Vagueness – a mathematician's perspective

Thomas Vetterlein
Institute for Knowledge-Based Mathematical Systems
Johannes Kepler University
Altenbergerstraße 69, 4040 Linz, Austria
and
Section for Medical Expert and Knowledge-Based Systems
Medical University of Vienna
Spitalgasse 23, 1090 Vienna, Austria
Thomas.Vetterlein@jku.at

Abstract

I comment on the question of how to deal appropriately with vague information by formal means. I am critical of attempts to endow vague concepts with semantics in analogy to crisp concepts that refer to a mathematical structure. I argue, to the contrary, that vagueness pertains to the variability of the models associated with a particular mode of perception; the crucial point from my perspective is that we may choose between structures of differing granularity.

I see the primary challenge as finding a solution to the problem of reasoning simultaneously at different levels of granularity – a problem to which a canonical solution is unlikely to exist. The idea of modelling vague concepts by fuzzy sets is defended, and additionally, two logical calculi of possible practical value are suggested.

1 Introduction

There are many different ways of interpreting vagueness in natural language – too many, it seems. There is little hope of overcoming the conceptual differences. Epistemicism, supervaluationism, contextualism, degree theory: each of these keywords refers to one or more of a variety of competing approaches to accounting for vagueness of expressions in natural language. Becoming familiar with even a single approach is a demanding task; a treatment of the topic may cover whole books. For the beginner, the diversity of the approaches and the sheer volume of material can easily be discouraging. Some efforts to systematise the field have recently been made; I refer to [Smi] for a compact and systematic presentation of presently important lines of research. A convergence to a generally accepted and easily comprehensible position cannot be observed.

A common goal of several approaches to vagueness is to find the correct way of arguing on the basis of vague concepts, ideally with the same preciseness and correctness with which we argue about crisp concepts in mathematical proofs. The question is posed: which propositions involving vague concepts are as undoubtedly correct as, say, the fact that for every natural number there is a number that is larger by one? The problem of an appropriate semantics is closely related. The question is raised: what semantics are as natural for vague concepts as natural numbers are for Peano arithmetic?

I doubt that developing a formalism to reason about vague concepts *correctly* is a reasonable goal. Our chances of finding a formalism for reasoning in the presence of vagueness are much better when our efforts focus on *adequacy* instead. Such formalisms are already available. Several reasoning methods are of practical relevance and not subject to the requirement to be in any sense correct. For example, below I mention two logics that can serve specific purposes in the context of vagueness. The formalisms developed for practical purposes are flexible, undogmatic, subject to improvement, and thus not of the same type as the formalisms suggested within a particular approach to vagueness. Since a range of pragmatic, practical approaches exists, we have little reason not to accept the situation as it is and to pursue ideal theoretical solutions of questionable feasibility. The goal of correctness does not even make sense.

The paper is organised as follows. In order to explain my point of view, I begin by looking further into the context of the problem and discuss topics that might, at first sight, seem unrelated. Vagueness concerns particular fundamental issues in which – in my opinion – progress must be made to allow progress in the discussion about vagueness.

Section 2 centres around a common characteristic of discussions about the nature of statements containing vague concepts. Often, utterances about properties of objects are dealt with on the one hand, and the characterisation of what these objects are actually like is treated on the other hand. The way we characterise external objects in natural language utterances and the actual nature of these objects are often treated separately and as two different topics. It is assumed that we speak about something that exists anyway. I oppose this *realistic* point of view with a sharply contrasting one: a purely *perception-based* account of reality that is restricted to the more fundamental aspect of perception and that rejects the idea that objects can be described, at least in principle, in full detail and fully correctly without taking into account the process of their observation.

Section 3 addresses three current approaches to vagueness: epistemicism [Wil1], supervaluationism [Fin], and contextualism [Sha1]. I devote particular attention to the position each approach adopts regarding the two afore-mentioned contrasting ways of understanding reality.

In Section 4, I make my critical stance against the realistic point of view explicit. I do not reject realism completely; my criticism refers to the vagueness debate.

A critical reflection on the three considered approaches to vagueness follows in Section 5. The criticism is extensive, but it does not affect each approach in its entirety.

In Section 6, an alternative approach is developed. I argue as follows. From the perception-based standpoint, viewing reality as the totality of what is around us is part of our thinking model. Indeed, this totality need not be regarded as determined by forces beyond the human scope. When using familiar mathematical structures to describe states or processes involving particular kinds of object, we just employ the thinking model associated with these states or processes. The structures representing properties of objects are assumed to reflect the ways in which we perceive objects, that is, how we are able to speak about an object at all. There is no obvious reason to assume that these structures have any status of existence other than being associated with our means of observation. As a consequence, the structures can be seen as something flexible, changing with the way we refer to an object under different conditions rather than being bound to an object in some absolute manner. In particular, different levels of granularity may give rise to different descriptions, and different descriptions lead to different models. I argue that a formal treatment of vagueness must face the challenge of providing a single formalism that copes with a variety of models differing in granularity, for instance, a coarse-grained level in combination with a fine-grained level model.

I show that defining a generally applicable theoretical framework is not a reasonable aim and stress the value of pragmatic approaches dealing with vagueness, of which I provide two examples. In Section 7, I defend Zadeh's well-known model of vague concepts [Zad1] and mention fuzzy sets as a possible – and surely not the worst – approach. In Section 8, I present further alternatives: I will outline two logics based on the idea that reasoning should be stable under small changes: the Logic of Approximate Entailment [Rus, DPEGG, GoRo] and the Logic of Strong Entailment [EGRV].

2 Two contrasting understandings of reality

The ongoing discussion about vagueness has reached a stage clearly beyond the simple initial observation that numerous expressions in natural language do not allow sharp delimitation of their meaning when we consider the full range of objects to which they might refer. Early contributions to the modern discussion, such as by M. Black [Bla], were far from the intricate analyses offered at present. However, reading [Bla] already leads to non-trivial questions, and navigating the labyrinth of recent contributions is surely an even greater challenge.

A reasonable systematisation of the problems involved and the solutions proposed is desirable. In this section I specify a – somewhat uncommon – guiding criterion for examining approaches to vagueness. I do not start with the Sorites paradox and examine which of the apparently contradictory statements is kept and which is rejected. I also do not follow Smith's classification [Smi, Ch. 2], which relates directly formal mathematics and natural language and which cannot capture approaches that reject this procedure. The question that I find most significant and which additionally serves to delineate my own opinion is how comprehensively the notion of reality is taken.

