LINZ 2005

26th Linz Seminar on Fuzzy Set Theory

Abstracts

Fuzzy Logics and Related Structures

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Abstracts

Siegfried Gottwald, Petr Hájek,
Ulrich Höhle, Erich Peter Klement
Editors

LINZ 2005

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FUZZY LOGICS AND RELATED STRUCTURES

ABSTRACTS

Siegfried Gottwald, Petr Hájek, Ulrich Höhle, Erich Peter Klement
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Since their inception in 1979 the Linz Seminars on Fuzzy Sets have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work.

LINZ 2005 will be already the 26th seminar carrying on this tradition, will be devoted to the mathematical aspects of "Fuzzy Logics and Related Structures". As usual, the aim of the Seminar is an intermediate and interactive exchange of surveys and recent results.

Siegfried Gottwald
Petr Hájek
Ulrich Höhle
Erich Peter Klement

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Herbrand's Theorem and the Skolemization of Prenex Fragments

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It is well known, that Skolem functions in the usual sense are not admissible for logics, where the evaluation of quantifiers cannot be reduced to the evaluation w.r.t. critical objects. For example, the existence of suprema (infima) of subsets of the truth values in absence of maxima (minima) is already an obstacle.

We show for the prenex fragment the following relation to Herbrand's Theorem:

Theorem 1. *Let $\mathcal{L}$ be a logic satisfying the following properties:*

1. $\models_{\mathcal{L}} Q \vee P \;\Rightarrow \models_{\mathcal{L}} P \vee Q$ *(commutativity of $\vee$)*
2. $\models_{\mathcal{L}} (Q \vee P) \vee R \;\Rightarrow \models_{\mathcal{L}} Q \vee (P \vee R)$ *(associativity of $\vee$)*
3. $\models_{\mathcal{L}} Q \vee P \vee P \;\Rightarrow \models_{\mathcal{L}} Q \vee P$ *(idempotency of $\vee$)*
4. $\models_{\mathcal{L}} P(y) \;\Rightarrow \models_{\mathcal{L}} \forall x [P(x)]^{(y)}$
5. $\models_{\mathcal{L}} P(t) \;\Rightarrow \models_{\mathcal{L}} \exists x P(x)$
6. $\models_{\mathcal{L}} \forall x (P(x) \vee Q^{(x)}) \;\Rightarrow \models_{\mathcal{L}} (\forall x P(x)) \vee Q^{(x)}$
7. $\models_{\mathcal{L}} \exists x (P(x) \vee Q^{(x)}) \;\Rightarrow \models_{\mathcal{L}} (\exists x P(x)) \vee Q^{(x)}$.

Let $\exists \overline{x} P^F(\overline{x})$ be the Skolem form of $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$. For all tuples of terms $\overline{t_1}, \ldots, \overline{t_m}$ of the Herbrand universe of $P^F(\overline{x})$

$$\models_{\mathcal{L}} \bigvee_{i=1}^{m} P^F(\overline{t_i}) \quad \Rightarrow \quad \models_{\mathcal{L}} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n).$$

We also discuss conditions, which allow the derivation of Herbrand's Theorem from the admissability of Skolem functions and apply the results to various sublogics of t-norm based logics. Finally we discuss alternatives to the usual Skolem functions, which might admit Skolemization/De-Skolemization, when the usual Skolem functions are not admissible

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Relations in Higher-order Fuzzy Logic I, II

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A theory of fuzzy relations is an important part of any theory intended to provide a formal framework for fuzzy mathematics. In [2], Henkin-style higher-order fuzzy logic is introduced and proposed as a foundational theory for fuzzy mathematics. In these two talks, we investigate the properties of fuzzy relations within its formal framework.

We follow closely the methodology of [1]. Therefore the notions introduced here are inspired by (and deduced from) the corresponding notions of classical mathematics. Sometimes they coincide with already known notions in fuzzy literature. However, our approach is usually more general (we work in arbitrary fuzzy logic), more expressive (we deal with the *graded* properties of fuzzy relations, as in [4]), and the proofs are more elegant (resembling the classical proofs). The subsequent talk [3] given by Ulrich Bodenhofer will show links between our approach and the more traditional ones.

1 Higher-order fuzzy logic

For convenience, we first reproduce basic definitions of higher-order fuzzy logic.

Definition 1 (Henkin-style second-order fuzzy logic) *Let $\mathcal{F}$ be a fuzzy logic which extends* BLΔ. *The Henkin-style second-order fuzzy logic over $\mathcal{F}$ is a theory over multi-sorted first-order $\mathcal{F}$ with sorts for objects (lowercase variables) and classes (uppercase variables). Both of the sorts subsume subsorts for n-tuples, for all $n \geq 1$. Apart from the obvious necessary function symbols and axioms for tuples (tuples equal iff their respective constituents equal), the only primitive symbol is the membership predicate $\in$ between objects and classes. The axioms for $\in$ are the following:*

1. *The comprehension axioms $(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi)$, φ not containing X, which enable the (eliminable) introduction of comprehension terms $\{x \mid \varphi\}$ with axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (where φ may be allowed to contain other comprehension terms).*
2. *The extensionality axiom $(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \to X = Y$.*

Convention 2 *The usual precedence of connectives is assumed. The formulae $(\forall x)(x \in X \to \varphi)$ and $(\exists x)(x \in X \mathbin{\&} \varphi)$ are abbreviated $(\forall x \in X)\varphi$ and $(\exists x \in X)\varphi$, respectively (similar notation can be used for defined binary predicates). The formulae $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (n times) are abbreviated φ^n. Furthermore, $x \notin X$ is shorthand for $\neg(x \in X)$, and similarly for other binary relational symbols. An alternative notation for $x \in A$ and $\langle x_1, \ldots, x_n\rangle \in R$ is simply Ax and $Rx_1 \ldots x_n$, respectively.*

Definition 3 (Henkin-style higher-order fuzzy logic) *Henkin-style fuzzy logic of higher orders is obtained by repeating the previous definition on each level of the type hierarchy. Obviously, defined symbols of any type can then be shifted to all higher types as well. (Consequently, all theorems are preserved by uniform upward type-shifts.) Types may be allowed to subsume all lower types.*

Henkin-style fuzzy logic $\mathcal{F}$ of order n will be denoted by $\mathcal{F}_n$, the whole hierarchy by $\mathcal{F}_\omega$. The types of terms are either denoted by a superscripted parenthesized type (e.g., $X^{(3)}$), or understood from the context.

It should be stressed that despite the name, Henkin-style higher-order fuzzy logics are *theories* over *first-order* fuzzy logics (see [5]).

Definition 4 (Fuzzy class operations and relations) *In $\mathcal{F}_2$, the following elementary fuzzy set operations can be defined:*

$$\begin{array}{rll}
\emptyset =_{\mathrm{df}} & \{x \mid 0\} & \textit{empty class}\\
\mathrm{V} =_{\mathrm{df}} & \{x \mid 1\} & \textit{universal class}\\
\mathrm{Ker}(X) =_{\mathrm{df}} & \{x \mid \Delta(x \in X)\} & \textit{kernel}\\
\mathrm{Supp}(X) =_{\mathrm{df}} & \{x \mid \nabla(x \in X)\} & \textit{support}\\
\backslash X =_{\mathrm{df}} & \{x \mid x \notin X\} & \textit{complement}\\
X \cap Y =_{\mathrm{df}} & \{x \mid x \in X \,\&\, x \in Y\} & \textit{intersection}\\
X \cup Y =_{\mathrm{df}} & \{x \mid x \in X \vee x \in Y\} & \textit{union}\\
X \setminus Y =_{\mathrm{df}} & \{x \mid x \in X \,\&\, x \notin Y\} & \textit{difference}
\end{array}$$

Definition 5 (Fuzzy class operations and relations) *Further we define the following elementary relations between fuzzy sets:*

$$\begin{array}{rll}
\mathrm{Hgt}(X) \equiv_{\mathrm{df}} & (\exists x)(x \in X) & \textit{height}\\
\mathrm{Norm}(X) \equiv_{\mathrm{df}} & (\exists x)\Delta(x \in X) & \textit{normality}\\
\mathrm{Crisp}(X) \equiv_{\mathrm{df}} & (\forall x)\Delta(x \in X \vee x \notin X) & \textit{crispness}\\
\mathrm{Fuzzy}(X) \equiv_{\mathrm{df}} & \neg\mathrm{Crisp}(X) & \textit{fuzziness}\\
\mathrm{Ext}_E(X) \equiv_{\mathrm{df}} & (\forall x,y)(Exy \,\&\, x \in X \to y \in X) & \textit{E-extensionality}\\
X \subseteq Y \equiv_{\mathrm{df}} & (\forall x)(x \in X \to x \in Y) & \textit{inclusion}\\
X \approx Y \equiv_{\mathrm{df}} & (\forall x)(x \in X \leftrightarrow x \in Y) & \textit{equality}\\
X \parallel Y \equiv_{\mathrm{df}} & (\exists x)(x \in X \,\&\, x \in Y) & \textit{compatibility}
\end{array}$$

We shall freely use all elementary theorems on these notions which follow from the metatheorems proved in [2], and thus can be checked by simple propositional calculations.

