The construction of left-continuous t-norms: a geometric approach in two dimensions

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Abstract

Each t-norm can be identified with its Cayley tomonoid, which consists of pairwise commuting order-preserving functions from the real unit interval to itself. Cayley tomonoids provide an easily manageable, yet versatile tool for the construction of t-norms. To give evidence to this claim, we review and reformulate several construction methods that are known in the literature. We adopt, on the one hand, a geometric point of view. Manipulations with t-norms have often been inspired by their three-dimensional graphs; by means of Cayley tomonoids, the process gains the character of putting together the pieces of a jigsaw puzzle. We consider, on the other hand, construction methods in their algebraic context. We show that those constructions that correspond to certain tomonoid extensions can be described on the basis of Cayley tomonoids in a particularly transparent way.

1 Introduction

The significance of left-continuous triangular norms, or l.-c. t-norms for short, lies in the fact that these operations are the natural choice to interpret the conjunction in fuzzy logic [Haj1]. Indeed, fuzzy logic deals with graded properties, and the canonical set of truth degrees is the real unit interval. Furthermore, a conjunction is commonly supposed to be associative, commutative, neutral w.r.t. a true statement, and in both arguments order-preserving. Accordingly, the conjunction in fuzzy logic is typically interpreted by a t-norm. In addition, the implication connective is interpreted by the residuum corresponding to the conjunction. To ensure the existence of a residuum, the t-norm must be left-continuous.

The family of left-continuous t-norms is complex. A comprehensive survey of the large diversity of different t-norms is the monograph [KMP]; overview articles are, e.g.,

[Mes1] and [Fod2]. For the lack of a systematic approach, research on t-norms has over many years not followed specific principles, but produced rather heterogeneous results. Increasingly complicated t-norms were found, and various methods to construct new t-norms from given ones were established. Results were often inspired by geometrical considerations; in fact, a t-norm can be identified with its graph, which is an object in three-dimensional space.

In this paper, it is our first purpose to advertise a representation of t-norms that is an eligible alternative to the common way of illustrating this type of binary operation. As a t-norm is associative and has the neutral element 1, it makes the real unit interval into a monoid. In semigroup theory, it is a well-known fact that any monoid can be represented by a monoid under composition of mappings; see, e.g., [ClPr]. This is Cayley's representation theorem, which was originally formulated for groups but can be generalised to monoids without problems. The idea is simple. With each element we associate the mapping that acts on the monoid by multiplication from the left (or from the right); such a mapping is called a translation. Moreover, the monoidal operation is the function composition and the monoidal identity is the identity mapping.

In the context of t-norms, the concept of a translation is actually not unknown; a translation is nothing else but a "vertical cut", because it arises geometrically from cutting the three-dimensional graph along a vertical plane. What just might be uncommon is the idea to view vertical cuts as elements of a monoid.

For a t-norm, the monoid of translations is called its Cayley tomonoid. A Cayley tomonoid consists of order-preserving mappings from the real unit interval to itself, the largest of which is the identity and smallest of which is the constant 0 mapping. It comes thus in a triangular shape. To get a first impression, we show in Figure 1 the Cayley tomonoids of the standard continuous t-norms.

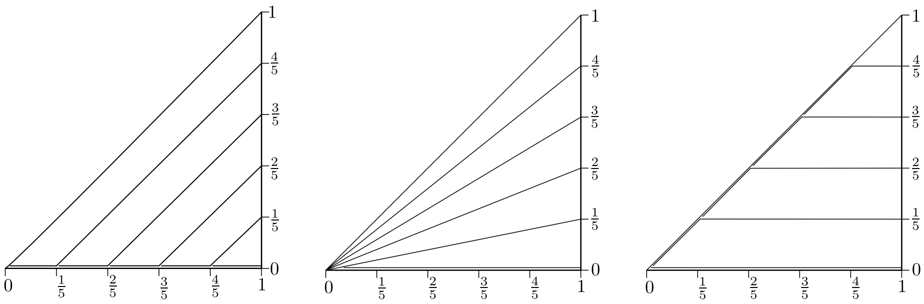

Figure 1: The Cayley tomonoids of the Łukasiewicz, product, and Gödel t-norm, respectively.

To depict a Cayley tomonoid we get by with two dimensions. In fact, it arises from the three-dimensional graph of the t-norm by a vertical projection. In this respect, there is another approach complementary to ours. Rather than looking at the graph "from the side" and working with translations, Maes and De Baets have proposed to look at the graph "from above" and to work with the so-called contour lines [MaBa]. The contour lines arise from the residual implication of a t-norm, rather than the t-norm itself, by fixing one argument.

When reducing the dimension of representation to two, one might wonder how the crucial property of t-norms, their associativity, is accounted for. In fact, understanding associativity as a symmetry property requires to rise the dimension and to work in four-dimensional space [Jen3]. In the present context, the property accounting for both the associativity and the commutativity of the represented operation is the commutativity of the translations: to apply a translation to the result of another one, or to proceed the other way round, does not make a difference.

The second purpose of this paper involves a look back to the development of the theory of t-norms. Our intention is to present a number of known construction methods for t-norms in a simple and unified way. Namely, we employ Cayley tomonoids in order to point out the essential features from a geometric viewpoint. But rather than working in three-dimensional space, we work in a triangular plane; constructing t-norm is then, intuitively, like assembling a jigsaw puzzle from triangular and rectangular pieces. We treat in this way ordinal sums, the rotation [Jen1], the rotation-annihilation [Jen2], the triple rotation [MaBa], and the H-transform [Mes2].

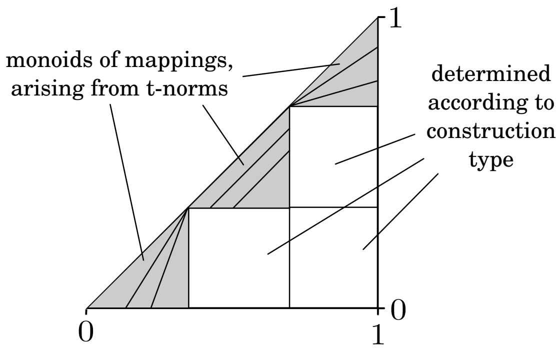

Figure 2: Idea underlying the constructions discussed in this paper. The basic constituents are monoids under composition of mappings, that is, Cayley tomonoids of t-norms or their modifications (highlighted in grey). These sets of mappings are placed into the new Cayley tomonoid as shown. Moreover, the mappings can be joined or separately extended; accordingly, the remaining rectangular sections are determined.

In each case, we follow the same "recipe"; see Figure 2. We first place sets of mappings, triangular in shape, along the identity line. Inside the upper-most triangle, there is a Cayley tomonoid, possibly restricted to a left-open interval. Inside each of the remaining triangles, there is either a Cayley tomonoid as well or its modification, like for instance a reflected Cayley tomonoid. The new translations are in this way specified in parts. We have to decide if the parts within different triangular sections belong to different translations of the new t-norm or if they are to be joined. Based on this decision, there may be exactly one possibility to fill the rectangular sections and to complete the creation of the new Cayley tomonoid; and if not, there still may be a canonical way to do so.

The third and last purpose of this paper is a repeated review of common t-norm construction methods. This time, however, we take up the considerable progress of recent times in the theory of residuated lattices [GJKO]. For a formulation of t-norm constructions in an algebraic style, see, e.g., [NEG2]. Here, our concern is to draw the bridge from the afore-mentioned representation by Cayley tomonoids to the algebraic context.

On the basis of the insights that we can gain by means of common algebraic techniques, the structure of left-continuous t-norms should actually no longer be viewed as mysterious. The t-norm monoids in question are residuated and can thus be viewed as MTL-algebras. It is well-known that the quotients of MTL-algebras correspond to their filters [BlTs, NEG1]. Tomonoids possess more congruences; nonetheless the collection of those quotients that are induced by filters provides a suitable framework to explore their structure. As we deal with a total order, this collection is naturally endowed with a total order as well.

We are thus suggested to understand a t-norm monoid as the final element of a chain of increasingly complex tomonoids. The sole knowledge about the chain of quotients is certainly not yet very satisfactory. But facts from pure algebra can sometimes be illuminated on the basis of a representation that makes their meaning intuitively accessible. Needless to say which tool we consider as well suited to get an idea of how the formation of more and more complex tomonoids proceeds. In fact, given the Cayley tomonoid of any negative, commutative tomonoid, it is straightforward to detect all its quotients induced by filters. But the crucial point is the converse direction. The possibility to describe a tomonoid via its Cayley tomonoid turns out to be particularly valuable when we want to determine those tomonoids whose quotient by a filter is a given one. We speak about extensions then. In specific cases, it is possible to describe extensions in a systematic way, and this is conveniently done by specifying the Cayley tomonoid of the new tomonoid in a piecewise manner.

We have developed this approach in our previous paper [Vet3]. For the purpose of the present paper, it is enough to recall the indicated framework and to compile the essential facts. On this basis, we review several t-norm construction methods that are accessible by algebraic methods and identify them as certain extensions.

The paper is structured as follows. We define in Section 2 Cayley tomonoids of t-norms and we introduce into the "two-dimensional" style of t-norm representation. In the intermediate Section 3, we introduce the reflection of Cayley tomonoids. We proceed in Section 4 by establishing basic ways of composing several, possibly modified, Cayley tomonoids to construct new, more complex t-norms. In Section 5, we turn to algebra: we review the algebraic background of our considerations and explain a main challenge of t-norm theory – the extension of tomonoids. In Section 6, we see that several construction methods are tomonoid extensions of a particularly simple type. We summarise the main features of our approach once again in Section 7.

2 Triangular norms and Cayley tomonoids

We are concerned in this paper with binary operations on the real unit interval specified as follows.

Definition 2.1. Let $[0,1]$ be the real unit interval endowed with its natural order. An operation $\odot\colon [0,1]^2 \to [0,1]$ is called a *triangular norm*, or *t-norm* for short, if, for any $a,b,c \in [0,1]$, (i) $(a \odot b) \odot c = a \odot (b \odot c)$, (ii) $a \odot b = b \odot a$, (iii) $a \odot 1 = a$, and (iv) $a \leq b$ implies $a \odot c \leq b \odot c$. The structure $([0,1]; \leq, \odot, 1)$ is then called the *t-norm monoid* based on $\odot$. If, in addition, $\lim_{x \nearrow b} a \odot x = a \odot b$ for each $a \in [0,1]$ and $b \in (0,1]$, we call $\odot$ *left-continuous*, abbreviated *l.-c.*

The defining properties of t-norms are chosen to fit to the idea of a conjunction in many-valued logic. Only the requirement of left continuity might look somewhat arbitrary. Its significance lies in the fact that it ensures the existence of a residual implication.

We will occasionally use the implication $\to$ associated with a l.-c. t-norm $\odot$; we have $a \to b = \max\,\{c \in [0,1]\colon c \odot a \leq b\},\ a,b \in [0,1]$. We will also sometimes need the residual negation $\neg$, defined by $\neg a = a \to 0,\ a \in [0,1]$.

T-norm monoids belong to the following class of structures, whose definition collects precisely the properties of t-norms.

Definition 2.2. An algebra $(\mathcal{L}; \odot, 1)$ is a *monoid* if, for all $a,b,c \in \mathcal{L}$, (i) $(a \odot b) \odot c = a \odot (b \odot c)$ and (ii) $a \odot 1 = a$. A total order $\leq$ on a monoid $\mathcal{L}$ is called *translation-invariant* if, for all $a,b,c \in \mathcal{L}$, $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$. A structure $(\mathcal{L}; \leq, \odot, 1)$ is a *totally ordered monoid*, or *tomonoid* for short, if $(\mathcal{L}; \odot, 1)$ is a monoid and $\leq$ is a translation-invariant total order.

Moreover, a tomonoid $\mathcal{L}$ is *commutative* if $a \odot b = b \odot a$ for all $a,b \in \mathcal{L}$. $\mathcal{L}$ is called *negative* if $a \leq 1$ for all $a \in \mathcal{L}$. $\mathcal{L}$ is called *quantic* if (i) the suprema of all non-empty subsets exist and (ii) for any elements a, b_ι, $\iota \in I$, of $\mathcal{L}$ we have

$$a \odot \bigvee_\iota b_\iota \;=\; \bigvee_\iota (a \odot b_\iota) \quad \text{and} \quad (\bigvee_\iota b_\iota) \odot a \;=\; \bigvee_\iota (b_\iota \odot a).$$

It is easily checked that a binary operation $\odot$ on $[0,1]$ is a t-norm if and only if $([0,1]; \leq, \odot, 1)$ is a negative, commutative tomonoid. Moreover, a t-norm is left-continuous if and only if this tomonoid is quantic.

The abbreviation “tomonoid” is taken from [EKMMW]. Moreover, we have chosen the notion “quantic” because of its close connection to the defining properties of quantales [Ros].

We will adopt an indirect viewpoint on t-norms. Rather than working with t-norm monoids directly, we will work with monoids of mappings.

By a *real interval*, we mean an interval of the form (a,b), $(a,b]$, $[a,b)$, or $[a,b]$ for some $a,b \in \mathbb{R}$ such that $a < b$. Let R be a real interval; then we always assume R to be endowed with its natural order. Furthermore, the notion “left-continuous” applies to mappings on real intervals in the common way. That is, $\lambda\colon R \to R$ is *left-continuous* if, for each $r \in R$ that is not the smallest element of R, we have $\lim_{x \nearrow r} \lambda(x) = \lambda(r)$.