I pick out two contrasting points of view regarding reality, but do not claim that these are the only possible ones. The positions do not originate from the discussion about theories of reality, to which I do not intend to contribute. They represent two extremes that exclude each other but are both characterised by a high degree of coherence. I must add that I am probably not capable of a fair presentation, because, naturally, I propagate only one point of view in the present context. Nevertheless, I do not want to claim that the other point of view is generally inappropriate; it has its applications. Its popularity, either in the pure form discussed here or in a weakened form, is, however, an obstacle to the discussions about vagueness.

On the one hand, reality can be understood in a comprehensive way: the number of facts considered as real can be extended to the maximum possible. This probably leads to what is called naive realism, which states that the world as we see and feel it exists independently of us; everything, except ourselves and what we actively influence, would have the same status of existence in our absence. In particular, a unique flow of history is assumed. This means, for instance, that everything that research has derived about the time before and after the existence of life on earth is taken literally. Furthermore, the role of an observer entering the world is to describe what exists and happens around him. Accordingly, an observer can make true judgements about the world. His observations are not considered as observations relative to him, and – provided that no errors occur – the content of the observations has a status of absoluteness.

I do not suggest that this view be labelled "naive"; I simply call this understanding of reality the *realism-based* viewpoint. Its main characteristic is that two constituents are distinguished and kept separate: the world and its observation by us.

On the other hand, we may confine ourselves solely to the latter constituent: the aspect of observation, or – more generally – of perception. This leads to a minimalist understanding of reality hereafter called the *perception-based* viewpoint. Only the very fact that we are capable of perception and of relating perceptions made at different times is considered as real. The world is then identified with the totality of experienced perceptions. This view of reality does not encompass everything that ever happened or will happen in the future but can, if we want, be related to a single individual. The role of the former constituent, the "world", is modified accordingly. The reference to objects, their properties, and their development is understood as our way of describing what we perceive.

Both views of reality suggest a particular understanding of the role of natural language and a particular understanding of mathematical logic. Let us extend both the realism-based and the perception-based viewpoint in order to include a suitable characterisation of natural and formal language. I do not claim that the additional elements are implied by necessity; I assume only that we are led to a coherent picture in both cases.

In this contribution, natural language is understood as just the part of language that refers to what we perceive as being and happening in the world. According to the realism-based viewpoint, natural language serves to tell what the world is like. It enables us to make judgements about an existing structure that can be true or false, determined by facts associated with properties of the world and accordingly called "extralingual". It is then natural to draw a close analogy between natural language and the formal languages used in mathematics. Also the latter usually refer to a certain structure and allow statements about it that can be true or false. The difference between the two types of language, however, is well known. Mathematical propositions are, with reference to a model, either true or false; and exactly one of these possibilities applies. The inability of natural language to provide the same conceptual clarity as mathematics is sometimes considered a deficiency.

According to the perception-based viewpoint, the significance of utterances in natural language is restricted. Language is assumed to enable us to systematise, and thus express, our perceptions. Language is not assumed to inform about independently holding properties of the objects it describes. Furthermore, descriptions are typically realised by comparison, by pointing out similarities or differences. Natural-language concepts serve to identify or distinguish different situations in a specific respect; natural language enables us to communicate perceptible similarities or differences.

Next, let us consider formal reasoning methods; let us examine how logic, as the foundation of mathematics, may be characterised when adopting one of the two basic points of view outlined above. According to the realism-based viewpoint, mathematics serves to develop systematically theories of the structures that appear in reality. In this sense, mathematics is a tool for describing the structure of the world. Logic provides the methodology for deriving knowledge in a proper way. In logic, the laws of truth are investigated: logic shows how to derive from one truth something else that is equally true.

Following these ideas, formal statements can be related directly to facts, and, in contrast to natural language, the clean logical foundation helps us to avoid imprecise statements and prevents us from erroneous conclusions. The exact relationship between natural and formal languages certainly calls for clarification; vagueness causes a problem that needs to be tackled.

The perception-based viewpoint has a different understanding of mathematics, and in particular logic. When choosing a formal language and axioms, we begin with a specific aspect under which a particular kind of situation can be described by perceptual means, for instance, the spatial extension of an object or the temporal extension of a process. In order to formalise reasoning, we do not choose a formal language and axioms by means of a straightforward translation of the corresponding natural-language statements. To apply the mathematical method, we take an indirect route: we first determine a model associated with the considered aspect. It is this model to which the formal language, the axioms, and derivation rules refer. The base set of the model represents circumstances that differ only in the selected aspect. Furthermore, the model includes the mutual relationships between these circumstances with respect to the selected aspect. Accordingly, a model is a finite first-order structure, or a method to construct successively an increasingly larger finite first-order structure, leading to a countably infinite structure.

Thus, constructing a mathematical model means compiling systematically all possible variations of a situation with respect to the considered aspect. In particular, the procedure relies on a perceptual ability of ours. In the case of an infinite construction, the possible variations are enumerated systematically, our capacity to imagine being necessarily involved as well.

As a basic example, let us consider the spatial extension as an aspect with respect to which objects can differ observably. Considering "size" just as the notion that distinguishes the smaller from the larger leads to a dense linear order that has a lower and no upper bound. The formal language and axioms to reason about "size" may then refer to this structure, and our derivation rules must be sound with respect to it.[1]

This concludes the specification of the two extreme poles with respect to which I intend to position my own approach and particular existing ones to vagueness.

[1] Incidentally, we may of course define equally well a structure without reference to any perception. Mathematics can be done on the basis of arbitrary axioms which a priori do not refer to any known structure. Indeed, in very few areas of mathematics proper understanding of the results is thus impossible. However, the value of such research is questionable.

3 Epistemicism, supervaluationism, and contextualism

The key question is how to deal formally with vague statements. In the absence of a satisfactory solution, it is not surprising that no standard approach exists and that different approaches are usually incompatible. Let us outline the characteristic features of three well-known lines of research.

Epistemicism belongs to the best-known approaches to vagueness. Timothy Williamson is one of its proponents; his position is described in [Wil2], and a comprehensive treatment can be found in [Wil1].

In [Wil2], the reader is asked to consider an example of a vague property, namely the property of a person to be "thin". Following Williamson, this property divides, under given circumstances, humans into two groups: those who are thin and those who are not. A "thin" person is said to be a member of the former group. Thus, "thin" divides the entities in question into two parts. Epistemicism places great emphasis on this fact, which is referred to as bivalence.

Williamson considers thinness to be dependent on a person's waist size compared to the average waist size of the rest of the population. However, this measure varies continuously, and no point of division is identifiable. Indeed, it may happen that a person, for example "TW" (the author of [Wil1, Wil2]), cannot be categorised as belonging to either of the two groups. The dilemma of vagueness becomes apparent: if we assume that "TW is thin" and "TW is not thin" are both false statements, we are faced with a contradiction, given that the two statements "TW is thin" and "TW is not thin" suggest a bivalent classification.