Definition 6 *The union and intersection of a class of classes are the functions $\bigcup^{(n+3)}$ and $\bigcap^{(n+3)}$, respectively, assigning a class $A^{(n+1)}$ to a class of classes $\mathcal{A}^{(n+2)}$ and defined as follows:*

$$\bigcup \mathcal{A} =_{\mathrm{df}} \{x \mid (\exists A \in \mathcal{A})(x \in A)\}$$
$$\bigcap \mathcal{A} =_{\mathrm{df}} \{x \mid (\forall A \in \mathcal{A})(x \in A)\}$$

Definition 7 *In $\mathcal{F}_2$, we define the following operations:*

$$\begin{array}{rl}
X \times Y =_{\mathrm{df}} & \{\langle x,y\rangle \mid x \in X \,\&\, y \in Y\}\\
\mathrm{Dom}(R) =_{\mathrm{df}} & \{x \mid \langle x,y\rangle \in R\}\\
\mathrm{Rng}(R) =_{\mathrm{df}} & \{y \mid \langle x,y\rangle \in R\}\\
R''A \equiv_{\mathrm{df}} & \{x \mid (\exists y)(y \in A \,\&\, Ryx)\}\\
R \circ S =_{\mathrm{df}} & \{\langle x,y\rangle \mid (\exists z)(\langle x,z\rangle \in R \,\&\, \langle z,y\rangle \in S)\}\\
R^{-1} =_{\mathrm{df}} & \{\langle x,y\rangle \mid \langle y,x\rangle \in R\}\\
\mathrm{Id} =_{\mathrm{df}} & \{\langle x,y\rangle \mid x = y\}
\end{array}$$

We can also define the usual properties of relations:

$$\begin{array}{rll}
\mathrm{Ext}_E(R) \equiv_{\mathrm{df}} & (\forall x,x',y,y')(Exx' \mathbin{\&} Eyy' \mathbin{\&} Rxy \to Rx'y') & \textit{E-extensionality} \\
\mathrm{Refl}(R) \equiv_{\mathrm{df}} & (\forall x)(Rxx) & \textit{reflexivity} \\
\mathrm{Sym}(R) \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \to Ryx) & \textit{symmetry} \\
\mathrm{Trans}(R) \equiv_{\mathrm{df}} & (\forall x,y,z)(Rxy \mathbin{\&} Ryz \to Rxz) & \textit{transitivity} \\
\mathrm{Asym}_E(R) \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \mathbin{\&} Ryx \to Exy) & \textit{E-antisymmetry}
\end{array}$$

Antisymmetry and other properties that classically refer to identity are defined here w.r.t. some relation E (usually an equality) in order to avoid the crispness of $=$. We adopt the convention that the index E can be dropped if $E = \mathrm{Id}$.

2 General properties of fuzzy relations

There are many theorems on relations easily provable in our theory (some of them we list below). It should be noticed that in our setting, the properties of relations (e.g., reflexivity) are graded. Thus the implications in the following theorems are generally stronger than the corresponding statements about entailment.

Theorem 8 *The following properties of relations are provable in $\mathcal{F}_2$:*

1. $\mathrm{Refl}(R) \leftrightarrow \mathrm{Id} \subseteq R$
2. $\mathrm{Sym}(R) \leftrightarrow R^{-1} \subseteq R$
3. $\mathrm{Trans}(R) \leftrightarrow R \circ R \subseteq R$
4. $\mathrm{Refl}(R) \to R \subseteq R \circ R$
5. $\mathrm{Trans}(R) \mathbin{\&} \mathrm{Trans}(Q) \to \mathrm{Trans}(R \cap Q)$
6. $R \subseteq S \to (R \circ T \subseteq S \circ T) \wedge (T \circ R \subseteq T \circ S)$

Thus every relation 1. is reflexive to the same degree as it contains identity, 2. is symetric to the same degree as it contains its own inverse, 4. is contained in the composition with itself at least in the degree of its reflexivity, etc.

Theorem 9 *For an arbitrary binary relation R and arbitrary classes A, B we have:*

1. $A \subseteq B \to R''A \subseteq R''B$
2. $\mathrm{Refl}(R) \to A \subseteq R''A$
3. $\mathrm{Trans}(R) \to \mathrm{Congr}_R(R''A)$
4. $A \subseteq B \mathbin{\&} \mathrm{Congr}_R(B) \to R''A \subseteq B$
5. $\mathrm{Trans}(R) \to R''(R''A) \subseteq R''A$
6. $\mathrm{Refl}(R) \mathbin{\&} \mathrm{Congr}_R(A) \to R''A \approx A$

Theorem 10 *If $\mathcal{A}$ is a crisp system of classes, then* $(\forall X \in \mathcal{A})\mathrm{Congr}_E(X) \to \mathrm{Congr}_E(\bigcap \mathcal{A}) \wedge \mathrm{Congr}_E(\bigcup \mathcal{A})$.

3 Similarities and partitions

The notion of similarity is defined as usual (in analogy with classical mathematics it could also be called equivalence). Like all properties of fuzzy relations in our setting, it is a graded notion.

Definition 11 (Similarity) $\mathrm{Sim}(R) \equiv_{\mathrm{df}} \mathrm{Refl}(R) \mathbin{\&} \mathrm{Sym}(R) \mathbin{\&} \mathrm{Trans}(R)$

Similarities are closely related with fuzzy partitions, as the following theorems show. Notice the exponents in Theorems 13–15, which are caused by the non-contractivity of fuzzy logic.

Definition 12 *We define:*

$$
\begin{aligned}
[x]_\sim &\equiv_{\rm df} \{y \mid y \sim x\} \\
\mathrm{Cover}(\mathcal{X}) &\equiv_{\rm df} \bigcup \mathcal{X} = \mathrm{V} \\
\mathrm{Disj}(\mathcal{X}) &\equiv_{\rm df} X \parallel Y \rightarrow X \approx Y \\
\mathrm{Part}(\mathcal{X}) &\equiv_{\rm df} \mathrm{Crisp}(\mathcal{X}) \ \&\ \mathrm{Disj}(\mathcal{X}) \ \&\ \mathrm{Cover}(\mathcal{X}) \\
\mathrm{V}/\!\sim &\equiv_{\rm df} \{X \mid (\exists x)(X = [x]_\sim)\} \\
\sim_\mathcal{X} &\equiv_{\rm df} \{\langle x,y\rangle \mid (\exists X \in \mathcal{X})(x \in X \ \&\ y \in X)\}
\end{aligned}
$$

Theorem 13 *It is provable in* $\mathcal{F}_2$*:*

1. $\mathrm{Refl}(\sim) \rightarrow (\forall x,y)([x]_\sim \approx [y]_\sim \rightarrow x \sim y)$
2. $\mathrm{Trans}^2(\sim) \ \&\ \mathrm{Sym}(\sim) \rightarrow (\forall x,y)(x \sim y \rightarrow [x]_\sim \approx [y]_\sim)$
3. $\mathrm{Sim}(\sim) \ \&\ \mathrm{Trans}(\sim) \rightarrow (\forall x,y)([x]_\sim \approx [y]_\sim \leftrightarrow x \sim y)$

Theorem 14 *It is provable in* $\mathcal{F}_3$*:*

1. $\mathrm{Crisp}(\mathrm{V}/\!\sim)$
2. $\mathrm{Refl}(\sim) \rightarrow \mathrm{Cover}(\mathrm{V}/\!\sim)$
3. $\mathrm{Trans}^3(\sim) \ \&\ \mathrm{Sym}(\sim) \rightarrow \mathrm{Disj}(\mathrm{V}/\!\sim)$
4. $\mathrm{Sim}(\sim) \ \&\ \mathrm{Trans}^2(\sim) \rightarrow \mathrm{Part}(\mathrm{V}/\!\sim)$

Theorem 15 *It is provable in* $\mathcal{F}_3$*:*

1. $\mathrm{Sym}(\sim_\mathcal{X})$
2. $\mathrm{Crisp}(\mathcal{X}) \ \&\ \mathrm{Cover}^2(\mathcal{X}) \rightarrow \mathrm{Refl}(\sim_\mathcal{X})$
3. $\mathrm{Disj}(\mathcal{X}) \rightarrow \mathrm{Trans}(\sim_\mathcal{X})$
4. $\mathrm{Part}(\mathcal{X}) \ \&\ \mathrm{Cover}(\mathcal{X}) \rightarrow \mathrm{Sim}(\sim_\mathcal{X})$

Theorem 16 *Let us assume that* $\mathrm{Sim}(\sim)$ *and* $\mathrm{Part}(\mathcal{X})$*. Then we have*

1. $\mathrm{Sim}(\sim_\mathcal{X})$
2. $\mathrm{Part}(\mathrm{V}/\!\sim)$
3. $\sim\, = \,\sim_{\mathrm{V}/\sim}$
4. $\mathcal{X} = \mathrm{V}/\!\sim_\mathcal{X}$

4 Fuzzy orderings

The notion of fuzzy quasiordering is defined as usual, viz. as a reflexive transitive relation. A quasiordering is an E-ordering iff it is E-antisymetric:

Definition 17 (Quasiordering and ordering)

$$
\begin{aligned}
\mathrm{QOrd}(R) &\equiv_{\rm df} \mathrm{Refl}(R) \ \&\ \mathrm{Trans}(R) \\
\mathrm{Ord}_E(R) &\equiv_{\rm df} \mathrm{QOrd}(R) \ \&\ \mathrm{Asym}_E(R)
\end{aligned}
$$

Theorem 18 *Many properties of (quasi)orderings are provable in* $\mathcal{F}_3$*, e.g. the following:*

1. $\mathrm{QOrd}(R) \rightarrow R''(R''A) \approx R''A$
2. $\mathrm{QOrd}(R) \rightarrow R \circ R \approx R$

3. $\mathrm{QOrd}(R) \to R''A \approx \bigcap\{X \mid A \subseteq X \;\&\; \mathrm{Congr}_R(X)\}$

The following notions are most meaningful for (quasi)orderings. Nevertheless, the definitions can be formulated for just any relations and most of the results hold regardless of any properties of the relations involved. Let us fix an arbitrary relation $\leq$ and denote its converse by $\geq$.

Definition 19 *The upper and lower cone of a class A w.r.t. $\leq$ is defined as follows:*

$$A^{\triangle} =_{\mathrm{df}} \{x \mid (\forall a \in A)(x \geq a)\}$$
$$A^{\triangledown} =_{\mathrm{df}} \{x \mid (\forall a \in A)(x \leq a)\}$$

The usual definition of suprema and infima as least upper bounds and greatest lower bounds can then be formulated as follows (notice that they are fuzzy classes, since the property of being a supremum is graded):

Definition 20 *The class of suprema and infima of a class A w.r.t. $\leq$ are defined as follows:*

$$\mathrm{Sup}\,A =_{\mathrm{df}} A^{\triangle} \cap A^{\triangle\triangledown}$$
$$\mathrm{Inf}\,A =_{\mathrm{df}} A^{\triangledown} \cap A^{\triangledown\triangle}$$

Example 21 *$\bigcup \mathcal{A}$ is a supremum of $\mathcal{A}$ w.r.t. $\subseteq$. Similarly, $\bigcap \mathcal{A} \in \mathrm{Inf}_{\subseteq} \mathcal{A}$.*

We formulate the following theorems only for suprema, omitting their dual versions.

