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Abstracts

Valued Relations and Capacities in Decision Theory

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Since their inception in 1980, the *Linz Seminars on Fuzzy Set Theory* have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work.

Linz2001 is the 22$^{\text{nd}}$ seminar carrying on this tradition. *Linz2001* deals with the use of *Valued Relations and Capacities in Decision Theory*. It is the hope of the organizers that the talks will provide a mathematical setting for both the theory and the practice of decision under imprecision and uncertainty, multiple attribute decision making, group decision making and game theory.

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Quaternary Fuzzy Relations in Preference Modelling

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Abstract

In this paper we introduce quaternary fuzzy relations in order to describe difference structures. Three models are developed and studied, based on three different interpretations of an implication. Functional forms of the quaternary relation are determined by solutions of functional equations of the same type.

Keywords: measurement theory; difference structure; fuzzy quaternary relations; implications; t-norms; t-conorms; uninorms.

1 Introduction

Preference modelling is a fundamental step of (multi-criteria) decision making, operations research, social choice and voting procedures, and has been studied extensively for several years. Typically, three binary relations (strict preference, indifference, and incomparability) are built up as a result of pairwise comparison of the alternatives. Then a single reflexive relation (the weak preference, or large preference) is defined as the union of the strict preference and indifference relations. All the three previous binary relations can be expressed in terms of the large preference in a unique way. Therefore, it is possible (and in fact, this is typical) to start from a reflexive binary relation , and build up strict preference, indifference and incomparability from it.

Some important classes of binary preferences have also been studied with respect to their representation by a real function of the alternatives [7]. As an illustration, consider a finite set of alternatives A and a binary relation P on A. Then there is a real-valued function f on A satisfying

$$aPb \quad \Leftrightarrow \quad f(a) > f(b) \tag{1}$$

if and only if P is asymmetric and negatively transitive. Such a P is called a *strict weak order*. If P is strict preference then a function f satisfying (1) is called a *utility function* [7]. In this situation we have an *ordinal scale*: transformations $\varphi : f(A) \to \mathbb{R}$ where $\varphi \circ f$ is also satisfies (1) are the strictly increasing functions.

The representation (1) arises in the measurement of temperature if P is interpreted as *"warmer then"*. According to the previous result, temperature is an ordinal scale — although it is well known

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that temperature is an interval scale. There is no contradiction: one can obtain this result by using judgments of comparative temperature *difference*.

To make this precise, one should introduce a quaternary relation D on a set A of objects whose temperatures are being compared. The relation $abDuv$ is interpreted as the difference between the temperature of a and the temperature of b is judged to be greater than that between the temperature of u and the temperature of v. We would like to find a real-valued function f on A such that for all $a,b,u,v \in A$ we have

$$abDuv \quad \Leftrightarrow \quad f(a) - f(b) > f(u) - f(v). \tag{2}$$

The main aim of the present paper is to study whether it is possible to extend, in a rational way, this approach to the use of a *quaternary fuzzy relation* on A. Note that the classical binary preference theory has successfully been extended in [4], and developed significantly further since that time (see the overview [1]).

The paper is organized as follows. In the next section we briefly summarize some results on difference measurement, especially on the representation (2). Some of these observations will guide us in the study of fuzzy extensions in Section 3. We will deal with the non-strict version W of D and investigate three models based on different forms of fuzzy implications. The functional form of an appropriate fuzzy difference operator will be given through solving some functional equation in each case. Using these forms we can characterize fuzzy extensions of (2). In Section 4 we study the strict quaternary relation D and the indifference E, based on W. We close the paper with concluding remarks.

We would like to emphasize that the study and the results in the present paper mean only the first steps in the topic. Any comments and suggestions are welcome.

2 Difference measurement

In classical measurement theory the following situation has been studied in full details. Let A be a set and D be a quaternary relation on A. In addition to the temperature interpretation, $abDuv$ makes sense also in preference: I like a over b more than I like u over v. We write $D(a,b,u,v)$ or, equivalently, $abDuv$. If the representation (2) holds then it is called *(algebraic) difference measurement*.

A representation theorem is known for (2). Here we do not go into details, the interested reader can find more results in [7].

We introduce two quaternary relations E and W based on D as follows:

$$\begin{aligned} abEuv \quad &\Leftrightarrow \quad [\text{not } abDuv \text{ and not } uvDab], \\ abWuv \quad &\Leftrightarrow \quad [abDuv \text{ or } abEuv]. \end{aligned}$$

Notice that in case of (2) we have

$$abEuv \quad \Leftrightarrow \quad f(a) - f(b) = f(u) - f(v), \tag{3}$$

$$abWuv \quad \Leftrightarrow \quad f(a) - f(b) \geq f(u) - f(v). \tag{4}$$

Closing this section, we collect here some properties valid in the classical case. We would like to keep as many as possible for the fuzzy extension in the next section.

For all $a,b,u,v,x,y \in A$ we have

W1. $W(a,b,u,v) = W(a,u,b,v)$

W2. $W(a,b,u,v) = W(v,u,b,a)$

W3. $W(a,b,x,x) = W(a,b,y,y)$

W4. $W(a,b,u,v)$ implies $W(a,b,x,y)$ or $W(x,y,u,v)$.

3 Fuzzy extensions

Our aim is to extend (4) to allow W to be a quaternary *fuzzy* relation. We fuzzify W and not D because of technical reasons on one hand. On the other hand, this way we can follow traditions in building preferences starting from a weak relation, and defining its strict part later on (see [4]).

Therefore, let I be any fuzzy implication, and Δ be any fuzzy difference operator.

At this formulation stage we require only the following general properties of I and Δ:

I1. I is a function from $[0,1]^2$ to $[0,1]$;

I2. I is nonincreasing in the first argument;

I3. I is nondecreasing in the second place;

I4. $I(0,0) = I(0,1) = I(1,1) = 1$, $I(1,0) = 0$ (that is, I an implication on $\{0,1\}$).

D1. Δ is a function from $\mathbb{R}^2$ to $[0,1]$;

D2. Δ is nondecreasing in the first argument;

D3. Δ is nondecreasing in the second place.

Then, define W by

$$W(a,b,u,v) = I(\Delta(f(u), f(v)), \Delta(f(a), f(b))), \tag{5}$$

for any $a,b,u,v \in A$.

Notice that in the classical case (4) we have

$$I(u,v) = \begin{cases} 1 & \text{if } u \leq v, \\ 0 & \text{if } u > v. \end{cases} \quad , \qquad \text{and} \qquad \Delta(x,y) = x - y,$$

where u,v and x,y can be any real number, not necessarily restricted to be in $[0,1]$.

In the sequel we study three models of the implication I. In any case, we restrict our investigations to implications defined from continuous Archimedean t-norms or t-conorms, or representable uninorms.

3.1 Model 1: The use of R-implications

We do recall only some essential parts from the theory of t-norms and related operations. We refer to the recent book [6] to find detailed results and proofs.

Suppose now that $I = I_T$ is the R-implication defined from a continuous Archimedean t-norm T. An additive generator of T is denoted by t. Then I_T has the following functional form:

$$I_T(x,y) = t^{-1}(\max\{t(y) - t(x), 0\}). \tag{6}$$

Therefore, the quaternary fuzzy relation W is given by

$$W(a,b,u,v) = t^{-1}(\max\{t(\Delta(f(a), f(b))) - t(\Delta(f(u), f(v))), 0\}). \tag{7}$$

It is obvious now that property W1 is formulated as follows:

$$\Delta(f(u), f(v)) \leq \Delta(f(a), f(b)) \quad \Leftrightarrow \quad \Delta(f(b), f(v)) \leq \Delta(f(a), f(u)),$$

and for $\Delta(f(u), f(v)) > \Delta(f(a), f(b))$ we have

$$t(\Delta(f(a), f(b))) - t(\Delta(f(u), f(v))) = t(\Delta(f(a), f(u))) - t(\Delta(f(b), f(v))).$$

In order to avoid complicated and heavy notation, we use simply the letters a, b, u, v, etc. to denote the function values $f(a), f(b), f(u), f(v)$. This can be done without any confusion.

Thus, the last equation can also be written as

$$t(\Delta(a,b)) + t(\Delta(b,v)) = t(\Delta(a,u)) + t(\Delta(u,v)). \tag{8}$$

For obtaining the general solution of this equation for Δ, we apply the following theorem. The functional equation (9) and its solution plays a key role in the paper. The proof of the following result will be included in the full version of this paper.

Theorem 1. *The general solution of*

$$F(x,y) + F(y,z) = F(x,u) + F(u,z) \tag{9}$$

is

$$F(x,y) = h(y) - h(x) + C, \tag{10}$$

where h is any real function and C is any constant. □

Since equation (8) is just that type in the theorem, we get the following result about Δ and W.

Theorem 2. *Assume that the implication I is represented by equation (6), where t is any additive generator of a continuous Archimedean t-norm. Define a quaternary fuzzy relation W by (7). Then W satisfies condition W1 (i.e., $W(a,b,u,v) = W(a,u,b,v)$) if and only if Δ is of the following form:*

$$\Delta(a,b) = t^{-1}(h(b) - h(a) + C), \tag{11}$$

where h is an appropriately chosen non-decreasing function and C is any positive constant.

In this case the quaternary relation W can be written as follows:

$$W(a,b,u,v) = t^{-1}(\max\{h(u) - h(v) - h(a) + h(b), 0\}). \tag{12}$$

□

We would like to illustrate how to choose function h and the constant C "appropriately". Because (8) holds, we can apply Theorem 1 with $F(x,y) = t(\Delta(x,y))$. Thus, we must have

$$t(\Delta(a,b)) = h(b) - h(a) + C.$$

We have to guarantee that this equation is solvable. That is, the value $h(b) - h(a) + C$ must be in the range of the additive generator t. That range is either the set of non-negative real numbers, or an interval of $[0,\omega]$, with a finite ω.

In any case, let $\alpha < \beta$ be real numbers, and choose $C := \beta - \alpha$. Let h be a function from $\mathbb{R}$ to the bounded interval $[\alpha,\beta]$. This choice squezees the value of $h(b) - h(a) + C$ in the interval $[0,2C]$.

If the range of t is the set of non-negative real numbers then C can be any positive number. If the range of t is $[0,\omega]$, then any $C \leq \omega/2$ is a good choice.

Notice the effect of the choice of C to the membership values (preference intensities). If the range of t is $[0,\omega]$, then $C < \omega/2$ implies we cannot reach zero degree of preference. This is always the case when $t(1) = +\infty$. The lowest membership degree can be arbitrarily close to zero, but it is always positive. Such positive relations play a key role in some classes of fuzzy weak orders studied in [3].

Fortunately, the quaternary relation W defined in the previous theorem satisfies all the four properties analogous to the classical case, as we state it now.

Theorem 3. *The quaternary fuzzy relation W defined in (12) satisfies all the following properties:*

FW1. $W(a,b,u,v) = W(a,u,b,v)$

FW2. $W(a,b,u,v) = W(v,u,b,a)$

FW3. $W(a,b,x,x) = W(a,b,y,y)$

FW4. $W(a,b,u,v) \leq \max\{W(a,b,x,y), W(x,y,u,v)\}$.

$\square$

Example. We would like to show an example. Consider the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max\{x+y-1,0\}$, which has an additive generator $t(x) = 1-x$, so the inverse is $t^{-1}(x) = 1-x$ ($x \in [0,1]$). The range of t is $[0,1]$, so let $\alpha = 0$, $\beta = 1$, $C = 1/2$, and

$$h(x) = \frac{e^x}{1+e^x} \quad (x \in \mathbb{R}).$$

Then, the quaternary fuzzy relation W has the following form:

$$W(a,b,u,v) = 1 - \max\left\{\frac{e^u}{1+e^u} - \frac{e^v}{1+e^v} - \frac{e^a}{1+e^a} + \frac{e^b}{1+e^b}, 0\right\}.$$

3.2 Model 2: The use of S-implications

Another broad class of fuzzy implications is based on a t-conorm S and a strong negation N:

$$I_{S,N}(x,y) = S(N(x),y) \quad (x,y \in [0,1]). \tag{13}$$

With the help of an S-implication, we can define the quaternary relation W as follows:

$$W(a,b,u,v) = S(N(\Delta(u,v)),\Delta(a,b)). \tag{14}$$

Suppose that S is a continuous Archimedean t-conorm with additive generator s (that is, we have $S(x,y) = s^{-1}(\min\{s(x)+s(y),s(1)\})$, and $N(x) = \varphi^{-1}(1-\varphi(x))$. Then (14) can be rewritten as

$$\begin{aligned} W(a,b,u,v) &= s^{-1}(\min\{s(N(\Delta(u,v)))+s(\Delta(a,b)),s(1)\}) \\ &= s^{-1}(\min\{s(\varphi^{-1}(1-\varphi(\Delta(u,v))))+s(\Delta(a,b)),s(1)\}) \\ &= s^{-1}(\min\{g(1-\Gamma(u,v))+g(\Gamma(a,b)),s(1)\}), \end{aligned} \tag{15}$$

where $g(x) = s(\varphi^{-1}(x))$ and $\Gamma(a,b) = \varphi(\Delta(a,b))$.

Now we formulate again property FW1 (see in Theorem 3) with the actual functional form of W:

$$\min\{g(1-\Gamma(u,v))+g(\Gamma(a,b)),s(1)\} = \min\{g(1-\Gamma(b,v))+g(\Gamma(a,u)),s(1)\}. \tag{16}$$

From this equality it follows that $g(1-\Gamma(u,v))+g(\Gamma(a,b)) < s(1)$ if and only if $g(1-\Gamma(b,v))+g(\Gamma(a,u)) < s(1)$. In this case (16) reduces to

$$g(1-\Gamma(u,v))+g(\Gamma(a,b)) = g(1-\Gamma(b,v))+g(\Gamma(a,u)). \tag{17}$$

Theorem 4. *Suppose s is an additive generator of a continuous Archimedean t-norm, and W is represented as in (14). Then property FW1 implies that*

$$\Delta(a,b) = s^{-1}(h(b)-h(a)+C), \tag{18}$$

where h is an appropriately chosen non-increasing function and C is a positive constant.

In this case

$$W(a,b,u,v) = s^{-1}(\min\{sNs^{-1}(h(v)-h(u)+C)+h(b)-h(a)+C,s(1)\}). \tag{19}$$

□

As we stated, the form of W given in (19) is only necessary for having property FW1.

Theorem 5. *Assume that conditions of Theorem 4 hold. Then the quaternary fuzzy relation W defined by (19) satisfies FW1 if and only if $s(1) < \infty$, and the strong negation N is generated also by s.*

If $s(1) = 1$ then we have

$$W(a,b,u,v) = s^{-1}(\min\{1-h(v)+h(u)+h(b)-h(a),1\}). \tag{20}$$

□

Comparing the formula (20) with the one coming from R-implications in equation (12), one can see that they are different in general, even in the case when S and T are duals (i.e., when $s(x) = t(1-x)$). However, in case of Example 1 we have $t(x) = 1-x$, so $s(x) = x$, and thus $s^{-1}(x) = x = 1-t^{-1}(x)$. It means that in this particular case both functional forms of the quaternary relation W coincide.

3.3 Model 3: the implication comes from a representable uninorm

In our third model we start from a representable uninorm (see [5]), and use its residual implication [2] (which is indeed an implication satisfying properties I1–I4) in the definition of W.

Let us recall that a *uninorm* is a function $U : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, nondecreasing, and has a neutral element $e \in [0,1]$ (i.e., $U(e,x) = x$ for all $x \in [0,1]$).

Representable uninorms can be obtained as follows. Consider $e \in]0,1[$ and a strictly increasing continuous $[0,1] \to \overline{\mathbb{R}}$ mapping g with $g(0) = -\infty$, $g(e) = 0$ and $g(1) = +\infty$. The binary operator U defined by

$$U(x,y) = g^{-1}(g(x) + g(y)), \quad \text{if } (x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\},$$

and either $U(0,1) = U(1,0) = 0$, or $U(0,1) = U(1,0) = 1$, is a uninorm with neutral element e (called *representable uninorm*). The function g is called an *additive generator* of U.

In case of a uninorm U, the residual operator I_U can be defined by

$$I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}.$$

In some cases (for instance, when U is representable) I_U is an implication.

It is easily seen that in case of a representable uninorm U with additive generator function g the residual implication I_U is of the following form [2]:

$$I_U(x,y) = \begin{cases} g^{-1}(g(y) - g(x)) & \text{if } (x,y) \in [0,1]^2 \setminus \{(0,0),(1,1)\} \\ 1 & \text{, otherwise} \end{cases} . \tag{21}$$

Then, the quaternary fuzzy relation W can be introduced as follows:

$$W(a,b,u,v) = \begin{cases} 1 & \text{if } (\Delta(a,b),\Delta(u,v)) \in \{(0,0),(1,1)\}, \\ g^{-1}(g(\Delta(a,b)) - g(\Delta(u,v))) & \text{otherwise.} \end{cases} . \tag{22}$$

Then, condition FW1 implies the functional equation

$$g(\Delta(a,b)) + g(\Delta(b,v)) = g(\Delta(a,u)) + g(\Delta(u,v)), \tag{23}$$

similarly to the previous two occurrences of the same type.