The specific feature of epistemicism is that the picture of the two groups into which humans are divided by the property "thin" is not revised when it comes to borderline cases; bivalence is preserved at all costs. The trick is to distinguish two levels: at the first level we find what we as humans may observe, or in some way conclude, and thus know; at the second level we have what remains hidden to us, what can by no means be revealed and thus remains, as a matter of principle, unknown to us. Epistemicism claims that, depending on the context, a hard cutoff between the thin and non-thin people exists but that we have no means of finding it. An explanation for this ignorance is also provided: our ability to conclude something requires a margin for error. What applies can be detected only if it could also be detected under slightly changed circumstances. This is why we can in fact conclude and state correctly less than actually applies. In particular, "TW" is either thin or not thin, but nobody knows; in fact nobody can know.

It is not difficult to tell which understanding of reality fits best with epistemicism. The fact that the realism-based viewpoint, together with its above-outlined extension with regard to natural language and mathematics, fits perfectly signifies the one and only advantage of this approach: its great coherence. If the world is given and fixed, and if language serves to specify what is and what happens in the world, each property denoted by a natural-language expression should have an actual, though possibly not experienceable, meaning in the sense that the property should apply or not apply in all circumstances. Otherwise, the expression would be without definite reference or even constitute – as Williamson puts it when considering a strict view – "mere noise" [Wil2, Sec. 1].

Kit Fine's paper [Fin] led to the so-called supervaluationist approach. Like epistemicism, supervaluationism also seeks to save bivalence when reasoning in the presence of vagueness. In contrast to epistemicism, however, so-called truth-value gaps are allowed; a vague property may apply, not apply, or does not possess a truth status. Thus, in the associated formal setting, a vague proposition is assigned "true", "false", or no truth value. In supervaluationism, the pursuit of bivalence does not lead to the claim that vague properties are actually crisp, but to the consideration of all possibilities to "sharpen" the property in an acceptable way. In particular, not one specific crisp property is singled out, but all acceptable sharpenings are considered. "Truth" is finally identified with "supertruth", which roughly means that a statement is true under all the acceptable sharpenings. As a particular argument in favour of this approach, the formalism is claimed to deal properly with so-called penumbral connections; for instance, the statement "K. is tall or K. is not tall" is assigned "true" even if K. is a borderline case for "tall".

The supervaluationist approach does not suggest an answer to the question of which theory of reality is appropriate. An incompatibility with the perception-based viewpoint, as regards the understanding of natural language, can nevertheless be detected. A precisification associates a natural-language expression with a fine-grained scale of distinctions and specifies a region on this scale. This procedure leaves the realm of what the expression has to say. In fact, a precisification does not model the natural-language expression itself but goes declaredly beyond it. This idea is not in line with the perception-based viewpoint; rather, a model of a natural-language expression is required to represent the corresponding perception. If the expression offers a rough classification, then its model must also be based on this rough classification. Finer distinctions require information which is not specifiable by means of the expression; hence, its use is inappropriate.

A remarkable counterpoint to the preceding two theories is Stewart Shapiro's contextualism, which is summarised in [Sha2] and explained in detail in [Sha1]. According to this approach, and as in supervaluationism, formalised vague statements are in one of three states – true, false, or undetermined. Partial evaluations arise, but – unlike in supervaluationism – they are not to be extended into total evaluations. Furthermore, the evaluations are bound to a given moment of a conversation. Thus, unlike both in supervaluationism and in epistemicism, the context does not arise as an element that must be taken into account additionally and increases the complexity of the approach, but is the core of the theory. The evaluations assign truth values only to as many properties as can be discussed at a time by humans, and it is then no problem to postulate a tolerance principle, according to which practically indistinguishable circumstances must be judged the same way.

According to a complaisant interpretation, contextualism is in accordance with the perception-based viewpoint outlined above. In fact, the question whether a vague property applies or not is asked only for a specific utterance of a specific speaker. Furthermore, the truth status of a proposition is kept flexible; it can change at any moment, depending on how the conversation proceeds. However, according to Shapiro, the decision on the applicability of a vague property is not made spontaneously on the basis of the speaker's impression. The decision is rather thought of as a function of everything that could be associated with a context, for instance, the state of the conversation, example cases, and contrasting cases. A further incompatibility is the fact that Shapiro uses his model to solve the Sorites paradox by means of a so-called forced march, which forces speakers to decide on truth or falsity in borderline cases, that is, in cases where by definition the perception suggests neither possibility.

Contextualism does not depend on a definite position on the question of how to understand reality. We cannot say that it exhibits elements of the realism-based viewpoint – provided that we leave [Sha1, Ch. 7] out of consideration. An interpretation in a framework of a perception-based approach would be possible. However, this possibility is not taken into consideration.

4 The realism-based viewpoint: an obstacle

I now return to the general level at which I started the discussion and reconsider the two contrasting understandings of reality. This section identifies the problems we face when adopting the extreme form of realism which I called realism-based viewpoint. I propagate the perception-based viewpoint as the line to be followed. My considerations affect all three approaches discussed in the preceding section and lead to basic criticism, although to a different degree in each case. I confine myself here to general arguments; critical points that are specific to each approach follow in the next section.

Note that, in this contribution, I only address the vagueness debate, and I do not claim that a broad interpretation of the notion of reality is bad in principle. To give an example where it is appropriate, I may mention any of the attempts to systematise parts of natural language. By insisting on a perception-based view these attempts might become unnecessarily complicated. For example, terminologies used in specific scientific fields have been reviewed systematically within the framework of so-called realism-based ontology [MuSm]. Among its underlying principles we also find those that do not appear in the best light in the present contribution. The different scope given, the approach can be seen as appropriate though.

The current discussions on vagueness involve more basic issues than those considered by the proponents of realism-based ontology. At the latest when addressing the often-posed question of where the "source" of vagueness is located, general features of language and of "the world" must be considered.

To be able to argue as flexibly as possible, we should consider the smallest number of facts necessary as unchangeably fixed. In particular, we should consider not more than the absolute minimum as part of reality. The realism-based viewpoint represents the opposite. There is probably an infinite number of possibilities to model human perceptions. There is, moreover, the standard way of doing so. The standard model is the basis of natural language and assumes an observer-independent world consisting of objects whose properties change over time. The realism-based viewpoint takes the standard model for granted. Owing to this limitation, all flexibility is relinquished when we consider our existence not from our everyday perspective but from a meta-level, as is done in discussions about vagueness.