Theorem 22 *The following are theorems of $\mathcal{F}_2$:*

1. *$A \subseteq B \to B^{\triangle} \subseteq A^{\triangle}$ (antitony of cones)*
2. *$A \subseteq A^{\triangle\triangledown}$ (closure)*
3. *$A^{\triangle} = A^{\triangle\triangledown\triangle}$ (stability)*
4. *$(A \subseteq B \;\&\; x \in \mathrm{Sup}\,A \;\&\; y \in \mathrm{Sup}\,B) \to y \leq x$ (antitony of suprema)*
5. *$(x \in \mathrm{Sup}\,A \;\&\; y \in \mathrm{Sup}\,A) \to (x \leq y \;\&\; y \leq x)$ (uniqueness)*

Corollary 23 1*-true suprema w.r.t. $\leq$ are E-unique if $\leq$ is antisymmetric w.r.t. E. If further $\Delta Exy \leftrightarrow x = y$, the unique element of* $\mathrm{Ker}(\mathrm{Sup}\,A)$ *can be called* the *supremum of A and denoted by* $\sup A$.

Example 24 *The suprema w.r.t. $\subseteq$ are $\approx$-uniquely determined. Due to the extensionality axiom, the element of the kernel of $\mathrm{Sup}_{\subseteq} \mathcal{A}$ is unique w.r.t. identity. Thus $\bigcup \mathcal{A} = \sup_{\subseteq} \mathcal{A}$.*

Theorem 25 $\mathrm{Sup}\,A = \mathrm{Inf}\,A^{\triangle}$

Corollary 26 *If there is an element of $\mathrm{Ker}(\mathrm{Inf}\,A)$ for all $A \in \mathcal{A}$, then there is an element of $\mathrm{Ker}(\mathrm{Sup}\,A)$ for all $A \in \mathcal{A}$ as well. In other words, the completeness of $\mathcal{A}$ w.r.t. infima entails its completeness w.r.t. suprema.*

Example 27 *Due to Example 24, the power class $\mathcal{P}(A) =_{\mathrm{df}} \{X \mid X \subseteq A\}$ is a complete lattice w.r.t. $\subseteq$. Similarly, due to Example 24 and Theorem 10, the class $\{X \mid \Delta\,\mathrm{Ext}_E\,X\}$ is a complete lattice w.r.t. $\subseteq$.*

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If–then Rules from Data Tables with Fuzzy Attributes

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Introduction and problem setting Tabular data describing objects and their attributes represents a basic form of data. Recently, methods for obtaining so-called association rules (particular if-then rules) from tabular data became popular, see [1] and also [11]. In our paper (extended abstract), we are interested in if-then rules from data with fuzzy attributes: rows and columns of data table correspond to objects $x \in X$ and attributes $y \in Y$, respectively. Table entries $I(x,y)$ are truth degrees to which object x has attribute y. We are interested in rules of the form "if A then B" ($A \Rightarrow B$), where A and B are collections of attributes, with the meaning: if an object has all the attributes of A then it has also all attributes of B. In crisp case, these rules were thoroughly investigated, see e.g. [8] and [7] for further information and references. Our aim is basically to look at such if-then rules from the point of view of fuzzy logic. Our motivation is the following: (1) in practice, attributes are usually fuzzy rather than bivalent; (2) non-logical attributes (like age, etc.) can be scaled to fuzzy attributes; (3) to investigate connections with related methods for processing of data with fuzzy attributes, particularly with formal concept analysis, e.g. [2, 3, 12]; (4) our results can be seen as preliminary results for a rigorous approach to mining association rules and related methods [8, 7] from the point of view of fuzzy logic.

We discuss the following topics: a tractable definition of if-then rules $A \Rightarrow B$ and their semantics (validity degree etc.); directly related mathematical structures; the notion of semantic entailment of if-then rules; non-redundant bases of all valid rules; algorithms for generating bases. We omit examples due to a limited scope.

Preliminaries As a structure of truth degrees we use a so-called complete residuated lattice with a truth-stressing hedge (shortly, a hedge), i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice; $\langle L, \otimes, 1\rangle$ is a commutative monoid; $\otimes$ and $\rightarrow$ satisfy the so-called adjointness property, see [2, 9]; and ${}^* : L \rightarrow L$ satisfies $1^* = 1$, $a^* \leq a$, $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, $a^{**} = a^*$ for each $a, b \in L$. Elements a of L are called truth degrees, $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [9, 10]. For each $\mathbf{L}$, two boundary cases of hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization: $1^* = 1$, and $a^* = 0$ for $a \neq 1$.

We use usual notions like $\mathbf{L}$-set (i.e. a fuzzy set with truth degrees in $\mathbf{L}$), etc., see e.g. [2]. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree $S(A,B) = \bigwedge_{u \in U}(A(u) \rightarrow B(u))$, which generalizes the classical subsethood relation $\subseteq$. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$.

Fuzzy attribute implications *Fuzzy attribute implication* (*over attributes* Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). The intended meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has also all attributes from B".

For an $\mathbf{L}$-set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M: $||A \Rightarrow B||_M = S(A,M)^* \rightarrow S(B,M)$. If M is the fuzzy set of all attributes of an object x, then

$||A \Rightarrow B||_M$ is the truth degree to which $A \Rightarrow B$ holds for x. Fuzzy attribute implications can describe particular dependencies. Let X and Y be sets of objects and attributes, respectively, I be an $\mathbf{L}$-relation between X and Y, i.e. I is a mapping $I\colon X \times Y \to L$. $\langle X,Y,I\rangle$ is called a *data table with fuzzy attributes.* $\langle X,Y,I\rangle$ represents a table which assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x,y) \in L$ to which object x has attribute y.

For fuzzy sets $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ we define fuzzy sets $A^{\uparrow} \in \mathbf{L}^Y$ and $B^{\downarrow} \in \mathbf{L}^X$ by $A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x)^* \to I(x,y))$, and $B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y))$. We put $\mathcal{B}(X^*,Y,I) = \{\langle A,B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \,|\, A^{\uparrow} = B, B^{\downarrow} = A\}$ and define $\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_1 \supseteq B_2$). Operators ${}^{\downarrow},{}^{\uparrow}$ form so-called Galois connection with hedge, see [5]. $\langle \mathcal{B}(X^*,Y,I),\leq\rangle$ is called a *fuzzy concept lattice* induced by $\langle X,Y,I\rangle$, $\langle A,B\rangle$ of $\mathcal{B}(X^*,Y,I)$ are interpreted as concepts (clusters) hidden in data (A and B are called the *extent* and the *intent* of $\langle A,B\rangle$) Furthermore, $\leq$ models the subconcept-superconcept hierarchy—concept $\langle A_1,B_1\rangle$ is a subconcept of $\langle A_2,B_2\rangle$ iff each object from A_1 belongs to A_2 (dually for attributes).

Now we define validity of fuzzy attribute implications in data. First, for a set $\mathcal{M} \subseteq \mathbf{L}^Y$ we define a degree $||A \Rightarrow B||_{\mathcal{M}} \in L$ to which $A \Rightarrow B$ holds in $\mathcal{M}$ by $||A \Rightarrow B||_{\mathcal{M}} = \bigwedge_{M\in\mathcal{M}} ||A \Rightarrow B||_M$. Having $\langle X,Y,I\rangle$, let $I_x \in \mathbf{L}^Y$ ($x \in X$) be defined by $I_x(y) = I(x,y)$ for each $y \in Y$. That is, I_x is the $\mathbf{L}$-set of all attributes of $x \in X$ (a row corresponding to x). A *degree* $||A \Rightarrow B||_{\langle X,Y,I\rangle} \in L$ *to which* $A \Rightarrow B$ *holds in* (*each row of*) $\langle X,Y,I\rangle$ is defined by $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{M}}$, where $\mathcal{M} = \{I_x \,|\, x \in X\}$. Denote $\mathrm{Int}(X^*,Y,I) = \{B\in\mathbf{L}^Y | \langle A,B\rangle \in \mathcal{B}(X^*,Y,I) \text{ for some } A\}$ the set of all intents of concepts of $\mathcal{B}(X^*,Y,I)$. Then, we can consider the degree $||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$ to which $A \Rightarrow B$ is true in the system of all intents. The following theorem shows basic relationships.

Theorem 1. $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = S(B,A^{\downarrow\uparrow})$ *for each* $A \Rightarrow B$. □

Implication bases The set of all attribute implications which are true in data is huge. For instance, it contains all trivially valid rules like $A \Rightarrow A$. To have only a reasonably large set of interesting rules and still not to lose anything, one can proceed via a notion of semantic entailment and consider only a non-redundant base of all implications true in data. Let T be a set of fuzzy attribute implications. $M \in \mathbf{L}^Y$ is called a *model* of T if $||A \Rightarrow B||_M = 1$ for each $A \Rightarrow B \in T$. The set of all models of T is denoted by $\mathrm{Mod}(T)$. A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ *semantically follows* from T is defined by $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Mod}(T)}$. T is called *complete* (in $\langle X,Y,I\rangle$) if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$ for each $A \Rightarrow B$. If T is complete and no proper subset of T is complete, then T is called a *non-redundant basis*. The following assertion shows that the models of a complete set of fuzzy attribute implications are exactly the intents of the corresponding fuzzy concept lattice.

Theorem 2. T *is complete iff* $\mathrm{Mod}(T) = \mathrm{Int}(X^*,Y,I)$. □

We are interested in finding non-redundant bases. First, a non-redundant basis T is a minimal set of implications which conveys, via the notion of semantic entailment, information about validity of attribute implications in $\langle X,Y,I\rangle$. In particular, attribute implications which are true (in degree 1) in $\langle X,Y,I\rangle$ are exactly those which follow (in degree 1) from T. Second, non-redundant bases are promising candidates for being the minimal complete sets of attribute implications which describe the concept intents (and consequently, the whole fuzzy concept lattice). Namely, concept intents are models of a non-redundant basis.

Algorithm for getting non-redundant bases Given $\langle X,Y,I\rangle$, $\mathcal{P} \subseteq \mathbf{L}^Y$ (a system of $\mathbf{L}$-sets of attributes) is called a *system of pseudo-intents* of $\langle X,Y,I\rangle$ if for each $P \in \mathbf{L}^Y$:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P.$$

If * is globalization and if Y is finite, then for each $\langle X,Y,I\rangle$ there exists a unique system of pseudo-intents (this is not so for the other hedges in general). From now on, let $\langle X,Y,I\rangle$ be finite and let $\mathbf{L}$ be finite and linearly ordered. We can prove the following theorem (cf. [8, 7]).