Theorem 6. *Assume that the implication I is represented by equation (21), where g is any additive generator of a representable uninorm. Define a quaternary fuzzy relation W by (22). Then W satisfies condition FW1 (i.e., $W(a,b,u,v) = W(a,u,b,v)$) if and only if Δ is of the following form:*

$$\Delta(a,b) = g^{-1}(h(b) - h(a) + C), \tag{24}$$

where h is any non-increasing function and C is any constant.

In this case the quaternary relation W can be written as follows:

$$W(a,b,u,v) = g^{-1}(h(u) - h(v) - h(a) + h(b)). \tag{25}$$

We emphasize that in the present case the equation

$$g(\Delta(a,b)) = h(b) - h(a) + C$$

has solution without any restriction to h or C, because the range of g is $\mathbb{R}$.

4 Strict preference and indifference

There exist axiomatic approaches to defining preference structures when we use binary fuzzy relations, see [4]. We try to apply those results in the present environment of quaternary fuzzy relations.

We start with a simple observation. According to the semantical meaning of $abWuv$ in the crisp case, it is obvious that W can be considered as a *binary relation* on $A \times A$. Thus, our task is easily completed.

First of all, recognize that W (as a binary relation on $A \times A$) is strongly complete: for any $a, b, u, v \in A$ we have either $abWuv$, or $uvWab$. Therefore, any axiomatization leads to the following unique formula for the strict preference D and indifference E (see [4]):

$$\begin{aligned} D(a,b,u,v) &= N(W(u,v,a,b)), \\ E(a,b,u,v) &= \min(W(a,b,u,v), W(u,v,a,b)), \end{aligned}$$

where N is a strong negation.

5 Concluding remarks

We have developed three approaches to quaternary fuzzy relations modelling difference measurement. Three simple formulas have been obtained which may be useful and attractive also in applications. We hope that the study can also be applied to fuzzy weak orders, where representations analogous to (1) could be proved only in two particular cases.

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Linearity Axioms for Fuzzy Orderings: Concepts, Properties, and Difficulties

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The linearity/completeness of orderings is a fundamental property not only in pure mathematics, but also in preference modeling, since it corresponds, in a more general setting, to the absence of incomparability of preference relations.

In the crisp case, an ordering $\preceq$ is called *linear* if and only if, for all $x,y \in X$,

$$(x \preceq y) \vee (y \preceq x). \tag{1}$$

The above axiom is just a simple formulation of a property which has a much deeper meaning in logical and algebraic terms. In particular, there are three essential aspects of relationship between orderings and linear orderings:

(i) Every ordering can be represented as the intersection of linear orderings.
(ii) There is a one-to-one correspondence between linearity and maximality with respect to inclusion, i.e. an ordering is linear if and only if there exists no larger ordering.
(iii) Every ordering can be linearized (Szpilrajn's Theorem [6]).

The fuzzy community has witnessed several approaches to generalizing the concept of completeness to fuzzy relations. This contribution is devoted to a detailed study of the two most common classes of approaches with respect to the three fundamental properties mentioned above, where we consider the general case of fuzzy orderings admitting vague equality [1, 5].

Firstly, many authors have fuzzified (1) by replacing the crisp disjunction with a t-conorm (usually called *S-completeness* [3]):

$$S\big(R(x,y),R(y,x)\big) = 1$$

Here the case $S = \max$ plays a specifically important role [2, 3, 8]:

Secondly, it is possible to reformulate (1) such that only the implication (and implicitly the negation) is involved:

$$\big((x \preceq y) \to \mathbf{0}\big) \to (y \preceq x). \tag{2}$$

Replacing the crisp implication by a fuzzy implication I yields the second class of approaches [5]. We will refer to this property as *I-completeness* in the following:

$$I\big(R(x,y),0\big) \leq R(y,x)$$

We are going to formulate and prove the following assertions:

a) Property (i) is preserved even for the strongest axiom—max-completeness in the following sense: any T-E-ordering can be represented as the intersection of max-complete relations [7]. The possibility that these relations are T-E-orderings themselves, however, can only be maintained under the condition that the Szpilrajn Theorem holds.

b) The Szpilrajn Theorem is not necessarily fulfilled for the t-conorm-based family of axioms, mostly only under unacceptably strong assumptions [4, 8].

c) If the underlying t-norm T (which is used for defining antisymmetry and transitivity) is left-continuous, then I-completeness with respect to the residual implication of T allows to fulfill the Szpilrajn Theorem [5].

d) In case that the underlying t-norm is not left-continuous, maximality and a corresponding Szpilrajn Theorem do not even make sense.

e) Under the assumptions of c), the following chain of implications holds:

$$\text{max-completeness} \implies \text{maximality} \implies I\text{-completeness}$$

For the special case that the underlying t-norm is nilpotent, there are also correspondences between S-completeness and I-completeness. However, neither S-completeness nor I-completeness have a one-to-one correspondence to maximality.

f) Maximality cannot be expressed by a property which only involves pairs of values (i.e. an expression with only two free variables). More specifically, in the crisp case, the global property of maximality can be characterized by a criterion which is defined locally—for pairs of elements. In the fuzzy case, this characterization does not hold anymore.

We conclude that *it is not possible to formulate a generalized axiom of linearity which can be expressed in a simple form like (1) or (2) and preserves all three fundamental properties.*

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Comparability Measure

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The introduction of the outranking concept plays an important role in the history of multi-criteria decision procedure, because the preference relation turns into the central point and the incomparability is also taken into consideration.

The influence of this view gives new perspective on different areas of the decision theory. The valued preference modeling, strict preference, indifference and incomparability are the components. The theoretical results of Fodor show how we can get these components from a weak preference relation.

In our work we show that incomparability has a quite different nature. Getting the preference relation from aggregation of the preference components (responsible for different characteristics of the object) it can be viewed as an integral of a function regarding to its importance factor. We show that the incomparability can be measured by the sharpness of the above mentioned function and it is quite different from the preference measure.

Weighted Triangular Conorms

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6 Introduction

Weights (or criteria importances) play an important role in decision making. Symmetric aggregation operators [13, 15, 17] usually model the unanimity (non-discriminative) decision making [5]. Incorporating of weights into symmetric aggregation operator based decision making was discussed in several papers. Recall here [4, 24, 25, 29, 2] among others. As a most prominent example we can take the arithmetic mean M and the related weighted mean W.

7 Continuous t–conorms with weights

Continuous Archimedean t–conorm $S : [0,1]^2 \longrightarrow [0,1]$ is related to an additive generator $g : [0,1] \longrightarrow [0,+\infty]$, for more details see [11]. Let $\mathbf{w} = (w_1,\dots,w_n)$, $n \in \mathbb{N}$, be a fixed weighting vector, $\mathbf{w} \in [0,1]^n$, $(x_1,\dots,x_n) \in [0,1]^n$, $\sum\limits_{i=1}^{n} w_i = 1$. The related weighted t–conorm $S_{\mathbf{w}} : [0,1]^n \longrightarrow [0,1]$ is given by

$$S_{\mathbf{w}}(x_1,\dots,x_n) = S(x_1^{(w_1)},\dots,x_n^{(w_n)}), \tag{1}$$

where the "powers" $x^{(w)}$ are given by $x^{(w)} = g^{-1}(wg(x))$. Note that then, e.g., $S(\overbrace{x^{(1/n)},\dots,x^{(1/n)}}^{n-times}) = x$ for all $x \in [0,1]$.

Evidently (1) can be rewritten into

$$S_{\mathbf{w}}(x_1,\dots,x_n) = g^{-1}(\sum_{i=1}^{n} w_i g(x_i)), \tag{2}$$

i.e., $S_{\mathbf{w}}$ is a weighted arithmetic mean [5]. For a general continuous t–conorm S, we still can apply formula (1) to introduce a weighted t–conorm, however, we have to define properly the power $x^{(w)}$. This can be done as follows: for any $x, w \in [0,1]$, $x^{(w)}=$

$$sup\left(z \in [0,1] \mid \exists i, j \in \mathbb{N}, \frac{i}{j} < w, u \in [0,1], S(\overbrace{u, \ldots, u}^{j-times}) < x; z = S(\overbrace{u, \ldots, u}^{i-times})\right). \tag{3}$$

Easily, we can check that for S related to g the formula (3) gives the above mentioned $x^{(w)}$.

Definition 1. Let S be a continuous t–conorm, $n \in \mathbb{N}$ and $\mathbf{w} = (w_1, \ldots, w_n)$ a weighting of vector. The weighted mean $S_{\mathbf{w}} : [0,1]^n \longrightarrow [0,1]$ is defined as $S_{\mathbf{w}}(x_1, \ldots, x_n) = S(x_1^{(w_1)}, \ldots, x_n^{(w_n)})$, where for $x, w \in [0,1]$, $x^{(w)}$ is defined by (3).

The structure of a general continuous t–conorm S as given in [18, 11] (in the form with additive generators fully describing the relevant Archimedean summands) allows to determine the weighted aggregation operator related to S.

Theorem 2. *Let $S = (< a_k, b_k, g_k > \mid k \in \mathcal{K})$ be a continuous t–conorm, $n \in \mathcal{N}, n \geq 2$, $\mathbf{w} = (w_1, \ldots, w_n)$ and $(x_1, \ldots, x_n) \in [0,1]^n$. Then*

$$S_{\mathbf{w}}(x_1, \ldots, x_n) = S(x_1^{(w_1)}, \ldots, x_n^{(w_n)}) =$$

$$\begin{cases} g_k^{-1}\left(\sum\limits_{i=1}^{n} w_i g_k(\max(a_k, x_i))\right), & if \quad \max(x_i \mid w_i > 0) \in]a_k, b_k] \\ \max_{\mathbf{w}}(x_1, \ldots, x_n), & else. \end{cases} \tag{4}$$

We will present several examples. Note only that the weighted maximum in our approach is given by

$$\max_{\mathbf{w}}(x_1, \ldots, x_n) = \max(x_i \mid w_i > 0) = \max(H(w_i, x_i)), \tag{5}$$

where $H : [0,1^2] \longrightarrow [0,1]$ is given by

$$H(a,b) = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a > 0. \end{cases}$$

Recall that standard approach to weighted maximum [4, 25] is similar with only exception that H is replace by some t–norm T. Note that H is a non–decreasing associative operation with infinitely many left neutral elements.

We will discuss also some properties of weighted t–conorms (continuity, idempotency). Note that we can extend our approach also to the case of non–negative weights with sum exceeding 1 and hence to generalize weighted t–conorm as discussed by Dubois and Prade in [4].

8 Ordinal scales

Approach proposed in the previous part can be applied also in the case of aggregation of values from a given finite ordinal scale C. We will extend the approach of Godo and Torra [9], taking into account arbitrary non–negative weights with sum exceeding 1. Several examples will be shown and some properties will be discussed.

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Possibilistic Markov Decision Processes

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Abstract

In this presentation we propose a synthesis of recent works concerning a qualitative approach, based on possibility theory, to multi-stage decision under uncertainty. Our framework is a qualitative possibilistic counterpart to *Markov decision processes* (MDP), for which we propose dynamic programming-like algorithms. The classical MDP algorithms and their possibilistic counterparts are then experimentally compared on a family of benchmark examples. Finally, we also explore the case of partial observability, thus providing qualitative counterparts to the *partially observable Markov decision processes* (POMDP) framework.

Detailed abstract

For a few years, there has been a growing interest in the Artificial Intelligence community towards the foundations and computational methods of decision making under uncertainty. This is especially relevant for applications to planning, where a suitable sequence of decisions is to be found, starting from a description of the initial world, of the available decisions and their effects, and of the goals to reach. Several authors have thus proposed to integrate some parts of decision theory into the planning paradigm; but up to now, they have focussed on "classical" models for decision making, based on *Markov decision processes* (MDP) (where actions are stochastic and the satisfaction of agents expressed by a numerical, additive utility function), and its implementation, *dynamic programming*. However, transition probabilities for representing the effects of actions are not always available, especially in Artificial Intelligence applications where uncertainty is often ordinal, qualitative. The same remark applies to utilities: it is often more adequate to represent preference over states simply with an ordering relation rather than with additive utilities. Recently, several authors have advocated this qualitative view of decision making and have proposed qualitative versions of decision theory, together with suitable logical languages for expressing preferences, namely, [2], [8], [3].

The latter propose a qualitative utility theory based on possibility theory, where preferences and uncertainty are both qualitative. [4] have extended this work to sequential, finite horizon decision making and have proposed possibilistic counterparts of the well known *Bellman's equations* [1]. This gave rise to the definition of the *Possibilistic Markov Decision Processes* framework, which was extended to infinite-horizon and partial observability in [5]. In [7], experimental comparisons were led between classical and possibilistic MDP algorithms. The present talk is based on a recent article [6] which summarizes the [4] early work, together with the results of [7] [5].

I will first briefly overview the *Markov Decision Processes* framework, and the three most well known algorithms for solving MDPs. Then, I will review in more details [3]'s *Qualitative Utility Theory*, and its extension to multi-stage decision theory. Then, I will describe an experimental comparison of stochastic and possibilistic MDP algorithms and finally, I will give results about the extension of possibilistic MDPs to *partially observable environments*.

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$(\mathbf{S},\mathbf{U})$-Integral-Based Aggregation Operators

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For a fixed $n \in \mathbb{N}$, an aggregation operator $A : [0,1]^n \longrightarrow [0,1]$ is a non-decreasing mapping preserving minimal and maximal elements (see, e.g., [Klir & Folger 1988, Kolesárová & Komorníková 1999]).

Several aggregation operators are closely related to some types of integrals: weighted means correspond to the Lebesgue integral, OWA-operators are special Choquet integrals, and both the max-min- and min-max-weighted means are related to the Sugeno integral.

To cover almost all known Lebesgue type integrals, the so-called (S,U)-integral was introduced [Klement *et al.* 2000a], where $S : [0,1]^2 \longrightarrow [0,1]$ is a continuous t-conorm and $U : [0,1]^2 \longrightarrow [0,1]$ is a left-continuous uninorm with neutral element $e \in]0,1[$ or a left-continuous t-norm which is conditionally distributive over S, i.e., for all $x,y,z \in [0,1]$ with $S(y,z) < 1$ we have

$$U(x, S(y,z)) = S(U(x,y), U(x,z)).$$

For more details concerning t-norms, t-conorms and uninorms we refer to [Klement *et al.* 2000b, Yager & Rybalov 1996].

In our context, the (S,U)-integral is defined with respect to an S-measure m on the finite set $X = \{1,\dots,n\}$, i.e., to a mapping $m : 2^X \longrightarrow [0,1]$ such that, for all $I \subseteq X$, we have

$$m(I) = \mathsf{S}_{i \in I}\, m(\{i\}).$$

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With each such S-measure $m : 2^X \longrightarrow [0,1]$ we can associate an operator $A : [0,1]^n \longrightarrow [0,1]$ via the (S,U)-integral, i.e. ,by

$$A(\mathbf{x}) = \int_X^{(S,U)} \mathbf{x}\, dm$$

or, equivalently, by

$$A(x_1,\ldots,x_n) = \underset{i=1}{\overset{n}{\mathsf{S}}}\, U(x_i, m(\{i\})). \tag{1}$$

Each operator $A : [0,1]^n \longrightarrow [0,1]$ defined via (1) is left-continuous and non-decreasing, and satisfies $A(0,\ldots,0) = 0$. It is an aggregation operator whenever $A(1,\ldots,1) = 1$, i.e., if $m(X) \geq e$.

Proposition 1. *Let S be a continuous t-conorm, U a left-continuous t-norm or a left-continuous uninorm with neutral element $e \in \,]0,1[$ which is conditionally distributive over S, let $m : 2^X \longrightarrow [0,1]$ be an S-measure, and let $A : [0,1]^n \longrightarrow [0,1]$ be the operator defined by (1).*

(i) If U is continuous (possibly up to the points $(0,1)$ and $(1,0)$) then the operator A is continuous.

(ii) A is horizontally additive [Benvenuti et al. *2001], i.e., for each $I \subseteq X$ and for each $\mathbf{x} \in [0,1]^n$ we have*

$$A(\mathbf{x}) = S(A(\mathbf{x}_I), A(\mathbf{x}_{\complement I})),$$

where the vector $\mathbf{x}_I$ coincides with $\mathbf{x}$ in all coordinates which belong to I and has value 0 in all other coordinates.

(iii) A is idempotent and U-homogeneous if and only if

$$m(X) = \underset{i=1}{\overset{n}{\mathsf{S}}}\, m(\{i\}) = e$$

and if, for each $\varepsilon \in \,]0,1[$,

$$\underset{i=1}{\overset{n}{\mathsf{S}}}\, (1-\varepsilon) \cdot m(\{i\}) < e.$$

(iv) A is symmetric only if $m(\{i\}) = m(\{j\})$ for all $i, j \in X$.

