Defuzzification using Steiner points

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Abstract

A defuzzification function assigns to each fuzzy set a crisp value in a way that this value may intuitively be understood as the "centre" of the fuzzy set.

In the present paper, this vague concept is put into a mathematically rigorous form. To this end, we proceed analogously to the case of sharply bordered subsets, for which the Steiner point is frequently used. The function assigning to each convex subset its Steiner point is characterised by three properties; here, we study functions whose domains consist of fuzzy sets and which fulfil analogous properties.

Although uniqueness can no longer be achieved, we give a complete characterisation of what we call Steiner points of fuzzy sets.

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space, where $n \geq 2$, and let $\mathcal{K}^n$ be the set of all convex bodies of $\mathbb{R}^n$. Intuitively, the transition from $\mathbb{R}^n$ to $\mathcal{K}^n$ may be viewed as an intermediate step towards the fuzzification of $\mathbb{R}^n$. For, assume that rather than knowing about some precise location as described by a point of $\mathbb{R}^n$, you only know about bounds of this location in every direction; the area of space specified this way is an intersection of halfspaces, and in fact a convex body.

Let us assume that a convex body represents some value together with information about its impreciseness. The question naturally arises which value could be meant, that is, if every set $A \in \mathcal{K}^n$ may be reasonably viewed as a set around some central element $s(A) \in A$. The mapping $s \mapsto s(A)$ could then be considered as a "defuzzification" function.

Let us collect the minimal requirements which such a function $s\colon\ \mathcal{K}^n \to \mathbb{R}^n$ should fulfil. Probably most importantly, the point $s(A)$ should not depend on where and how A is positioned in space. Namely, s should be equivariant with respect to Euclidean isometries, which means that $s(\tau A) = \tau s(A)$ for any isometry τ. Moreover, s should respect the structure which $\mathcal{K}^n$ carries, both the linear and the topological one. Indeed, $\mathcal{K}^n$ is endowed with the pointwise addition generalising the addition on $\mathbb{R}^n$; and $\mathcal{K}^n$ is endowed with the topology induced by the Hausdorff metric; cf. e.g. [Sch2]. So s should be compatible with the addition, and s should be continuous.

These three properties are fulfilled by the function s which associates with each convex body its Steiner point; we give the definition at the beginning of Section 2. It was an open question for many years if there is any other such function, and it turned out that this is not the case; the Steiner point is unambiguously defined by the mentioned three conditions [She, Sch1].

Let us next consider the more general situation that our base set not only contains sharply limited subsets of a real Euclidean space, but also sets with unsharp boundaries. The fuzzy analogue of $\mathcal{K}^n$ is $\mathcal{E}^n$, the set of normal, support-bounded, upper semicontinuous, and fuzzy-convex fuzzy sets; cf. [DiKl]. We wonder what kind of "defuzzification" function exists on $\mathcal{E}^n$ with values in $\mathbb{R}^n$, such that conditions analogous to those characteristic for the Steiner point of convex bodies hold.

Steiner points on the space $\mathcal{E}^n$ of fuzzy sets will again be assumed to be equivariant with respect to Euclidean isometries, in a sense analogous to the crisp case. Furthermore, they are supposed to respect the structure inherent in $\mathcal{E}^n$. Indeed, the linear structure of $\mathcal{K}^n$ generalises in a straightforward way to $\mathcal{E}^n$; see e.g. [DiKl]. For the topology, however, we have to make a choice; in this paper, we will take the d_∞-metric on $\mathcal{E}^n$; cf. [Hei], or again [DiKl]. The conditions for functions $S\colon\ \mathcal{E}^n \to \mathbb{R}^n$ may then be formulated in complete analogy to the crisp case.

We succeeded to determine all functions fulfilling the three assumed properties; this is the main result of this paper. As our formulation already suggests, there is more than one; so uniqueness is lost. The representation of these functions will moreover show that it is difficult to make a canonical choice.