Theorem 3. *Let $\mathcal{P}$ be a system of pseudo-intents and put $T = \{P \Rightarrow P^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\}$. Then* (i) *$T$ is non-redundant basis; if * is globalization then T is minimal;* (ii) *if * is globalization then there is an* $\mathbf{L}$*-closure operator cl_{T^*} such that $\mathcal{P} \cup \mathrm{Int}(X^*,Y,I)$ is the set of all fixpoints of cl_{T^*}.* □

The previous theorem showed that for * being the globalization, we can get all intents and all (pseudo) intents (of a given data table with fuzzy attributes) by computing the fixpoints of cl_{T^*}. This can be done with polynomial time delay using an extension of Ganter's algorithm [4] as follows:

Input: data table with fuzzy attributes $\langle X,Y,I\rangle$, $\mathbf{L}$ finite linearly ordered with globalization.
Output: $\mathrm{Int}(X^*,Y,I)$ (set of all intents), $\mathcal{P}$ (set of all pseudo-intents).

$B := \emptyset$
if $B = B^{\downarrow\uparrow}$: *add* B *to* $\mathrm{Int}(X^*,Y,I)$; *else*: *add* B *to* $\mathcal{P}$
while $B \neq Y$:
  $T := \{P \Rightarrow P^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\}$
  $B := B^+$ (B^+ is the lectically smallest fixpoint of cl_{T^*} which is a successor of B)
  if $B = B^{\downarrow\uparrow}$: *add* B *to* $\mathrm{Int}(X^*,Y,I)$; *else*: *add* B *to* $\mathcal{P}$

Reduction to the crisp case A data table with fuzzy attributes can be transformed to a data table with crisp attributes. An interesting question is that of the relationship between the validity of attribute implications in the corresponding tables. The details follow. For a data table $\langle X,Y,I\rangle$ with fuzzy attributes we consider a data table $\langle X, Y \times L, I'\rangle$ with crisp attributes, where $I' \in \mathbf{2}^{X\times(Y\times L)}$ and $\langle x, \langle y,a\rangle\rangle \in I'$ iff $a \leq I(x,y)$ for each $x \in X$ and $\langle y,a\rangle \in Y \times L$ (see also [6]). For $A \in \mathbf{2}^Y$ denote by $\lceil A \rceil$ an $\mathbf{L}$-set $\lceil A \rceil = \{\langle y,a\rangle \in Y \times L | a \leq A(y)\}$. Using some technical results concerning relationships of validity of attribute implications in $\langle X,Y,I\rangle$ and the naturally corresponding implications in $\langle X, Y \times L, I'\rangle$, one can show:

Theorem 4. *Suppose $\mathcal{P}$ is a system of pseudo-intents of $\langle X, Y \times L, I'\rangle$.*
Then $T_c = \{\lceil P \rceil \Rightarrow \lceil P \rceil^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\}$ is complete with respect to data table $\langle X,Y,I\rangle$. □

Theorem 4 enables us to find a complete set of fuzzy attribute implications as follows. For an input data table $\langle X,Y,I\rangle$ we compute its crisp counterpart $\langle X, Y \times L, I'\rangle$, then we can use the classical algorithm for getting pseudo-intents of $\langle X, Y \times L, I'\rangle$. Finally, we construct T_c using the pseudo-intents of $\langle X, Y \times L, I'\rangle$ as shown in Theorem 4. T_c need not be non-redundant. In fact, T_c can be considerably greater than the minimal non-redundant basis which can be computed directly from the input data table with fuzzy attributes. To sum up, from the user's viewpoint, both data tables $\langle X,Y,I\rangle$ and $\langle X, Y \times L, I'\rangle$ represent the same information—the concepts extracted from both data tables are in one-to-one correspondence. However, the minimal basis of $\langle X, Y \times L, I'\rangle$ cannot be turned into a non-redundant basis of $\langle X,Y,I\rangle$ by the $\lceil \cdots \rceil$ operator. The size of a minimal basis of $\langle X, Y \times L, I'\rangle$ is equal or greater to the size of a minimal basis of $\langle X,Y,I\rangle$. This feature can be surprising, but it has the following (informal) explanation: the basis of $\langle X, Y \times L, I'\rangle$ contains implications describing, among the dependencies between attributes, the dependencies between truth degrees. In the basis of $\langle X,Y,I\rangle$, such dependencies are determined automatically by the chosen $\mathbf{L}$. Therefore, working directly in fuzzy setting is beneficial.

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Relations in Higher-order Fuzzy Logic III

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This contribution is the third and last of a series of talks on relations in higher-order fuzzy logic. The first two [5] have introduced the logical framework (see also [3, 4]) along with a set of basic results that, at first glance, look very familiar. These results, however, have been developed from a much more general basis. Their proofs have been devised independently and resemble closer to proofs known from classical theory than to proofs of results existing in fuzzy set theory.

The purpose of this contribution is to establish links between existing results in fuzzy set theory and the results contained in [5]. Moreover, we provide an interpretation to which extent this new framework really adds value and an educated guess what its potential impact on the further development of theory may be.

Links to Existing Concepts and Results

Graded properties of fuzzy relations

In [5, Section 1], the definitions of the five properties E-extensionality, reflexivity, symmetry, transitivity, and E-antisymmetry (cf. Def. 6) are most crucial. Looking as traditional definitions at first glance, the expressions $\mathrm{Ext}_E(R)$, $\mathrm{Refl}(R)$, $\mathrm{Sym}(R)$, $\mathrm{Trans}(R)$, and $\mathrm{Asym}_E(R)$ are not crisp, but may be true to some degree. An approach in this direction has already been introduced by Gottwald [12, 13] and later on picked up by Jacas and Recasens [16]. These works have in common that they are not based on a general logical framework, but on triangular norms on the unit interval (note, however, that Gottwald uses notations that are inspired by formal logic, similar to the terminology introduced in [5]).

The property of extensionality, to our best knowledge, has only been considered in a crisp way so far [14, 17, 18]. The three properties $\mathrm{Refl}(R)$, $\mathrm{Sym}(R)$, and $\mathrm{Trans}(R)$ appear in [12, 13, 16], at least under the restrictions stated above.

The property $\mathrm{Asym}_E(R)$ is different from the one introduced by Gottwald [12, 13] who starts from Zadeh's definition of antisymmetry [19], but with a general t-norm instead of the minimum. The definition of $\mathrm{Asym}_E(R)$ is inspired by the similarity-based approach to fuzzy orderings (see, e.g., [1, 15] and other publications) and trivially coincides with Gottwald's definition if $E = \mathrm{Id}$. Note that the definition of $\mathrm{Asym}_E(R)$ appears in [16], interestingly, without any reference to the similarity-based approach to fuzzy orderings.

In [5, Section 2], several basic results about relations in higher-order fuzzy logic are provided. The crisp counterparts of the assertions comprised in Theorems 2 and 3 are well-known and can be found in any textbook that contains an adequately deep introduction to fuzzy relations (e.g. [11]). Assertions 1.–3. and 6. from Theorem 2 are also known in the graded framework (see, e.g., [13, Sections 18.4 and 18.6]). The fact that intersections and unions of extensional fuzzy sets are again extensional is also well-known [2, 17], its graded generalization in [5, Theorem 4] constitutes a new finding.

Similarities and partitions

In [5, Section 3], a first step towards a graded theory of equivalence relations and partitions is taken. The degree to which a relation is a similarity, denoted $\text{Sim}(R)$, is defined in the same way as in [13] (again note the difference that Gottwald restricts to the unit interval equipped with a t-norm). The concept of a graded fuzzy partition that is built up on this basis can be considered an entirely new concept. The degree of disjointness $\text{Disj}(\mathcal{X})$ is a straightforward generalization of the disjointness criterion that is well-known from literature [7, 10, 17, 18] (in our notation, being equivalent to $\text{Disj}(\mathcal{X}) = 1$). The degree $\text{Part}(\mathcal{X})$ to which a class of classes $\mathcal{X}$ is a partition is a straightforward generalization of the concept of a T-partition introduced in [7] (being put in a wider context in [10]).

Results like the ones from Theorem 5 are available in [13, Section 18.6, p. 466]. Moreover, crisp counterparts of these assertions and the ones from Theorem 6 occur in literature (see [7, 10, 17, 18] and several others), although the graded framework gives these theorems a rather different flavor. Theorems 7 and 8 closely resemble to some results known from literature [7, 10, 17]. In these papers, however, slightly different ways to construct an equivalence relation from a partition are employed than the relation $\sim_{\mathcal{X}}$ which is only guaranteed to be a fuzzy equivalence relation in the traditional sense (in our framework, being equivalent to $\text{Sim}(\sim_{\mathcal{X}}) = 1$) if $\text{Part}(\mathcal{X}) = 1$ [5, Theorem 8].

Fuzzy orderings and lattice operations

Finally, in [5, Section 4], a graded concept of fuzzy orderings is introduced in line with the similarity-based approach to fuzzy orderings [1, 15]. Gottwald [12, 13] uses the same techniques to define a graded concept of fuzzy partial ordering, but with respect to the crisp equality and not with reference to a fuzzy equivalence relation. Theorem 9 lists results that are well-known in the classical non-graded theory of fuzzy quasiorderings, but new in a graded framework. Assertion 1. is a graded version of the idempotence of the full image with respect to a fuzzy quasiordering [2]. Assertion 2. is a well-known correspondence (see, e.g., [11]). As also known from the classical non-graded theory [2], Assertion 3. is a graded generalization of the fact that the full image of a fuzzy class A with respect to a fuzzy quasiordering R is uniquely represented as the intersection of all R-extensional super-classes of A.

Definitions 10 and 11 can be considered as a starting point towards a general graded theory of fuzzy lattices. The definitions of upper and lower cones, suprema and infima, respectively, appear in the same way as in [6, 8, 9]. Some of the assertions of Theorem 10 are similarly contained in [6].

