WEAK EFFECT ALGEBRAS

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Abstract. Weak effect algebras are based on a commutative, associative and cancellative partial addition; they are moreover endowed with a partial order which is compatible with the addition, but in general not determined by it. Every BL-algebra, i.e. the Lindenbaum algebra of a theory of Basic Logic, gives rise to a weak effect algebra; to this end, the monoidal operation is restricted to a partial cancellative operation.

We examine in this paper BL-effect algebras, a subclass of the weak effect algebras which properly contains all weak effect algebras arising from BL-algebras. We describe the structure of BL-effect algebras in detail. We thus generalise the well-known structure theory of BL-algebras.

Namely, we show that BL-effect algebras are subdirect products of linearly ordered ones and that linearly ordered BL-effect algebras are ordinal sums of generalised effect algebras. The latter are representable by means of linearly ordered groups.

1. Introduction

BL-algebras are the algebraic counterpart of Hájek's Basic Logic, the many-valued logic based on continuous t-norms and their residua [Haj, CEGT]. The structure of BL-algebras is well-known: They are subdirect products of linearly ordered BL-algebras, and the latter are, in a certain transparent way, composed from MV- and product algebras (see [AgMo]).

Interestingly, BL-algebras may be identified with a certain kind of partial algebra, which very much reminds of effect algebras [Vet1]. Effect algebras have been studied in a completely different context; they describe the inner structure of the set of effects in quantum mechanics [FoBe]. They are based on a partial addition, which is cancellative and which determines a bounded partial order in the natural way.

In [Vet1], weak effect algebra were introduced as a common generalisation of effect algebras and BL-algebras. To this end, the effect algebra's partial order was added as an own relation, and the assumption was dropped that this order is uniquely determined by the partial addition. So in a weak effect algebra, from $a \leq b$ it does in general not follow that $a + x = b$ for some element x, although the converse is true.

The prototypical examples of weak effect algebras arise from BL-algebras; the construction is, roughly speaking, the following. BL-algebras are based on two operations, a conjunction $\odot$ and an implication $\rightarrow$. First of all, there is no loss of information when we drop one of them, for example $\rightarrow$. Second, $\odot$ is in general

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not cancellative; but we may restrict $\odot$ to a partial operation which is defined only for the extremal cases; the resulting operation is then cancellative, and the original operation may easily be recovered. Now, putting the "cancellative part" of $\odot$, the partial order, and the constants together, we get a weak effect algebra.

The aim of this paper is to examine BL-effect algebras, which are special weak effect algebras. Among them, we find in particular all the algebras which correspond to BL-algebras. However, a simple example shows that BL-effect algebras are still strictly more general than BL-algebras.

We proceed as follows. We show in a first step that BL-effect algebras are subdirect products of linearly ordered BL-effect algebras. To this end, we introduce the necessary concepts of congruences and ideals of weak effect algebras. We then turn to the linear case. We prove that linearly ordered BL-effect algebras are ordinal sums of generalised effect algebras [HePu]. The latter are known to be isomorphically embeddable into the positive cone of linearly ordered abelian groups (see [DvPu]).

So the picture is in the end similar to case of BL-algebras, whose structure theory is contained as a special case. We note that in the present case, the hard step is the first one: the subdirect representation theorem; the second one, which caused so much trouble in the case of BL-algebras, is rather easy.

In a concluding section, we point out the significance of our results with respect to the general problem how to classify residuated lattices (that is, integral, commutative residuated ℓ-monoids). We outline a possible approach to the latter problem, an approach which is exemplified by the procedure in this paper.

2. Weak effect algebras

Effect algebras were originally introduced to describe the internal structure of the set of Hilbert space effects, which in turn model possibly unsharp quantum-physical propositions [FoBe]. As opposed to that, BL-algebras are algebras of propositions of Basic Logic, the logic of continuous t-norms and their residua, introduced by Hájek [Haj]. In [Vet1], we combined both notions; we generalised effect algebras so as to get a class of partial algebras which is large enough to comprise also BL-algebras.

Definition 2.1. A *weak effect algebra* is a structure $(E; \leq, +, 0, 1)$ such that the following conditions hold:

(E1) $(E; \leq, 0, 1)$ is a poset with the smallest element 0 and the largest element 1.

(E2) $+$ is a partial binary operation such that for any $a, b, c \in E$

  (a) $(a+b)+c$ is defined iff $a+(b+c)$ is defined, and in this case $(a+b)+c = a+(b+c)$;

  (b) $a+0$ is always defined and equals a;

  (c) $a+b$ is defined iff $b+a$ is defined, and in this case $a+b = b+a$.

(E3) If, for $a, b, c \in E$, $a+c$ and $b+c$ are defined, then $a \leq b$ if and only if $a+c \leq b+c$.

(E4) If, for $a, b \in E$, $a \leq b$, then there is a largest element $\bar{a} \leq a$ such that $\bar{a} + x = b$ for some $x \in L$.

It is easily seen that effect algebras are identifiable with those weak effect algebras for which the following holds: For any a, b such that $a \leq b$, there is an x such that $a + x = b$.

Note that (E3) implies the cancellativity of the partial addition. So if for some pair a, b of elements of a weak effect algebra such that $a \leq b$, there is an element x such that $a + x = b$, then x is uniquely determined; we will denote this element by $b - a$.

In the sequel, we will make use of the usual convention that a statement containing some partial operation is to be read as: The involved terms are defined, and the statement holds.