Remark 1.1 *Although not much recognised by practitioners, the Steiner point has unique advantageous properties as a point representing the position of a body. There are many situations when a body grows uniformly in all directions. This is for example the case of tumours, bacterial colonies, crystals, and the like. Also errors of observation can cause the same effect; for instance defocusing, bias of a measuring method, and many image processing techniques like mathematical morphology may result in a body whose shape differs from the original one by a constant in each direction.*

A natural idea is to describe the position of a body by its centre of gravity. However, for non-symmetric shapes it is not stable under uniform growth in all directions. In contrast to this, the Steiner point is preserved. This is because such a growth corresponds to the sum (in fact, convolution) of the original body with a ball centered at the origin of coordinates. Thus if we want to describe the position of a body by a point which is invariant under growth, the Steiner point is the unique solution.

One might object that the Steiner point is defined only for convex sets. However, its definition is applicable to non-convex bodies as well; in this case, we get the Steiner point of the convex hull of the original set. So we do not make use of the information about the non-convex part, but the original aim of stability under growth remains fulfilled.

Real objects have often unsharp boundaries (or our method of observation gives unsharp results) which can naturally be represented by fuzzy sets. To extend the positioning technique to this case, it is desirable to generalise the Steiner point to fuzzy sets.

2 The Steiner point of convex bodies

Let us fix some $n \geq 2$. By $\mathcal{K}^n$, we denote the set of convex bodies of $\mathbb{R}^n$, that is, the set of non-empty compact convex subsets of $\mathbb{R}^n$. The set $\mathcal{K}^n$ is endowed with a linear structure; the addition of two subsets and the multiplication of a subset by a positive real are defined pointwise. We furthermore endow $\mathcal{K}^n$ with the Hausdorff metric d_H.

We may identify $\mathcal{K}^n$ in a pleasant way with a positive cone of a Banach space as follows. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $\mathcal{C}(S^{n-1})$ be the space of continuous functions from S^{n-1} to $\mathbb{R}$, endowed with the supremum norm. Now, for any $A \in \mathcal{K}^n$, we define by

$$h_A\colon\ S^{n-1} \to \mathbb{R},\quad e \mapsto \max\{(a,e)\colon\ a \in A\}$$

the *support function* of A, see e.g. [Sch2]; here $(\cdot,\cdot)$ denotes the usual inner product of $\mathbb{R}^n$. Every support function is continuous, that is, in $\mathcal{C}(S^{n-1})$. Addition and multiplication by positive reals in $\mathcal{K}^n$ correspond to the same operations on the respective support functions. Moreover, the Hausdorff metric on $\mathcal{K}^n$ coincides with the metric of $\mathcal{C}(S^{n-1})$, i.e. the supremum metric. We will denote the set of all support functions of convex bodies by $\mathcal{S}^n$.

The investigations of this paper are based on the following facts [She, Sch1].

Definition 2.1 The *Steiner point* of $A \in \mathcal{K}^n$ is defined by

$$s(A) = \tfrac{1}{V(B^n)} \int_{S^{n-1}} h_A(e)\ e\ d\lambda(e),$$

where $e \in S^{n-1}$ varies over the unit vectors of $\mathbb{R}^n$, λ is the Lebesgue measure on S^{n-1}, and $V(B^n)$ is the volume of the unit ball B^n of $\mathbb{R}^n$.

Notice that $s(A) \in A$.

In the sequel, $\mathbb{R}^n$ is always assumed to be endowed with the Euclidean metric and the topology induced by it.

Furthermore, we will have to refer to several special types of Euclidean isometries of $\mathbb{R}^n$. By a *rotation*, we will always mean a proper rotation, that is, an isometry leaving the origin fixed and continuously connected to the identity. By a *reflection*, we will always mean a reflection leaving the origin fixed, that is, an involutive isometry whose set of fix points is a hyperplane containing the origin. Finally, by a *rigid motion* we will mean an isometry composed of rotations and translations.

The following theorem is for the case $n = 2$ due to Shephard [She] and for the case $n \geq 3$ due to Schneider [Sch1].

Theorem 2.2 *Let $s'\colon\ \mathcal{K}^n \to \mathbb{R}^n$ have the following properties:*

(S1) *For any $A, B \in \mathcal{K}^n$, $s'(A + B) = s'(A) + s'(B)$.*

(S2) *For $A \in \mathcal{K}^n$ and any rigid motion τ, we have $s'(\tau A) = \tau s'(A)$.*

(S3) *s' is continuous.*

Then $s' = s$.