Definition 2.3. Let $(R;\leq)$ be a real interval, and let Φ be a set of order-preserving mappings from R to R. We denote by $\leq$ the pointwise order on Φ, by $\circ$ the function composition, and by id_R the identity mapping on R. Assume that (i) $\leq$ is a total order on Φ, (ii) Φ is closed under $\circ$, and (iii) $id_R \in \Phi$. Then we call $(\Phi;\leq,\circ,id_R)$ a *composition tomonoid* on R.

It is straightforward to check that a composition tomonoid is in fact a tomonoid.

Let us introduce the following properties of a composition tomonoid $(\Phi;\leq,\circ,id_R)$ on an interval R:

(C1) $\circ$ is commutative.

(C2) id_R is the top element.

(C3) Every $\lambda \in \Phi$ is left-continuous.

(C4) Φ is closed under pointwise calculated suprema of non-empty subsets.

(C5) R has a top element 1, and for each $a \in R$ there is a unique $\lambda \in \Phi$ such that $\lambda(1) = a$.

The following proposition is an adapted version of Cayley's representation theorem for monoids [ClPr].

Proposition 2.4. *Let $\odot$ be a t-norm. For each $a \in [0,1]$, we define*

$$\lambda_a\colon [0,1] \to [0,1], \quad x \mapsto x \odot a, \tag{1}$$

and let $\Lambda = \{\lambda_a\colon a \in [0,1]\}$. Then $(\Lambda;\leq,\circ,id_{[0,1]})$ is a composition tomonoid on $[0,1]$ fulfilling the properties (C1)*,* (C2)*, and* (C5)*. Moreover,*

$$\pi\colon [0,1] \to \Lambda, \quad a \mapsto \lambda_a \tag{2}$$

is an isomorphism between $([0,1];\leq,\odot,1)$ and $(\Lambda;\leq,\circ,id_{[0,1]})$.

If $\odot$ is left-continuous, Λ fulfils also (C3) *and* (C4)*.*

Proof. The proof does not involve difficulties; cf. [Vet1, Vet3]. □

Definition 2.5. Let $\odot$ be a t-norm. For any $a \in [0,1]$, the mapping λ_a defined by (1) is called the *translation* by a. Moreover, the composition tomonoid $(\Lambda;\leq,\circ,id_{[0,1]})$ as specified in Proposition 2.4 is called the *Cayley tomonoid* of $\odot$.

Note that, to keep notation simple, we do not include the reference to the t-norm $\odot$ into the symbols λ_a, Λ. The t-norm to which we refer will always be clear from the context.

By Proposition 2.4, each t-norm $\odot$ can be identified with its Cayley tomonoid Λ. The latter consists of order-preserving functions, which are left-continuous if $\odot$ is left-continuous, and which are continuous if $\odot$ is continuous. Any two of them are comparable. That is, endowed with the pointwise order, Λ is totally ordered, in fact order isomorphic to the real unit interval. The bottom element of Λ is the constant 0 mapping, which we denote by $\bar{0}_{[0,1]}$. The top element of Λ is $id_{[0,1]}$. Λ is complete, but infima and suprema are in general not calculated pointwise. However, if $\odot$ is left-continuous, the pointwise calculated supremum of an arbitrary set of mappings in Λ is again in Λ and consequently its supremum. If $\odot$ is even continuous, the same applies to infima. Finally, the multiplication in $[0,1]$ by $\odot$ corresponds to the function composition in Λ. That is, for $a, b, c \in [0,1]$ such that $c = a \odot b$, we have $\lambda_c = \lambda_a \circ \lambda_b$.

The Cayley tomonoid is a set of functions from $[0,1]$ to $[0,1]$ that is unique for each t-norm. It is natural to ask if this set is characterised by the properties indicated in Proposition 2.4. This is indeed the case. There are even redundancies: (C5) implies (C2); moreover, (C1), (C3), and (C5) imply (C4).

Proposition 2.6. *Let Λ be a composition tomonoid on $[0,1]$ such that* (C1) *and* (C5) *hold. Then there is a unique t-norm $\odot$, defined by*

$$a \odot b \;=\; \lambda(b) \quad \textit{where } \lambda \in \Lambda \textit{ is such that } \lambda(1) = a,$$

such that $(\Lambda; \leq, \circ, id_{[0,1]})$ is the Cayley tomonoid of $\odot$.

If $\mathcal{L}$ fulfils also (C3)*, $\odot$ is left-continuous.*

We summarise that each (left-continuous, continuous) t-norm can be identified with a composition tomonoid on $[0,1]$ consisting of pairwise commuting (left-continuous, continuous) mappings from $[0,1]$ to $[0,1]$ such that for any $a \in [0,1]$ exactly one element of Λ maps 1 to a.

We provide a couple of examples. We have already mentioned the standard continuous t-norms; let us make up for their definition. In the sequel, $\wedge$ and $\vee$ will denote the minimum and maximum operation, respectively, if applied to real numbers. For $a, b \in [0,1]$, let

$$\begin{aligned} a \odot_1 b &= (a + b - 1) \vee 0, \\ a \odot_2 b &= a \cdot b, \\ a \odot_3 b &= a \wedge b; \end{aligned}$$

this is the **Łukasiewicz t-norm**, the **product t-norm**, and the **Gödel t-norm**, respectively. Their Cayley tomonoids are shown in Figure 1. The plots show in each case the translations by 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and 1. We note that the plots in this paper have only schematic character.

As our first example of a non-continuous t-norm, let us visualise the **drastic t-norm**:

$$a \odot_4 b \;=\; \begin{cases} a \wedge b & \text{if } a = 1 \text{ or } b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Cayley tomonoid of the t-norm $\odot_4$ consists, apart from $id_{[0,1]}$, of functions mapping the whole interval $[0,1)$ to 0; cf. Figure 3. Apparently, $\odot_4$ is not left-continuous.

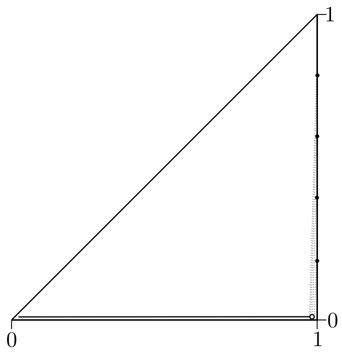

Figure 3: The drastic t-norm $\odot_4$

An example of a non-continuous, but left-continuous t-norm is the **nilpotent minimum t-norm**; cf. Figure 4 (left). This t-norm was defined in [Fod1], and a class of related t-norms is studied in [CEGM]. Let

$$a \odot_5 b \;=\; \begin{cases} a \wedge b & \text{if } a + b > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

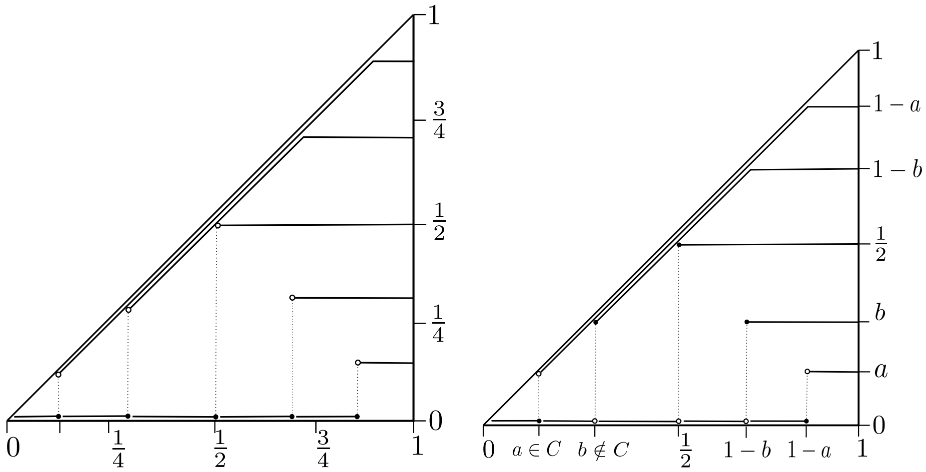

Figure 4: Left: The nilpotent minimum t-norm $\odot_5$. Right: The possibly non-measurable t-norm $\odot_6$.

The t-norm $\odot_5$ is left-continuous. For the last example of our introductory section, we modify $\odot_5$ and demonstrate that the Cayley tomonoid of a t-norm that is neither left- nor right-continuous and possibly not even Borel measurable can still be a quite intuitive representation. A non-measurable t-norm was first proposed in [Kle]; the following example is taken from [KMP, Ex. 3.75].

Let $C \subseteq (0, \frac{1}{2})$ and let

$$a \odot_6 b \;=\; \begin{cases} 0 & \text{if } a + b < 1, \quad \text{or } a + b = 1 \text{ and } a \wedge b \in C \\ a \wedge b & \text{otherwise.} \end{cases}$$

Then $\odot_6$ is Borel measurable if and only if C is Borel measurable. Moreover, to specify the Cayley tomonoid, we need to distinguish five different cases; cf. Figure 4 (right). To determine a translation λ_a, we have to distinguish whether (i) $a < \frac{1}{2}$ and $a \in C$, or (ii) $a < \frac{1}{2}$ and $a \notin C$, or (iii) $a = \frac{1}{2}$, (iv) $a > \frac{1}{2}$ and $1 - a \in C$, or (v) $a > \frac{1}{2}$ and $1 - a \notin C$.

For what follows, we underline that our analysis aims at a characterisation of t-norms up to isomorphism. Here, two t-norms $\odot$ and $\odot'$ are called isomorphic if the t-norm monoids $([0,1]; \leq, \odot, 1)$ and $([0,1]; \leq, \odot', 1)$ are isomorphic. This in turn means that there is an order automorphism $\iota \colon [0,1] \to [0,1]$ such that $a \odot' b = \iota(\iota^{-1}(a) \odot \iota^{-1}(b))$ for $a, b \in [0,1]$.

3 Reflection of Cayley tomonoids

We have seen that each t-norm corresponds to its Cayley tomonoid, which is a composition tomonoid acting on the real unit interval $[0,1]$. In its graphical representation, a Cayley tomonoid has a triangular shape, containing the identity line as its top element and the zero line as its bottom element.

As a preparation for what follows, we discuss in this subsection a specific way in which we can modify the Cayley tomonoid of a t-norm: its reflection along the line $x \mapsto 1-x$.

We will denote the standard negation on $[0,1]$ by $\sim$, that is, we define $\sim \colon [0,1] \to [0,1], \;\; x \mapsto 1 - x$.

Definition 3.1. Let $\lambda \colon [0,1] \to [0,1]$ be order-preserving, left-continuous, and below $id_{[0,1]}$. Then we call

$$\lambda^\star \colon [0,1] \to [0,1], \quad a \mapsto \min \{x \colon \lambda(\sim x) \;\leq\; \sim a\}$$

the *reflection* of λ. Moreover, for a set Λ of such functions, we call $\Lambda^\star = \{\lambda^\star \colon \lambda \in \Lambda\}$ the reflection of Λ.

We note that Definition 3.1 is conceptually related to the notion of a pseudoinverse; cf. [KMP, Sec. 3.1].

We list some properties of the reflection operation.

Lemma 3.2. *Let $\kappa, \lambda \colon [0,1] \to [0,1]$ be order-preserving, left-continuous, and below $id_{[0,1]}$. Then we have:*

(i) *For any $a, b \in [0,1]$,*

$$\lambda^\star(a) \leq b \quad \textit{if and only if} \quad \lambda(\sim b) \leq \sim a. \tag{4}$$

(ii) *Also* $\lambda^\star$ *is order-preserving, left-continuous, and below* $\mathbf{id}_{[0,1]}$.

(iii) $(\lambda^\star)^\star = \lambda$.

(iv) $\kappa^\star \circ \lambda^\star = (\lambda \circ \kappa)^\star$.

Furthermore, let $\lambda_\iota \colon [0,1] \to [0,1]$, $\iota \in I$, *be order-preserving, left-continuous, below* $\mathbf{id}_{[0,1]}$, *and pairwise comparable. Then we have:*

(v) $(\bigvee_\iota \lambda_\iota)^\star = \bigvee_\iota \lambda_\iota^\star$, *where the suprema are calculated pointwise.*

Proof. (i) is obvious.

(ii) It is clear from the definition of λ that $\lambda^\star$ is order-preserving. From (4) we conclude that $\lambda^\star$ preserves arbitrary suprema; hence $\lambda^\star$ is left-continuous. (4) also implies $\lambda^\star(a) \leq a$ for any a, that is, $\lambda^\star$ is below $id_{[0,1]}$.

(iii)-(v) follows again from (4). □

To see that the reflection operation can indeed be interpreted as a reflection, let us identify left-continuous functions with their graphs in which "jumps" are filled by vertical lines; cf. Figure 5.

Definition 3.3. Let $\lambda \colon [0,1] \to [0,1]$ be order-preserving, left-continuous, and below $id_{[0,1]}$. Then we call

$$G(\lambda) \;=\; \{\langle a,b\rangle \colon \lambda(a) \leq b, \text{ and } b \leq \lambda(a') \text{ for each } a' > a\}$$

the *connected graph* of λ.