We next recall how we may associate a BL-effect algebras with a BL-algebra. For a more intuitive presentation of this construction, see [Vet2].

Definition 2.2. Let $(L; \leq_{\mathrm{BL}}, \odot, \Rightarrow, 0_{\mathrm{BL}}, 1_{\mathrm{BL}})$ be a BL-algebra. Set

$$a \leq b \quad \text{if} \quad b \leq_{\mathrm{BL}} a,$$
$$a \oplus b = a \odot b, \qquad a \ominus b = b \Rightarrow a,$$
$$0 = 1_{\mathrm{BL}}, \qquad 1 = 0_{\mathrm{BL}}.$$

Then we call $(L; \leq, \oplus, \ominus, 0, 1)$ a *dual BL-algebra.*

Furthermore, we define a partial addition $+$ on L as follows: For $a, b \in L$, let $a + b = a \oplus b$ if a is the smallest element x such that $x \oplus b = a \oplus b$ and b is the smallest element y such that $a \oplus y = a \oplus b$; else, we let $a + b$ undefined. We call $+$ the *partial addition associated to* $\oplus$.

So what we do with a BL-algebra is: The order is reversed, which means just a change of notation; and the total operation $\oplus$, which corresponds to the original $\odot$, is restricted to the pairs of elements which are minimal among those whose sum is the same.

We next describe the transition from a partial algebra to a total one.

Definition 2.3. Let $(E; \leq, +, 0, 1)$ be a weak effect algebra. Let us call the partial addition $+$ *extendible* to the total operation $\oplus$ if

$$a \oplus b \overset{\text{def}}{=} \max \{a' + b' \colon\ a' \leq a,\ b' \leq b \text{ and } a' + b' \text{ is defined}\} \tag{1}$$

exists for any $a, b \in L$. In this case, we call $\oplus$ the *total addition associated to* $+$.

We will establish the exact connection between BL-algebras and weak effect algebras; Theorem 2.5 is a slightly modified version of [Vet1, Theorems 4.2, 4.3].

Definition 2.4. A *BL-effect algebra* is a weak effect algebra $(E; \leq, +, 0, 1)$ such that the following conditions hold:

- (E5) $(E; \leq)$ is a lower semilattice.
- (E6) For any $a, b, c \in E$ such that $c \leq a + b$, there are $a_1 \leq a$ and $b_1 \leq b$ such that (α) $c = a_1 + b_1$ and (β) $a_1 = a$ in case $c \geq a$.
- (E7) For any $a, b \in E$, there are a_1, a_2, b_1, b_2 such that $a = a_1 + a_2$, $b = b_1 + b_2$ and $a_1 \leq b$, $b_1 \leq a$ and $a_2 \wedge b_2 = 0$.
- (E8) For any a, b, c, x, y such that $a + x = b + y = c$, there is a z such that $(a \wedge b) + z = c$.

Theorem 2.5. (I) *Let $(L; \leq, \oplus, \ominus, 0, 1)$ be a dual BL-algebra, and let $+$ be the associated partial addition. Then $(L; \leq, +, 0, 1)$ is a BL-effect algebra. Moreover, the total addition associated to $+$ exists and coincides with $\oplus$.*

(II) *Let* $(L; \leq, +, 0, 1)$ *be a BL-effect algebra, and let* $+$ *be extendible to* $\oplus$*. Let* $\ominus$ *be the residual of* $\oplus$*. Then* $(L; \leq, \oplus, \ominus, 0, 1)$ *is a dual BL-algebra. Moreover, the partial addition associated to* $\oplus$ *coincides with* $+$*.*

The following example shows that there are BL-effect algebras not arising from BL-algebras.

Example 2.6. Let π be any irrational positive real number, and let $E = \{a \in \mathbb{R}\colon a \in \mathbb{Q} \text{ and } 0 \leq a < \pi, \text{ or } a = \pi\}$. Let $+$ on E be the addition of real numbers whenever the sum is in E. Then $(E; +, 0, \pi)$ is a weak effect algebra.

The partial addition on E is not extendible; the maximum (1) does not exist if the sum of the two elements $a, b \in E$ exceeds π. So E does not arise from a (dual) BL-algebra.

We conclude the section by establishing some basic properties of BL-effect algebras.

The key property assumed in [Rav] to represent effect algebras in the positive cone of partially ordered groups, is the Riesz decomposition property. We will use it in the present context formally unchanged.

Definition 2.7. A weak effect algebra is said to fulfil the *Riesz decomposition property*, or (RDP) for short, if for all a, b, c, d such that $a + b = c + d$ there are e_1, e_2, e_3, e_4 such that the scheme

$$\begin{array}{ccccc} e_1 & e_2 & \rightarrow & a \\ e_3 & e_4 & \rightarrow & b \\ \downarrow & \downarrow & & \\ c & d & & \end{array} \tag{2}$$

holds. Here, by the scheme (2) to hold, we mean that any column or line adds up to what the arrow points to.