The remaining part of this section contains auxiliary results which will be needed in the sequel. The first, crucial Lemma 2.3 is an extended version of [Sch1, Lemma 2]. We will have to reproduce Schneider's proof partly, but for full details and for left out parts, we refer to [Sch1]. We will furthermore use the idea of the proof of [Groe, Theorem 4.6.1].

In what follows, if τ is a rotation or a reflection and $f \in \mathcal{C}(S^{n-1})$, we will denote by τf the left translate of f, that is, $(\tau f)(e) = f(\tau^{-1}(e))$.

A spherical harmonic of dimension n and degree $d = 0, 1, \ldots$ is the restriction to the unit sphere S^{n-1} of a polynomial $\mathbb{R}^n \to \mathbb{R}$ which is harmonic and homogeneous of degree d. By $\mathcal{H}$, we will denote the subspace of $\mathcal{C}(S^{n-1})$ consisting of finite sums of spherical harmonics. The space $\mathcal{H}$ is dense in $\mathcal{C}(S^{n-1})$ [Sch1, Lemma 1]. For more information on spherical harmonics in the present context, we recommend [Groe].

Lemma 2.3 *Let $t\colon\ \mathcal{H} \to \mathbb{R}^n$ be a linear function such that $t \circ \tau = \tau \circ t$ for all rotations τ. Then there is a real number κ_t and a rotation ρ_t such that*

$$t(f) = \kappa_t\ \rho_t\left(\int_{S^{n-1}} f(e)\ e\ d\lambda(e)\right), \tag{1}$$

Moreover, $\rho_t = id$ if (i) *$n \geq 3$ or* (ii) *$n = 2$ and t commutes also with all reflections.*

Proof. Recall first that if f is a spherical harmonic of degree $d \neq 1$, then $\int_{S^{n-1}} f(e)\ e\ d\lambda(e) = 0$. So we have to show that $t(f) = 0$ in this case and that (1) holds if f is of degree 1.

A spherical harmonic f of degree 0 is a constant; so in this case, $t(f) = t(\tau f) = \tau(t(f))$ for all rotations, whence $t(f) = 0$.

For each spherical harmonic f of degree 1, there is a $c \in \mathbb{R}^n$ such that $f = f_c$, where $f_c(e) = (c, e)$. So $T : \mathbb{R}^n \to \mathbb{R}^n,\ c \mapsto t(f_c)$ is a linear function such that, for every rotation τ, $T(\tau(c)) = t(f_{\tau(c)}) = t(\tau f_c) = \tau(t(f_c)) = \tau(T(c))$, because $f_{\tau(c)}(e) = (\tau(c), e) = (c, \tau^{-1}(e)) = f_c(\tau^{-1}(e)) = (\tau f_c)(e)$ for $e \in S^{n-1}$. If now $n = 2$, it follows $T = \kappa_t' \rho_t$ for some $\kappa_t' \in \mathbb{R}$ and a rotation ρ_t. If then t commutes with reflections, so does T, whence ρ_t equals *id* or $-$*id*. Moreover, if $n \geq 3$, we have $T = \kappa_t'$ *id* for some $\kappa_t' \in \mathbb{R}$. Using the fact that $c = n \int_{S^{n-1}} (c, e)\ e\ d\lambda(e)$, equation (1) follows for $\kappa_t = \frac{\kappa_t'}{n}$.

Next, let f be a spherical harmonic of degree $d \geq 2$. For the proof that $t(f) = 0$ if $n \geq 3$, we refer to [Sch1, Lemma 2]. Let $n = 2$. Denoting by $\omega(e)$ the angle corresponding to $e \in S^1$, we have $f(e) = a \cos d\,\omega(e) + b \sin d\,\omega(e)$, where $a, b \in \mathbb{R}$. Let τ be the rotation by $\frac{2\pi}{d}$; then $\tau t(f) = t(\tau f) = t(f)$, whence $t(f) = 0$. $\square$

We will now denote by $\mathcal{K}_0^n$ the set of convex bodies of $\mathbb{R}^n$ containing 0.

