Vagueness: where degree-based approaches are useful, and where we can do without

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Abstract

The vagueness of a property becomes apparent when more than one level of granularity is addressed in a discourse. For instance, the adjective "tall" can be used to distinguish just two kinds of persons – those who are tall as opposed to those who are not. This coarse distinction contrasts with any finer one, in particular with the finest possible one, on which we specify sizes, beyond all limits of precision, by real numbers. We understand vagueness as a relative notion, causing a problem when information on a coarse level is to be transferred to a fine level.

Under this viewpoint it is no problem to accept that, depending on the application, reasoning under vagueness may require different formal frameworks. In this paper, we consider the same kind of vague properties in two different contexts. We first discuss a version of the sorites paradox. In this case, it is necessary to combine reasoning on two levels of granularity in a single formalism. Following established practice, we choose an approach based on numerical degrees. Second, we consider generalised Aristotelian syllogisms. In this case, degree-based solutions are unnecessarily rich in structure and are not found adequate. We give preference to a formalism that stays entirely on the coarse level of argumentation.

We conclude that there is no reason to call for a uniform formalism to cope with the problem of vagueness. In particular, degree-based approaches are usually applicable, but there can be simpler alternatives. Different problems may call for different solutions, and choosing diversity does not mean that we approach the problem of vagueness incoherently.

1 Introduction

Vagueness in natural language is a topic that has been studied intensively in recent times. A variety of approaches has been developed to deal with vague information in practical applications. Usually, pragmatism dominates over well-founded principles. It seems that in most cases Zadeh's fuzzy set model is employed, whose popularity is in spite of all criticism unbroken. For a general overview of fuzzy set theory see, e.g., [DuPr2]. Quite independently from the efforts to cope with vagueness in practical respects, approaches have been developed to understand this feature of natural language on a foundational level. An intensive debate in philosophy is ongoing. In this case, the picture is not dominated by a single idea. An overview of approaches proposed in recent years can be found, e.g., in [Smi].

Vagueness is usually associated with the existence of borderline cases. It pertains to properties such that there is a continuous transition from the objects fulfilling the property to the objects not fulfilling the property. For instance, the adjective "large" refers vaguely to the size of objects, because objects vary in size continuously. As a consequence, a set of objects specified by an expression like "large" is not sharply delimitable. A so-called borderline-large object is an object which is not large enough to be called "large" and the same time not small enough to be called "not large".

This paper is meant as a contribution to the general problem of how to reason with vague information by formal means. We pick up a particular aspect of the discussion; namely, we support our negative attitude to the frequently expressed concern of finding a single appropriate method to reason in the presence of vagueness.

Our considerations are based on the recent paper [Vet], in which we develop a particular standpoint with regard to the vagueness debate. Our approach is called *perceptionalism* and relies on the conviction that it is our perceptions that are to be considered as the basic constituents of reality, rather than the ready-made world consisting of moving particles and existing independently of a human observer. We view properties like "small" and "large" as describing the observation of objects in a comparative way rather than the objects themselves. It is then just natural that descriptions of the same objects can be made in different frameworks and may in particular refer to different levels of granularity. For formal reasoning under vagueness, the challenge is to find models for the varying levels of granularity relevant in a particular context. This is easy if there is only one level involved, but tricky if more than one level is to be taken into account.

In the philosophical debate, efforts are frequently devoted to the aim to find the one and only way to argue about vague properties "correctly". This aim is not in accordance with the point of view adopted in [Vet]. In fact there is no reason to expect that under different circumstances similar ways of formal reasoning are appropriate.

The present paper discusses two examples: the sorites paradox, and generalised Aristotelian syllogisms. In both cases, we deal with roughly specified proportions of a whole. Reasoning frameworks are already available for both considered situations. Here we are interested in the particular feature by which the two cases differ. For, the difference suggests to proceed formally in totally different ways.

The paper is organised as follows. The first half of the paper is devoted to the sorites paradox. We comment on the variety of approaches to the paradox in Section 2. We then develop a formalism to cope with the paradox on the basis of the Logic of Approximate Entailment (LAE) [Rod]. Section 3 provides a general introduction to LAE; in Section 4, we incorporate elements of first-order logic; and we finally explain in Section 5 how the resulting formalism applies to the sorites paradox.

In the second part of the paper, we discuss generalised Aristotelian syllogisms. In

Section 6, we again review existing approaches. Then we make clear how a logic to deal with the syllogisms should look like when based on our ideas, and we propose a specific such logic in Section 7. How this logic applies to the syllogisms is shown in Section 8. Section 9 contains a summary of the paper and some concluding remarks.

2 The sorites paradox

It is amazing in how many scientific contributions the sorites paradox is dealt with. The paradox is in fact an essential part of the debate on vagueness.

The sorites paradox is highly relevant as with practically any vague property we may associate one of its instances. We choose here a version in accordance with the setting of the second part of this paper.

Consider a teapot with a volume of 1 litre which is leaky. Assume that when the pot is filled with tea every second one drop of tea leaks out of it. Assume furthermore that one drop has a volume of 0.02 millilitres. In this setting, the sorites argument goes as follows.

(A) When the leaky teapot is nearly full, so it is one second later.

(B) Filled with 48000 drops of tea, the teapot is nearly full.

Consequently,

(C) the empty teapot is nearly full.

As a paradox, it actually does not represent acceptable reasoning, but rather a failure of reasoning; a formal framework is requested to overcome the deficiency. An abundance of suggestions has been made.

In mathematics, degree-based approaches dominate. In fact, soon after L. A. Zadeh in his seminal paper [Zad1] established the notion of a fuzzy set, J. A. Goguen suggested a framework for reasoning on the basis of a continuous set of truth degrees [Gog]. Speaking in modern terminology, his inference method is based on product logic, that is, the fuzzy logic based on the product of real numbers to interpret the conjunction and the corresponding residuum to interpret the implication. In [HaNo], V. Novák and P. Hájek followed similar lines, using the nowadays well established frameworks for fuzzy logic. Further related papers include [Pao, WaVe, Pel, KeVa].

On the philosophical side, the picture is non-uniform. A solution of the sorites paradox is a "test case" for all approaches viewing vagueness from a fundamental perspective. I shall mention a few important references: for epistemicism, see [Wil, Ch. 7, App.]; for supervaluationism, see [Fin]; for contextualism, see [Sha]. To understand the solution of the sorites paradox from a philosophical viewpoint we would be required to get familiar with one of the theories of vagueness; we cannot address such a demanding task here. We restrict to the remark that none of the mentioned approaches are compatible with the idea of using degrees, as proposed, e.g., in [HaNo].

In order to justify our own choice how to deal with sorites-like arguments, we must review our approach developed in [Vet]. First to mention, there is a feature common in the debate among philosophers which in our opinion prevents an easy solution. The discussion is dominated by a realistic world view: it seems to be normal to assume that everything tangible in the world is determined by something outside the influence of humans. It is then natural to understand statements of natural language as telling something about the world, whose existence is assumed to be decoupled from its observation. Consequently, there exist independent criteria according to which natural-language statements are true or false. Just like in predicate logic, truth conditions are asked for. Solutions along such lines are complicated.

