

On the informal level, the relation $\sim$ should be thought of identifying elements of equal size w.r.t. some measure; the typical example will be given below. In particular, when considering a Boolean algebra as a system of (sharp) propositions, an s-equivalence relation may serve to identify two propositions of equal expressive strength – the letter 's' refers to the word 'strength'. Concerning the interpretation of fuzzy logics along these lines, see [Vet1].

Given an s-equivalence relation $\sim$ on an algebra L, we will in the sequel denote the $\sim$-equivalence class of some element a by $[a]$, and $[L] = \{[a]\colon\; a \in L\}$ will be the quotient set w.r.t. $\sim$. – Let us see first how the partial order $\leq$ on L is related to the relation $\precsim$.

Lemma 3.2 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L. Let $a, b, c \in L$. Then $a \precsim b \precsim c$ and $a \leq c$ imply that there is a $b' \sim b$ such that $a \leq b' \leq c$.*

Proof. Let $a \precsim b \precsim c$. Then, by (B1), there are $a' \sim a$ and $b' \sim b$ such that $a' \leq b' \leq c$. Let $d = a \setminus b'$. By (B2), $a \setminus a' = (a \vee a') \setminus a' \sim (a \vee a') \setminus a = a' \setminus a$; so $d \precsim a' \setminus a$ and $d' \leq a' \setminus a$ for some $d' \sim d$. Let $b'' = (b' \setminus d') \vee d$. Then $a \leq b'' \leq c$ and, by (B3), $b'' \sim b' \sim b$. □

It follows that an s-equivalence relation of a Boolean algebra L is a congruence of L seen as bounded poset. The induced order will be denoted by $\leq$ as well.

Lemma 3.3 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L. Then, by setting*

$$[a] \leq [b] \quad \textit{if} \quad a \precsim b \quad \textit{for } a, b \in L$$

$([L]; \leq, [0], [1])$ is a bounded poset, and $\iota\colon\; L \to [L],\;\; a \mapsto [a]$ is a surjective homomorphism of bounded posets.

Proof. By definition, $a \leq b$ implies $a \precsim b$ for $a, b \in L$. So reflexivity of $\precsim$ is immediate, and for the transitivity see the first line of the proof of Lemma 3.2. Furthermore, $a \precsim b \precsim a$ implies by Lemma 3.2 that $a \leq b' \leq a$ for some $b' \sim b$, whence $a \sim b$. It follows that $[L]$ is a bounded poset. The last statement is obvious. □

Furthermore, (B2) implies that $\sim$ is compatible with the complementation; we have $a \sim b$ iff $\neg a \sim \neg b$. However, an s-equivalence relation is in general not compatible with the lattice operations. We should rather have the picture in mind that $\sim$ identifies elements of equal extension.

Example 3.4 Let $(L; \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. Moreover, let $\mu\colon\; L \to [0, 1]$ be a measure on L in the real unit interval $[0, 1]$; this means (i) $\mu(a \vee b) = \mu(a) + \mu(b)$ for $a, b \in L$ such that $a \perp b$, and (ii) $\mu(1) = 1$. Call the measure μ *homogeneous* if for any $a, b \in L$ such that $\mu(a) \leq \mu(b)$, there is an $a' \leq b$ such that $\mu(a') = \mu(a)$.

Now, for $a, b \in L$, define $a \sim b$ if $\mu(a) = \mu(b)$. It is evident that $\sim$ is an s-equivalence relation on L if μ is homogeneous. Note that $\sim$ has the additional property that no a is equivalent to any element strictly below a. The equivalence classes are in a one-to-one correspondence with the range of μ; in particular, the induced order is total.

We devote the remainder of this section to the question to what extent this example applies in the general case. For the further procedure in this paper, however, we will not make use of this information; we just like to suggest an appropriate intuitive picture associated with the formalism. The reader not interested in these considerations may continue with Section 4.

Definition 3.5 Let L be a Boolean algebra, and let $(G; \wedge, \vee, +, 0, u)$ – denoted by (G, u) for short – be a unital Abelian ℓ-group, that is, a lattice-ordered Abelian group endowed with a strong unit $u \in G^+$. Then a map $\mu\colon\ L \to G$ is called a *G-valued measure* on L if (i) $\mu(a \vee b) = \mu(a) + \mu(b)$ for $a, b \in L$ such that $a \perp b$, and (ii) $\mu(1) = u$.

We will associate a group-valued measure to a Boolean algebra endowed with an s-equivalence relation. To this end, we need some preparatory material. The notion of an effect algebra is due to Foulis and Bennett [FoBe]. For more information on these algebras and also a shorter axiom system, we refer to the monograph [DvPu].

Definition 3.6 An effect algebra is a structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0, 1 are constants such that the following holds:

(E1) $(a+b)+c$ is defined iff $a+(b+c)$ is defined, and in this case $(a+b)+c = a+(b+c)$;

(E2) $a+b$ is defined iff $b+a$ is defined, and in this case $a+b = b+a$.

(E3) If $a+c$ and $b+c$ are defined and coincide, then $a = b$.

(E4) $a+0$ is defined for every a and equals a;

(E5) If $1+a$ is defined, then $a = 0$;

(E6) For every a there is a b such that $a+b = 1$.

In addition, we define $a \leq b$ if $a+c = b$ for some c. Furthermore, we denote the complement of a specified by (E6), which is unique by (E3), by $\neg a$.

The relation $\leq$ makes an effect algebra a bounded poset, 0 being the bottom and 1 being the top element.

Effect algebras are modelled upon intervals in partially ordered groups.

Definition 3.7 Let (G, u) be a unital Abelian ℓ-group. Let

$$G[0,u] = \{a \in G\colon\ 0 \leq a \leq u\}$$

be the *unit interval* of (G, u). Define on $G[0, u]$ the partial addition $+$ as follows: For $a, b \in G[0, u]$, let $a+b$ be defined if $a+b \leq u$, and let $a+b$ in this case coincide with the group sum; otherwise, let $a + b$ be undefined. Then $(G[0, u]; +, 0, u)$ is called an *interval effect algebra*.

It is straightforward that an interval effect algebra is an effect algebra. Conversely, how to characterize the interval effect algebra among the effect algebras algebraically is still an open problem. We have the following sufficient condition [Rav]; the notation is chosen in accordance with [DvVe1].

Definition 3.8 Let $(E; +, 0, 1)$ be an effect algebra. We say that E fulfills the *strong Riesz decomposition property*, or (RDP$_2$) for short, if for any $a, b, c, d \in E$ such that $a + b = c + d$ there are $e_1, \ldots, e_4$ such that

$$\begin{array}{cccc} e_1 & e_2 & \rightarrow & c \\ e_3 & e_4 & \rightarrow & d \\ \downarrow & \downarrow & & \\ a & b. & & \end{array} \tag{1}$$

By the scheme (1) to hold, we mean that each row and column adds up to what the respective arrow points to.

The following theorem, and even a slightly stronger version of it, is due to K. Ravindran [Rav]. A proof can also be found in [DvVe2].

Theorem 3.9 *Every effect algebra fulfilling* (RDP$_2$) *is an interval effect algebra.*

Theorem 3.10 *Let $\sim$ be an s-equivalence relation on the Boolean algebra L. Then there is a unital Abelian ℓ-group (G, u) and a G-valued measure $\mu\colon\ L \to G[0, u]$ on L such that, for all $a, b \in L$, $\ \mu(a) = \mu(b)$ iff $a \sim b$.*

Proof. On $[L]$, we may because of the axiom (B3) define a partial addition as follows. For $a, b \in L$, let $[a] + [b] = [a' \vee b']$ in case $a' \perp b'$ for some $a' \sim a$ and $b' \sim b$; else let the sum be undefined. We claim that then $([L]; +, [0], [1])$ is an effect algebra fulfilling (RDP$_2$).