Example 2. (i) Let S be a nilpotent t-conorm with normed additive generator $g : [0,1] \longrightarrow [0,1]$ and define $A : [0,1]^n \longrightarrow [0,1]$ by

$$A(x_1,\ldots,x_n) = g^{-1}\Big(\min\Big(\sum_{i=1}^{n} g(m(\{i\})) \cdot g(x_i), 1\Big)\Big).$$

Then A is an aggregation operator only if

$$\sum_{i=1}^{n} g(m(\{i\})) \geq 1, \tag{2}$$

and A is a weighted quasi-arithmetic mean without annihilator if

$$\sum_{i=1}^{n} g(m(\{i\})) = 1. \tag{3}$$

(ii) Let S be a strict t-conorm with additive generator $g:[0,1] \longrightarrow [0,\infty]$ and define $A:[0,1]^n \longrightarrow [0,1]$ by

$$A(x_1,\dots,x_n) = g^{-1}\Big(\sum_{i=1}^{n} g(m(\{i\}))\cdot g(x_i)\Big).$$

Then A is an aggregation operator if $m(X) > 0$, and, provided we have $g(m(\{i\})) > 0$ for each $i \in X$, A is a weighted quasi-arithmetic mean with annihilator 1 if (3) holds.
In particular, if S equals the probabilistic sum $S_{\mathbf{P}}$ (an additive generator thereof being given by $g(x) = -\log(1-x)$), if $n = 2$ and if we put $g(m(\{1\})) = g(m(\{2\})) = \frac{1}{3}$, then we get

$$A(x_1,x_2) = 1-(1-x_1)^{\frac{1}{3}}\cdot(1-x_2)^{\frac{1}{3}}, \tag{4}$$

and A is a symmetric aggregation operator with annihilator 1 which is continuous but not idempotent.

(iii) Consider the ordinal sums $S = (\langle \frac{1}{2},1,S_{\mathbf{L}}\rangle)$ and $U = (\langle \frac{1}{2},1,T_{\mathbf{P}}\rangle)$ (observe that U is a t-norm), and define $A:[0,1]^n \longrightarrow [0,1]$ by

$$A(x_1,\dots,x_n) = \begin{cases} \max\limits_{i\in\{1,\dots,n\}}(\min(x_i, g(m(\{i\})))) \\ \qquad \text{if } x_i \le \frac{1}{2} \text{ or } g(m(\{i\})) \le \frac{1}{2} \text{ for all } i \in X, \\ \min\Big(\frac{1}{2}+\frac{1}{2}\sum\limits_{i=1}^{n}(2\max\limits_{i\in\{1,\dots,n\}}(x_i,\frac{1}{2})-1) \\ \qquad \cdot(2\max\limits_{i\in\{1,\dots,n\}}(g(m(\{i\})),\frac{1}{2})-1),1\ Big) \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Then A is an aggregation operator only if

$$\sum_{g(m(\{i\}))>\frac{1}{2}} (g(m(\{i\}))-\tfrac{1}{2}) \ge \tfrac{1}{2},$$

and it is idempotent if

$$\sum_{g(m(\{i\}))>\frac{1}{2}} (g(m(\{i\}))-\tfrac{1}{2}) = \tfrac{1}{2},$$

In particular, if $n = 2$ and $g(m(\{1\})) = g(m(\{2\})) = \frac{3}{4}$ then we get

$$A(x_1,x_2) = \begin{cases} \max(x_1,x_2) & \text{if } x_1 \le \frac{1}{2} \text{ and } x_2 \le \frac{1}{2}, \\ \frac{2x_1+1}{4} & \text{if } x_1 > \frac{1}{2} \text{ and } x_2 \le \frac{1}{2}, \\ \frac{2x_2+1}{4} & \text{if } x_1 \le \frac{1}{2} \text{ and } x_2 > \frac{1}{2}, \\ \frac{x_1+x_2}{2} & \text{if } x_1 > \frac{1}{2} \text{ and } x_2 > \frac{1}{2}, \end{cases}$$

i.e., A is a continuous, idempotent, symmetric aggregation operator which is an upper ordinal sum in the sense of [Mesiar & De Baets 2000].

Note that the underlying t-conorm S can be read as an (S,U)-integral with respect to the S-measure m specified by $m(\{i\}) = e$ for each $i \in X$.

Observe also that, given an aggregation operator based on the (S,U)-integral, the dual aggregation operators are related to the t-norm which is dual to the t-conorm S (e.g., the operator dual to (4) given by $A(x_1,x_2) = (x_1 \cdot x_2)^{\frac{1}{3}}$ corresponds to the product t-norm $T_{\mathbf{P}}$).

All aggregation operators mentioned here can be understood as t-conorms/t-norms with implemented weights of single inputs (criteria).

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Generalized Dimension Function

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Real decision making problems do need methodologies for a better understanding rather than choice proposals. In this sense, geometrical representation will always play a key role. Classical Dimension Theory (Dushnik-Miller, 1941) seems to be a natural possibility in order to get some hint about underlying criteria, whenever basic information is given in terms of crisp preference relations being partial order sets. Such an approach presents well known algorithmic problems (Yannakakis, 1982), partially solved in Yáñez-Montero (2000).

When dealing with valued preference relations (Zadeh, 1971) we can consider the *dimension function* which evaluates the dimension for all its α-cuts, once such an α-cut is a crisp partial order set (see Montero-Yáñez-Cutello, 1998, where such an approach was applied to max-min transitive valued preference relations). Some alternative approaches in the literature do impose other formal assumptions, but they are based upon underlying representation which appears to be too difficult to be managed by decision makers.

In this paper we generalize the approach given in Montero-Yáñez-Cutello (1998), by considering a general representation for arbitrary crisp preference relations: any strict preference relation can be represented in terms of unions and intersections of linear orders (see González-Ríos, 1997, but also Fodor-Roubens, 1995). Meanwhile non comparability is explained by the intersection operator, *inconsistencies* (i.e., symmetry and non transitivity) will be associated to the union operator.

In fact: let X be a finite set of alternatives, and let us consider $\mathcal{C} = \{L/L$ linear order on X $\}$. Then for every non-reflexive crisp binary relation R on X there exist a family of linear orders $\{L_{st}\}_{s,t} \subset \mathcal{C}$ such that $R = \cup_s \cap_t L_{st}$.

Hence, given X a finite set of alternatives, the *generalized dimension* of a crisp binary relation R can be defined as the minimum number of different linear orders, L_{st}, such that $R = \cup_s \cap_t L_{st}$.

It is obvious that our generalized dimension will be the classical dimension when restricted to partial order sets. Of course, practical implementation of generalized dimension presents analogous criticism to searching classical dimension: its algorithmic complexity. However, a bound for this new concept can be obtained by a combination of algorithms presented in González-Ríos (1999) and Yáñez-Montero (2000).

Anyway, this approach will then lead to a *generalized dimension function* showing the generalized dimension for every α-cut, $\alpha \in (0,1]$, no matter our valued preference relation μ is max-min transitive or not. In case our valued preference is max-min transitive, there is a threshold α_2 between inconsistency and incomparability: there will be no union operator in the above representation if and only if $\alpha > \alpha_2$. In general, other critical values can be considered in order to get a better understanding of the complexity of our valued preference relation.

In this way, we shall find different representations for different decision maker attitudes. In fact, different people with the same valued preference relation μ, if forced to be crisp, can face different crisp problems depending on their exigency level: if the decision maker does not take into account low intensities (high α), alternatives are easily incomparable (no alternative is better enough than the other according to any underlying criteria); if the decision maker is sensible to low intensities (low α), formal cycles will be frequent; within an appropriate range, a linear order may appear, or perhaps decision maker is just transitive, or non transitive but without formal cycles.

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Inductive Inference: An Axiomatic Approach

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A predictor is asked to rank eventualities according to their plausibility, based on past cases. We assume that she can form a ranking given any memory that consists of finitely many conceivable cases. Mild consistency requirements on these rankings imply that they have a numerical representation via a matrix assigning numbers to eventuality-case pairs, as follows. Given a memory, each eventuality is ranked according to the sum of the numbers in the row, corresponding to this eventuality, over all the columns, corresponding to cases in memory. This rule generalizes ranking by empirical frequencies, as well as kernel methods for non-parametric estimation of density functions. Interpreting this result for the ranking of theories or hypotheses, rather than of specific eventualities, it is shown that one may ascribe to the predictor subjective conditional probabilities of cases given theories, such that her rankings of theories agree with their likelihood functions

The Scheme of Fuzzy Dominance

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In this paper we generalize the duality scheme in [Castagnoli & Li Calzi, 1997] introducing the definition of Şpseudo-adjointŤ, using the semi-rings of fuzzy measures and fuzzy integrals of Choquet and Sugeno instead of the vector spaces of measures with sign and Lebesgue integrals.

Keywords. categories, fuzzy measures, fuzzy integrals, functional analysis.

Aspects of Information Aggregation and Fusion

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We introduce a more general type of OWA operator called the Induced Ordered Weighted Averaging (IOWA) Operator. These operators take as their argument pairs, called OWA pairs, in which one component is used to induce an ordering over the second components which are then aggregated. A number of different decision making situations are shown to be representable in this framework, notable among these are nearest neighbor methods. We then extend this technique to the Choquet integral and introduce the idea of the induced Choquet aggregation. We then consider ordinal environments and provide a generalization of the Sugeno integral.

Diversification, Convex Preferences and Non-Empty Core in the Choquet Expected Utility Model

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We show, in the Choquet expected utility model, that preference for diversification, that is, convex preferences, is equivalent to a concave utility index and a convex capacity. We then introduce a weaker notion of diversification, namely “sure diversification". We show that this implies that the core of the capacity is non-empty. The converse holds under concavity of the utility index, which is itself equivalent to the notion of comonotone diversification, that we introduce. In an Anscombe-Aumann setting, preference for diversification is equivalent to convexity of the capacity and preference for sure diversification is equivalent to non-empty core. In the expected utility model, all these notions are equivalent and are represented by the concavity of the utility index.

Keywords: Diversification,Choquet expected utility, Capacity, Convex preferences, Core.

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Refinement of Minimum-Based Ordering in between Discrimin and Leximin

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Abstract: The discrimin ordering between two vectors of evaluations of equal length amounts to apply the minimum-based ordering, once identical components having the same rank in the two vectors have been deleted. This short, informal, note suggests extensions of the idea underlying the discrimin ordering, where *subsets* of components in the two vectors are judged to be equivalent (up to a permutation) and can thus be ignored in the comparison.

1. Background

Let $\mathbf{u} = (u_1, ..., u_i, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_i, ..., v_n)$ be two vectors having the *same length* to be compared. The u_i's and v_i's are assumed to belong to a linearly ordered scale, e.g. [0, 1], or a finite subset of it, such as {0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1} for instance. The u_i's and the v_i's can be thought of as the values of criteria is for two distinct objects, assuming the commensurateness of the criteria.

Then the discrimin ordering (Fargier *et al*., 1993 ; Dubois *et al*., 1996, 1997) can be defined in the following way. Let $D_1(\mathbf{u},\mathbf{v}) = \{i, \text{ such that } u_i = v_i\}$.

Then $\mathbf{u} >_{\text{discrimin}} \mathbf{v} \Leftrightarrow \min_{j \notin D_1(\mathbf{u},\mathbf{v})} u_j > \min_{j \notin D_1(\mathbf{u},\mathbf{v})} v_j$.

The discrimin ordering clearly refines the minimum-based ordering and the Pareto ordering:

$\mathbf{u} >_{\text{Pareto}} \mathbf{v} \Leftrightarrow \forall i\ u_i \geq v_i \text{ and } \exists_j\ u_j > v_j$;

$\mathbf{u} >_{\min} \mathbf{v} \Leftrightarrow \min_i u_i > \min_i v_i$.

Note that in the above definition of discrimin, only identical components having the same *rank* (i.e. corresponding to the same criterion) and the same value are ignored in the comparison.

The leximin ordering, which refines the discrimin ordering, is also based on the idea of deleting identical elements when comparing the two vectors, but *once* the two vectors have been reordered in a non-decreasing order.

Thus with classical discrimin, comparing
$\mathbf{u}$ = (.2, .5, .3, .4, .8) and

$\mathbf{v}$ = (.2, .3, .5, .6, .8)
amount to compare vectors **u'** and **v'** where

$\mathbf{u'}$ = (.5, .3, .4)
$\mathbf{v'}$ = (.3, .5, .6)

since $u_1 = v_1$ = .2 and $u_5 = v_5$ =.8. Thus, $\mathbf{u} =_{min} \mathbf{v}$ [1] . We still have $\mathbf{u} =_{discrimin} \mathbf{v}$, since $\mathbf{u'} =_{min} \mathbf{v'}$, but $\mathbf{u'} <_{leximin} \mathbf{v'}$ and then $\mathbf{u} <_{leximin} \mathbf{v}$.

2. Extending the discrimin: the idea

Classical discrimin is based on the elimination of identical singletons at the same places in the comparison process of the two sequences. More generally, we can work with 2 elements subsets which are identical *and* pertain to the same pair of criteria.

Namely in the above example, we may consider that (.5, .3) and (.3, .5) are "equilibrating" each other. Note that it supposes that the two corresponding criteria have the same importance. Then we delete them, and we are led to compare

$\mathbf{u''}$ = (.4)
$\mathbf{v''}$ = (.6).

Let us take another example:

$\mathbf{u}_2$ = (.5, .4, .3, .7, .9)
$\mathbf{v}_2$ = (.3, .9, .5, .4, 1)

then we would again delete (.5, .3) with (.3, .5) yielding:

$\mathbf{u'}_2$ = (.4, .7, .9)
$\mathbf{v'}_2$ = (.9, .4, 1).

Note that in this example we do not simplify .4, .9 with .9, .4 since they do not pertain to the same pair of criteria.

Moreover, simplifications can take place only one time. Thus, if the vectors are of the form $\mathbf{u}$ = (x, y, x, s) and $\mathbf{v}$ = (y, x, y, t) (with min (x, y) $\leq$ min (s, t) in order to have the two vectors min-equivalent), we may either delete components of ranks 1 and 2, or of ranks 2 and 3, leading in both cases to compare (x, s) and (y, t), and to consider the first vector as smaller in the sense of the order 2-discrimin, as soon as x < min (y, s, t).

We can now introduce the definition of the (order) 2-discrimin. Let us build a set $D_2(\mathbf{u},\mathbf{v})$ as {i,j, such that $u_i = v_j$ and $u_j = v_i$ and if there are several such pairs, they have no common components} $\cup$ {k, $u_k = v_k$}. Then the 2-discrimin is just the minimum-based ordering once components corresponding to pairs in $D_2(\mathbf{u},\mathbf{v})$ are deleted. Note that $D_2(\mathbf{u},\mathbf{v})$ is not always unique as shown by the above example. However this does not affect the result of the comparison of the vectors after the deletion of the components in $D_2(\mathbf{u},\mathbf{v})$ as it can be checked from the above formal example, since the minimum aggregation is not sensitive to the place of the terms. The 2-discrimin also includes the deletion of identical components as in the ordinary discrimin, since it would be strange to delete pairs of identical values in the comparison but not single identical values (which may blur the comparison).

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[1] If $\mathbf{u} \leq_x \mathbf{v}$ and $\mathbf{v} \leq_x \mathbf{u}$, we write $\mathbf{u} =_x \mathbf{v}$ for x = min, discrimin or leximin.

The 2-discrimin refines the discrimin ordering. It deletes pairs of values which play a neutral role in the aggregation, but which may lead to ties if these values are not ignored. Indeed in the first example $\mathbf{u} =_{discrimin} \mathbf{v}$, while $\mathbf{u} \leq_{2\text{-discimin}} \mathbf{v}$.

3. Order k - discrimin

Clearly we can work with 3 elements-sets as well or more generally with k elements-sets. The two sub-vectors should be identical up to a permutation. This means that for the corresponding subset of criteria, any symmetrical combination of the three (or more generally k) evaluations yields the same result for each vector. Then these components can be ignored in the comparison of the vectors.

For instance, the two vectors
$\mathbf{u}_3$ = (.3, .5, .2, 1, .6)
$\mathbf{v}_3$ = (.6, .3, .4, .7, .5)
can be simplified into
$\mathbf{u'}_3$ = (.2, 1)
$\mathbf{v'}_3$ = (.4, .7).
since (0.3, 0.5, 0.6) and (0.6, 0.3, 0.5) are equivalent up to a permutation on the set of ranks (1, 2, 5). It can be done for larger sets as well.

Let us call the classical discrimin ’1-discrimin’, and the others ’2-discrimin’, ’3-discrimin’, and so on. For 3-discrimin, we have to build a set
D3($\mathbf{u}$,$\mathbf{v}$) = {i,j,k, $\exists$ a permutation σ defined on{i,j,k}, $u_i = v_{\sigma(i)}$ and $u_j = v_{\sigma(j)}$ and $u_k = v_{\sigma(k)}$ and if there are several such permutations they do not overlap} $\cup$ $D_2(\mathbf{u},\mathbf{v})$.

Note that 3-discrimin incorporates 2-discrimin and 1-discrimin. Again there may exist several ways of building $D_3(\mathbf{u},\mathbf{v})$. Then the situation may become more intricate than with the 2-discrimin as shown by the following example:

Let $\mathbf{u}$ = (0.3, 0.5, 0.8, 1., 0.6, 0.7, 0.5)
$\mathbf{v}$ = (0.6, 0.3, 0.4, 0.7, 0.5, 0.6, 0.7)
There exist two overlapping permutations :
(0.3, 0.5, 0.6) with (0.6, 0.7, 0.5)
and (0.6, 0.3, 0.5) with (0.5, 0.6, 0.7).
Ignoring the values involved in the first permutation, we obtain $\mathbf{v} <_{min} \mathbf{u}$ since 0.4 < 0.5, while using the second permutation, we get $\mathbf{v} =_{min} \mathbf{u}$ since both vectors of remaining values lead to the same minimum 0.3. This points out that the definition of the 3-discrimin should be further refined by choosing the permutation, which lead to a discriminant situation if possible. Note that this permutation should involve the minimum value of the vectors, to be of interest.

