Totally ordered monoids based on triangular norms

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Abstract

A totally ordered monoid, or tomonoid for short, is a monoid together with a translation-invariant (i.e., compatible) total order. We consider in this paper tomonoids fulfilling the following conditions: the multiplication is commutative; the monoidal identity is the top element; all non-empty suprema exist; and the multiplication distributes over arbitrary suprema. The real unit interval endowed with its natural order and a left-continuous t-norm is our motivating example. A t-norm is a binary operation used in fuzzy logic for the interpretation of the conjunction.

Given a tomonoid of the indicated type, we consider the chain of its quotients induced by filters. The intention is to understand the tomonoid as the result of a linear construction process, leading from the coarsest quotient, which is the one-element tomonoid, up to the finest quotient, which is the tomonoid itself. Consecutive elements of this chain correspond to extensions by Archimedean tomonoids. If in this case the congruence classes are order-isomorphic to real intervals, a systematic specification turns out to be possible.

In order to deal with tomonoids and their quotients in an effective and transparent way, we follow an approach with a geometrical flavor: we work with transformation monoids, using the Cayley representation theorem. Our main results are formulated in this framework. Finally, a number of examples from the area of t-norms are included for illustration.

1 Introduction

Totally ordered monoids, or tomonoids as we say shortly [9], have been studied in several different fields for different motivations. For example, translation-invariant total orders on $\mathbb{N}^n$, i.e. tomonoid structures on $\mathbb{N}^n$, play an important role in computational mathematics and in particular in connection with Gröbner bases of polynomial ideals; see, e.g., [6]. For an early overview of the topic, we refer to [14]. A paper of more recent times, which has considerably influenced the present paper, is [9].

Furthermore, tomonoids are of particular importance in many-valued logics. In these logics, the set of truth values is usually the real unit interval. Furthermore, a binary connective serving as the logical "and" is required to be associative, commutative, and in both arguments isotone. Accordingly, the conjunction is commonly interpreted by a triangular norm, or t-norm for short. This in turn means that the real unit interval endowed with the natural order $\leq$, a t-norm $\odot$, and the constant 1 is a commutative tomonoid.

A lot of research has been devoted to t-norms and in particular to left-continuous t-norms. Numerous examples and methods of their construction have been found and their theory has been developed in several directions. For an overview, see, e.g., [21]; for a detailed discourse, see [20]. A systematic account of t-norms, addressing the general case rather than special classes or even special operations, has still not been established. The present work is meant as a contribution in this direction.

We are interested in tomonoids that are commutative, negative and quantic. In fact, tomonoids arising from t-norms fulfil these properties. Negativity means that the monoidal identity is the top element; and quanticity means that non-empty suprema exist and the multiplication distributes over suprema. We note that the tomonoids under consideration are closely related to quantales [25]. Besides, they can be viewed as residuated lattices [15], in fact as almost complete totally ordered basic semihoops [8]. The implication, however, which is included in the signature of residuated structures, is not considered as useful for the present approach and serves us as an auxiliary operation only.

Nonetheless, our starting point is a basic result on residuated lattices: the latter's quotients are in one-to-one correspondence with their normal convex subalgebras [1]. In particular, the quotients of basic semihoops are in one-to-one correspondence with their filters. Moreover, the set of filters of a totally ordered basic semihoop, ordered by set-theoretical inclusion, is a chain. Applied to the present context, we conclude that we can associate with each tomonoid the chain of quotients that are induced by filters.

However, it is not straightforward to understand the structure of a tomonoid on the basis of its quotients. The fact that the filters form a chain suggests a linear construction process. The quotient induced by the improper filter, which comprises the whole tomonoid, is the one-element tomonoid; the smaller a filter is, the more complex the induced quotient becomes; and the quotient induced by the one-element filter is the tomonoid itself. However, given a tomonoid $\mathcal{P}$, it is not at all obvious to see how $\mathcal{P}$ can be extended: it is not at all clear how to determine those tomonoids $\mathcal{L}$ such that $\mathcal{P}$ is a quotient of $\mathcal{L}$.

The question of how to determine the extensions of a given tomonoid is a central issue in the present paper. Our approach to tackle the problem relies on the regular representation of monoids [5], extended in the straightforward way to the case that a total order is present. This idea has already been applied in our previous work [26]. Namely, a commutative tomonoid can be identified with the tomonoid of its (right or left inner) translations, which we call *Cayley tomonoid*; here, a translation is the mapping from the tomonoid to itself resulting from multiplication with a fixed element. We are led to a monoid of pairwise commuting order-preserving mappings on a totally ordered set.

We note that semigroups of order-preserving mappings have been the topic of a number of papers, e.g., [24, 11]. The semigroups of mappings arising in the present context, however, have rather specific properties and require a particular analysis. Moreover, we note that several of our results could be formulated in the framework of S-posets; see, e.g., [10, 3].

The Cayley tomonoid is well-suited to a study of the structural properties of the underlying tomonoid. We do not just have a representation of the tomonoid under consideration, but we can also "read off" all its quotients. But most important, we are provided with a special means to describe extensions. We restrict here to what we call Archimedean extensions, which correspond to the indecomposable steps in the above-mentioned linear construction process. Under the assumption that the classes of the congruence leading from an extended tomonoid to the original one are ordered like real intervals, Archimedean extensions can be fully described. To this end, we specify how the Cayley tomonoid of the extended tomonoid arises from the Cayley tomonoid of the original one.

We proceed as follows. After Section 2 provides some basic definitions, Section 3 is devoted to the quotients by filters of tomonoids. Section 4 explains how tomonoids and their quotients are represented by Cayley tomonoids. Next, in Section 5, we consider the chain of quotients as a whole. In Section 6, we focus on standard Archimedean extensions, which allow a detailed description. Some concluding words are provided in Section 7.

2 Tomonoids and quantic tomonoids

Our topic is the following class of structures.

Definition 2.1. An algebra $(\mathcal{L}; \odot, 1)$ is a *monoid* if (i) $\odot$ is an associative binary operation and (ii) 1 is neutral w.r.t. $\odot$. A total order $\leq$ on a monoid $\mathcal{L}$ is called *translation-invariant* if, for any $a, b, c \in \mathcal{L}$, $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$. A structure $(\mathcal{L}; \leq, \odot, 1)$ such that $(\mathcal{L}; \odot, 1)$ is a monoid and $\leq$ is a translation-invariant total order on $\mathcal{L}$ is a *totally ordered monoid*, or *tomonoid* for short.

As a reference on tomonoids, we recommend the comprehensive paper [9]. It must furthermore be stressed that more general structures than those considered in this paper have been studied by several authors. We refer to the survey [2] for results on semigroups (without a restriction from the outset) endowed with a translation-invariant preorder (rather than a total order). We note that in the literature on semigroups, what we call "translation-invariant" is often simply called "compatible".

In the sequel, we will refer to a totally ordered set $(A; \leq)$ as a *toset*. We call A *almost complete* if every non-empty subset has a supremum; and we call A *conditionally* *complete* if every non-empty subset that possesses an upper bound has a supremum.

Definition 2.2. A tomonoid $(\mathcal{L}; \leq, \odot, 1)$ is called *commutative* if the monoidal operation $\odot$ is commutative. $\mathcal{L}$ is called *negative* if the neutral element 1 is the top element. $\mathcal{L}$ is called *quantic* if (i) $\mathcal{L}$ is almost complete and (ii) for any elements a, b_ι, $\iota \in I$, of $\mathcal{L}$ we have

$$a \odot \bigvee_\iota b_\iota \;=\; \bigvee_\iota (a \odot b_\iota) \quad \text{and} \quad (\bigvee_\iota b_\iota) \odot a \;=\; \bigvee_\iota (b_\iota \odot a).$$

We note that tomonoids are often understood to be commutative, like for instance in [9]. Furthermore, if the tomonoid is written additively, the dual order is used; accordingly, negativity is then referred to as positivity. Finally, we have chosen the term "quantic", because the conditions defining quanticity come very close to the properties characterising quantales [25].

The tomonoids considered in this paper are commutative, negative and quantic. We abbreviate these three properties by "q.n.c.".

A q.n.c. tomonoid does not necessarily possess a bottom element. If not, we can add an additional element with this role in the usual way.

Definition 2.3. Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. Let $\mathcal{L}^0 = \mathcal{L}$ if $\mathcal{L}$ has a bottom element. Otherwise, let $\mathcal{L}^0$ arise from $\mathcal{L}$ by adding a new element 0; we extend then the total order to $\mathcal{L}^0$ requiring $0 \leq a$ for any $a \in \mathcal{L}^0$, and we extend $\odot$ to $\mathcal{L}^0$ letting $0 \odot a = a \odot 0 = 0$ for any $a \in \mathcal{L}^0$.

Obviously, for any q.n.c. tomonoid $\mathcal{L}$, $\mathcal{L}^0$ is a q.n.c. tomonoid again. Moreover, the total order of $\mathcal{L}^0$ is complete, that is, all suprema and consequently also all infima exist.

A *subtomonoid* of a q.n.c. tomonoid $\mathcal{L}$ is a submonoid F of $\mathcal{L}$ together with the total order restricted from $\mathcal{L}$ to F.

By an *interval* of a q.n.c. tomonoid $\mathcal{L}$, we mean a non-empty convex subset of $\mathcal{L}$. Let J be an interval of $\mathcal{L}$. As $\mathcal{L}^0$ is complete, J possesses a lower boundary $u = \inf J \in \mathcal{L}^0$ and an upper boundary $v = \sup J \in \mathcal{L}$. We will denote an interval J in the usual way by means of its boundaries: by (u, v), $(u, v]$, $[u, v)$, or $[u, v]$, depending on whether or not u and v belong to J.

A *homomorphism* between tomonoids is defined as usual. An *epimorphism* is a surjective homomorphism. We will furthermore use the following definition. A mapping $\chi \colon A \to B$ between tosets A and B is called *sup-preserving* if, whenever the supremum of elements $a_\iota \in A$, $\iota \in I \neq \emptyset$, exists in A, then $\chi(\bigvee_\iota a_\iota)$ is the supremum of $\chi(a_\iota)$, $\iota \in I$, in B. Obviously, a sup-preserving mapping is order-preserving.

Finally, a remark is in order concerning the relationship of the structures studied in this paper to quantales. Recall that a quantale is a complete lattice endowed with an associative binary operation distributing from both sides over arbitrary suprema [25]. Moreover, a quantale is strictly two-sided if the top element is the monoidal identity; totally ordered if the underlying lattice is; commutative if the monoidal operation is commutative. This means that q.n.c. tomonoids that possess a bottom element are exactly the strictly two-sided, totally ordered, commutative quantales. Moreover, homomorphisms of quantales are sup-preserving by definition [25]. We could conclude that the categorical framework of quantales is for us the most appropriate one. Unfortunately, other facts prevent this choice. The formation of quotients could not be done in the same way as we propose below.

Triangular norms

We are interested in tomonoids of the following form. Let $[0,1]$ be the real unit interval; let $\leq$ be its natural order; and let $\odot$ be a binary operation on $[0,1]$ making $([0,1]; \leq, \odot, 1)$ a negative, commutative tomonoid. Then $\odot$ is called a *triangular norm*, or *t-norm* for short.

Furthermore, a function $\varphi\colon [0,1] \to [0,1]$ is called *left-continuous* if $\lim_{x \nearrow a} \varphi(x) = \varphi(a)$ for any $a \in (0,1]$. A t-norm $\odot$ is called left-continuous if, for each $a \in [0,1]$, the function $[0,1] \to [0,1], \;\; x \mapsto x \odot a$ is left-continuous. Obviously, a t-norm $\odot$ is left-continuous if and only if the tomonoid $([0,1]; \leq, \odot, 1)$ is quantic.

This case is considered here. For a left-continuous t-norm $\odot$, we call $([0,1]; \leq, \odot, 1)$ the *t-norm monoid* based on $\odot$. In other words, t-norm monoids are those q.n.c. tomonoids whose universe is the real unit interval endowed with the natural order.

In fuzzy logic, t-norms play an essential role; they interpret the conjunction [16]. The best known examples are the following three. The *Łukasiewicz t-norm* is based on the additive structure of the reals and is defined as follows:

$$a \odot_1 b \;=\; (a + b - 1) \vee 0, \tag{1}$$

where $a, b \in [0,1]$. The *product t-norm* makes use of the multiplicative structure of the reals:

$$a \odot_2 b \;=\; a \cdot b. \tag{2}$$

Finally, the *Gödel t-norm* just relies on the total order of the reals:

$$a \odot_3 b \;=\; a \wedge b. \tag{3}$$

All these three t-norms are, when considered as two-place real functions, continuous. In fact, any continuous t-norm arises from the mentioned ones by means of an ordinal sum construction; see, e.g., [20]. Hence the structure of t-norm monoids based on continuous t-norms is well known.

On the basis of the Łukasiewicz t-norm, we may also define the following set of finite tomonoids. For $n \geq 2$, we call

$$L_n \;=\; \{0, \tfrac{1}{n-1}, \ldots, \tfrac{n-2}{n-1}, 1\}.$$

the n-element *Łukasiewicz chain*. Endowed with the natural order, the operation (1), and the constant 1, L_n is a q.n.c. tomonoid, in fact a subtomonoid of $([0,1]; \leq, \odot_1, 1)$.

As a first example of a t-norm that is left-continuous but not continuous, we may mention the so-called nilpotent minimum t-norm [12]:

$$a \odot_4 b \;=\; \begin{cases} a \wedge b & \text{if } a + b > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

In contrast to the continuous case, the set of left-continuous t-norms has repeatedly been considered as very intransparent. Research has in fact mainly focused on the generation of examples or on methods of constructing new left-continuous t-norms from given ones. Also the present work might support the impression that the theory of left-continuous t-norms requires more general tools than the theory of continuous t-norms.

Residuation

At some places in the sequel, it will be practical to have an additional binary operation available that is definable in a q.n.c. tomonoid. Residuation is a concept well-known in logic in general and in fuzzy logic in particular [16, 15]. In fuzzy logic, the conjunction is typically interpreted by a left-continuous t-norm; the implication connective is then interpreted according to the following definition.

Definition 2.4. A negative, commutative tomonoid $(\mathcal{L}; \leq, \odot, 1)$ is called *residuated* if there is a binary operation $\to$ on $\mathcal{L}$ such that $\odot$ and $\to$ form an adjoint pair, that is, for any $a, b, c \in \mathcal{L}$, we have

$$a \odot b \leq c \quad \text{if and only if} \quad a \leq b \to c. \tag{5}$$

We note that, for a residuated negative, commutative tomonoid $\mathcal{L}$, the operation $\to$ is determined by (5) uniquely; in fact, we have then

$$a \to b \;=\; \max \{c \in \mathcal{L} \colon a \odot c \leq b\}, \qquad a, b \in \mathcal{L}. \tag{6}$$

It is easily checked that in a q.n.c. tomonoid, the maximum in (6) always exists, and it follows that each q.n.c. tomonoid is residuated. We will use the operation $\to$ in the sequel occasionally.

The concept of residuation leads us to structures prominent in fuzzy logic. A *basic semihoop* [8] is a structure $(\mathcal{L}; \wedge, \vee, \odot, \to, 1)$ such that $(\mathcal{L}; \odot, 1)$ is a commutative monoid, $(\mathcal{L}; \wedge, \vee, 1)$ is a lattice with top element 1, the pair of operations $\odot$ and $\to$ fulfil (5), and prelinearity holds, that is, $(a \to b) \vee (b \to a) = 1$ for any $a, b \in \mathcal{L}$.

For totally ordered basic semihoops, the prelinearity condition is redundant. Hence we can view residuated negative, commutative tomonoids as the same as totally ordered basic semihoops.