Proposition 2.8. *Every BL-effect algebra fulfils* (RDP)*.*

Proof. Assume $a + b = c + d$. Because $c \leq a + b$, there are by (E6) $e_1 \leq a$ and $e_3 \leq b$ such that $c = e_1 + e_3$. From $e_1 \leq a \leq e_1 + e_3 + d$, we conclude also by (E6) that $a = e_1 + e_2$ for some e_2; and similarly, we find an e_4 such that $b = e_3 + e_4$. It follows $c = e_2 + e_4$ by cancellation. □

Lemma 2.9. *In a BL-effect algebra, the following holds.*

(i) *Let* $a + b = c + d$*. Then* $a \leq c$ *if and only if* $b \geq d$*.*

(ii) *Let* $c - a$ *and* $c - b$ *exist. Then also* $c - (a \wedge b)$ *exists, and* $c - (a \wedge b) = (c - a) \vee (c - b)$*.*

Proof. (i) If $a \leq c$, it follows by (E6) that $c = a + b_1$ for some $b_1 \leq b$. So $d \leq b$ follows by cancellation.

(ii) By (E8), $c - (a \wedge b)$ exists. By (i), we have $c - (a \wedge b) \geq c - a, c - b$. Let $x \geq c - a, c - b$. Let $x' = x \wedge c$, and let $x'' \leq x'$ be the largest element such that $x'' + y = c$ for some y; then $x \geq x'' \geq c - a, c - b$. By (i), we further conclude $y \leq a, b$, so $y \leq a \wedge b$, hence $x \geq x'' = c - y \geq c - (a \wedge b)$. □

3. Congruences of weak effect algebras

Even for an effect algebra, it is difficult to give the exact conditions that an equivalence relation does not only allow the formation of a quotient algebra, but that this quotient is also again an effect algebra; see e.g. [GuPu]. For weak effect algebras, the various special properties related to the partial order cause further difficulties.

Like in the case of effect algebras, our definition of a congruence is such that we may at least form a quotient algebra; this quotient algebra, however, is in general not again a weak effect algebra.

Definition 3.1. Let $(E; \leq, +, 0, 1)$ be a weak effect algebra. Then an equivalence relation $\sim$ is called a *congruence* of E if the following holds:

(C1) For all a, a', b, b' such that $a \sim a'$, $b \sim b'$ and $a+b$, $a'+b'$ exist, $a+b \sim a'+b'$.
(C2) For all a, b, b' such that $a \leq b$ and $b \sim b'$, there is an $a' \sim a$ such that $a' \leq b'$.
(C3) For all a, a', b such that $a \sim a'$ and $a \leq b \leq a'$, we have $b \sim a$.

$\sim$ being a congruence of E, we set $[a]_\sim \overset{\text{def}}{=} \{a' \colon a' \sim a\}$ for $a \in E$, and $[E]_\sim \overset{\text{def}}{=} \{[a]_\sim \colon a \in E\}$. We define

$$[a]_\sim \leq [b]_\sim \quad \text{if} \quad a' \leq b' \text{ for some } a' \sim a,\ b' \sim b$$

for $a, b \in E$, and we let $+$ be the partial binary operation on $[E]_\sim$ such that

$$\begin{array}{c} [a]_\sim + [b]_\sim \text{ is defined} \\ \text{and equals } [c]_\sim \end{array} \quad \text{iff} \quad \begin{array}{c} \text{for some } a' \sim a \text{ and } b' \sim b, \\ a' + b' \text{ is defined and } a' + b' \sim c. \end{array}$$

Then $([E]_\sim; \leq, +, [0]_\sim, [1]_\sim)$ is called the *quotient algebra* of E induced by the congruence $\sim$.

Note that only (C1) in this definition makes sure that the quotient algebra's partial operation $+$ is defined unambigously. Moreover, (C2) and (C3) guarantee that the quotient algebra is by $\leq$ partially ordered:

Lemma 3.2. *Let $(E; \leq, +, 0, 1)$ be a weak effect algebra, and let $\sim$ be a congruence of E. Then $[E]_\sim$ is by $\leq$ partially ordered, and bounded by $[0]_\sim$ and $[1]_\sim$.*

Proof. Let $a, b, c \in E$ such that $[a]_\sim \leq [b]_\sim$ and $[b]_\sim \leq [c]_\sim$. Then by (C2), there are $b' \sim b$ and $a' \sim a$ such that $b' \leq c$ and $a' \leq b'$; it follows $[a]_\sim \leq [c]_\sim$. Similarly, from $[a]_\sim \leq [b]_\sim$ and $[b]_\sim \leq [a]_\sim$ it follows $a' \leq b'$ and $b' \leq a$ for some $a' \sim a$, $b' \sim b$; so $[a]_\sim = [b]_\sim$ by (C3). Reflexivity of $\leq$ is obvious. So $\leq$ is a partial order on E, and $[0]_\sim$ is clearly the smallest element and $[1]_\sim$ the largest element. □

The numerous conditions needed to ensure that a congruence is structure-preserving are listed in the following theorem. Fortunately, the situation will not remain that complicated when we will have to do with BL-effect algebras.

Theorem 3.3. *Let $(E; \leq, +, 0, 1)$ be a weak effect algebra, and let $\sim$ be a congruence of E fulfilling the following conditions:*

(C4) *For $a, b, c \in E$ such that $c \sim a + b$, there are $a \sim a'$ and $b \sim b'$ such that $c = a' + b'$.*
(C5) *For a, a', b, b', c, c' such that $a \sim a'$, $b \sim b'$, $c \sim c'$, if $a \leq b$ and $a' + c$, $b' + c'$ exist, then there are $a'' \sim a$, $c'' \sim c$ such that $a'' \leq b'$ and $a'' + c''$ and $b' + c''$ exist.*

(C6) *For* a, b, c, c' *such that* $c \sim c'$, *if* $a + c \leq b + c'$, *then* $a' \leq b'$ *for some* $a' \sim a$ *and* $b' \sim b$.