It is not difficult to see that if $([a] + [b]) + [c]$ is defined, there are pairwise disjoint elements $a' \sim a$, $\ b' \sim b$, and $c' \sim c$; the associativity of $+$ follows. The commutativity of $+$ is obvious; the cancellativity holds by (B2); and by (B3), we have $[a] + [0] = [a]$. If moreover $[1] + [a]$ is defined, then $b \perp a'$ for some $b \sim 1$ and $a' \sim a$, so $a' \leq \neg b \sim 0$, that is, $a \precsim 0$ and so $[a] = [0]$. Finally, $[a] + [\neg a] = [1]$ holds obviously. We have proved that $[L]$ is an effect algebra.

Let now $a, b, c, d \in L$ such that $[a] + [b] = [c] + [d]$. Then we may assume w.l.o.g. that $a \perp b$ and $c \perp d$. It follows $c' \leq a \vee b$ for $c' \sim c$ and $d' = (a \vee b) \setminus c' \sim d$. We then have $a \vee b = c' \vee d'$, so we may write $[a] = [a \wedge c'] + [a \wedge d']$ and express similarly also $[b]$, $[c]$, and $[d]$. (RDP_2) follows.

By Theorem 3.9, $([L]; +, [0], [1])$ is an interval effect algebra. Putting $\mu(a) = [a]$ for any $a \in L$, the assertions follow. $\square$

As a corollary, we have that there exists an $\mathbb{R}$-valued measure on L, that is, a measure on L in the usual sense, which is constant on the $\sim$-equivalence classes. Indeed, an Abelian ℓ-group G is representable, and it follows that every unital Abelian ℓ-group has at least one state. This means that there is a homomorphism s from the unital ℓ-group G to $(\mathbb{R}; \wedge, \vee, 0, 1)$, and $\mu_0 = s \circ \mu$ gives a measure. However, μ_0 will in general not map distinct equivalence classes to distinct reals. – We note that the existence of a real-valued measure can also be derived by means of Tarski's Theorem [Tar, Satz 1.58].

4 The algebra associated to an s-equivalence relation on a Boolean algebra

We are concerned with a Boolean algebra L endowed with an s-equivalence relation $\sim$, and we are going to explore the internal structure of the set $[L]$ of $\sim$-equivalence classes. We will compile a set of operations definable on $[L]$ under one further suitable assumption. The idea how to choose operations for $[L]$ is simple: We will connect pairs of equivalence classes elementwise by some Boolean operation.

Definition 4.1 Let $\sim$ be an s-equivalence relation on a Boolean algebra $(L; \wedge, \vee, \neg, 0, 1)$. Consider the bounded poset $([L]; \leq, [0], [1])$. If, for every $a, b \in L$, the set $\{[a' \vee b'] \colon\ a' \sim a,\ b' \sim b\}$ contains a smallest element, we say that $[L]$ is *residuable*.

In other words, we require that for every pair of elements a, b of the Boolean algebra there are $\bar{a} \sim a$ and $\bar{b} \sim b$ such that $\bar{a} \vee \bar{b} \precsim a' \vee b'$ for any $a' \sim a$ and $b' \sim b$. Note that the elements $\bar{a}$ and $\bar{b}$ are in general not uniquely determined by a and b, but only the equivalence class of the supremum $\bar{a} \vee \bar{b}$.

Lemma 4.2 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L. Let $a, a', b, b' \in L$ be such that $a \sim a',\ \ b \sim b'$. Then $a \wedge b \precsim a' \wedge b'$ if and only if $a \vee b \succsim a' \vee b'$. In particular, $a \wedge b \sim a' \wedge b'$ if and only if $a \vee b \sim a' \vee b'$.*

Moreover, $a \precsim b$ if and only if $\neg b \precsim \neg a$.

Proof. We first prove that $a \wedge b \sim a' \wedge b'$ iff $a \vee b \sim a' \vee b'$. Let $a \wedge b \sim a' \wedge b'$. Then $a \setminus b = a \setminus (a \wedge b) \sim a' \setminus (a' \wedge b') = a' \setminus b'$ by (B2), and similarly $b \setminus a \sim b' \setminus a'$. So we conclude $a \vee b = (a \setminus b) \vee (a \wedge b) \vee (b \setminus a) \sim a' \vee b'$ by (B3).

Conversely, let $a \vee b \sim a' \vee b'$. Then $a \setminus b \sim a' \setminus b'$ by (B2), and $a \wedge b = a \setminus (a \setminus b) \sim a' \setminus (a' \setminus b') = a' \wedge b'$.

Now, to prove the first equivalence claimed, assume $a \wedge b \precsim a' \wedge b'$. Because $a' \wedge b' \precsim a \vee b$, there is by Lemma 3.2 a $c \sim a' \wedge b'$ such that $a \wedge b \leq c \leq a \vee b$. Furthermore, let $b'' \sim b$ be such that $c \leq b'' \leq c \vee b$, and $a'' \sim a$ be such that $c \leq a'' \leq c \vee a$. Then $a'' \vee b'' \leq a \vee b$, and $a'' \wedge b'' = c$. From the first part, we have $a'' \vee b'' \sim a' \vee b'$. So $a \vee b \succsim a' \vee b'$.

Conversely, let $a \vee b \succsim a' \vee b'$. Then there is a $d \sim a' \vee b'$ such that $a \wedge b \leq d \leq a \vee b$. Moreover, there are $a'' \sim a$ and $b'' \sim b$ such that $a \wedge d \leq a'' \leq d$ and $b \wedge d \leq b'' \leq d$. Then $a'' \vee b'' = d$ and $a'' \wedge b'' \geq a \wedge b$. It follows from the first part that $a'' \wedge b'' \sim a' \wedge b'$. So $a \wedge b \precsim a' \wedge b'$.

The last statement is clear from (B2). $\square$

We next turn to the existence of infima and suprema in the poset $[L]$. As usual $[a] \wedge [b]$ and $[a] \vee [b]$ denote the greatest lower bound and smallest upper bound of $[a]$ and $[b]$, respectively, in case they exist.

Lemma 4.3 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L. Let $a, b \in L$. Then* (i) $[a \vee b] = \min\{[a' \vee b']\colon\ a' \sim a,\ b' \sim b\}$ *if and only if* (ii) $[a \vee b] = [a] \vee [b]$ *if and only if* (iii) $[a \wedge b] = \max\{[a' \wedge b']\colon\ a' \sim a,\ b' \sim b\}$ *if and only if* (iv) $[a \wedge b] = [a] \wedge [b]$.

In particular, if $[L]$ is residuable, then $[L]$ is lattice-ordered.

Proof. Assume (i). We have $[a \vee b] \geq [a], [b]$, and from $[x] \geq [a], [b]$, it follows $x \geq a', b'$ for some $a' \sim a$ and $b' \sim b$, whence by assumption $x \geq a' \vee b' \succsim a \vee b$, that is, $[x] \geq [a \vee b]$. (ii) follows.

Conversely, assume (ii). For $a' \sim a$ and $b' \sim b$, we have $a' \vee b' \succsim a, b$, that is, $[a' \vee b'] \geq [a], [b]$. By assumption, this means $[a' \vee b'] \geq [a \vee b]$, and (i) is proved.

The equivalence of (iii) and (iv) holds by duality. Finally, (i) and (iii) are equivalent by Lemma 4.2. $\square$

Note that if $[L]$ is residuable, then $\{[a' \vee b']\colon\ a' \sim a,\ b' \sim b\}$ contains a smallest and $\{[a' \wedge b']\colon\ a' \sim a,\ b' \sim b\}$ contains a greatest element, and both these elements are represented by the same pair $a' \sim a$ and $b' \sim b$.

We see next that a further pair $a'' \sim a$ and $b'' \sim b$ represents both the maximum of the former set and the minimum of the latter set.

Lemma 4.4 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L. Let $a, b \in L$. Then the following statements are equivalent:*

(i) $[a \wedge b] = \min\{[a' \wedge b']\colon\ a' \sim a,\ b' \sim b\}$,

(ii) $[a \vee b] = \max\{[a' \vee b']\colon\ a' \sim a,\ b' \sim b\}$.