For the case of an almost complete order, this correspondence specialises to the structures considered in this paper. If $\mathcal{L}$ is a q.n.c. tomonoid, let $\wedge, \vee$ be the lattice operations and let $\to$ be defined by (6); then $(\mathcal{L}; \wedge, \vee, \odot, \to, 1)$ is an almost complete totally ordered basic semihoop. Conversely, if $\mathcal{L}$ is an almost complete totally ordered basic semihoop, then $(\mathcal{L}; \leq, \odot, 1)$ is a q.n.c. tomonoid.

We finally note that *MTL-algebras* [7] are basic semihoops with a bottom element. Thus q.n.c. tomonoids with a bottom element correspond to complete totally ordered MTL-algebras.

3 Filters and quotients of tomonoids

In this section we discuss the formation of quotients of tomonoids. The notion of a filter will be of central interest.

Definition 3.1. Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. Then a *filter* of $\mathcal{L}$ is a subtomonoid $(F; \leq, \odot, 1)$ of $\mathcal{L}$ such that $f \in F$ and $g \geq f$ imply $g \in F$.

By the *trivial* tomonoid, we mean the one-element tomonoid, consisting of 1 alone. Each non-trivial tomonoid $\mathcal{L}$ possesses at least two filters: $\{1\}$, the *trivial* filter, and $\mathcal{L}$, the *improper* filter.

We note that filters on semigroups have been studied in a more general context than the present one. For instance, in [19], the lattices of filters of semigroups endowed with a translation-invariant preorder are discussed.

In what follows, we always tacitly assume a q.n.c. tomonoid $\mathcal{L}$ to be a subset of $\mathcal{L}^0$. In particular, infima are always meant to be calculated in $\mathcal{L}^0$, and the element 0, which is possibly not contained in $\mathcal{L}$, may be used to specify an interval of $\mathcal{L}$.

Definition 3.2. Let F be a filter of a q.n.c. tomonoid $\mathcal{L}$. Let d be the infimum of F in $\mathcal{L}^0$; then we call d the *boundary* of F. If d belongs to F, we write $F = d^{\leq}$; if d does not belong to F, we write $F = d^{<}$.

Moreover, let $d \in \mathcal{L}^0$ be the boundary of a filter. If there are two filters whose boundary is d, we call d a *double* boundary. Otherwise we call d a *simple* boundary.

Note that each filter of a q.n.c. tomonoid $\mathcal{L}$ is of the form $d^{<} = (d, 1]$ for some $d \in \mathcal{L}^0$, or $d^{\leq} = [d, 1]$ for some $d \in \mathcal{L}$. Thus each filter F is uniquely determined by its boundary d together with the information whether or not d belongs to F.

It is moreover clear that each filter of a q.n.c. tomonoid is again a q.n.c. tomonoid.

Lemma 3.3. *Let $\mathcal{L}$ be a q.n.c. tomonoid, and let $d \in \mathcal{L}$.*

(i) $d^{\leq}$ *is a filter if and only if d is idempotent.*

(ii) $d^{<}$ *is a filter if and only if $d \neq 1$, $d = \bigwedge_{a>d} a$, and $d < a \odot b$ for all $a, b > d$.*

Proof. (i) $[d, 1]$ is a filter if and only if $[d, 1]$ is closed under multiplication if and only if $d \odot d = d$, that is, if d is idempotent.

(ii) Let $d^<$ be a filter. Then $d < 1$ because each filter contains 1; $d = \inf d^< = \inf (d, 1] = \bigwedge_{a>d} a$; and $(d, 1]$ is closed under multiplication, that is, $a \odot b > d$ for each $a, b > d$.

Conversely, let $d \neq 1$ such that $d = \bigwedge_{a>d} a$ and $d < a \odot b$ for any $a, b > d$. Then $\{a \in \mathcal{L} \colon a > d\}$ is a filter, whose infimum is d, that is, which equals $d^<$. $\square$

We now turn to quotients of tomonoids.

Definition 3.4. Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. An equivalence relation $\sim$ on $\mathcal{L}$ is called a *tomonoid congruence* if (i) $\sim$ is a congruence of $\mathcal{L}$ as a monoid and (ii) the $\sim$-classes are convex. We endow then the quotient $\langle \mathcal{L} \rangle_\sim$ with the total order given by

$$\langle a \rangle_\sim \leq \langle b \rangle_\sim \text{ if } a' \leq b' \text{ for some } a' \sim a \text{ and } b' \sim b$$

for $a, b \in \mathcal{L}$, with the induced operation $\odot$, and with the constant $\langle 1 \rangle_\sim$. The resulting structure $(\langle \mathcal{L} \rangle_\sim; \leq, \odot, \langle 1 \rangle_\sim)$ is called a *tomonoid quotient* of $\mathcal{L}$.

Note that because we work with a total order, for two $\sim$-classes of a tomonoid quotient we have $\langle a \rangle_\sim < \langle b \rangle_\sim$ if and only if $a' < b'$ for all $a' \sim a$ and $b' \sim b$.

In this paper, we focus on quotients induced by a filter.

Definition 3.5. Let F be a filter of a q.n.c. tomonoid $\mathcal{L}$. For $a, b \in \mathcal{L}$, let

$$\begin{aligned} a \sim_F b \quad & \text{if } a = b, \\ & \text{or } a < b \text{ and there is a } f \in F \text{ such that } b \odot f \leq a, \\ & \text{or } b < a \text{ and there is a } f \in F \text{ such that } a \odot f \leq b. \end{aligned}$$

Then we call $\sim_F$ the *congruence induced by F*.

Lemma 3.6. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid, and let $(F; \leq, \odot, 1)$ be a filter of $\mathcal{L}$. Then the congruence induced by F is a tomonoid congruence, and $\langle \mathcal{L} \rangle_{\sim_F}$ is negative and commutative.*

Proof. It is easily checked that $\sim_F$ is compatible with $\odot$ and that the $\sim_F$-classes are convex. Clearly, negativity and commutativity are preserved. $\square$

Definition 3.7. Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid, and let $(F; \leq, \odot, 1)$ be a filter of $\mathcal{L}$. Let $\sim_F$ be the congruence induced by F. We will refer to the $\sim_F$-classes as *F-classes* and we denote them by $\langle \cdot \rangle_F$. Similarly, let $\mathcal{P}$ be the quotient of $\mathcal{L}$ by $\sim_F$. Then we refer to $\mathcal{P}$ as the *quotient of $\mathcal{L}$ by F* and we denote it by $\langle \mathcal{L} \rangle_F$.

We furthermore call in this case $\mathcal{L}$ an *extension of $\mathcal{P}$ by F*, and we refer to F as the *extending* tomonoid.

By Lemma 3.6, tomonoid congruences preserve commutativity and negativity. We now see that the same applies to quanticity.

Lemma 3.8. *Let $\mathcal{L}$ be a q.n.c. tomonoid, and let F be a filter of $\mathcal{L}$. Then also $\langle \mathcal{L} \rangle_F$ is quantic. Moreover, let $a_\iota \in \mathcal{L}$, $\iota \in I$, be such that among $\langle a_\iota \rangle_F$, $\iota \in I$, there is no largest element; then*

$$\bigvee_\iota \langle a_\iota \rangle_F \;=\; \langle \bigvee_\iota a_\iota \rangle_F. \tag{7}$$

Proof. We drop in this proof the subscript "F". Let $a_\iota \in \mathcal{L}$, $\iota \in I$, and assume that the $\langle a_\iota \rangle$ do not possess a largest element. Let $a = \bigvee_\iota a_\iota$. Then $\langle a \rangle \geq \langle a_\iota \rangle$ for all ι. Moreover, let $b \in \mathcal{L}$ be such that $\langle b \rangle \geq \langle a_\iota \rangle$ for all ι. Then b is not equivalent to any a_ι, hence $\langle b \rangle > \langle a_\iota \rangle$; consequently $b > a_\iota$ for all ι, so that $b \geq a$ and $\langle b \rangle \geq \langle a \rangle$. Thus (7) holds, and it also follows that $\langle \mathcal{L} \rangle$ is almost complete.

To show that $\odot$ distributes over suprema in $\langle \mathcal{L} \rangle$, let $b_\iota \in \mathcal{L}$, $\iota \in I$, and $a \in \mathcal{L}$. Assume first that the elements $\langle a \odot b_\iota \rangle$, $\iota \in I$, do not possess a maximal element. Then also the $\langle b_\iota \rangle$ do not possess a maximal element, and (7) implies

$$\langle a \rangle \odot \bigvee_\iota \langle b_\iota \rangle \;=\; \bigvee_\iota (\langle a \rangle \odot \langle b_\iota \rangle). \tag{8}$$

Assume second that the $\langle a \odot b_\iota \rangle$ possess the maximal element $\langle a \odot b_\kappa \rangle$, but that the $\langle b_\iota \rangle$ do not possess a maximal element. Let $\iota \in I$ such that $\langle b_\iota \rangle > \langle b_\kappa \rangle$. Then $a \odot b_\iota \sim a \odot b_\kappa$, and we have $a \odot b_\kappa \leq a \odot \inf \langle b_\iota \rangle = a \odot \bigwedge_{f \in F} (b_\iota \odot f) \leq \bigwedge_{f \in F} (a \odot b_\iota \odot f) = \inf \langle a \odot b_\iota \rangle = \inf \langle a \odot b_\kappa \rangle \leq a \odot b_\kappa$. We conclude that $a \odot b_\iota = a \odot b_\kappa$ for any $\iota \in I$ such that $b_\iota > b_\kappa$. Thus $a \odot \bigvee_\iota b_\iota = a \odot b_\kappa$. By (7), $\langle a \rangle \odot \bigvee_\iota \langle b_\iota \rangle = \langle a \odot \bigvee_\iota b_\iota \rangle = \langle a \odot b_\kappa \rangle = \bigvee_\iota (\langle a \rangle \odot \langle b_\iota \rangle)$, and (8) is proved.

Assume third that the $\langle b_\iota \rangle$, $\iota \in I$, possess the maximal element $\langle b_\kappa \rangle$. Then $\langle a \odot b_\kappa \rangle$ is maximal among the $\langle a \odot b_\iota \rangle$. Then obviously, (8) holds as well. □

Thus quotients of q.n.c. tomonoids by filters are q.n.c. tomonoids again. However, we will see below in an example that these congruences do not necessarily preserve suprema. In fact, (7) does in general not hold if the a_ι belong to the same F-class. Thus our quotient is not necessarily a quotient in the sense of quantale theory – which would naturally require the preservation of suprema [25].

Depending on the type of filter, we can characterise the elements of a quotient induced by a filter as follows.

Lemma 3.9. *Let $\mathcal{L}$ be a q.n.c. tomonoid.*

(i) *Let e be idempotent. Then each $e^{\leq}$-class is of the form $[u, v]$ for some $u, v \in \mathcal{L}$ such that $u \leq v$. The class of 1 is $e^{\leq} = [e, 1]$.*

Moreover, let P consist of all smallest elements of the $e^{\leq}$-classes. Then P is closed under $\odot$, and $(P; \leq, \odot, e)$ is isomorphic to the quotient $\mathcal{L}_{e^{\leq}}$.

(ii) *Let $d^{<}$ be a filter. Then each $d^{<}$-class is of the form (u, v), $(u, v]$, $[u, v)$, or $[u, v]$ for some $u, v \in \mathcal{L}^0$ such that $u < v$, or $\{u\}$ for some $u \in \mathcal{L}$. The class of 1 is $d^{<} = (d, 1]$.*

Proof. (i) Let $a \in \mathcal{L}$. For $b \leq a$ we have $b \sim_{e^{\leq}} a$ if and only if $a \odot e \leq b$, hence the smallest element of $\langle a \rangle_{e^{\leq}}$ is $a \odot e$. Similarly, for $b \geq a$ we have $b \sim_{e^{\leq}} a$ if and only if $b \odot e \leq a$; hence, by quanticity, $\langle a \rangle_{e\leq}$ possesses a largest element. Thus $\langle a \rangle_{e\leq}$ is of the indicated form.

The last part follows again from the fact that, for each $a \in \mathcal{L}$, the smallest element of $\langle a \rangle_{e\leq}$ is $a \odot e$.

(ii) This is clear by the completeness of $\mathcal{L}^0$. □

A concluding question might be: What about quotients that are not induced by filters? In the case of basic semihoops, there are no more; quotients and filters are in one-to-one correspondence then. For q.n.c. tomonoids, Rees quotients modulo order ideals preserve the structure as well; cf. [9]. Here, we consider the totality of quotients induced by filters, which are pairwise comparable; additional quotients would lead to a different picture. Considering other kinds of quotients could well be worthwhile in connection with Archimedean extensions as discussed below – an aspect with which, however, we do not deal here.

4 The Cayley tomonoid

Any monoid can be identified with a monoid under composition of mappings – namely, with the set of mappings acting on the monoid by left or right multiplication. This is the regular representation [5], which is due to A. Cayley for the case of groups. If the monoid is commutative, any two mappings commute. Moreover, the presence of a translation-invariant total order on the monoid means that the mappings are order-preserving. We have made use of the connection between t-norms and monoids of pairwise commuting order-preserving mappings in [26] and will do so here as well.

Definition 4.1. Let $(R; \leq)$ be a toset, and let Φ be a set of order-preserving mappings from R to R. We denote by $\leq$ the pointwise order on Φ, by $\circ$ the functional composition, and by id_R the identity mapping on R. Assume that (i) $\leq$ is a total order on Φ, (ii) Φ is closed under $\circ$, and (iii) $id_R \in \Phi$. Then we call $(\Phi; \leq, \circ, id_R)$ a *composition tomonoid* on R.

It is easily checked that a composition tomonoid is in fact a tomonoid.

A composition tomonoid Φ on a toset R will be called *isomorphic* to another composition tomonoid Ψ on a toset S if there is an order isomorphism $\iota\colon R \to S$ such that $\Psi = \{\iota \circ \lambda \circ \iota^{-1} \colon \lambda \in \Phi\}$.

Let us introduce the following properties of a composition tomonoid $(\Phi; \leq, \circ, id_R)$ on a toset R:

(C1) $\circ$ is commutative.

(C2) id_R is the top element.

(C3) R is conditionally complete, and every $\lambda \in \Phi$ is sup-preserving.

(C4) Φ is almost complete, and suprema are calculated pointwise.

(C5) R has a top element 1, and for each $a \in R$ there is a unique $\lambda \in \Phi$ such that $\lambda(1) = a$.

Proposition 4.2. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. For each $a \in \mathcal{L}$, put*

$$\lambda_a \colon \mathcal{L} \to \mathcal{L}, \quad x \mapsto x \odot a, \tag{9}$$

and let $\Lambda = \{\lambda_a \colon a \in \mathcal{L}\}$. Then $(\Lambda; \leq, \circ, id_{\mathcal{L}})$ is a composition tomonoid on $\mathcal{L}$ fulfilling the properties (C1)–(C5)*. Moreover,*

$$\pi \colon \mathcal{L} \to \Lambda, \quad a \mapsto \lambda_a \tag{10}$$

is an isomorphism of the tomonoids $(\mathcal{L}; \leq, \odot, 1)$ and $(\Lambda; \leq, \circ, id_{\mathcal{L}})$.

Proof (sketched). For $a, b \in \mathcal{L}$, we have $a \leq b$ if and only if $\pi(a) \leq \pi(b)$; $\pi(a \odot b) = \pi(a) \circ \pi(b)$; and $\pi(1) = id_{\mathcal{L}}$. Hence Λ is a composition tomonoid. Moreover, (C1) holds because $\odot$ is commutative, and (C2) holds because 1 the top element of $\mathcal{L}$; (C3) holds because $\mathcal{L}$ is quantic; and (C5) is obvious. It also follows that π is an isomorphism of tomonoids.