(C7) *Let* $a \leq b$, *and let* $\bar{a} \leq a$ *be largest such that* $\bar{a} + x = b$ *for some* x. *Then from* $c \leq a$ *and* $c' + y = b$ *for* $c' \sim c$, *it follows* $c'' \leq \bar{a}$ *for some* $c'' \sim c$.

Then the quotient algebra $([E]_\sim; \leq, +, [0]_\sim, [1]_\sim)$ *is again a weak effect algebra.*

Proof. (E1) holds by Lemma 3.2.

To show (E2), assume that $([a]_\sim + [b]_\sim) + [c]_\sim$ exists. Then by (C4), this sum equals $[(a' + b') + c']_\sim$ for some $a' \sim a$, $b' \sim b$, $c' \sim c$, so (E2)(a) follows. (E2)(b) and (E2)(c) are obvious.

For the proof of (E3), let us assume that $[a]_\sim + [c]_\sim$ and $[b]_\sim + [c]_\sim$ exist. We may further assume that $a + c$ and $b + c'$ exist for some $c' \sim c$. If now $[a]_\sim \leq [b]_\sim$, then $a' \leq b'$ for some $a' \sim a$, $b' \sim b$, and it follows by (C5) that $a'' + c'' \leq b + c''$ for some $a'' \sim a$, $c'' \sim c$; thus, we have $[a]_\sim + [c]_\sim \leq [b]_\sim + [c]_\sim$. Conversely, from $[a]_\sim + [c]_\sim \leq [b]_\sim + [c]_\sim$ it follows by (C4) that $a' + c'' \leq b' + c'''$ for some $a' \sim a$, $b' \sim b$, $c'', c''' \sim c$, and further by (C6) that $a'' \leq b''$ for some $a'' \sim a$ and $b'' \sim b$; thus, $[a]_\sim \leq [b]_\sim$.

Finally, let $[a]_\sim \leq [b]_\sim$; we may assume $a \leq b$. Let $\bar{a}$ be the largest element below a such that $\bar{a} + x = b$ for some x. Assume further $[c]_\sim \leq [a]_\sim$ and $[c]_\sim + [y]_\sim = [b]_\sim$ for some c, y; we claim that then $[c]_\sim \leq [\bar{a}]_\sim$, which proves (E4). Indeed, by (C4), $c' + y' = b$ for some $c' \sim c$, $y' \sim y$, so by (C2) and (C7), $c'' \leq \bar{a}$ for some $c'' \sim c$. □

The notion of a homomorphism of partial algebras is used here as follows [Gra].

Definition 3.4. Let $(E; \leq, +, 0_E, 1_E)$ and $(F; \leq, +, 0_F, 1_F)$ be weak effect algebras. A mapping $\varphi\colon\ E \to F$ is called a *homomorphism* if, for all $a, b \in E$, the following holds: (i) if $a \leq b$, then $\varphi(a) \leq \varphi(b)$; (ii) if $a + b$ is defined, then so is $\varphi(a) + \varphi(b)$, and $\varphi(a + b) = \varphi(a) + \varphi(b)$; (iii) $\varphi(0_E) = 0_F$ and $\varphi(1_E) = 1_F$.

Moreover, a homomorphism $\varphi\colon\ E \to F$ is called *full* if, for all $a, b \in E$, $\varphi(a) + \varphi(b)$ is defined if and only if $a' + b'$ is defined for some $a', b' \in E$ such that $\varphi(a') = \varphi(a)$ and $\varphi(b') = \varphi(b)$.

Proposition 3.5. *Let* $(E; \leq, +, 0_E, 1_E)$ *be a weak effect algebra, and let* $\sim$ *be a congruence such that* $[E]_\sim$ *is again a weak effect algebra. Then the canonical embedding* $\iota\colon\ E \to [E]_\sim,\ \ a \mapsto [a]_\sim$ *is a full homomorphism.*

4. The subdirect representation of BL-effect algebras

We show in this section that any BL-effect algebra is a subdirect product of linearly ordered BL-effect algebras. Although certain modifications are necessary, it is possible to proceed in the standard way.

Definition 4.1. Let $(E; \leq, +, 0, 1)$ be a weak effect algebra. We call $I \subseteq E$ an *ideal* of E if (α) $a \leq r$ and $r \in I$ imply $a \in I$ and (β) $r, s \in I$ such that $r + s$ exists implies $r + s \in I$.

I being an ideal of E, we set

$$a \sim_I b \quad \text{if} \quad a - r = b - s \text{ for some } r, s \in I.$$

Remarkably, we do not have to make any further assumption on an ideal of a BL-effect algebra that it induces a congruence preserving all axioms. We divide the proof into two parts.

Lemma 4.2. *Let I be an ideal of a BL-effect algebra E. Then $\sim_I$ is a congruence of E, and the quotient algebra $([E]_{\sim_I}; \leq, +, [0]_{\sim_I}, [1]_{\sim_I})$ is a weak effect algebra.*

Proof. The relation $\sim_I$ is clearly reflexive and symmetric. To see that it is transitive, let $a \sim_I b$ and $b \sim_I c$, that is, let $a - r = b - s$ and $b - t = c - u$ for $r, s, t, u \in I$. Then we have $(a - r) + s = (c - u) + t$; applying (RDP) to this equation givens $(a - r) - e_2 = (c - u) - e_3$ for certain $e_2 \leq t$, $e_3 \leq s$, which means $e_2, e_3 \in I$. We conclude $a - (r + e_2) = c - (u + e_3)$, so $a \sim_I c$.