We reject the idea that natural language is subject to truth conditions referring to an independent, all-encompassing world. The reference to fixed structures is rather a feature of propositions of formal languages; formal languages are associated with abstract mathematical structures. Instead, we view utterances in natural language as describing perceptional impressions. Saying, for instance, "K. is tall", I do not express an independently holding truth; I am not, correctly or falsely, claiming that K.'s precise size is larger than the medium size of the people under consideration. In fact, the concept of a precise size does not correspond to any observation but is a mathematical notion. Instead, I express my impression that K. is taller than most people I have presently in mind.

But observations are bound to the observer's perspective and in particular to a certain level of granularity. Observations have a comparative character and a comparison can be performed in more or less detail. That is, to observe means to classify and the classification can be done at a scale with more or less many entries. When saying "K. is tall", I could have in mind simply a two-element scale, distinguishing between "tall" and "not tall". When saying "K. is around 180 cm tall", I could have in mind a scale distinguishing differences of 5 cm. The concept of total preciseness is a mathematical limit construction, based on the fact that observations of a given precision can always be imagined to be even more precise.

From this perspective, vagueness is a relative notion; an expression is vague relative to a finer scale than the one associated to it in a given context. Thus vagueness becomes problematic in situations where we switch between levels of granularity. In the present discussions typically one coarse level together with the finest possible one is considered: a natural language expression, like "tall", together with a mathematical structure, like the positive reals. The challenge is then to deal with the coarse notion within the fine structure. In the general case, the task is to define a common refinement and to embed into it all occurring notions.

Thus in the presence of vague properties, our primary task is to deal coherently with different levels of granularity. Several formal approaches that correspond to this challenge have been developed. Not all of them aim at a formal treatment of vagues properties but have been created for other motivations. Let us mention a few well-known research lines.

As a model that seems to correspond to our concerns quite directly, we may consider partitions of the universe of discourse of varying coarseness. This idea has been developed within Z. Pawlak's theory of rough sets [Paw1, Paw2]. The rough set model has rarely been considered in connection with vagueness, an exception being, for instance, [BiSt]. Further examinations in this direction seem to be worthwhile.

Another idea of how to bring together different levels of granularity makes use of degrees of compatibility. To this end, we first choose a set of continuously variable degrees, usually the real unit interval. Given an element of a coarse scale, we then specify its compatibility with each of the elements of the fine scale by choosing an element of the set of degrees. For instance, the coarse scale may consist of the elements "small", "medium-sized", and "large", and the fine scale may be $\mathbb{R}^+$. Then, to each of the three natural-language expressions and each positive real number, we may assign a real between 0 ("totally incompatible") and 1 ("perfectly compatible").

This is the idea underlying Zadeh's notion of a fuzzy set [Zad1]. That is, it is the idea on which the most popular practical approach to vagueness is based. Reasoning has been formalised, for instance, within t-norm based many-valued logic [HaNo]. Among philosophers, the idea has often been considered as well, but rarely complaisantly; an exception is [Smi].

Finally, we may mention similarity-based reasoning. We may decide to allow the use of notions referring to a coarse level also within a fine level, but then only in the prototypical cases. Moreover, we may endow the fine structure with a similarity relation; a non-prototypical element has then a certain resemblance with the closest prototype. This again leads to the possibility to say that a coarse property holds to a certain degree.

For instance, the property "small" may be modelled by a crisp set of positive reals, namely those which clearly represent a small size, together with a similarity relation on the set of positive reals. Then we can assign to any precise size $r \in \mathbb{R}^+$ its similarity with the set of sizes prototypical for "small". We are again led to a fuzzy set, and conversely, any fuzzy set may interpreted in this way; see, e.g., [DuPr1].

One framework for reasoning about similarities is due to G. Gerla [Ger]. A universe of discourse is endowed with a fuzzy equivalence relation, and truth degrees refer to similarities. Another approach which uses a similarity relation not on a universe of discourse but directly on the set of propositions has been proposed by M. Ying [Ying].

The approach presented in this paper follows the ideas of similarity-based reasoning. Our priority is a clear and well-justifiable framework, even if, admittedly, the calculus to which we are led is not particularly elegant. The topic of the subsequent chapters will be approximate reasoning in the sense of Ruspini [Rus].

3 The Logic of Approximate Entailment

A framework for reasoning which allows conclusion to be drawn even if they are only approximately correct is due to E. Ruspini [Rus]. His proposal has been elaborated in a series of further papers from a logical point of view [DPEGG, EGGR, GoRo]. In particular, the Logic of Approximate Entailment, or LAE for short, was introduced in R. O. Rodríguez' thesis [Rod]. Our aim is to demonstrate that LAE is able to cope with the vagueness of natural-language expressions.

LAE makes use of a continuous set of degrees, just like fuzzy logic. Unlike fuzzy logic, LAE does not suffer from conceptual arbitrariness; all what LAE requires is (the generalisation of) a metric space. LAE is based on the notion of similarity, just like the approach of Ying. Unlike Ying's logic, however, similarity refers in LAE to the underlying universe of discourse and no special decision is needed concerning compound propositions.

In this section, we shall shortly specify LAE. Our universe of discourse is a non-empty set W, representing the variety of possible situations which we are going to consider. The elements of W are commonly called *possible worlds*. Properties will be modelled by (crisp) subsets of W.

In classical propositional logic, the consequence relation between propositions is reflected by the subsethood relation. In approximate reasoning, we deal with the case that propositions imply each other only approximately; accordingly, a mean is provided to express that the subsethood relation holds only to a certain degree. Namely, the set of worlds W is endowed with a metric, or more generally, with a similarity relation s. We assume that a given world may more or less resemble to another one; s maps pairs of worlds to a number between 0 and 1 and expresses in this way a degree of similarity between them.

For the sequel, let $[0, 1]$ be the real unit interval and let $\odot\colon [0, 1]^2 \to [0, 1]$ be a fixed t-norm.

Definition 3.1. Let W be any non-empty set. A function $s\colon W \times W \to [0, 1]$ is called a *similarity relation* on W w.r.t. $\odot$ if, for any $u, v, w \in W$,

(S1) $s(u, u) = 1$ (*reflexivity*),

(S2) $s(u, v) = 1$ implies $u = v$ (*separability*),

(S3) $s(u, v) = s(v, u)$ (*symmetry*),

(S4) $s(u, v) \odot s(v, w) \leq s(u, w)$ (*$\odot$-transitivity*).

In this case, we call (W, s) a *similarity space*.

The value 1 is used exactly in case of coincidence, as expressed by the reflexivity and separability of s. Furthermore, similarity is supposed to be a symmetric notion. Finally, $\odot$-transitivity can be viewed as a triangle inequality, where the connecting operation is allowed to be any prior chosen t-norm.