Let us recall the effect of a realism-based view on the interpretation of natural language. It is often stated – possibly with a negative undertone – that natural language does not allow the same kind of semantics as a formal language. In particular, it is stated that natural language lacks the precision we find in mathematics. It is, however, interesting to imagine what precision in natural language would actually mean. Assume that "K. is tall", stated in the context of a conversation, means that K.'s height is lower-bounded by some precise value. This interpretation is odd simply because there is no way of making the associated observation. Not even the best instruments would help, and, at a certain level of preciseness, such statements are even unreasonable in the context of modern physical theories. Even without reference to possible methods of measurement, the interpretation of "tall" in terms of precise values is odd; we would express more than we can tell by looking at K. and estimating her height.

Some authors go so far as to claim that vagueness reflects a kind of deficiency of language. It might be beyond doubt that not all our perceptions can be expressed to others by means of language. However, the converse idea that language could be such that we can express more than we perceive has no basis – simply because we do not have anything more than our perceptions. Williamson's statement that words in natural language mean that something holds true or not but we have no way of knowing is an assumption whose meaning is void.

The weakness of the realism-based viewpoint becomes evident when we consider the relationship between natural and formal language. The realist understanding puts emphasis on analogies. In fact, a property expressed by natural language is regularly called a "predicate" – a notion which, in my opinion, should better remain the preserve of mathematical logic. A predicate in first-order logic refers to a formal language that is interpreted in a formal structure. A vague property refers to a way of expressing what we observe. Even if both these aspects are related, they are definitely not the same. I criticise the tendency to treat natural and formal languages analogously. The point is of general significance: the problem of inappropriate use of formal reasoning techniques is present in all three approaches mentioned above.

The analogous treatment of natural and formal language becomes obvious when a statement is translated "into symbolic form" without specifying the formal framework; both language and derivation rules are either tacitly understood or a matter of discussion, and the semantics is regularly neglected. Williamson's article [Wil2, Sec. 1] is an example of such a style of reasoning; a particularly extreme case is an article by G. Evans [Eva].

Generally, there are two acceptable approaches to logic – a syntactic and a model-theoretic one. The syntactic approach views a logic as a calculus that specifies how formal statements can be derived from other formal statements. If the content of the formal statements is unclear, one certainly must take care not to do anything more than recombine strings of symbols.

A logical calculus contributes to a better understanding of a problem in a particularly transparent way if we start with its semantics. In this case, it is ensured that we know how to interpret the results derived by formal means. The relationship between formal propositions and the addressed situation is mediated by the structure on which the semantics is based.

Both approaches to logic may be justifiable. The former is presently more popular, although the latter is definitely of higher value. It is, however, unacceptable that in many cases neither of these lines is followed, as in the examples mentioned above.

By adopting the perception-based viewpoint I follow the principle of assuming not more than necessary in any case. Not even allowing perceptions to be part of reality would mean denying that anything can be taken for real; such an absurd position would block the current discussion and probably others as well.

From the minimalist perspective, we easily observe that considerations that depend on a wide notion of reality are susceptible to encountering pseudo-problems. A problem statement can either be rendered precise or lacks content. We can only make a question precise if we can reduce it to the domain of perceptions, that is, if we can reformulate it in terms of notions ultimately related to perceptions. This principle is seldom applied in discussions about vagueness.

Consider, for example, the common question of what the "source of vagueness" is like. Unless the epistemicist's answer is accepted, an argumentation may proceed as follows. The set of objects to which a specific property expressed in natural language applies is typically not sharply delimitable. This suggests that vague properties are actually not suitable to express facts about the world; at least they are not capable of doing so in the same neat way as predicates in mathematics. The question is then why; it is not obvious where this problem originates. Since it is assumed that there is the world on the one hand and our natural-language statements about it on the other hand, the reason must apparently be located in exactly one of these domains. Hence, either the world is – in some mysterious sense – the carrier of vagueness, or the so-called semantics of natural language has features different from usual first-order semantics. Unless the former possibility is taken seriously, vagueness is understood as the problem of associating semantics with natural-language expressions, in analogy to predicates of a first-order language. As language is assumed to express true facts, there should moreover exist a canonical solution rather than a variety of equally acceptable ones: the correct semantics are to be discovered.

This train of thought may sound so convincing that we might immediately want to start the search for the one and only semantics. However, we should not be dazzled by arguments that cannot even be subjected to criticism because they are void. The distinction between the "world" and our statements about it is not unreasonable by itself; it is well applicable, provided that we identify the "world" as an overall model of our sensual experiences. However, general statements about the "world" then refer to a model. Such statements can not necessarily be reformulated in terms of what gives rise to the model, that is, in terms of our sensual experiences. A statement for which a reformulation is impossible does not tell us anything; it just reveals the arbitrariness of our model.

Examples in which a reduction to the level of perceptions is impossible can be found in the above considerations. In particular, we cannot state precisely what it means to locate the source of vagueness, what it means that the world itself is vague, or what it means that natural language could be not vague. At least I do not see a way of doing so. Arguments based on such questions or assumptions lack content.

5 Three untenable theories of vagueness

Let us now examine the three mentioned approaches to vagueness individually in a critical way.

In a framework based on the perception-based view there is certainly no hope of keeping any feature of epistemicism. To make clear that the differences are irreconcilable, I just mention Williamson's use of the notion "knowledge". He maintains that opponents of his theory say that in borderline cases of a property there is nothing to know concerning the question whether the property applies or not. I go one step further and claim that talking of knowledge is generally inadequate, not only in borderline cases. If I say “K. is tall”, I express my impression that most people I have seen before are smaller, and there is no relationship to knowledge. Williamson’s understanding of “knowledge” and “ignorance” is pre-empiricist and thus unacceptable.

Although supervaluationism seems to be based on less arbitrary assumptions, the approach is also not acceptable. I have already commented that the very idea of a precisification is not well compatible with the perception-based viewpoint. Moreover, the overall motivation is questionable: to cope with what Fine calls penumbral connections. Fine claims that “K. is tall” and “K. is not tall” are exhaustive statements. Yes, they are – at the moment at which they are made because then we have a two-element scale in mind, and we classify people into just two groups. However, we no longer do so when considering somebody who does not fit into the scheme. The existence of penumbral connections is doubtful. Compare also the criticism of Smith [Smi, Sec. 2.4].

Contextualism is an approach with potential. However, if it is taken as a means to model the mechanism through which we come to the conclusions “tall” or “not tall”, its best features are overlooked. The dynamic character of language as regards the fluctuating distinction between “tall” and “not tall” is well taken into account. However, the dynamic character of language as regards spontaneous introduction of new concepts in order to adapt dynamically the view on some topic, to integrate new details, and, in particular, to increase precision is not considered, or at least not systematically.