We next show that all the properties (C1)–(C7) hold for $\sim_I$. By Theorem 3.3, $[E]_{\sim_I}$ will then be proved to be a weak effect algebra.

From $a \sim_I a'$, $b \sim_I b'$ and the existence of $a + b$, $a' + b'$, it easily follows $a + b \sim_I a' + b'$, which is (C1).

From $a \leq b$ and $b - r = b' - s$ for $r, s \in I$, it follows $a \leq (b' - s) + r$, so by (E6) $a = a' + a_r$ for $a' \leq b' - s$ and $a_r \leq r$; so $a' \sim_I a$ and $a' \leq b'$. This is (C2).

Let $a - r = a' - s$ for $r, s \in I$, and $a \leq b \leq a'$. Then $a - r \leq b \leq (a - r) + s$, and it follows by (E6) that $b = (a - r) + t$ for some $t \leq s$. So $b \sim_I a$, and (C3) is proved.

Let now $c \sim_I a + b$, that is, $c - s = (a + b) - r$ for some $r, s \in I$. Then we have $a + b = r + (c - s)$, and by (RDP) we conclude that $c - s = a' + b'$ for $a' \sim_I a$, $b' \sim_I b$. So $c = a' + b' + s$; this proves (C4).

To see (C5), let $a \leq b$ and let $a' + c$, $b' + c'$ exist, where $a \sim_I a'$, $b \sim_I b'$, $c \sim_I c'$. Then by (C2) $a'' \leq b'$ for an $a'' \sim_I a'$. Let $a' - r = a'' - s$, $b - t = b' - u$, $c - v = c' - w$ for $r, s, t, u, v, w \in I$. Then $a' - r \leq b'$, and $(a' - r) + (c' - w)$ as well as $b' + (c' - w)$ exist; (C5) is shown.

Let now $c \sim_I c'$ and $a + c \leq b + c'$. Let $c - r = c' - s$ for $r, s \in I$; then $a + (c' - s) \leq b + c'$, so $a \leq b + s$. This proves (C6).

Finally, let $a \leq b$, let $\bar{a} \leq a$ be largest such that $\bar{a} + x = b$ for some x, and let $c \leq a$ and $c' + y = b$ for some $c' \sim_I c$. From $c - r = c' - s$ for $r, s \in I$, we get $(c - r) + s + y = b$, so $c - r \leq \bar{a}$, and also (C7) is proved. ☐

Theorem 4.3. *Let I be an ideal of a BL-effect algebra E. Then $\sim_I$ is a congruence of E, and the quotient algebra $([E]_{\sim_I}; \leq, +, [0]_{\sim_I}, [1]_{\sim_I})$ is again a BL-effect algebra.*

Proof. By Lemma 4.2, $\sim_I$ is a congruence of E such that $[E]_{\sim_I}$ is a weak effect algebra. To see that $[E]_{\sim_I}$ is actually a BL-effect algebra, it remains to prove the axioms (E5)–(E8).

We first show that any equivalence class is closed under infima. Indeed, let $a \sim_I a'$, that is, $a - r = a' - s$ for $r, s \in I$. Then $a - r \leq a \wedge a' \leq a' = (a - r) + s$; so because (E6) holds in E, $a \wedge a' = (a - r) + t$ for some $t \in I$.

We next show that, for a, b,

$$[a \wedge b]_\sim = [a]_\sim \wedge [b]_\sim, \tag{3}$$

which implies (E5). We clearly have $[a \wedge b]_\sim \leq [a]_\sim, [b]_\sim$. Let c be such that $[c]_\sim \leq [a]_\sim, [b]_\sim$; then by (C2), $c' \leq a$ and $c'' \leq b$ for some $c', c'' \sim_I c$, so $c' \wedge c'' \leq a \wedge b$ and $[c]_\sim = [c' \wedge c'']_\sim \leq [a \wedge b]_\sim$. So (3) follows.

To see (E6), let $[c]_\sim \leq [a]_\sim + [b]_\sim$. We may assume that $a + b$ exists and also, by (C2) and (C4), that $c \leq a + b$ holds. The first part of (E6) now easily follows. If even $[a]_\sim \leq [c]_\sim$, then we have $a' \leq c \leq a + b$ for $a' \sim_I a$. From $a - r = a' - s$, $r, s \in I$, we get $a - r \leq c \leq (a - r) + r + b$, from which $c = (a - r) + b_1$ for some $b_1 \leq b + r$ follows. This proves the second part of (E6).

Finally, for any pair of elements a, b, there are a_1, a_2, b_1, b_2 such that $a = a_1 + a_2$, $b = b_1 + b_2$, $a_1 \leq b$, $b_2 \leq a$, $a_2 \wedge b_2 = 0$. We then have the same relations among the respective equivalence classes; this follows in the last case from (3). So also (E7) holds.

(E8) follows from (C4) and (3). □

Definition 4.4. An ideal I of a weak effect algebra E is called *prime* if for any $a, b \in E$ such that $a \wedge b = 0$, either $a \in I$ or $b \in I$.