Moreover, if $[L]$ *is residuable, then for any* $a, b \in L$*, the set* $\{[a' \wedge b']\colon\ a' \sim a,\ b' \sim b\}$ *has a minimum.*

Proof. The first part holds by Lemma 4.2.

Let $[L]$ be residuable. Then the set $\{[a' \wedge \neg b']\colon\ a' \sim a,\ b' \sim b\} = \{[a' \wedge c]\colon\ a' \sim a,\ c \sim \neg b\}$ has a greatest element; so there are $\bar{a} \sim a$ and $\bar{b} \sim b$ such that $\bar{a} \wedge \neg\bar{b} \gtrsim a' \wedge \neg b'$ for all $a' \sim a$ and $b' \sim b$. It follows $\bar{a} \wedge \bar{b} = \bar{a} \searrow (\bar{a} \wedge \neg\bar{b}) \precsim a' \searrow (a' \wedge \neg b') = a' \wedge b'$ for all $a' \sim a$ and $b' \sim b$. □

We next turn to pointwise connection of two equivalence classes by the difference operation.

Lemma 4.5 *Let* $\sim$ *be an s-equivalence relation on a Boolean algebra* L*. Let* $a, b \in L$*. If* $[a \vee b] = [a] \vee [b]$*, then* $[a \searrow b] = \min\{[a' \searrow b']\colon\ a' \sim a,\ b' \sim b\}$*.*

Proof. Assume $[a \vee b] = [a] \vee [b]$, and let $a' \sim a$ and $b' \sim b$. Then $a' \wedge b' \precsim a \wedge b$ by Lemma 4.3. So by (B1) and (B2), $a \searrow b = a \searrow (a \wedge b) \precsim a' \searrow (a' \wedge b') = a' \searrow b'$. □

Definition 4.6 Let $\sim$ be an s-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Then we define for $a, b \in L$

$$\begin{aligned} [a] \oplus [b] &= \max\{[a' \vee b']\colon\ a' \sim a,\ b' \sim b\}, \\ [a] \ominus [b] &= \min\{[a' \searrow b']\colon\ a' \sim a,\ b' \sim b\}, \end{aligned}$$

and we let $\mathit{0} = 0$ and $\mathit{1} = 1$. We call the structure $([L]; \wedge, \vee, \oplus, \ominus, \mathit{0}, \mathit{1})$ the *ambiguity algebra* associated to the pair $(L, \sim)$.

Example 4.7 Let L be the Boolean algebra generated by the closed subintervals of the real unit interval by means of intersection, union, and complementation. For $a, b \in L$, define $a \sim b$ if the Borel measures of a and b coincide. Then $[L]$ can be identified with $[0, 1]$, ordered in the natural way. Moreover, $\oplus$ is the Łukasiewicz t-conorm, $\ominus$ is the truncated difference. So the $([L]; \wedge, \vee, \oplus, \ominus, \mathit{0}, \mathit{1})$ is the Łukasiewicz t-conorm algebra.

The proof of the following lemma contains the central argument on which this paper is based.

Lemma 4.8 *Let* $\sim$ *be an s-equivalence relation on a Boolean algebra* L *such that* $[L]$ *is residuable. Then the ambiguity algebra* $([L]; \wedge, \vee, \oplus, \ominus, \mathit{0}, \mathit{1})$ *associated to* $(L, \sim)$ *is a dual residuated lattice.*

Proof. $([L]; \wedge, \vee, \mathit{0}, \mathit{1})$ is a bounded lattice by Lemma 4.3.

Note next that $d \sim a \vee b$ implies $d = a' \vee b'$ for some $a' \sim a$ and $b' \sim b$. So, for any $a, b, c \in L$, we have $([a] \oplus [b]) \oplus [c] = \max \{a' \vee b' \vee c' \colon\ a' \sim a,\ b' \sim b,\ c' \sim c\}$, from which the associativity of $\oplus$ is evident. It is clear that $\oplus$ is commutative; and because $a \vee b \sim a$ for $b \sim 0$, we have $[a] \oplus \mathit{0} = [a]$. So $([L]; \oplus, \mathit{0})$ is a commutative monoid.

Furthermore, $[a] \leq [b] \oplus [c]$ holds if and only if $a' \leq b' \vee c'$ for some $a' \sim a$, $b' \sim b$, $c' \sim c$. And $[a] \ominus [b] \leq [c]$ if and only if $a' \setminus b' \leq c'$ for some $a' \sim a$, $b' \sim b$, $c' \sim c$. Since $\vee$ and $\setminus$ form an adjoint pair in the Boolean algebra L, we conclude that $\oplus$ and $\ominus$ form an adjoint pair in $[L]$. □

We continue to prove that the algebra just proved to be a dual residuated lattice is actually a dual MV-algebra.

Lemma 4.9 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Let $a \in L$. Then $\mathit{1} \ominus [a] = [\neg a]$.*

Proof. For any $b \sim 1$ and $a' \sim a$, we have $a' \vee b \sim 1$, so $b \setminus a' = (a' \vee b) \setminus a' \sim 1 \setminus a = \neg a$ by (B2). The assertion is now evident from the definition of $\ominus$. □

Lemma 4.10 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Let $a, b \in L$. Then $(([a] \ominus [b]) \oplus [b]) = [a] \vee [b]$.*

Proof. We may assume that $[a \vee b] = [a] \vee [b]$. By Lemma 4.5, $[a] \ominus [b] = [a \setminus b]$. Furthermore, $a \setminus b \perp b$; so by Lemma 4.2, $(a \setminus b) \vee b \gtrsim c' \vee b'$ for all $c' \sim a \setminus b$ and $b' \sim b$. So $[a \setminus b] \oplus [b] = [(a \setminus b) \vee b] = [a \vee b]$, and the assertion follows. □

We are ready to state the main theorem of this section.

Theorem 4.11 *Let $\sim$ be an s-equivalence relation on a Boolean algebra L such that $[L]$ is residuable. Then the ambiguity algebra $([L]; \wedge, \vee, \oplus, \ominus, \mathit{0}, \mathit{1})$ associated to $(L, \sim)$ is a dual MV-algebra.*

Proof. That $[L]$ is a dual MV-algebra follows from Lemmas 4.8, 4.9, and 4.10. □

Let us finally see how large the class of representable MV-algebras is at least.

Example 4.12 We generalize and refine Example 4.7. Let $(G; \leq, +, 0, u)$ be a unital totally ordered Abelian group. Let L consist of the finite unions of half-open intervals $[g, h)$, where $g, h \in G$ such that $0 \leq g < h \leq u$. Then $(L; \cap, \cup, \complement, \varnothing, [0, u))$ is obviously a Boolean algebra, where $\cap$ is the intersection, $\cup$ is the union, and $\complement A = [0, u) \backslash A$. Given $A \in L$, let $A = \dot\bigcup_i [g_i, h_i)$ be represented by disjoint non-empty intervals, and let $\mu(A) = \Sigma_i (h_i - g_i)$. Obviously, this defines a G-valued measure $\mu\colon\ L \to G[0, u]$. It is furthermore not difficult to check that the ambiguity algebra $[L]$ associated to $(L, \sim)$ is isomorphic to the dual MV-algebra arising from the unit interval $G[0, u]$ in the way described in Theorem 2.2.

Theorem 4.13 *Let $(M; \wedge, \vee, \oplus, \ominus, 0, 1)$ be an MV-algebra which is the direct product of totally ordered MV-algebras. Then there is an s-equivalence relation on a Boolean algebra such that $[L]$ is residuable and such that the ambiguity algebra associated to $(L, \sim)$ is isomorphic to M.*

Proof. By Example 4.12, the statement holds for all totally ordered MV-algebra. It is furthermore clear how this example is to be generalized to represent any direct product of totally ordered MV-algebras. □

So in particular, all totally ordered MV-algebras are representable in the way shown by Theorem 4.11. Moreover, any MV-algebra is a subalgebra of an MV-algebra representable this way.