Finally, in all three approaches we find the idea that vague statements express truths about the world in the sense of the realism-based viewpoint. Accordingly, all three approaches do not separate cleanly the formal from the informal level of argumentation but follow the unfortunate practise of identifying natural language directly with a formal language. Williamson argues explicitly in this way. Fine begins his paper [Fin] with the questions “What is the correct logic of vagueness?” and “What are the correct truth conditions for a vague language?” Shapiro’s realistic understanding of natural language becomes evident at the latest when he discusses in length the question whether vagueness originates from language or from the world [Sha1, Ch. 7].

Asking for truth conditions easily leads to unfounded argumentation. Sometimes the formal language is extended by a meta-language containing a so-called truth predicate. Lacking any kind of semantics or proof system, reasoning in this pseudo-formal construct may become prone to erratic speculation. In [Wil2, Sec. 1], for instance, the so-called T-scheme is used but then found to require a verbose justification; classical logic is used, but then specific laws are called into question; the idea that we might do better with intuitionistic logic is proposed. The overall impression is that not even the author takes a result derived from such an unstable basis seriously.

6 Shifting granularities

What is implied when I say “the blackboard is flat”? Understanding this statement as a description of what the world is like at a specific location at a specific time indeed requires truth conditions. Note that truth conditions are loaded with meaning: they tell whether the statement in question applies or does not apply under all possible circumstances, possibly including all places in the world and any time in history and in the future. My statement, if found true, would then reveal that an evaluation of the truth conditions leads to a positive result. Is this the implication of my saying "the blackboard is flat"?

No. When I utter this sentence, I do not refer, consciously or unconsciously, to a universe of possible circumstances. Giving this sentence a meaning does not require a sophisticated theory. The utterance is based on my image of a flat surface as opposed to a curved one. I call the blackboard flat because I observe that it is flat, and this in turn means that my observation fits with the picture of flatness that I have in mind. As a result, the utterance evokes a corresponding picture in the imagination of the person to whom I speak.

My impression can, of course, change later; on closer inspection I may find that the blackboard actually has a dent and is thus not flat. This discovery does not imply that my first statement was wrong; the two statements just report two successive impressions which do not agree. In particular, the "world" and its properties need not be involved to understand what it means to communicate that something is "flat". To develop a theory of "flatness" does not mean to analyse the "world" but to analyse impressions differing in definite respects.

The viewpoint I propagate considers human utterances as expression of perceptions in an absolute or relative way, and not as descriptions of how the world "really is". My approach to a formal treatment of vague information incorporates this idea and thus relies entirely on the perception-based viewpoint. It is little surprise that there is no associated canonical formal framework. Vagueness concerns the very process of defining a formal framework, and the approach can just offer general guidelines of how to use formal techniques to reason about vague properties.

Restricting reality to perceptions leads to a three-part picture: (i) our perceptions, (ii) the models we associate with selected types of perceptions, and (iii) the statements referring to the models. The world as it is then understood is a world of perceptions and is not directly accessible by the statements about it: a model lies in between.

Let a specific way of viewing a certain situation be given, for instance, a set of persons distinguished by size. Recall that the perception-based viewpoint is incompatible with direct translation of content to be modelled into a formal language. Instead, we must first associate a structure with the considered situation, then we can think of a way to reason about this model. When creating a model, we may observe that there is no canonical way to proceed; a certain amount of freedom is given. In case of "size", we must decide how fine the distinctions reflected in the model should be.

From mathematical practise, we are used to putting everything into a model that is imaginable with respect to a specific way of looking at things. Our model is in this case typically the result of an unbounded process and hence infinite. In particular, to model the notion of "size" it is common to use the set of positive rationals or reals. Whereas the construction method is based on a type of perception, not all elements of the result, the finished model, correspond to particular perceptions. In fact, we cannot distinguish between arbitrarily close sizes. The models we typically use in mathematics are finer than anything distinguishable.

In contrast, a natural-language expression such as "tall" is not appropriately modelled within an infinite structure but within a structure containing only what is distinguishable at the moment at which it is uttered. The utterance “K. is tall” could well refer to the structure consisting just of “small”, “medium-sized”, and “tall” and be endowed, so to say, with the natural order. The perception-based viewpoint does not regard the statement “K. is tall” as a consequence of the absolute fact that K. has a specific height. Rather, it is understood as the description of an observation relative to other observations. The speaker might refer to the many people he has seen before, he might consider a specific group of people, and one or more persons might be particularly weighted in this group: according to his impression, K. is considerably taller than the middle-sized people he has in mind. We are led to the three-element linear order as the appropriate model of “size” in this case.

I conclude that there need not be a unique model associated with a particular type of perception. Various alternatives might exist, and if there is a choice, one cannot claim that a particular choice describes the situation sufficiently or even correctly.

Freedom in formalisation even applies at the propositional level. Consider classical propositional logic (CPL). Sometimes CPL is viewed as the unquestionable basis of everything. However, CPL is the logic of true and false, no more or less. In everyday life, CPL is fundamental because it is the standard way of bringing structure into a variety of actual or possible situations; we choose properties that hold in some and do not hold in the other cases. When applying CPL, the reference of the yes-no propositions is essential for interpreting the results. The insight into the given problem that is achieved depends directly on the assignments made. However, these assignments may well be subject to modification, and modified assignments could well lead to different insights.

There need not arise a unique structure in a given context: in particular, different levels of granularity are possible. Vagueness reflects the fact that different points of view of the same object may lead to different structures, to coarser or to finer ones. The process of modelling is generally flexible, so in particular we can choose models of different granularity. Combining the associated structures into a single model may be seen as an uncommon, but certainly not an unfeasible, task in mathematical modelling.

Developing a formalism is not a challenge but just usual mathematical work if limited to one level of granularity. When regarding “tall” as “distinctly taller than middle-sized”, we can model this property by the top element of a three-element linear order, and its vagueness is irrelevant. If the three categories chosen are found sufficient to describe the persons in question, no problem arises. In fact, it does not even make sense to classify “tall” as vague in this case because “vague” is a relative notion, and a finer level of perception must necessarily be specified. The mathematical theory of a three-element linear order does not cause problems. In the case of the finest level of granularity, the relevant theory can be that of rational numbers, which is less straightforward but well developed.