Lemma 4.5. *Let E be a BL-effect algebra.*

(i) *Let I be a prime ideal. Then $[E]_{\sim_I}$ is linearly ordered.*

(ii) *For any $a \neq b$, there is a prime ideal I such that $a \not\sim_I b$.*

Proof. (i) Let $a, b \in E$, and let $a = a_1 + a_2$, $b = b_1 + b_2$ be decomposed according to (E7). Then either $a_2 \in I$, implying $a \sim_I a_1 \leq b$ and $[a]_\sim \leq [b]_\sim$; or $b_2 \in I$, implying $[b]_\sim \leq [a]_\sim$.

(ii) $I = \{0\}$ is an ideal such that $a \not\sim_I b$. Let I be an ideal which is maximal w.r.t. the property $a \not\sim_I b$. Assume $c \wedge d = 0$ and $c, d \notin I$. In view of (E6), $I_c = \{c_1 + \ldots + c_k + e\colon\ c_1, \ldots, c_k \leq c,\ e \in I\}$ is the ideal generated by I and c; similarly, also I_d is given. By the maximality of I, we have $a \sim_{I_c} b$ and $a \sim_{I_d} b$; this means that $a - r = b - s$ and $a - t = b - u$ for $r, s \in I_c$ and $t, u \in I_d$. By Lemma 2.9, we have $a - (r \wedge t) = (a - r) \vee (a - t) = (b - s) \vee (b - u) = b - (s \wedge u)$. But $r = c_1 + \ldots + c_k + e$ and $t = d_1 + \ldots + d_l + f$ for certain $c_1, \ldots, c_k \leq c$, $d_1, \ldots, d_l \leq d$, $e, f \in I$, and by (E6) we conclude $r \wedge t \in I$. Similarly, also $s \wedge u \in I$. So $a \sim_I b$, a contradiction. □

We have the following notions of subalgebras and subdirect representations of weak effect algebras [Gra].

Definition 4.6. Let $(E; \leq, +, 0, 1)$ and $(F; \leq', +', 0, 1)$ be weak effect algebras such that F is a subset of E containing the constants and $\leq'$ is the order of E restricted to F. Assume furthermore that if $a +' b$ is defined in F, then also $a + b$ is defined in E, in which case $a +' b = a + b$. Then $(F; \leq, +', 0, 1)$ is called a *weak subalgebra* of E. If, in addition, for $a, b \in F$, $a +' b$ is defined in F if and only if $a + b$ is defined in E and lies in F, then $(F; \leq, +', 0, 1)$ is called a *relative subalgebra* of E.

Definition 4.7. Let $(E_\iota; \leq_\iota, +_\iota, 0_\iota, 1_\iota)$, $\iota \in I$, be weak effect algebras. Let $E = \Pi_\iota E_\iota$ be the cardinal product of the E_ι, $\iota \in I$, endowed with the pointwise order $\leq$, with the partial addition $+$ defined whenever performable in all components, and with the constants $0 = (0_\iota)_\iota$ and $1 = (1_\iota)_\iota$. Then the weak effect algebra $(E; \leq, +, 0, 1)$ is called the *direct product* of the E_ι, $\iota \in I$.

Moreover, a relative subalgebra F of E such that the projections $\pi_\iota\colon\ F \to E_\iota$ are surjective, is called a *subdirect product* of the E_ι.

Theorem 4.8. *Any BL-effect algebra is the subdirect product of linearly ordered BL-effect algebras.*

Proof. Let E be a BL-effect algebra, and let φ be the canonical mapping from E to the direct product of all quotient algebras arising from prime ideals. The latter are linearly ordered by Lemma 4.5(i). Then φ is by Theorem 4.3 and Proposition 3.5 a homomorphism, which, by Lemma 4.5(ii), is furthermore injective. It follows that E is a weak subalgebra of the direct product of the linearly ordered quotients induced by the prime ideals.

It remains to show that E even a relative subalgebra. So assume that, for $a, b, c \in E$, $[a]_I + [b]_I = [c]_I$ holds for all prime ideals I; we have to show that then, in E, $a + b$ exists and equals c. Now, for an ideal I, $[a]_I + [b]_I = [c]_I$ means $a' + b' \sim_I c$ for some $a' \sim_I a$ and $b' \sim_I b$. By (C4), we may even assume equality here, that is, $a' + b' = c$. Thus there are $r, s \in I$ such that $a' = (a - r) + s$, whence we may replace a' by an element below a. So, for any prime ideal I, there is an $a_I \leq a$ such that $a_I \sim_I a$ and $c = a_I + b_I$ for some b_I.

Let now $\bar{a} \leq a$ be the maximal element such that $\bar{a} + \bar{b} = c$ for some $\bar{b} \in E$. Then $a_I \leq \bar{a} \leq a$ and hence, by (C3), $\bar{a} \sim_I a$ for all I. It follows $\bar{a} = a$ and $\bar{b} = b$. □

5. Linearly ordered BL-effect algebras

Our next question is if linearly ordered BL-effect algebras are constructed in a similar way from simpler algebras, as this is the case for BL-algebras. The answer is positive.

The constituents are, as to be expected, of a more general nature than in the case of BL-algebras. However, they may be described neatly in a uniform way: as generalised effect algebras. The latter were introduced in [HePu]; they resemble effect algebras, but they are not assumed to have a largest element.