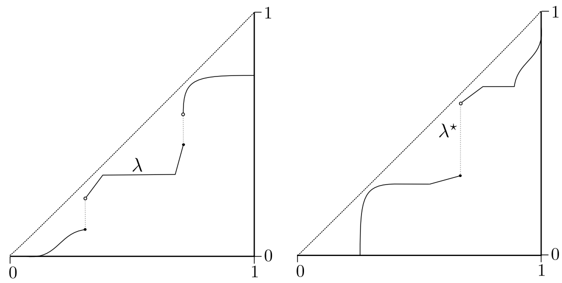

Figure 5: The reflection operation.

Lemma 3.4. *Let* $\lambda \colon [0,1] \to [0,1]$ *be order-preserving, left-continuous, and below* $\mathbf{id}_{[0,1]}$. *Then, for each* $a, b \in [0,1]$, $\langle a,b\rangle \in G(\lambda)$ *if and only if* $\langle \sim b, \sim a\rangle \in G(\lambda^\star)$.

Proof. Let $\langle a,b\rangle \in G(\lambda)$. Then $\lambda(a) \leq b$, and $x > a$ implies $b \leq \lambda(x)$. The former condition implies $\lambda^\star(\sim b) \leq \sim a$ by (4).

The latter condition means that $\lambda(x) < b$ implies $x \leq a$, that is, $\lambda(\sim x) < b$ implies $\sim a \leq x$. Hence for all $y < b$, $\lambda(\sim x) \leq y$ implies $\sim a \leq x$, and we conclude $\sim a \leq \min\, \{x\colon \lambda(\sim x) \leq y\} = \lambda^\star(\sim y)$ for all $y < b$. Thus $\sim a \leq \lambda^\star(y)$ for all $y > \sim b$. The proof is complete that $\langle \sim b, \sim a\rangle \in G(\lambda^\star)$.

The converse direction follows from Lemma 3.2(iii). $\square$

Let $\Lambda = \{\lambda_t \colon t \in [0,1]\}$ be the Cayley tomonoid associated with a l.-c. t-norm. The reflection of Λ consists then of the functions given by

$$\lambda_t^\star(a) \;=\; \sim(t \to \sim a) \tag{5}$$

for each $t, a \in [0,1]$. We next see that $\Lambda^\star = \{\lambda_t^\star \colon t \in [0,1]\}$ is a composition tomonoid, and even a Cayley tomonoid provided that the residual negation is involutive.

Theorem 3.5. *Let Λ be the Cayley tomonoid of a l.-c. t-norm $\odot$. Then $(\Lambda^\star; \leq, \circ, id_{[0,1]})$ is a composition tomonoid fulfilling* (C1)–(C4)*. If the residual negation $\neg$ of $\odot$ is involutive, $\Lambda^\star$ fulfils also* (C5) *and $\Lambda^\star$ is the Cayley tomonoid of a l.-c. t-norm as well.*

Moreover,

$$\Lambda \to \Lambda^\star, \quad \lambda \mapsto \lambda^\star$$

is an isomorphism between $(\Lambda; \leq, \circ, id_{[0,1]})$ and $(\Lambda^\star; \leq, \circ, id_{[0,1]})$.

Proof. By Lemma 3.2, $\Lambda^\star$ is a composition tomonoid fulfilling (C1)–(C4) and $\lambda \mapsto \lambda^\star$ is an isomorphism.

For any $a \in [0,1]$, we have $\lambda_a^\star(1) = \min\,\{x\colon \lambda_a(\sim x) = 0\} = \sim \max\,\{x\colon \lambda_a(x) = 0\} = \sim \neg a$. Let $\neg$ be involutive. Then $\sim \circ \neg$ is an order automorphism, and it follows that (C5) holds in this case as well. $\square$

Our last proposition is devoted to the case that the reflection operation is the identity.

Proposition 3.6. *Let Λ be a composition tomonoid on $[0,1]$ such that* (C2)*,* (C3)*, and* (C5) *hold and $\lambda^\star = \lambda$ for each $\lambda \in \Lambda$. Then* (C1) *holds as well, and consequently, Λ is the Cayley tomonoid of a l.-c. t-norm.*

Proof. The function composition is commutative by Lemma 3.2(iv). $\square$

4 Construction of t-norms out of Cayley tomonoids

In this section, we explore the possibilities of composing new Cayley tomonoids from given ones. Figure 2 illustrates what we have in mind; the general scheme can be described as follows. We partition the real unit interval into subintervals, and with each subinterval, we associate a composition tomonoid acting on it. Then either (A) independently for each subinterval, we extend the mappings on it to the whole unit interval; or (B) each new translation arises from selecting one mapping per subinterval and joining them. We will come along a number of construction methods that are well-known in the literature.

Composition tomonoids acting in series

Our first construction method follows option (A). After partitioning $[0, 1]$ and associating a composition tomonoid to each subinterval, we extend each mapping to the whole unit interval as follows: the values below the subinterval are mapped to themselves and the value above it are mapped to the same element as its right border.

The result is illustrated in Figure 6; this is the best-known construction method, the ordinal sum. For the ordinal sum of partially ordered semigroups; see [Fuc]. For the present context, see, e.g., [KMP, Thm. 3.43].

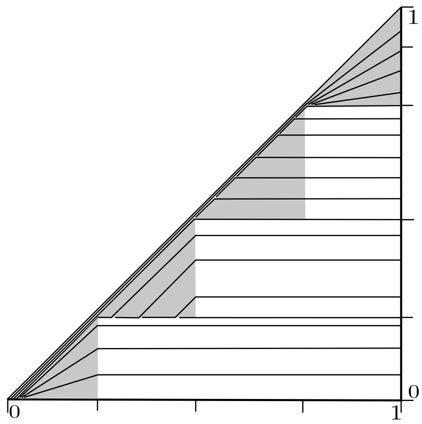

Figure 6: The ordinal sum construction. The figure shows the ordinal sum of a product, a Łukasiewicz, a Gödel, and again a product t-norm.

Here as well as in the sequel, we need to remap composition tomonoids from $[0, 1]$ to some other real interval $[a, b]$, $a < b$. We introduce the following abbreviation. For $a, b \in [0, 1]$ such that $a < b$, let $\tau\colon [a, b] \to [0, 1]$, $x \mapsto \frac{x-a}{b-a}$. For any function $\lambda\colon [0, 1] \to [0, 1]$, we then put $\lambda^{[a,b]} = \tau^{-1} \circ \lambda \circ \tau$. For any set Λ of functions from $[0, 1]$ to $[0, 1]$, we write $\Lambda^{[a,b]} = \{\lambda^{[a,b]} \colon \lambda \in \Lambda\}$. We use an analogous notation also for the case that the domains are left-open, right-closed intervals.

Theorem 4.1. *Let $[u_\iota, v_\iota)$, $\iota \in I$, be pairwise disjoint subintervals of $[0, 1]$, and for each ι, let Λ_ι be the Cayley tomonoid of a l.-c. t-norm. For $t \in [0, 1]$ such that $t \in [u_\iota, v_\iota)$ for some $\iota \in I$, let*

$$\lambda_t(x) \;=\; \begin{cases} x & \text{if } x < u_\iota, \\ \lambda^{[u_\iota,v_\iota]}(x) & \text{if } x \in [u_\iota, v_\iota], \text{ where } \lambda \in \Lambda_\iota \text{ is such that } t = \lambda^{[u_\iota,v_\iota]}(v_\iota); \\ t & \text{if } x > v_\iota; \end{cases} \tag{6}$$

and for $t \in [0, 1]$ such that $t \notin [u_\iota, v_\iota)$ for all $\iota \in I$, let

$$\lambda_t(x) \;=\; \begin{cases} x & \text{if } x \le t, \\ t & \text{if } x > t. \end{cases} \tag{7}$$

Then $\{\lambda_t \colon t \in [0,1]\}$ *is the Cayley tomonoid of a l.-c. t-norm as well.*

Proof. The properties (C1)–(C5) are immediate. □

Composition tomonoids acting on $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ in parallel

All following constructions take option (B): we let the translations of the new Cayley tomonoid act on subintervals in parallel. There will be only two subintervals, that is, we partition $[0,1]$ into a pair of subintervals whose common boundary will be $\frac{1}{2}$.

We then must decide if $\frac{1}{2}$ belongs to the lower or upper interval. We consider first the case that our partition consists of the subsets $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$.

For two composition tomonoids $(\Phi; \leq, \circ, id_R)$ and $(\Psi; \leq, \circ, id_S)$, a *homomorphism* $h \colon \Phi \to \Psi$ is a mapping such that, for any $\kappa, \lambda \in \Phi$, $\kappa \leq \lambda$ implies $h(\kappa) \leq h(\lambda)$, furthermore $h(\kappa \circ \lambda) = h(\kappa) \circ h(\lambda)$, and $h(id_R) = id_S$. A homomorphism is called *sup-preserving* if it preserves arbitrary non-empty suprema.

Proposition 4.2. *Let* Φ *be a composition tomonoid on* $(\frac{1}{2}, 1]$ *such that* (C1)–(C5) *hold; let* Ψ *be a composition tomonoid on* $[0, \frac{1}{2}]$ *such that* (C1)–(C4) *hold; and assume that there is a sup-preserving homomorphism* $h \colon \Phi \to \Psi$*. For* $t \in (\frac{1}{2}, 1]$*, let*

$$\lambda_t(x) \;=\; \begin{cases} h(\varphi)(x) & \text{if } x \leq \frac{1}{2}, \\ \varphi(x) & \text{if } x > \frac{1}{2}, \end{cases} \quad \text{where } \varphi \in \Phi \text{ is such that } \varphi(1) = t; \tag{8}$$

and for $t \in [0, \frac{1}{2}]$*, let*

$$\lambda_t(x) \;=\; \begin{cases} 0 & \textit{if } x \leq \frac{1}{2}, \\ h(\varphi)(t) & \textit{if } x > \frac{1}{2}, \end{cases} \quad \text{where } \varphi \in \Phi \text{ is such that } \varphi(1) = x. \tag{9}$$

Then $\{\lambda_t \colon t \in [0,1]\}$ *is the Cayley tomonoid of a l.-c. t-norm.*

Proof. We have to prove that $\Lambda = \{\lambda_t \colon t \in [0,1]\}$ is a composition tomonoid fulfilling (C1), (C3), and (C5).

We first show that Λ consists of order-preserving and left-continuous mappings. Let $t \in (\frac{1}{2}, 1]$. Let $\varphi_t \in \Phi$ such that $\varphi_t(1) = t$. By (8), $\lambda_t|_{[0,\frac{1}{2}]} = h(\varphi_t)$ and $\lambda_t|_{(\frac{1}{2},1]} = \varphi_t$; thus λ_t is order-preserving and left-continuous. Let now $t \in [0, \frac{1}{2}]$. By (9), $\lambda_t(x) = 0$ for $x \leq \frac{1}{2}$, and $\lambda_t(x) = h(\varphi_x)(t)$ for $x > \frac{1}{2}$. Let $x_\iota > \frac{1}{2}$, $\iota \in I$; then $\lambda_t(\bigvee_\iota x_\iota) = h(\varphi_{\bigvee_\iota x_\iota})(t) = h(\bigvee_\iota \varphi_{x_\iota})(t) = (\bigvee_\iota h(\varphi_{x_\iota}))(t) = \bigvee_\iota h(\varphi_{x_\iota})(t) = \bigvee_\iota \lambda_t(x_\iota)$. Here, we have used the fact that Φ fulfils (C5), h preserves arbitrary suprema, and Ψ fulfils (C4). We have proved that λ_t is order-preserving and left-continuous for all t.

It is easily checked that $\lambda_s(x) \leq \lambda_t(x)$ for any $s, t, x \in [0,1]$ such that $s \leq t$. Moreover, $\lambda_1 = id_{[0,1]}$. Thus Λ is totally ordered and $id_{[0,1]}$ is the top element.

For $t \in (\frac{1}{2}, 1]$, we have $\lambda_t(1) = \varphi_t(1) = t$; for $t \in [0, \frac{1}{2}]$, we have $\lambda_t(1) = h(\varphi_1)(t) = h(id_{(\frac{1}{2},1]})(t) = id_{[0,\frac{1}{2}]}(t) = t$ as well. We conclude that Λ fulfils condition (C5).

We next prove that Λ is closed under $\circ$ and that $\circ$ is commutative. It will then follow that Λ is a composition tomonoid fulfilling (C1), (C3), and (C5), hence a Cayley tomonoid.

Let $s, t \in [0,1]$. If $s, t \leq \frac{1}{2}$, we have $\lambda_s \circ \lambda_t = \lambda_t \circ \lambda_s = \bar{0}_{[0,1]} = \lambda_0$.

Let $s, t > \frac{1}{2}$. Let $u = \varphi_s(t)$. Then $\varphi_u(1) = u = (\varphi_s \circ \varphi_t)(1)$, hence $\varphi_u = \varphi_s \circ \varphi_t$ because Φ fulfils (C5). For $x \in (\frac{1}{2}, 1]$, we have $(\lambda_s \circ \lambda_t)(x) = (\varphi_s \circ \varphi_t)(x) = \varphi_u(x) = \lambda_u(x)$; and for $x \in [0, \frac{1}{2}]$, we have $(\lambda_s \circ \lambda_t)(x) = (h(\varphi_s) \circ h(\varphi_t))(x) = h(\varphi_s \circ \varphi_t)(x) = h(\varphi_u)(x) = \lambda_u(x)$. Thus $\lambda_s \circ \lambda_t = \lambda_u \in \Lambda$, and we similarly argue to see that $\lambda_t \circ \lambda_s = \lambda_u$ as well.