Conclusion and Outlook

The question remains what kind of value is added by basing a theory of fuzzy relations on the fuzzy class theory as introduced in [4, 5]. First of all, the framework discussed here is well-founded and general. Proofs in this framework are still concise, elegant, and expressive — which is remarkable in light of the fact that all properties of fuzzy relations are graded. Note that Gottwald states in [13, Section 18.6, p. 465] that the development of a full-fledged graded theory of fuzzy equivalence relations and orderings is an open issue. Although the results presented in [5] can only be considered as a good starting point, we strongly believe that this framework has the potential to solve that open issue. The elegance and conciseness of the approach not only allows to generate shorter proofs of many known results in a routine manner. Overcoming the technicality and clumsiness of the classical theory of fuzzy relations may also open the field for discovering completely new results — that is no serious scientific statement based on clear evidence, but a strong belief it is indeed.

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Linear Space of Fuzzy Vectors*

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Abstract. We present several possible representations of fuzzy vectors motivated by the aim to make the operations with them practically computable. Further, we study the possibility of finding one point representing a fuzzy vector in such a way that the algebraic operations are preserved. This should generalize the notion of Steiner centroid known for crisp convex sets. Surprisingly, it is difficult to obtain uniqueness in the fuzzified approach.
Keywords: Convex set, support function, Steiner point, convex fuzzy set, fuzzy vector, defuzzification, image processing, biomedical imaging, magnetic resonance, computer tomography.

1 Introduction

Fuzzy vectors are a multi-dimensional generalization of fuzzy numbers and intervals representing vague quantities. They admit arithmetical operations (inherited from the algebra of convex sets) which introduce a linear structure and embed the space of fuzzy vectors as a positive cone in a linear space. For practical implementation of the algebra of fuzzy vectors, the notion of support function appears useful. It allows to represent fuzzy vectors by functions on the unit ball (or the product of an interval and the unit sphere). Linear operations with fuzzy vectors then correspond to pointwise operations on the support functions which are easier to compute. The use of support functions is also useful for optimization in the space of fuzzy vectors [2].

If we want to represent a compact convex subset of $\mathbb{R}^n$ by a single element, usually the Steiner centroid (Steiner point) is chosen. The function which associates with every compact convex set its Steiner centroid is continuous (w.r.t. the Hausdorff metric) and preserves the linear structure and all isometries. These properties uniquely characterize the Steiner centroid [7, 5]. Trying to extend the Steiner centroid to fuzzy vectors, we find out that the above properties do not determine it uniquely. E.g., they are satisfied for any weighted average of the Steiner centroids of level sets.

The Steiner centroid can be considered a defuzzification method. Its properties may be useful in medical imaging, where also the fuzzified version is desirable as a tool describing blurred images.

2 Computing in the space of fuzzy vectors via support functions

Let us outline a way of doing computations with fuzzy vectors via computations in Lebesgue vector spaces $\mathcal{L}^p$. We fix an $n \in \mathbb{N}$. By $\mathcal{K}^n$, we denote the set of compact convex subsets of $\mathbb{R}^n$ with the

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usual (Minkowski) addition and multiplication by positive reals. A *fuzzy n-vector* (*fuzzy vector* for short) is a function $u\colon \mathbb{R}^n \to [0,1]$ such that for each $\alpha \in (0,1]$, the upper level set (α-cut) $[u]_\alpha := \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$ is nonempty, closed, convex, and the set $\mathrm{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$ is bounded (hence compact) [2,3]. We denote by $\mathcal{E}^n$ the collection of all fuzzy n-vectors. In particular for $n = 1$, the set $\mathcal{E}^1$ consists of the so-called fuzzy numbers or fuzzy intervals. The natural extension of algebraic operations from $\mathbb{R}^n$ to fuzzy vectors was introduced in [4]. If $u,v \in \mathcal{E}^n$ and $c \in \mathbb{R}_+$, then there exist unique fuzzy vectors, denoted by $u+v$, respectively cu, such that, for all $\alpha \in (0,1]$,

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha\,,\ [cu]_\alpha = c\,[u]_\alpha\,. \tag{1}$$

Determining analytically (using the Zadeh's extension principle) the results of the operations defined by (1) amounts to finding the functions

$$(u+v)(x) = \sup\{u(y)+v(z) : y,z \in \mathbb{R}^n, y+z = x\} \tag{2}$$

$$(cu)(x) = \sup\{u(y) : y \in \mathbb{R}^n, cy = x\}\,. \tag{3}$$

The use of formulas (1)–(3) is generally difficult. An alternative method is based on support functions. We first formulate it for classical convex compact sets.

Let B^n, resp. S^n, be the unit ball, resp. the unit sphere, in $\mathbb{R}^n$. For any $A \in \mathcal{K}^n$, we define its *support function*, $h_A\colon S^n \to \mathbb{R}$, by

$$h_A(x) = \max\,\{\langle a,x\rangle : a \in A\}$$

(see e.g. [6]). Addition and multiplication by positive reals in $\mathcal{K}^n$ correspond to the same (pointwise) operations on the support functions. These can be computed easier than the operations on convex sets.

Following Diamond and Kloeden [3], the *support function* of a fuzzy vector $u \in \mathcal{E}^n$ is the function $H_u\colon (0,1] \times B^n \to \mathbb{R}$ defined by

$$H_u(\alpha,x) = \sup\{\langle a,x\rangle : a \in [u]_\alpha\}\,. \tag{4}$$

This notion differs from the Bobylev's basic idea [1] which uses the function $\overline{H}_u\colon B^n \to \mathbb{R}$ defined by

$$\overline{H}_u(x) = \sup\{\langle a,x\rangle : a \in [u]_{||x||}\}\,. \tag{5}$$

Alternatively, one may consider the function $\hat{H}_u\colon B^n \to \mathbb{R}$, where

$$\hat{H}_u(x) = \sup\{\langle a,x\rangle : a \in [u]_{1-||x||}\}\,, \tag{6}$$

which has also some advantageous properties. It is clear that H_u can be uniquely recovered from any of these representations.

It can be easily seen that $H_u(\alpha,\cdot)$ is exactly the support function $h_{[u]_\alpha}$. Again, the linear operations on $\mathcal{E}^n$ correspond the pointwise operations on the support functions. We obtain the following characterization:

Theorem 1. *A function $\varphi\colon (0,1] \times B^n \to \mathbb{R}$ is the support function of a fuzzy vector $u \in \mathcal{E}^n$ if and only if it is bounded and, for each $(\alpha,z) \in (0,1] \times B^n$, it satisfies the following conditions:*

$$c \in [0,1] \implies \varphi(\alpha,cz) = c\,\varphi(\alpha,z)\,, \tag{7}$$

$$x,y,x+y \in B^n \implies \varphi(\alpha,x+y) \leq \varphi(\alpha,x)+\varphi(\alpha,y)\,. \tag{8}$$

$$\varphi(\cdot,z) \text{ is nonincreasing and continuous from the left}\,. \tag{9}$$

3 Steiner centroids of convex sets and fuzzy vectors

The *Steiner centroid* of $A \in \mathcal{K}^n$ is defined by

$$s(A) = \frac{1}{V(B^n)} \int_{S^n} h_A(e)\, e \, d\lambda(e)\,,$$

where λ is the Lebesgue measure on S^n and $V(B^n)$ is the volume of the unit ball B^n. Notice that $s(A) \in A$.

Theorem 2. *Let $s'\colon \mathcal{K}^n \to \mathbb{R}^n$ have the following properties:*

(S1) *For any $A, B \in \mathcal{K}^n$, $s'(A+B) = s'(A) + s'(B)$.*

(S2) *For $A \in \mathcal{K}^n$ and any Euklidean isometry τ of $\mathbb{R}^n$, we have $s'(\tau A) = \tau s'(A)$.*

(S3) *s' is continuous.*

Then $s' = s$.

This theorem was proved by Shephard [7] for $n = 2$ and generalized by Schneider [5].
Inspired by the latter theorem, we propose the following general definition:

Definition 1. *Let us call a function $S\colon \mathcal{E}^n \to \mathbb{R}^n$ a* Steiner centroid *if it has the following properties:*

(SF1) *For any $v, w \in \mathcal{E}^n$, $S(v+w) = S(v) + S(w)$.*

(SF2) *For any Euklidean isometry τ of $\mathbb{R}^n$ and any $v \in \mathcal{E}^n$, we have $S(\tau v) = \tau S(v)$, where $\tau v = v \circ \tau^{-1}$ (v being seen as a membership function from $\mathbb{R}^n$ to $[0,1]$).*

(SF3) *S is continuous.*

Example 1. Let $\mu \in L^2((0,1])$ be a function such that $\int_0^1 \mu(\alpha)\, d\alpha = 1$. Define for $v \in \mathcal{E}^n$

$$S_\mu(v) = \int_0^1 s([v]_\alpha)\, \mu(\alpha)\, d\alpha,$$

where s is the classical Steiner centroid of the level set $[v]_\alpha$. Then S_μ is a Steiner centroid.

We see that a Steiner centroid of fuzzy vectors is not defined unambiguously by the properties (SF1)–(SF3). It is amazingly difficult to impose further properties on S to obtain uniqueness; it is an open question if this is possible in some reasonable (well motivated) way. At least we have proved that Example 1 reflects the most general case.

Remark 1. It is desirable to represent a solid by some typical point characterizing its position. The center of gravity seems to be the most natural choice. In contrast to it, the Steiner centroid represents rather the center of gravity of the *boundary* of the solid. In view of Th. 2, only the Steiner centroid preserves the convex arithmetic. This property might be useful in image processing and biomedical applications. Let us consider a (convex) non-symmetric solid. When it grows by a constant value on each side of its boundary, the new shape is obtained by a Minkowski sum of the old shape and a ball of the respective diameter. The center of gravity changes, but the Steiner centroid is preserved. Thus Steiner centroids may be natural in processing medical images obtained by magnetic resonance or computer tomography. The use of Steiner centroids is restricted to convex solids. In the non-convex case, one still might use the Steiner centroid of the convex hull.

Our fuzzification of the Steiner centroid can be considered a multi-dimensional defuzzification technique. The above motivation becomes even more natural when we use fuzzy sets for description of blurred images.