Definition 5.1. A *generalised effect algebra* is a structure $(E; \leq, +, 0)$ with the following properties:

(GE1) $(E; \leq, 0)$ is a poset with a smallest element 0.

(GE2) $+$ is a partial binary operation such that for any $a, b, c \in E$

- (a) $(a+b)+c$ is defined iff $a+(b+c)$ is defined, and in this case $(a+b)+c = a + (b + c)$;
- (b) $a + 0$ is always defined and equals a;
- (c) $a + b$ is defined iff $b + a$ is defined, and in this case $a + b = b + a$.

(GE3) If, for $a, b, c \in E$, $a + c$ and $b + c$ are defined, then $a + c = b + c$ implies $a = b$.

(GE4) For any $a, b \in E$, $a \leq b$ if and only if $a + c = b$ for some $c \in E$.

A generalised effect algebra with a largest element is called an *effect algebra*.

So effect algebras are exactly the weak effect algebras in which (GE4) holds; note that this is in accordance with the remark following Definition 2.1. Furthermore, a generalised effect algebra without largest element may be enlarged by a new such element, and becomes then also a weak effect algebra fulfilling (GE4). We conclude that the essential new condition in Definition 5.1 is the axiom (GE4), saying that the partial order is determined by the addition.

We recall the following fact about the structure of generalised effect algebras; see [DvGr] or [DvVe].

Definition 5.2. Let $(E; \leq, +, 0)$ be a generalised effect algebras and $(G; \leq, +, 0)$ a po-group. A mapping $\varphi\colon\ E \to G^+$ is called a *homomorphism* if, for all $a, b \in E$, the following holds: (i) if $a \leq b$, then $\varphi(a) \leq \varphi(b)$; (ii) if $a + b$ is defined, then $\varphi(a + b) = \varphi(a) + \varphi(b)$; (iii) $\varphi(0) = 0$. In case that, moreover, $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$, we call φ an *isomorphic embedding* of E into G.

Theorem 5.3. *Let $(E; \leq, +, 0)$ be a generalised effect algebra; let $(E; \leq)$ be a lower semilattice; and assume that for any a, b, c such that $c \leq a + b$, we have $c = a_1 + b_1$*

for some $a_1 \leq a$ *and* $b_1 \leq b$. *Then* E *may be isomorphically embedded into an abelian* ℓ*-group* $(G; \leq, +, 0)$*, such that the image of* E *is a convex subset of* G^+ *containing* 0.

In particular, any linearly ordered generalised effect algebra may be isomorphically embedded into a linearly ordered abelian group $(G; \leq, +, 0)$*, such that the image is a convex subset of* G^+ *containing* 0.

We next describe how BL-effect algebras may be constructed from a collection of generalised effect algebras. This construction is completely analogous to the ordinal sum construction linearly ordered BL-algebra are based on; cf. e.g. [AgMo].

Definition 5.4. Let $(I; \leq)$ be a linear order with a largest element κ; for any $\iota \in I$, let E_ι be a generalised effect algebra, and let E_κ be an effect algebra. Let $E'_\iota = E_\iota \setminus \{0\}$ for any ι, and let $E = \dot{\bigcup}_\iota E'_\iota \cup \{\bar{0}\}$, where $\bar{0}$ is a new element. For $a, b \in E$, set $a \leq b$ if either $a, b \in E'_\iota$ for some ι and $a \leq b$ holds in E_ι, or $a \in E'_{\iota_1}$ and $b \in E'_{\iota_2}$ such that $\iota_1 < \iota_2$, or $a = \bar{0}$. Let $\bar{1}$ be the one element of E_κ.

We finally define the partial binary operation $+$ as follows; let $a, b \in E$. If $a, b \in E'_\iota$ for some ι such that $a + b$ exists, let $a + b$ be defined as in E_ι; let $a + \bar{0} = \bar{0} + a = a$ for any $a \in E$; and let $a + b$ in other cases undefined. Then the structure $(E; \leq, +, \bar{0}, \bar{1})$ is called the *ordinal sum* of the algebras E_ι, $\iota \in I$.

Lemma 5.5. *An ordinal sum of generalised effect algebras is a weak effect algebra.*

An ordinal sum of linearly ordered generalised effect algebras is a BL-effect algebra, which is also linear ordered.

Proof. This is a simple check of the axioms. □

Now we are ready to formulate the main result of this section.

Theorem 5.6. *Every linearly ordered BL-effect algebra is the ordinal sum of linearly ordered generalised effect algebras.*

Proof. Let E be a linearly ordered BL-effect algebra. For $a, b \in E$ such that $0 < a \leq b$, set $z(a, b)$ in case that there is some x such that $a + x = b$. We will show that (i) from $0 < a \leq b \leq c$ and $z(a, b)$ as well as $z(b, c)$, it follows $z(a, c)$; and (ii) from $0 < a \leq b \leq c$ and $z(a, c)$, it follows $z(a, b)$ and $z(b, c)$. It will then follows that $E \setminus \{0\}$ consists of disjunct convex subsets which are, together the zero element, generalised effect algebras and such that addition is undefined between elements from different such subsets. So taking into account that the generalised effect algebra containing the 1 element will necessarily be an effect algebra, the assertion will be proved.

(i) holds trivially.