5 Conditions implying residuability for Boolean algebras

The condition that the quotient w.r.t. a horizontal equivalence on a Boolean algebra is residuable, is of a rather abstract nature. We wonder if there are easy conditions implying this property. We will show here that when assuming the Boolean algebra to be separable and complete and furthermore the relation $\precsim$ to be order-continuous, we only need to require the lower directedness of the sets of joins of the elements of two equivalence classes.

We will call any poset *separable* if it contains a countable dense subset.

Definition 5.1 Let L be a complete separable Boolean algebra, and let $\sim$ be an s-equivalence relation on L. We say that $\sim$ is *normal* if the following conditions are fulfilled:

(B4) Let $a_0 \geq a_1 \geq \ldots$ and $b \precsim a_i$ for every i. Then $b \precsim \bigwedge_i a_i$.

(B5) For any $a, b \in L$, the set $\{[a' \vee b'] \colon\ a' \sim a,\ b' \sim b\}$ is lower directed.

Lemma 5.2 *Let $\sim$ be a normal s-equivalence relation on a complete separable Boolean algebra L.*

(i) *Let $a_0 \leq a_1 \leq \ldots$ and $b \succsim a_i$ for every i. Then $b \succsim \bigvee_i a_i$.*

(ii) *Let $a_0 \leq a_1 \leq \ldots$. Then $b = \bigvee_i a_i$ implies $[b] = \bigvee_i [a_i]$.*

(iii) *$[L]$ is separable.*

Proof. (i) By (B4), we have $\neg b \precsim \bigwedge_i \neg a_i$. So $b \succsim \bigvee_i a_i$ as asserted.

(ii) This is easily derived from part (i).

(iii) Let $\{e_i \colon\ i < \omega\}$ be a dense subset of L. Given $a \in L$, we have $a = \bigvee_j e_{i_j}$ for certain $i_1, \ldots < \omega$. Then $a = \bigvee_j (e_{i_1} \vee \ldots \vee e_{i_j})$, so by part (ii), $[a] = \bigvee_j [e_{i_1} \vee \ldots \vee e_{i_j}]$ in $[L]$. It follows that the equivalence classes of suprema of finitely many of the e_i are dense in $[L]$. □

Theorem 5.3 *Let $\sim$ be a normal s-equivalence relation on a complete separable Boolean algebra L. Then $[L]$ is residuable.*

Proof. Let $a, b \in L$. We will show that there are $\bar{a} \sim a$ and $\bar{b} \sim b$ such that $\bar{a} \wedge \bar{b} \succsim a' \wedge b'$ for all $a' \sim a$ and $b' \sim b$; residuability then follows by Lemma 4.3.

Because $[L]$ is separable by Lemma 5.2(iii), the set $K = \{[a' \wedge b'] \colon\ a' \sim a,\ b' \sim b\}$ contains a countable subset $K_0 = \{[a_i \wedge b_i] \colon\ i < \omega\}$ such that the sets of upper bounds of K and of K_0 coincide.

By (B5), there are $a' \sim a$ and $b' \sim b$ such that $a' \vee b' \precsim a_0 \vee b_0, a_1 \vee b_1$. In view of Lemma 3.2, let $c \sim a' \vee b'$ such that $a_0 \leq c \leq a_0 \vee b_0$; let $b_1' \sim b$ be such that $b_0 \wedge c \leq b_1' \leq c$. Then $a_0 \vee b_1' \leq a_0 \vee b_0$ and $a_0 \wedge b_1' \geq a_0 \wedge b_0$. Moreover, because $c = a_0 \vee (b_0 \wedge c) \leq a_0 \vee b_1' \leq c$, we have $a_0 \vee b_1' \sim a' \vee b'$ and, by Lemma 4.2, $a_0 \wedge b_1' \sim a' \wedge b' \succsim a_1 \wedge b_1$. So we may replace $[a_1 \wedge b_1]$ by $[a_0 \wedge b_1']$ in K_0 without changing the set of upper bounds of K_0.

Continuing in the same way, we get a chain $a_0 \vee b_0 \geq a_0 \vee b_1' \geq a_0 \vee b_2' \geq \ldots$ and $a_0 \wedge b_0 \leq a_0 \wedge b_1' \leq \ldots$. Letting $d = \bigwedge_i (a_0 \vee b_i')$, we have $d \setminus a \precsim b \precsim d$; so $d = a_0 \vee \bar{b}$ for some $\bar{b} \sim b$. Then $a_0 \vee \bar{b} \leq a_0 \vee b_i'$ and consequently $a_0 \wedge \bar{b} \geq a_0 \wedge b_i'$ for every i. The assertion follows. □

6 s-Equivalence relations on Browerian algebras

In this and the next subsection, we will generalize the framework developed so far; rather than relying on Boolean algebras, we will use distributive lattices with the additional property that the difference of any pair of elements exists. The procedure requires more care than in the previous case; our notions need to be adapted and the proofs become more involved. However, as a result, we are able to offer a possibility to represent a class of residuated lattices which is large enough at least to include the (totally ordered) BL-algebras, which are associated to Hájek's Basic Fuzzy Logic BL.

A Browerian algebra is a bounded lattice $(L; \wedge, \vee, 0, 1)$ such that

$$a \setminus b = \min\{x \colon\ b \vee x \geq a\},$$

that is, the difference of a and b, exists for every pair $a, b \in L$. This means that $\vee$ and $\setminus$ are an adjoint pair. More information on these algebras can be easier found when considering the dual notion, which are Heyting algebras; see for instance [RaSi].

For elements a, b of a Browerian algebra, we will say that a is *pseudoorthogonal* to b and write $a \perp\!\!\!\perp b$ if $a \setminus b = a$. Furthermore, we will say that a and b are *orthogonal* and write $a \perp b$ if $a \perp\!\!\!\perp b$ and $b \perp\!\!\!\perp a$. Note that $\perp\!\!\!\perp$ is in general not symmetric, whereas by construction $\perp$ is.

Lemma 6.1 *Let $(L; \wedge, \vee, 0, 1)$ be a Browerian algebra. Let $a, b, c \in L$. Then:*

(i) *$a \perp\!\!\!\perp b$ implies $a \perp\!\!\!\perp c$ for $c \leq b$.*
In general, $a \setminus b \perp\!\!\!\perp b$.

(ii) *We have*

$$\begin{aligned}
&a \vee b = (b \setminus a) \vee a = (a \setminus b) \vee (b \setminus a) \vee (a \wedge b), \\
&a \setminus b = a \setminus (a \wedge b) = (a \vee b) \setminus b, \\
&(a \setminus b) \setminus c = a \setminus (b \vee c).
\end{aligned}$$

Moreover, if the supremum $\bigvee_\iota a_\iota$ exists in L, then $(\bigvee_\iota a_\iota) \setminus b = \bigvee_\iota (a_\iota \setminus b)$.

We next present the adapted version of Definition 3.1 for the case of Browerian algebras.

Definition 6.2 Let $(L; \wedge, \vee, 0, 1)$ be a Browerian algebra. Let $\sim$ be an equivalence relation $\sim$ on L, and write $a \precsim b$ if $a' \leq b'$ for some $a' \sim a$ and $b' \sim b$. $\sim$ is called a *s-equivalence relation* on L if the following conditions hold:

(H1) If $a \precsim b \precsim c$ and $a \leq c$, then there is a $b' \sim b$ such that $a \leq b' \leq c$.

(H2) Let $a \sim a'$, $b \sim b'$. Then $a \wedge b \sim a' \wedge b'$ if and only if $b \setminus a \sim b' \setminus a'$.

(H3) Let $a \sim a'$, $b \sim b'$. Then $a \perp\!\!\!\perp b$, $a' \perp\!\!\!\perp b'$ implies $a \vee b \sim a' \vee b'$.