Moreover, let us consider two vectors of the form
$\mathbf{u}$ = (a x b b s)
$\mathbf{v}$ = (b a x a t).
Simplifying (a, x, b) with (b, a, x) leads to compare (b, s) with (a, t), while simplifying the components for i = 1 and i = 4 (i.e., (a, b) with (b, a)) would lead to compare (x, b, s) with (a, x, t). This shows (consider the case where x < min(a, b, s, t) that when triples and pairs are overlapping,

the deletion of the triples can be more efficient in the comparison (as soon as we are using the 3-discrimin and not only the 2-discrimin).

However they cannot exist two overlapping permutations which, after ignoring their components in the min-based comparison, would lead to opposite orderings (namely **u** > **v** and **v** > **u**). This holds for the k-dicrimin as well. This can be seen by considering vectors of the form

u = (t a b s x t y) and

v = (b s a t s x z)

where y and z are such that $\mathbf{u} =_{\min} \mathbf{v}$. Simplifying by the first four components leads to compare min(x, t, y) with min(s, x, z), while deleting (s, x, t) with (t, s, x) leads to the comparison of min(t, a, b, y) with min(b, s, a, z). Clearly the two comparisons cannot disagree with each other (i.e. cannot lead to **u > v** and **v > u** respectively). This remark could be further generalized by replacing some of the above vector components by sub-vectors. In case of several overlaps, this analysis can be iterated.

So the generalized procedure for applying k-discrimin is to look for permutations of orders 1, 2, …, k and to explore the different possibilities in case of overlapping permutations until a discriminating one is found.

k discrimin and leximin

Besides, note that 'k-discrimin' amounts to a limited leximin on k-long subsequences. Thus 'k-discrimin' provides orderings in between discrimin and leximin. However the n-discrimin ordering may remain less discriminating than the leximin ordering as shown by the following example.

u = (0.4, 0.5, 0.3, 0.7, 0.6)

v = (0.6, 0.3, 0.5, 0.4, 1).

Then $\mathbf{u} <_{\text{leximin}} \mathbf{v}$ (since 0.7 < 1),

while $\mathbf{u} =_{\text{5-discrimin}} \mathbf{v}$ (since (0.4, 0.7, 0.6) and (0.6, 0.4, 1) have the same minimum), since 5-discrimin and 2-discrimin are equivalent here.

4. Concluding remarks

Discrimin and more generally k-discrimin, which takes advantage of the perfect identity of the components, are well in the spirit of discrete scales.

We may think of further generalizing the above approach in the following way. We can simplify vectors to be compared by deleting components i and j, if for some meaningful aggregation function f, we have $f(u_i, u_j) = f(u_j, v_j)$, here extending the 2-discrimin. For instance, we may simplify (0.4, 0.6) with (0.5, 0.5) because they have the same average inside vectors which are originally min-equivalent (i.e. such as $\min_k u_k = \min_k v_k$). For instance, considering the two vectors **u** = (0.3, 0.4, 0.6, 0.8) and **v** = (0.3, 0.5, 0.5, 0.4), this may lead (together with ordinary discrimin) to find **v** < **u,** while $\mathbf{u} =_{\text{discrimin}} \mathbf{v}$. However, this may be delatable in a situation where **u** is to be compared to **v’** = (0.3, 0.5, 0.5, 0.7) since it would lead to **v’** < **u** while $\mathbf{u} <_{\text{discrimin}} \mathbf{v'}$! This points out that this may be used only for vectors which are discrimin equivalent. Moreover, the non-unicity of the extension of $D_2(\mathbf{u},\mathbf{v})$ may also create difficulties for some functions f.

Another extension of discrimin ordering have been recently proposed by Dubois and Fortemps (1999) for vectors of unequal lengths leading to the general notion of delocalized discrimin where vectors can be completed by introducing the top element of the scale, i. e., 1 here, in places in between components of the original vectors. In that extension, components of the two vectors having the same rank, e.g. u_i, and v_i, may have different ranks once the '1' are added. This clearly corresponds to an extension which differs from the above approach. We may think of combining the two ideas.

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On Concordance Rules Based on Non-Additive Measures: An Axiomatic Approach

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In multiattribute decision theory, various models have been proposed to evaluate and compare the alternatives and to support the decision making process. For that purpose, the mathematical representation of DM's preferences is a key issue. This modelling tasks consists in representing a preference relation $\succsim$, given on the multiattribute space $X = X_1 \times \ldots \times X_n$ by a *decision rule* defining the preference $x \succsim y$, for any pair of alternative (x,y), as a function of their vectors of attributes values $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ in X. Basically, one can distinguish two different approaches to define $\succsim$:

The Aggregate then Compare approach (AC). This approach is exemplified by the works of Fishburn [9], Keeney and Raiffa [12]. It consists in summarizing each vector x by a quantity $u(x)$ (the *utility* of x) representing the subjective value of x for the Decision Maker. This utility is obtained by aggregation of marginal utilities $u_j(x_j)$, very often a weighted sum. Denoting ψ the aggregation operator, the preference relation $\succsim$ is defined by:

$$x \succsim y \Leftrightarrow \phi(\psi(u_1(x_1),\ldots,u_n(x_n)),\psi(u_1(x_1),\ldots,u_n(x_n))) \geq 0$$

where ϕ is a comparison function. A classical choice for ϕ is $\phi(x,y) = x-y$, leading to complete and transitive preferences. Function ψ is very often a weighted sum, thus leading to the so-called "additive utility model". Many representation theorems for such an additive model have been proposed in the literature on transitive conjoint measurement, see *e.g.* [14, 15, 21]. More generally, one can possibly use another Choquet integral, allowing positive and negative synergies between criteria (see [10, 11], and [18] for representation results), or in a more qualitative framework, a Sugeno integral [19, 4].

The Compare then Aggregate approach (CA). This approach consists in comparing, for any pair (x,y) in X^2 and each attribute $j = 1,\ldots,n$, the attribute values x_j and y_j so as to decide whether x is at least as good as y according to the j^{th} component. This yields n preference indices $\phi_j(x,y)$. These indices are then aggregated before performing the following preference test:

$$x \succsim y \Leftrightarrow \psi(\phi_1(x,y),\ldots,\phi_n(x,y)) \geq 0$$

Choosing ϕ as: $\phi_j(x,y) = 1$ if $u_j(x_j) > u_j(y_j)$, $\phi_j(x,y) = 0$ if $u_j(x_j) = u_j(y_j)$, $\phi_j(x,y) = -1$ otherwise and $\psi(\alpha_1,\ldots,\alpha_n) = \sum_{j=1}^{n} \mu(j)\alpha_j$ leads to the *additive concordance rule*. When $\mu(j)$ are all equal, we obtain a well known majority rule where $x \succsim y$ iff a majority of attributes is concordant with this preference. This rule is also used in Electre-like methods [16, 20, 17] for multicriteria decision making, when $\mu(j), j = 1,\ldots,n$ represent the weights of criteria (this implicitly defines an additive measure of importance represented by $\mu(A) = \sum_{j\in A} \mu(j)$ for any subset A of attributes). The ordinal nature of the CA approach is worthwhile noticing. It indeed amounts to constructing n preference *relations* $\succsim_j, j = 1,\ldots,n$ (characterised by functions ϕ_j by $x \succsim_j y \Leftrightarrow \phi_j(x_j,y_j) \geq 0$) which are then aggregated to form the overall preference *relation* $\succsim$. Additive concordance rules generally lead to non-transitive preference models. One can find representation results linked with particular concordance rules in the literature on non-transitive conjoint measurement, see *e.g.* [7, 8, 2, 1, 6].

Generalised Concordance Rules. The additive concordance rules introduced above can be cast in a more general setting. First a preference relation $\succsim_j$ is supposed to exist on each attribute range X_j. It can be derived from the marginal utility functions if any (then $x_j \succsim_j y_j \Leftrightarrow u_j(x_j) \geq u_j(y_j)$) or introduced as such from scratch by the decision maker. Let $\succ_j$ and $\sim_j$ denote the strict preference and the indifference relations derived from $\succ_j$. The following coalition of attributes derives from the marginal preferences:

$$C_{\succsim}(x,y) = \{j \in N, x_j \succsim_j y_j\}$$

$C_{\succsim}(x,y)$ is the set of criteria where x is as least as good as y. Finally, assume an importance relation $\succsim_I$ exists on 2^N, whereby $A \succsim_I B$ means that the group of attributes A is as least as important as the group B. It can be derived from the importance function if any (then $A \succsim_I B \Leftrightarrow \mu(A) \geq \mu(B)$) or introduced as such from scratch by the decision maker. Such a relation is supposed to be reflexive, and monotonic:

$$A \succsim_I B \Rightarrow A \cup C \succsim_I B \quad \text{and} \quad A \succsim_I B \cup C \Rightarrow A \succsim_I B$$

This property is satisfied if $\succsim_I$ derives from a capacity μ ($A \succsim_I B \Leftrightarrow \mu(A) \geq \mu(B)$). Importance relations derived from additive capacities also obey the following property of *preadditivity*:

$$\forall A,B,C \subseteq N, \quad (A \cap (B \cup C) = \emptyset \Rightarrow (B \succsim_I C \Leftrightarrow A \cup B \succsim_I A \cup C))$$

However, it is well known that preadditivity of $\succsim_I$ does not imply that it is representable by an additive capacity (see [13]). Let us now define generalized concordance rules:

Definition 1. A *generalized concordance rule* defines a preference relation $\succsim$ on X from the relations $\succsim_j$ on X_j, $\forall j = 1,\ldots,n$ and the relation $\succsim_I$ on 2^N as follows:

$$x \succsim y \Leftrightarrow C_{\succsim}(x,y) \succsim_I C_{\succsim}(y,x) \qquad (GCR)$$

This definition is a MCDM counterpart to (and a generalization of) the "lifting rule" proposed by [3, 5] for decision under uncertainty. When $\succsim_I$ (resp. $\succsim_j$) derive from a capacity function μ (resp. a utility function u_j), or equivalently they are weak orders (and thus always representable by capacity and utility functions), the previous rule becomes:

$$x \succsim y \Leftrightarrow \mu(C_{\succsim}(x,y)) \geq \mu(C_{\succsim}(y,x))$$

The additive concordance rule is recovered when μ is an additive capacity.

Our study is dedicated to the *GCR* decision rules defined on a finite cartesian product X. After discussing the use of measurement scales in the AC approach, we emphasize the interest of ordinal approaches to preference modelling, and especially the *GCR* rules. Using simple examples, we illustrate the descriptive ability of such rules and the potential interest of particular instances obtained from non-additive capacity functions. Then we investigate preference structures which can be represented by a rule of type *GCR* and propose a characterization result. Finally, we investigate the subclass of *GCR* rules compatible with *quasi-transitive* preferences. It is shown that such rules are based on a hierarchy of oligarchies of attributes. More precisely, there exists a group of attributes that unanimously decides on the preference, indifference/incomparability between two alternatives. In case of indifference, the decision is possibly left to another less powerful oligarchy and so on. Such structures are illustrated by non-additive concordance rules based on possibility and necessity measures.

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The Choquet Integral versus the Šipoš Integral, Choquet Expected Utility versus Prospect Theory, and Concave/Convex versus Cavex Measures: Three Logically Independent, but Practically Related, Debates

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The Choquet integral has been well known as a method for integration with respect to nonadditive and fuzzy measures (for the latter it was first proposed by Ulrich Höhle, 1982). Less known is the Šipoš integral, a variation which may be more natural due to its symmetry around zero. In mathematical decision theory, Choquet expected utility, named after and based on the Choquet integral, has received much attention. Remarkably and unknown to its inventors Kahneman and Tversky, prospect theory is a natural generalization of the Šipoš integral rather than of the Choquet integral, with the special mathematical role of the zero outcome in the Šipoš integral reflected by the special psychological role of the zero outcome in human decision making. Whereas prospect theory is prevailing over Choquet expected utility in empirical applications, its mathematical theory remains underdeveloped today.

Empirical studies have revealed not only a special role of the zero outcome, but also the interest of new conditions of fuzzy measures. Not concavity or convexity (also called superadditivity), but a mix of these two, cavexity, seems to be the central condition in human decision making. The condition reflects a decreased degree of sensitivity, rather than a general aversion or pessimism, towards vague information. The vaguer the information is the closer it should be to fifty-fifty, rather than to logically impossible.

This lecture will present theoretical foundations for fuzzy measures to reflect the conditions mentioned above.

Decision Making Based on Hybrid Probability-Possibility Measure

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Pseudo-analysis is based on pseudo-operations which enables the extension of the classical analysis, see [1, 9, 10, 11, 12, 14, 15, 16, 17, 18]. We shall use a part of pseudo-analysis to extend the classical von Neumann Morgenstern utility theory [13], which is based on the probability. First of all we are using results on special pseudo-additive measures: max-measures, see [3, 5, 7, 9, 11].

A basic notion in probability theory is independence. The main issue in probabilistic independence is the existence of special events $A_1, \ldots, A_n$ such that $P(A_1 \cap \cdots \cap A_n) = \prod_{i=1}^n P(A_i)$. Such events are called independent events. In order to preserve the computational advantages of independence, any operation $*$ for which it could be established that $P(A_1 \cap \cdots \cap A_n) = *_{i=1}^n P(A_i)$, would do. However the Boolean structure of sets of events and the additivity of the probability measure, impose considerable constraint on the choice of operation $*$. In the paper [6] was studied the possible operations $*$ when changing P for a pseudo-additive (decomposable) measure based on a t-conorm S. A first remark is that it is natural to require that $*$ be a continuous triangular norm. It turns out by [6] that the only reasonable pseudo-additive measures admitting of an independence-like concept, are based on conditionally distributive pairs (S, T) (see [9], condition (CD)) of conorms and t-conorms, namely:

(a) probability measures (and $* =$ product);

(b) possibility measures (and $*$ is any t-norm);

(c) suitably normalized hybrid set-functions m such that there is $a \in]0,1[$ which gives for A and B disjoint

$$m(A \cup B) = \begin{cases} m(A) + m(B) - a & \text{if } m(A) > a, m(B) > a, \\ \max(m(A), m(B)) & \text{otherwise,} \end{cases}$$

and for separability:

$$m(A \cap B) = \begin{cases} a + \frac{(m(A)-a)(m(B)-a)}{1-a} & \text{if } m(A) > a, m(B) > a, \\ a \cdot T_1(\frac{m(A)}{a}, \frac{m(B)}{a}) & \text{if } m(A) \le a, m(B) \le a, \\ \min(m(A), m(B)) & \text{otherwise,} \end{cases}$$

Remark. Cox's well-known theorem, see [2, 19], which justifies the use of probability for treating uncertainty, was discussed in many papers. It is clear that the family of pseudo-additive measures exhibited in this paper is worth studying in Cox relaxed framework, since a natural form of conditioning can be defined on the basis of the triangular norm in the pair (S,T) satisfying (CD), leading to an almost regular independence notion exhibited here.

Suppose m is a S-measure on $X = \{x_1, x_2, x_3\}$ and $m_i = m(\{x_i\})$. Suppose we want to decompose the ternary tree into the binary tree so that they are equivalent. Then the reduction of lottery property enforces the following equations

$$S(v_1, v_2) = 1, \quad T(\mu, v_1) = m_2, \quad T(\mu, v_2) = m_3,$$

where T is the triangular norm that expresses separability for S-measures. The first condition expresses normalization (with no truncating effect for t-conorm S allowed). If these equations have unique solutions, then by iterating this construction, any distribution of a S-measure can be decomposed into a sequence of binary lotteries. This property is basic in probability theory since it explains why probability trees can be used as a primitive notion for developing the notion of probability after Shafer [20]. Turning the S-measure into a sequence of binary trees leads to the necessity of solving the following system of equations

$$\alpha_1 = T(\mu, v_1),\ \alpha_2 = T(\mu, v_2),\ S(v_1, v_2) = 1 \tag{1}$$

for given α_1 and α_2. Assuming that $T_1 = \min$ we have solved (1) completely in [6] and exhibited the analytical forms of (μ, v_1, v_2).

We define the set $\Phi_{S,a}$ of ordered pairs (α, β) in the following way

$$\Phi_{S,a} = \{(\alpha,\beta) \mid (\alpha,\beta) \in]a,1[, \alpha + \beta = 1 + a\}$$

$$\cup \{(\alpha,\beta) \mid \min(\alpha,\beta) \le a, \max(\alpha,\beta) = 1\}.$$

A *hybrid mixture set* is a quadruple $(\mathcal{G}, M, T, S)$ where $\mathcal{G}$ is a set, (S,T) is a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (CD) and $M : \mathcal{G}^2 \times \Phi_{S,a} \to \mathcal{G}$ is a function (hybrid mixture operation) given by

$$M(x,y;\alpha,\beta) = S(T(\alpha,x), T(\beta,y)).$$

It is enough to restrict to the case $(< S_M, S_L >, < T_1, T_P >)_a$. Then it is easy to verify that M satisfies the axioms **M1-M5** on $\Phi_{S,a}$, see [6].