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Adding Modalities to Fuzzy Logics

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1 Introduction

Substructural logics may, broadly speaking, be characterized as logics where structural rules *fail*. For example, relevance logics [1] lack the weakening rule "from *A* and *B*, derive *A*", while Linear logic [9] lacks also the contraction rule "from *A*, derive *A* and *A*". However, although structural rules fail in general in these logics, it is often assumed that they hold in particular cases. Hence, in Linear logic weakening and contraction rules hold for formulae distinguished by the unary operators ! or ?, and in relevance logics weakening holds for the premises of so-called enthymematic implications. General programs giving structural rules for distinguished formulae have been presented by Restall [17] in an axiomatic and algebraic framework, and Gore [10] in a Display logic framework, using *modalities*. In these approaches, formulae for which structural rules apply are identified (similiarly to Linear logic) using modal operators e.g. as $\Box A$ where A is a formula. This allows embeddings of logics with structural rules into weaker logics with modalities e.g. of intuitionistic logic into linear logic, and facilitates an analysis of logical consequence within the language of the logic.

t-Norm based fuzzy logics (see [11] for details) are particular cases of *contraction-free* substructural logics, while *uninorm* based fuzzy logics (introduced recently in [13]) are logics where also the weakening rule may fail. This perspective is emphasized by the presentation of many of these logics as substructural logics in the framework of *hypersequents*, a generalization of Gentzen sequents consisting of a multiset (interpreted as a disjunction) of sequents, see [2, 4, 8] for details. Indeed such logics may be viewed as known substructural logics embedded in the hypersequent framework and extended with an external structural rule called communication that acts on more than one component at a time. In this work we present the basic ideas for a general methodology for adding Linear logic and S4 type modalities to fuzzy logics in the hypersequent framework, giving here as examples Monoidal t-norm logic **MTL**, Involutive t-norm logic **IMTL**, Uninorm logic **UL**, Involutive uninorm logic **IUL** and Gödel logic **G**. Additionally, calculi for logics with Delta-like connectives (see e.g. [3] for details) may be obtained by adding a further rule for $\Box$. By providing also a general cut-elimination method, we ensure that such calculi are analytic, and that adding modalities is conservative with respect to the original logic. We also investigate the *semantics* of these logics, both the algebraic semantics obtained by adding interior operators to residuated lattices, and the standard semantics where all connectives, including modalities, are interpreted as functions on the real unit interval $[0,1]$. We note that semantically, modalities are closely related to, but not the same as, the storage operators introduced by Montagna for t-norm based fuzzy logics in [16].

2 Proof Theory

Below we introduce axiomatizations for a number of fuzzy logics, based on a language with binary connectives $\odot$, $\to$, $\wedge$, and $\vee$, constants $\bot$, f and t, and defined connectives $A \leftrightarrow B =_{def} (A \to B) \wedge (B \to A)$, $\neg A =_{def} A \to f$ and $\top =_{def} \neg\bot$.

Definition 1. Uninorm logic **UL** *consists of the following axioms and rules:*

(U1) $(A \to B) \to ((B \to C) \to (A \to C))$
(U2) $(A \to (B \to C)) \to (B \to (A \to C))$
(U3) $((A \odot B) \to C) \leftrightarrow (A \to (B \to C))$
(U4) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(U5) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(U6) $((A \to B) \wedge t) \vee ((B \to A) \wedge t)$
(U7) $\bot \to A$
(U8) $A \to A$
(U9) $A \to (A \vee B)$
(U10) $B \to (A \vee B)$
(U11) $(A \wedge B) \to A$
(U12) $(A \wedge B) \to B$
(U13) $A \leftrightarrow (t \to A)$

$$(mp)\ \frac{A \to B \quad A}{B} \qquad\qquad (adj)\ \frac{A \quad B}{A \wedge B}$$

Axiomatizations for other fuzzy logics may be introduced as extensions of **UL** *e.g.*[3]

$$(INV)\ \neg\neg A \to A \qquad (W)\ A \to (B \to A) \qquad (ID)\ (A \odot A) \leftrightarrow A$$

$$\mathbf{IUL} = \mathbf{UL} + (INV) \qquad \mathbf{MTL} = \mathbf{UL} + (PRL) \qquad \mathbf{G} = \mathbf{MTL} + (ID)$$
$$\mathbf{IMTL} = \mathbf{MTL} + (INV) \qquad \mathbf{IUML} = \mathbf{IUL} + (ID)$$

A proof theoretic characterization of these logics is obtained by generalizing the notion of a Gentzen sequent to that of a *hypersequent*: a multiset of sequents (pairs of multisets of formulae) interpreted disjunctively and written:

$$\Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$$

If Δ_i contains at most one formula for $i = 1, \ldots, n$ then the hypersequent is *single-conclusion*, otherwise it is *multiple-conclusion*. Like sequent calculi, hypersequent calculi consist of axioms, logical rules and structural rules. Axioms for all the logics defined here are as follows:

$$(ID)\ A \Rightarrow A \qquad (f,l)\ f \Rightarrow \qquad (t,r) \Rightarrow t \qquad (\bot)\ \Gamma, \bot \Rightarrow \Delta \qquad (\top)\ \Gamma \Rightarrow \top, \Delta$$

Logical rules for connectives are the same as those in sequent calculi for substructural logics, except that a "side-hypersequent" may also occur, denoted here by G.

$$(t,l)\ \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, t \Rightarrow \Delta} \qquad (f,r)\ \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow f, \Delta}$$

$$(\to,l)\ \frac{G|\Gamma_1 \Rightarrow A, \Delta_1 \quad G|\Gamma_2, B \Rightarrow \Delta_2}{G|\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2} \qquad (\to,r)\ \frac{G|\Gamma, A \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \to B, \Delta}$$

$$(\odot,l)\ \frac{G|\Gamma, A, B \Rightarrow \Delta}{G|\Gamma, A \odot B \Rightarrow \Delta} \qquad (\odot,r)\ \frac{G|\Gamma_1 \Rightarrow A, \Delta_1 \quad G|\Gamma_2 \Rightarrow B, \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow A \odot B, \Delta_1, \Delta_2}$$

[3] Noting that for **MTL** and **IMTL** these axiomatizations are equivalent to those given in a more restricted language in [7].

$$(\wedge_i, l)_{i=1,2} \quad \frac{G|\Gamma, A_i \Rightarrow \Delta}{G|\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\wedge, r) \quad \frac{G|\Gamma \Rightarrow A, \Delta \quad G|\Gamma \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\vee, l) \quad \frac{G|\Gamma, A \Rightarrow \Delta \quad G|\Gamma, B \Rightarrow \Delta}{G|\Gamma, A \vee B \Rightarrow \Delta} \qquad (\vee_i, r)_{i=1,2} \quad \frac{G|\Gamma \Rightarrow A_i, \Delta}{G|\Gamma \Rightarrow A_1 \vee A_2, \Delta}$$

Structural rules are divided into two categories. *Internal* rules deal with formulae within components as in sequent calculi, and include a distinguished "cut" rule corresponding to the transitivity of deduction:

$$(WL) \quad \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta} \qquad (WR) \quad \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow A, \Delta}$$

$$(CL) \quad \frac{G|\Gamma, A, A \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta} \qquad (CR) \quad \frac{G|\Gamma \Rightarrow \Delta, A, A}{G|\Gamma \Rightarrow \Delta, A}$$

$$(CUT) \quad \frac{G|\Gamma_1, A \Rightarrow \Delta_1 \quad G|\Gamma_2 \Rightarrow A, \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

External rules manipulate whole components; for example the external weakening and contraction rules (EW) and (EC) add and remove components as follows:

$$(EW) \quad \frac{G}{G|\Gamma \Rightarrow \Delta} \qquad (EC) \quad \frac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}$$

The crucial external structural rule from the point of view of fuzzy logics, however, is the following *communication rule* which allows interaction between sequents:

$$(COM) \quad \frac{G|\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1 \qquad G|\Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}$$

Definition 2. **GIUL** *and* **GUL** *consist of the multiple-conclusion and single-conclusion versions respectively of the rules given above, excluding all internal structural rules save* (CUT)*. Calculi for other fuzzy logics are then defined as follows:*

GIUML *is* **GIUL** + (CL), (CR)	**GMTL** *is* **GUL** + (WL)
GIMTL *is* **GIUL** + (WL), (WR)	**GG** *is* **GMTL** + (CL)

We now turn our attention to adding modalities to fuzzy logics, extending our language with the unary connective $\Box$. For a logic **GL**, **GLS4** is obtained by adding the following rules, familiar from sequent calculi for Linear logic and the modal logic **S4**:

$$(\Box, l) \quad \frac{G|\Gamma, A \Rightarrow \Delta}{G|\Gamma, \Box A \Rightarrow \Delta} \qquad (\Box, r) \quad \frac{G|\Box\Gamma \Rightarrow C}{G|\Box\Gamma \Rightarrow \Box C}$$

We may (as for Linear logic) add structural rules applying only to boxed formulae, e.g.

$$(\Box WL) \quad \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, \Box A \Rightarrow \Delta} \qquad (\Box CL) \quad \frac{G|\Gamma, \Box A, \Box A \Rightarrow \Delta}{G|\Gamma, \Box A \Rightarrow \Delta} \qquad (\Box S) \quad \frac{G|\Box\Gamma, \Pi \Rightarrow \Sigma}{G|\Box\Gamma \Rightarrow |\Pi \Rightarrow \Sigma}$$

$(\Box WL)$ and $(\Box CL)$ allow the weakening and contraction of boxed formulae on the left respectively, while $(\Box S)$ ensures that boxed formulae obey the law of excluded middle. In this framework, we are able to embed certain logics into others. As an example, consider the embedding $p^* = \Box p$ for all atoms p, $(A\#B)^* = \Box(A^*\#B^*)$ for all binary connectives #. It is easy to show for example that $\Rightarrow A$ is derivable in **GG** iff $\Rightarrow A^*$ is derivable in **GMTLS4** $+ (\Box CL)$ and that $\Rightarrow A$ is derivable in **GMTL** iff $\Rightarrow A^*$ is derivable in **GUS4** $+ (\Box WL)$. Other embeddings of logics with single-conclusion calculi into logics with multiple-conclusion calculi may also be given.