One may think of a similarity space as a generalised metric space. Indeed, let (W, s) be a similarity space and put $d(u, v) = 1 - s(u, v)$ for $u, v \in W$. Then (S4) translates to $d(u, w) \leq d(u, v) \oplus d(v, w)$, where $\oplus$ is the t-conorm associated to $\odot$. If $\odot$ is the Łukasiewicz t-norm, then the triangle inequality in the usual sense holds and d is indeed a metric, which is bounded by 1. Furthermore, if $\odot$ is the product t-norm, d is isomorphic to a metric as well, which in this case is unbounded, having $\mathbb{R}^+ \cup \{\infty\}$ as its range.

Propositions will be modelled classically: by sets of possible worlds, that is, by subsets of W. For $A \subseteq W$ and $t \in [0,1]$ we define

$$U_t(A) \;=\; \{w \in W\colon s(w,a) \geq t \text{ for some } a \in A\}$$

to be the t-neighbourhood of A. Let now $A, B \subseteq W$ model the two propositions φ and ψ, respectively. Then we say that φ approximately implies ψ to the degree t if

$$A \subseteq U_t(B).$$

The propositional logic LAE is defined model-theoretically as follows. For an axiomatisation of LAE, we refer to [Rod, EGRV].

Definition 3.2. The *propositional formulas* of LAE are built up from a countable set of *variables* $\varphi_1, \ldots$ and the *constants* $\bot, \top$ by means of the binary operators $\wedge$ and $\vee$ and the unary operator $\neg$. A *conditional formula* of LAE is a triple consisting of two propositional formulas φ and ψ as well as a value $t \in [0,1]$, denoted by

$$\varphi \stackrel{t}{\Rightarrow} \psi.$$

Let (W,s) be a similarity space. An *evaluation* for LAE is a structure-preserving mapping e from the set of propositional formulas to the Boolean algebra of subsets of W. We say that a conditional formula $\varphi \stackrel{t}{\Rightarrow} \psi$ is *satisfied* if

$$e(\varphi) \subseteq U_t(e(\psi)).$$

A *theory* of LAE is a set of conditional formulas. We say that a theory $\mathcal{T}$ *semantically entails* a conditional formula $\varphi \stackrel{t}{\Rightarrow} \psi$ if any evaluation satisfying all elements of $\mathcal{T}$ also satisfies $\varphi \stackrel{t}{\Rightarrow} \psi$.

Aiming at a formal treatment of vague properties, we will in the next section extend the setting introduced so far.

4 Logic of Approximate Entailment for a first-order structure

The Logic of Approximate Entailment has been developed as a propositional logic, propositions being modelled by subsets of a similarity space (W,s). Heading towards applications, we next consider the case that our set of worlds possesses a first-order structure. We wish to formalise statements of the form that first-order propositions imply other such propositions only approximately.

We note that first-order logic for metric spaces has been studied in recent years. The concerns are somewhat different, but the setting has some resemblance to ours. See [BBHU] and the references given there.

Our idea is the following. Let $\varphi(x)$ and $\psi(x)$ be formulas of a first-order language with the only free variable x. We would like to express that $\varphi(x)$ implies $\psi(x)$ approximately. An interpretation being given, $\varphi(x)$ and $\psi(x)$ can be identified with two subsets A and B of the universe. Accordingly,

$$\varphi(x) \stackrel{t}{\Rightarrow} \psi(x)$$

will be defined to hold exactly if A is contained in B to the degree t in the sense of LAE. This in turn means that

$$\text{there is a } y \in U_t(x) \text{ such that } \varphi(x) \to \psi(y), \tag{1}$$

where $\to$ is the usual implication of classical propositional logic.

Also constants will be defined to be flexible in meaning. Namely, let c be a constant; then

$$\varphi(c) \stackrel{t}{\Rightarrow} \psi(c)$$

will be defined to mean

$$\text{there is a } y \in U_t(c) \text{ such that } \varphi(c) \to \psi(y). \tag{2}$$

For the rest of the section, let us fix a triple $(W, s, \mathcal{L})$; here, (W, s) is a similarity space and $\mathcal{L}$ is a first-order language not containing constants.

Our calculus will be associated with $(W, s, \mathcal{L})$. Namely, we consider theories in the language $\mathcal{L}$ which is to be interpreted in a model with the base set W. To this end, we add constants $\bar{w}$ for each $w \in W$, interpreted by w itself. Moreover, we will add binary relations R_t for each $t \in [0, 1]$, interpreted by those pairs $v, w \in W$ that are similar to the degree at least t.

The *structured Logic of Approximate Entailment* associated with $(W, s, \mathcal{L})$, or sLAE for short, is defined as follows.

Definition 4.1. Let $\mathcal{L}_c$ be the result of adding to $\mathcal{L}$ a constant $\bar{w}$ for each $w \in W$ and additional binary relations R_t for each $t \in [0, 1]$.

An *ordinary formula* of sLAE is a formula of $\mathcal{L}_c$. A *conditional formula* of sLAE is a triple consisting of a pair φ, ψ of ordinary formulas and a number $t \in [0, 1]$, denoted by

$$\varphi \stackrel{t}{\Rightarrow} \psi.$$

By a *formula,* we mean either an ordinary or a conditional formula.

An interpretation of an ordinary formula in a model under an assignment of the free variables is defined as usual, but such that the following holds. The model's base set is W; for $w \in W$, the constant $\bar{w}$ is interpreted by w; for $v, w \in W$, $R_t(\bar{v}, \bar{w})$ is satisfied if and only if $s(v, w) \geq t$.

Let furthermore φ and $\psi(d_1, \ldots, d_n)$ be ordinary formulas, where $d_1, \ldots, d_n$, $n \geq 0$, are exactly all the constants and free variables of the latter formula, and let $t \in [0, 1]$. Then $\varphi \stackrel{t}{\Rightarrow} \psi(d_1, \ldots, d_n)$ is satisfied if so is the ordinary formula

$$\varphi \to \exists y_1 \ldots \exists y_n (R_t(y_1, d_1) \wedge \ldots \wedge R_t(y_n, d_n) \wedge \psi(y_1, \ldots, y_n)), \tag{3}$$

where $y_1, \ldots, y_n$ are variables distinct from every $d_1, \ldots, d_n$. In case $n = 0$, (3) is understood as $\varphi \to \psi$.

A *theory* is a set of formulas of sLAE. *Semantic entailment* of a formula by a theory is defined as expected.

The axiomatisation of sLAE does not cause difficulties. The only peculiarity is the fact that the elements of the base set are requested to be in one-to-one correspondence with the constants. This is not a problem in the finite case; in the infinite case, however, an infinitary rule must be assumed.