Lemma 2.4 *Let* $t :\ \mathcal{K}_0^n \to \mathbb{R}^n$ (i) *preserve sums,* (ii) *commute with rotations, and* (iii) *be continuous. Then for some constant* $\kappa_t \in \mathbb{R}$ *and a rotation* ρ_t *of* $\mathbb{R}^n$

$$t(A) = \kappa_t\ \rho_t\Bigg(\int_{S^{n-1}} h_A(e)\ e\ d\lambda(e)\Bigg), \tag{2}$$

where h_A *is the support function of* $A \in \mathcal{K}_0^n$.

Moreover, $\rho_t =$ *id if* (i) $n \geq 3$ *or* (ii) $n = 2$ *and* t *commutes also with all reflections.*

Proof. Note that $\mathcal{K}_0^n$ is closed under sums and multiplication by positive reals. Moreover, t preserves sums by assumption, consequently also the multiplication by positive rationals, and so by continuity also the multiplication by positive reals. Let $\mathcal{S}_0^n = \{h_A :\ A \in \mathcal{K}_0^n\} \subseteq \mathcal{S}^n$.

Let $\mathcal{H}_0 = \mathcal{S}_0^n \cap \mathcal{H}$. Then $\mathcal{H}_0$ is dense in $\mathcal{S}_0^n$. Indeed, given an $A \in \mathcal{K}_0^n$ and $\varepsilon > 0$, there is some $B \in \mathcal{K}^n$ such that $h_B \in \mathcal{H}$ and $d(A, B) < \varepsilon$; and B can be moved by a vector of length less than ε to a set C containing the origin; then $h_C \in \mathcal{H}_0$ and $d(B, C) < \varepsilon$.

Let $t_\mathcal{H}: \mathcal{H}_0 \to \mathbb{R}^n, \; h_A \mapsto t(A)$. For $h \in \mathcal{H}_0$ and a positive constant c, we have $t_\mathcal{H}(h) = t_\mathcal{H}(h+c)$; indeed, for any rotation τ we have $\tau\, t_\mathcal{H}(c) = t_\mathcal{H}(\tau c) = t_\mathcal{H}(c)$, whence $t_\mathcal{H}(c) = 0$. Furthermore, if $f \in \mathcal{H}$, there is a $c \geq 0$ such that $f + c \in \mathcal{H}_0$; indeed, by [Sch2, Lemma 1.7.9], there is $c' \geq 0$ such that $f + c'$ is a support function, and consequently there is some $c \geq c'$ such that $f + c$ is the support function of a set containing the origin. It follows that we may extend $t_\mathcal{H}$ to the whole space $\mathcal{H}$ unambiguously by setting $t_\mathcal{H}(f) = t_\mathcal{H}(f+c)$, where for each $f \in \mathcal{H}$ we choose a $c \geq 0$ sufficiently large.

The extended function $t_\mathcal{H}: \mathcal{H} \to \mathbb{R}^n$ is easily seen to be a vector space homomorphism, which moreover commutes with rotations. By Lemma 2.3, we conclude $t_\mathcal{H}(f) = \kappa_t \;\; \rho_t(\int_{S^{n-1}} f(e) \;\; e \;\; d\lambda(e))$ for any $f \in \mathcal{H}$, where $\kappa_t \in \mathbb{R}$ and ρ_t is a rotation. This proves (2) for all A whose support functions are in $\mathcal{H}$. So the assertion follows by the continuity of t.

In case that $n = 2$ and t commutes with reflections, $t_\mathcal{H}$ also commutes with reflections. So again by Lemma 2.3, we conclude $\rho_t = id$, as well as in the case $n \geq 3$. □

3 Steiner points of fuzzy sets

Our considerations will refer to the set $\mathcal{E}^n$ of fuzzy sets over the $\mathbb{R}^n$ as defined by Diamond and Kloeden in [DiKl]. The space $\mathcal{E}^n$ contains by definition all maps $\bar{u}$ from $\mathbb{R}^n$ to the real unit interval $[0,1]$ such that (i) $\bar{u}$ attains the value 1 at some point $x \in \mathbb{R}^n$, (ii) the support of $\bar{u}$ is bounded, (iii) $\bar{u}$ is upper semicontinuous, and (iv) $\bar{u}$ is fuzzy-convex. For details on these notions, we refer to [DiKl].