Vagueness comes into play when we switch from a given level of granularity to finer ones. Whenever a borderline-tall person comes up in a conversation about small and tall persons, the speakers are forced to refine the scale. The interesting question is then how the entities of the previous scale translate into those of the new one or – more formally speaking – how the old scale is embedded in the new one. I presume that, after switching to the finer scale, the role of “tall” is narrower than before because “tall” is now understood as not applying to the borderline-tall person. However, examining how the transition works exactly is not a case for speculation but for empirical tests in the field of experimental psychology.

A particularly interesting question is how the coarse model, such as the three-element one, relates to the finest possible model that is usually used in mathematics. This question is most closely related to what discussions on vagueness usually focus on. It is a special case of the challenge of formalising reasoning under vagueness: how to reason simultaneously on a coarse level and the finest possible one.

We conclude by considering the Sorites paradox:

If n grains of sand form a heap, then so do $n-1$.
$10,000$ grains form a heap.
Consequently, one single grain forms a heap.

Let us try to determine, on the basis of the perception-based viewpoint, what the problem is.

It is assumed that there is a collection of grains in front of us. For reasoning, an aspect is chosen with respect to which the situation is described. The Sorites paradox deals with the size of the collection of grains; this is the chosen aspect.

There are, possibly among others, two different ways of describing the grains in this respect. On the one hand, two collections of grains may differ by exactly one grain. This observation lets us realise that the grains can be counted. Thus, we may state that there are exactly n grains in front of us, where n runs, say, from 1 to $10,000$. On the other hand, we may simply ponder whether the grains in front of us form a heap or not. In the first case we distinguish between $10,000$ situations, in the latter case between two. A model for the first case may be the natural numbers from 1 to $10,000$ endowed with the successor relation; for the latter case we may choose the two-element Boolean algebra.

For each of the two choices, we may now speak and make conclusions about the set of grains. The quality of our statements will differ. In the first case, we may address questions depending on single grains, for instance, whether the totality of grains can be divided into seven collections of equal size. In the second case, we may address questions concerning the totality of grains, for instance, whether an ant crawling behind the grains can be seen or not.

The Sorites argument makes simultaneous use of both ways of describing the situation. Two models are referred to at the same time, and elements from distinct models are put into relation. Moreover, all elements of the models are taken into account.

However, if we want to consider both situations together in a consistent manner, we must provide a common model: we must merge the two structures. We need to define a common refinement: the coarser structure – heap, non-heap – needs to be embedded in the finer one – $1, \ldots, 10,000$. Consistency can be expected only by reference to one structure; the entanglement of references to two different structures causes the paradox.

The embedding can probably be achieved by a variety of methods. One possibility of saving the notion "heap"/"non-heap" and making it accessible to the fine scale, is extending it by adding a degree [HaNo].

This is the "solution" to the paradox. The question why we are "taken in" by the paradox can be answered as follows. Saying that a specific number of grains form a heap evokes a picture of a heap in our mind. The exact number does not matter because it is not part of the image. Furthermore, the picture does not change when removing a single grain, and so we agree with the first statement. On the other hand, the number $10,000$ seems so large to us that putting together that many grains presumably results in a heap. Finally, a single grain does not evoke a picture of a heap.

According to the perception-based view, coping with vagueness essentially requires combining models. Generally applicable solutions to the problem of how to deal formally with vagueness, however, are not obvious.

7 Fuzzy sets

The last two sections present specific models of situations in which vagueness is to be taken into account. Neither are they "theories of vagueness" nor do they claim to be. They address selected aspects and are of practical rather than dogmatic quality.

Degree-based approaches, although of some practical relevance, seem to be of limited popularity in discussions on vagueness; an exception is [Smi]. From the perception-based viewpoint, the idea is reasonable.

When the vagueness of a property denoted by an expression in natural language is problematic, at least two levels of granularity are involved. Recall the previous example. When calling somebody "tall" or "short", we make the distinction between, say, short, middle-sized, and tall persons; formally speaking, we use the three-element linear order. Alternatively, when we intend to distinguish between any two people who differ in size, we are led to a structure such as the rational numbers, based on an infinite iteration of the idea that lengthy objects of given sizes can be concatenated and split into equal parts. The tricky question is how to treat both levels of granularity in a single formal calculus.

Let us focus on the common special case: we intend to include one coarse level and the finest possible level in the analysis. To each level we may associate a structure; I hereafter refer to them simply as the coarse and the fine model, respectively. The task is to interrelate them, that is, to describe how an element of one structure relates to an arbitrary element of the other one. This equates to the question of how well two such elements fit to each other, taking into account the content they represent.

Thus, we must specify how an entity from the finest possible level, such as a precise size, fits to an entity of the coarse level, such as "tall". There are several degrees of fit, and the set of degrees is a bounded linear order. As the transition between the two extremes, *not fitting at all* to *perfectly fitting*, is smooth, the linear order should be dense. Using the real unit interval $[0, 1]$ as a set of truth degrees – the usual choice in fuzzy set theory – is thus reasonable, although the rational unit interval would be more appropriate.

We should then ask which significance individual degrees have. In particular, $[0, 1]$ is not only a linear order but also an additive structure. Recall that notions such as "tall" are used when directly observing the object in question, and the speaker's opinion is the determining factor. A speaker using "tall" or "not tall" decides spontaneously that "tall" fits well or does not fit at all; the obvious cases, modelled by 0 and 1, are determined by a speaker's rough impression. In borderline cases, a speaker would use neither "tall" nor "not tall". The question is then whether any specific intermediate truth degree reflects the situation appropriately, and the answer depends on whether the speaker is able to decide spontaneously on a number between 0 and 1.

We see that the relationship between the coarse and the fine model does not stand on firm ground, and nothing can be done about it because the speaker's spontaneity is involved. Judgements such as "tall" and "middle-sized" might even not be reproducible by the same speaker. It is moreover unclear if the intermediate truth values can be used to measure a degree to which some property applies. If we are pedantic, we may add that the relationship of our fine scale to the object in question is also not clear because the former is over-precise. As the fine scale we may, for instance, use the positive rationals; the size of a given person can, however, not be associated with a precise number in a well-defined way.

The effect of the last point is, of course, tiny compared to the remaining uncertainties and can thus be neglected. Furthermore, the question whether real numbers are a suitable choice for truth degrees can only be clarified in experiments. Probably the earliest investigation in this direction is described in [HeCa]. The result is amazing and confirms that the very idea of using fuzzy sets to model natural-language expressions is reasonable; the participants in the experiments assigned truth degrees in a largely consistent way. Experiments of this kind suggest the overall conclusion that the fuzzy set model is an appropriate choice as a model of vague concepts.