Definition 4.2. Axioms and rules of sLAE are those of first-order logic for $\mathcal{L}_c$ as well as the following ones.

$$\text{(L1)}\quad \bar{v} \neq \bar{w}, \quad \text{where } v, w \in W \text{ such that } v \neq w$$

$$\text{(L2a)}\quad R_t(\bar{v}, \bar{w}), \quad \text{where } v, w \in W \text{ such that } s(v, w) \geq t,$$
$$\text{(L2b)}\quad \neg R_t(\bar{v}, \bar{w}), \quad \text{where } v, w \in W \text{ such that } s(v, w) < t$$

$$\text{(L3)}\quad \frac{\varphi(\bar{w}) \text{ for all } w \in W}{\forall x\, \varphi(x)}$$

$$\text{(L4a)}\quad \frac{\varphi \stackrel{t}{\Rightarrow} \psi(d_1, \ldots, d_n)}{\varphi \to \exists y_1 \ldots \exists y_n (R_t(y_1, d_1) \wedge \ldots \wedge R_t(y_n, d_n) \wedge \psi(y_1, \ldots, y_n))},$$
$$\text{(L4b)}\quad \frac{\varphi \to \exists y_1 \ldots \exists y_n (R_t(y_1, d_1) \wedge \ldots \wedge R_t(y_n, d_n) \wedge \psi(y_1, \ldots, y_n))}{\varphi \stackrel{t}{\Rightarrow} \psi(d_1, \ldots, d_n)},$$

where $d_1, \ldots d_n$ are the free variables and the constants occurring in $\psi(d_1, \ldots, d_n)$

Provability of a formula from a theory is defined as expected.

We have the following completeness theorem.

Theorem 4.3. *Let $\mathcal{T}$ be a consistent theory of* sLAE*, and let α be formula of* sLAE*. Then $\mathcal{T}$ proves α if and only if $\mathcal{T}$ semantically entails α.*

Proof. The soundness is clear; this is the "only if" part.

By rules (L4a) and (L4b), we may assume that a theory consists of ordinary formulas only. Indeed, as these rules are the only ones where conditional formulas appear, we may assume that conditional formulas do not appear in proofs except (L4a) at the beginning and (L4b) at the end.

Let $\mathcal{T}$ not prove α. Then $\mathcal{T} \cup \{\neg\alpha\}$ is consistent. We may extend $\mathcal{T} \cup \{\neg\alpha\}$ to contain a set of witnesses, which due to rule (L3) may all be chosen as $\bar{w}$ for some $w \in W$. Thus $\mathcal{T} \cup \{\neg\alpha\}$ has a model each of whose elements interprets a constant. By rule (L1), the base set is actually in a one-to-one correspondence with W. By rules (L2a) and (L2b), the R_t, $t \in [0, 1]$, correspond to the similarity relation s in the prescribed way. The proof of the "if" part is complete. ☐

5 The logic sLAE for the sorites paradox

In which sense can we expect a "solution" of the sorites paradox by formal means?

Let us note first that the sentences (A), (B), (C) constituting the paradox all refer to one and the same object: a partially filled teapot. However, some parts refer to a coarse level of granularity and some to a fine level. In particular, "the teapot is nearly full" and "the teapot is empty" are statements on the coarse level; "the teapot contains 48000 drops of tea" and "the teapot contains one drop less than before" refer to the fine level. In other words, we use for the same thing two different models. According to the coarse model, the teapot can be, say, empty, well filled, nearly full, or full; only a few distinctions are made. In contrast, the fine model distinguishes between 50 001 situations, taking into account differences of filling degrees that are as small as the volume of a single drop of liquid.

The sequence (A)–(C) shows that a mixture of arguments referring to two different models can lead to confusion. The sorites paradox is in fact a great illustration of the fact that a coherent argumentation needs to be conducted on a single level of granularity. An entanglement of different levels of granularities is prone to inconsistency.

To reveal where the argument (A)–(C) is corrupted, we must ensure that that all statements refer to a single model, namely a sufficiently fine-grained one. If necessary, properties must be redefined, further differentiated, or made compatible with the chosen model in another way. Reasoning with regard to a single model cannot be paradoxical, provided that the arguments are sound.

Thus all we have to do is to determine a fine-grained model and apply our calculus sLAE to it. The apparent contradiction will necessarily disappear. In this sense we can expect a solution of the paradox. We note that any calculus similar to sLAE could be taken as well.

As our set of worlds we take $G = \{0, 1, 2, \ldots, 50000\}$; each $n \in G$ represents the number of drops contained in the teapot under consideration, or equivalently the proportion $\frac{n}{50000}$ of liquid within the total volume of the teapot. Furthermore, we define

$$s(m,n) \;=\; \frac{\min\{m,n\}}{\max\{m,n\}} \tag{4}$$

for each two distinct values $m, n \in G$ and $s(n,n) = 1$ for $n \in G$. Then (G, s) is a similarity space, s being a similarity relation with respect to the product t-norm. Furthermore, we endow G with the function $S\colon G \to G$ which maps each $n < 50000$ to $n+1$ and 50000 to itself.

We next need to take the property "nearly full" into consideration. We deal with this coarse notion on the fine level by associating with it a set of prototypes. Let H be a unary relation; $H(n)$ is to express that filled with n drops of tea the teapot is nearly full. H is assumed to be chosen such that for any n fulfilling $H(n)$, we can clearly say that a teapot filled with n drops is nearly full.

We note that there is no canonical choice for the interpretation of H; *the* set of prototypes of a coarse-grained notion does not exist. The property "nearly full" refers to a coarser model than the one we deal with, so that it does not make sense to associate a specific crisp subset to it. Actually, we will not do so – in the sense that we leave open how H is interpreted. We do assume that $H(\overline{48000})$ is true, nothing else; so any interpretation of H such that 48000 is among the elements for which H holds will do.

Let us now ask which conditional formulas with $H(\overline{48000})$ as its antecedent TSR can prove. We have $H(Sn) \to R_{s(n,Sn)}(n, Sn) \wedge H(Sn)$, hence

$$H(Sn) \to \exists m(R_{s(n,Sn)}(n, m) \wedge H(m)),$$

that is, since $s(n, Sn) = \frac{n}{n+1}$,

$$H(Sn) \stackrel{\frac{n}{n+1}}{\Rightarrow} H(n);$$

for example, $H(\overline{48000}) \stackrel{0.999979}{\Rightarrow} H(\overline{47999})$. Similarly, we may relate $H(\overline{48000})$ to $H(\overline{1})$ or even $H(\overline{0})$:

$$H(\overline{48000}) \stackrel{0.00002}{\Rightarrow} H(\overline{1}),$$
$$H(\overline{48000}) \stackrel{0}{\Rightarrow} H(\overline{0}).$$

As expected, reasoning in this framework is no longer paradoxical.

Let us conclude the first half of this paper with some additional remarks. Let us point out how our approach to vagueness helps to explain why we are taken aback by the sorites paradox, that is, why we actually believe that there is contradiction to be resolved.