The intended interpretation of a Browerian algebra endowed with an s-equivalence relation is similar as before. A Browerian algebra is isomorphic to a subset algebra, and so we may consider it as a system of sharp propositions, and we may think of an s-equivalence relation as identifying propositions of equal expressive strength.

We have by (H1) that $\precsim$ induces a partial order, also denoted by $\leq$, on the set $[L]$ of equivalence classes of an s-equivalence relation, and $\iota\colon L \to [L]$, $a \mapsto [a]$ is a surjective homomorphism of bounded posets.

We note that if $\sim$ is an s-equivalence relation on a Boolean algebra L according to Definition 3.1, then the relation $\sim$ on L, viewed as a Browerian algebra, fulfills (H1)–(H3), that is, is an s-equivalence relation in the sense of Definition 6.2 as well.

Lemma 6.3 *Let $\sim$ be an s-equivalence relation on the Browerian algebra L. Let $a, b, c \in L$.*

(i) *Let $c \sim a \vee b$. Then there are $a' \sim a$ and $b' \sim b$ such that $c = a' \vee b'$.*

(ii) *Let $a' \sim a$ and $b' \sim b$ such that $a \vee b \sim a' \vee b'$. Then $a \perp\!\!\!\perp b$ if and only if $a' \perp\!\!\!\perp b'$. In particular, $a \perp b$ if and only if $a' \perp b'$.*

Proof. (i) By (H1) and (H2), we choose $a' \sim a$ and $b' \sim b$ such that $a' \leq c$ and $c \setminus a' \leq b' \leq c$. Then $c = a' \vee b'$.

(ii) Assume $a \perp\!\!\!\perp b$. Then $a' \sim a = a \setminus b = (a \vee b) \setminus b \sim (a' \vee b') \setminus b' = a' \setminus b' \leq a'$ by (H2); so $a' \setminus b' = a'$ by Lemma 7.9, that is, $a' \perp\!\!\!\perp b'$. □

We suggested above to imagine $\sim$ as a relation identifying elements of equal size. To a certain extent, this picture can be applied to the present context as well. The following example is the typical one.

Example 6.4 Let $(I; \leq)$ be a totally ordered set. For every $\iota \in I$, let $(L_\iota; \wedge, \vee, 0, 1)$ be a Boolean algebra. Define the *ordinal sum* of these algebras as follows (caution: this definition is not exactly the same as e.g. in [Fuc]). Let L consist of those $(a_\iota)_{\iota \in I} \in \prod_{\iota \in I} L_\iota$ such that, for $\kappa \in I$, $a_\kappa > 0$ implies $a_\iota = 1$ for all $\iota < \kappa$. Endow L with the pointwise order. Then L is obviously a lattice with smallest the element $(0)_\iota$ and the largest element $(1)_\iota$. It is straightforward that L is actually a Browerian algebra.

Assume now in addition that, for every ι, $\mu_\iota \colon L_\iota \to [0, 1]$ is a homogeneous measure on L_ι. Define $(a_\iota)_\iota \sim (b_\iota)_\iota$ if $\mu_\iota(a_\iota) = \mu_\iota(b_\iota)$ for all ι. We may check that $\sim$ is an s-equivalence relation on L. The induced order is total.

In the rest of this section, we develop an analogue of Theorem 3.10. Again, this material will afterwards not be used.

Definition 6.5 Let L be a Browerian algebra, and let $(G; \wedge, \vee, +, 0)$ be an Abelian ℓ-group. Then a map $\mu \colon L \to G$ is called a *G-valued measure* on L if $a \perp b$ implies $\mu(a \vee b) = \mu(a) + \mu(b)$.

Note that we do not work with unital ℓ-groups here.

Lemma 6.6 *Let L be a Browerian algebra, and let $a, b, c, d \in L$ be such that $a \perp b$, $c \perp d$, and $a \vee b = c \vee d$. Then there exist pairwise orthogonal elements $e_1, \ldots, e_4 \in L$ such that $a = e_1 \vee e_2$, $b = e_3 \vee e_4$, $c = e_1 \vee e_3$, $d = e_2 \vee e_4$.*

Proof. Let $e_1 = c \setminus b$, $e_2 = c \setminus a$, $e_3 = d \setminus b$, $e_4 = d \setminus a$.

Then $e_1 = c \setminus b = ((c \vee d) \setminus d) \setminus b = (c \vee d) \setminus (b \vee d) = (a \vee b) \setminus (b \vee d) = ((a \vee b) \setminus b) \setminus d = a \setminus d$. Similarly, $e_2 = b \setminus d$, $e_3 = a \setminus c$, $e_4 = b \setminus c$.

It follows $e_1 \setminus e_2 = (c \setminus b) \setminus (b \setminus d) = c \setminus b = e_1$; similarly $e_2 \setminus e_1 = e_2$; so $e_1 \perp e_2$. Analogously, we see that $e_i \perp e_j$ for every distinct pair i, j.

Finally, we have $e_1 \vee e_2 = (c \setminus b) \vee (b \setminus d) \geq c \setminus d = c \geq e_1 \vee e_2$, because $c \perp\!\!\perp d$; so $e_1 \vee e_2 = c$. Analogously, we derive the claimed equations for a, b, d. $\square$

We next introduce the appropriate algebraic tool needed in the sequel; the notion of an effect algebra is, unfortunately, too special. We use the following straightforward generalization [HePu].

Definition 6.7 A *generalized effect algebra*, or *GE-algebra* for short, is a structure $(E; +, 0)$, where $+$ is a partial binary operation and 0 is a constant such that the axioms (E1)–(E4) of effect algebras are fulfilled and furthermore the following one:

(GE) If $a + b$ is defined and equals 0, then $a = b = 0$.

In addition, we define $a \leq b$ if $a + c = b$ for some c.

Again, $\leq$ defines a partial order on a GE-algebra, 0 being the bottom element. Note that effect algebras may be considered GE-algebras possessing a top element.

GE-algebras are modelled upon subsets of positive cones of partially ordered groups.

Definition 6.8 Let G be an Abelian ℓ-group. Let $E \subseteq G^+$ be such that (i) $0 \in E$ and (ii) if $a, b \in E$ and $a \leq b$, then $b - a \in E$. Let $+$ be the partial addition on E which is the restriction of the group addition to those pairs of elements whose sum is in E. Then $(E; +, 0)$ is called a *group-representable GE-algebra*.

Clearly, a group-representable GE-algebra is a GE-algebra. For the converse direction, we have the following theorem, which is a straightforward generalization of Theorem 3.9. For a proof, see e.g. [DvPu], or see the similar theorem in [DvVe3], whose proof needs to be only slightly modified. The property (RDP$_2$) is defined in exact analogy to the case of effect algebras.

Theorem 6.9 *Every GE-algebra fulfilling* (RDP$_2$) *is a group-representable GE-algebra.*

We may formulate the announced result.

Theorem 6.10 *Let $\sim$ be an s-equivalence relation on the Browerian algebra L. Then there is an Abelian ℓ-group G and a G-valued measure $\mu\colon L \to G$ on L such that, for all $a, b \in L$, $\mu(a) = \mu(b)$ iff $a \sim b$.*

Proof. Consider the partial algebra $([L]; +, 0, 1)$, where we put $[a]+[b] = [a' \vee b']$ if $a' \perp b'$ for some $a' \sim a$ and $b' \sim b$, and let else $[a] + [b]$ undefined. By (H3), this definition is unambiguous.

Assume that $([a] + [b]) + [c]$ is defined; this means that $d \perp c'$ for a $c' \sim c$ and a $d \sim a' \vee b'$ such that $a' \perp b'$ for a pair $a' \sim a$ and $b' \sim b$. By Lemma 6.3, there is an orthogonal pair $a'' \sim a$ and $b'' \sim b$ such that $d = a'' \vee b''$. It is not difficult to check that $b'' \perp c'$ and $a'' \perp b'' \vee c'$. So (E1) follows.