These conditions take a particularly easy form when we switch from the "vertical" to the "horizontal" viewpoint; and this is what we will do here. Namely, given some $\bar{u} \in \mathcal{E}^n$, let u be the function associating to each $\alpha \in (0,1]$ its α-level set, given by $[\bar{u}]^\alpha = \{x \in \mathbb{R}^n : \; \bar{u}(x) \geq \alpha\}$, and to 0 its support $[\bar{u}]^0$, which is the closure of $\{x \in \mathbb{R}^n : \; \bar{u}(x) > 0\}$. Then u is a mapping from $[0,1]$ to non-empty compact convex subsets of $\mathbb{R}^n$, that is, to $\mathcal{K}^n$. With respect to the topology of $\mathcal{K}^n$ defined above and the partial order of $\mathcal{K}^n$ given by inclusion, u is decreasing, left-continuous on $(0,1]$

and continuous at 0.

We denote by $\mathcal{F}^n$ the set of all functions from $[0,1]$ to $\mathcal{K}^n$ which are (i) decreasing and (ii) left-continuous on $(0,1]$ and continuous at 0. The mapping $\bar{u} \mapsto u$ defines a one-to-one correspondence between $\mathcal{E}^n$ and $\mathcal{F}^n$; see [NeRa], or [DiKl, Prop. 6.1.6]. In this paper, we will deal with $\mathcal{F}^n$ exclusively: From now, by a fuzzy set we will *always* mean an element of $\mathcal{F}^n$ rather than of $\mathcal{E}^n$.

Like $\mathcal{K}^n$, the set $\mathcal{F}^n$ bears a natural linear and metric structure. Namely, addition and multiplication by positive reals are defined pointwise; see [Ngu]. Moreover, we endow $\mathcal{F}^n$ with the supremum metric d_∞ [DiKl]: For $u, v \in \mathcal{F}^n$, we let $d_\infty(u,v) = \sup_{0 \leq \alpha \leq 1} d_H(u(\alpha), v(\alpha))$.

Although we will not make use of it, we include the following note. As we know how to embed the set $\mathcal{K}^n$ of convex bodies into a Banach space, it is immediate how we may represent $\mathcal{F}^n$ as a positive cone of a Banach space as well. As the smallest possible one, we may consider $\mathcal{R}([0,1], \mathcal{C}(S^{n-1}))$, the space of regulated and left-continuous functions from $[0,1]$ to $\mathcal{C}(S^{n-1})$, endowed with the supremum norm. Recall that a function is called *regulated* if at any point $\alpha \in [0,1]$ both the left and the right limit exist; see e.g. [Hoe]. A fuzzy set $u \in \mathcal{F}^n$ may then be identified with the element $[0,1] \to \mathcal{C}(S^{n-1}), \;\; \alpha \mapsto h_{u(\alpha)}$ of $\mathcal{R}([0,1], \mathcal{C}(S^{n-1}))$.

We will now formulate conditions for a "defuzzification" function on $\mathcal{F}^n$ in analogy to the conditions appearing in Theorem 2.2. We will add one basic further property explicitly: A Steiner point should never be located outside the support of a fuzzy set.

In the sequel, for a fuzzy set $u \in \mathcal{F}^n$ and a rigid motion τ, we set $\tau u : [0,1] \to \mathcal{K}^n, \;\; \alpha \mapsto \tau(u(\alpha))$. That is, expressed for the corresponding element $\bar{u} \colon \mathbb{R}^n \to [0,1]$ of $\mathcal{E}^n$, we have $\tau\bar{u} = \bar{u} \circ \tau^{-1}$.

Definition 3.1 A function $S \colon \mathcal{F}^n \to \mathbb{R}^n$ is called a *Steiner point* if it has the following properties:

(SF0) For any $u \in \mathcal{F}^n, \; S(u) \in u(0)$.

(SF1) For any $u, v \in \mathcal{F}^n, \; S(u+v) = S(u) + S(v)$.

(SF2) For $u \in \mathcal{K}^n$ and any rigid motion τ, we have $S(\tau u) = \tau S(u)$.

(SF3) S is continuous.