A natural-language utterance evokes a picture in our imagination; this picture is simple, containing only those details that matter. In general, we use the coarsest possible model to comprehend facts. A statement like "the pot is nearly full" is not true or false depending on a specific proportion of the liquid in the total volume, but represents a rough picture. Assuming that a teapot is nearly full, we do not figure out the precise bounds of filling degrees between which the notion "nearly full" is correctly applicable and conclude that the actual filling degree is within these bounds. We rather imagine a teapot with an amount of tea which gives rise to the statement "nearly full", as opposed to, say, "empty" or "well filled" on the one hand and "full" on the other hand. Our model in use is a coarse one, distinguishing between, say, four cases.

We further imagine that by the removal of a single drop the picture of a nearly full teapot remains the same. It is clear to us that although we can observe that one drop leaks out of the teapot we cannot observe the resulting difference. It follows that we agree with statement (A). When hearing the second part of (A), we might certainly extend our model – but only by one element, hence not significantly: we now deal with the properties, say, "empty", "well filled", "nearly full minus one drop", "nearly full" and "full". Furthermore, 48000 is a number that is smaller but quite close to 50000; accordingly it seems appropriate to say that the teapot is nearly full when containing this number of drops. Hence we agree with (B) as well. Finally, the two coarse notions "empty" and "nearly full" are distinct. This fact given, we disagree with (C).

We summarise that when thinking through the three sentences separately, we are not suggested to use any other model than a coarse one. On the contrary, whenever notions like “nearly full” occur, which are not usable on the fine level, we are bound to stay at the coarse level.

We are taken aback by the paradox because we ignore this restriction to the coarse level when putting (A) and (B) together. (A) and (B) tell us that the teapot is in a certain state and that this state is preserved after a second. Thus we reason that this state is preserved forever and we make the surprising conclusion that “empty” and “nearly full” actually coincide. Having agreed with (A) at the coarse level, we apparently do not notice that we switch to the fine level: the repeated application of (A) works only in a fine-grained model, as 48001 different filling degrees are involved. Consequently, we do not notice the problem of the argumentation: the filling degree is specified by the expression “nearly full” and thus refers exclusively to a coarse level. In short, when concluding (C) from (A) and (B), we ignore that “nearly full” is a notion that at the relevant fine level is simply not specified.

At last, we may ask for a reason of this ignorance. This question is certainly inappropriate; we have to accept it as a fact; still, we may wonder under which circumstances we are led to confusion. We note that the conclusion (C) from (A) and (B) could indeed correctly be drawn with any other property than “nearly full”, provided it is compatible with the relevant model G. For instance, when we replace “nearly full” by “red”, the paradoxical nature of the argument disappears; our set of world would then be $G \times \{\text{red}, \text{ not red}\}$. Thus the point seems to be that when drawing the conclusion (C), we assume the teapot to be in a specific state and we do not care about the fact that this state is characterised in a way not compatible with the refined model on which our argument relies.

6 Generalised Aristotelian syllogisms

We now turn to the second topic of this paper. We continue discussing the formal treatment of expressions specifying proportions in a rough way. However, we will do so in a different context.

The Aristotelian syllogisms represent a particular kind of reasoning, which, although originating from the fourth century B.C., reminds in a remarkable way of modern logic. An example is the following:

$$\frac{\text{No } X \text{ are } M \quad \text{All } Y \text{ are } M}{\text{No } X \text{ are } Y} \tag{5}$$

Here M, X, Y are to be understood as properties of the elements of a universe of discourse. The proposition “No X are M” is to be understood as “No element with property X has property M”, and similarly for the remaining two propositions.

As a natural framework, we may choose an abstract set endowed with one-placed relations. To formalise a syllogism, it is then enough to consider first-order logic, the language consisting of unary relations. See, e.g., [CCM].

Generalised Aristotelian syllogisms have been considered in the work of A. De Morgan [Mor]. The idea is to deal not only with the “particular” and the “universal”, that is, with single or all elements of some universe, but also with sets whose extent is characterised by natural-language expressions like “few”, “many”, or similarly. In other words, not only existential and universal quantification is considered, but also what is called intermediate quantification. An example is the following, which is cited, like the other syllogisms mentioned in this paper, from [Nov2]:

$$\frac{\text{Most } X \text{ are } M \quad \text{All } M \text{ are } Y}{\text{Many } X \text{ are } Y} \tag{6}$$

The attribute “many” denotes a roughly quantified proportion and thus may be considered as inherently vague. Vagueness concerns at least two levels of granularity and then usually a rough and a fine one. Here, to argue at the fine level means that we take into account every single element of the set under consideration; just like in the case discussed in the previous sections, it means to deal with exact proportions. At the coarse level, in contrast, we just make the distinction between, say, “none”, “few”, “many”, “most”, and “all”.

Statements like those in (6) have been considered in a formal setting by many authors. Zadeh develops in [Zad2] a framework to deal with expressions of the type “few”, “many”, or the like; his approach is based on fuzzy sets. The line was taken up by Novák, who has developed the appropriate logical framework [Nov2]. His Theory of Intermediate Quantifiers extends Fuzzy Type Theory, a generalisation of (classical) type theory [Nov1]. Furthermore, P. L. Peterson’s monograph [Pet] is devoted to the topic.

A completely different approach relies on probability theory. This might sound surprising, but has led to reasonable results. In fact, a statement like “most X are M” may be read as “the probability that an arbitrary (i.e. randomly chosen) M is an X is high”. The paper [DGMP] is based on this idea. To cope with rough specifications like “high”, the authors work with intervals of probabilities. Furthermore, the work of D. G. Schwartz on the topic is inspired by the probabilistic approach [Sch1, Sch2, Sch3]. The calculus presented in [Sch2] is remarkable in that no explicit probability values appear.

Here, we present an approach which is characterised by its simplicity and scantness. Vagueness concerns different levels of granularity; but not in all arguments involving vague properties other levels than the coarse one are considered. The Aristotelian syllogisms provide easy examples of reasoning on a single coarse level. We may check, for instance, that the inference (6) does not refer to single elements of the universe of discourse but just to roughly described subsets.

We suggest a calculus referring only to one coarse level. Dealing with a single model, we just proceed according to common mathematical practice. If our calculus looks uncommon, the reason might be that we are, as mathematicians, accustomed to arguing with regard to the finest possible level.

7 A logic for generalised Aristotelian syllogisms

Our abstract object under consideration is an algebra of finite sets. We use the common first-order language describing relationships between sets, omitting however a top element: $\subseteq$, $\cap$, $\cup$, $\setminus$, $\emptyset$. We furthermore deal with the size of sets; we use the binary relation $\sim$ to denote equal cardinalities. We finally use specialised versions of the subsethood relation, expressing not only subsethood but also roughly the proportion of a subset in the set of reference: $\overset{\text{few}}{\subset}$, $\overset{\text{many}}{\subset}$, $\overset{\text{most}}{\subset}$, $\overset{\text{n.a.}}{\subset}$. The formulas

$$A \overset{\text{few}}{\subset} B, \quad A \overset{\text{many}}{\subset} B, \quad A \overset{\text{most}}{\subset} B, \quad A \overset{\text{n.a.}}{\subset} B$$

mean that $A \subseteq B$ and the proportion of A within B is quite small but not zero, considerably large, larger than a half, or very high but not one, respectively. Referring to the informal language of (5) and (6), the statements "A few / Many / Most / Nearly all B are A" are expressed by $A \cap B \overset{\text{few}}{\subset} B$, $A \cap B \overset{\text{many}}{\subset} B$, $A \cap B \overset{\text{most}}{\subset} B$, or $A \cap B \overset{\text{n.a.}}{\subset} B$, respectively.