The commutativity of $+$ is obvious; so also (E2) holds. If $[a] + [c] = [b] + [c]$, then $a' \vee c' \sim b' \vee c''$ for $a' \sim a$, $b' \sim b$, and $c' \sim c'' \sim c$ such that $a' \perp c'$ and $b' \perp c''$, whence $a' = (a' \vee c') \setminus c' \sim (b' \vee c'') \setminus c'' = b'$, and (E3) follows. (E4) is trivial.

To see (GE), assume $[a] + [b] = 0$. This means $a' \vee b' \sim 0$ for some $a' \sim a$ and $b' \sim b$, and it follows $a' \sim b' \sim 0$; hence $a = b = 0$. So $[L]$ is a generalized effect algebra.

We claim that $[L]$ fulfills (RDP_2). So let $a, b, c, d \in L$ be such that $[a] + [b] = [c] + [d]$. We may assume $a \perp b$ and $c \perp d$. By Lemma 6.3, there is an orthogonal pair $c' \sim c$ and $d' \sim d$ such that $a \vee b = c' \vee d'$. So (RDP_2) follows by Lemma 6.6.

By Theorem 6.9, $([L]; +, 0)$ is isomorphically embeddable into an Abelian ℓ-group. Putting $\mu(a) = [a]$ for each $a \in L$, the assertions follow. □

We think that the fact that we may associate with a Browerian algebra a GE-algebra which even fulfills the strong version of the Riesz decomposition property, is interesting independently from the actual context of this paper. However, we should stress that the value of this theorem must not be overestimated. For, there is no guarantee that the the GE-algebra provides any useful information on the Browerian algebra from which it is constructed; it may happen that there are not sufficiently many additions defined. Consider, for instance, an ordinal sum of Boolean algebras, as described in Example 6.4; in this case, the associated GE-algebra does not reflect the ordinal sum construction at all.

7 The algebra associated to an s-equivalence relation on a Browerian algebra

We are henceforth concerned with the quotient of a Browerian algebra with respect to an s-equivalence relation. We are going to endow this set with the structure of a residuated lattice, analogous to the case that the underlying poset is a Boolean algebra. As to be expected without the self-duality of Boolean algebras, we will have to require stronger conditions to obtain similar results. We are then again led to a class of residuated lattices, which is larger than before and still interesting from the point of view of logics.

Definition 7.1 Let $\sim$ be an s-equivalence relation on a Browerian algebra $(L; \wedge, \vee, \neg, 0, 1)$. Consider the bounded poset $([L]; \leq, \mathit{0}, \mathit{1})$. If, for every $a, b \in L$, the set $\{[a' \vee b'] \colon\ a' \sim a,\ b' \sim b\}$ contains a smallest and a greatest element, we say that $[L]$ is *residuable*.

Lemma 7.2 *Let* $\sim$ *be an s-equivalence relation on a Browerian algebra* L*. Let* $a \sim a',\ \ b \sim b'$*. Then* $a \wedge b \sim a' \wedge b'$ *if and only if* $a \vee b \sim a' \vee b'$*.*

Proof. We have $a \vee b = (a \setminus b) \vee b$ and $a \setminus b \perp\!\!\perp b$. So from $a \wedge b \sim a' \wedge b'$, it follows $a \setminus b \sim a' \setminus b'$ by (H2), and $a \vee b \sim a' \vee b'$ by (H3).

Conversely, assume $a \vee b \sim a' \vee b'$. Then $a \setminus b = (a \vee b) \setminus b \sim (a' \vee b') \setminus b' = a' \setminus b'$ by (H2); and $a \wedge b \sim a' \wedge b'$ again by (H2). $\square$

Lemma 7.3 *Let* $\sim$ *be an s-equivalence relation on a Browerian algebra* L*. Let* $a, a', b, b' \in L$ *be such that* $a \sim a',\ \ b \sim b'$*. Then* $a \wedge b \precsim a' \wedge b'$ *if and only if* $a \vee b \succsim a' \vee b'$*.*

Proof. Assume $a \wedge b \precsim a' \wedge b'$. Let $c \sim a' \wedge b'$ be such that $a \wedge b \leq c \leq a$, and let $b'' \sim b$ be such that $c \leq b'' \leq b \vee c$. Then $a \wedge b'' \leq a \wedge (b \vee c) = (a \wedge b) \vee c = c \leq a, b''$, that is, $c = a \wedge b''$. By Lemma 7.2, it follows $a' \vee b' \sim a \vee b'' \leq a \vee b$.

Conversely, assume $a \vee b \succsim a' \vee b'$. Let $e \sim a' \vee b'$ be such that $a \leq e \leq a \vee b$, and let $b'' \sim b$ be such that $b \wedge e \leq b'' \leq e$. Then $a \vee b'' \leq e = e \wedge (a \vee b) = a \vee (b \wedge e) \leq a \vee b''$, that is, $e = a \vee b''$. By Lemma 7.2, it follows $a' \wedge b' \sim a \wedge b'' \geq a \wedge b$. $\square$

Lemma 7.4 *Let* $\sim$ *be an s-equivalence relation on a Browerian algebra* L*. Let* $a, b \in L$*. Then* (i) $[a \vee b] = \min \{[a' \vee b'] \colon\ a' \sim a,\ b' \sim b\}$ *if and only if* (ii) $[a \vee b] = [a] \vee [b]$ *if and only if* (iii) $[a \wedge b] = \max \{[a' \wedge b'] \colon\ a' \sim a,\ b' \sim b\}$ *if and only if* (iv) $[a \wedge b] = [a] \wedge [b]$*.*

In particular, if $[L]$ *is residuable, then* $[L]$ *is lattice-ordered.*

Proof. We proceed just like in the proof of Lemma 4.3 to see that (i) is equivalent to (ii). Dually, we see that (iii) is equivalent to (iv).

The equivalence of (i) and (iii) follows by Lemma 7.3. $\square$

Lemma 7.5 *Let* $\sim$ *be an s-equivalence relation on a Browerian algebra* L*. Let* $a, b \in L$*. If* $[a \vee b] = [a] \vee [b]$*, then* $[a \setminus b] = \min \{[a' \setminus b'] \colon\ a' \sim a,\ b' \sim b\}$*.*

Proof. Let $a' \sim a$ and $b' \sim b$. By Lemma 7.4, there is a $d \sim a' \vee b'$ such that $d \geq a \vee b$. Then, by (H2), $a' \setminus b' \sim d \setminus b \geq (a \vee b) \setminus b = a \setminus b$. $\square$

So we may define the operations $\oplus$ and $\ominus$ as in Section 4.

Definition 7.6 Let $\sim$ be an s-equivalence relation on a Browerian algebra L such that $[L]$ is residuable. Then we define for $a, b \in L$

$$\begin{aligned} [a] \oplus [b] &= \max \{[a' \vee b']:\ a' \sim a,\ b' \sim b\}, \\ [a] \ominus [b] &= \min \{[a' \setminus b']:\ a' \sim a,\ b' \sim b\}, \end{aligned}$$

and we let $\mathit{0} = [0]$ and $\mathit{1} = [1]$. We call the structure $([L]; \wedge, \vee, \oplus, \ominus, \mathit{0}, \mathit{1})$ the *ambiguity algebra* associated to the pair $(L, \sim)$.

To see that this definition is correct, recall that $\oplus$ is well-defined because $[L]$ is residuable, and $\ominus$ is well-defined due to residuability, Lemma 7.4 and Lemma 7.5.

Theorem 7.7 *Let $\sim$ be an s-equivalence relation on a Browerian algebra L such that $[L]$ is residuable. Then the ambiguity algebra $([L]; \wedge, \vee, \oplus, \ominus, \mathit{0}, \mathit{1})$ associated to $(L, \sim)$ is a dual residuated lattice.*

Proof. This is proved similarly to Lemma 4.8. □

Our next question is if we may sharpen the conditions of an s-equivalence relation so as to make the representation of BL-algebras possible.

Definition 7.8 Let L be a Browerian algebra L, and let $\sim$ be an s-equivalence relation on L. Then we say that $[L]$ is *divisible* if the following condition holds:

(Div) For any $a \in L$, $a \sim 0$ implies $a = 0$.