Finally, let $s \leq \frac{1}{2}$ and $t > \frac{1}{2}$. For $x \in (\frac{1}{2}, 1]$, we have $(\lambda_s \circ \lambda_t)(x) = \lambda_s(\varphi_t(x)) = \lambda_{\varphi_t(x)}(s) = (\lambda_t \circ \lambda_x)(s) = \lambda_t(\lambda_s(x)) = (\lambda_t \circ \lambda_s)(x)$. Moreover, let $u = \lambda_t(s)$; then $\lambda_u(x) = \lambda_x(u) = (\lambda_x \circ \lambda_t)(s) = \lambda_t(\lambda_x(s)) = (\lambda_t \circ \lambda_s)(x)$. Thus $\lambda_s \circ \lambda_t = \lambda_t \circ \lambda_s = \lambda_u \in \Lambda$. □

In order to apply Proposition 4.2 to the construction of t-norms, we are required to provide a composition tomonoid on $(\frac{1}{2}, 1]$ fulfilling (C1)–(C5). Recall that a t-norm $\odot$ is said to be *without zero divisors* if $a \odot b = 0$ implies $a = 0$ or $b = 0$. In this case, all translations distinct from $\bar{0}_{[0,1]}$ map $(0, 1]$ to itself. We write in this case

$$\Lambda_{\setminus 0} \;=\; \{\lambda|_{(0,1]} \colon \lambda \in \Lambda \setminus \{\bar{0}_{[0,1]}\}\}.$$

We moreover need a second function algebra and a homomorphism from $\Lambda_{\setminus 0}$ to it. Two possibilities are immediate: we use Λ again; or we use $\Lambda^\star$, the reflected Cayley tomonoid. The next two theorems make use of these possibilities.

Theorem 4.3. *Let Λ be the Cayley tomonoid of a left-continuous t-norm without zero divisors. Let $\Phi = \Lambda_{\setminus 0}^{(\frac{1}{2},1]}$; let $\Psi = \Lambda^{[0,\frac{1}{2}]}$; and let $h \colon \Phi \to \Psi$ such that, for each $\lambda \in \Lambda \setminus \{\bar{0}_{[0,1]}\}$, ${\lambda|_{(0,1]}}^{(\frac{1}{2},1]}$ is mapped to $\lambda^{[0,\frac{1}{2}]}$. Then there is a l.-c. t-norm whose translations are given by* (8) *and* (9).

Proof. Φ, Ψ, and $h \colon \Phi \to \Psi$ fulfil the conditions of Proposition 4.2. □

In the sequel, we will shorten the definitions of example t-norms using its commutativity. That is, whenever definitions are formally incomplete, they are completable by commutativity.

The following example of Theorem 4.3 is based on the product t-norm. The t-norm is shown in Figure 7 (left):

$$a \odot_7 b \;=\; \begin{cases} 2ab - a - b + 1 & \text{if } a, b > \frac{1}{2}, \\ a(2b-1) & \text{if } a \leq \frac{1}{2}, \text{ and } b > \frac{1}{2}, \\ 0 & \text{if } a, b \leq \frac{1}{2}. \end{cases}$$

$\odot_7$ differs from a t-norm proposed by Hájek [Haj2] in that it is composed from only two rather than countably infinitely many constituents; cf. Section 6 below.

The topic of the second theorem is S. Jenei's rotation of a t-norm without zero divisors [Jen1].

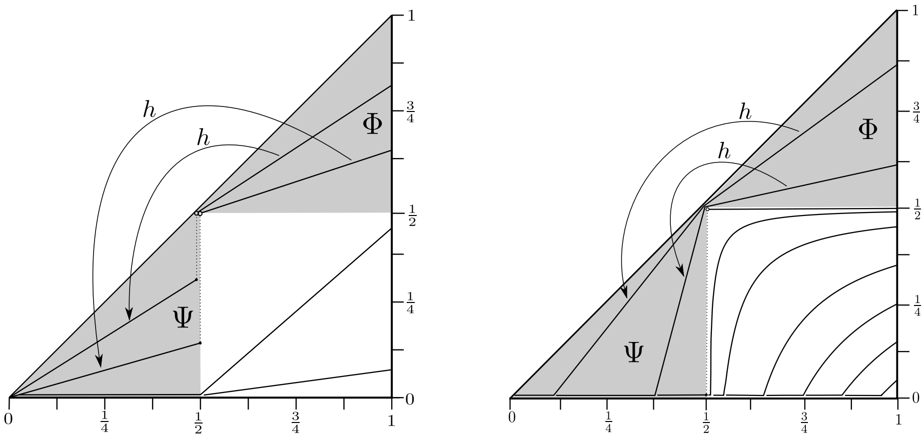

Figure 7: Two sets of functions (highlighted in grey) act on separate intervals in parallel. Left: $\odot_7$, a modified t-norm of Hájek. Right: The rotated product t-norm $\odot_8$.

For a unary or binary operation $\square$ on the real unit interval, we will write $\square^\sharp$ for $\square$ scaled to the interval $[\frac{1}{2}, 1]$. For instance, given $\odot\colon [0,1]^2 \to [0,1]$, we define $a \odot^\sharp b = \tau(\tau^{-1}(a) \odot \tau^{-1}(b))$ for $a, b \in [\frac{1}{2}, 1]$, where $\tau\colon [\frac{1}{2}, 1] \to [0,1], \quad x \mapsto 2x - 1$.

Theorem 4.4. *Let Λ be the Cayley tomonoid of a left-continuous t-norm without zero divisors. Let $\Phi = \Lambda_{\setminus 0}^{(\frac{1}{2},1]}$; let $\Psi = \Lambda^{\star[0,\frac{1}{2}]}$; and let $h\colon \Phi \to \Psi$ be such that, for each $\lambda \in \Lambda \setminus \{\bar{0}_{[0,1]}\}$, $\lambda|_{(0,1]}{}^{(\frac{1}{2},1]}$ is mapped to $\lambda^{\star[0,\frac{1}{2}]}$. Then there is a l.-c. t-norm $\odot_r$ whose translations are given by* (8) *and* (9)*.*

$\odot_r$ is moreover given according to the following formula:

$$a \odot_r b \;=\; \begin{cases} a \odot^\sharp b & \textit{if } a, b > \frac{1}{2}, \\ \sim(b \to^\sharp \sim a) & \textit{if } a \le \frac{1}{2} \textit{ and } b > \frac{1}{2}, \\ 0 & \textit{if } a, b \le \frac{1}{2}. \end{cases} \tag{10}$$

Proof. By assumption, Φ fulfils (C1)-(C5). By Theorem 3.5, Ψ is a composition tomonoid fulfilling (C1)–(C4) and h is a sup-preserving homomorphism. Thus Proposition 4.2 applies.

(10) is clear from (5). ☐

Applying Theorem 4.4 to the product t-norm, we get the Cayley tomonoid depicted in Figure 7 (right):

$$a \odot_8 b \;=\; \begin{cases} 2ab - a - b + 1 & \text{if } a, b > \frac{1}{2}, \\ \frac{a+b-1}{2b-1} & \text{if } a \le \frac{1}{2},\ b > \frac{1}{2}, \text{ and } a + b > 1, \\ 0 & \text{if } a + b \le 1. \end{cases}$$

Composition tomonoids acting on $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$ in parallel

We next consider the possibility to include the marginal point $\frac{1}{2}$ of our two intervals into the upper one; our partition will thus consist of $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$. It is immediate that we cannot proceed like in Proposition 4.2. The left-continuity implies that the translations are uniquely determined at the point $\frac{1}{2}$ by the lower component and thus might collide with their definition on the upper interval.

Under certain circumstances, a construction is still possible. The problem is to ensure left-continuity at the common boundary point $\frac{1}{2}$. If the upper component is the Cayley tomonoid of a t-norm with zero divisors, it contains functions from $[\frac{1}{2}, 1]$ to $[\frac{1}{2}, 1]$ mapping some $x > \frac{1}{2}$ to $\frac{1}{2}$. Let us check under which circumstances it is possible to define the translations of a new Cayley tomonoid in a way that the single element $\frac{1}{2}$ is not necessarily mapped to $\frac{1}{2}$, but possibly to smaller values.

Certain restrictions are immediate. Let $\kappa, \lambda \colon [0,1] \to [0,1]$ be two functions which are order-preserving, left-continuous, below $id_{[0,1]}$, and such that $\kappa < \lambda$. Assume further that there is a $\frac{1}{2} < c < 1$ such that $\kappa(x) = \frac{1}{2}$ for $x \in (\frac{1}{2}, c]$ and $\kappa(x) > \frac{1}{2}$ for $x \in (c, 1]$. Similarly, assume that there is a $\frac{1}{2} < c' < 1$ such that $\lambda(x) = \frac{1}{2}$ for $x \in (\frac{1}{2}, c']$ and $\lambda(x) > \frac{1}{2}$ for $x \in (c', 1]$. Then we observe:

(i) Assume that $c' < c$ and $\lambda(\frac{1}{2}) < \frac{1}{2}$. Then κ and λ do not commute.

(ii) Assume $\kappa(\frac{1}{2}) < \lambda(\frac{1}{2})$. Then κ and λ again do not commute.

Hence if κ and λ are elements of the Cayley tomonoid associated with a t-norm, we have $\kappa(\frac{1}{2}) = \lambda(\frac{1}{2})$, and either this value equals $\frac{1}{2}$ or $c = c'$. The following proposition concerns the latter possibility.

Proposition 4.5. *Let Φ be a composition tomonoid on $[\frac{1}{2}, 1]$ such that conditions* (C1)–(C5) *hold; let Ψ be a composition tomonoid on $[0, \frac{1}{2}]$ such that* (C1)–(C4) *hold; and assume that there is a sup-preserving homomorphism $h \colon \Phi \to \Psi$. Assume furthermore that there are $0 \le d < \frac{1}{2} < c < 1$ such that the following holds: for each $\varphi \in \Phi$, either $\varphi(x) > \frac{1}{2}$ for all $x \in (\frac{1}{2}, 1]$ and $h(\varphi)(\frac{1}{2}) = \frac{1}{2}$, or $\{x \in [\frac{1}{2}, 1] \colon \varphi(x) = \frac{1}{2}\} = [\frac{1}{2}, c]$ and $h(\varphi)(\frac{1}{2}) = d$, or $\varphi(x) = \frac{1}{2}$ for all $x \in [\frac{1}{2}, 1]$ and $h(\varphi)(\frac{1}{2}) = 0$. Then there is a l.-c. t-norm whose translations are given by* (8) *and* (9).

Proof. Note first that if, for any $\varphi \in \Phi$, either $\varphi(x) > \frac{1}{2}$ for all $x \in (\frac{1}{2}, 1]$ or φ is constant $\frac{1}{2}$, the proposition holds by Proposition 4.2. Assume now that there is at least one $\varphi \in \Phi$ such that $\varphi(x) = \frac{1}{2}$ for $x \le c$ and $\varphi(x) > \frac{1}{2}$ for $x > c$.

Let $\lambda_t, \in [0,1]$, be given according to (8) and (9). Like in the proof of Proposition 4.2, we conclude that every λ_t is order-preserving and left-continuous, $\lambda_t(1) = t$, and $\Lambda = \{\lambda_t \colon t \in [0,1]\}$ is totally ordered and has the top element $\lambda_1 = id_{[0,1]}$.

Note next that we have, for any $t > c$, $\lambda_t(d) = d$, $\lambda_t(\frac{1}{2}) = \frac{1}{2}$, $\lambda_t(x) > \frac{1}{2}$ for $x \in (\frac{1}{2}, c]$, and $\lambda_t(x) > c$ for $x \in (c, 1]$. Moreover, for $\frac{1}{2} < t \le c$ we have $\lambda_t(\frac{1}{2}) = d$; $\lambda_t(x) = \frac{1}{2}$ for $x \in (\frac{1}{2}, c]$; and $\lambda_t(x) > \frac{1}{2}$ for $x \in (c, 1]$. Finally, $\lambda_{\frac{1}{2}}(x) = 0$ for $x \in [0, \frac{1}{2}]$, $\lambda_{\frac{1}{2}}(x) = d$ for $x \in (\frac{1}{2}, c]$, and $\lambda_{\frac{1}{2}}(x) = \frac{1}{2}$ for $x \in (c, 1]$.

Given these facts, we show that Λ is closed under $\circ$ and $\circ$ is commutative. Let $s, t > \frac{1}{2}$. If then $s > c$ or $t > c$, we show similarly to the proof of Proposition 4.2 that $\lambda_s \circ \lambda_t = \lambda_t \circ \lambda_s = \lambda_{\varphi_s(t)} \in \Lambda$. Furthermore, if then $s, t \leq c$, we have $\lambda_s \circ \lambda_t = \lambda_t \circ \lambda_s = \lambda_{\frac{1}{2}} \in \Lambda$.

Let $s \leq \frac{1}{2}$ and $t > \frac{1}{2}$, or $s, t \leq \frac{1}{2}$. Then we proceed like in the proof of Proposition 4.2 to see that $\lambda_s \circ \lambda_t = \lambda_t \circ \lambda_s \in \Lambda$. □

We can apply Proposition 4.5 to t-norms in two ways. The following theorem covers the remaining case of S. Jenei's rotation construction [Jen1]. To this end, we choose for the upper component Φ the Cayley tomonoid of a t-norm and for the lower component Ψ its reflection. We put a peculiar restriction on the used t-norm; let Λ be its Cayley tomonoid. Then, for any non-zero $\lambda \in \Lambda$, either $\lambda(x) > 0$ for each $x > 0$, or $\lambda(x) = 0$ iff $x \in [0, c]$, where $c > 0$ is a fixed value. The residual negation has then the range $\{0, c, 1\}$.