Let (S,T) be a pair of continuous t-conorm and t-norm, respectively, of the form $(< S_M, S_L >, < T_1, T_P >)_a$. Let u_1, u_2 be two utilities taking values in the unit interval $[0,1]$ and let μ_1, μ_2 be two degrees of plausibility from $\Phi_{S,a}$. Then we define *the optimistic hybrid utility function* by means of the hybrid mixture as

$$U(u_1, u_2; \mu_1, \mu_2) = S(T(u_1, \mu_1), T(u_2, \mu_2)).$$

In the paper [6] it is examined in details this utility function. Although the above description of optimistic hybrid utility is rather complex, it can be easily explained, including the name optimistic, see [6]. Putting together the results of this paper, the utility of a n-ary lottery can be computed by decomposing the S- measure into a sequence of binary trees and applying the above computation scheme for hybrid utility recursively from the bottom to the top of the binary tree expansion.

More details and proofs of theorems stated in this paper can be found in [6].

Open problem. Find a corresponding axiomatization for hybrid utility as was done for classical utility theory in [13] and possibility utility theory in [5].

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Pareto-Optimality and Qualitative Aggregation Structures

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Among non-additive, ordinal methods for criteria aggregation and decision under uncertainty, some have their origin in an approach first proposed by Bellman and Zadeh in 1970. Instead of maximising sums of degrees of satisfaction pertaining to various criteria, they proposed to maximise the minimum of such degrees, thus leading to a calculus of fuzzy constraints, for instance [1]. Unfortunately, rankings of solutions using such qualitative techniques are usually rather coarse. This drawback seems to undermine the merits of qualitative techniques, whose appeal is to obviate the need for quantifying utility functions. Worse, some of the generally not unique maximin optimal solutions, may fail to be Pareto Optimal. Besides, other well-behaved aggregation operations on finite ordinal scales seem to be constant on significant subsets of their domains [4], which make these aggregations not so attactive in practice.

This work starts an investigation of some limitations of finitely-scaled methods for criteria aggregation, and a search for remedies to these limitations. Given a finite, totally ordered set $(X, \geq)$ with top 1 and bottom 0, consider an aggregation function $f : X^n \rightarrow X$, which, by definition, is increasing in the wide sense and such that $f(1, 1 \ldots 1) = 1$ and $f(0, \ldots 0) = 0$. It can be shown that maximising f over a subset $S \subseteq X^n$ of n-tuples generally leads to a maximising set that contains non Pareto - optimal solutions. This fact is rather unsurprising since using f as a ranking function comes down to sorting $|X|^n$ elements into $|X|$ sets of equally ranked n-tuples.

Clearly it shows that the discriminating power of qualitative aggregation operations is bound to be unacceptably weak and intuitively debatable. One way out of this difficulty may be to use functions from X^n to a bigger finite scale Y. However , this idea is not satisfactory from a practical point of view since the combinatorics of functions from X^n to Y become rapidly prohibitive as Y is bigger, and are thus much higher than those of functions from X^n to X.

When f = min, the natural way to tackle the problem has been to introduce relations that naturally refine the min-ordering, and restore the Pareto optimality of the selected maximal solutions. Such relations are the discrimin ordering and the leximin ordering [2, 3]. The discrimination power of the latter is maximal, i.e. it is equal to that of the most discriminating symmetric aggregation operations.

In this work, we try to generalise this refinement technique to more general families of aggregation operation. We restrict to the case of symmetric functions. Consider a family $\{f^p\}_p$ of symmetric functions $X^p \to X$. For any positive integer p, f^p is supposed to be

-) *extensively preferentially consistent* with f^{p-1}: $\forall$ u $\in$ X,

$f^{p-1}(x_1 \dots x_{p-1}) \geq f^{p-1}(y_1 \dots y_{p-1})$ imply $f^p(x_1 \dots x_{p-1}, u) \geq f^p(y_1 \dots y_{p-1}, u)$.

-) *globally strictly monotone* :

if $x_i > y_i$ $\forall$ i = 1, p, then $f^p(\boldsymbol{x}) > f^p(\boldsymbol{y})$ where $\boldsymbol{x}$ and $\boldsymbol{y} \in X^n$.

These conditions look natural in the scope of applications. The first condition is a weak form of preferential independence. They are satisfied by the minimum, the maximum (but not other order-statistics). By convention f^1 is the identity function on X. Call $\{f^1 \dots f^p \dots\}$ a qualitative aggregation structure. We use the nottion f when the number of arguments is not emphasized. The generalisation of discrimin and leximin to such aggregation structures is as follows:

Discri – f :Let $D(\boldsymbol{x}, \boldsymbol{y}) = \{i, x_i \neq y_i\}$ be the discriminating set of components for $\boldsymbol{x}$ and $\boldsymbol{y}$. Then define $\boldsymbol{x} \geq_{\text{discri f}} \boldsymbol{y} \Leftrightarrow f(\{x_i, i \in D(\boldsymbol{x}, \boldsymbol{y})\}) \geq f(\{y_i, i \in D(\boldsymbol{x}, \boldsymbol{y})\})$

Lexi – f : Let $\boldsymbol{x} \in X^n$, and let $V(\boldsymbol{x}) = \{x \in X, \exists\, i \in \{1\dots n\}, x_i = x\}$ be the set of distinct values in the vector $\boldsymbol{x}$. Let $k_{\boldsymbol{x}}(x)$ = number of times the value x appears in $\boldsymbol{x}$. Let $M(\boldsymbol{x})$ be the multi-set induced by $\boldsymbol{x}$: $\forall\, \boldsymbol{x} \in X$, the degree of "membership" of $\boldsymbol{x}$ to $M(\boldsymbol{x})$ is $k_{\boldsymbol{x}}(x)$. Let $M(\boldsymbol{x}) - M(\boldsymbol{y})$ be the multi-set with membership function $\max(0, k_{\boldsymbol{x}} - k_{\boldsymbol{y}})$. Denoting $\Sigma_{x \in X} k_{\boldsymbol{x}}(x)$ the cardinality of $M(\boldsymbol{x})$. It is obvious that $|M(\boldsymbol{x})| = n = |M(\boldsymbol{y})|$, $\forall\, \boldsymbol{x}, \boldsymbol{y} \in X^n$. Hence $|M(\boldsymbol{x}) - M(\boldsymbol{y})| = |M(\boldsymbol{y}) - M(\boldsymbol{x})|$. Then define : $\boldsymbol{x} \geq_{\text{lexi-f}} \boldsymbol{y} \Leftrightarrow f(M(\boldsymbol{x}) - M(\boldsymbol{y})) \geq f(M(\boldsymbol{y}) - M(\boldsymbol{x}))$

It can be shown that under mild conditions such as global monotonicity and extensive preferential consistency, lexi-f and discri-f maximal solutions are indeed Pareto-optimal, and that the corresponding ordering of solutions is quite discriminant.

Globally strictly monotone aggregation functions on X are easily proved to be idempotent on finite scales, since if $0 = x_1 < x_2 < \dots < x_m = 1$ it follows that $f(x_i \dots x_i) = x_i$. It rules out the Archimedean t-norms and conorm–like operations on finite sets [4]. Moreover, the only associative idempotent aggregation operations different from min and max, the α-medians ($f(x, y) = \text{median}(\alpha, x, y)$) are generally not globally strictly monotone, since they are constant on large subsets of X^n. The above extensions of leximin and discrimin orderings to these operations thus do not possess enough discrimination power; they can only be refined by directly adopting Pareto-ordering on the ranges where these aggregation functions are constant .

The simplest non trivial example of aggregation structure is for X = {1, 2, 3}. Adopting the lexi-f^2 ordering does not leave many degrees of freedom: one must indicate the relative position of $f^2(1, 3)$ and $f^2(2, 2)$. If $f^2(1, 3) > f^2(2, 2)$, this is the leximax ordering. If $f^2(1, 3) < f^2(2, 2)$, this is the leximin ordering. If $f^2(1, 3) = f^2(2, 2)$, this is a kind of ordinal average (which is less discriminant). With three arguments, the ordering of 3-tuples is fixed by further positioning (2, 2, 2) with respect to (1, 1, 3) and (1, 3, 3) (note that $(1, 3, 3) >_{\text{lexi-f}} (1, 1, 3)$ in any case). The lexi-f positions of (1, 2, 2) with respect to (1, 1, 3), (1, 2, 3) with respect to (2, 2, 2), and (2, 2, 3) with respect to (1, 3, 3), are enforced by the position of (2, 2) with respect to (1, 3). If $f^2(1, 3) > f^2(2, 2)$, then $f^3(1, 3, 3) \geq f^3(2, 2, 2)$ and only the relative position of (2, 2, 2) and (1, 1, 3) is left open. If $f^2(2, 2) > f^2(1, 3)$, then $f^3(2, 2, 2) \geq f^3(1, 1, 3)$ and only the relative position of (2, 2, 2) and (1, 3, 3) is left open.

The generation of complete preorderings of tuples of elements from a finite ordered scale in agreement with Pareto-ordering and symmetry has been considered in Moura-Pires and Prade[5] in the scope of fuzzy constraint satisfaction problems. A natural question is whether any such complete preorderings of X^n can be obtained as a lexi-f ordering for some qualitative aggregation structure $\{f^1 \dots f^n\}$, and more generally, can be generated by a small number of extra constraints on the relative positioning of a few tuples. Unfortunately the answer for lexi-f orderings is negative. A counterexample is obtained using a 4-element scale X = {0, 1, 2, 3}. Then f^2 is characterized by the relative positionings of (0, 2) w.r.t. (1, 1), (1, 3) w.r.t. (2, 2), and (0, 3) w.r.t. (1, 1) and (2, 2). However, using $f : X^2 \varnothing X$, $f^2(0, 3)$ $\{f^2(0, 0), f^2(1, 1), f^2(2, 2), f^2(3, 3)\}$. Neither the discri-f nor the lexi-f extension, nor even the adding of Pareto-ordering itself can generate the complete preorderings such that (1, 1) < (0, 3) < (2, 2). There are 12 total orderings which are Pareto-compatible and respect symmetry in this example, and only 8 of them can be generated as a lexi-f ordering via an aggregation structure $X^n \varnothing X$. Generating the other total orderings requires a function $X^2 \varnothing Y$ where Y has 5 levels.

Another property which may simplify the study of qualitative aggregation structures is the following regularity : let (i, j) X^2 = {0, 1, 2, …, n};

if $f^2(i, j) > f^2(i + 1, j - 1)$, then $f^2(i + 1, j + 1) > f^2(i + 2, j)$.

The combinatorics of such regular aggregations functions look moderate and deserve further exploration. However the above study has exhibited some intrinsic limitations of the otherwise appealing finite setting for criteria aggregation using a single finite scale, whereby concise representations and functions having good algebraic properties turn out to lack expressivity, even under natural lexicographic-like extensions .

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Symmetric and Asymmetric Sugeno Integrals

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In decision making, either in the uncertainty, risk, or multicriteria framework, it may be useful to use negative numbers for describing scores or utilities. In multicriteria decision making, a negative score means that the value of the attribute is less than an a priori fixed level of neutrality or indifference, making the frontier between good and bad scores (bipolar sales). In decision under uncertainty or risk, negative utilities are interpreted as losses, while positive utilities are gains.

Cumulative prospect theory [5] is based on this distinction of gains and losses, and there an extension of the Choquet integral to negative values is introduced for representing natural behaviours. Mathematically speaking, this extension is called the symmetric integral by Denneberg [1], an integral introduced already by Šipoš [6] in 1979. The main property of this integral, which we denote by the functional $\check{\mathcal{C}}\mu$, μ being the fuzzy measure or capacity, is:

$$\check{\mathcal{C}}_\mu(-f) = -\check{\mathcal{C}}_\mu(f)$$

for any act f. By contrast, the usual definition of the Choquet integral fulfills $\mathcal{C}_\mu(-f) = -\mathcal{C}_{\overline{\mu}}(f)$, where $\overline{\mu}$ is the conjugate fuzzy measure.

In multicriteria decision making on bipolar scales, the use of the symmetric and asymmetric (Choquet) integrals have been studied by the author [3, 4].

The preceding discussion was held in the case where utilities and scores are on a true numerical scale (either interval or ratio scale). It is however often the case in decision making that available information is of qualitative nature. For example, in multicriteria decision making problems, qualitative scales such as (bad, acceptable, good) are given to the decision maker in order to put scores for each criteria. In this situation, the use of the Choquet integrals —and hence its extensions to negative numbers— is forbidden, since a interval or a ratio scale is assumed. Then, the Sugeno integral, defined only with minimum and maximum, appears to be the right counterpart for the ordinal case. It has been axiomatised by Dubois et al., in the framework of decision under uncertainty [2].

The question arises now, where it is possible to define a symmetric and an asymmetric extension of the Sugeno integral, in order to derive models similar to cumulative prospect theory in an ordinal framework. This paper addresses this topic, and begins by the introduction of an appropriate ordinal scale, where the notion of negative number is introduced.

We build from an ordinal scale E^+ with least element $\mathbb{0}$ a symmetrical scale E^-, and introduce two ordinal operators $\varovee$ and $\varowedge$, extending maximum and minimum on $E = E^+ \cup E^-$ so that to have

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an algebraic structure similar to that of a ring. In fact $(E, \varovee, \varowedge)$ fulfills all properties of a ring except associativity and distributivity which lives only on E^+ and E^- separately.

With this structure, it becomes possible to define a symmetric Sugeno integral $\check{\mathcal{S}}_\mu$, which fulfills $\check{\mathcal{S}}_\mu(-f) = -\check{\mathcal{S}}_\mu(f)$. Also, taking the ordinal Möbius transform, the expression of the symmetric Sugeno integral w.r.t. the Möbius transform is similar to the one obtained for the Šipoš integral.

Lastly, we focus on the multicriteria decision making framework with bipolar ordinal scale. We show how to build a preference modelling based on the symmetric Sugeno integral.

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Sugeno Integral and Decision Rule Representation for Multicriteria Sorting Problems and Decisions under Risk

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We consider a multicriteria sorting problem consisting in assignment of some actions to some predefined and preference-ordered decision classes. The actions are described by a finite set of criteria. The sorting task is usually performed using one of three preference models: discriminant function (as in scoring methods, discriminant analysis, UTADIS), outranking relation (as in ELECTRE TRI) or "*if ..., then ...* " decision rules that involve partial profiles on subset of criteria and dominance relation on these profiles. A challenging problem in multicriteria sorting is the aggregation of ordinal criteria. To handle this problem some max-min aggregation operators have been considered, with the most general one - the fuzzy integral proposed by Sugeno (1974). We show that the decision rule model has some advantages over the integral of Sugeno. More generally, we consider the multicriteria sorting problem in terms of conjoint mesurement and prove a representation theorem stating an equivalence of a very simple cancellation property ensuring monotonicity of the aggregation, a general discriminant function and a specific outranking relation, on the one hand, and a decision rule model on the other hand. With respect to Sugeno integral, there is a positive and a negative interpretation of our result. Positive interpretation says that any preference model expressed in terms of the Sugeno integral can be represented by a set of specific decision rules called single graded decision rules. Negative interpretation says that not all preference models represented by a set of general(i.e. multi-grade) decision rules can be represented also in terms of the Sugeno integral. Therefore, the decision rule model has a larger applicability than the Sugeno integral,which is a clear advantage. In our opinion there is also another advantage of the decision rule model, which is perhaps more important for multicriteria decision aiding: the decision rule model expresses the preference model in much more intelligible terms than the Sugeno integral.

Moreover, we consider a more general decision rule model based on the rough sets theory being one of emerging methodologies for extraction of knowledge from data. The advantage of the rough set approach in comparison to competitive methodologies is the possibility of handling inconsistent data that are often encountered in preferential information, due to hesitation of decision makers, unstable character of their preferences, imprecise or incomplete information and the like. We propose a general model of conjoint measurement that, using the basic concepts of the rough set approach (lower and upper approximation), is able to represent these inconsistencies by a specific discriminant function.

We show that these inconsitencies can also be represented in a meaningful way by *“if . . . , then . . . ”* decision rules induced from rough approximations.

Finally, we use these theoretical results to open a new avenue for applications of the rough set concept to decision support. We consider the classical problem of decision under risk proposing a rough set model based on stochastic dominance. We start with the case of traditional additive probability distribution over the set of states of the world, however, the model is rich enough to handle non-additive probability distributions and even qualitative ordinal distributions. The rough set approach gives a representation of decision maker’s preferences in terms of "ifĚ, then..." decision rules induced from rough approximations of sets of exemplary decisions. We prove that also in this case decision rule representation is based on a set of very few and simple axioms and it is more general than the Sugeno integral.

Semiotical Foundation of Multicriteria Preference Aggregation

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Abstract

In this communication, we investigate the semiotical foundation of the logical approach to multicriteria preference aggregation promoted since 1969 by Bernard Roy (see [9]). We first show the denotational isomorphism which exists between his concordance principle and the proceeding of balancing reasons as promoted by C.S. Peirce. This result illustrates the split truth versus falseness denotation installed by the concordance principle. Taking furthermore support on the Peircian distinction between credibility and state of belief concerning a preferential assertion, we propose a semiotical foundation for the numerical determination of the truth assessment knowledge carried by the family of criteria. This approach, first makes apparent the semiotical requirements guaranteeing the coherence of a family of criteria, but also allows to extend the concordance principle in order to support pairwise overlapping criteria, incomplete performance tableaux and/or ordinal criteria weights.