Accepting this conclusion, I find that the degree-based approach to vagueness is well in line with the standpoint propagated in this contribution. It is, however, important to remark that the justification of the fuzzy set model strongly relies on its flexibility. A particular fuzzy set may be justified as a model of an expression used by a speaker in a particular context in a particular conversation. A general model of the same expression is subject to indeterminateness, which can be reduced if the context is well specified, but never eliminated. However, this indeterminateness need not be seen as a serious drawback; it is a natural consequence of our task of combining two levels of granularity. On the one hand, we deal with the content of utterances and not with measurement results. On the other hand, we use a model whose base set consists of entities that are even more precise than any measurement device.

Criticisms of degree-based approaches to vagueness may be rooted in the fact that fuzzy sets are not as rigidly connected to anything tangible as, say, real numbers. From the realism-based viewpoint, it is natural to require that a fuzzy set be uniquely determined. From the perception-based viewpoint, this is, however, unfeasible and contradicts the role of a fuzzy set as a model of a natural-language expression. A unique fuzzy set associated with a concept such as "tall" would imply a definite relationship between this concept and the underlying fine model. If it were possible to assign a particular role to the value 0.666 in the image of a fuzzy set modelling "tall", it would be possible to define a particular role for the precise size mapping to this particular value. Thus, the absurd consequence would be that "tall" determines the role of specific elements of the fine structure.

I have argued in favour of the classical fuzzy set model; indeed, fuzzy sets are useful tools for embedding a coarse model in a fine one. My defence ends here. The development of fuzzy set theory has been motivated by more ambitious aims than modelling individual natural-language expressions. It is another issue how to use fuzzy sets to formalise reasoning under vagueness. In this respect, results are not yet convincing.

Although fuzzy logics in the sense of [Haj] can be regarded as logics of fuzzy sets, and although a very prosperous research field has developed around fuzzy logics, it remains unclear how fuzzy logic relates to vagueness. Much has been written on this topic; I will add one example that offers both hope and disappointment. The idea of defining a logic of a collection of fuzzy sets with pointwise defined logical operations is defendable. A further result presented in [HeCa] suggests that when connecting expressions referring to "size" by "or" or "not", the corresponding fuzzy sets connect like the pointwise maximum or the pointwise standard negation, respectively. This result is encouraging. However, it does not support Gödel logic with standard negation. Apart from the fact that [HeCa] shows only that fuzzy sets referring to the same aspect, namely "size", can be combined, the question of the role of the residual implication remains open.

We conclude that fuzzy sets are a natural tool for interrelating coarse and fine concepts. However, when formalising reasoning under vagueness, the methodology of fuzzy logic, in particular of fuzzy logic in the sense of [Haj], has limited value. From the perception-based viewpoint, the problem is easily determined: for inference at the coarse level the fine model should play no role. If we are to formalise reasoning as we ourselves do it, the fine model is out of place; contrary to a common claim [Zad2], fuzzy sets are of little help in emulating what we conclude ourselves from statements involving vague concepts. If we are to reason about concepts that belong to a coarse level, reasoning should also take place at this coarse level. An appropriate approach should offer an inference with respect to coarse structures, which should, however, be allowed to vary in granularity.

8 Logics for reasoning with tolerance

Methods to cope with vagueness have been developed on the basis of practical needs in several fields. A controlling device based on vague specifications, a clinical guideline based on vague conditions, an expert system based on vague notions – in all these cases the duality of a rough and a fine scale appears and a practical solution is required to master the inconvenient side of vagueness.

It is certainly regrettable that methods are often chosen ad hoc and are not justified on the basis of clear principles. However, as far as vagueness is concerned, we cannot expect to be able to develop a logic dealing with vagueness under all circumstances and in the only appropriate way. The problem must be faced for each application separately and can hardly be solved once and for all. There is no difficulty in using different approaches for different applications; and when dealing with a specific problem it is not a deficiency to be unable to single out a single ideal formalism.

This section describes two logics that address one particular aspect arising in connection with vagueness: they are, in a certain sense, tolerant with regard to small changes. These approaches can of course not generally compete with fuzzy set theory. They are quite application-specific, but with regard to the problem considered they are rather satisfying.

We deal with crisp properties, but a similarity relation allows expressing that two properties, although otherwise generally unrelated, resemble each other. A formalism for approximate reasoning was originally proposed by E. Ruspini [Rus], and a variety of associated logics were subsequently studied [DPEGG, GoRo]. I mention two logics that arose from this line: the Logic of Approximate Entailment (LAE) [Rod] and the Logic of Strong Entailment (LSE) [EGRV]. I define them semantically. Axiomatisations of these, or closely related, logics can be found in the referenced articles.

LAE is a propositional logic. Our model reflects the fine level; a system of yes-no properties is modelled by a Boolean algebra $\mathcal{B}$ of subsets of a set W. Furthermore, W is endowed with a similarity relation, that is, with a mapping $s\colon W \times W \to [0,1]$ such that, for any $p, q, r \in W$, $s(p,q) = 1$ iff $p = q$, $s(p,q) = s(q,p)$, and $s(p,r) \geq s(p,q) \odot s(q,r)$, where $\odot$ is a fixed t-norm. For $d \in [0,1]$, we define $U_d(A) = \{p \in W\colon\ s(p,q) \geq d \text{ for some } q \in A\}$ to be the d-neighbourhood of an $A \in \mathcal{B}$.

The language of LAE comprises variables $\varphi_1, \varphi_2, \ldots$ and constants $\top, \bot$, and propositions are built up from the atoms by means of the operations $\wedge, \vee, \neg$. An implication is a triple consisting of two propositions α, β and a value $d \in [0,1]$, denoted by

$$\alpha \overset{d}{\to} \beta. \tag{1}$$

Propositions are evaluated by elements of $\mathcal{B}$ such that connectives are preserved. An implication (1) is satisfied by an evaluation v if $v(\alpha) \subseteq U_d(v(\beta))$. The notion of semantic entailment of an implication by a set of implications is defined in the straightforward way.

LSE is defined similarly, but an implication is denoted by

$$\alpha \overset{d}{\Rightarrow} \beta,$$

and is satisfied by an evaluation v if $U_d(v(\alpha)) \subseteq v(\beta)$.

Both logics deal with "tolerance". Intuitively speaking, in LAE, $\alpha \overset{d}{\to} \beta$ means that α implies β only approximately; there is a proposition α' that is similar to α to a degree $\geq d$ and implies β. Furthermore, if a proposition α'' is similar to α to a degree $\geq e$, the conclusion is possible that $\alpha \overset{d \odot e}{\to} \beta$. In LSE, $\alpha \overset{d}{\Rightarrow} \beta$ means that α implies β and that this is even the case for all propositions α' that are similar to α to a degree $\geq d$.