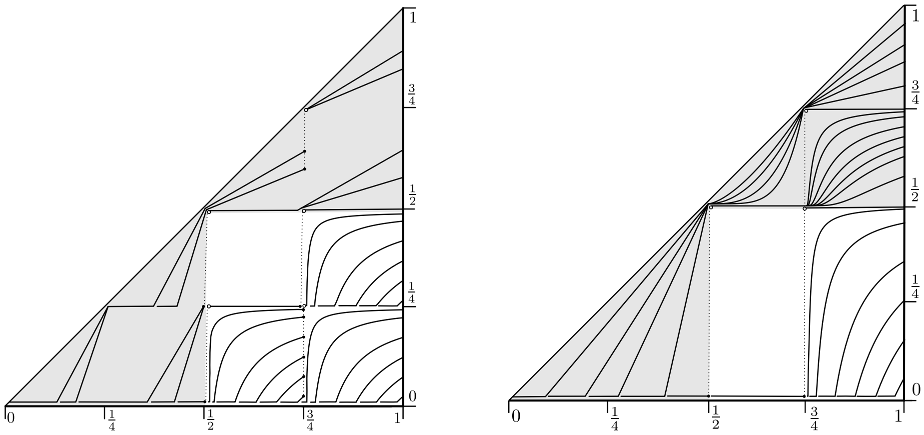

Figure 8: Composition of Cayley tomonoids on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ according to Proposition 4.5. Left: the t-norm $\odot_9$. Right: the t-norm $\odot_{10}$.

Theorem 4.6. *Let Λ be the Cayley tomonoid of a l.-c. t-norm such that the range of its residual negation $\neg$ is exactly three-element. Let $\Phi = \Lambda^{[\frac{1}{2},1]}$ and $\Psi = \Lambda^{\star[0,\frac{1}{2}]}$. Let $h\colon \Phi \to \Psi$ by such that, for each $\lambda \in \Lambda$, $\lambda^{[\frac{1}{2},1]}$ is mapped to $\lambda^{\star[0,\frac{1}{2}]}$. Then there is a l.-c. t-norm whose translations are given by* (8) *and* (9).

Proof. Proposition 4.5 is applicable. □

An example of Theorem 4.6 is the following t-norm, which is shown in Figure 8 (left):

$$a \odot_9 b \;=\; \begin{cases} 4ab - 3a - 3b + 3 & \text{if } a, b > \frac{3}{4}, \\ 4ab - 3a - 2b + 2 & \text{if } \frac{1}{2} < a \leq \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ \frac{a+2b-2}{4b-3} \vee \frac{1}{4} & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ \frac{a+b-1}{4b-3} \vee 0 & \text{if } a \leq \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ \frac{1}{2} & \text{if } \frac{1}{2} < a, b \leq \frac{3}{4}, \\ \frac{a+b-1}{4b-2} \vee 0 & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \\ 0 & \text{if } a \leq \frac{1}{4} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \\ 0 & \text{if } a, b \leq \frac{1}{2}. \end{cases}$$

Second, Proposition 4.5 includes another application to t-norms; consider the case $d = 0$. The lower component can in this case not be derived from the upper component in a straightforward way. We provide an example only; the following t-norm is shown in Figure 8 (right):

$$a \odot_{10} b \;=\; \begin{cases} 4ab - 3a - 3b + 3 & \text{if } a, b > \frac{3}{4}, \\ \frac{1}{4}(4a-2)^{\frac{1}{4b-3}} + \frac{1}{2} & \text{if } \frac{1}{2} < a \leq \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ \frac{a+2b-2}{4b-3} \vee 0 & \text{if } a \leq \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ \frac{1}{2} & \text{if } \frac{1}{2} < a, b \leq \frac{3}{4}, \\ 0 & \text{if } a \leq \frac{1}{2} \text{ and } b \leq \frac{3}{4}. \end{cases}$$

Composition tomonoids merged

The two preceding constructions were based on the following idea: we partitioned $[0, 1]$ into two subintervals, we chose a composition tomonoid on the lower subinterval and a further one on the upper subinterval, and we joined the mappings pairwise. It turned out the that latter composition tomonoid could not be chosen arbitrarily; it consisted either of mappings on $(\frac{1}{2}, 1]$, or it consisted of mappings on $[\frac{1}{2}, 1]$ subjected to the requirement that the set of points mapping to $\frac{1}{2}$ was of a quite specific form.

Next, we consider again composition tomonoids on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$, but our assumptions will be quite different. Namely, we consider Cayley tomonoids such that for each $c \in [0, 1]$ there is a unique $t \in [0, 1]$ such that $\{x \in [0, 1] \colon \lambda_t(x) = 0\} = [0, c]$. In other words, the residual negation of the t-norm will be involutive, and we will in fact assume that the residual negation equals the standard negation, that is $\neg a = \sim a$ for all $a \in [0, 1]$. It is immediate from (5) that then the reflection $\lambda^\star$ of any translation λ is λ itself. In particular, the reflection of the Cayley tomonoid Λ of a l.-c. t-norm is again Λ.

A natural idea is then to compose four copies of the Cayley tomonoid in the way indicated in Figure 9 (left). The result is a construction called *triple rotation* by K. Maes and B. De Baets [MaBa]. Namely, the upper triangle contains the Cayley tomonoid as usual. The lower triangle contains its reflection, that is, the Cayley tomonoid once again. The translations by values above $\frac{1}{2}$ are determined on $[0, \frac{1}{2}]$ by the latter; but they are determined by the former only for those $x \in (\frac{1}{2}, 1]$ that are mapped to values strictly greater than $\frac{1}{2}$. For the remaining values, an idea comes into play suggested by the name "triple" rotation: there is a third copy of the Cayley tomonoid, this time rotated by $180°$. Finally, the translations by values below $\frac{1}{2}$ represent the Cayley tomonoid another time.

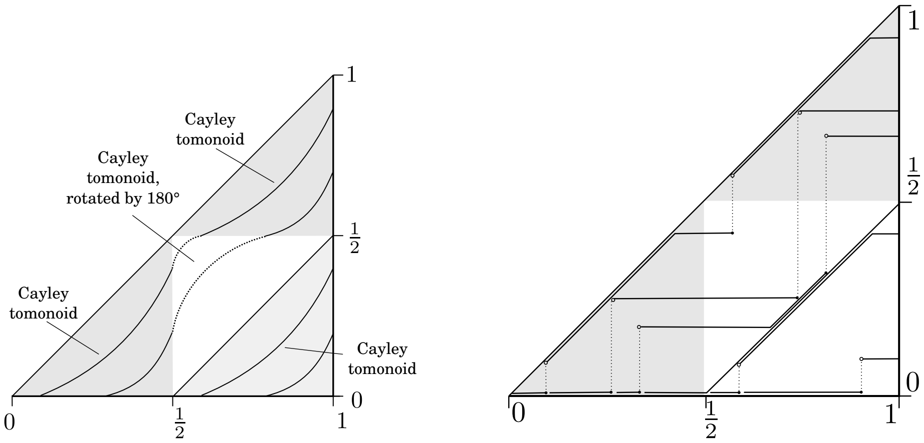

Figure 9: The triple rotation. Left: the idea. Right: $\odot_{11}$, the application to the nilpotent minimum.

If the result of this construction process is to be a left-continuous t-norm again, some technical problems need to be overcome. In particular, a translation rotated by $180°$ will be right- rather than left-continuous. The solution to this problem according to [MaBa] is to use right limits. The *companion* of a l.-c. t-norm $\odot$ is defined as follows:

$$a \tilde{\odot} b \;=\; \lim_{x \searrow b} a \odot x, \quad a, b \in [0,1).$$

For a l.-c. t-norm $\odot$ such that $\neg$ is the standard negation, we can now define:

$$a \odot_t b \;=\; \begin{cases} \sim (b \rightarrow^\sharp \sim a) & \text{if } a \leq \frac{1}{2} \text{ and } b > \frac{1}{2} \\ a \odot^\sharp b & \text{if } a, b > \frac{1}{2} \text{ and } a \odot^\sharp b > \frac{1}{2} \\ \sim (\neg^\sharp a \,\tilde{\odot}^\sharp\, \neg^\sharp b) & \text{if } a, b > \frac{1}{2} \text{ and } a \odot^\sharp b = \frac{1}{2} \\ 0 & \text{if } a, b \leq \frac{1}{2}. \end{cases} \tag{11}$$

The question then still remains: when is $\odot_t$ actually a left-continuous t-norm? In the following theorem we will provide sufficient conditions. For necessary and sufficient conditions, we refer to [MaBa].

Theorem 4.7. *Let Λ be the Cayley tomonoid of a t-norm $\odot$ such that $\neg$ is the standard negation. Then $\odot_t$ is a l.-c. t-norm if the following conditions are fulfilled for all $a, b, c \in [0,1)$:*

(TR1) $a \tilde{\odot} b \;=\; b \tilde{\odot} a$;

(TR2) $\neg c \rightarrow (a \tilde{\odot} b) \;=\; \neg b \rightarrow (a \tilde{\odot} c)$ *if* $\neg a \leq b, c$;

(TR3) $\neg c \to (a \mathbin{\tilde{\odot}} b) \; = \; (\neg c \to a) \mathbin{\tilde{\odot}} b \;\; \textit{if} \;\; c < \neg a \leq b.$

Proof. It is easily checked that 1 is neutral w.r.t. $\odot_t$ and that the mapping $x \mapsto x \odot_t a$ is for every a order-preserving. Furthermore, by (TR1), $\tilde{\odot}$ is commutative, and it follows that $\odot_t$ is commutative. Moreover, $\tilde{\odot}$ is, by construction, right-continuous in its right argument, and it follows that $\odot_t$ is left-continuous.

It remains to check the associativity of $\odot_t$. Let $a, b, c \in [0,1]$ such that $b < c$. We will prove $(a \odot_t b) \odot_t c = (a \odot_t c) \odot_t b$.

Case 1: $a \leq \frac{1}{2}$. Here, we can proceed like in case of Theorem 4.4.

Case 2: $a > \frac{1}{2}$ and $b, c \leq \frac{1}{2}$. Then $(a \odot_t b) \odot_t c = (a \odot_t c) \odot_t b = 0$.

Case 3: $a, c > \frac{1}{2}, \; b \leq \frac{1}{2}$, and $a \odot^\sharp c = \frac{1}{2}$. Then $a \odot_t c \leq \frac{1}{2}$, hence $(a \odot_t c) \odot_t b = 0$. Furthermore, $a \odot_t b = \sim (a \to^\sharp \sim b)$, hence $(a \odot_t b) \odot_t c = \sim (c \to^\sharp (a \to^\sharp \sim b)) = \sim ((a \odot^\sharp c) \to^\sharp \sim b)) = 0$ because $a \odot^\sharp c = \frac{1}{2} \leq \sim b$.

Case 4: $a, c > \frac{1}{2}, b \leq \frac{1}{2}$, and $a \odot^\sharp c > \frac{1}{2}$. Then $(a \odot_t b) \odot_t c = \sim (c \to^\sharp (a \to^\sharp \sim b)) = \sim ((a \odot^\sharp c) \to^\sharp \sim b)) = (a \odot_t c) \odot_t b$.

Case 5: $a, b, c > \frac{1}{2}$ and $a \odot^\sharp c = \frac{1}{2}$. Because $b < c$, we then have $a \odot^\sharp b = \frac{1}{2}$ as well, and we get $(a \odot_t b) \odot_t c = \sim (c \to^\sharp ((a \to^\sharp \frac{1}{2}) \mathbin{\tilde{\odot}^\sharp} (b \to^\sharp \frac{1}{2})))$. Moreover, $a \odot_t c = \sim ((a \to^\sharp \frac{1}{2}) \mathbin{\tilde{\odot}^\sharp} (c \to^\sharp \frac{1}{2})) \leq \frac{1}{2}$, hence we have $(a \odot_t c) \odot_t b = \sim (b \to^\sharp ((a \to^\sharp \frac{1}{2}) \mathbin{\tilde{\odot}^\sharp} (c \to^\sharp \frac{1}{2})))$. Thus it is our aim to prove that

$$c \to (\neg a \mathbin{\tilde{\odot}} \neg b) \; = \; b \to (\neg a \mathbin{\tilde{\odot}} \neg c)$$

holds for any $a, b, c > 0$ such that $a \odot b = a \odot c = 0$. But this follows from (TR2).