In order to describe the preferences a decision maker might express concerning a given set of decision actions, Roy (see [10, 11]) considers essentially all multiple pragmatic consequences they involve. The elaboration of a consequence spectra follows precise methodological requirements that are:

- an *intelligibility principle*: Its components must gather as directly as possible all imaginable consequences such that the decision maker is able to understand them with respect to each of the underlying preference dimension;
- an *universality principle*: The components must ideally cover all preference dimensions that reflect fundamental and unanimous outranking judgments concerning the set of all decision actions.

On this basis, Roy constructs a *family of criteria functions* allowing partial truth assessment of pairwise outranking situations. A global (universal) outranking assertion is thus *truth or falseness warranted from multiple points of view* depending on the decomposition of the cloud of consequences into separated preference dimensions.

Aggregating these partial truth assessments is then achieved via the *concordance principle*, where the *credibility* of the corresponding universal outranking situation is computed as the sum of the relative weight of the subset of criteria confirming truthfulness of the corresponding assertion. If a majority of criteria is *concordant* about supporting this truthfulness, the outranking situation may be affirmed to be more or less *true* depending on the effective majority it obtained. The *credibility* of an universal outranking situation is computed as the sum of the relative weight of the subset of criteria confirming truthfulness of the corresponding assertion.

As a first result we show the denotational isomorphism which exists between this concordance principle and the *proceeding of balancing reasons* as promoted by C.S. Peirce (see [8]). This result formally justifies the split truth versus falseness denotation installed by the concordance principle, a logical denotation appearing as powerful natural fuzzification of Boolean Logic (see Bisdorff [6]). Indeed, the algebraic framework of the credibility calculus (see Bisdorff & Roubens [1, 2]), coupled to its split logical denotation, allows us to solve selection, ranking and clustering problems (see Bisdorff [3, 5, 6, 7]) directly on the base of a more or less credible pairwise outranking relation without using intermediate cut techniques as is usual in the classic Electre methods (see Roy & Bouyssou [11]).

Now, to give adequate results, this *concordance principle* imposes necessary coherence properties on the underlying family of criteria such as: – *Exhaustivity* of the family of criteria, – *Cohesion* between local preferences, modelled at the level of the individual criterion, and global preferences modelled by the whole family of criteria, – *Non-redundancy* of the criteria. Instead of studying, as is usual, the logical consequence of these coherence axioms on the class of representable global outrankings, we focus here our attention to the logical antecedent of these coherence axioms. Therefore, we explore the relationship between the family of criteria and its semiotical interpretation in terms of its underlying cloud of consequences.

As a second result, we reformulate the concordance principle and a new version of the coherence axioms of the family of criteria is presented. Indeed, the *elementary semiotical reference* associated with each individual criterion function allows a clear partial truth assessment. In case of *mutual exclusiveness* and *universal closure* of such elementary semiotical references, universal outranking assertions may be truth assessed through a weighted mean of credibilities, formally equivalent to the proceeding of balancing reasons and, following our first result, to the classic concordance principle.

This semiotical reformulation of the concordance principle finally allows to identify possible origins for incoherence of the family of criteria and as a third result, we propose three extensions to the classic concordance principle: – *overlapping criteria*, – *incomplete* performance tableaux, and finally – *ordinal* importance weights.

An extended version of this communication has been submitted for publication ([7]).

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On a Sorting Procedure in the Presence of Qualitative Interacting Points of View

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Abstract

We present a sorting procedure for the assignment of alternatives to graded classes. The available information is given by partial evaluations of the alternatives on ordinal scales representing interacting points of view and a subset of prototypic alternatives whose assignment is imposed beforehand. The partial evaluations of each alternative are embedded in a common interval scale by means of commensurateness mappings, which in turn are aggregated by the discrete Choquet integral. The behavioral properties of this Choquet integral are then measured through importance and interaction indices.

Keywords: multi-attribute decision-making, ordinal data, interacting points of view, Choquet integral.

10 Introduction

In this paper we use the discrete Choquet integral as a discriminant function in ordinal multiattribute sorting problems in the presence of interacting (dependent) points of view. The technique we present is due to Roubens [14] and proceeds in two steps: a pre-scoring phase determines for each point of view and for each alternative a net score (the number of times a given alternative beats all the other alternatives minus the number of times that this alternative is beaten by the others) and is followed by an aggregation phase that produces a global net score associated to each alternative. These global scores are then used to assign the alternatives to graded classes.

The fuzzy measure linked to the Choquet integral can be learnt from a subset of alternatives (called prototypes) that are assigned beforehand to the classes by the decision maker. This leads to solving a linear constraint satisfaction problem whose unknown variables are the coefficients of the fuzzy measure.

Once a fuzzy measure (compatible with the available information on prototypes) is found, it is useful to interpret it through some behavioral parameters. We present the following two types of parameters:

1. The importance indices, which make it possible to appraise the overall importance of each point of view and each combination of points of view,

2. The interaction indices, which measure the extent to which the points of view interact (positively or negatively).

11 An ordinal sorting procedure

Let A be a set of q potential alternatives, which are to be assigned to disjoint classes, and let $N = \{1,\dots,n\}$ be a label set of points of view to satisfy. For each point of view $i \in N$, the alternatives are evaluated according to a s_i-point ordinal performance scale; that is, a totally ordered set

$$X_i := \{g_1^i \prec_i g_2^i \prec_i \cdots \prec_i g_{s_i}^i\}.$$

We assume that each alternative $x \in A$ can be identified with its corresponding profile

$$(x_1,\dots,x_n) \in \mathop{\times}_{i=1}^{n} X_i =: X,$$

where, for any $i \in N$, x_i represents the partial evaluation of x related to point of view i. In other words, each alternative is completely determined from its partial evaluations.

Through this identification, we clearly have

$$A \subseteq X \quad \text{and} \quad q \leqslant \prod_{i=1}^{n} s_i\,.$$

For any $x_i \in X_i$ and any $y_{-i} \in X_{-i} := \times_{j\in N\setminus\{i\}} X_j$, we set

$$x_i y_{-i} := (y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_n) \in X.$$

Now, consider a partition of X into m nonempty classes $\{Cl_t\}_{t=1}^m$, which are increasingly ordered; that is, for any $r,s \in \{1,\dots,m\}$, with $r > s$, the elements of Cl_r have a better comprehensive evaluation than the elements of Cl_s.

We also set

$$Cl_r^{\geqslant} := \bigcup_{t=r}^{m} Cl_t \qquad (r = 1,\dots,m).$$

The *sorting* problem we actually face consists in partitioning the elements of A into the classes $\{Cl_t\}_{t=1}^m$. Since A is given, the problem amounts to identifying the classes themselves as a partition of X.

Greco *et al.* [5, Theorem 2.1] proved a nice representation theorem stating the equivalence between a very simple cancellation property and a general discriminant function. As we have assumed beforehand that each set X_i is endowed with a total order $\succcurlyeq_i$, we present here a slightly modified version of their result.

Theorem 1. *The following two assertions are equivalent:*

1. *For all $i \in N$, $t \in \{1,\dots,m\}$, $x_i, x_i' \in X_i$, $y_{-i} \in X_{-i}$, we have*

$$x_i' \succcurlyeq_i x_i \text{ and } x_i y_{-i} \in Cl_t \;\Rightarrow\; x_i' y_{-i} \in Cl_t^{\geqslant}.$$

2. *There exist*
   - *functions $g_i : X_i \to \mathbb{R}$ $(i \in N)$, increasing, called criteria,*

- *a function* $f : \mathbb{R}^n \to \mathbb{R}$*, increasing in each argument, called discriminant function,*
- $m-1$ *ordered thresholds* $\{z_t\}_{t=2}^m$ *satisfying*

$$z_2 \leqslant z_3 \leqslant \cdots \leqslant z_m$$

such that, for any $x \in X$ *and any* $t \in \{2,\ldots,m\}$*, we have*

$$f[g_1(x_1), g_2(x_2), \ldots, g_n(x_n)] \geqslant z_t \quad \Leftrightarrow \quad x \in Cl_t^{\geqslant}.$$

Theorem 1 states that, under a simple condition of monotonicity, it is possible to find a discriminant function that strictly separates the classes $Cl_1, \ldots, Cl_m$ by thresholds. This result is very general and imposes no particular forms to criteria and discriminant functions.

For a practical use of this result and in order to produce a meaningful result, Roubens [14] restricted the family of possible discriminant functions to the class of n-place Choquet integrals and the criteria functions to normalized scores. We now present the sorting procedure in this particular case.

11.1 Normalized scores as criteria

In order to locate x_i in the scale X_i we define a mapping $\mathrm{ord}_i : A \to \{1,\ldots,s_i\}$ as

$$\mathrm{ord}_i(x) = r \quad \Leftrightarrow \quad x_i = g_r^i.$$

For each point of view $i \in N$, the order $\preccurlyeq_i$ defined on X_i can be characterized by a valuation $R_i : A \times A \to \{0,1\}$ defined as

$$R_i(x,y) := \begin{cases} 1, & \text{if } x_i \succcurlyeq_i y_i\,, \\ 0, & \text{otherwise.} \end{cases}$$

>From each of these valuations we determine a partial *net score* $S_i : A \to \mathbb{R}$ as follows:

$$S_i(x) := \sum_{y \in A} [R_i(x,y) - R_i(y,x)] \qquad (x \in A).$$

In the particular case where

$$A = \mathop{\times}_{i=1}^{n} X_i\,,$$

then it is easy to see that

$$S_i(x) = q\left(\frac{2\,\mathrm{ord}_i(x) - 1}{s_i} - 1\right) \qquad (i \in N).$$

Indeed, there are $(\mathrm{ord}_i(x) - 1)\,q/s_i$ alternatives $y \in A$ such that $x_i \succ_i y_i$ and $(s_i - \mathrm{ord}_i(x))\,q/s_i$ alternatives $y \in A$ such that $y_i \succ_i x_i$.

The integer $S_i(x)$ represents the number of times that x is preferred to any other alternative minus the number of times that any other alternative is preferred to x for point of view i.

On can easily show that the partial net scores identify the corresponding partial evaluations. That is,

$$x_i \succcurlyeq_i y_i \quad \Leftrightarrow \quad S_i(x) \geqslant S_i(y). \tag{1}$$

Thus aggregating the partial evaluations of a given alternative amounts to aggregating the corresponding partial scores. This latter aggregation makes sense since, contrary to the partial evaluations, the partial scores are *commensurable*, that is, each partial score can be compared with any other partial score, even along a different point of view.

Clearly, the partial scores are defined according to the same interval scale. As positive linear transformations are meaningful with respect to such a scale, we can normalize these scores so that they range in the unit interval. We thus define normalized partial scores $S_1^N, \dots, S_n^N$ as

$$S_i^N(x) := \frac{S_i(x) + (q-1)}{2(q-1)} \in [0,1] \qquad (i \in N).$$

Throughout the paper, we will use the notation $S^N(x) := (S_1^N(x), \dots, S_n^N(x))$.

11.2 The Choquet integral as a discriminant function

As mentioned in the beginning of this section, the partial scores of a given alternative x can be aggregated by means of a Choquet integral [1], namely

$$\mathcal{C}_v(S^N(x)) := \sum_{i=1}^{n} S_{(i)}^N(x) \, [v(A_{(i)}) - v(A_{(i+1)})]$$

where v represents a fuzzy measure on N; that is, a monotone set function $v : 2^N \to [0,1]$ fulfilling $v(\emptyset) = 0$ and $v(N) = 1$. This fuzzy measure merely expresses the importance of each subset of points of view. The parentheses used for indices represent a permutation on N such that

$$S_{(1)}^N(x) \leqslant \cdots \leqslant S_{(n)}^N(x),$$

and $A_{(i)}$ represents the subset $\{(i), \dots, (n)\}$.

We note that for additive measures ($v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$) the Choquet integral coincides with the usual discrete Lebesgue integral and the set function v is simply determined by the importance of each point of view: $v(1), \dots, v(n)$. In this particular case

$$\mathcal{C}_v(S^N(x)) = \sum_{i=1}^{n} v(i) \, S_i^N(x) \qquad (x \in A),$$

which is the natural extension of the *Borda score* as defined in voting theory if alternatives play the role of candidates and points of view represent voters.

If points of view cannot be considered as being independent, importance of combinations $S \subseteq N$, $v(S)$, has to be taken into account.

Some combinations of points of view might present a positive interaction or *synergy*. Although the importance of some points of view, members of a combination S, might be low, the importance of a pair, a triple,..., might be substantially larger and $v(S) > \sum_{i \in S} v(i)$.

In other situations, points of view might exhibit negative interaction or *redundancy*. The union of some points of view do not have much impact on the decision and for such combinations S, $v(S) < \sum_{i \in S} v(i)$. In this perspective, the use of the Choquet integral is recommended.

The Choquet integral presents standard properties for aggregation (see [3, 8]): it is continuous, non decreasing, located between min and max.

We will now indicate an axiomatic characterization of the class of all Choquet integrals with n arguments. This result is due to Marichal [8]. Let e_S denote the characteristic vector of S in $\{0,1\}^n$, i.e., the vector of $\{0,1\}^n$ whose ith component is one if and only if $i \in S$.

Theorem 2. *The operators $M_v : \mathbb{R}^n \to \mathbb{R}$ (v being a fuzzy measure on N) are*

- *linear w.r.t. the fuzzy measures, that is, there exist 2^n functions $f_T : \mathbb{R}^n \to \mathbb{R}$ $(T \subseteq N)$, such that*

$$M_v = \sum_{T \subseteq N} v(T)\, f_T\,,$$

- *non decreasing in each argument,*
- *stable for the admissible positive linear transformations, that is,*

$$M_v(r x_1 + s, \ldots, r x_n + s) = r M_v(x_1, \ldots, x_n) + s$$

for all $x \in \mathbb{R}^n$, $r > 0$, $s \in \mathbb{R}$,

- *properly weighted by v, that is,*

$$M_v(e_S) = v(S),$$

if and only if $M_v = \mathcal{C}_v$ for all fuzzy measure v on N.

This important characterization clearly justifies the way the partial scores have been aggregated.

The first axiom is proposed to keep the aggregation model as simple as possible. The second axiom says that increasing a partial score cannot decrease the global score. The third axiom only demands that the aggregated value is stable with respect to any change of scale. Finally, assuming that the partial score scale is embedded in $[0,1]$, the fourth axiom suggests that the weight of importance of any subset S of criteria is defined as the global evaluation of the alternative that completely satisfies points of view S and totally fails to satisfy the others.

The fourth axiom is fundamental. It gives an appropriate definition of the weights of subsets of points of view, interpreting them as global evaluation of particular alternatives.

The major advantage linked to the use of the Choquet integral derives from the large number of parameters $(2^n - 2)$ associated with a fuzzy measure but this flexibility can be also considered as a serious drawback when assessing real values to the importance of all possible combinations. We will come back to the important question the next section.

Let v be a fuzzy measure on N. The Möbius transform of v is a set function $m : 2^N \to \mathbb{R}$ defined by

$$m(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|}\, v(T) \qquad (S \subseteq N).$$

This transformation is invertible and thus constitutes an equivalent form of a fuzzy measure and v can be recovered from m using

$$v(S) = \sum_{T \subseteq S} m(T) \qquad (S \subseteq N).$$

This transformation can be used to redefine the Choquet integral without reordering the partial scores:

$$\mathcal{C}_v(S^N(x)) = \sum_{T\subseteq N} m(T) \bigwedge_{i\in T} S_i^N(x).$$

A fuzzy measure v is k-additive [3] if its Möbius transform m satisfies $m(S) = 0$ for S such that $|S| > k$ and there exists at least one subset S such that $|S| = k$ and $m(S) \neq 0$.

Thus, k-additive fuzzy measures can be represented by at most $\sum_{i=1}^{k} \binom{n}{i}$ coefficients.

For a k-additive fuzzy measure,

$$\mathcal{C}_v(S^N(x)) = \sum_{\substack{T\subseteq N\\ |T|\leqslant k}} m(T) \bigwedge_{i\in T} S_i^N(x).$$

In order to assure boundary and monotonicity conditions imposed on v, the Möbius transform of a k-additive fuzzy measure must satisfy:

$$\begin{cases} m(\emptyset) = 0, \quad \displaystyle\sum_{\substack{T\subseteq N\\ |T|\leqslant k}} m(T) = 1 \\ \displaystyle\sum_{\substack{T|i\in T\subseteq S\\ |T|\leqslant k}} m(T) \geqslant 0, \quad \forall S\subseteq N,\, \forall i\in S \end{cases}$$

11.3 Assessment of fuzzy measures

Assume that all the alternatives of $A \subseteq X$ are already sorted into classes $Cl_1, \ldots, Cl_m$. In some particular cases there exist a fuzzy measure v on N and $m-1$ ordered thresholds $\{z_t\}_{t=2}^{m}$ satisfying

$$z_2 \leqslant z_3 \leqslant \cdots \leqslant z_m$$

such that for any $x \in A$, and any $t \in \{2, \ldots, m\}$, we have

$$\mathcal{C}_v(S^N(x)) \geqslant z_t \quad \Leftrightarrow \quad x \in Cl_t^{\geqslant}.$$

Of course, if such a fuzzy measure does exist then the thresholds may be defined by

$$z_t := \min_{x\in Cl_t^{\geqslant}} \mathcal{C}_v(S^N(x)) \qquad (t = 2, \ldots, m).$$

Conversely, the knowledge of the fuzzy measure v associated to the sorting problem completely determines the assignment.