To show (C4), let $a_\iota \in \mathcal{L}$, $\iota \in I$, and $x \in \mathcal{L}$. Then $\bigvee_\iota \lambda_{a_\iota}(x) = \bigvee_\iota (x \odot a_\iota) = x \odot \bigvee_\iota a_\iota = \lambda_{\bigvee_\iota a_\iota}(x)$. Thus the pointwise calculated supremum of λ_{a_ι}, $\iota \in I$, is a mapping contained in Λ. □

Definition 4.3. Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. For each $a \in \mathcal{L}$, the mapping λ_a defined by (9) is called the *translation* by a. Furthermore, the tomonoid $(\Lambda; \leq, \circ, id_{\mathcal{L}})$, consisting of all translations as specified in Proposition 4.2, is called the *Cayley tomonoid* of $\mathcal{L}$.

The isomorphism (10) is a particularly simple way to represent a q.n.c. tomonoid as a monoid of order-preserving mappings on a toset. Representations of partially ordered monoids by order-preserving mappings on posets have otherwise been studied in a more general context; the notion of an S-poset was introduced by Fakhruddin in [10]. We refer to [3] and the references given there.

In view of the isomorphism of a q.n.c. tomonoid with its Cayley tomonoid, we wish to characterise Cayley tomonoids as special composition tomonoids. A subset of the properties indicated in Proposition 4.2 turns out to be sufficient.

Proposition 4.4. *Let $(\mathcal{L}; \leq)$ be an almost complete toset with the top element* 1*, and let Λ be a composition tomonoid on $\mathcal{L}$ such that* (C1)*,* (C3)*, and* (C5) *hold. Then also* (C2) *and* (C4) *hold. Moreover, there is a unique binary operation $\odot$ on $\mathcal{L}$, namely,*

$$a \odot b \;=\; \lambda(b) \quad \text{where } \lambda \in \Lambda \text{ is such that } \lambda(1) = a,$$

such that $(\mathcal{L}; \leq, \odot, 1)$ is a q.n.c. tomonoid and $(\Lambda; \leq, \circ, id_{\mathcal{L}})$ is its Cayley tomonoid.

Proof. See, e.g., [26, Thm. 2.3] and its proof. □

In particular, each left-continuous t-norm can be identified with a monoid under composition of pairwise commuting, order-preserving, and left-continuous functions from $[0, 1]$ to $[0, 1]$ such that for any $a \in [0, 1]$ exactly one of them maps 1 to a.

Example 4.5. *The Cayley tomonoids of the t-norm monoids based on the standard t-norms are shown in Figure* 1*. A selection of translations are indicated in a schematic way.*

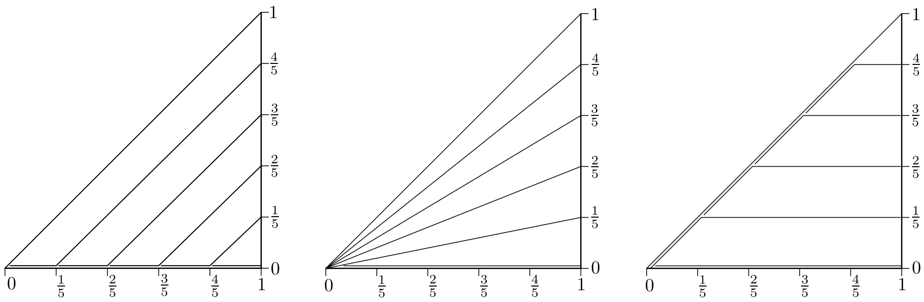

Figure 1: The Łukasiewicz, product, and Gödel t-norm.

Quotients and Cayley tomonoids

Our next aim is to see how quotients of tomonoids are represented by means of Cayley tomonoids.

We will use the following notation and conventions. Let $\mathcal{L}$ be a q.n.c. tomonoid and let $\mathcal{P}$ be the quotient of $\mathcal{L}$ by the filter F. Then Λ will always denote the Cayley tomonoid of $\mathcal{L}$. Furthermore, let $R \in \mathcal{P}$; then R will always be considered as a subset of $\mathcal{L}$, namely as a class of the congruence on $\mathcal{L}$ that yields $\mathcal{P}$.

For any $f \in F$, λ_f maps R to itself. We write $\lambda_f^R \colon R \to R$ for λ_f with its domain and range being restricted to R, and we put $\Lambda^R = \{\lambda_f^R \colon f \in F\}$.

Moreover, let $R \in \mathcal{P}$ and $T \in \mathcal{P} \backslash \{F\}$, and let $S = R \odot T$. Then for any $t \in T$, λ_t maps R to S. We write $\lambda_t^{R,S} \colon R \to S$ for λ_t with its domain restricted to R and its range restricted to S, and we put $\Lambda^{R,S} = \{\lambda_t^{R,S} \colon t \in T\}$.

Finally, we denote a function that maps all values of a set A to the single value b by $c^{A,b}$.

The following two lemmas describe the sets Λ^R and $\Lambda^{R,S}$, respectively. Figure 2 shows the situation schematically.

Lemma 4.6. *Let $\mathcal{P}$ be the quotient of the q.n.c. tomonoid $(\mathcal{L}; \leq, \odot, 1)$ by the non-trivial filter F of $\mathcal{L}$.*

(i) *The top element of $\mathcal{P}$ is F. Let $u = \inf F$; then $u < 1$, and F is one of $(u, 1]$ or*

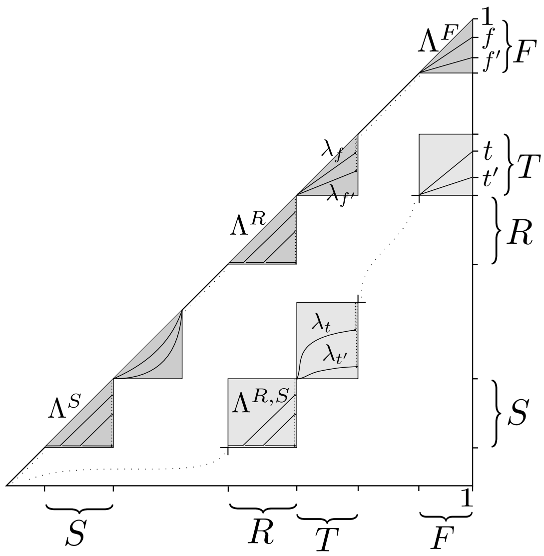

Figure 2: Schematic drawing of the quotient of a q.n.c. tomonoid $\mathcal{L}$ by a filter F. We see, on the one hand, the Cayley tomonoid of $\mathcal{P}$, among whose elements we have S, R, T, and the top element F. In light grey, the translation by T is shown; in dark grey, the translation by F, i.e. the identity function on $\mathcal{P}$, is shown. We see, on the other hand, the Cayley tomonoid of $\mathcal{L}$, among whose elements we have $t', t, f', f, 1$. The translations by these five elements, belonging to the Cayley tomonoid of $\mathcal{L}$, are indicated as well.

$[u, 1]$. *Moreover,* $(\Lambda^F; \leq, \circ, \mathrm{id}_F)$ *is the Cayley tomonoid of* F*, that is, a composition tomonoid on* F *fulfilling* (C1)–(C5). *Furthermore, we have:*

(a) *Let* $f \in F$. *If* $F = [u, 1]$, $\lambda^F_f(u) = u$*; if* $F = (u, 1]$, $\bigwedge_{g \in F} \lambda^F_f(g) = u$. *Moreover,* $\lambda^F_f(1) = f$.

(b) *If* $F = [u, 1]$, Λ^F *has the bottom element* $\mathrm{c}^{F,u}$.

Finally,

$$\pi \colon F \to \Lambda^F, \quad f \mapsto \lambda^F_f$$

is an isomorphism between $(F; \leq, \odot, 1)$ *and* $(\Lambda^F; \leq, \circ, \mathrm{id}_F)$.

(ii) *Let* $R \in \mathcal{P}$ *be distinct from* F. *Let* $u = \inf R$ *and* $v = \sup R$. *If* $u = v$*, then* $R = \{u\}$ *and* $\lambda^R_f(u) = u$ *for any* $f \in F$.

Assume now $u < v$. *Then* R *is one of* (u, v), $[u, v)$, $(u, v]$, *or* $[u, v]$. *Moreover,* $(\Lambda^R; \leq, \circ, \mathrm{id}_R)$ *is a composition tomonoid on* R *fulfilling* (C1)–(C4) *as well as the following properties:*

(c) *Let* $f \in F$. *If* $u \in R$, $\lambda^R_f(u) = u$*; if* $u \notin R$, $\bigwedge_{r \in R} \lambda^R_f(r) = u$. *Moreover, if* $v \notin R$, $\bigvee_{r \in R} \lambda^R_f(r) = \lambda_f(v) = v$.

(d) *If $R = [u, v]$, Λ^R has the bottom element $\mathsf{c}^{R,u}$. If $R = [u, v)$, then $\mathsf{c}^{R,u} \notin \Lambda^R$.*

Finally,

$$\varrho\colon F \to \Lambda^R, \quad f \mapsto \lambda_f^R \tag{11}$$

is a sup-preserving epimorphism from $(F; \leq, \odot, 1)$ to $(\Lambda^R; \leq, \circ, \mathsf{id}_R)$.

Proof. (i) Here, Proposition 4.2 is applied to the q.n.c. tomonoid F.

(a) Let $f \in F$. If $u \in F$, clearly $\lambda_f^F(u) = u$. If $u \notin F$, we have $u \leq \bigwedge_{g \in F} \lambda_f^F(g) \leq \bigwedge_{g \in F} g = u$, that is, $\bigwedge_{g \in F} \lambda_f^R(g) = u$. Clearly, $\lambda_f(1) = f \odot 1 = f$.

(b) If $u \in F$, $\lambda_u^F = \mathsf{c}^{F,u}$ is the bottom element of Λ^F.

(ii) The case that R is a singleton is trivial. Assume $u < v$.

The fact that Λ^R is a composition tomonoid fulfilling (C1)–(C4) and that ϱ, defined by (11), is a sup-preserving epimorphism follows from Proposition 4.2.

(c) Let $f \in F$. We see like in the proof of (a) that $\lambda_f^R(u) = u$ if $u \in R$, and $\bigwedge_{r \in R} \lambda_f^R(r) = u$ otherwise. Moreover, if $v \notin R$, then $\lambda_f(v) \notin R$ and consequently $r < \lambda_f(v) \leq v$ for any $r \in R$, that is, $\lambda_f(v) = v$.

(d) Assume $u \in R$. If R has a largest element v as well, $v \odot z = u$ for some $z \in F$, and hence $\mathsf{c}^{R,u} = \lambda_z^R \in \Lambda^R$. If R does not contain its supremum v, then by (c), $\bigvee_{r \in R} \lambda_f^R(r) = v$ for any $f \in F$, and it follows $\mathsf{c}^{R,u} \notin \Lambda^R$. □

In what follows, we call a pair A, B of elements of the q.n.c. tomonoid $\mathcal{P}$ *$\odot$-maximal* if A is the largest element X such that $X \odot B = A \odot B$, and B is the largest element Y such that $A \odot Y = A \odot B$. In other words, A, B is a $\odot$-maximal pair if and only if $A = B \to C$ and $B = A \to C$, where $C = A \odot B$.

Lemma 4.7. *Let $\mathcal{P}$ be the quotient of the q.n.c. tomonoid $(\mathcal{L}; \leq, \odot, 1)$ by the non-trivial filter F of $\mathcal{L}$. Let $R, T \in \mathcal{P}$ such that $T < F$, and let $S = R \odot T$.*

(i) *Let R, T be $\odot$-maximal. Then $S < R$. Let $u = \inf R$, $v = \sup R$, $u' = \inf S$, and $v' = \sup S$. If $u = v$, then $R = \{u\}$, $u' \in S$, and $\lambda_t^{R,S}(u) = u'$ for all $t \in T$. If $u' = v'$, then $S = \{u'\}$ and $\lambda_t^{R,S} = \mathsf{c}^{R,u'}$ for all $t \in T$.*

Assume now $u < v$ and $u' < v'$. If then $u \in R$, we have $u' \in S$. Moreover, $\Lambda^{R,S} = \{\lambda_t^{R,S} \colon t \in T\}$ is a set of mappings from R to S with the following properties:

(a) *R and S are conditionally complete, and for any $t \in T$, $\lambda_t^{R,S}$ is sup-preserving.*

(b) *Let $t \in T$. If $u \in R$, $\lambda_t^{R,S}(u) = u'$; if $u \notin R$, $\bigwedge_{r \in R} \lambda_t^{R,S}(r) = u'$.*

(c) *Under the pointwise order, $\Lambda^{R,S}$ is totally ordered.*

(d) *Let $K \subseteq \Lambda^{R,S}$ such that $\bigvee_{\lambda \in K} \lambda(r) \in S$ for all $r \in R$. Then the pointwise calculated supremum of K is in $\Lambda^{R,S}$.*

(e) *If $u' \in S$ and $v \in R$, $\Lambda^{R,S}$ has the bottom element $c^{R,u'}$. If $u' \in S$ and $v \notin R$, then either $\Lambda^{R,S} = \{c^{R,u'}\}$ or $c^{R,u'} \notin \Lambda^{R,S}$. If $v \notin R$ and $v' \in S$, then $u' \in S$ and $\Lambda^{R,S} = \{c^{R,u'}\}$.*

(f) *For any $t \in T$ and $f \in F$, $\lambda_f^S \circ \lambda_t^{R,S}$ and $\lambda_t^{R,S} \circ \lambda_f^R$ are in $\Lambda^{R,S}$ and coincide.*

Finally,

$$\tau\colon T \to \Lambda^{R,S}, \quad t \mapsto \lambda_t^{R,S} \tag{12}$$

is a sup-preserving mapping from T to $\Lambda^{R,S}$ such that, for any $f \in F$ and $t \in T$,

$$\tau(\lambda_f^T(t)) = \lambda_f^S \circ \tau(t) = \tau(t) \circ \lambda_f^R. \tag{13}$$

(ii) *Let R, T not be $\odot$-maximal. Then S contains a smallest element u', and $\lambda_t^{R,S} = c^{R,u'}$ for all $t \in T$.*

Proof. (i) We clearly have $S \leq R$. If $S = R$, the maximal element Y such that $R \odot Y = R \odot T$ would be F, in contradiction to the assumptions that $T < F$ and R, T is a $\odot$-maximal pair. Thus $S < R$.

We consider first the case that R is a singleton, that is, $R = \{u\}$. Then $u \odot f = u$ for all $f \in F$. Let $t \in T$; then $\lambda_t^{R,S}(u) \odot f = u \odot t \odot f = u \odot t = \lambda_t^{R,S}(u)$ for any $f \in F$; hence $u' \in S$ and $\lambda_t^{R,S}(u) = u'$.

The case that S is a singleton is trivial.

Assume now $u < v$ and $u' < v'$. Let $u \in R$. Then $\lambda_t^{R,S}(u) = u' \in S$ for any $t \in T$. Indeed, we again have $u \odot f = u$ and consequently $\lambda_t^{R,S}(u) \odot f = \lambda_t^{R,S}(u)$ for any $f \in F$.

(a), (c), (d), and the fact that τ, defined by (12), is sup-preserving follow from Proposition 4.2.

(b) Let $t \in T$. If $u \in R$, we have seen above that $\lambda_t^{R,S}(u) = u'$. If $u \notin R$, choose some $\tilde{r} \in R$; then $\bigwedge_{r \in R} \lambda_t^{R,S}(r) = \bigwedge_{f \in F} \lambda_t^{R,S}(\tilde{r} \odot f) = \bigwedge_{f \in F} (\lambda_t^{R,S}(\tilde{r}) \odot f) = \inf S = u'$.

(e) Let $u' \in S$ and $v \in R$. Then, for an arbitrary $\tilde{t} \in T$, $\lambda_{\tilde{t}}(v)$ and u' are both in the congruence class S, whose smallest element is u'. Thus, for some $f \in F$, we have $\lambda_{\tilde{t}}(v) \odot f = u'$, and consequently $\lambda_t^{R,S} = c^{R,u'}$, where $t = \tilde{t} \odot f \in T$.