We will next see how we may generate typical examples of Steiner points. Note first that, for a fixed α, the function $S\colon\ \mathcal{F}^n \to \mathbb{R}^n$ which simply maps each fuzzy set to the Steiner point of its α-level set fulfils all conditions (SF0)–(SF3) – and is thus a Steiner point. This already implies that we have more than one possibility to define a Steiner point.

For the general case, we need some preparations. We shall call $\mu\colon\ [0,1] \to [0,1]$ a *measure function* if (i) μ is increasing, that is, $\mu(\alpha) \leq \mu(\beta)$ if $\alpha < \beta$ and (ii) $\mu(0) = 0$ and $\mu(1) = 1$. Furthermore, let us call a finite sequence $D = (\alpha_0, \ldots, \alpha_k)$ a *division* of $[0,1]$ if $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$, where $|D| = k \geq 1$. Let $\mathcal{D}$ be the collection of all divisions of $[0,1]$, and endow $\mathcal{D}$ with the inclusion as a partial order; $\mathcal{D}$ then becomes a directed set.

We define the integral of a regulated function $f:\ [0,1] \to \mathbb{R}^n$ over $[0,1]$ w.r.t. a measure function μ as follows:

$$\int_{[0,1]} f(\alpha)\ d\mu(\alpha) = \lim_{D\in\mathcal{D}} \sum_{i=0}^{|D|-1} f(\xi_i)(\mu(\alpha_{i+1}) - \mu(\alpha_i)),$$

where $D = (\alpha_0, \ldots, \alpha_{|D|})$ and $\xi_i \in (\alpha_i, \alpha_{i+1})$ for $0 \leq i < |D|$. This limit exists and is called the *interior integral*. For further information, we refer to [Hoe].

Proposition 3.2 *Let $\mu\colon\ [0,1] \to [0,1]$ be a measure function. For $u \in \mathcal{F}^n$, let*

$$S_\mu(u) = \int_{[0,1]} s(u(\alpha))\ d\mu(\alpha), \tag{3}$$

where s is the Steiner point of crisp sets. Then S_μ is a Steiner point.

In the one-dimensional case, a function very similar to S_μ as defined by (3) has been proposed for defuzzification; Dubois and Prade introduced in [DuPr] a defuzzification method called *averaging level cuts* (ALC). Moreover, in [Ous], it is observed that in contrast to the centre of gravity and certain further defuzzification methods, this one is compatible with the linear operations. Another defuzzification function which is still similar to ours, is due to Filev and Yager; see [FiYa, Yag]. All the above papers refer only to the one-dimensional case. In [RoSp], Roventa and Spircu treat also multidimensional fuzzy sets and introduce a formula based on an integral over level sets (so-called *distribution averaging*). The use of Steiner points seems to be a novelty in our approach.

Here, we will show that S_μ as given by (3) is the most general example of a Steiner point. In other words, S_μ, where μ varies over the measure functions, represent exactly those defuzzification methods which are not only compatible with the linear operations, but also equivariant with respect to Euclidean isometries, continuous, and such that the crisp values are within the support of each fuzzy set.

To impose further properties on a Steiner point in addition to (SF0)–(SF3) to obtain uniqueness is amazingly difficult; it is an open question if this is possible in some reasonable, well motivated way.

4 Characterisation of Steiner points – the case of step fuzzy sets

Before considering the problem how to characterise the Steiner points defined in the last section, we will solve a simplified version. Namely, we will assume that our fuzzy universe contains only fuzzy sets which are piecewise constant. According to the usual ("vertical") viewpoint, these are fuzzy sets which map from $\mathbb{R}^n$ to certain finitely many values from $[0,1]$ only. We will call them step fuzzy sets.

Definition 4.1 Let $D = (\alpha_0, \ldots, \alpha_k)$ be a division of $[0,1]$. Then we call a fuzzy set $u \in \mathcal{F}^n$ a *D-step fuzzy set* if it is constant on $[\alpha_0, \alpha_1], (\alpha_1, \alpha_2], \ldots, (\alpha_{k-1}, \alpha_k]$, respectively. We denote by $\mathcal{F}^n_D$ the set of all D-step fuzzy sets.

Furthermore, by a *step fuzzy set* we mean a fuzzy set which is a D-step fuzzy set for some division D of $[0,1]$.