Case 6: $a, b, c > \frac{1}{2}$, $a \odot^\sharp b = \frac{1}{2}$, and $a \odot^\sharp c > \frac{1}{2}$. Then we have $(a \odot_t b) \odot_t c = \sim (c \to^\sharp ((a \to^\sharp \frac{1}{2}) \mathbin{\tilde{\odot}^\sharp} (b \to^\sharp \frac{1}{2})))$, $(a \odot_t c) \odot_t b = \sim (((a \odot^\sharp c) \to^\sharp \frac{1}{2}) \mathbin{\tilde{\odot}^\sharp} (b \to^\sharp \frac{1}{2}))$. Thus it is our aim to prove that

$$c \to (\neg a \mathbin{\tilde{\odot}} \neg b) \; = \; (c \to \neg a) \mathbin{\tilde{\odot}} \neg b$$

holds for any $a, b, c > 0$ such that $a \odot b = 0$, and $a \odot c > 0$. But this follows from (TR3). $\square$

As an example, consider the nilpotent minimum t-norm $\odot_5$, see (3). $\odot_5$ fulfils the conditions (TR1)–(TR3), and the application of Theorem 4.7 results in the t-norm defined as follows; see Figure 9 (right):

$$a \odot_{11} b \; = \; \begin{cases} 0 & \text{if } a + b \leq 1, \\ a & \text{if } a + b > 1, \;\; a + \frac{1}{2} \leq b, \text{ and } b > \frac{3}{4} \\ & \text{or } a + b > \frac{3}{2}, \;\; a \leq b, \text{ and } b > \frac{3}{4}, \\ a - \frac{1}{2} & \text{if } 1 < a + b \leq \frac{3}{2}, \;\; b < a, \text{ and } \frac{1}{2} < b \leq \frac{3}{4} \\ & \text{or } a + b > 1, \;\; a \leq b + \frac{1}{2}, \text{ and } \frac{1}{4} < b \leq \frac{1}{2}. \end{cases}$$

5 Quotients and extensions of totally ordered monoids

So far, we have exploited the fact that t-norms correspond to their Cayley tomonoids, and we have shown possibilities how to modify and to compose these sets of mappings to derive new t-norms. The constructions were based on geometrical considerations, and in case of any problems a solution was found in an ad-hoc manner.

In the rest of this paper, we have a more systematic look a t-norms. We chose already at the beginning a suitable algebraic framework; we work with quantic, negative, commutative tomonoids. We note that we could equally well work with MTL-algebras; however, the additional implication would be more an obstacle than a help. Moreover, t-norm monoids are quantales; however, the framework of quantales is too narrow for our purposes.

We begin by reviewing quotients of tomonoids. Our structures possess a total order relation, which needs to taken into special account. We note that, to get the same result, we could replace the total order by lattice operations and employ the usual algebraic notion of a congruence.

Definition 5.1. Let $(\mathcal{L}; \leq, \odot, 1)$ be a negative, commutative tomonoid. An equivalence relation $\sim$ on $\mathcal{L}$ is called a *tomonoid congruence* if (i) $\sim$ is a congruence of $\mathcal{L}$ as a monoid and (ii) the $\sim$-classes are convex.

Given a tomonoid congruence $\sim$ on $\mathcal{L}$, we denote the quotient of $\mathcal{L}$ by $\sim$ with $\langle \mathcal{L} \rangle_\sim$. We endow $\langle \mathcal{L} \rangle_\sim$ with the total order according to

$$\langle a \rangle_\sim \leq \langle b \rangle_\sim \quad \text{if} \quad a \sim b \quad \text{or} \quad a' < b' \text{ for any } a' \sim a \text{ and } b' \sim b,$$

where $a, b \in \mathcal{L}$; with the induced operation $\odot$ according to

$$\langle a \rangle_\sim \odot \langle b \rangle_\sim \;=\; \langle a \odot b \rangle_\sim,$$

where $a, b \in \mathcal{L}$; and with the constant

$$\mathbf{1} \;=\; \langle 1 \rangle_\sim.$$

The structure $(\langle \mathcal{L} \rangle_\sim; \leq, \odot, \mathbf{1})$ is the *tomonoid quotient* of $\mathcal{L}$ by $\sim$.

Let $\sim$ be a tomonoid congruence on a tomonoid $\mathcal{L}$. If $\mathcal{L}$ is complete the $\sim$-classes are intervals, that is, of the form (a, b), $[a, b)$, or $(a, b]$ for some $a, b \in \mathcal{L}$ such that $a < b$, or $[a, b]$ for some $a, b \in \mathcal{L}$ such that $a \leq b$. It is moreover immediate that $\langle \mathcal{L} \rangle_\sim$ is again a tomonoid. In addition, if $\mathcal{L}$ is commutative, so is $\langle \mathcal{L} \rangle_\sim$; if $\mathcal{L}$ is negative, so is $\langle \mathcal{L} \rangle_\sim$; and if $\mathcal{L}$ is complete, so is $\langle \mathcal{L} \rangle_\sim$.

A common way to construct quotients of partially ordered algebras makes use of filters. Here, we are interested exclusively in quotients of tomonoids arising in this way. We note, however, that there may exist more. For instance, the *Rees quotient*, which identifies all elements below a given element, is a type of congruence not included in the present discussion. We also note that in the case of residuated lattices we obtain in an analogous way all congruences [BlTs].

Definition 5.2. Let $(\mathcal{L}; \leq, \odot, 1)$ be a negative, commutative tomonoid. Then a *filter* of $\mathcal{L}$ is a submonoid F of $\mathcal{L}$ such that $a \in F$ and $b \geq a$ imply $b \in F$. Let then, for $a, b \in \mathcal{L}$,

$$
\begin{aligned}
a \sim_F b \quad & \text{if } a = b, \\
& \text{or } a < b \text{ and there is a } c \in F \text{ such that } b \odot c \leq a, \\
& \text{or } b < a \text{ and there is a } c \in F \text{ such that } a \odot c \leq b.
\end{aligned}
$$

Then we call $\sim_F$ the *congruence induced by* F.

Given a negative, commutative tomonoid, there are always the following filters: $\{1\}$, the *trivial* filter, and $\mathcal{L}$, the *improper* filter.

By a subtomonoid of a tomonoid, we mean a submonoid together with the restricted total order. Each filter F of a negative, commutative tomonoid $\mathcal{L}$, endowed with the total order restricted to F, the monoidal operation, and the monoidal identity, is a subtomonoid. F is negative and commutative as well, and if $\mathcal{L}$ is, in addition, quantic, so is F.

Congruences of negative, commutative tomonoids induced by filters are in fact tomonoid congruences. In the present context, it is worth noting that such congruences preserve also the property of being quantic.

Lemma 5.3. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a quantic, negative, commutative tomonoid, and let $(F; \leq, \odot, 1)$ be a filter of $\mathcal{L}$. Then the congruence induced by F is a tomonoid congruence, and $\langle \mathcal{L} \rangle_\sim$ is a quantic, negative, commutative tomonoid again.*

Proof. It is easily checked that $\langle \mathcal{L} \rangle_\sim$ is a tomonoid, which is moreover commutative and negative. For the proof that $\langle \mathcal{L} \rangle_\sim$ is also quantic, see [Vet3, Lemma 3.8]. ☐

To simplify notation, we will refer to the congruence of a tomonoid induced by a filter F also by the symbol F. That is, we call the $\sim_F$-classes simply *F-classes* and we denote them by $\langle \cdot \rangle_F$. Similarly, the quotient of $\mathcal{L}$ by $\sim_F$ will be called the *quotient of $\mathcal{L}$ by F* and we denote it by $\langle \mathcal{L} \rangle_F$. Furthermore, we call $\mathcal{L}$ an *extension of $\langle \mathcal{L} \rangle_F$ by F*, where F is the *extending* tomonoid.

We will from now on use the notion of a Cayley tomonoid in a more general way. Namely, we define the Cayley tomonoid of any quantic, negative, commutative tomonoid just in the same way as for t-norm monoids. We then observe that the correspondence of t-norm monoids with their Cayley tomonoids generalises to arbitrary quantic, negative, commutative tomonoids.

As our next step, we shall see that any quotient of a t-norm monoid by a filter can be detected from its Cayley tomonoid. What we detect is in fact the Cayley tomonoid of the quotient.

Definition 5.4. Let Λ be the Cayley tomonoid of a l.-c. t-norm $\odot$. Let F be a filter of the t-norm monoid $([0,1]; \leq, \odot, 1)$. We call

$$\gamma_F \colon [0,1] \to [0,1], \quad x \mapsto \bigwedge_{f \in F} \lambda_f(x),$$

the *cut* associated with F.

The cut γ_F serves to detect the F-classes in the following way.

Lemma 5.5. *Let $\odot$ be a l.-c. t-norm, and let F be a filter. Then γ_F is constant on each F-class. Indeed, we have*

$$\gamma_F(x) \;=\; \inf \langle x \rangle_F, \quad x \in [0,1].$$

Conversely, let R be a maximal interval on which γ_F is constant. If R is left-open, R is an F-class. If R possesses a minimal element u, either R is an F-class or $\{u\}$ and $R \backslash \{u\}$ are F-classes, depending on whether or not there is an $r \in R \backslash \{u\}$ and an $f \in F$ such that $\lambda_f(r) = u$.

Proof. $\gamma_F(x)$ is the infimum of $x \odot f$, $f \in F$, that is, $\inf \langle x \rangle_F$. In particular, γ_F is constant on $\langle x \rangle_F$.

For the second part, let R be as indicated, and let $u = \inf R$. Then each two elements of $R \backslash \{u\}$ are F-equivalent. Moreover, by the first part, any $x \notin R$ is not F-equivalent to any element of R. Hence R is one F-class, or $R \backslash \{u\}$ and $\{u\}$ are two F-classes. The first possibility applies if and only if there is an $f \in F$ and $r \in R \backslash \{u\}$ such that $f \odot r = u$. □

We now explain how the quotient of a tomonoid can be derived from its Cayley tomonoid. Let F be a filter of a t-norm monoid. Figures 10–12 provide examples.

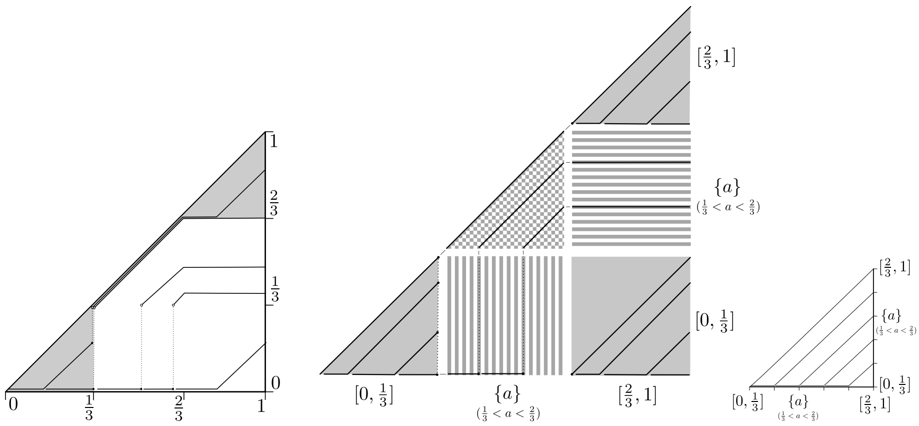

Figure 10: A t-norm monoid and its quotient by a filter. The left plot shows the Cayley tomonoid of the rotation-annihilation of two Łukasiewicz t-norms [Jen2]; the filter $[\frac{2}{3}, 1]$ is highlighted in grey. The middle plot shows, in an "exploded view", the congruence classes. The right plots show the quotient by the filter.

First case: The filter F possesses the minimal element d; cf. Figure 10. Then d is an idempotent, that is, $d \odot d = d$, and $F = [d, 1]$. The cut associated with F is then

$\gamma_F = \lambda_d \in \Lambda$. Being the translation by an idempotent, λ_d has a step-wise shape: for any $x \in [0,1]$, $c = \lambda_d(x) < x$ implies that λ_d is constant c on $[c, x]$, cf. [Vet1].

The translations $\lambda_f,\ f \in F$, are all those that are above λ_d. Moreover, the congruence classes are the maximal intervals $[u, v]$ such that λ_d is constant on $[u, v]$.

The t-norm monoid depicted in Figure 10 (left) possesses the filter $F = [\frac{2}{3}, 1]$. The translations by the elements of F are, accordingly, all those above the stepped mapping $\lambda_{\frac{2}{3}}$. Moreover, the F-classes are $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ as well as the singletons $\{a\}$, $\frac{1}{3} < a < \frac{2}{3}$.

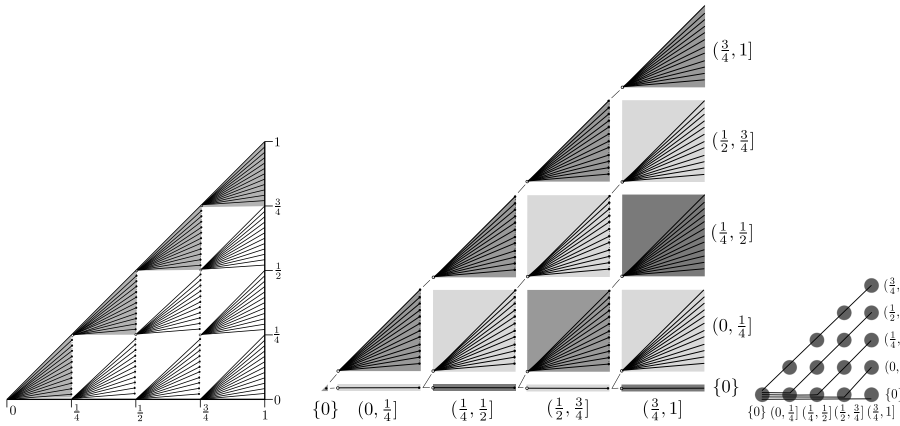

Figure 11: A (modified) t-norm of Hájek and the filter $(\frac{3}{4}, 1]$.