Moreover, we say that $[L]$ is *prelinear* if the following condition holds:

(Pre) Let $a, b \in L$ be such that $a \wedge b \gtrsim a' \wedge b'$ for all $a' \sim a$ and $b' \sim b$. Then $c_1 \sim c_2$, $c_1 \leq a \setminus b$ and $c_2 \leq b \setminus a$ implies $c_1 = c_2 = 0$.

The condition (Div) may informally be seen as the condition that no non-zero element represents falsity. Moreover, note that $[L]$ is prelinear if $[a \vee b] = [a] \vee [b]$ implies $[a \setminus b] \wedge [b \setminus a] = \mathit{0}$.

The condition (Div) has the following immediate consequence.

Lemma 7.9 *Let $\sim$ be an s-equivalence relation on the Browerian algebra L such that* (Div) *hold. Let $a, b \in L$. Then $a \sim b$ and $a \leq b$ imply $a = b$.*

Proof. By (H2), $b \setminus a \sim a \setminus a = 0$, so the assertion follows by (Div). □

We shall see that (Div) implies the divisibility of an ambiguity algebra associated to a Browerian algebra.

Lemma 7.10 *Let $\sim$ be an s-equivalence relation on a Browerian algebra L such that $[L]$ is residuable and divisible. Let $a, b \in L$ be such that $a \perp\!\!\perp b$. Then $[a] \oplus [b] = [a \vee b]$.*

Proof. Let $\bar{a} \sim a$ and $\bar{b} \sim b$ be such that $[\bar{a} \vee \bar{b}] = [a] \oplus [b]$. Then in particular, $\bar{a} \vee \bar{b} \succsim a \vee b$. Let $d \sim a \vee b$ be such that $\bar{a} \leq d \leq \bar{a} \vee \bar{b}$, and $b' \sim b$ such that $d \wedge \bar{b} \leq b' \leq d$. Then we have $\bar{a} \vee b' = d$ and $\bar{a} \wedge b' \geq \bar{a} \wedge \bar{b}$.

So $\bar{a} \setminus \bar{b} = \bar{a} \setminus (\bar{a} \wedge \bar{b}) \geq \bar{a} \setminus (\bar{a} \wedge b') = \bar{a} \setminus b' = d \setminus b' \sim a \setminus b = a \succsim \bar{a} \setminus \bar{b}$, that is, $\bar{a} \setminus \bar{b} \sim \bar{a}$. By Lemma 7.9, $\bar{a} \perp\!\!\perp \bar{b}$ and hence $\bar{a} \vee \bar{b} \sim a \vee b$ by (H3). $\square$

Lemma 7.11 *Let $\sim$ be an s-equivalence relation on a Browerian algebra L such that $[L]$ is residuable and divisible. Let $a, b \in L$. Then*

$$([a] \ominus [b]) \oplus [b] = [a] \vee [b]. \tag{2}$$

Proof. By replacing a and b by equivalent elements if necessary, we may, in view of Lemma 7.4, assume that $[a \vee b] = [a] \vee [b]$. Then, by Lemma 7.5, $[a] \ominus [b] = [a \setminus b]$. By Lemma 7.10, $[a \setminus b] \oplus [b] = [a \vee b]$. So the assertion is proved. $\square$

We proceed taking into account the condition (Pre), and we arrive at the main theorem of the present section.

Theorem 7.12 *Let $\sim$ be an s-equivalence relation on a Browerian algebra L such that $[L]$ is residuable, divisible and prelinear. Then the ambiguity algebra $([L]; \wedge, \vee, \oplus, \ominus, 0, 1)$ associated to $(L, \sim)$ is a BL-algebra.*

Proof. Divisibility holds by Lemma 7.11.

To see prelinearity, let $a, b \in L$. By replacing a and b by equivalent elements if necessary, we may assume that $[a \vee b] = [a] \vee [b]$. But then $[a] \ominus [b] = [a \setminus b]$ and $[b] \ominus [a] = [b \setminus a]$ by Lemma 7.5. So $([a] \ominus [b]) \wedge ([b] \ominus [a]) = 0$ by (Pre). $\square$

Example 7.13 We shall combine Example 6.4 with Example 4.12. Let $(I; \leq)$ be a totally ordered set. For every $\iota \in I$, let $(G_\iota; \leq, +_\iota, 0_\iota)$ be a totally ordered Abelian group, and choose $0 \leq u_\iota \leq \infty$. Let L_ι consist of (i) the finite unions of half-open intervals $[g, h)$, where $g, h \in G_\iota$ such that $0 \leq g < h \leq u_\iota$ and (ii) the set $[0, u_\iota)$.

Let now L contain all $(A_\iota)_\iota \in \prod_\iota L_\iota$ such that either $A_\iota = \varnothing$ for all ι, or there is a smallest $\kappa \in I$ such that $A_\kappa \neq \varnothing$, in which case (i) $A_\kappa \subsetneqq G_\kappa^+$ and (ii) $A_\iota = [0, u_\iota)$ for all $\iota < \kappa$. Endow L with the pointwise order; then $(L; \wedge, \vee, (\varnothing)_\iota, ([0, u_\iota))_\iota)$ is a Browerian algebra.

For every ι, define $\mu_\iota : L_\iota \to G_\iota^+ \cup \{\infty\}$ by $\mu_\iota(A) = \Sigma_i (h_i - g_i)$, supposed that A is the union of the disjoint intervals $[g_i, h_i)$. For $(A_\iota)_\iota, (B_\iota)_\iota \in L$, let $(A_\iota)_\iota \sim (B_\iota)_\iota$ if $\mu_\iota(A_\iota) = \mu_\iota(B_\iota)$ for all ι.

Note that L has a greatest element exactly if I has a greatest element τ and $u_\tau < \infty$. Assume that this is the case. Then the ambiguity algebra associated to $(L, \sim)$ is a BL-algebra. In view of Theorem 2.2, we readily see that all totally ordered BL-algebras can be constructed this way.

Theorem 7.14 *Let $(M; \wedge, \vee, \oplus, \ominus, 0, 1)$ be a BL-algebra which is the direct product of totally ordered BL-algebras. Then there is an s-equivalence relation on a Browerian algebra such that $[L]$ is residuable and such that the ambiguity algebra associated to $(L, \sim)$ is isomorphic to M.*

Proof. This is clear from Example 7.13 and its obvious generalization. $\square$

So again we see that in particular, all totally ordered BL-algebras are representable by our method, and any BL-algebra is a subalgebra of a BL-algebra representable in this way.

8 Conditions implying residuability for Browerian algebras

We again wonder if there are conditions for s-equivalence relations on Browerian algebras such that residuability of the quotient set is automatic. We will consider dual locales, and we will require order continuity of the $\precsim$-relation.

Recall that a *dual locale* is a structure $(L; \wedge, \vee, 0, 1)$ such that (i) L is a complete lattice, 0 being the bottom and 1 being the top element, and (ii) $\vee$ distributes over arbitrary infima, that is,

$$a \vee \bigwedge_{\iota \in I} b_\iota = \bigwedge_{\iota \in I} (a \vee b_\iota) \text{ for } a, b_\iota \in L, \ \iota \in I.$$

Note that every dual locale is a Browerian algebra. For detailed information on locales – the notion dual to a dual locale –, we refer to [Joh].

Definition 8.1 Let L be a separable dual locale, and let $\sim$ be an s-equivalence relation on L. We say that $\sim$ is *normal* if the following conditions are fulfilled:

(H4) (i) Let $a_0 \geq a_1 \geq \ldots$ and $b \precsim a_i$ for every i. Then $b \precsim \bigwedge_i a_i$.
(ii) Let $a_0 \leq a_1 \leq \ldots$ and $b \succsim a_i$ for every i. Then $b \succsim \bigvee_i a_i$.