Next, let $u' \in S$ and $v \notin R$. For any $t, t' \in T$ such that $t \sim_F t'$, we have $\lambda_t(v) \sim_F \lambda_{t'}(v)$. Consequently, either $\lambda_t(v) \in S$ for all $t \in T$, or $\lambda_t(v) \notin S$ for all $t \in T$. Furthermore, from $v \notin R$ it follows $v \odot f = v$ and thus $\lambda_t(v) \odot f = v \odot t \odot f = v \odot t = \lambda_t(v)$ for all $t \in T$ and $f \in F$. We conclude that, in the former case, $\lambda_t(v) = u'$ for any $t \in T$, that is, $\Lambda^{R,S} = \{c^{R,u'}\}$. In the latter case, $v' \leq \lambda_t(v) = \bigvee_{r \in R} \lambda_t^{R,S}(r) \leq v'$, that is, $\lambda_t(v) = v'$ for all $t \in T$, and $c^{R,u'} \notin \Lambda^{R,S}$.

Finally, let $v \notin R$ and $v' \in S$. Let $t \in T$. Then $\lambda_t(v) = \bigvee_{r \in R} \lambda_t^{R,S}(r) \in S$ and $\lambda_t(v) \odot f = v \odot t \odot f = v \odot t = \lambda_t(v)$ for any $f \in F$; thus $\lambda_t(v) = u' \in S$, that is, $\lambda_t^{R,S} = c^{R,u'}$, and we conclude again $\Lambda^{R,S} = \{c^{R,u'}\}$.

(f) Let $t \in T,\ f \in F$, and $r \in R$. We have $(\lambda_f^S \circ \lambda_t^{R,S})(r) = r \odot t \odot f = \lambda_{t \odot f}^{R,S}(r) = \lambda_{f \odot t}^{R,S}(r) = r \odot f \odot t = (\lambda_t^{R,S} \circ \lambda_f^R)(r)$.

Furthermore, $\tau(\lambda_f^T(t))(r) = \lambda_{t \odot f}^{R,S}(r) = r \odot t \odot f$, and also (13) follows. The proof of part (i) is complete.

(ii) Consider first the case that there is an $R' > R$ such that $R' \odot T = S$. Let $r \in R$, $t \in T$, and $r' \in R'$. Then $r < r' \odot f$ for any $f \in F$, and consequently $r \odot t \leq r' \odot f \odot t$ for any $f \in F$. As $r' \odot t \in S$, we conclude that $r \odot t$ is the smallest element of S, that is, $\lambda_t^{R,S}(r) = r \odot t = u'$, where $u' = \inf S \in S$.

Similarly, we argue in the case that there is a $T' > T$ such that $R \odot T' = S$. Let $r \in R$, $t \in T$, and $t' \in T'$. Then $t < t' \odot f$ for any $f \in F$, and consequently $r \odot t \leq r \odot t' \odot f$ for any $f \in F$. We conclude again that $u' = \inf S \in S$ and $\lambda_t^{R,S}(r) = r \odot t = u'$. $\square$

Referring to Figure 2, let us rephrase in an informal way the information that the above two lemmas provide. Let $\mathcal{L}$ be a q.n.c. tomonoid, F a filter of $\mathcal{L}$, and $\mathcal{P}$ the quotient of $\mathcal{L}$ by F.

In Figure 2, the two axes represent $\mathcal{L}$. The axes are partitioned into subintervals, the F-classes, each of which is an element of $\mathcal{P}$. Thus the Cayley tomonoid of $\mathcal{P}$ consists of mappings from subintervals to subintervals; accordingly, each translation is represented by rectangles and triangles. The identity on $\mathcal{P}$ is shown as well as the translation by an element T of $\mathcal{P}$.

The Cayley tomonoid of $\mathcal{L}$ is inserted into this picture as follows. Given an element T of $\mathcal{P}$, each translation $\lambda_t,\ t \in T$, traverses all rectangles and triangles of which the translation by T is composed. We may say that the latter "splits up" to the "bundle" of translations λ_t, $t \in T$.

Lemmas 4.6 and 4.7 describe the Cayley tomonoid of $\mathcal{L}$ by focusing separately on each triangular and rectangular section. Lemma 4.6 is concerned with the triangular sections. Part (i) is devoted to the set Λ^F of mappings from F to F. Λ^F, located in the uppermost triangle, is the Cayley tomonoid of F. Part (ii) describes the set Λ^R of mappings from R to R, where $R \in \mathcal{P} \backslash \{F\}$. Λ^R, located in the triangle associated with R, is a composition tomonoid, and there is a surjective homomorphism $\varrho\colon F \to \Lambda^R$. Finally, let $S = R \odot T$, where T is not the top element of $\mathcal{P}$; Lemma 4.7 is concerned with the set $\Lambda^{R,S}$ of mappings from R to S. If R, T is not $\odot$-maximal, $\Lambda^{R,S}$ is trivial by part (ii). Otherwise, part (i) applies: $\Lambda^{R,S}$, located in the rectangle associated with R and S, is then a toset of order-preserving mappings; and there is an order-preserving mapping $\tau\colon T \to \Lambda^{R,S}$ commuting with the action of F in the sense of (13).

We provide in the sequel several examples of t-norms illustrating these facts. We note that some definitions are involved. To keep them as short as possible, we will in general not provide full specifications, but assume commutativity to be used to cover all cases.

Example 4.8. *Let us consider the following t-norm:*

$$a \odot_5 b \;=\; \begin{cases} 4ab - 3a - 3b + 3 & \text{if } a, b > \frac{3}{4}, \\ 4ab - 3a - 2b + 2 & \text{if } \frac{1}{2} < a \le \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a - b + 1 & \text{if } \frac{1}{4} < a \le \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a & \text{if } a \le \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ 2ab - a - b + \frac{3}{4} & \text{if } \frac{1}{2} < a, b \le \frac{3}{4}, \\ ab - \frac{1}{2}a - \frac{1}{4}b + \frac{1}{8} & \text{if } \frac{1}{4} < a \le \frac{1}{2} \text{ and } \frac{1}{2} < b \le \frac{3}{4}, \\ 0 & \text{if } a \le \frac{1}{4} \text{ and } \frac{1}{2} < b \le \frac{3}{4}, \text{ or } a, b \le \frac{1}{2}. \end{cases} \tag{14}$$

$\odot_5$ *is a modification of a t-norm defined by Hájek in* [17]. *The t-norm monoid* $([0,1]; \le, \odot_5, 1)$ *possesses the filter* $F = (\frac{3}{4}, 1]$*, and the* F*-classes are* $\{0\}$, $(0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, *and* $(\frac{3}{4}, 1]$. *The quotient by* F *is isomorphic to* L_5*, the five-element Łukasiewicz chain. An illustration can be found in Figure* 3.

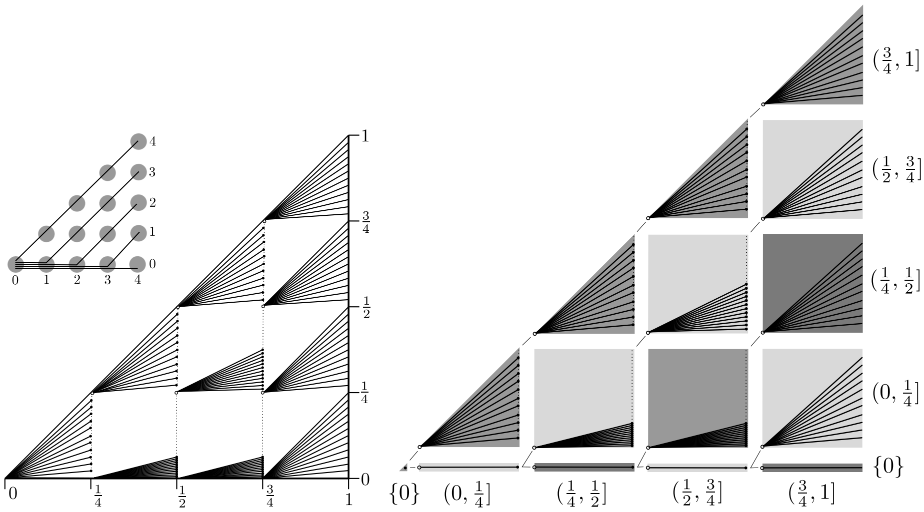

Figure 3: The Cayley tomonoids of the t-norm monoid $([0,1]; \le, \odot_5, 1)$ and its five-element quotient L_5 is shown on the left. On the right, we find an "exploded view" of the Cayley tomonoid of $([0,1]; \le, \odot_5, 1)$; for better visibility, the congruence classes are separated by margins.

5 The chain of quotients of a tomonoid

Each filter of a q.n.c. tomonoid induces a quotient. We now turn to the question of what we can say about the collection of quotients as a whole.

Each filter of a q.n.c. tomonoid $\mathcal{L}$ is of the form $d^{\le} = [d, 1]$ or $d^{<} = (d, 1]$, where d is its boundary; it follows that for any two filters, one is included in the other one. In other words, the set of all filters is totally ordered by set-theoretical inclusion.

Definition 5.1. Let $\mathcal{L}$ be a q.n.c. tomonoid. We denote the set of all filters of $\mathcal{L}$ by $\mathcal{F}$, and we endow $\mathcal{F}$ with the set-theoretical inclusion $\subseteq$ as a total order.

We next note that we could regard tomonoids also as algebras; we could replace the total order relation by the lattice operations. Then the following lemma would be a corollary of the Second Isomorphism Theorem of Universal Algebra; see, e.g., [4].

Lemma 5.2. *Let $\mathcal{L}$ be a q.n.c. tomonoid, and let F and G be filters of $\mathcal{L}$ such that $F \subseteq G$. Then $\langle G \rangle_F$ is a filter of $\langle \mathcal{L} \rangle_F$, and $\langle \mathcal{L} \rangle_G$ is isomorphic to the quotient of $\langle \mathcal{L} \rangle_F$ by $\langle G \rangle_F$.*

Proof. We claim that, for $a, b \in \mathcal{L}$, $a \sim_G b$ if and only if $\langle a \rangle_F \sim_{\langle G \rangle_F} \langle b \rangle_F$. Indeed, assume $a \leq b$; then $a \sim_G b$ if and only if there a $g \in G$ such that $b \odot g \leq a$. Since F is a filter contained in G, the latter holds if and only if there are a $g \in G$ and an $f \in F$ such that $b \odot g \odot f \leq a$ if and only if $\langle b \odot g \rangle_F \leq \langle a \rangle_F$ for some $g \in G$ if and only if $\langle b \rangle_F \odot \langle g \rangle_F \leq \langle a \rangle_F$ for some $\langle g \rangle_F \in \langle G \rangle_F$ if and only if $\langle a \rangle_F \sim_{\langle G \rangle_F} \langle b \rangle_F$.

It follows that we can define

$$\varphi\colon \langle \mathcal{L} \rangle_G \to \langle \langle \mathcal{L} \rangle_F \rangle_{\langle G \rangle_F}, \quad \langle a \rangle_G \mapsto \langle \langle a \rangle_F \rangle_{\langle G \rangle_F}$$

and that φ is a bijection. Moreover, φ preserves $\odot$ and is an order-isomorphism. The lemma follows. $\square$

Lemma 5.2 is the basis of our loose statement that a q.n.c. tomonoid is the result of a linear construction process. In general, this process does not proceed in a step-wise fashion. But we can speak about a single step if there is a pair of successive filters; let us consider this case now.

For an element a of a tomonoid and $n \geq 1$, we write a^n for $a \odot \ldots \odot a$ (n factors).

Definition 5.3. A q.n.c. tomonoid $\mathcal{L}$ is called *Archimedean* if, for each $a, b \in \mathcal{L}$ such that $a < b < 1$, we have $b^n \leq a$ for some $n \geq 1$.

An extension of a q.n.c. tomonoid by an Archimedean tomonoid is called *Archimedean*.

We recall in this context that two elements a, b of a negative, commutative tomonoid $\mathcal{L}$ are called *Archimedean equivalent* if $a = b$, or $a < b$ and there is an $n \geq 1$ such that $b^n \leq a$, or $b < a$ and there is an $n \geq 1$ such that $a^n \leq b$. This is an equivalence relation whose classes are called the *Archimedean classes* of $\mathcal{L}$. Note then that $\mathcal{L}$ is Archimedean if and only if $\mathcal{L}$ possesses at most *two* Archimedean classes. In fact, the top element alone always forms one class, and $\mathcal{L}$ is Archimedean exactly if the remaining elements form one further class. See, e.g., [13].

For two filters $F, G \in \mathcal{F}$, we will write $F \subset\!\!\cdot\; G$ to express that G is the immediate successor of F, that is, G is the next smallest filter to F. We can formulate a first description of the chain of quotients as follows.

Theorem 5.4. *Let $\mathcal{L}$ be a q.n.c. tomonoid. Then we have:*

(i) *The largest and smallest elements of $\mathcal{F}$ are $\mathcal{L}$ and $\{1\}$, respectively. Moreover, $\langle\mathcal{L}\rangle_{\mathcal{L}}$ is the trivial tomonoid, and $\langle\mathcal{L}\rangle_{\{1\}}$ is isomorphic to $\mathcal{L}$.*

(ii) *For each $F \in \mathcal{F}\backslash\{\mathcal{L}\}$ such that F is not an immediate predecessor, $\sim_F = \bigcap_{G \supset F} \sim_G$.*

(iii) *For each $F \in \mathcal{F}\backslash\{\{1\}\}$ such that F is not an immediate successor, $\sim_F = \bigcup_{G \subset F} \sim_G$.*

(iv) *For each $F, G \in \mathcal{F}$ such that $F \subset\!\!\cdot\; G$, $\langle\mathcal{L}\rangle_F$ is an Archimedean extension of $\langle\mathcal{L}\rangle_G$.*

Proof. (i) The largest filter is $\mathcal{L}$, and the quotient $\langle\mathcal{L}\rangle_{\mathcal{L}}$ is one-element, that is, trivial. The smallest filter is $\{1\}$, and the quotient $\langle\mathcal{L}\rangle_{\{1\}}$ has singleton classes only, that is, coincides with $\mathcal{L}$.

(ii) Let $F \in \mathcal{F}$ such that F is neither $\mathcal{L}$ nor the predecessor of another filter. As $\mathcal{F}$ is closed under arbitrary intersections, we then have $F = \bigcap_{G \supset F} G$. Let $a, b \in \mathcal{L}$ such that $a \leq b$. We have to show that $a \sim_F b$ if and only if, for each $G \supset F$, $a \sim_G b$. Clearly, $a \sim_F b$ implies $a \sim_G b$ for each $G \supset F$. Conversely, assume $a \sim_G b$ for each $G \supset F$. Then for each $G \supset F$ there is a $g_G \in G$ such that $b \odot g_G \leq a$. It follows $b \odot f \leq a$, where $f = \bigvee_{G \supset F} g_G \in F$, hence $a \sim_F b$.

(iii) Let $F \in \mathcal{F}$ such that F is neither $\{1\}$ nor the successor of another filter. As $\mathcal{F}$ is closed under arbitrary unions, $F = \bigcup_{G \subset F} G$ then. For $a \leq b$, we have $a \sim_F b$ if and only if $b \odot f \leq a$ for some $f \in F$ if and only if $b \odot f \leq a$ for some $f \in G$ such that $G \subset F$ if and only if $a \sim_G b$ for some $G \subset F$.

(iv) Let $F, G \in \mathcal{F}$ such that $F \subset\!\!\cdot\; G$. By Lemma 5.2, $\langle\mathcal{L}\rangle_G$ is then isomorphic to the quotient of $\langle\mathcal{L}\rangle_F$ by the filter $\langle G\rangle_F$.