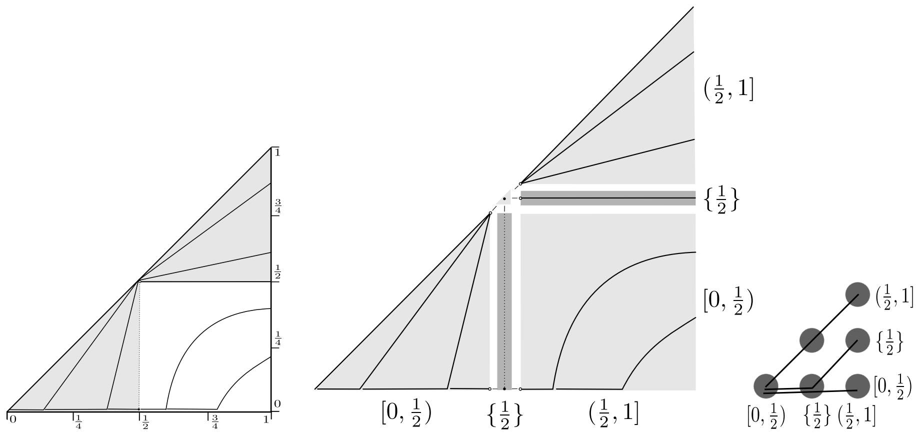

Figure 12: The rotation of the product t-norm [Jen1] and the filter $(\frac{1}{2}, 1]$.

Second case: The filter F does not possess a minimal element; cf. Figures 11 and 12. Then γ_F has the property that, for any $x \in [0,1]$, $\lambda_d(x) = c < x$ implies that λ_d

is constant c on $(c, x]$. Thus γ_F has a step-wise shape also in this case, although the intervals on which γ_F is constant might be left-open as well as right-open.

γ_F, although not in general an element of Λ, is comparable with any element of Λ, and the translations λ_f, $f \in F$, are all those that are above γ_F. Moreover, the F-classes can be determined from γ_F according to Lemma 5.5.

The t-norm monoid in Figure 11 (left) has the filter $F = (\frac{3}{4}, 1]$. Its associated cut γ_F proceeds along the lower border of the area highlighted in grey. γ_F is constant on $[0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, and $(\frac{3}{4}, 1]$. The latter three intervals are left-open, hence F-classes. In case of the first one, we notice that for no $r \in (0, \frac{1}{4}]$ there is an $f > \frac{3}{4}$ such that $\lambda_f(r) = 0$. Consequently, $\{0\}$ and $(0, \frac{1}{4}]$ are two F-classes.

Figure 12 (left) shows a t-norm monoid with the filter $F = (\frac{1}{2}, 1]$. γ_F is constant on $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$. The former interval is one F-class because in this case there is for any $r \in (0, \frac{1}{2})$ some $f > \frac{1}{2}$ such that $\lambda_f(r) = 0$. The latter interval, in contrast, consists of the F-classes $\{\frac{1}{2}\}$ and $(\frac{1}{2}, 1]$.

Conclusion: Having determined a filter, we may draw a grid over the Cayley tomonoid along the boundaries of the F-classes; see the middle plots in Figures 10–12. Associating with the translations bundlewise the traversed rectangles and triangles, we obtain a new, coarser set of mappings: the Cayley tomonoid of the quotient by F; see the right plots in Figures 10–12.

Thus, geometrically speaking, a quotient of a t-norm monoid gives rise to a partitioning of the Cayley tomonoid into triangular and rectangular sections. The following lemma describes the Cayley tomonoid with respect to this partitioning; for each triangle and each rectangle, we consider those parts of the translations that traverse this area.

We will use the following notation. Let $\odot$ be a l.-c. t-norm and let $\mathcal{P}$ be the quotient of $\mathcal{L}$ by the filter F. Λ denotes, as before, the Cayley tomonoid of $\odot$. Furthermore, any $R \in \mathcal{P}$ will be considered as a subinterval of $[0, 1]$, namely as a class of the congruence on $[0, 1]$ that induces $\mathcal{P}$.

For any $f \in F$, λ_f maps R to itself. We write $\lambda_f^R \colon R \to R$ for λ_f with its domain and range being restricted to R, and we put $\Lambda^R = \{\lambda_f^R \colon f \in F\}$. These are the "triangles".

Moreover, let $R, S, T \in \mathcal{P}$ be such that $R \odot T = S < R$. Then for any $t \in T$, λ_t maps R to S. We write $\lambda_t^{R,S} \colon R \to S$ for λ_t with its domain restricted to R and its range restricted to S, and we put $\Lambda^{R,S} = \{\lambda_t^{R,S} \colon t \in T\}$. These are the "rectangles".

Finally, we denote a function that maps all values of a set A to the single value b by $c^{A,b}$.

Lemma 5.6. *Let $\odot$ be a l.-c. t-norm, let F be a filter of $([0, 1]; \leq, \odot, 1)$, and let $\mathcal{P}$ be the quotient of $[0, 1]$ by F.*

(i) *The top element of $\mathcal{P}$ is F. Let $u = \inf F$; then F is one of $(u, 1]$ or $[u, 1]$.*

Moreover, $(F; \leq, \odot, 1)$ is a subtomonoid of $([0, 1]; \leq, \odot, 1)$, and $(\Lambda^F; \leq, \circ, id_F)$ is its Cayley tomonoid. That is, Λ^F is a composition tomonoid on F fulfilling (C1)–(C5)*. For each $f \in F$, λ_f^F is left-continuous, order-preserving, and such*

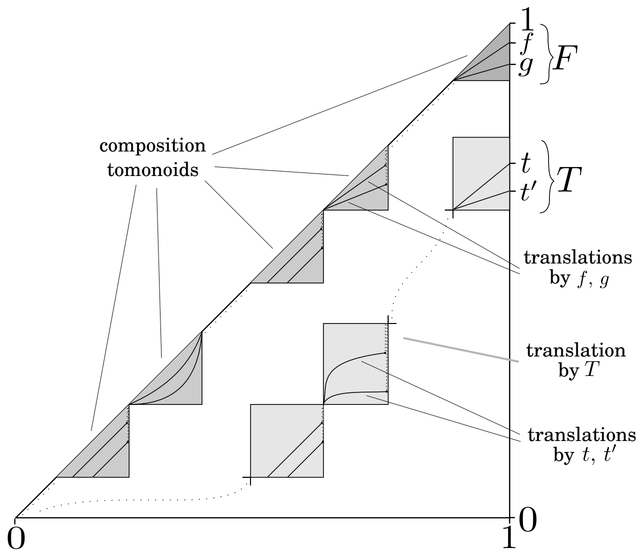

Figure 13: Schematic drawing of a quotient $\mathcal{P}$ of a t-norm monoid. The top element of $\mathcal{P}$ is the filter F. Moreover, the element T of $\mathcal{P}$ is shown and the translation by T is indicated. Similarly, the elements $f, g, t, t' \in \mathcal{L}$ are depicted and the translations by them.

that $\lim_{x \searrow u} \lambda_f(x) = u$ *and* $\lambda_f^F(1) = f$.

Finally,

$$\pi \colon F \to \Lambda^F, \quad f \mapsto \lambda_f^F$$

is an isomorphism between $(F; \leq, \odot, 1)$ *and* $(\Lambda^F; \leq, \circ, \mathrm{id}_F)$.

(ii) *Let* $R \in \mathcal{P}$ *be distinct from* F. *Let* $u = \inf R$ *and* $v = \sup R$. *If* $u = v$, *then* $R = \{u\}$ *and* $\lambda_f^R(u) = u$ *for any* $f \in F$. *Assume now* $u < v$. *Then* R *is one of* (u, v), $[u, v)$, $(u, v]$, *or* $[u, v]$.

Moreover, $(\Lambda^R; \leq, \circ, \mathrm{id}_R)$ *is a composition tomonoid on* R *fulfilling* (C1)–(C4). *For each* $f \in F$, λ_f^R *is left-continuous, order-preserving, and such that* $\lim_{x \searrow u} \lambda_f(x) = u$; *moreover, if* $v \notin R$, $\lambda_f(v) = v$.

Finally,

$$\varrho \colon F \to \Lambda^R, \quad f \mapsto \lambda_f^R$$

is a surjective sup-preserving homomorphism from $(F; \leq, \odot, 1)$ *to* $(\Lambda^R; \leq, \circ, \mathrm{id}_R)$.

(iii) *Let* R *and* S *be distinct elements of* $\mathcal{P}$ *such that* $R \odot T = S$, *where* $T = R \to S$. *Let* $u = \inf R$, $v = \sup R$, $u' = \inf S$, $v' = \sup S$. *If* $u = v$, *then* $R = \{u\}$, $u' \in S$, *and* $\lambda_t^{R,S}(u) = u'$ *for any* $t \in T$. *If* $u' = v'$, *then* $S = \{u'\}$ *and* $\lambda_t^{R,S} = \mathrm{c}^{R,u'}$ *for any* $t \in T$. *Assume now* $u < v$ *and* $u' < v'$.

Then $\Lambda^{R,S} = \{\lambda_t^{R,S} \colon t \in T\}$ is totally ordered w.r.t. the pointwise order, and closed under pointwise calculated suprema of upper-bounded non-empty subsets. For each $t \in T$, $\lambda_t^{R,S}$ is left-continuous, order-preserving, and such that $\lim_{x \searrow u} \lambda_t(x) = u'$. For any $t \in T$ and $f \in F$,

$$\lambda_f^S \circ \lambda_t^{R,S} \;=\; \lambda_t^{R,S} \circ \lambda_f^R \;\in \Lambda^{R,S}. \tag{12}$$

Finally,

$$\tau \colon T \to \Lambda^{R,S}, \;\; t \mapsto \lambda_t^{R,S}$$

is a sup-preserving mapping from T to $\Lambda^{R,S}$ such that $\tau(\lambda_f^T(t)) = \lambda_f^S \circ \tau(t) = \tau(t) \circ \lambda_f^R$ for any $f \in F$ and $t \in T$.

(iv) *Let R and S be distinct elements of $\mathcal{P}$ and assume that there is a $T < R \to S$ such that $R \odot T = S$. Then $u' = \inf S \in S$ and, for all $t \in T$, $\lambda_t^{R,S} = c^{R,u'}$.*

Proof. See [Vet3, Lemmas 4.6, 4.7]. □

Lemma 5.6 describes how a Cayley tomonoid is composed of its parts; Figure 13 provides an illustration. Let $([0,1]; \leq, \odot, 1)$ be a t-norm monoid and F its filter F. The upper-most triangle contains $(\Lambda^F; \leq, \circ, id_F)$, which is simply the Cayley tomonoid of the subtomonoid $(F; \leq, \odot, 1)$. By adding a bottom element if necessary, F becomes isomorphic to a t-norm monoid. Thus, Λ^F is, or arises from, the Cayley tomonoid of a l.-c. t-norm.

Furthermore, let R be any other F-class and assume that R is not a singleton. The triangle associated with R contains $(\Lambda^R; \leq, \circ, id_R)$, which is a further composition tomonoid. Λ^R fulfils the properties (C1)–(C4), but in contrast to Λ^F, (C5) need not hold. Moreover, Λ^R is not necessarily isomorphic to F, but a homomorphic image: there is a surjective homomorphism from F to Λ^R.

The remaining parts of the Cayley tomonoid of $\odot$ are the rectangular sections. Let R and S be distinct F-classes such that $R = S \odot (R \to S)$ and assume that R and S are not singletons. Then the rectangle associated with R and S contains $\Lambda^{R,S}$, a set of mappings that depends on Λ^R and Λ^S according to (12). As shown in [Vet3], in the specific case that F is an archimedean tomonoid, the commutativity condition (12) alone is sufficient to determine $\Lambda^{R,S}$ to a great extent.

6 Construction of t-norms as extensions

Lemma 5.6 proposes a way of decomposing a given Cayley tomonoid. On the basis of the picture provided by Lemma 5.6, we next intend to define ways of composing new Cayley tomonoids out of suitable constituents. In other words, we intend to explore how a given tomonoid can be extended. We address in this way the core problem of the theory of negative, commutative tomonoids, which is certainly demanding. But there are cases that do not cause any difficulties and we will consider a few of them in this section.

For compact specifications, we introduce the following mode of expression. Let a tomonoid $\mathcal{L}$ be an extension of a tomonoid $\mathcal{P}$ by a further tomonoid F. An element R of the quotient $\mathcal{P}$ is then identified with a subinterval of $\mathcal{L}$. If R is a singleton, we say that R is left *unexpanded*. Otherwise, we will *assign* to R the composition tomonoid Λ^R, together with a surjective homomorphism from Λ^F to Λ^R.

Let us first extend a t-norm monoid, and let us do so in the easiest possible way. The top element is always expanded, and we assume that all remaining elements are left unexpanded. We are led to the construction of a two-component ordinal sum; see Theorem 4.1.

Theorem 6.1. *Let $\odot$, $\hat{\odot}$ be two t-norms. Then there is, up to isomorphism, a unique extension $\mathcal{L}$ of the t-norm monoid $([0,1];\leq,\odot,1)$ such that 1 is assigned the Cayley tomonoid of $([0,1];\leq,\hat{\odot},1)$; and each $c \in [0,1)$ is left unexpanded.*

Moreover, $\mathcal{L}$ is isomorphic to the t-norm monoid constructed from $\odot$ and $\hat{\odot}$ according to Theorem 4.1 ("ordinal sum of $\odot$ and $\hat{\odot}$, with the natural order of $\{0,1\}$").