(H5) For any $a, b \in L$, the sets $\{[a \wedge b] \colon\ a' \sim a,\ b' \sim b\}$ and $\{[a \vee b] \colon\ a' \sim a,\ b' \sim b\}$ are lower directed.

Lemma 8.2 *Let $\sim$ be a normal s-equivalence relation on a separable dual locale L.*

(i) *Let* $a_0 \leq a_1 \leq \ldots$. *Then* $b = \bigvee_i a_i$ *implies* $[b] = \bigvee_i [a_i]$.

(ii) $[L]$ *is separable.*

Proof. This is proved like Lemma 5.2(ii),(iii). Instead of Lemma 5.2(i), (H4)(ii) is used. □

In several steps, we show that normality of the s-equivalence relation implies residuability of the quotient set.

Lemma 8.3 *Let* $\sim$ *be a normal s-equivalence relation on a separable dual locale* L. *Let* $a_1, a_2, b_1, b_2 \in L$ *be such that* $a_1 \sim a_2$ *and* $b_1 \sim b_2$. *Then there is a* $b_1' \sim b_1$ *such that* $a_1 \wedge b_1 \leq a_1 \wedge b_1'$, $a_1 \vee b_1' \leq a_1 \vee b_1$ *and* $a_2 \wedge b_2 \precsim a_1 \wedge b_1'$, $a_1 \vee b_1' \precsim a_2 \vee b_2$.

Proof. By (H5), there are $a_3 \sim a_1$ and $b_3 \sim b_1$ such that $e_3 = a_3 \vee b_3 \precsim a_1 \vee b_1, a_2 \vee b_2$. By (H1), there is an $e_1 \sim e_3$ such that $a_1 \leq e_1 \leq a_1 \vee b_1$; and there is a $b_1' \sim b_1$ such that $b_1 \wedge e_1 \leq b_1' \leq e_1$.

Then $a_1 \vee b_1' = e_1 \leq a_1 \vee b_1$; and $a_1 \wedge b_1 = a_1 \wedge b_1 \wedge e_1 \leq a_1 \wedge b_1'$. Moreover, we have $a_1 \vee b_1' \precsim a_2 \vee b_2$ and, by Lemma 7.3, $a_2 \wedge b_2 \precsim a_1 \wedge b_1'$. The assertion is proved. □

Lemma 8.4 *Let* $\sim$ *be a normal s-equivalence relation on a separable dual locale* L. *Let* $a, b \in L$. *Then* $\{[a' \vee b']\colon\ a' \sim a,\ b' \sim b\}$ *has a smallest element.*

Proof. It suffices to show that there are $\bar{a} \sim a$ and $\bar{b} \sim b$ such that $\bar{a} \wedge \bar{b} \succsim a' \wedge b'$ for all $a' \sim a$ and $b' \sim b$; the assertion then follows by Lemma 7.4.

By Lemma 8.2(ii), $K = \{[a' \wedge b']\colon\ a' \sim a,\ b' \sim b\}$ contains a countable subset $K_0 = \{[a_i \wedge b_i]\colon\ i < \omega\}$ such that the upper bounds of K and K_0 coincide.

So by Lemma 8.3, there are $b_1' \sim b_2' \sim \ldots \sim b$ such that $a_0 \vee b_0 \geq a_0 \vee b_1' \geq a_0 \vee b_2' \geq \ldots$ and $a_0 \wedge b_0 \leq a_0 \wedge b_1' \leq \ldots$ and the upper bounds of $\{[a_0 \wedge b_i']\colon\ i < \omega\}$ and those of K coincide.

Let $d = \bigwedge_i (a_0 \vee b_i')$. Then we have $d \setminus a_0 \precsim b$ and by (H4)(i) $b \precsim d$. So $d = a_0 \vee \bar{b}$ for some $\bar{b} \sim b$. Moreover, $a_0 \vee \bar{b} \leq a_0 \vee b_i'$ and consequently $a_0 \wedge \bar{b} \succsim a_0 \wedge b_i'$ for every i. The assertion follows. □

Lemma 8.5 *Let* $\sim$ *be a normal s-equivalence relation on a separable dual locale* L. *Let* $a_1, a_2, b_1, b_2 \in L$ *be such that* $a_1 \sim a_2$ *and* $b_1 \sim b_2$. *Then there is a* $b_1' \sim b_1$ *such that* $a_1 \wedge b_1' \leq a_1 \wedge b_1$, $a_1 \vee b_1 \leq a_1 \vee b_1'$ *and* $a_1 \wedge b_1' \precsim a_2 \wedge b_2$, $a_2 \vee b_2 \precsim a_1 \vee b_1'$.

Proof. By (H5), there are $a_3 \sim a_1$ and $b_3 \sim b_1$ such that $d_3 = a_3 \wedge b_3 \precsim a_1 \wedge b_1$, $a_2 \wedge b_2$. Let $d_1 \sim d_3$ be such that $d_1 \leq a_1 \wedge b_1$ and, in view of Lemma 7.2, let $e_1 \sim e_3 = a_3 \vee b_3$ be such that $e_1 \geq a_1 \vee b_1$. Then $e_1 \setminus a_1 \sim e_3 \setminus a_3$ by (H2), and $e_1 \setminus a_1 \perp\!\!\!\perp d_1$, $e_3 \setminus a_3 \perp\!\!\!\perp d_3$. Let $b'_1 = (e_1 \setminus a_1) \vee d_1$; then $b'_1 \sim b_3$ by (H3), and $e_1 = a_1 \vee b'_1$. Finally, $a_1 \wedge b'_1 \geq d_1 \sim a_1 \wedge b'_1$ by Lemma 7.2, whence $d_1 = a_1 \wedge b'_1$ by Lemma 7.9. So all assertions follow. □

Lemma 8.6 *Let $\sim$ be a normal s-equivalence relation on a separable dual locale L. Let $a, b \in L$. Then $\{[a' \vee b']\colon\ a' \sim a,\ b' \sim b\}$ has a greatest element.*

Proof. By Lemma 8.2(ii), $K = \{[a' \vee b']\colon\ a' \sim a,\ b' \sim b\}$ contains a countable subset $K_0 = \{[a_i \vee b_i]:\ i < \omega\}$ such that the upper bounds of K and K_0 coincide.

So by Lemma 8.5, there are $b'_1 \sim b'_2 \sim \ldots \sim b$ such that $a_0 \vee b_0 \leq a_0 \vee b'_1 \leq a_0 \vee b'_2 \leq \ldots$ and $a_0 \wedge b_0 \geq a_0 \wedge b'_1 \geq \ldots$ and the upper bounds of $\{[a_0 \vee b'_i]\colon\ i < \omega\}$ and K coincide.

Let $d = \bigvee_i (a_0 \vee b'_i)$. Then we have by (H4) $d \setminus a_0 = \bigvee_i ((a_0 \vee b'_i) \setminus a_0) \precsim b$. So $d = a_0 \vee \bar{b}$ for some $\bar{b} \sim b$. We have $a_0 \vee \bar{b} \geq a_0 \vee b'_i$ for every i; so the assertion follows. □

Theorem 8.7 *Let $\sim$ be a normal s-equivalence relation on a separable dual locale L. Then $[L]$ is residuable.*

Proof. This is the content of Lemmas 8.4 and 8.6. □

9 Conclusion

We have proposed a way to represent a certain subclass of dual residuated lattices, including the direct products of totally ordered dual MV- or BL-algebras. Namely, a Boolean algebra or a Browerian algebra, respectively, is endowed with an equivalence relation subject to conditions which are chosen in accordance with the case that elements of equal size with regard to some measure are identified. We saw that, under a natural assumption, the set of equivalence classes bears the structure of an MV- or a dual residuated lattice, respectively. In particular, any totally ordered dual BL-algebra arises this way, and any dual BL-algebra is a subalgebra of an algebra arising this way.

It is not clear how the class of algebras arising by the proposed construction can be characterized. On the other hand, it would be worth to explore possibilities to make the formalism more flexible, to capture a larger class.

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