Assume that $\langle G\rangle_F$ is not Archimedean. Then there is a filter H of $\langle G\rangle_F$ such that $\{\langle 1\rangle_F\} \subset H \subset \langle G\rangle_F$. But then $\bigcup H$ is a filter of $\mathcal{L}$ such that $F \subset \bigcup H \subset G$, a contradiction. $\square$

Theorem 5.4 involves the whole set of quotients induced by filters. Our next aim is to simplify the description by not taking every single filter into account, but to combine some of them into groups. Furthermore, we will work with the boundaries of the filters rather than the filters themselves.

Definition 5.5. Let $\mathcal{L}$ be a q.n.c. tomonoid. We denote by D the subset of $\mathcal{L}^0$ consisting of all filter boundaries, and we endow D with the total order of $\mathcal{L}^0$. Moreover, we denote by E the subset of D consisting of all idempotent elements of $\mathcal{L}$.

Lemma 5.6. *Let $\mathcal{L}$ be a q.n.c. tomonoid.*

(i) *D is a subset of $\mathcal{L}^0$ that contains 0 and 1 and is closed under arbitrary infima and suprema. If $d \neq 1$ and $d = \bigwedge\{d' \in D : d' > d\}$, then $d^<$ is a filter. If $d \neq 0$ and $d = \bigvee\{d' \in D : d' < d\}$, then $d \in E$ and $d^{\leq}$ is a filter.*

(ii) *Each simple boundary d such that $d^<$ is a filter possesses an immediate predecessor in D.*

(iii) *E is a subset of $\mathcal{L}$ that contains 1 and is closed under arbitrary suprema.*

Proof. (i) $\mathcal{L}$ is a filter and the infimum of $\mathcal{L}$ in $\mathcal{L}^0$ is 0, the bottom element of $\mathcal{L}^0$; hence $0 \in D$. Furthermore, $\{1\}$ is a filter; hence $1 \in D$.

The intersection as well as the union of a set of filters is again a filter. In particular, the infima and suprema of arbitrary sets of boundaries are again boundaries.

Assume next that d_ι, $\iota \in I$, and d are filter boundaries such that $d_\iota < d$ and $\bigvee_\iota d_\iota = d$. Then $d_\iota \leq d \odot d$ for each ι, hence $d = \bigvee_\iota d_\iota \leq d \odot d \leq d$, and it follows $d \in E$.

(ii) Let d be such that $(d, 1]$ is a filter but $[d, 1]$ is not. Let F be the filter generated by $[d, 1]$ (that is, the smallest filter containing $[d, 1]$). Then F is the smallest filter properly containing $(d, 1]$, and the boundary d' of F is strictly smaller than d. Thus d' is the immediate predecessor of d in D.

(iii) Clearly, 1 is idempotent. Moreover, $E \subseteq D$ by Lemma 3.3(i). Hence it follows from part (i) that the supremum of a set of idempotent boundaries is again idempotent. □

We first observe that at double boundaries, extensions by the smallest non-trivial tomonoid occur. Note that there is only one two-element q.n.c. tomonoid.

Lemma 5.7. *Let $\mathcal{L}$ be a q.n.c. tomonoid. Let d be a double boundary. Then $\langle \mathcal{L} \rangle_{d^<}$ is an extension of $\langle \mathcal{L} \rangle_{d^\leq}$ by the two-element tomonoid.*

Proof. $\langle d^\leq \rangle_{d^<}$ is two-element; thus the claim follows from Lemma 5.2. □

We next associate with each filter boundary only one filter.

Definition 5.8. Let $\mathcal{L}$ a q.n.c. tomonoid. For each $d \in D$, let $F(d) = d^\leq$ if $d \in E$ and otherwise $F(d) = d^<$. Furthermore, let $\sim_d = \sim_{F(d)}$ and $\mathcal{L}_d = \langle \mathcal{L} \rangle_{F(d)}$.

Let us now consider the immediate successor relation among filter boundaries.

Definition 5.9. Let $c, d \in D$ such that $c < d$ and there is no element of D strictly between c and d. Then we say that $\langle c, d \rangle$ is a pair of *successive boundaries*.

For successive boundaries $\langle c, d \rangle$, $F(c)$ is either the smallest filter properly containing $F(d)$ or otherwise the second smallest filter properly containing $F(d)$. In the former case, the extension is Archimedean. In the latter case, an Archimedean extension follows an extension by the two-element tomonoid, and we shall combine these two extensions into a single one.

Definition 5.10. A q.n.c. tomonoid $\mathcal{L}$ is called *quasi-Archimedean* if $\mathcal{L}$ possesses a bottom element 0 and $\mathcal{L} \backslash \{0\}$ is an Archimedean subtomonoid of $\mathcal{L}$.

An extension of a q.n.c. tomonoid by a quasi-Archimedean tomonoid is called *quasi-Archimedean*.

In a quasi-Archimedean q.n.c. tomonoid $\mathcal{L}$, the 0 and the 1 elements alone form two Archimedean classes, and there is at most one further Archimedean class. We note that a t-norm is commonly called Archimedean if the t-norm monoid based on it is, according to our terminology, Archimedean or quasi-Archimedean [20].

Lemma 5.11. *Let $\mathcal{L}$ be a q.n.c. tomonoid and let $\langle c, d\rangle$ be a pair of successive boundaries. If then c is a single boundary, $\mathcal{L}_d$ is an Archimedean extension of $\mathcal{L}_c$. If c is a double boundary, $\mathcal{L}_d$ is a quasi-Archimedean extension of $\mathcal{L}_c$.*

Proof. Let c be a single boundary. Then $F(d) \subset\!\!\cdot\; F(c)$, thus $\mathcal{L}_d$ is an Archimedean extension of $\mathcal{L}_c$ by Theorem 5.4(iv).

Let c be a double boundary. By Lemma 5.2, $\mathcal{L}_d$ is an extension of $\mathcal{L}_c$ by $\langle F(c)\rangle_{F(d)}$. Furthermore, we have $F(d) \subset\!\!\cdot\; (c, 1] \subset\!\!\cdot\; [c, 1] = F(c)$. From the fact that $(c, 1]$ is a filter, it follows that $\langle F(c)\rangle_{F(d)}$ and $\langle (c, 1]\rangle_{F(d)}$ are filters differing only by the bottom element $\langle c\rangle_{F(d)} = \{c\}$. Moreover, $\langle (c, 1]\rangle_{F(d)}$ is Archimedean, hence $\langle F(c)\rangle_{F(d)}$ is quasi-Archimedean. □

We now turn to the case that an interval of a q.n.c. tomonoid consists of idempotents. We will associate with such an interval a single extension.

Definition 5.12. Let $\mathcal{L}$ be a q.n.c. tomonoid, and let J be a maximal interval of $\mathcal{L}$ contained in E. If J is not a singleton, J is called an *interval of idempotents*.

Since the set of idempotents is closed under suprema and the set of boundaries is closed under infima, an interval of idempotents is of the form $[d, e]$ or $(d, e]$ for some $d \in D$ and $e \in E$ such that $d < e$. It is moreover clear that two intervals of idempotents are disjoint and do not even share a boundary.

Definition 5.13. A q.n.c. tomonoid $(\mathcal{L}; \leq, \odot, 1)$ is called a *semilattice* if $a \odot b = a \wedge b$ for all $a, b \in \mathcal{L}$.

An extension of a q.n.c. tomonoid by a semilattice is called a *semilattice* extension.

Lemma 5.14. *Let $\mathcal{L}$ be a q.n.c. tomonoid. Let $(d, e]$ be an interval of idempotents. Then $\mathcal{L}_e$ is an extension of $\mathcal{L}_d$ such that the extending filter is isomorphic to the semilattice $((d, e]; \wedge, \leq, e)$.*

Similarly, let $[d, e]$ be an interval of idempotents. Then $\mathcal{L}_e$ is an extension of $\mathcal{L}_d$ such that the extending filter is isomorphic to the semilattice $([d, e]; \wedge, \leq, e)$.

Proof. We recall first that the multiplication of an idempotent with a larger element results in the idempotent. Indeed, if e is idempotent and $e \leq a$, then $e = e \odot e \leq e \odot a \leq e \odot 1 = e$ and hence $e \odot a = e$.

Assume now that $(d, e]$ is an interval of idempotents. By Lemma 5.2, $\mathcal{L}_d$ is isomorphic to the quotient of $\mathcal{L}_e$ by $\langle d^<\rangle_{e\leq}$. By Lemma 3.9(i), $\langle d^<\rangle_{e\leq}$ is in turn isomorphic to $\{a \odot e \colon a > d\} = (d, e]$, endowed with $\leq$ and $\odot$ restricted to $(d, e]$, and with the constant e. For $a, b \in (d, e]$, we have $a \odot b = a \wedge b$; hence $(d, e]$ is a semilattice.

The second part is shown similarly. □

We now refine Theorem 5.4 using a restricted set of filters.

Definition 5.15. Let $\mathcal{L}$ be a q.n.c. tomonoid. Let C be the set of all filter boundaries d such that if d belongs to an interval J of idempotents, either $d = \inf J$ or $d = \sup J$.

We note that Lemma 5.6(i) holds for C as well.

Lemma 5.16. *Let $\mathcal{L}$ be a q.n.c. tomonoid. Then C is a subset of $\mathcal{L}^0$ that contains 0 and 1 and is closed under arbitrary infima and suprema. If $d \neq 1$ and $d = \bigwedge\{d' \in C \colon d' > d\}$, then $d^<$ is a filter. If $d \neq 0$ and $d = \bigvee\{d' \in C \colon d' < d\}$, then $d^\leq$ is a filter.*

Proof. Clearly, $0, 1 \in C$. Furthermore, let d be the infimum, but not an element, of a subset of C. Then $d \in D$ by Lemma 5.6(i). Assume now that d belongs to the interval J of idempotents. Then, for each $c \in C$, we have $c \leq \inf J \leq d$ or $\sup J \leq c$; since by assumption $d = \inf\{c \in C \colon d < c\}$, it follows $d = \sup J$, that is, $d \in C$. Similarly, we argue for suprema in C.

For the rest, we argue as in the proof of Lemma 5.6(i). □

Theorem 5.17. *Let $\mathcal{L}$ be a q.n.c. tomonoid.*

(i) *$\mathcal{L}_0$ is the trivial tomonoid, and $\mathcal{L}_1$ is isomorphic to $\mathcal{L}$.*

(ii) *Let $c, d \in C$ such that c is a single boundary and $\langle c, d\rangle$ is a pair of successive boundaries. Then $\mathcal{L}_d$ is an Archimedean extension of $\mathcal{L}_c$.*

(iii) *Let $c, d \in C$ such that c is a double boundary and $\langle c, d\rangle$ is a pair of successive boundaries. Then $\mathcal{L}_d$ is a quasi-Archimedean extension of $\mathcal{L}_c$.*

(iv) *Let $d, e \in C$ such that $(d, e]$ is an interval of idempotents. Then $\mathcal{L}_e$ is a semilattice extension of $\mathcal{L}_d$, the extending semilattice being isomorphic to $((d, e]; \leq, \wedge, e)$.*

(v) *Let $d, e \in C$ such that $[d, e]$ is an interval of idempotents. Then $\mathcal{L}_e$ is a semilattice extension of $\mathcal{L}_d$, the extending semilattice being isomorphic to $([d, e]; \leq, \wedge, e)$.*

(vi) *Let $d \in C \backslash \{0\}$ such that $d = \bigvee_{c \in C,\, c<d} c$. Then $\sim_d = \bigcap_{c \in C,\, c<d} \sim_c$.*

(vii) *Let $d \in C \backslash \{1\}$ be a single boundary such that $d = \bigwedge_{c \in C,\, c>d} c$. Then $\sim_d = \bigcup_{c \in C,\, c>d} \sim_c$.*

(viii) *Let $d \in C \backslash \{1\}$ be a double boundary such that $d = \bigwedge_{c \in C,\, c>d} c$. Then $\sim_{d^<} = \bigcup_{c \in C,\, c>d} \sim_c$. Moreover, $\langle \mathcal{L} \rangle_{d^<}$ is an extension of $\mathcal{L}_d$ by the two-element tomonoid.*

Proof. (i) $F(0) = \mathcal{L}$ and $F(1) = \{1\}$, thus the claim holds by Theorem 5.4(i).

(ii), (iii) hold by Lemma 5.11.

(iv), (v) hold by Lemma 5.14.

(vi) By Lemma 5.16, $F(d) = d^{\leq} = \bigcap_{c \in C,\ c<d} F(c)$. Thus the proof can be done as in Theorem 5.4(ii).

(vii) $d^{<}$ is a filter by Lemma 5.16, and $F(d) = d^{<} = \bigcup_{c \in C,\ c>d} F(c)$. Thus the proof can be done as in Theorem 5.4(iii).

(viii) Again, $d^{<} = \bigcup_{c \in C,\ c>d} F(c)$; thus the first claim is proved like part (vii). The second claim holds by Lemma 5.7. □

We now discuss a number of examples to illustrate Theorem 5.17. In line with our objectives, all examples are t-norm monoids.

Example 5.18. *We first review the three standard t-norms, defined by* (1)–(3)*. The associated Cayley tomonoids are depicted in Figure* 1*.*

The first t-norm monoid is $([0,1]; \leq, \odot_1, 1)$*, where* $\odot_1$ *is the Łukasiewicz t-norm. It has the filters* $0^{\leq}$ *and* $1^{\leq}$ *and does not possess any filter apart from the trivial and the improper one. Consequently, also the set of filter boundaries is the smallest possible one:* $D = \{0,1\}$*. Moreover,* 0 *is a simple boundary. Thus, according to Theorem* 5.17(ii)*, this t-norm monoid is an Archimedean extension of the trivial tomonoid. In fact,* $([0,1]; \leq, \odot_1, 1)$ *is an Archimedean tomonoid.*

In the case of the t-norm monoid that is based on the product t-norm $\odot_2$*, we again have* $D = \{0,1\}$*. This time, however,* 0 *is a double boundary; there is one proper, non-trivial filter, namely,* $0^{<}$*. Thus* $([0,1]; \leq, \odot_2, 1)$ *is a quasi-Archimedean extension of the trivial tomonoid. The intermediate step leads to the two-element tomonoid; the* $0^{<}$*-classes are* $\{0\}$ *and* $(0,1]$*.*

The third t-norm is the Gödel t-norm $\odot_3$*. Here, every element is idempotent and thus a filter boundary;* $D = E = [0,1]$*. Moreover, each boundary apart from* 1 *is a double boundary. We have* $C = \{0,1\}$*, and according to Theorem* 5.17(v)*, this t-norm monoid is a semilattice extension of the trivial tomonoid. In fact,* $([0,1]; \leq, \odot_3, 1)$ *is a semilattice.*

We continue providing two less trivial examples.