Axiomatizations for fuzzy logics with modalities are given as follows:

Definition 3. **LS4** *is* **L** *extended with the following axioms and rule:*

$$(T_\Box)\ \Box A \to A \qquad (\to_\Box)\ \Box(A \to B) \to (\Box A \to \Box B)$$
$$(\vee_\Box)\ \Box(A \vee B) \to (\Box A \vee \Box B) \qquad (4_\Box)\ \Box A \to \Box\Box A$$
$$(\wedge_\Box)\ (\Box A \wedge \Box B) \to \Box(A \wedge B)$$

$$(nec)\ \frac{A}{\Box A}$$

Corresponding axioms for $(\Box WL)$, $(\Box CL)$ *and* $(\Box S)$ *are respectively:*

$$(ID_\Box)\ \Box A \leftrightarrow (\Box A \odot \Box A) \qquad (W_\Box)\ A \to (\Box B \to A) \qquad (S_\Box)\ \Box A \vee \neg \Box A$$

Theorem 1. $\Rightarrow A$ *is derivable in* **GLS4** *plus arbitary structural rules for* $\Box$ *iff* A *is derivable in* **LS4** *plus the corresponding axioms.*

The crucial result for the proof theory of fuzzy logics with modalities is as follows:

Theorem 2. *Cut-elimination holds for* **GLS4** *plus arbitary structural rules for* $\Box$.

This theorem is proved in a general way by imposing *conditions* (similar to those given in [6]) on the rules of the calculi (met by all those given above), and then giving a Schütte-Tait style cut-elimination proof using the invertibility of at least one of the premises in each application of (CUT). Note that as important consequences of this result, we obtain the subformula property for all our logics, and also that the extensions of fuzzy logics with modalities are conservative.

3 Semantics

We begin by introducing an algebraic semantics for the logics defined in the previous section, the idea being to consider particular classes of residuated lattices where the modal operator is interpreted by an *interior operator* I.

Definition 4. *A* lattice ordered monoid with interior operator *(*I-l-monoid *for short) is a system* $\mathcal{A} = \langle A, \odot, \vee, \wedge, I, t\rangle$ *such that* $\langle A, \odot, t\rangle$ *is a commutative monoid,* $\langle A, \vee, \wedge\rangle$ *is a lattice, the operation* $\odot$ *commutes with existing suprema, i.e., if* $\sup(X) \in A$ *for* $X \subseteq A$*, then* $y \odot \sup(X) = \sup(y \odot X)$ *for all* $y \in A$*, and* I *is a unary operation which satisfies the following conditions:*

$$(1)\ I(x) \le x \qquad (4)\ I(x \wedge y) = I(x) \wedge I(y)$$
$$(2)\ I(t) = t \qquad (5)\ I(x \vee y) = I(I(x) \vee I(y))$$
$$(3)\ I(I(x)) = I(x) \qquad (6)\ I(I(x) \odot I(y)) = I(x) \odot I(y)$$

A residuated lattice with interior operator *(*I-r-lattice *for short) is a system* $\mathcal{A} = \langle A, \odot, \to, \vee, \wedge, I, t\rangle$ *such that* $\langle A, \odot, \vee, \wedge, I, t\rangle$ *is an I-l-monoid, and* $\to$ *is the* residuum *of* $\odot$*, i.e.* $z \le x \to y$ *iff* $x \odot z \le y$ *for all* $x,y,z \in A$.

An I-l-monoid or I-r-lattice is bounded *if it has a minimum element* $\bot$ *wrt the lattice order,* integral *if* t *is the top element, and* idempotent *if all its elements are idempotents. An I-r-lattice is* dualizing *if there exists an element* f *such that* $(x \to f) \to f = x$ *for all* $x \in A$*, and* prelinear *if for all* $x, y \in A$*:*

$$((x \to y) \wedge t) \vee ((y \to x) \wedge t) = t \quad \textit{and} \quad I(x \vee y) = I(x) \vee I(y)$$

- *A* ULS4 algebra *is a bounded prelinear I-r-lattice.*
- *An* IULS4 algebra *is a dualizing ULS4 algebra.*
- *An* MTLS4 algebra *is an integral ULS4 algebra.*
- *An* IMTLS4 algebra *is a dualizing MTLS4 algebra.*
- *An* IUMLS4 algebra *is an idempotent IULS4 algebra.*
- *A* GS4 algebra *is an idempotent MTLS4 algebra.*

Further possible conditions for the interior operator are, for all $x \in A$*:*

$$(id_I)\ I(x) = I(x) \odot I(x) \qquad (w_I)\ I(x) \leq t \qquad (s_I)\ t \leq I(x) \vee (I(x) \to f)$$

Note that $I(\bot) = \bot$ in every ULS4 algebra, and that $I(x \to y) \leq I(x) \to I(y)$ is valid in every I-r-lattice. It is straightforward to show that the logics of the previous section are sound and complete with respect to the algebras defined above. Moreover, *prelinear* I-r-lattices are characterized in particular by the following property:

Theorem 3. *Every prelinear I-r-lattice is isomorphic to a subdirect product of a family of linearly ordered I-r-lattices.*

Clearly this implies that the logics are complete with respect to the appropriate class of linearly ordered I-r-lattices. In fact we would like to prove something stronger, i.e. that the logics are complete with respect to algebras based on the *real numbers*.

Definition 5. *A* standard prelinear I-r-lattice *is a prelinear I-r-lattice with a lattice reduct that is a convex subset S of* $\mathbb{R}$ *(the reals). A* standard ULS4 algebra *is a ULS4 algebra with a lattice reduct that is the real interval* $[0,1]$*. A prelinear I-r-lattice is* superstandard *if it is standard and I is left-continuous, that is, if for all* $X \subseteq S$ *with an upper bound in S,* $I(\sup(X)) = \sup(I(X))$.

Note that the monoid operation in a standard ULS4 algebra must be a *left-continuous uninorm*, i.e., a left-continuous weakly increasing commutative and associative binary operation with unit $t \in [0,1]$. Similarly, the monoid operation in a standard MTLS4 algebra must be a *left-continuous t-norm*.

Our goal here is to prove standard and superstandard completeness for fuzzy logics with modalities, i.e. we want to prove that a logic is complete with respect to the appropriate class of standard or superstandard algebras. To this end, we investigate an alternative presentation of I-r-lattices:

Lemma 1. *(i) Let* $\mathcal{A}$ *be an I-l-monoid or I-r-lattice, and let* $O = \{x \in \mathcal{A} : x = I(x)\}$*. Then O is the domain of a submonoid O of* $\mathcal{A}$ *closed under the lattice operations such that for all* $a \in \mathcal{A}$*, the set* $O_a = \{o \in O : o \leq a\}$ *has supremum* $I(a) \in O$*.*

(ii) Let $\mathcal{A}$ *be a lattice ordered commutative monoid or commutative residuated lattice, and let O be a subset of the domain of* $\mathcal{A}$ *closed under the monoid and lattice operations, and such that for all* $a \in \mathcal{A}$*, the set* $O_a = \{o \in O : o \leq a\}$ *has a supremum which belongs to O. Then letting for all* $a \in \mathcal{A}$*,* $I(a) = \sup(O_a)$*, the operator I makes* $\mathcal{A}$ *a I-l-monoid (aI-r-lattice respectively). Moreover,* $I(x) = x$ *iff* $x \in O$*.*

Hence I-r-lattices can be presented as residuated lattices with a privileged set O, called an *open system*, that satisfy the conditions of Lemma 1. The use of open systems allows us to prove a completion result which extends the well known completion results for residuated lattices:

Theorem 4. *Let $\mathcal{A}$ be a linearly ordered I-l-monoid. Then $\mathcal{A}$ embeds into a complete linearly ordered I-r-lattice $\hat{\mathcal{A}}$ by an embedding Φ which preserves the suprema and the residuals existing in $\mathcal{A}$. Moreover, the construction preserves integrality and boundedness, that is, if $\mathcal{A}$ is integral (bounded), then so is $\hat{\mathcal{A}}$.*

The use of open systems offers us a nice characterization of linearly ordered I-r-lattices with a left-continuous interior operation.

Lemma 2. *The following are equivalent for linearly and densely ordered I-r-lattices.*

(i) The operator I is left-continuous.
(ii) The open system O is densely ordered.

We now consider standard completeness for two particular fuzzy logics with modalities. First, observe that the superstandard completeness of **MTLS4** is an easy consequence of the following theorem.

Theorem 5. *Every finite or countable linearly ordered MTLS4 algebra embeds into a superstandard MTLS4 algebra.*

The proof is similar to that given in the standard completeness proof for **MTL** by Jenei and Montagna [12]. However, some care is needed in order to ensure that the embedding preserves the interior operator I and that I can be forced to be left-continuous.

Theorem 6. **MTLS4** *is complete with respect to the class of superstandard MTLS4 algebras.*

Our second example is the extension of **MTLS4** with contraction for modal formulae, condition (id_I), for which we have the following result.

Theorem 7. **MTLS4** *plus $(\Box ID)$ is complete with respect to the class of all superstandard MTLS4 algebras whose open system entirely consists of idempotents (or alternatively, satisfying (id_I)).*

4 Open questions and work in progress

The following problems are currently the subject of active research:

1. We intend to continue our investigations into the standard and superstandard completeness of the fuzzy logics with modalities defined above. Consider for example the logic **ULS4** plus $(\Box ID)$, $(\Box W)$, and the following density rule, for $p \notin \Gamma \cup \{A \to B, C\}$:

$$\frac{\Box\Gamma \vdash (A \to p) \vee (p \to B) \vee C}{\Box\Gamma \vdash (A \to B) \vee C}$$

   We intend to prove that this logic is sound and complete with respect to the class of standard ULS4 algebras with an open system consisting of idempotent elements below the unit of the monoid.
2. We intend to investigate a new method for constructing left-continuous t-norms. Let $\star$ be a continuous t-norm, and let O be a densely ordered subset of $[0,1]$ closed under taking suprema. The operation $\circ$ defined by $x \circ y = I(x) \star I(y)$ for an interior operator I, is a left-continuous, but in general, not continuous t-norm. It is interesting to investigate left-continuous t-norms constructed using this method.