We next proceed as usual in mathematics: we assemble the relationships holding between the chosen predicates. Recall that we wish to stay on the coarse level; we will not consider the set under consideration elementwise and define relations like $\overset{\text{few}}{\subset}$ by fuzzy percentages or similarly. We will rather treat these relations in the same way as when we reason with them; indeed, when reasoning with expressions like "a few", we do not calculate percentages.

As we will model certain expressions of natural language, our axioms below do not have a definite character; we do not claim that these axioms are the only acceptable ones. They represent a specific understanding of the involved notions, and if understood differently they would have to be formalised differently. The possible imperfectness of our axioms lies in the nature of the representation of natural language by formal means.

When formulating axioms to characterise properties expressed in natural language, we focus on the typical circumstances under which expressions are used; we consider the picture which comes spontaneously to our mind when we think about the expressions to be formalised. Rather than checking systematically all possible cases involved, we choose our axioms on the basis of prototypical situations. We note that it would actually be problematic to agree with some generalised Aristotelian syllogisms if validity was checked in a systematic way rather than according to typical cases. For instance, in (6) X is not meant to represent a very small set of individuals. Moreover, other syllogisms turn out to be incorrect if the involved sets are empty.

Let us now comment on the basic decisions underlying the axioms below. The expression "a few" represents a small but non-zero portion of a set; we accept then any of its non-empty subsets also as small. The expression "many" may be formalised in a "pragmatic" or "semantic" way; that is, we may reflect the way in which the expression "many" is actively used or the way we would answer to the question if there are "many". In accordance with the example (6), we will chose the "semantic" interpretation; that is, the relation $A \overset{\text{many}}{\subset} B$ will be stable under an enlargement of A within

B. Furthermore, “many” represents a strictly larger proportion than “a few”; and more than a half elements represent always “many” elements. Finally, a subset is defined to contain “nearly all” elements exactly if its complement contains “a few” elements.

We furthermore have to decide if, given a set, subsets of a specific category should always exist or not. It does not make sense to postulate the existence of a subset containing, e.g., “many” elements for small sets of reference. We restrict to the following requirement: if some proper subset of a set represents “many” element, then also a subset with “a few” elements exists.

We specify now the first-order *Theory of Syllogistic Reasoning*, or TSR for short. Recall that generalised Boolean algebras are sectionally complemented distributive lattices [Gra]; they differ from Boolean algebras only in that the top element is omitted.

Definition 7.1. The language of TSR consists of the binary relations $\subseteq, \overset{\text{few}}{\subset}, \overset{\text{many}}{\subset}, \overset{\text{most}}{\subset}, \overset{\text{n.a.}}{\subset}$ as well as $\sim$, the binary functions $\cap, \cup, \setminus$, and the constant $\emptyset$. TSR contains the theory of generalised Boolean algebras, written in the language $\subseteq, \cap, \cup, \setminus, \emptyset$, and the following axioms (where the universal quantification over the free variables is understood).

Axioms for size:

$$\text{(S1)}\ \ A \sim A \qquad \text{(S2)}\ \ (A \sim B)\ \rightarrow\ (B \sim A)$$

$$\text{(S3)}\ \ (A \sim B) \wedge (B \sim C)\ \rightarrow\ (A \sim C)$$

$$\text{(S4)}\ \ (A \preccurlyeq B) \vee (B \preccurlyeq A) \qquad \text{(S5)}\ \ (A \sim \emptyset)\ \leftrightarrow\ (A = \emptyset)$$

$$\text{(S6)}\ \ (A \sim B) \wedge (A \subseteq B)\ \rightarrow\ (A = B)$$

$$\text{(S7)}\ \ (A \sim C) \wedge (B \sim D) \wedge (A \cap B = \emptyset) \wedge (C \cap D = \emptyset)\ \rightarrow\ (A \cup B\ \sim\ C \cup D)$$

Here, $A \preccurlyeq B$ is an abbreviation of $\exists C((A \sim C) \wedge (C \subseteq B))$.

General axioms for proportions:

$$\text{(P1)}\ \ (A \overset{\star}{\subset} B)\ \rightarrow\ (\emptyset \subset A) \wedge (A \subseteq B)$$

$$\text{(P2)}\ \ (A \overset{\star}{\subset} C) \wedge (B \subseteq C) \wedge (A \sim B)\ \rightarrow\ (B \overset{\star}{\subset} C)$$

Here, $A \subset B$ is an abbreviation of $(A \subseteq B) \wedge \neg(A = B)$. Moreover, $\overset{\star}{\subset}$ is to be uniformly replaced by one of $\overset{\text{few}}{\subset}, \overset{\text{many}}{\subset}, \overset{\text{most}}{\subset}, \overset{\text{n.a.}}{\subset}$.

Axioms for “few”:

$$\text{(few1)}\ \ (\emptyset \subset A) \wedge (A \subseteq B) \wedge (B \overset{\text{few}}{\subset} C)\ \rightarrow\ (A \overset{\text{few}}{\subset} C)$$

$$\text{(few2)}\ \ (A \overset{\text{few}}{\subset} B) \wedge (B \subseteq C)\ \rightarrow\ (A \overset{\text{few}}{\subset} C)$$

Axioms for “many”:

$$\text{(many1)}\ \ (A \overset{\text{many}}{\subset} C) \wedge (A \subseteq B) \wedge (B \subseteq C)\ \rightarrow\ (B \overset{\text{many}}{\subset} C)$$

$$\text{(many2)}\ \ (A \overset{\text{many}}{\subset} C) \wedge (A \subseteq B) \wedge (B \subseteq C) \ \rightarrow\ (A \overset{\text{many}}{\subset} B)$$

$$\text{(many3)}\ \ (A \overset{\text{many}}{\subset} B) \ \rightarrow\ \neg(A \overset{\text{few}}{\subset} B)$$

$$\text{(many4)}\ \ (B \overset{\text{many}}{\subset} C) \wedge (B \subset C) \ \rightarrow\ \exists A((A \subset B) \wedge (A \overset{\text{few}}{\subset} C))$$

Axioms for "most":

$$\text{(most1)}\ \ (A \overset{\text{most}}{\subset} B) \ \leftrightarrow\ (A \subseteq B) \wedge (B \backslash A \prec A)$$

$$\text{(most2)}\ \ (A \overset{\text{most}}{\subset} B) \ \rightarrow\ (A \overset{\text{many}}{\subset} B)$$

Here, $A \prec B$ is an abbreviation of $\exists C((A \sim C) \wedge (C \subset B))$.