Example 5.19. *The nilpotent minimum t-norm* $\odot_4$ *is defined by* (4)*; the associated Cayley tomonoid is depicted in Figure* 4 *(left).*

Here, the filter boundaries are $D = \{0\} \cup [\frac{1}{2}, 1]$*, and the idempotents are* $E = \{0\} \cup (\frac{1}{2}, 1]$*. For each* $\frac{1}{2} < d < 1$*, the* $d^{\leq}$*-classes are* $[0, 1-d]$*, the singletons* $\{a\}$ *for each* $1-d < a < d$*, and* $[d,1]$*. Similarly, the* $d^{<}$*-classes are* $[0, 1-d)$*, the singletons* $\{a\}$ *for each* $1-d \leq a \leq d$*, and* $(d,1]$*. Furthermore,* $\frac{1}{2}$ *is a simple boundary, and the* $\frac{1}{2}^{<}$*-classes are* $[0, \frac{1}{2})$*,* $\{\frac{1}{2}\}$*, and* $(\frac{1}{2}, 1]$*.*

In this case, we have $C = \{0, \frac{1}{2}, 1\}$*. The quotient* $[0,1]_{\frac{1}{2}} = \langle [0,1] \rangle_{\frac{1}{2}^{<}}$ *is isomorphic to* L_3*, the three-element Łukasiewicz chain, and thus a finite Archimedean extension of the trivial tomonoid. Moreover,* $([0,1]; \leq, \odot_4, 1)$ *itself is a semilattice extension of* L_3*, the semilattice being* $((\frac{1}{2}, 1]; \leq, \wedge, 1)$*.*

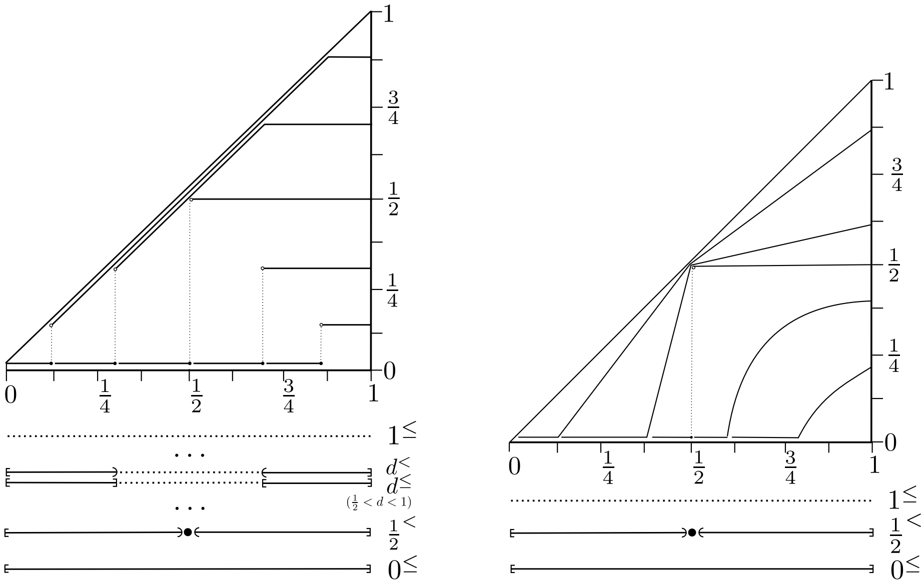

Figure 4: Left: The nilpotent minimum t-norm $\odot_4$. Right: The rotated product t-norm, $\odot_6$. The lower parts show, respectively, the filters together with the classes of the induced quotients; a dotted line denotes singleton classes.

Example 5.20. *Our last example is the so-called rotated product t-norm* [18]*; cf. Figure* 4 *(right). Let*

$$a \odot_6 b \;=\; \begin{cases} 2ab - a - b + 1 & \text{if } a, b > \frac{1}{2}, \\ \frac{a+b-1}{2b-1} & \text{if } a \leq \frac{1}{2},\ b > \frac{1}{2}, \text{ and } a + b > 1, \\ 0 & \text{if } a + b \leq 1. \end{cases}$$

Note that only one translation is not continuous, namely $\lambda_{\frac{1}{2}}$, which has only one point of discontinuity, namely $\frac{1}{2}$.

The t-norm monoid based on $\odot_6$ has one non-trivial, proper filter, $\frac{1}{2}^{<}$, hence we have $C = D = \{0, \frac{1}{2}, 1\}$. The quotient $[0,1]_{\frac{1}{2}} = \langle [0,1] \rangle_{\frac{1}{2}^{<}}$ is again isomorphic to L_3. Moreover, the t-norm monoid itself is an Archimedean extension of L_3, the extending filter being isomorphic to the left-open real interval endowed with the multiplication of reals.

Example 5.20 shows that the natural homomorphism from a tomonoid to a quotient induced by a filter does not in general preserve suprema. The "problem" is the presence of a congruence class of the form of a right-open interval. In fact, the congruence induced by $\frac{1}{2}^{<}$ has the classes $[0, \frac{1}{2})$, $\{\frac{1}{2}\}$, and $(\frac{1}{2}, 1]$; we have $\sup\, [0, \frac{1}{2}) = \frac{1}{2} \notin [0, \frac{1}{2})$. This situation can also be seen from the fact that $\Lambda^{(\frac{1}{2},1],[0,\frac{1}{2})}$ does not contain its supremum.

6 Standard Archimedean extensions

The objective of this paper is to better understand the structure of q.n.c. tomonoids in general and of t-norm monoids in particular. So far, we have dealt with the chain of quotients induced by filters. This way we might have gained the idea that a q.n.c. tomonoid arises from a sequence of extensions, and the length of this sequence depends on the number of filters. If the chain of filters has a simple structure, as for instance in the case that there are only finitely many filters, this idea is actually accurate. In general, however, we must be aware of the fact that the chain that can have a complicated structure; we may think, for instance, of the Cantor set.

In the present section, we have a closer look at a well-behaved case. We consider again successive elements of the chain of quotients. We have seen that extensions are in this case Archimedean. We will establish that, under an additional assumption, such extensions can be described in a systematic way.

In what follows, a *real interval* is a set of the form (a, b), $[a, b)$, $(a, b]$ for some $a, b \in \mathbb{R}$ such that $a < b$, or $[a, b]$ for some $a, b \in \mathbb{R}$ such that $a \leq b$. A real interval not consisting of only one element is called *proper*.

Definition 6.1. Let $\mathcal{P}$ be the quotient of the q.n.c. tomonoid $\mathcal{L}$ by an Archimedean filter such that the following condition holds:

(E) Each congruence class is order-isomorphic to a real interval.

Then we call $\mathcal{P}$ a *standard Archimedean quotient* of $\mathcal{L}$, and we call $\mathcal{L}$ a *standard Archimedean extension* of $\mathcal{P}$.

Given a q.n.c. tomonoid $\mathcal{P}$, we are going to determine the standard Archimedean extensions $\mathcal{L}$ of $\mathcal{P}$. Lemmas 4.6 and 4.7 provide us the guidelines. Namely, we will specify sectionwise the Cayley tomonoid Λ of $\mathcal{L}$; for each pair R and S of congruence classes, we specify the elements of Λ restricted in domain to R and in range to S.

Let us recall from Section 4 which constituents are needed to describe Λ; cf. Figure 5. We will use the same notation. Providing in each line the reference to the relevant theorem or proposition following below, the listing may serve as a guide through this section.

- The Cayley tomonoid Λ^F of the extending filter F: Proposition 6.3.
- For each $R \in \mathcal{P}$ such that $R < F$,
  - the composition tomonoid Λ^R: Theorem 6.6;
  - and the epimorphism $F \to \Lambda^R,\ \ f \mapsto \lambda_f^R$: Proposition 6.7.
- For each pair $R, S \in \mathcal{P}$ such that $S = R \odot T$ for some $T \in \mathcal{P} \backslash \{F\}$,
  - the set of mappings $\Lambda^{R,S}$: Proposition 6.8;
  - and the mapping $T \to \Lambda^{R,S},\ \ t \mapsto \lambda_t^{R,S}$: Proposition 6.9.

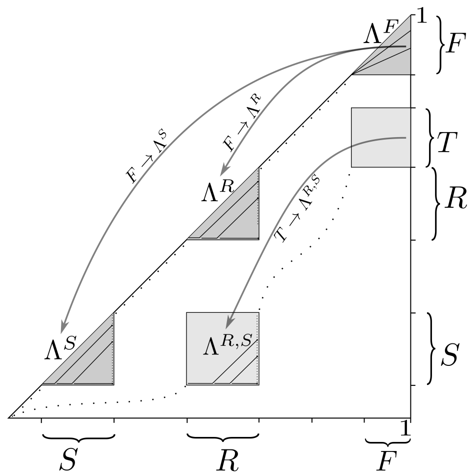

Figure 5: The elements needed to determine the Cayley tomonoid of a standard Archimedean extension.

We need a few additional notions. A *non-minimal* element in a toset A is meant to be any $x \in A$ if A does not possess a smallest element, and any $x \in A \backslash \{u\}$ if A has the smallest element u. Furthermore, let χ be an order-preserving mapping from A to another toset B. Then we call $\{x \in A\colon \chi(x)$ is non-minimal in $B\}$ the *support* of χ. Note that the support of χ is the whole set A if B does not possess a smallest element, and that the support of χ is empty if and only if B possesses a smallest element u and $\chi = \mathrm{c}^{A,u}$.

Our first step is to specify the extending filters for standard Archimedean extensions.

Definition 6.2. The tomonoid $([0,1]; \leq, \odot_1, 1)$, where $\odot_1$ is defined by (1), is called the *Łukasiewicz tomonoid*.

The tomonoid $((0,1]; \leq, \odot_2, 1)$, where $\odot_2$ is the real multiplication – cf. (2) –, is called the *product tomonoid*.

Proposition 6.3. *Let $\mathcal{P}$ be a standard Archimedean quotient of the q.n.c. tomonoid $\mathcal{L}$ by the filter F. Then $(F; \leq, \odot, 1)$ is isomorphic either to the Łukasiewicz tomonoid or to the product tomonoid.*

Proof. By Lemma 4.6(i) and condition (E), F is order-isomorphic to a right-closed real interval.

Assume first that F possesses a smallest element. Then $(F; \leq, \odot, 1)$ is a q.n.c. tomonoid such that F is order-isomorphic to the real unit interval; that is, F is isomorphic to a t-norm monoid. Let $\odot$ be the t-norm. By assumption, the t-norm monoid is Archimedean. By [20, Proposition 2.16], $\odot$ is continuous, and the Archimedean property implies that $\odot$ is isomorphic to the Łukasiewicz t-norm.

Assume second that F does not have a smallest element. Then $(F^0; \leq, \odot, 1)$ is a q.n.c. tomonoid such that F^0 is order-isomorphic to the real unit interval; that is, F^0 is isomorphic to a t-norm monoid. Let $\odot$ be the t-norm. The t-norm monoid is quasi-Archimedean. Again, by [20, Proposition 2.16], $\odot$ is continuous, and the quasi-Archimedean property implies that $\odot$ is isomorphic to the product t-norm. $\square$

Our next aim is to specify the sets of mappings Λ^R, where R is any element of a standard Archimedean quotient. The subsequent lemma serves as a preparation.

For tosets A and B that are order-isomorphic to real intervals, *continuity* of a mapping from A to B will have the obvious meaning.

Lemma 6.4. *Let $\mathcal{P}$ be a standard Archimedean quotient of the q.n.c. tomonoid $\mathcal{L}$. Let $R \in \mathcal{P}$, and assume that R is not a singleton. Then $(\Lambda^R; \leq, \circ, \mathrm{id}_R)$ is a composition tomonoid on R fulfilling* (C1)–(C4). *Moreover, the following holds:*

(C6) *Any $\lambda \in \Lambda^R$ is continuous.*

(C7) *For any $\lambda \in \Lambda^R \backslash \{id_R\}$ and any non-minimal element r of R, $\lambda(r) < r$.*

(C8) *Λ^R is order-isomorphic to one of the real intervals $(0, 1]$ or $[0, 1]$. Moreover, Λ^R is order-isomorphic to $[0, 1]$ if and only if R is itself isomorphic to a closed real interval.*

Proof. By condition (E), R is order-isomorphic to a real interval. Furthermore, by Proposition 6.3, the extending filter F is isomorphic to the Łukasiewicz or to the product tomonoid.

By Lemma 4.6, Λ^R fulfils (C1)–(C4). Thus we only have to prove (C6)–(C8). We will first show (C7) as well as a strengthened form of (C7), then (C6), and finally (C8).

(C7) Let $f \in F \backslash \{1\}$ and let $r \in R$. Assume that $\lambda_f^R(r) = r \odot f = r$. Then $r \odot f^n = r$ for any $n \geq 1$, and since F is Archimedean, it follows that $r \odot g = r$ for all $g \in F$; thus r is the smallest element of the congruence class R. We conclude that if r is not the smallest element of R, then $\lambda_f^R(r) < r$.

We next prove:

($\star$) For any $\lambda \in \Lambda^R \backslash \{id_R\}$ and any $r \in R$ that is neither the smallest nor the largest element of R, $\bigwedge_{x>r} \lambda(x) < r$.

Let $f \in F \backslash \{1\}$ and let $r \in R$ be neither the smallest nor the largest element of R. As F is isomorphic to the Łukasiewicz or to the product tomonoid, there is a $g \in F$ such that $f \leq g^2 < g < 1$. Assume that $\lambda_g^R(x) = x \odot g > r$ for all $x \in R$ such that $x > r$; then $x \odot g^n > r$ for any $n \geq 1$, and since F is Archimedean, it further follows $x \odot h > r$ for all $h \in F$, in contradiction to the fact that x and r are in the same congruence class R. Hence there is an $x \in R$ such that $x > r$ and $\lambda_g^R(x) = x \odot g \leq r$. As r is non-minimal and λ_g is not the identity, we conclude by (C7) that $\lambda_f^R(x) = x \odot f \leq x \odot g \odot g \leq r \odot g < r$. The proof of ($\star$) is complete.

(C6) Let $f \in F$ and assume that λ^R_f is discontinuous at $r \in R$. Note that then $f < 1$ and r is neither the smallest nor the largest element of R. Let $p = \lambda^R_f(r)$ and $q = \bigwedge_{x>r} \lambda^R_f(x)$; then $p < q < r$ by ($\star$). By (C4) and (C7), we may choose a $\lambda \in \Lambda^R$ such that $p < \lambda(q) < q$ and $q < \lambda(r) < r$. By ($\star$), there is an $x > r$ such that $\lambda(x) \leq r$. Then $\lambda^R_f(\lambda(x)) \leq \lambda^R_f(r) = p$ and $\lambda(\lambda^R_f(x)) \geq \lambda(q) > p$, a contradiction.

(C8) Let $f, g \in F$ be such that $f < g$ and λ^R_g has a non-empty support. Let $h \in F$ be such that $f = g \odot h$; then $h < 1$. We conclude from (C7) that $\lambda^R_f = \lambda^R_h \circ \lambda^R_g < \lambda^R_g$.

Consider the case that all λ^R_f, $f \in F$, have a non-empty support. Then $\varrho\colon F \to \Lambda^R,\ \ f \mapsto \lambda^R_f$ is injective and consequently bijective. Moreover, $\lambda^R_f \circ \lambda^R_f < \lambda^R_f$ for all $f \in F\backslash\{1\}$, that is, Λ^R does not possess a smallest element. As ϱ is an order isomorphism, also F does not possess a smallest element. Thus F is isomorphic to the product tomonoid, and we conclude that Λ^R is order-isomorphic to the real interval $(0, 1]$.

Consider next the case that there is an $f \in F$ such that λ^R_f has an empty support. Then $u = \inf R \in R$ and $\lambda^R_f = c^{R,u} \in \Lambda^R$. By Lemma 4.6(ii), ϱ is sup-preserving; hence there is a largest $d \in F$ such that $\lambda^R_d = c^{R,u}$. Note that $d < 1$ then. We conclude that ϱ is bijective when restricted to $[d, 1]$, and Λ^R is order-isomorphic to the real interval $[0, 1]$.

Finally, by Lemma 4.6(ii)(d), $u = \inf R \in R$ and $c^{R,u} \in \Lambda^R$ if and only if R is of the form $[u, v]$. The second part of (C8) follows. □

We will see that a composition tomonoid Λ^R is isomorphic to one of the following four.

Definition 6.5. (i) Let Φ consist of the functions $\lambda_t\colon [0, 1] \to [0, 1],\ \ x \mapsto (x + t - 1) \vee 0$ for each $t \in [0, 1]$. Then $(\Phi; \leq, \circ, id_{[0,1]})$ is called the *Łukasiewicz composition tomonoid*.