Proof. It is evident that $\mathcal{L}$ coincides with the result of Theorem 4.1 applied to the Cayley tomonoids associated with $\odot$ and $\hat{\odot}$.

Note that in this case all rectangles are degenerated, that is, one element in height. Consequently, all translations must be constant outside the "triangular parts". $\square$

Repeated application of Theorem 6.1 leads to finite ordinal sums of t-norm monoids.

We continue with the rotation construction according to Theorem 4.4. This construction was described from an algebraic perspective, e.g., in [NEG2]. It is straightforward to see that this construction corresponds to a particular extension. In this as well as in some cases below, we extend a finite MV-algebra. For $n \geq 2$, the n-element Łukasiewicz chain is $Ł_n = \{0, \frac{1}{n-1}, \ldots, 1\}$, endowed with the natural order and the monoidal operation given by $a \odot b = (a+b-1) \vee 0$.

Furthermore, let $\lambda\colon (0,1] \to (0,1]$ be order-preserving, left-continuous, and below $id_{(0,1]}$; then we define its reflection $\lambda^\star\colon [0,1) \to [0,1)$ analogously to Definition 3.1. Note that Lemma 3.2 applies in this case as well.

Theorem 6.2. *Let $\odot$ be a t-norm without zero divisors. Then there is, up to isomorphism, a unique extension $\mathcal{L}$ of $Ł_3$ such that 1 is assigned the Cayley tomonoid Λ of $((0,1];\leq,\odot,1)$; $\frac{1}{2}$ is left unexpanded; and 0 is assigned the reflection $\Lambda^\star$ of the Cayley tomonoid of $((0,1];\leq,\odot,1)$, together with $\Lambda \to \Lambda^\star,\quad \varphi \mapsto \varphi^\star$.*

$\mathcal{L}$ is then isomorphic to the t-norm monoid constructed according to Theorem 4.4 ("rotation of $\odot$").

Proof. The translations by elements of the extending filter are given according to Lemma 5.6(ii) and thus in accordance with (8). Furthermore, the single "rectangular section" is uniquely determined by (12) in Lemma 5.6(iii). We conclude that $\mathcal{L}$ is constructed just like in Theorem 4.4. $\square$

Theorem 6.2 describes an extension where only the top element 1 and the bottom element 0 are expanded. Let us once again consider an extension of this type, but this time starting from a t-norm monoid.

Theorem 6.3. *Let $\odot$ and $\hat{\odot}$ be l.-c. t-norms. Then there is, up to isomorphism, a unique extension $\mathcal{L}$ of the t-norm monoid $([0,1]; \leq, \odot, 1)$ such that 1 is assigned the Cayley tomonoid Λ of $([0,1]; \leq, \hat{\odot}, 1)$; c is left unexpanded for each $c \in (0,1)$; and 0 is assigned the reflection $\Lambda^\star$ of Λ, together with $\Lambda \to \Lambda^\star, \quad \varphi \mapsto \varphi^\star$.*

Proof. The translations by elements of the extending filter are given according to Lemma 5.6(ii). The remaining translations are uniquely determined. □

Example 6.4. *In Theorem 6.3, let $\odot$ be the Łukasiewicz t-norm. Then $\mathcal{L}$ is isomorphic to the so-called rotation-annihilation of $\hat{\odot}$ [Jen2]; cf. Figure 10.*

Composition tomonoids acting in parallel, once more

We finally turn to a construction method that was to our knowledge not motivated geometrically. Nonetheless, there is a close connection to Section 4. In fact, we have another case of composition tomonoids acting on subintervals in parallel, and this time the number of subintervals is arbitrary. The problem of how to fill the "rectangular" parts becomes then non-trivial. However, assuming that all composition tomonoids coincide and that furthermore the set of subintervals has itself the structure of a negative, commutative tomonoid, which is moreover weakly cancellative, there is a straightforward possibility. We define it in the present setting.

Namely, we consider a particularly obvious way of extending a tomonoid $\mathcal{L}$ by a further tomonoid F: by the lexicographical product $\mathcal{L} \times_{\text{lex}} F$. This is possible if $\mathcal{L}$ is cancellative, or at least weakly cancellative. The idea leads to A. Zemánková's H-transforms [Mes2]; a further paper on the topic is [JeMo].

Definition 6.5. Let $(\mathcal{L}; \leq, \odot, 1)$ be a tomonoid. $\mathcal{L}$ is called *cancellative* if, for any $a, b, c \in \mathcal{L}, \ a \odot b = a \odot c$ implies $b = c$.

Moreover, $\mathcal{L}$ is called *weakly cancellative* if either $\mathcal{L}$ does not possess a smallest element and is cancellative, or $\mathcal{L}$ possesses the smallest element 0 and, for any $a, b, c \in \mathcal{L}$, $a \odot b = a \odot c > 0$ implies $b = c$.

In [Mes2], only cancellative tomonoids were considered; the generalisation to the weakly cancellative case is, however, straightforward. We note that the notion of weak cancellativity was actually introduced for MTL-algebras [MNH].

Theorem 6.6. *Let $(W; \leq, \odot, 1)$ and $(V; \leq, \odot, 1)$ be quantic, negative, commutative tomonoids, and assume that W is weakly cancellative and any subset of W has a maximal element. Let $W^\star$ arise from W by dropping its zero element if present. Let $\mathcal{L} = (W^\star \times V) \cup \{0\}$, where 0 is a new element. Endow $\mathcal{L}$ with the total order $\leq$ such that 0 is the bottom element and such that for $(v,a), (w,b) \in \mathcal{L}$*

$$(v,a) \leq (w,b) \text{ if } v < w, \text{ or } v = w \text{ and } a \leq b.$$

Endow $\mathcal{L}$ with the binary operation $\odot$ such that 0 is an absorbing element and such that for $(w,a),(v,b) \in \mathcal{L}$

$$(w,a) \odot (v,b) \;=\; \begin{cases} 0 & \textit{if } w \odot v \textit{ is the zero element of } W, \\ (w \odot v,\; a \odot b) & \textit{otherwise.} \end{cases}$$

Then $(\mathcal{L}; \leq, \odot, (1,1))$ is a quantic, negative, commutative tomonoid.

Proof. For non-zero elements $(u,a),(v,b),(w,c)$ of $\mathcal{L}$, we have $(u,a) \odot (v,b) \odot (w,c) = (u \odot v \odot w, a \odot b \odot c)$ if $u \odot v \odot w$ is not the zero element of W and otherwise 0. It follows that $\odot$ is on $\mathcal{L}$ associative. Clearly, $(1,1)$ is neutral w.r.t. $\odot$, and $\odot$ is commutative.

Assume next $(u,a) < (v,b)$ and $(u,a) \odot (w,c) > 0$. Then either $u < v$, in which case $u \odot w < v \odot w$ and thus $(u,a) \odot (w,c) = (u \odot w, a \odot c) < (v \odot w, b \odot c) = (v,b) \odot (w,c)$ because of the cancellativity in W. Or $u = v$ and $a < b$, in which case $(u,a) \odot (w,c) = (u \odot w, a \odot c) = (v \odot w, a \odot c) \leq (v \odot w, b \odot c) = (v,b) \odot (w,c)$. We conclude that $\odot$ is order-preserving.

Obviously, $\mathcal{L}$ is a negative, commutative tomonoid, and $\mathcal{L}$ is complete. Let (w,a), $(v_\iota, b_\iota) \in \mathcal{L} \backslash \{0\}$, $\iota \in I$, such that $v \odot w_\iota$ is not the zero element of W for at least one ι. By (iii), there is a $\kappa \in I$ such that $v_\kappa \geq v_\iota$ for all ι. Let $I_\kappa = \{\iota \in I \colon v_\iota = v_\kappa\}$. By (i), we get $\bigvee_{\iota \in I} (v_\iota, b_\iota) = \bigvee_{\iota \in I_\kappa} (v_\kappa, b_\iota) = (v_\kappa, \bigvee_{\iota \in I_\kappa} b_\iota)$ and, by (ii), $(w,a) \odot \bigvee_{\iota \in I} (v_\iota, b_\iota) = (w \odot v_\kappa, w \odot \bigvee_{\iota \in I_\kappa} b_\iota) = (w \odot v_\kappa, \bigvee_{\iota \in I_\kappa} w \odot b_\iota)$. Furthermore, by weak cancellativity in W, $w \odot v_\kappa$ is the maximal element among $w \odot v_\iota$, $\iota \in I$. Hence $\bigvee_{\iota \in I} (w \odot v_\kappa, w \odot b_\iota) = (w \odot v_\kappa, \bigvee_{\iota \in I_\kappa} w \odot b_\iota)$ as well. The proof is complete that $\mathcal{L}$ is quantic. ☐

As an example, let $V = (0,1]$, endowed with the usual product of reals, and let $W = Ł_5$, the five-element Łukasiewicz chain. Then we get a quantic, negative, commutative tomonoid; up to isomorphism, we get the t-norm monoid displayed in Figure 11.

This t-norm is similar to the one defined by Hájek in [Haj2]. We obtain the original one by replacing $Ł_5$ by $(\mathbb{Z}^-; \leq, +, 0)$. We note that there are further extensions of $Ł_5$ by $((0,1]; \leq, \cdot, 1)$; in fact, the sets $\Lambda^{R,S}$, where R and S are distinct congruence classes, may be chosen strictly smaller.

The construction can be generalised involving more than two, or even countably many, tomonoids.

Theorem 6.7. *For each $i < \omega$, let $(W_i; \leq, \odot, 1)$ be a negative, commutative tomonoid such that W_i is cancellative and each subset of W_i has a maximal element. Put $\mathcal{L} = \prod_i W_i \cup \{0\}$, where 0 is a new element. Endow $\mathcal{L}$ with the total order $\leq$ such that 0 is the smallest element and such that otherwise the lexicographical order applies. Endow $\mathcal{L}$ with the binary operation $\odot$ such that 0 is an absorbing element and such that otherwise the componentwise multiplication applies. Then $(\mathcal{L}; \leq, \odot, (1,1))$ is a quantic, negative, commutative tomonoid.*

Proof. It is obvious that $\mathcal{L}$ is a commutative monoid. We prove similarly to the proof of Theorem 6.6 that $\odot$ is translation-invariant.

Also the proof that $\mathcal{L}$ is quantic follows the lines of the proof of Theorem 6.6. We only show how suprema are determined. Let $(v_{\iota i})_i \in \mathcal{L}$, $\iota \in I$. Let $\kappa_1 \in I$ be such that $v_{\kappa_1 1}$ is maximal in $\{v_{\iota 1} \colon \iota \in I\}$. Let $\kappa_2 \in I$ be such that $v_{\kappa_2 2}$ is maximal in $\{v_{\iota 2} \colon \iota \in I \text{ and } v_{\iota 1} = v_{\kappa_1 1}\}$. Continuing in this way, we generate the element $(v_{\kappa_i i})_i$ of $\mathcal{L}$, which is the supremum of $(v_{\iota i})_i \in \mathcal{L}$, $\iota \in I$. $\square$

As an example, let $W_i = \mathbb{Z}^-$ for each i, endowed with the usual addition of integers. Then $\prod_i W_i \cup \{0\}$ is order-isomorphic to the real unit interval; thus we get a t-norm. This t-norm was defined in [Mes1] and further studied in [Smu]. Endowing $W_i = \mathbb{Z}^-$ with an arbitrary tomonoid structure leads to what is called an iteration of the general H-translation in [Mes2].

7 Conclusion

This paper is devoted to a new approach to the investigation of t-norms. We may well call it geometric although it makes no use of the usual tool to visualise a t-norm – its three-dimensional graph. To reduce the picture to two dimensions, we could, for instance, view the graph "from above" and consider the horizontal cuts, or contour lines; on this idea Maes and De Baets's work is based [MaBa]. What we propose here is to view the graph "from the side": we study the set of vertical cuts. This is simply the set of functions from $[0, 1]$ to $[0, 1]$ arising from $\odot$ by keeping one of its arguments fixed.

The advantage of this approach is not only the reduction to two dimensions. Without further ado we get close to well established methods in algebra. First of all, "vertical cuts" are translations as known in the theory of semigroups, and the set of all translations together with the function composition is known as Cayley's representation. Second, the most basic way of investigating an ordered monoid is to consider its quotients, and the quotients of tomonoids are obvious from its Cayley tomonoid.

We have reformulated several construction methods for t-norms first in the framework of Cayley tomonoids and then, provided this was possible, in an algebraic framework. In the latter case, we have pointed out that an n-component ordinal sum correspond to $n-1$ extensions of a t-norm monoid such that the top element is expanded; the rotation of a t-norm without zero divisors is an extension of the three-element Łukasiewicz chain; the rotation-annihilation is an extension of a t-norm monoid such that the top and the bottom elements are expanded; and the H-transform of t-norms is an extension of weakly cancellative tomonoids.

We have applied a modest piece of the general theory of quantic, negative, commutative tomonoids. This field in turn still offers a large potential for further research. The theory of t-norms has long been developed under the point of view of real functions. At present it seems more promising to proceed in a contrasting manner, namely, to choose an algebraic framework and to consider the finite structures. Our forthcoming paper [Vet2], for instance, is on finite negative, commutative tomonoids; further efforts towards a systematisation of these totally ordered algebras could be rewarding.

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