Axioms for "nearly all":

$$\text{(n.a.)}\ \ (A \overset{\text{n.a.}}{\subset} B) \ \leftrightarrow\ (B \backslash A \overset{\text{few}}{\subset} B)$$

We are led to the following models.

Definition 7.2. An *SR-algebra* is a structure $(\mathcal{S}; \wedge, \vee, -, 0, \hookleftarrow, \overset{\text{few}}{<}, \overset{\text{many}}{<}, \overset{\text{most}}{<}, \overset{\text{n.a.}}{<})$ subject to the following requirements.

(SR1) $(\mathcal{S}; \wedge, \vee, -, 0)$ is a generalised Boolean algebra such that each $A \in \mathcal{S}$ is the supremum of finitely many atoms. We denote by $\operatorname{card} A$ the number of atoms below an $A \in \mathcal{S}$.

(SR2) For $A, B \in \mathcal{S}$, $A \hookleftarrow B$ holds if and only if $\operatorname{card} A = \operatorname{card} B$.

(SR3) Let $B \in \mathcal{S}$. Let $n = \operatorname{card} B$, and let h be smallest such that $2 \cdot h > n$. Then we have for any $A \leq B$:

(i) $A \overset{\text{most}}{<} B$ if and only if $\operatorname{card} A \geq h$.

(ii) There is a j such that $1 \leq j \leq h$ and we have:

$$A \overset{\text{many}}{<} B \text{ if and only if } \operatorname{card} A \geq j.$$

If $n \geq 2$, then j is such that $1 < j \leq h$.

(iii) If $n > 2$, there is an i such that $1 \leq i < j$ and we have:

$$A \overset{\text{few}}{<} B \text{ if and only if } 1 \leq \operatorname{card} A \leq i.$$

If $n \leq 2$, then $A \overset{\text{few}}{<} B$ implies $\operatorname{card} A = 1$ and $n = 2$.

(iv) $A \overset{\text{n.a.}}{<} B$ if and only if $B - A \overset{\text{few}}{<} B$.

(SR4) Let $A, B, C \in \mathcal{S}$ such that $\emptyset \subset A \subseteq B \subseteq C$. Then:

(i) $A \overset{\text{few}}{<} B$ or $B \overset{\text{few}}{<} C$ implies $A \overset{\text{few}}{<} C$.

(ii) $A \overset{\text{many}}{<} C$ implies $A \overset{\text{many}}{<} B$ and $B \overset{\text{many}}{<} C$.

Lemma 7.3. *Each SR-algebra is a model of* TSR*, where* $\subseteq$, $\overset{\textit{few}}{\subset}$, $\overset{\textit{many}}{\subset}$, $\overset{\textit{most}}{\subset}$, $\overset{\textit{n.a.}}{\subset}$, $\sim$, $\cap$, $\cup$, $\setminus$, $\emptyset$ *are interpreted by* $\leq$, $\overset{\textit{few}}{<}$, $\overset{\textit{many}}{<}$, $\overset{\textit{most}}{<}$, $\overset{\textit{n.a.}}{<}$, $\leftharpoondown\!\!\rightharpoondown$, $\wedge$, $\vee$, $-$, 0*, respectively.*

We have the following completeness theorem.

Theorem 7.4. *Let* $\mathcal{T}$ *be a consistent finite theory extending* TSR *and let* φ *be a formula. Then* $\mathcal{T}$ *proves* φ *if and only if any interpretation in an SR-algebra satisfying* $\mathcal{T}$ *satisfies* φ*.*

Proof. The "only if" part is clear.

To see the "if" part, assume that $\mathcal{T}$ does not prove φ. Let $(\mathcal{S}; \subseteq, \overset{\text{few}}{\subset}, \overset{\text{many}}{\subset}, \overset{\text{most}}{\subset}, \overset{\text{n.a.}}{\subset}, \sim, \cap, \cup, \setminus, \emptyset)$ be a finite model of $\mathcal{T} \cup \{\neg\varphi\}$. By assumption $(\mathcal{S}; \cap, \cup, \setminus, \emptyset)$ is a generalised Boolean algebra. (SR1) is proved. We may assume that $\mathcal{S}$ is a collection of subsets of a finite set X. W.l.o.g. we may furthermore assume that $\mathcal{S}$ is actually the power set of X.

Let $A, B \in \mathcal{S}$. We next prove (SR2). Let A and B be of equal cardinality. If both are empty, $A \sim B$ by (S1). If $a \in A$ and $b \in B$, we have $\{a\} \sim \{b\}$, because otherwise (S4) would imply $\{a\} \sim \emptyset$ or $\{b\} \sim \emptyset$, in contradiction to (S5). Hence for non-empty A and B, $A \sim B$ follows from (S7). For the converse direction, let A be of strictly smaller size than B. Let $C \subset B$ be of the same cardinality like A. Then $A \sim C$. By (S2) and (S3), $A \sim B$ would imply $C \sim B$, in contradiction to (S6).

We now turn to (SR3). Fix some $B \in \mathcal{S}$, and let $A \subseteq B$. If $B = \emptyset$, we have $\neg(A \overset{\star}{\subset} B)$ by (P1), in accordance with (SR3). Let us assume that B is non-empty.

By (P2), the relations $A \overset{\star}{\subset} B$ depend only on the cardinality of A. Let $n = \operatorname{card} B$, and let h be minimal such that $2 \cdot h > n$. Then the subsets of B of size $\geq h$ are those containing a strict majority of elements; hence (SR3)(i) follows from (most1).

Furthermore, by (most2), (many1), and (P1) there is a $1 \leq j \leq h$ such that subsets of B of size $\geq j$ contain "many" elements; this is the first half of (SR3)(ii). If B is at least two-element and $j = 1$, B would contain a proper one-element subset with "many" elements, in contradiction to (many4) and (P1). So $j > 1$ in this case, and (SR3)(ii) follows.

Let B be at least three-element. Then B contains a proper subset with "many" elements. By (many4), (many3), and (few1), the first half of (SR3)(iii) follows. Let B at most two-element and assume $A \overset{\text{few}}{\subset} B$. By (P1), A and B are non-empty. By (most1) and (most2), $B \overset{\text{many}}{\subset} B$, and by (many3), $\neg(B \overset{\text{few}}{\subset} B)$. Hence $\emptyset \subset A \subset B$, that is, A is one-element and B is two-element. (SR3)(iii) is shown.

Finally, (SR3)(iv) is clear by (n.a.). So (SR3) is proved.

(SR4)(i) holds by (few1) and (few2). (SR4)(ii) holds by (many1) and (many2).

Thus $\mathcal{S}$ is an SR-algebra, and the proof is complete. $\square$

Our calculus refers to an algebra of subsets, endowed with the usual set-theoretical operations as well as binary relations to compare cardinalities. The same is also true for most other approaches, with the only difference that sometimes a fuzzified version of the relations is used. So how does our calculus actually differ from other ones?