(ii) Let Φ consist of the functions $\lambda_t\colon (0, 1] \to (0, 1],\ \ x \mapsto t \cdot x$ for each $t \in (0, 1]$. Then $(\Phi; \leq, \circ, id_{(0,1]})$ is called the *product composition tomonoid*.

(iii) Let Φ consist of the functions $\lambda_t\colon [0, 1) \to [0, 1),\ \ x \mapsto \frac{(x+t-1)\vee 0}{t}$ for each $t \in (0, 1]$. Then $(\Phi; \leq, \circ, id_{[0,1)})$ is called the *reversed product composition tomonoid*.

(iv) Let Φ consist of the functions $\lambda_t\colon (0, 1) \to (0, 1),\ \ x \mapsto x^{\frac{1}{t}}$ for each $t \in (0, 1]$. Then $(\Phi; \leq, \circ, id_{(0,1)})$ is called the *power composition tomonoid*.

A composition tomonoid on a toset R that is isomorphic to one of these four will be called a *standard composition tomonoid*.

A graphical representation of the four standard composition tomonoids is given in Figure 6. Note that the key property in which they differ is their base set – the real unit interval with, without the left, right margin.

The following theorem improves a central result of [26].

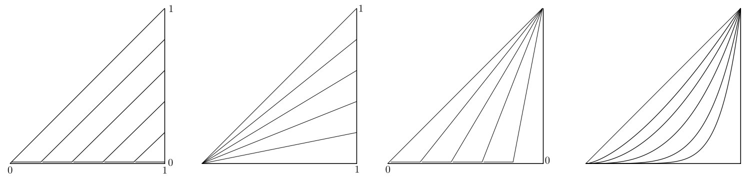

Figure 6: The standard composition tomonoids.

Theorem 6.6. *Let $\mathcal{P}$ be a standard Archimedean quotient of the q.n.c. tomonoid $\mathcal{L}$. Let $R \in \mathcal{P}$, and assume that R is not a singleton. Then $(\Lambda^R; \leq, \circ, id_R)$ is a standard composition tomonoid.*

In fact, if then R has a smallest and a largest element, Λ^R is isomorphic to the Łukasiewicz tomonoid. If R has a largest but no smallest element, Λ^R is isomorphic to the product tomonoid. If R has a smallest but no largest element, Λ^R is isomorphic to the reversed product tomonoid. If R has no smallest and no largest element, Λ^R is isomorphic to the power tomonoid.

Proof. By condition (E), we can assume that R is a real interval with the boundaries 0 and 1. Note that the claim follows from Proposition 6.3 in case that R is the top element of $\mathcal{P}$.

By Lemma 6.4, $(\Lambda^R; \leq, \circ, id_R)$ is a composition tomonoid fulfilling (C1)–(C4) and (C6)–(C8). By (C6), each $\lambda \in \Lambda^R$ is continuous. Moreover, by Lemma 4.6(ii)(c), if $0 \notin R$, the right limit of λ at 0 is 0, and if $1 \notin R$, the left limit of λ at 1 is 1.

Let $\bar{\lambda}\colon [0,1] \to [0,1]$ be the continuous extension of $\lambda \in \Lambda^R$ to $[0,1]$, and let $\bar{\Lambda} = \{\bar{\lambda}\colon \lambda \in \Lambda^R\} \cup \{c^{[0,1],0}\}$. Then $(\bar{\Lambda}; \leq, \circ, id_{[0,1]})$ is still a composition tomonoid fulfilling (C1)–(C4), and (C6). Moreover, the following holds:

(C7') For any $\lambda \in \bar{\Lambda} \backslash \{id_{[0,1]}\}$ and any $r \in (0,1)$, $\lambda(r) < r$.

(C8') $\bar{\Lambda}$ is order-isomorphic to $[0,1]$.

In fact, (C7') is implied by the fact that Λ^R fulfils (C7).

To see (C8'), assume first that $R = [0,1]$. Then, by (C8), Λ^R is order-isomorphic to $[0,1]$, and $c^{[0,1],0} \in \Lambda^R$ by Lemma 4.6(ii)(d). Hence also $\bar{\Lambda}$ is order-isomorphic to $[0,1]$. Assume second that R is distinct from $[0,1]$. Then, by (C8), Λ^R is order-isomorphic to $(0,1]$, and again by Lemma 4.6(ii), either $0 \notin R$ or else $c^{R,0} \notin \Lambda^R$. We conclude that $\bar{\Lambda}$ is order-isomorphic to $[0,1]$ in this case as well.

By [26, Theorem 5.11], the claim follows. □

We conclude that the triangular sections of the Cayley tomonoid of our standard Archimedean extension $\mathcal{L}$ are easily described: up to isomorphism, there are only four possibilities, in case of the top element even only two.

Theorem 6.6 describes each composition tomonoid Λ^R separately. Each element of Λ^R is the restriction of a translation λ_f, where f is an element of the extending filter F, to R. It remains to determine which mapping in Λ^R belongs to which element of F. Our next proposition states that this is easy: the epimorphism $F \to \Lambda^R,\ \ f \mapsto \lambda_f^R$ is uniquely determined by one non-trivial assignment.

Proposition 6.7. *Let* $(\Phi; \leq, \circ, id_R)$ *be a standard composition tomonoid on a toset* R*; let* $(F; \leq, \odot, 1)$ *be either the product or the Łukasiewicz tomonoid; let* $\tilde{f} \in F \backslash \{1\}$ *be non-minimal, and let* $\tilde{\lambda} \in \Phi \backslash \{id_R\}$ *have a non-empty support. Then there is at most one sup-preserving epimorphism* $\varrho\colon F \to \Phi$ *such that* $\varrho(\tilde{f}) = \tilde{\lambda}$.

Proof. Let $n \geq 1$. As F is the product or the Łukasiewicz tomonoid, and $\tilde{f}$ is a non-minimal element of it, there is a unique $f_n \in F$ such that $f_n^n = \tilde{f}$. Similarly, Φ is a standard composition tomonoid, and $\tilde{\lambda}$ is a non-minimal element of it; it is readily checked that in each of the four possible cases there is a unique $\lambda_n \in \Phi$ such that $\lambda_n^n = \tilde{\lambda}$.

It follows that any homomorphism mapping $\tilde{f}$ to $\tilde{\lambda}$ must map f_n to λ_n. As ϱ is supposed to be a sup-preserving homomorphism, the claim follows. ☐

We now turn to the sets of mappings $\Lambda^{R,S}$. As already announced in [26], these sets are largely determined by Λ^R and Λ^S. In fact, given Λ^R and Λ^S there is not much room for variation. Figure 7 is intended to give an impression of the situation; it illustrates that the mapping $\lambda_t^{R,S}$ from $\Lambda^{R,S}$ is uniquely determined by its value at the single point $r \in R$.

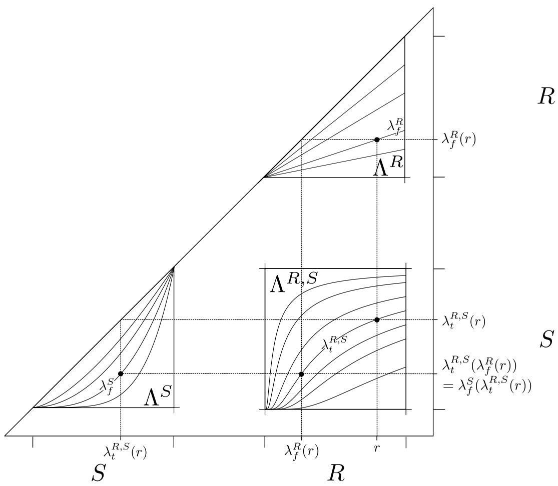

Figure 7: The mapping $\lambda_t^{R,S}$ is determined by its value at a single point. The figure shows how its value at r determines its value at $\lambda_f^S(r)$.

Proposition 6.8. *Let* Φ *be a standard composition tomonoid on the toset* R *and let* Ψ *be a standard composition tomonoid on the toset* S*. Furthermore, let* $(F; \leq, \odot, 1)$

be either the product or the Łukasiewicz tomonoid, and assume that there are epimorphisms $F \to \Phi, \ \ f \mapsto \varphi_f$ and $F \to \Psi, \ \ f \mapsto \psi_f$. Let

$$\Xi \;=\; \{\xi\colon R \to S : \ \text{ for all } f \in F,\ \xi \circ \varphi_f = \psi_f \circ \xi\}. \tag{15}$$

Moreover, let Ξ' be a set of mappings from R to S such that (1) *for any $\xi \in \Xi'$ and $f \in F, \ \ \psi_f \circ \xi$ and $\xi \circ \varphi_f$ coincide and are in Ξ',* (2) *if the pointwise calculated supremum of a subset of Ξ' exists, it is in Ξ', and* (3) *if S has a smallest element u', $c^{R,u'} \in \Xi'$. Then either $\Xi' = \Xi$ or there is a $\zeta \in \Xi$ such that $\Xi' = \{\xi \in \Xi\colon \xi \leq \zeta\}$.*

Proof. If $u' \notin S$ and Ξ is empty, or $u' \in S$ and Ξ contains only $c^{R,u'}$, we are done. Assume now that these possibilities do not apply. We first prove two properties of Ξ.

($\star$) For any $r \in R$ and any non-minimal $s \in S$, there is at most one $\xi \in \Xi$ such that $\xi(r) = s$.

To see ($\star$), let $\xi, \upsilon \in \Xi$ be such that $\xi(r) = \upsilon(r) = s$. For any $r' < r$, there is an $f \in F$ such that $\varphi_f(r) = r'$, because Φ is a standard composition tomonoid. Thus $\xi(r') = \xi(\varphi_f(r)) = \psi_f(\xi(r)) = \psi_f(\upsilon(r)) = \upsilon(\varphi_f(r)) = \upsilon(r')$. For any $r' > r$, there an $f \in F$ such that $\varphi_f(r') = r$, thus $\psi_f(\xi(r')) = \xi(\varphi_f(r')) = \xi(r) = \upsilon(r) = \upsilon(\varphi_f(r')) = \psi_f(\upsilon(r'))$, and since Ψ is a standard composition tomonoid and s is non-minimal, it follows $\xi(r') = \upsilon(r')$ again. We conclude $\xi = \upsilon$.

($\star\star$) Let $\xi, \upsilon \in \Xi$ have a non-empty support. Then there is an $f \in F$ such that either $\xi = \upsilon \circ \varphi_f$ or $\upsilon = \xi \circ \varphi_f$; in particular, ξ and υ are comparable.

To see ($\star\star$), let $r \in R$ be in the support of both ξ and υ. Assume $\xi(r) \leq \upsilon(r)$, and let $f \in F$ be such that $(\upsilon \circ \varphi_f)(r) = \psi_f(\upsilon(r)) = \xi(r)$. Note that $\upsilon \circ \varphi_f \in \Xi$. Then it follows by ($\star$) that $\xi = \upsilon \circ \varphi_f$. Similarly, $\upsilon(r) \leq \xi(r)$ implies that there is an $f \in F$ such that $\upsilon = \xi \circ \varphi_f$.

Let now Ξ' be a set of functions from R to S with the indicated properties. By (1), $\Xi' \subseteq \Xi$. We claim that $\xi \in \Xi'$, $\upsilon \in \Xi$, and $\upsilon \leq \xi$ imply $\upsilon \in \Xi'$. Indeed, in this case either S has the smallest element u' and $\upsilon = c^{R,u'}$; then $\upsilon \in \Xi'$ by (3). Or υ has a non-empty support; by ($\star\star$), then $\upsilon = \xi \circ \varphi_f$ for some $f \in F$; hence $\upsilon \in \Xi'$ by (1).

Assume now that Ξ' is a proper subset of Ξ. As Ξ is totally ordered, any element of $\Xi \backslash \Xi'$ is an upper bound of Ξ'; hence the pointwise supremum ζ of Ξ' exists. Furthermore, $\zeta \in \Xi'$ by (2), hence also $\zeta \in \Xi$, and we conclude $\Xi' = \{\xi \in \Xi\colon \xi \leq \zeta\}$. ☐

Again, let $\mathcal{L}$ be a standard Archimedean extension of the q.n.c. tomonoid $\mathcal{P}$. The triangular sections of the Cayley transform of $\mathcal{L}$ are derivable from Theorem 6.6, and they are connected to translations by elements of the extending filter F according to Proposition 6.7. We conclude from Proposition 6.8 that then also the rectangular sections are determined, up to the choice of largest mappings.

Let $R, S, T \in \mathcal{P}$ such that $S = R \odot T$ and $T < F$, and let us explain how $\Lambda^{R,S}$ is determined by Λ^R and Λ^S. If R, T are not $\odot$-maximal, Lemma 4.7(ii) applies. In this case, $\Lambda^{R,S}$ consists of a single mapping, namely, $c^{R,u'}$, where u' is the smallest element of S. Assume that R, T is a $\odot$-maximal pair. If then $u' = \inf S \in S$, it still might be the case that $\Lambda^{R,S} = \{c^{R,u'}\}$; cf. Lemma 4.7(i)(e). Assume that $\Lambda^{R,S}$

contains at least two elements. If then $u' \in S$, we know from Lemma 4.7(i)(e) that $c^{R,u'} \in \Lambda^{R,S}$ if and only if $v = \sup R \in R$.

Let us now apply Proposition 6.8 to the composition tomonoids $\Phi = \Lambda^R$ and $\Psi = \Lambda^S$ and the mappings $F \to \Lambda^R,\ \ f \mapsto \lambda^R_f$ and $F \to \Lambda^S,\ \ f \mapsto \lambda^S_f$. We first observe that $\Lambda^{R,S}$ fulfils conditions (1) and (2) on Ξ', but not necessarily (3). Put $\Xi' = \Lambda^{R,S}$ if $u' \notin S$ and otherwise $\Xi' = \Lambda^{R,S} \cup \{c^{R,u'}\}$. Then Ξ' fulfils all three conditions (1)–(3), and $\Lambda^{R,S}$ arises from Ξ' by removing $c^{R,u'}$ if $u' \in S$ and $v \notin R$. Next, let Ξ be defined by (15). By Proposition 6.8, then either $\Xi' = \Xi$ or $\Xi' = \{\xi \in \Xi \colon \xi \leq \zeta\}$ for some $\zeta \in \Xi$.

Similar to the case of Theorem 6.6, Proposition 6.8 describes the sets $\Lambda^{R,S}$ separately. It remains to determine which mapping in $\Lambda^{R,S}$ belongs to which translation. Similarly as in case of Proposition 6.7, the mapping $T \to \Lambda^{R,S},\ \ t \mapsto \lambda^{R,S}_t$ is uniquely determined by a single assignment.

Proposition 6.9. *Let R, Φ, S, Ψ, F, as well as the mappings $f \mapsto \varphi_f$ and $f \mapsto \psi_f$ be as in Proposition* 6.8*, and let Ξ be defined by* (15)*. Let X be a further standard composition tomonoid on the toset T, and let $F \to X,\ \ f \mapsto \chi_f$ be a sup-preserving epimorphism. Let $\tilde{t} \in T$ be non-minimal, and let $\tilde{\xi} \in \Xi$ have a non-empty support. Then there is at most one mapping $\tau \colon T \to \Xi$ such that $\tau(\chi_f(t)) = \psi_f \circ \tau(t)$ for any $t \in T$ and $\tau(\tilde{t}) = \tilde{\xi}$.*

Proof. Assume that there are two mappings τ_1 and τ_2 as required. Let $t > \tilde{t}$; and let $\xi_1 = \tau_1(t),\ \ \xi_2 = \tau_2(t)$. Since X is a standard composition tomonoid, there is an $f \in F$ such $\chi_f(t) = \tilde{t}$. Then $\psi_f \circ \xi_1 = \psi_f \circ \tau_1(t) = \tau_1(\chi_f(t)) = \tau_1(\tilde{t}) = \tilde{\xi}$ and similarly $\psi_f \circ \xi_2 = \tilde{\xi}$. Let r be in the support of $\tilde{\xi}$; then $\psi_f(\xi_1(r)) = \psi_f(\xi_2(r))$ is non-minimal, and we conclude $\xi_1(r) = \xi_2(r)$. As in the proof of Proposition 6.8, it follows $\xi_1 = \xi_2$, that is, $\tau_1(t) = \tau_2(t)$.