Lemma 4.2 *Let $D = (\alpha_0, \ldots, \alpha_k)$ be a division of $[0,1]$. Let $S\colon\ \mathcal{F}^n_D \to \mathbb{R}^n$ be a function fulfilling the properties* (SF0)–(SF3) *of Definition* 3.1. *Then there are unique real numbers $\kappa_1, \ldots, \kappa_k$ such that $\kappa_1 + \ldots + \kappa_k = 1$ and* *for all* $u \in \mathcal{F}_D^n$

$$S(u) = \kappa_1 s(u(\alpha_1)) + \ldots + \kappa_k s(u(\alpha_k)). \tag{4}$$

Proof. For $1 \leq i \leq k$ and $A \in \mathcal{K}_0^n$, let $u_{i,A}$ be the fuzzy set which is constantly A on $[0, \alpha_i]$ and constantly $\{0\}$ on $(\alpha_i, 1]$. Then the mapping $\mathcal{K}_0^n \to \mathbb{R}^n, \;\; A \mapsto S(u_{i,A})$ preserves sums, commutes with rotations, and is continuous. So by Lemma 2.4, $S(u_{i,A}) = \bar{\kappa}_i \;\; \rho_i(\int_{S^{n-1}} h_A(e) \;\; e \;\; d\lambda(e)) = \bar{\kappa}_i \;\; \rho_i(s(A))$ for some $\bar{\kappa}_i \in \mathbb{R}$ and some rotation ρ_i.

In case $\bar{\kappa}_i = 0$ we may assume $\rho_i =$ *id*. Otherwise, $\rho_i =$ *id* is implied by (SF0). Indeed, consider the case that A is the straight line between the origin 0 and some point different from 0. Then $s(A)$ is the midpoint of A, that is, different from 0; and by (SF0), $S(u_{i,A})$ is on A. Consequently ρ_i must be the trivial rotation. It follows that, for any $A \in \mathcal{K}_0^n$ and $i = 1, \ldots k$, we have $S(u_{i,A}) = \bar{\kappa}_i \; s(A)$.

Next let u be any D-step fuzzy set; let $u(\alpha_i) = A_i$, $i = 1, \ldots, k$. Then $u + u_{1,A_2} + \ldots + u_{k-1,A_k} = u_{1,A_1} + \ldots + u_{k,A_k}$. Applying S to both sides, we get $S(u) = S(u_{1,A_1}) + S(u_{2,A_2}) - S(u_{1,A_2}) + \ldots + S(u_{k,A_k}) - S(u_{k-1,A_k}) = \bar{\kappa}_1 s(A_1) + (\bar{\kappa}_2 - \bar{\kappa}_1) s(A_2) + \ldots + (\bar{\kappa}_k - \bar{\kappa}_{k-1}) s(A_k)$. So (4) is proved for the case that $0 \in u(0)$.

For the general case, let us denote by $b_c \colon [0,1] \to \mathcal{K}^n$ the fuzzy set which is constantly B_c – the ball with diameter c centered at 0. Note that $S(b_c) = 0$ because of the rotation invariance of S. So given an arbitrary D-step fuzzy set u, choose $c \geq 0$ large enough such that $0 \in u(0) + b_c$; then $S(u) = S(u + b_c)$. Because the right-hand side of (4) does not change when replacing u by $u + b_c$, (4) follows.

Finally, let $c \in \mathbb{R}^n$ and denote by the singleton $\{c\}$ also the fuzzy set being constantly $\{c\}$. By translation invariance, we have $S(u) + S(\{c\}) = S(u + \{c\}) = S(u) + c$ for any $u \in \mathcal{F}^n$, so $S(\{c\}) = c$. It follows that $\kappa_1 + \ldots + \kappa_k = 1$; this completes the proof. $\square$

The preceeding Lemma 4.2 shows why we postulated that Steiner points should fulfil condition (SF0). We note that it still holds if we drop (SF0), but restrict to a dimension at least 3.

Next we will show a further consequence of (SF0): it causes all the coefficients in (4) to be non-negative.