In real situations, the assignment of all alternatives is not known but has to be determined. However, this assignment, or equivalently the fuzzy measure v, can be learnt from a reference set of prototypes, which have been sorted beforehand by the decision maker.

Practically, the decision maker is asked to provide a set of prototypes $P \subseteq A$ and the assignment of each of these prototypes to a given class; that is, a partition of P into prototypic classes $\{P_t\}_{t=1}^{m}$, where $P_t := P \cap Cl_t$ for all $t \in \{1, \ldots, m\}$. Here some prototypic classes may be empty.

As the Choquet integral is supposed to strictly separate the classes Cl_t, we must impose the following necessary condition

$$\mathcal{C}_v(S^N(x)) - \mathcal{C}_v(S^N(x')) \geqslant \varepsilon, \tag{2}$$

for each ordered pair $(x,x') \in P_t \times P_{t-1}$ and each $t \in \{2,\ldots,m\}$, where ε is a given strictly positive threshold.

These separation conditions, put together with the boundary and monotonicity constraints on the fuzzy measure, form a linear constraint satisfaction problem whose unknowns are the coefficients of the fuzzy measure. Thus the sorting problem consists in finding a feasible solution satisfying all these constraints. If ε has been chosen too big, the problem might have no solution. To avoid this, we can consider ε as a non-negative variable to be maximized. In this case its optimal value must be strictly positive for the problem to have a solution.

In the resolution of this problem, we use the principle of parsimony: If no solution is found for $k = 1$, we turn to $k = 2$. If no solution is still found, we turn to $k = 3$, and so on, until $k = n$. Notice however that an empty solution set for $k = n$ is necessarily due to an incompatibility between the assignment of the given prototypes and the assumption that the discriminant function is a Choquet integral.

Due to the increasing monotonicity of the Choquet integral, the number of separation constraints (2) can be reduced significantly. For example, if $x'' \in P_{t-1}$ is such that $\mathcal{C}_v(S^N(x')) \geqslant \mathcal{C}_v(S^N(x''))$ then, by transitivity, the constraint

$$\mathcal{C}_v(S^N(x)) - \mathcal{C}_v(S^N(x'')) \geqslant \varepsilon$$

is redundant.

Now, on the basis of orders $\succcurlyeq_i$ $(i \in N)$, we can define a dominance relation D on X as follows: For each $x,y \in X$,

$$xDy \;\;\Leftrightarrow\;\; x_i \succcurlyeq_i y_i \;\; \forall i \in N.$$

By (1), this is equivalent to

$$xDy \;\;\Leftrightarrow\;\; S_i^N(x) \geqslant S_i^N(y) \;\; \forall i \in N.$$

Being an intersection of complete orders, the binary relation D is a partial order, i.e., it is reflexive, antisymmetric, and transitive. Furthermore we clearly have

$$xDy \;\;\Rightarrow\;\; \mathcal{C}_v(S^N(x)) \geqslant \mathcal{C}_v(S^N(y)).$$

It is then useful to define, for each $t \in \{1,\ldots,m\}$, the set of non-dominating alternatives of P_t,

$$Nd_t := \{x \in P_t \mid \nexists x' \in P_t \setminus \{x\} : xDx'\},$$

and the set of non-dominated alternatives of P_t,

$$ND_t := \{x \in P_t \mid \nexists x' \in P_t \setminus \{x\} : x'Dx\},$$

and to consider only the constraint (2) for each ordered pair $(x,x') \in Nd_t \times ND_{t-1}$ and each $t \in \{2,\ldots,m\}$. The total number of separation constraints boils down to

$$\sum_{t=2}^{m} |Nd_t|\,|ND_{t-1}|.$$

Now, suppose that there exists a k-additive fuzzy measure v^* that solves the above problem. Then any alternative $x \in A$ will be assigned to

- the class Cl_t if

$$\min_{y \in Nd_t} \mathcal{C}_{v^*}(S^N(y)) \leqslant \mathcal{C}_{v^*}(S^N(x)) \leqslant \max_{y \in ND_t} \mathcal{C}_{v^*}(S^N(y)),$$

- one of the classes Cl_t or Cl_{t-1} if

$$\max_{y \in ND_{t-1}} \mathcal{C}_{v^*}(S^N(y)) < \mathcal{C}_{v^*}(S^N(x)) < \min_{y \in Nd_t} \mathcal{C}_{v^*}(S^N(y)).$$

12 Behavioral analysis of aggregation

Now that we have a sorting model for assigning alternatives to classes, an important question arises: How can we interpret the behavior of the Choquet integral or that of its associated fuzzy measure? Of course the meaning of the values $v(T)$ is not always clear for the decision maker. These values do not give immediately the global importance of the points of view, nor the degree of interaction among them.

In fact, from a given fuzzy measure, it is possible to derive some indices or parameters that will enable us to interpret the behavior of the fuzzy measure. These indices constitute a kind of *identity card* of the fuzzy measure. In this section, we present two types of indices: importance and interaction. Other indices, such as tolerance and dispersion, were proposed and studied by Marichal [6, 7].

12.1 Importance indices

The overall importance of a point of view $i \in N$ into a decision problem is not solely determined by the number $v(\{i\})$, but also by all $v(T)$ such that $i \in T$. Indeed, we may have $v(\{i\}) = 0$, suggesting that element i is unimportant, but it may happen that for many subsets $T \subseteq N$, $v(T \cup \{i\})$ is much greater than $v(T)$, suggesting that i is actually an important element in the decision.

Shapley [15] proposed in 1953 a definition of a coefficient of importance, based on a set of reasonable axioms. The *importance index* or *Shapley value* of point of view i with respect to v is defined by:

$$\phi(v, \{i\}) := \sum_{T \subseteq N \setminus \{i\}} \frac{(n - |T| - 1)! \, |T|!}{n!} [v(T \cup \{i\}) - v(T)]. \tag{3}$$

The Shapley value is a fundamental concept in game theory expressing a power index. It can be interpreted as a weighted average value of the marginal contribution $v(T \cup \{i\}) - v(T)$ of element i alone in all combinations. To make this clearer, it is informative to rewrite the index as follows:

$$\phi(v, \{i\}) = \frac{1}{n} \sum_{t=0}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{i\} \\ |T| = t}} [v(T \cup \{i\}) - v(T)].$$

Thus, the average value of $v(T \cup \{i\}) - v(T)$ is computed first over the subsets of same size t and then over all the possible sizes. Consequently, the subsets containing about $n/2$ points of view are the less important in the average, since they are numerous and a same point of view j is very often involved into them.

The use of the Shapley value in multicriteria decision making was proposed in 1992 by Murofushi [10]. It is worth noting that a basic property of the Shapley value is

$$\sum_{i=1}^{n} \phi(v,\{i\}) = 1.$$

Note also that, when v is additive, we clearly have $v(T \cup \{i\}) - v(T) = v(\{i\})$ for all $i \in N$ and all $T \subseteq N \setminus \{i\}$, and hence

$$\phi(v,\{i\}) = v(\{i\}), \quad i \in N. \tag{4}$$

If v is non-additive then some points of view are dependent and (4) generally does not hold anymore. This shows that it is sensible to search for a coefficient of overall importance for each point of view.

In terms of the Möbius representation, the Shapley value takes a very simple form [15]:

$$\phi(v,\{i\}) = \sum_{T \ni \{i\}} \frac{1}{|T|}\, m(T). \tag{5}$$

Now, the concept of importance index can be easily generalized to subsets of points of view. The *importance index* of subset $S \subseteq N$ with respect to v is defined by

$$\phi(v,S) := \sum_{T \subseteq N \setminus S} \frac{(n-|T|-|S|)!\,|T|!}{(n-|S|+1)!}\,[v(T \cup S) - v(T)].$$

This index, introduced by Marichal [9] as the *influence index* of points of view S, measures the overall importance of subset S of points of view.

In terms of the Möbius representation, it is given by

$$\phi(v,S) = \sum_{\substack{T \subseteq N \\ T \cap S \neq \emptyset}} \frac{1}{|T \setminus S| + 1}\, m(T).$$

It was shown [9] that this expression is also the average amplitude of the range of $\mathcal{C}_v$ that points of view S may control when assigning partial scores in $[0,1]$ to the points of view in $N \setminus S$ at random. That is,

$$\begin{aligned}
\phi(v,S) &= \int_0^1 \cdots \int_0^1 \Big[\lim_{\substack{x_j \to 1 \\ j \in S}} \mathcal{C}_v(x) - \lim_{\substack{x_j \to 0 \\ j \in S}} \mathcal{C}_v(x) \Big]\, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_{n-s}} \\
&= \int_{[0,1]^n} \Big[\lim_{\substack{x_j \to 1 \\ j \in S}} \mathcal{C}_v(x) - \lim_{\substack{x_j \to 0 \\ j \in S}} \mathcal{C}_v(x) \Big]\, \mathrm{d}x\,,
\end{aligned}$$

where $N \setminus S = \{i_1, \ldots, i_{n-s}\}$.

12.2 Interaction indices

Another interesting concept is that of *interaction* among points of view. We have seen that when the fuzzy measure is not additive then some points of view interact. Of course, it would be interesting to appraise the degree of interaction among any subset of points of view.

Consider first a pair $\{i, j\} \subseteq N$ of points of view. It may happen that $v(\{i\})$ and $v(\{j\})$ are small and at the same time $v(\{i, j\})$ is large. Clearly, the number $\phi(v, \{i\})$ merely measures the average contribution that point of view i brings to all possible combinations, but it gives no information on the phenomena of interaction existing among points of view.

Clearly, if the marginal contribution of j to every combination of points of view that contains i is greater (resp. less) than the marginal contribution of j to the same combination when i is excluded, the expression

$$[v(T \cup \{i, j\}) - v(T \cup \{i\})] - [v(T \cup \{j\}) - v(T)]$$

is positive (resp. negative) for any $T \subseteq N \setminus \{i, j\}$. We then say that i and j positively (resp. negatively) interact.

This latter expression is called the *marginal interaction* between i and j, conditioned to the presence of elements of the combination $T \subseteq N \setminus \{i, j\}$. Now, an interaction index for $\{i, j\}$ is given by an average value of this marginal interaction. Murofushi and Soneda [11] proposed in 1993 to calculate this average value as for the Shapley value. Setting

$$(\Delta_{ij}\, v)(T) := v(T \cup \{i, j\}) - v(T \cup \{i\}) - v(T \cup \{j\}) + v(T),$$

the *interaction index* of points of view i and j related to v is then defined by

$$I(v, \{i, j\}) := \sum_{T \subseteq N \setminus \{i,j\}} \frac{(n - |T| - 2)!\,|T|!}{(n-1)!} (\Delta_{ij}\, v)(T). \qquad (6)$$

It should be mentioned that, historically, the interaction index (6) was first introduced in 1972 by Owen (see Eq. (28) in [13]) in game theory to express a degree of complementarity or competitiveness between elements i and j.

The interaction index among a combination S of points of view was introduced by Grabisch [3] as a natural extension of the case $|S| = 2$. The *interaction index* of S ($|S| \geq 2$) related to v, is defined by

$$I(v, S) := \sum_{T \subseteq N \setminus S} \frac{(n - |T| - |S|)!\,|T|!}{(n - |S| + 1)!} (\Delta_S\, v)(T),$$

where we have set

$$(\Delta_S\, v)(T) := \sum_{L \subseteq S} (-1)^{|S| - |L|} v(L \cup T).$$

In terms of the Möbius representation, this index is written [3]

$$I(v, S) = \sum_{T \supseteq S} \frac{1}{|T| - |S| + 1} m(T), \qquad S \subseteq N. \qquad (7)$$

Viewed as a set function, it coincides on singletons with the Shapley value (3).

In terms of the Choquet integral, we have [4, Proposition 4.1]

$$\begin{aligned} I(v,S) &= \int_0^1 \cdots \int_0^1 (\Delta_S\, \mathcal{C}_v)(x)\, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_{n-s}} \\ &= \int_{[0,1]^n} (\Delta_S\, \mathcal{C}_v)(x)\, \mathrm{d}x, \end{aligned}$$

where $N \setminus S = \{i_1, \ldots, i_{n-s}\}$ and

$$(\Delta_S\, \mathcal{C}_v)(x) := \sum_{L \subseteq S} (-1)^{|S|-|L|} \lim_{\substack{x_i \to 1 \\ i \in L}} \lim_{\substack{x_j \to 0 \\ j \in S \setminus L}} \mathcal{C}_v(x).$$

It was proved in [4, Proposition 5.1] that the transformation (7) is invertible and its inverse is written as

$$m(S) = \sum_{T \supseteq S} B_{|T|-|S|}\, I(v,T), \qquad S \subseteq N, \tag{8}$$

where B_n is the nth Bernoulli number, that is the nth element of the numerical sequence $\{B_n\}_{n\in\mathbb{N}}$ defined recursively by

$$\begin{cases} B_0 = 1, \\ \sum\limits_{k=0}^{n} \binom{n+1}{k} B_k = 0, \qquad n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

13 Concluding remarks

We have described a sorting procedure which aggregates interacting points of view measured on qualitative scales. The aggregation function that is used is the discrete Choquet integral whose parameters are learnt form a reference set of alternatives.

The motivation of this approach is based mainly on a very general representation theorem pointing out the use of a discriminant function, but also on an axiomatic characterization of the class of Choquet integrals having a fixed number of arguments.

The use of some indices is proposed to appraise the overall importance of points of view as well as the interaction existing among them.

The next step will be to measure the quality of the sorting procedure with respect to the choice of the prototypic alternatives and their assignment. A research is now in progress along this line.

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How to Improve the Score of an Act in MCDM? Application to the Choquet Integral

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Consider a decision process in which n criteria must be taken into account. We assume that the scores of acts with respect to the n criteria are given directly in the commensurate satisfaction scale $[0,1]$. We assume furthermore that the decision maker preferences can be modeled by a global evaluation score $\mathcal{H}(x)$ for every act $x \in \Omega := [0,1]^n$. The interpretation of the aggregation function $\mathcal{H}$ in multicriteria decision making is often based on indices such as importance indices that measure the importance of one criterion. For instance, for the Choquet integral, the importance index is the so-called Shapley value. Here, we are interested in the case in which acts are persons (students, trainees,...). More precisely, we wish to determine on which criteria acts should be improved if we want their global evaluation to increase as much as possible.

Any index is always defined in a specific context, and should never be used outside the specific context in which it has been defined. This holds in particular for the Shapley value. Actually, this value is borrowed from game theory, which is not the concern of the decision maker. As a consequence, the Shapley value may not be the appropriate importance index in some circumstances. So, we introduce a new index of importance which represents the mean worth for acts to reach higher scores in a set of criteria. More precisely, we wish to define the *mean worth* $W_A(\mathcal{H})$ for acts to reach higher levels in the coalition A of criteria, subject to the evaluation function $\mathcal{H}$.

W_A can be seen as an operator from a functional space defined on Ω to $\mathbb{R}$. In order to work on a functional space that has nice properties, we consider $L^2(\Omega)$:

$$L^2(\Omega) = \left\{ \mathcal{H} : \Omega \to \mathbb{R} \ , \ \int_\Omega |\mathcal{H}(x)|^2 dx < \infty \right\} \ .$$

Consequently, we aim to construct the mean worth W_A as an operator from $L^2(\Omega)$ to $\mathbb{R}$. The *worth* $W_A(\mathcal{H})$ is defined with the help of four axioms:

- The first axiom (referred to as **Continuity (C)**) states that $\mathcal{H} \mapsto W_A(\mathcal{H})$ should be continuous.
- The second axiom (referred to as **Linearity (L)**) states that $\mathcal{H} \mapsto W_A(\mathcal{H})$ should be linear.
- The third axiom (referred to as **Step Evaluation (SE)**) considers the special family of evaluation functions that can take only values 0 and 1. For these $\{0,1\}$ evaluation functions, it is interesting for an act x to carry out a given improvement in criteria among the coalition A if and only if the global score of x is zero and the global score after the improvement becomes 1. We deduce that it is reasonable to define the worth W_A for these $\{0,1\}$ evaluation functions as being proportional

to the *number of situations* in which an act goes from global score 0 to 1 by improving only its scores in criteria among A.

- The last axiom (referred to as **Normalization (N)**) focuses on another special family of evaluation functions. This time, the weighted sums are considered. A natural expression of W_A comes up.

We have shown the following theorem.

Theorem 1. *W_A satisfies* **(L)**, **(C)**, **(SE)** *and* **(N)** *if and only if* $\forall \mathcal{H} \in L^2(\Omega)$,

$$W_A(\mathcal{H}) = 3 \cdot 2^{|A|} \int_{x \in \Omega} \int_{y_A \in [x_A, 1]} [\mathcal{H}(y_A, x_{-A}) - \mathcal{H}(x)] \, dx \, dy_A \ ,$$

where (y_A, x_{-A}) *denotes the act* $z \in \Omega$ *such that* $z_i = y_i$ *if* $i \in A$ *and* $z_i = x_i$ *otherwise, and the notation* $y_A \in [x_A, 1]$ *means that for any* $i \in A$, $y_i \in [x_i, 1]$.