To see (ii), assume $0 < a \leq b \leq c$ and $z(a, c)$, say $c = a + x$. Then by (E6), $b = a + y$ for some $y \leq x$, whence $z(a, b)$. Now, we have either $x \leq b$; in this case, $y \leq x \leq a + y$, whence $x = y + z$ for some z, and so $c = b + z$, that is, $z(b, c)$.

Or we have $b \leq x$. Then let $a' \leq b$ be the largest element such that $a' + x' = c$ for some x'; we have $a' \geq a > 0$. If then $x' \leq b$, we proceed as above to conclude $z(b, c)$. Else we have $a' \leq b \leq x' \leq c = a' + x'$. So $x' = a' + z$ for some z, hence $c = a' + a' + z$; but $a' + a' > a'$, and $a' + a'$ sums up with z to c, so $a' \leq b < a' + a'$. This in turn means $b = a' + t$ for some $t \leq a' \leq a' + t$, so $a' = t + u$ for some u, and $a' + a' = a' + t + u = b + u$, so $c = b + u + z$, that is, $z(b, c)$. □

As a corollary, we may describe the exact relation between linearly ordered BL-algebras and linearly ordered BL-effect algebras. Call a generalised effect algebra $(E; \leq, +, 0)$ *standard* if it is isomorphic to a group interval, that is, if it is either of the form $(G[0,u]; \leq, +, 0)$, where $+$ is defined whenever the sum is below u, or of the form $(G^+; \leq, +, 0)$, where $+$ is always defined.

Theorem 5.7. *A linearly ordered BL-effect algebra E corresponds to a dual BL-algebra (in the way described in Theorem 2.5) if and only if E is the ordinal sum of standard generalised effect algebras.*

Proof. By Theorem 2.5, a BL-effect algebra E corresponds to a (dual) BL-algebra iff the partial addition on E is extendible according to (1). Let E be a linearly ordered. By Theorem 5.6, E is then the ordinal sum of linearly ordered generalised effect algebras E_ι, $\iota \in I$. So E obviously corresponds to a BL-algebra iff, for each $\iota \in I$, the partial addition on E_ι is extendible to a total addition.

If E_ι is standard, then clearly, the addition is extendible. Assume that E_ι is not standard. We may identify E_ι with a subset of the positive cone of an abelian partially ordered group G. In G, E_ι is upper bounded, but has no maximal element; so for all pairs $a, b \in E_\iota$ such that $a + b > c$ for all $c \in E_\iota$, the maximum (1) does not exist. So in this case, the addition is not extendible. □

6. Outlook: Representation of residuated lattices by means of po-groups

In the concluding section, we address the general problem how to characterise residuated lattices in general. We assume that the latter are bounded and commutative; recall that we could also talk about integral, commutative residuated ℓ-monoids. For general information on these algebras, see e.g. [JiTs] and the references given there.

Dual BL-algebras are special residuated lattices, and we will describe the meaning of the results of the present paper for the analysis of the more general algebras. We will keep the exposition short; missing details are easily added.

Definition 6.1. Let $(L; \leq, \oplus, \ominus, 0, 1)$ be a residuated lattice, that is, let $(L; \leq, 0, 1)$ be a bounded lattice, let $(L; \oplus, 0)$ be a commutative monoid, and for any $a, b, c, \in L$, assume $a \leq b \oplus c$ iff $a \ominus b \leq c$. Derive the partial addition $+$ from $\oplus$ as described in Definition 2.2. We then call $(L; \leq, +, 0, 1)$ the *partial algebra associated to L*. If $(L; \leq, +, 0)$ is isomorphically embeddable into a partially ordered group $(G; \leq, +, 0)$, we call L *po-group representable*.

Theorem 6.2. *Any dual BL-algebra is po-group representable.*

Proof. Let $(L; \leq, +, 0)$ be the partial algebra associated to a dual BL-algebra. Then L is a BL-effect algebra, which is, by Theorem 4.8, a subdirect product of linearly ordered ones.

So we may restrict to the linear case. If L is linearly ordered, then, by Theorem 5.7, L is the ordinal sum of generalised effect algebras E_ι, $\iota \in I$, where I is endowed with a linear order. Each E_ι is in turn isomorphically embeddable into the positive cone of a linearly ordered group G_ι. Let G be the lexicographical product of G_ι, $\iota \in I$, based on the linear order of I. Identifying each G_ι with a subgroup of G in the natural way, we see that $(L; \leq, +, 0)$ is isomorphically embeddable into $(G; \leq, +, 0)$. □

Our question is: Under which conditions are residuated lattices po-group representable? In particular: Are residuated lattices po-group representable in general? The answer is in general negative; however, the class of po-group representable ones comprises numerous examples of MTL-algebras which are used in application as t-norms and which are not BL-algebras; cf. [Vet3].

Note that Theorem 6.2 might also be derivable from the representation theorem of dual BL-algebras [AgMo]. However, for more general residuated lattices like MTL-algebras, such theorems are not available; but we may still try to proceed like in the present paper, that is, associating with the residuated lattice a partial algebra and embedding it into a po-group. When knowing that a residuated lattice is po-group representable, the fundamental question what its structure is like, will certainly not yet be solved. But at least, it seems fair to say that we know more about the structure of po-groups than about residuated lattices.

Let us finally mention on-going work on ordered monoids, in particular the paper [EKMMW]. Here, the problem is raised when totally ordered monoids are homomorphic images of po-group cones of a particularly easy form. We could say that this approach is in a sense "opposite" to ours, although it is not excluded that both approaches can be brought onto a common line.

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