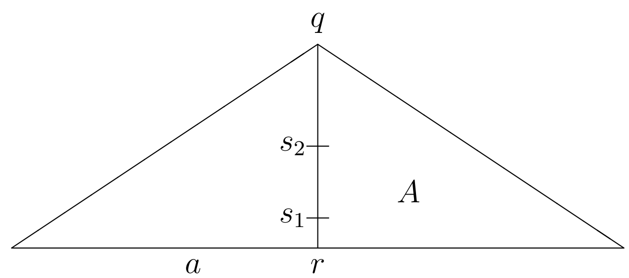
We will use the following fact. The Steiner point of a polytope A with vertices $p_1, \ldots, p_l$ is given as follows:

$$s(D) = \sum_{i=1}^{l} \psi(p_i, A)\; p_i, \tag{5}$$

where $\psi(p_i, A)$ is, for $i = 1, ..., l$, the external angle of A at p_i; this is the proportion of the area of the unit sphere which is taken by the normal vectors of the supporting planes of A at p_i. See e.g. [Grue].

Lemma 4.3 *Let $D = (\alpha_0, \ldots, \alpha_k)$ be a division of $[0, 1]$. Let $S\colon\ \mathcal{F}^n_D \to \mathbb{R}^n$ be a function fulfilling the properties* (SF0)–(SF3) *of Definition* 3.1. *Let $\kappa_1, \ldots, \kappa_k$ be the unique real numbers fulfilling* (4). *Then $\kappa_1, \ldots, \kappa_k \geq 0$.*

Proof. Without loss of generality, we may restrict to the case $n = 2$. Furthermore, assume that $k = 3$ and that $\kappa_1, \kappa_3 \geq 0$, but $\kappa_2 < 0$; the general case is not more difficult.



Let A be an equilateral triangle with base side a; let r be the middle point of a; and let q be the vertex of A opposite to a. Now let s_1 be the Steiner point of A, and let s_2 be located half-way between r and q. As a consequence of (5), the proportion η of the distances $s_1 r$ and $s_2 r$ can be made arbitrarily small; to this end, we keep the height of A fixed and make a as large as necessary.

Now, define $u :\ [0, 1] \to \mathcal{K}^2$ as follows. On $[0, \alpha_1]$, put u to A, so that $s(A) = s_1$. On $(\alpha_1, \alpha_2]$, put u to some B such that $s_1 \in B \subseteq A$ and $s(B) = s_2$. On $(\alpha_2, \alpha_3]$, put u to the singleton $\{s_1\}$. If η is sufficiently small, $s(u)$ is outside A. $\square$

5 Characterisation of Steiner points – the general case

We shall now state our main result: the representation of Steiner points.

Theorem 5.1 *Let $S\colon\ \mathcal{F}^n \to \mathbb{R}^n$ be a Steiner point. Then there is a measure function $\mu\colon\ [0,1] \to [0,1]$ such that for all $u \in \mathcal{F}^n$*

$$S(u) = \int_{[0,1]} s(u(\alpha))\ d\mu(\alpha). \tag{6}$$

Proof. For every $\alpha > 0$, let $\mu(\alpha)$ be the coefficient corresponding to the interval $[0,\alpha]$ according to Lemma 4.2. By (4), equation (6) holds for all step fuzzy sets.

Furthermore, let $u \in \mathcal{F}^n$ be arbitrary. Then u is by definition left-continuous and regular. Because s is continuous, the mapping $s \circ u\colon\ [0,1] \to \mathbb{R}^n,\quad \alpha \mapsto s(u(\alpha))$ is left-continuous and regular; so the integral (6) exists.

Moreover, it follows that u is the uniform limit of step fuzzy sets; see e.g. [Hoe]. So by the continuity of S as well as the integral, (6) holds for u. $\square$

We may summarise that any Steiner point of fuzzy sets may be seen as a weighted average of the Steiner points of the level sets of each fuzzy set.

We conclude our paper with one additional remark. It concerns the one-dimensional case, which was not treated in this paper. In the case $n = 1$, the only convex bodies are closed intervals. Their centres of gravity coincide with the Steiner points. Then the collection $\mathcal{F}^1$ consists of all fuzzy intervals, i.e., fuzzy sets whose levels sets are closed intervals. We again have some freedom in the choice of a Steiner point on $\mathcal{F}^1$. Moreover, the argument of Lemma 4.3 does not work and the coefficients κ_i may be also negative.

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