The second part of this paper concerns the application of theorem 1 to the Choquet integral. In particular, we compute the *worth* to reach higher levels in one attribute, and in a couple of attributes. The expression of $W_A(\mathcal{H})$ involves the fuzzy measure. Interestingly, this leads to quantities that are closely related to the Shapley indexes and the interaction indexes. We finally show how $W_A(\mathcal{H})$ can be used as a new way to interpret a Choquet integral.

Consensual Dynamics, Stochastic Matrices, Choquet Measures, and Shapley Aggregation

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Consider a set $N = \{1,...,n\}$ of interacting agents whose individual opinions are denoted by $\{x_i \in R,\ i \in N\}$. The interaction structure on N, representing the degree of reciprocal influence which the agents exert upon each other within the consensual dynamics, is expressed by the symmetric interaction matrix $\mathbf{V} = [v_{ij},\ i, j \in N]$, with interaction coefficients $v_{ij} = v_{ji} \in (0,1)$ for $i \neq j$ and $v_{ii} = 0$. The average value of the interaction coefficients between agent i and the remaining agents is denoted $v_i = \sum_{j\in N\setminus i} v_{ij}/(n-1)$, with $v_i \in (0,1)$.

In this paper the values of the interaction coefficients are assumed to be constant in time and can either be given exogenously, as in the traditional models of consensual dynamics (e.g. DeGroot 74), or computed endogenously in terms of the individual opinions x_i themselves. The endogenous definition of the interaction coefficients can be done in various ways. In the soft consensus model (see e.g. Fedrizzi et al. 99), for instance, the interaction coefficients v_{ij}, with $i \neq j$, are defined by filtering the square difference values $(x_i - x_j)^2$ with a decreasing sigmoid function $\sigma(t) = 1/(1+e^{\beta(t-\alpha)})$. As a result, agents with similar opinions ($(x_i - x_j)^2 < \alpha$) interact strongly, whereas agents with dissimilar opinions ($(x_i - x_j)^2 > \alpha$) interact weakly.

The interaction matrix $\mathbf{V}$ admits the usual graph representation: each node i represents an individual agent and encodes the corresponding opinion x_i, and each edge $\{i, j\}$ encodes the corresponding interaction coefficient v_{ij}. Notice that both the interaction matrix and the associated graph are symmetric (i.e. the edges are undirected).

The graph representation of the set N of interacting agents is the basis for the construction of the following Choquet measure $\mu : 2^N \longrightarrow [0,1]$. Let $S \subseteq N$ be a coalition of agents. The value $\mu(S)$ of the coalition $S \subseteq N$ is defined to be proportional to the sum of the edge values contained in the subgraph associated to S, $\mu(S) = \sum_{\{i,j\}\subseteq S} v_{ij}/\mathcal{N}$, where the normalization factor is given by $\mathcal{N} = \sum_{\{i,j\}\subseteq N} v_{ij}$. The measure μ satisfies the boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$, and is monotonic and superadditive (with null singletons).

The individual share of the value $\mu(S)$ of a coalition $S \subseteq N$ is given by $\tilde{\mu}(S) = \mu(S)/|S|$ and is related in an interesting way with the average edge value $v(S)$ within the coalition S, given by $v(S) = \mu(S)/\frac{1}{2}|S|(|S|-1)$. It follows that $\tilde{\mu}(S) = \frac{1}{2}(|S|-1)v(S)$. Notice that given two coalitions $T \subseteq S \subseteq N$ it is always the case that the value $\mu(S)$ of the larger coalition is $\geq$ than the value $\mu(T)$ of the smaller coalition, but it might not be the case that the individual share $\tilde{\mu}(S)$ in the larger coalition is also $\geq$ than the individual share $\tilde{\mu}(T)$ in the smaller coalition. In other words, the question of whether or not it is worthwhile to form a larger coalition depends crucially on the way the average edge value changes, even though there is an overall bias to extend the coalition given by the dependency on the cardinality.

The Möbius transform a of the measure μ is given by $a_\mu(\{i\}) = 0$, $a_\mu(\{i,j\}) = v_{ij}/\mathcal{N}$, with null higher order terms. The measure μ is therefore of the 2-additive type.

Using the Möbius transform, one can easily compute the Shapley interaction indices $I_\mu(\{i\}) = \phi_\mu(\{i\})$, $I_\mu(\{i,j\}) = v_{ij}/\mathcal{N}$, with null higher order terms. The Shapley power indices (first order terms) are given by $\phi_\mu(\{i\}) = \frac{1}{2}\sum_{j\in N\setminus i} v_{ij}/\mathcal{N}$.

Notice that, apart from the normalization factor $\mathcal{N}$, the Shapley interaction index $I_\mu(\{i,j\})$ coincides with the interaction coefficient v_{ij} while the Shapley power index $\phi_\mu(\{i\})$ is proportional to v_i, the average degree of interaction between agent i and the remaining agents. In the soft consensus model, therefore, the Shapley power index $\phi_\mu(\{i\})$ reflects the local degree of consensus around agent i and is thus a natural endogenous weight for a consensual aggregation of the n opinion values. This leads to the the Shapley aggregation operator $\Phi_\mu(x_1,\dots,x_n) = \sum_{i\in N} x_i\ \phi_\mu(\{i\})$. Naturally, two other interesting aggregation operators to consider are the Choquet operator $\mathcal{C}_\mu$ and its OWA core (defined by the symmetrized measure).

We now turn to the consensual dynamics issue and show that it shares significant aspects of the above discussion in terms of the Choquet measure μ and the Shapley aggregation Φ_μ.

We begin by defining the *context variables* $\overline{x}_i = \sum_{j\in N\setminus i} v_{ij}x_j/\sum_{k\in N\setminus i} v_{ik}$. The context variable $\overline{x}_i$ represents the context opinion as seen by agent i, i.e. the weighted average opinion of the remaining agents. Notice that the context weights correspond to a local normalization of the interaction coefficients between agent i and the remaining agents.

The context variables $\overline{x}_i$ have an interesting property and for that reason they play a central role in what follows. Their Shapley average value $\Phi_\mu(\overline{x}_1,\dots,\overline{x}_n) = \sum_{i\in N} \overline{x}_i\,\phi_\mu(\{i\})$ *coincides* with the Shapley average value of the standard variables x_i, that is $\Phi_\mu(x_1,\dots,x_n) = \sum_{i\in N} x_i\ \phi_\mu(\{i\})$. To reach this important result it is useful to rewrite the context variables as $\overline{x}_i = \sum_{j\in N\setminus i} v_{ij}x_j/(n-1)v_i$ and the Shapley power indices as $\phi_\mu(\{i\}) = (n-1)v_i/2\mathcal{N}$.

Consider now a stochastic matrix $\mathbf{C} = [c_{ij},\ i,j\in N]$ and the general convex linear dynamical law $\mathbf{x} \longmapsto \mathbf{x}' = \mathbf{C}\mathbf{x}$, where $\mathbf{C} \geq 0$, $\mathbf{C}\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T = (1\ \dots\ 1)$. We can also write $x_i \longmapsto x_i' = \sum_{j\in N} c_{ij}x_j$, where $c_{ij} \geq 0$ and $\sum_{j\in N} c_{ij} = 1$.

In each iteration the new opinion x_i' of agent i is a convex combination of his/her old opinion x_i and the old opinions $x_{j\neq i}$ of the remaining agents. The old opinions $x_{j\neq i}$ are weighted with the coefficients c_{ij} with $j \neq i$, which are the $n-1$ degrees of freedom of the convex combination associated with agent i. As a result, the weight of the old opinion x_i, i.e. the coefficient c_{ii}, is constrained to be one minus the sum of the remaining coefficients, $c_{ii} = 1 - \sum_{j\in N\setminus i} c_{ij}$.

We are interested in the following two different types of the dynamical law above. In the first case (inhomogeneous dynamics) the free coefficients c_{ij}, with $j\neq i$, are assumed to be proportional to the corresponding interaction coefficients v_{ij}, with $j\neq i$. In the second case (homogeneous dynamics), instead, the free coefficients c_{ij}, with $j\neq i$, are assumed to be proportional to a local normalization of the corresponding interaction coefficients v_{ij}, with $j\neq i$. More specifically, they are assumed to be proportional to $v_{ij}/\sum_{k\in N\setminus i} v_{ik}$, with $j\neq i$.

In the first case the local dynamics can be substantially faster or slower depending on the local average value of the interaction coefficients: large (small) interaction coefficients, i.e. high (low) Shapley power index, implies fast (slow) dynamics. Whereas in the second case this effect is compensated by the local normalization of the interaction coefficients.

- Inhomogeneous dynamics: in this case the transition coefficients are given by

$c_{ij} = \varepsilon v_{ij}/(n-1), \;\; i \neq j \qquad \sum_{j \in N \setminus i} c_{ij} = \varepsilon v_i \qquad c_{ii} = 1 - \varepsilon v_i \qquad \varepsilon \in [0,1]$

and the dynamical law can be written as

$x_i \;\longmapsto\; x'_i = (1 - \varepsilon v_i) x_i + \varepsilon v_i \bar{x} \qquad \varepsilon \in [0,1].$

For $\varepsilon \neq 0$, the stochastic transition matrix $\mathbf{C}$ is positive and thus irreducible. It follows (Frobenius theorem) that it has a simple maximal unit eigenvalue and a unique normalized positive eigenvector ψ, $\psi^T \mathbf{C} = \psi^T$. It is easy to check that $\psi_i = 1/n$, with $\sum_{i \in N} \psi_i = 1$.

Moreover, this dynamical law leaves $\psi^T \mathbf{x}$ invariant (the plain average) and, given that $\mathbf{C}$ is positive, it converges (see e.g. DeGroot 74) to the consensual solution $\mathbf{x}^\infty = (\psi^T \mathbf{x})\mathbf{1}$. In other words, the opinions x_i converge asymptotically to their plain average $\sum_{i \in N} \psi_i x_i = \sum_{i \in N} x_i / n$.

- Homogeneous dynamics : in this case the transition coefficients are given by

  $c_{ij} = \varepsilon v_{ij}/(n-1) v_i, \;\; i \neq j \qquad \sum_{j \in N \setminus i} c_{ij} = \varepsilon \qquad c_{ii} = 1 - \varepsilon \qquad \varepsilon \in [0,1]$

  and the dynamical law can be written as

  $x_i \;\longmapsto\; x'_i = (1 - \varepsilon) x_i + \varepsilon \bar{x} \qquad \varepsilon \in [0,1].$

  For $\varepsilon \neq 0$, the stochastic transition matrix $\mathbf{C}$ is positive, except for the case $\varepsilon = 1$ in which it is positive outside the main diagonal. In any case the matrix $\mathbf{C}$ is irreducible. It follows (Frobenius theorem) that it has a simple maximal unit eigenvalue and a unique normalized positive eigenvector φ, $\varphi^T \mathbf{C} = \varphi^T$. It is easy to check that $\varphi_i = \frac{1}{2} \sum_{j \in N \setminus i} v_{ij} / \mathcal{N} = \phi_\mu(\{i\})$, with $\sum_{i \in N} \varphi_i = 1$.

  Moreover, this dynamical law leaves $\varphi^T \mathbf{x}$ invariant (the Shapley average) and, given that $\mathbf{C}^2$ is positive (see note), it converges (see e.g. DeGroot 74) to the consensual solution $\mathbf{x}^\infty = (\varphi^T \mathbf{x})\mathbf{1}$. In other words, the opinions x_i converge asymptotically to their Shapley average $\sum_{i \in N} \varphi_i x_i = \sum_{i \in N} x_i \phi_\mu(\{i\})$.

  Note: the sole exception is the case $\varepsilon = 1$ for $n = 2$. In this case, the iterations of the dynamics simply exchange the opinions x_1 and x_2, and thus no consensus is ever reached.

Summarizing, we have discussed two types of consensual dynamics, both of which refer significantly to the notion of context variables. The first type produces the plain average as the asymptotic consensual opinion and, in an extended version (time dependent interaction coefficients), corresponds to the purely consensual component of the dynamics in the soft consensus model (see e.g. Fedrizzi et al. 99). The second type, on the other hand, produces the Shapley average as the asymptotic consensual opinion. In this way it provides a dynamical realization of the Shapley aggregation (which can be extended in the spirit of the soft consensus model) and connects nicely with the Choquet measure and game theoretical analysis discussed before.

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Monotonicity and its Cognates in the Theory of Choice

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The standard requirement of monotonicity of a voting procedure states that an improvement in the ranking of the winning alternative, ceteris paribus, in some voters' preference orderings should not make it non-winning. This property has an obvious counterpart in multi-criterion decision making contexts. A concept that is apparently closely linked to monotonicity is known as the participation axiom which requires that it should never be disadvantagous for a voter to abstain rather than to vote according to his/her preferences. A third related concept is vulnerability to preference truncation. This is satisfied by such procedures that make it advantageous for voters to always reveal their entire preference rankings. We discuss these requirements both in collective choice and multi-criterion choice settings. Particular attention is paid to conditions which guarantee the monotonicity, participation and invulnerability to the truncation paradox.

Investigation of Fuzzy Equilibrium Relations in Decision Making

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Game theory provides useful tools to investigate many economical and human behaviours where the notion of coalitions, equilibrium, conflict and so on play an important role.

In his attempt to extend classical binary relation to economics purposes, Yin Yang proposed an equilibrium relation where elements take their values in the triplet $-1, 0, 1$. Motivations for such development arise from the lack of binary and/or valued relations to model multi-agent word with a combination of interactive negative, neutral and positive relationships as well as the coexistence of positive and negative relationships. Further, a bridge to classical equivalence binary or valued relations can be constructed. A such representation gives rise to the notions of coalitions sets, harmony sets, conflict sets and bipolar partitionings. Particularly, each quantity (bipolar variable) should have both a negative pole and a positive pole capturing the positive side and the negative side of a relation separately as well as the coexistence of both sides in the combination.

This communication attempts to extend Yin Yang's proposal to the case of fuzzy valued relations where the notions of reflexivity, symmetry, transitivity, ordering, closure, etc. are re-interpreted. Particularly, positive and negative equivalence relations can be constructed. For instance, from an intuitive viewpoint, negative reflexivity can be used as a measure of self-adjustability to external and internal changes. This enables an element of a given set to be adjusted with an harmonic state. While positive reflexivity remains similar to the reflexivity of valued fuzzy relations. For decision making purposes, scoring procedures, usually mentioned in fuzzy literature are reviewed in the light of the basic principles of the fuzzy equilibrium relations. Construction of strict relations that will be used for the search of non-dominated and non-dominating alternatives is also proposed.

Games with Fuzzy Coalitions

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We present an axiomatization of the interaction value between players of fuzzy coalitions. It is based on three axioms :

Symmetry Axiom, Partition Axiom, Generalized Efficiency Axiom.

As a special case we get the classical interaction- and Shapley-index.

We give some details.
Let $X = \{1, \dots, n\}$ be a finite set and let
$S = \{i_1, \dots, i_s\} \subset X$ be a (classical) coalition. We put $s = |S|$.
Let V_n^s be the real vector space of all functions $f : [0,1]^n \to R$ whose partial derivatives are continuous and exist up to order s on the diagonal $\{t\xi_X : 0 \le t \le 1\}$ and vanish at 0 (here $\xi_S : X \to \{0,1\}$ is the characteristic function of $S, S \subset X$, so that ξ_X is identified with $(1, \dots, 1)$).

Moreover let M_n^s be the real vectorspace of all s-indexed n-dimensional matrices

$$A_n^s = (a_{i_1, i_2, \dots, i_s}) \tag{1}$$

where $1 \le i_1, i_2, \dots, i_s \le n$ and $a_{i_1, i_2, \dots, i_s} \in \mathbb{R}$ (so that M_n^s can be identified with $\mathbb{R}^{(n^s)}$).

We can show the following result :

Let $\varphi_n^s : V_n^s \to M_n^s, n \in N$.

Then (φ_n^s) is linear, continuous and satisfies the symmetry-, partition- and generalized efficiency axiom if and only if

$$\varphi_n^s(f) = \Big(\int_0^1 D_{i_1} D_{i_2} \cdots D_{i_s} f(t\xi_X) dt \Big). \tag{2}$$

The interpretation of the "diagonal formula" is as follows.
For a given level of membership $t \in [0.1]$ we consider the diagonal fuzzy coalition $t\xi_X = (t, \dots, t)$ in which each player participates equally. Then $D_{i_1} D_{i_2} \cdots D_{i_s} f(t\xi_X)$ is a measure of the loss of the coalition $S = \{i_1, \dots, i_s\}$ if it leaves the diagonal fuzzy coalition, and $\varphi_n^s(f)$ can be regarded as the average loss imposed on the players of S if they want to modify a diagonal coalition.

On Some Mathematical Problems in the Theory of Fuzzy Sets

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Today's Theory of Fuzzy Sets has many facets. Logical, mathematical and engineering research shaped this theory and transformed it into a powerful tool for solving down-to-earth problems in fields as diverse as image processing, speech recognition and decision making in economics. The aim of our presentation is to discuss several unsolved mathematical problems occurring from applications of the Theory of Fuzzy Sets. We will mostly focus on two questions: the Jordan decomposability and integral representation of the fuzzy measures seen in a game theoretical context and the efficient solvability of optimization problems in the space of fuzzy vectors seen as an instrument of rational decision making under uncertainty.

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