The embedding of a coarse structure in a fine one, which is necessary to deal formally with vague properties, is not fixed a priori and, in particular, tolerant of small changes. This latter aspect is taken into account by the two logics mentioned. LAE and LSE are no candidates for the ultimate logic of vagueness, but they take a relevant aspect into account.

9 Conclusion

The discussion about vagueness has become complicated; there are several competing research lines that can hardly be combined into a single one. Issues at increasingly fundamental levels are addressed. The modern discussion might have begun with some general considerations about natural language and formal methods, but nowadays, ideas such as "genuinely vague objects" appear in serious arguments.

I have sought to develop a view of the topic in which questions such as "Why is language vague?" do not play a role. Posing the question why natural language is vague makes no sense, simply because language cannot be different from what it is. In particular, I have argued that it is inappropriate to view vagueness as something originating from a deficiency. Instead, I have stressed the clear difference between a formal and a natural language.

I have reviewed the situation from a perspective which I originally adopted to cope with difficulties in a different field – the foundational debate in quantum physics. Although even quantum physicists express their results in terms of moving particles as little balls, quantum physics is best understood the minimalist way, that is, by means of purely statistical interpretation. The mechanistic perspective has, in my opinion, become obsolete since quantum theories emerged. The traditional point of view according to which the observer steps into the world to make observations of what is there independently, is outdated: the observation is the basis, and the observed object depends on its observation. This perspective reduces the role of mathematics to providing tools for developing formal models that fit the observations as well as possible, rather than tools for reasoning correctly about facts of the world.

I have widened this perspective to address not only specific physical phenomena but anything expressible in natural language. If we consider anything at all as part of reality, the set of perceptions experienced during our lives, is the bare minimum. Our language describes these perceptions, relating them to each other. Surely, a notion such as "size" gives rise to a formal structure; "size" can be represented by a linear order. Nevertheless, we use expressions such as "tall" to distinguish between different sizes or to identify similar sizes. The very reason for calling somebody "tall" is to communicate the impression that the person is taller than most other people considered in the given situation.

I have argued that vagueness reflects the fact that objects can be classified according to, say, their size at different levels of granularity. Vagueness is a relative notion; it concerns the problem that a rough classification cannot be refined in a canonical way. The expression "tall" is used to distinguish from "middle-sized"; the expression "1.8 m tall" is used to distinguish from "smaller than 1.75 m or taller than 1.85 m". The challenge is to deal with the two or more levels of granularity in a combined formalism.

There are several possibilities to do this practically, and specific solutions are typically undogmatic and imperfect. In fact, when evaluating a particular method, we must take into account that we deal with utterances in natural languages rather than with reproducible physical measurements; the evaluation is a task for experimental psychology.

The discussions on vagueness have developed their own dynamics, and presumptions are often made that are difficult to discern. It is my standpoint that the decision whether a topic is well defined depends on the possibility to reduce the problem to the level of perception. As long as the topic ultimately concerns perceptions, arguments can be discussed. If, however, a style is predominant in which speculation dominates, it is difficult to believe that the results are meaningful.

References

[Bla] M. Black, Vagueness: An exercise in logical analysis, *Philosophy of Science* **4** (1937), 427 - 455. Shortened reprint in [KeSm].

[DPEGG] D. Dubois, H. Prade, F. Esteva, P. Garcia, L. Godo. A logical approach to interpolation based on similarity relations, *Int. J. Approx. Reasoning* **17** (1997), 1 - 36.

[EGRV] F. Esteva, Ll. Godo, R. O. Rodríguez, T. Vetterlein, On the logics of similarity-based approximate and strong entailment, Actas del XV Congreso Español Sobre Tecnologías y Logica Fuzzy (Proceedings of the 15th Spanish Congress on Fuzzy Logic and Technology) (Huelva 2010); 187 - 192.

[Eva] G. Evans, Can there be vague objects?, *Analysis* **38** (1978), 208; reprinted in [KeSm].

[Fin] K. Fine, Vagueness, truth and logic, *Synthese* **30** (1975), 265 - 300; reprinted in [KeSm].

[GoRo] Ll. Godo, R. O. Rodríguez, Logical approaches to fuzzy similarity-based reasoning: an overview, in: G. Della Riccia et al. (eds.), “Preferences and similarities”, Springer-Verlag, Wien 2008; 75 - 128.

[Haj] P. Hájek, “Metamathematics of Fuzzy Logic”, Kluwer Acad. Publ., Dordrecht 1998.

[HaNo] P. Hájek, V. Novák, The sorites paradox and fuzzy logic, *Int. J. Gen. Syst.* **32** (2003), 373 - 383.

[HeCa] H. M. Hersh, A. Caramazza, A fuzzy set approach to modifiers and vagueness in natural language, *Journal of Experimental Psychology: General* **105** (1975), 254 - 276.

[Kee] R. Keefe, “Theories of Vagueness”, Cambridge University Press, Cambridge 2000.

[KeSm] R. Keefe, P. Smith (eds.), “Vagueness: a reader”, MIT Press, Cambridge 1999.

[MuSm] K. Munn, B. Smith, “Applied Ontology. An Introduction”, Ontos-Verlag, Frankfurt-Heusenstamm 2009.

[PlBu] T. Placek, J. Butterfield, “Non-Locality and Modality”, Proc. of the NATO Advanced Research Workshop on Modality, Probability, and Bell’s Theorem (Cracow, 19.-23.8.2001), Kluwer Acad. Publ., Dordrecht 2002.

[Rod] R. O. Rodríguez, “Aspectos formales en el Razonamiento basado en Relaciones de Similitud Borrosas”, Ph.D. Thesis, Technical University of Catalonia, 2002.

[Rus] E. Ruspini, On the semantics of fuzzy logic, *International Journal of Approximate Reasoning* **5** (1991), 45 - 88.

[Sha1] S. Shapiro, “Vagueness in Context”, Oxford University Press, Oxford 2006.

[Sha2] S. Shapiro, Reasoning with slippery predicates, *Stud. Log.* **90** (2008), 313 - 336.

[Smi] N. J. J. Smith, “Vagueness and Degrees of Truth”, Oxford University Press, Oxford 2008.

[Wil1] T. Williamson, “Vagueness”, Routledge, Abingdon 2005.

[Wil2] T. Williamson, Vagueness and ignorance, *Proc. of the Aristotelian Society, Supp.* **66** (1992), 145 - 162; reprinted in [KeSm].

[Zad1] L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338 - 353.

[Zad2] L. A. Zadeh, Fuzzy logic = computing with words, *IEEE Trans. Fuzzy Syst.* **4** (1996), 103 – 111.