Usually, the universe of discourse is considered in its full extent, and proportions are modelled numerically. In contrast, SR-algebras provide coarse-grained models of a theory under consideration, and proportions are dealt with in a qualitative way only. SR-algebras are isomorphic to algebras of subsets; when considered as such, their atoms contain in general, however, more than one element. Moreover, a relation like $\overset{\text{many}}{\subset}$ is not assigned a definite range of proportions; $\overset{\text{many}}{\subset}$ is only characterised by the properties that we think it should fulfil. SR-algebras should be thought of as models which do not involve more structure than necessary.

Consider the following simple example; cf. (6). Let M be the set of flying animal species, X the set of bird species, and Y the set of animal species with a maximal weight of under 20 kg. Let us describe the inclusions between these sets by the theory $\mathcal{T} = \{M \cap X \overset{\text{most}}{\subset} X,\ M \subseteq Y\}$, expressing that "*most bird species can fly*" and "*flying species are not heavier than 20 kg*". An SR-algebra modelling $\mathcal{T}$ consists, for instance, of subsets of $\{a, b, c, d, e, f\}$, where $M = \{a, b, c, e\}$, $X = \{a, b, c, d\}$, $Y = \{a, b, c, e, f\}$. Each atom $a, \ldots, g$ represents a smallest unit relevant for reasoning; each atom represents a set of equal cardinality which is however indeterminate. This model is particularly coarse-grained. But we would hardly consider a finer partition of the set of animal species to draw the conclusion in (6): "*many bird species are species with a maximal weight of under 20 kg.*"

8 TSR applied to generalised Aristotelian syllogisms

It is straightforward how generalised Aristotelian syllogisms are dealt with in the present framework. We show the details for the example (6) as well as for the following two:

$$\frac{\text{No } Y \text{ are } M \quad \text{All } X \text{ are } M}{\text{Most } X \text{ are not } Y} \qquad \frac{\text{Nearly all } M \text{ are } Y \quad \text{All } M \text{ are } X}{\text{Some } X \text{ are not } Y}$$

The translation of statements occurring in generalised syllogisms is in most cases straightforward. However, sometimes small details have to be taken into account which are essential for the validity of the argument. The shown examples are, however, uncritical in this respect. We just note that "some X" is to be understood as "at least one individual with property X". Moreover, the sets appearing in syllogisms are generally assumed to be non-empty; here it is necessary in one case to add this condition explicitly.

Theorem 8.1. *The following rules are derivable from* TSR*:*

$$\text{(AS1)}\ \frac{M \cap X \overset{\textit{most}}{\subset} X \quad M \subseteq Y}{Y \cap X \overset{\textit{many}}{\subset} X} \qquad \text{(AS2)}\ \frac{M \cap Y = \emptyset \quad X \subseteq M \quad X \neq \emptyset}{X \backslash Y \overset{\textit{most}}{\subset} X}$$

$$\text{(AS3)}\ \frac{M \cap Y \overset{\textit{n.a.}}{\subset} M \quad M \subseteq X}{X \backslash Y \neq \emptyset}$$

Proof. (AS1) From $M \cap X \overset{\text{most}}{\subset} X$ we derive $M \cap X \overset{\text{many}}{\subset} X$. From $M \cap X \subseteq Y \cap X \subseteq X$ it follows $Y \cap X \overset{\text{many}}{\subset} X$.

(AS2) We have $X \cap Y \subseteq M \cap Y = \emptyset$ and consequently $X \backslash Y = X$. From $X \neq \emptyset$ we derive $X \overset{\text{most}}{\subset} X$. The claim follows.

(AS3) $M \cap Y \overset{\text{n.a.}}{\subset} M$ is equivalent to $M \backslash Y \overset{\text{few}}{\subset} M$. It follows $M \backslash Y \neq \emptyset$. By $M \subseteq X$, we get $X \backslash Y \neq \emptyset$. □

We conclude by presenting one example of a syllogism where the translation to TSR does need special attention. Recall that we understand "nearly all" as the complement of a small non-zero proportion. On this fact the validity of (AS3) relies. Consider, however, the following syllogism:

$$\frac{\text{No } Y \text{ are } M \quad \text{All } X \text{ are } M}{\text{Nearly all } X \text{ are not } Y}$$

This syllogism is not compatible with axiom (n.a.). In fact, $M \cap Y = \emptyset$ and $X \subseteq M$ implies $X \backslash Y = X$, meaning that "All X are not Y". It furthermore follows $\neg(X \overset{\text{n.a.}}{\subset} X)$ and thus $\neg(X \backslash Y \overset{\text{n.a.}}{\subset} X)$, which is the negation of "Nearly all X are not Y".

However, the translation to the language of TSR can be chosen as follows.

Theorem 8.2. *The following rule is derivable from* TSR*:*

$$\text{(AS4)}\ \frac{M \cap Y = \emptyset \quad X \subseteq M \quad \exists Z (Z \overset{\textit{many}}{\subset} X)}{\exists Z ((Z \overset{\textit{n.a.}}{\subset} X) \wedge (Y \cap Z = \emptyset))}$$

Proof. From the assumptions, TSR proves $X \cap Y = \emptyset$. Furthermore, from $\exists Z (Z \overset{\text{many}}{\subset} X)$ it follows $\exists Z (Z \overset{\text{few}}{\subset} X)$ and consequently $\exists Z (Z \overset{\text{n.a.}}{\subset} X)$. The claim follows. □

9 Conclusion

We have addressed the problem how to reason formally in the presence of vagueness. In search of general guidelines that the formal treatment of vague properties should follow, we are faced, on the one hand, with the conceptual deficiencies of those approaches that are either chosen ad hoc to satisfy specific practical needs or guided by formal considerations without reference to the actual issue. On the other hand, we do not seriously expect that any of the approaches that deal with vagueness in foundational respects can be of any help. Consequently, new ways have to be explored, revealing new relevant aspects rather than insisting on conventional positions. Here, we object to one feature in the discussion about vagueness that calls for a revision: the opinion that it is a serious aim to establish the unique and "correct" way to reason about vagueness. We argue, in contrast, that the aim is appropriateness and not uniqueness; solutions might be individual and imperfect, but can still be appropriate.

We have given a pair of two examples showing that the diversity of formalisms dealing with vagueness need not imply an inconsistent view of vagueness. We rely on the approach called perceptionalism in [Vet]. We argue that vagueness is a challenge when reasoning refers to different levels of granularity.

In cases where two levels are to be merged, degree-based approaches are acceptable. An example is the well-known sorites paradox. We have proposed a particular formalism that uses truth degrees and is based on a comparatively clear concept. Namely, a modification of the Logic of Approximate Entailment is presented as an alternative to fuzzy logic.

In cases where only one level of granularity is involved we can proceed along common principles of mathematical modelling. We have illustrated how to avoid many-valuedness in such cases. The generalised Aristotelian syllogisms served as the example.

Both reasoning frameworks have formally nothing in common, still they are in line with a coherent interpretation of vagueness in natural language. Accepting this interpretation, much of vain effort to bring onto a common line what is different in nature could be avoided.

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