Let $t < \tilde{t}$. Then there is an $f \in F$ such that $\chi_f(\tilde{t}) = t$, and we have $\tau_1(t) = \tau_1(\chi_f(\tilde{t})) = \psi_f \circ \tau_1(\tilde{t}) = \psi_f \circ \tau_2(\tilde{t}) = \tau_2(\chi_f(\tilde{t})) = \tau_2(t)$. We conclude $\tau_1 = \tau_2$, and the claim follows. ☐

This concludes our specification of standard Archimedean extensions. We now demonstrate on the basis of some examples how Proposition 6.3, Theorem 6.6 and Propositions 6.7, 6.8, 6.9 can be used to determine the standard Archimedean extensions of a given tomonoid.

Example 6.10. *We first reconsider Example* 4.8*. We now adopt the opposite viewpoint: we consider the quotient, i.e., the Łukasiewicz chain L_5, and we explore its standard Archimedean extensions. We require the extended tomonoid to be composed of intervals as shown: we assign a singleton to the bottom element and left-open right-closed real intervals to the remaining four elements.*

By Proposition 6.3*, the extending tomonoid F is, up to isomorphism, the product or Łukasiewicz tomonoid. As F does not possess a smallest element, F is in fact isomorphic to the product tomonoid.*

By Theorem 6.6, $\Lambda^{(0,\frac{1}{4}]}$, $\Lambda^{(\frac{1}{4},\frac{1}{2}]}$, $\Lambda^{(\frac{1}{2},\frac{3}{4}]}$ *are all isomorphic to the product composition tomonoid.*

To determine the translations λ_t, $\frac{3}{4} < t < 1$, *it is by Proposition* 6.7 *sufficient to specify one of them. To this end, we choose one element distinct from the identity from each composition tomonoid* $\Lambda^{(0,\frac{1}{4}]}$, $\Lambda^{(\frac{1}{4},\frac{1}{2}]}$, $\Lambda^{(\frac{1}{2},\frac{3}{4}]}$, *and* $\Lambda^{(\frac{3}{4},1]}$, *and we require that these mappings arise from the same translation.*

Next, the sets $\Lambda^{R,S}$, *where* R *and* S *are among* $\{0\}$, $(0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, $(\frac{3}{4}, 1]$ *are to be determined. The case that* $\{0\}$ *occurs is trivial and covered by Lemma* 4.7(ii). *Let both* R *and* S *be distinct from* $\{0\}$*; then Proposition* 6.8 *applies. It is straightforward to calculate* Ξ *according to* (15) *from* Λ^R *and* Λ^S, *which are both product composition tomonoids. The actual set* $\Lambda^{R,S}$ *results from* Ξ *by determining a largest element* ζ.

Still given R *and* S, *it remains to determine the mapping* $T \to \Lambda^{R,S}, \ t \mapsto \lambda_t^{R,S}$, *where* $T = R \to S$. *By Proposition* 6.9 *it is sufficient to make one assignment; we require* $\lambda_t^{R,S} = \zeta$, *where* t *is the maximal element of* T.

A possible result of this construction is the t-norm $\odot_5$, *see Figure* 3 *and* (14).

Example 6.11. *We next construct the standard Archimedean extensions of a four-element tomonoid. The tomonoid to be extended is specified in Figure* 8 *(left upper corner) by means of its Cayley tomonoid. We assign to its elements the real intervals* $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3})$, $\{\frac{2}{3}\}$, *and* $(\frac{2}{3}, 1]$, *respectively.*

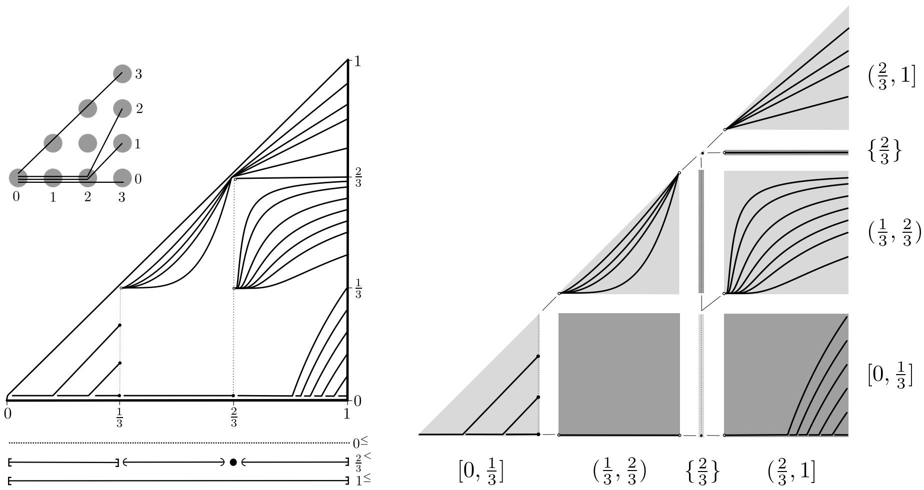

Figure 8: An extension of a four-element tomonoid.

The extending tomonoid has a left-open interval as its universe and is consequently again isomorphic to the product tomonoid.

Let us determine the triangular sections. The composition tomonoids $\Lambda^{[0,\frac{1}{3}]}$, *and* $\Lambda^{(\frac{1}{3},\frac{2}{3})}$ *are, according to Theorem* 6.6, *isomorphic to the Łukasiewicz and the power composition tomonoid, respectively. By Lemma* 4.7(ii), $\Lambda^{\{\frac{2}{3}\}}$ *consists of the mapping* *assigning $\frac{2}{3}$ to itself.*

We next determine an arbitrary translation λ_t, where $\frac{2}{3} < t < 1$, by choosing a mapping different from the identity from each of these three non-trivial composition tomonoids. Then the translations λ_t are uniquely determined for all $t \in (\frac{2}{3}, 1]$.

We now turn to the rectangular sections of the Cayley tomonoid to be constructed. The mappings in $\Lambda^{(\frac{2}{3},1],(\frac{1}{3},\frac{2}{3})}$ are determined by (15). *The whole set is needed in this case because, by condition* (C5), *for each $t \in (\frac{1}{3}, \frac{2}{3})$ there must be a translation mapping* 1 *to t. The situation is similar in the case of $\Lambda^{(\frac{2}{3},1],[0,\frac{1}{3}]}$.*

$\Lambda^{(\frac{1}{3},\frac{2}{3}),[0,\frac{1}{3}]}$ contains the constant 0 *mapping only; this is implied by Lemma* 4.7(i)(e). *Furthermore, by Lemma* 4.7(i), *$\Lambda^{\{\frac{2}{3}\},[0,\frac{1}{3})}$ consists of the single mapping assigning $\frac{2}{3}$ to* 0. *Finally, also $\Lambda^{(\frac{2}{3},1],\{\frac{2}{3}\}}$ is trivial, consisting of the constant $\frac{2}{3}$ mapping.*

The Cayley tomonoid is thus completely determined. The result is a t-norm like, e.g., the following one:

$$a \odot_7 b \;=\; \begin{cases} 3ab - 2a - 2b + 2 & \text{if } a, b > \frac{2}{3}, \\ \frac{1}{3}((3a-1)^{\frac{1}{3b-2}} + 1) & \text{if } \frac{1}{3} < a \leq \frac{2}{3} \text{ and } b > \frac{2}{3}, \\ (a + \frac{1}{3}\log_2(3b-2)) \vee 0 & \text{if } a \leq \frac{1}{3} \text{ and } b > \frac{2}{3}, \\ 0 & \text{if } a, b \leq \frac{2}{3}. \end{cases}$$

In our last example of a standard Archimedean extension, we consider a t-norm that, in a slightly modified form, was considered in [26] and found particularly peculiar. In fact, the framework developed in [26] was not sufficient to provide an interpretation.

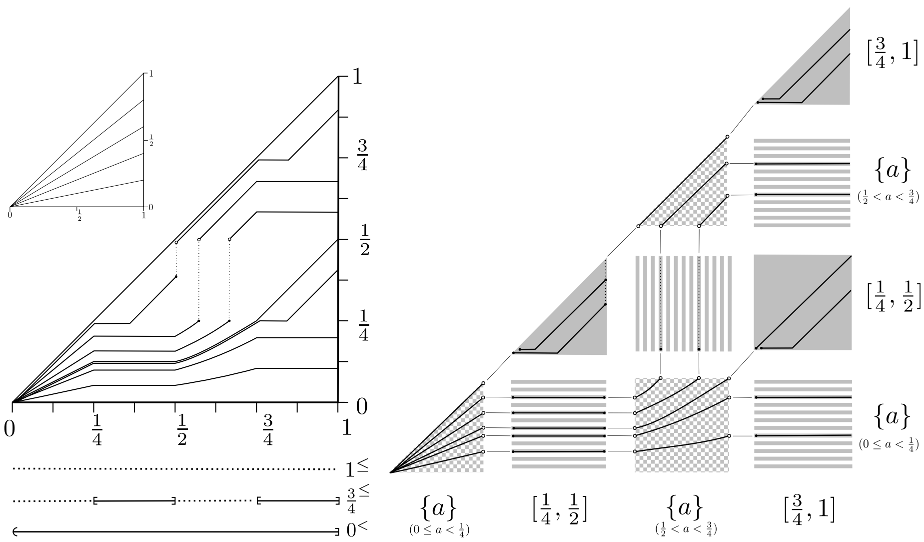

Figure 9: An extension of the product tomonoid.

Example 6.12. *We shall extend an infinite tomonoid: the t-norm monoid based on the product t-norm; cf. Figure* 9 *(left upper corner). We have to assign a real interval to each of the uncountably many elements of* $(0, 1]$. *We assign* $[\frac{3}{4}, 1]$ *to the top element and* $[\frac{1}{4}, \frac{1}{2}]$ *to* $\frac{1}{2}$, *and we assign singletons* $\{a\}$, *where* $0 \leq a < \frac{1}{4}$ *or* $\frac{1}{2} < a < \frac{3}{4}$, *to the remaining elements.*

The extending tomonoid has a smallest element and is thus isomorphic to the Łukasiewicz tomonoid. $\Lambda^{[\frac{1}{4},\frac{1}{2}]}$ *is isomorphic to the Łukasiewicz composition tomonoid as well. For* $0 \leq a < \frac{1}{4}$ *and* $\frac{1}{2} < a < \frac{3}{4}$, $\Lambda^{\{a\}}$ *contains the mapping that assigns* a *to* a.

Again, we have to choose one element from each composition tomonoid Λ^R. *If this set consists of one mapping only, we obviously do not have to do anything. We just have to assign an element* $t \in (\frac{3}{4}, 1)$ *to a mapping from* $\Lambda^{[\frac{1}{4},\frac{1}{2}]}$. *The choice is restricted by the requirement that the assignment must be extensible to an epimorphism from* $[\frac{3}{4}, 1]$ *to* $\Lambda^{[\frac{1}{4},\frac{1}{2}]}$.

Using the same argument as in Example 6.11, *we see that* $\Lambda^{[\frac{3}{4},1],[\frac{1}{4},\frac{1}{2}]}$ *is uniquely determined, and so are the remaining rectangular parts.*

The resulting extension is a t-norm monoid. If choosing the epimorphism from $[\frac{3}{4}, 1]$ *to* $\Lambda^{[\frac{1}{4},\frac{1}{2}]}$ *bijective, the t-norm might be defined as follows:*

$$a \odot_8 b = \begin{cases} (a+b-1) \vee \frac{3}{4} & \text{if } a, b > \frac{3}{4}, \\ a & \text{if } \frac{1}{2} < a \leq \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ (a+b-1) \vee \frac{1}{4} & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ a & \text{if } a \leq \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ 2^{4(a+b)-7} & \text{if } \frac{1}{2} < a, b \leq \frac{3}{4} \text{ and } a+b \leq \frac{5}{4}, \\ a+b-\frac{3}{4} & \text{if } \frac{1}{2} < a, b \leq \frac{3}{4} \text{ and } a+b > \frac{5}{4}, \\ 2^{4b-5} & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \\ 2^{4b-3}a & \text{if } a \leq \frac{1}{4} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \\ \frac{1}{8} & \text{if } \frac{1}{4} < a, b \leq \frac{1}{2}, \\ \frac{a}{2} & \text{if } a \leq \frac{1}{4} \text{ and } \frac{1}{4} < b \leq \frac{1}{2}, \\ 2ab & \text{if } a, b \leq \frac{1}{4}. \end{cases}$$

Finally, it seems appropriate to include the example of a t-norm monoid that involves an extension that is Archimedean but not standard Archimedean. Note that quotients of t-norm monoids by Archimedean filters are always standard Archimedean. Thus our example necessarily involves two quotients by non-trivial filters.

Example 6.13. *Figure* 10 *shows the following t-norm:*

$$a \odot_9 b \;=\; \begin{cases} 3ab - 2a - 2b + 2 & \text{if } a, b > \frac{2}{3}, \\ b & \text{if } a > \frac{2}{3} \text{ and } b \leq \frac{2}{3}, \\ 3ab - a - b + \frac{1}{3} & \text{if } \frac{1}{3} < a, b \leq \frac{2}{3}, \\ 0 & \text{if } a \leq \frac{1}{3} \text{ and } b \leq \frac{2}{3}. \end{cases}$$

Consider the quotient $[0, 1]_{\frac{2}{3}}$ *induced by the filter* $\frac{2}{3}^{<}$. *This is an Archimedean q.n.c. tomonoid and thus an Archimedean extension of the trivial tomonoid. However, this*

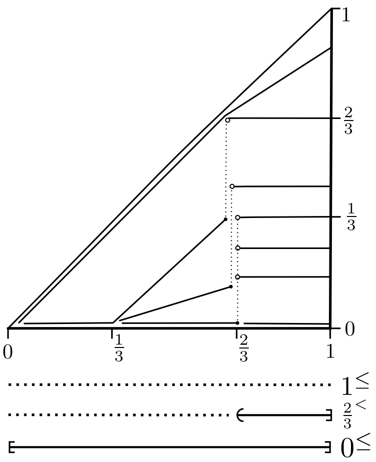

Figure 10: The t-norm $\odot_9$.

extension is not standard because $[0,1]_{\frac{2}{3}}$*, the only equivalence class, is not order isomorphic to a real interval. Consisting of the singletons* $\{a\}$*, where* $0 \leq a \leq \frac{2}{3}$*, and the interval* $(\frac{2}{3}, 1]$*, it is ordered like, say,* $[0,1] \cup \{2\}$.

7 Conclusion

We have considered a particular class of totally ordered monoids, namely those that arise from left-continuous t-norms. These tomonoids are quantic, negative, and commutative. Representing the chain of quotients of a q.n.c. tomonoid by means of its Cayley tomonoid gives rise to a convenient approach to the problem of how to describe this kind of tomonoids and how to bring order into the wide diversity of left-continuous t-norms. The chain of quotients of a t-norm monoid can have a complicated order structure. But a pair of successive elements in this chain corresponds to extensions of Archimedean tomonoids, and if the congruence classes are in this case ordered like a real interval, we are able to provide a systematic specification.

It remains an open problem to define effective ways of specifying extensions of tomonoids in general. In particular for the finite case, this is a research field in need of creative approaches. An approach totally different from the present one is described in [27]. Moreover, a useful framework could be what has been called the level-set approach by M. Petrík and P. Sarkoci [23].

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