3. We would like to prove the finite model property for **MTLS4** and possibly other fuzzy logics with modalities, along the lines of the recent work by Blok and Van Alten in e.g. [5].
4. We intend to develop *Kripke-style semantics* for fuzzy logics with modalities. As a starting point we may define for every I-l-monoid $\mathcal{M}$ with open system O and evaluation e in $\mathcal{M}$, the following Kripke model:
   $x \models p$ iff $x \leq e(p)$ for each atom p;
   $x \models A \odot B$ iff there are u, v such that $u \models A$, $v \models B$ and $x \leq u \odot v$.
   $x \models A \to B$ if for all y if $y \models A$, then $x \odot y \models B$.
   $x \models \Box A$ iff there is $z \in O$ such that $x \leq z$ and $z \models A$.
   We plan to investigate this semantics, focussing on the linearly ordered case.
5. Given an I-r-lattice $\mathcal{L}$, under suitable conditions its open system can be equipped with the structure of an I-r-lattice, which we call $\mathcal{L}^I$. We plan to investigate the relationship between $\mathcal{L}$ and $\mathcal{L}^I$. For example, when $\mathcal{L}$ ranges over a variety **V**, what is the variety generated by all $\mathcal{L}^I$?
6. So far we have considered only fuzzy logics that can be presented as hypersequent calculi with standard logical rules. We would like to investigate the addition of modalities to other fuzzy logics, in particular Łukasiewicz logic **Ł**, and Product logic ■, given hypersequent calculi with non standard logical rules in [15] and [14] respectively.

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On the Special Role of the Hamacher Product in Fuzzy Logics

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1 Introduction

In [2], Hájek studied several types of fuzzy logics and inspired an intensive development of this field. He started from the *basic logic*. In its standard model (with truth values from the real interval $[0,1]$), the conjunction & is interpreted by a continuous t-norm and the implication $\to$ by the corresponding residuum. The negation is introduced as a derived connective, $\neg\varphi = \varphi \to \mathbf{0}$, where $\mathbf{0}$ denotes the false statement. Further, he studied three extensions of the basic logic. The *Łukasiewicz logic* is based on the Łukasiewicz operations; its negation is involutive, $\neg\neg\varphi = \varphi$. This property is not possessed in further two logics, the *Gödel logic* and the *product logic*, where the conjunction in the standard model is interpreted by the minimum, resp. the algebraic product. The derived negation then corresponds to a fuzzy negation which is not involutive, therefore there is no disjunction playing a role dual to that of the conjunction. Inspired by this, *fuzzy logics with involutive negations* were defined and investigated in [1]. In this approach, an involutive negation $\sim$ is introduced as an additional unary connective, in general different from $\neg$.

One important difference seems not to have been paid sufficient attention: In the standard model of the product logic, the conjunction is interpreted by any strict t-norm, but without loss of generality we may restrict attention to the product t-norm only; all other standard models are isomorphic to this one. Nevertheless, when we introduce the involutive negation, we obtain (infinitely) many non-isomorphic standard models based on different strict t-norms.

In this contribution we study certain equalities in the strict basic involutive logic $SBL_\sim$ with fixed standard involutive negation $\sim$ interpreted by $\sim(x) = 1-x$, and we show that the standard interpretation of the conjunction is the Hamacher product $T_0^{\mathbf{H}}$, which is defined by

$$T_0^{\mathbf{H}}(x,y) = \frac{xy}{x+y-xy}$$

for all $(x,y) \in \,]0,1]^2$.

2 Basic logic and its extensions

For all details concerning the basic logic (BL) we refer to [2], and for the strict basic involutive logic ($SBL_{\sim}$) to [1]. Throughout the rest of this paper, $\sim$ is fixed to be the standard involutive negation, and the conjunction & is interpreted by a strict t-norm T, its dual t-conorm is denoted by S (for more details on t-norms and t-conorms see [3]).

In this paper we shall consider, for a given t-norm T, the one-place function $T^n\colon [0,1] \to [0,1]$ defined by $T^0(x) = 1$ and $T^n(x) = T(T^{n-1}(x), x)$ for $n \in \mathbb{N}$ (for a t-conorm S the function S^n is defined in complete analogy). An additive generator $t\colon [0,1] \to [0,\infty]$ of a strict t-norm T satisfying $t(0.5) = 1$ will be called a standardized additive generator of T. Then for the corresponding additive generator $s\colon [0,1] \to [0,\infty]$ of the dual strict t-conorm S we also get $s(0.5) = 1$.

We shall study the following conditions for a t-norm T, a t-conorm S, and a fixed number $n \in \mathbb{N}$, $n > 1$.

$$S^n \circ T^n = \mathrm{id} \qquad (\mathrm{C}_n)$$

$$\forall (x_1,\ldots,x_n) \in [0,1]^n : S(T^n(x_1),\ldots,T^n(x_n)) = T^n(S(x_1,\ldots,x_n)) \qquad (\mathrm{D}_n)$$

$$S^n \circ T^n = T^n \circ S^n \qquad (\mathrm{E}_n)$$

Equivalent formulations:

$$\forall (x,y) \in [0,1]^2 : (T(x,x) = y \iff S(y,y) = x) \qquad (\mathrm{C}'_2)$$

$$\forall (x,y) \in [0,1]^2 : (T^n(x) = y \iff S^n(y) = x) \qquad (\mathrm{C}'_n)$$

$$\forall (x,y,z) \in [0,1]^3 : (z = S(x,y) \iff T(z,z) = S(T(x,x),T(y,y))) \qquad (\mathrm{D}'_2)$$

$$\forall (x,y,z) \in [0,1]^3 : \big(z = S(x,y) \iff T^2(z) = S(T^2(x),T^2(y))\big) \qquad (\mathrm{D}''_2)$$

$$\forall (x_1,\ldots,x_n) \in [0,1]^n \ \forall z \in [0,1] :$$
$$(z = S(x_1,\ldots,x_n) \iff T^n(z) = S(T^n(x_1),\ldots,T^n(x_n))) \qquad (\mathrm{D}'_n)$$

Note that that the equivalence of (C_n) and (C'_n) as well as the equivalence of (D_n) and (D'_n) holds in our case, i.e., for a strict t-norm T and its dual S, but not in general for arbitrary t-norms and t-conorms.

3 Characterization of special strict t-norms

In this section we will characterize the t-norms satisfying (some of) the equalities mentioned above.

Theorem 1. *Let T be a strict t-norm and S its dual t-conorm and let t,s be their standardized additive generators. Let $n \in \mathbb{N}$, $n > 1$. Then the following are equivalent:*

- (C_n),
- (D_n),
- *the function $g = s \circ t^{-1}\colon [0,\infty] \to [0,\infty]$ satisfies $g(y) = n\,g(ny)$ for all $y \in [0,\infty]$.*

Although we are mainly interested in strict t-norms we include the following result which is valid for continuous t-norms.

Theorem 2. *Let (T,S) be a pair of a continuous t-norm and its dual t-conorm, respectively. We express T as a unique ordinal sum $(\langle a_\alpha, b_\alpha, T_\alpha\rangle)_{\alpha\in A}$ of continuous Archimedean t-norms. For each $\alpha \in A$, let $t_\alpha\colon [a_\alpha, b_\alpha] \to [0,\infty]$ be the standardized additive generator of T on $[a_\alpha, b_\alpha]$. Then T,S satisfy* (C_n) *for all $n \in \mathbb{N}$ if and only if*

$$\forall \alpha \in A\ \forall x \in\,]a_\alpha, b_\alpha[\, : t_\alpha(x) = \frac{1}{t_{\alpha^*}(1-x)}, \tag{1}$$

where $\alpha^* \in A$ such that $[a_\alpha, b_\alpha] = [1-b_{\alpha^*}, 1-a_{\alpha^*}]$.

Corollary 1. *Let (T,S) be a pair of a strict t-norm and its dual t-conorm, respectively. Then T,S satisfy* (C_n) *for all $n \in \mathbb{N}$ if and only if*

$$t(x) = \frac{1}{t(1-x)}, \tag{2}$$

where t is the standardized additive generator of T.

Definition 1. *A t-norm T is called a* nearly Hamacher t-norm *if it is isomorphic to the Hamacher product $T_0^{\mathbf{H}}$ such that the same isomorphism provides an isomorphism between the dual t-conorm S of T and the dual $S_0^{\mathbf{H}}$ of the Hamacher product $T_0^{\mathbf{H}}$.*

Lemma 1. *A continuous Archimedean t-norm T is nearly Hamacher if and only if its standardized additive generator $t\colon [0,1] \to [0,\infty]$ satisfies for each $x \in [0,1]$*

$$t(x) = \frac{1}{t(1-x)}.$$

Corollary 2. *Let T be a continuous Archimedean t-norm T and S its dual t-conorm. Then the following are equivalent:*

- *T and S satisfy* (C_n) *for all $n \in \mathbb{N}$;*
- *T and S satisfy* (D_n) *for all $n \in \mathbb{N}$;*
- *T is a nearly Hamacher t-norm.*

Observe that (C_n) as well as (D_n) implies (E_n).

Open Problem. Is the validity of (E_n) for all $n \in \mathbb{N}$ sufficient to characterize all nearly Hamacher t-norms?

4 Conclusion

As can be seen from Theorem 2, the equalities (C_n) and (D_n) can be studied within a broader framework of continuous t-norms. These equalities (without the assumption that T and S be dual) define varieties of $SBL_\sim$, among them we have infinitely many non-isomorphic cases. The results of a deeper investigation of these varieties from a logical point of view will be the topic of a forthcoming paper.

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Local MV-algebras and Their Representations

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Abstract. In this work we describe some properties of local MV-algebras (see [2], [7]) and we characterize a family of local algebras that generalize Komori algebras, by embedding them into algebras of *quasi-constant* functions.

Let $A = (A, \oplus, *, 0)$ be an MV-algebra. For any $a \in A$, the *order* of A, in symbols $ord(a)$, is the smallest natural number n such that $na = \underbrace{a \oplus \cdots \oplus a}_{n \text{ times}} = 1$. If no such n exists, then $ord(a) = \infty$ (see [2]).
An MV-algebra A is *local* if it satisfies one of the following equivalent conditions:

$i)$ for any $a \in A$, either $ord(a) < \infty$ or $ord(a^*) < \infty$,
$ii)$ the set $\{a \in A : ord(a) = \infty\}$ is a proper ideal of A
$iii)$ A has one and only one maximal ideal.

An MV-algebra A is *perfect* if for any $a \in A$, $ord(a) = \infty$ iff $ord(a^*) < \infty$. Clearly, every perfect algebra is local. The most important example of perfect MV-algebra, is Chang's algebra [1] $C = \{nc : n \in \omega\} \cup \{1 - nc : n \in \omega\}$ where $c^* = 1 - c$ and $ord(c) = \infty$ and $ord(1-c) < \infty$.
For any MV-algebra A, the *radical* of A (denoted by $Rad(A)$) is the intersection of all maximal ideals of A. Note that Chang algebra C is such that $Rad(C) = \{nc : n \in \omega\}$.
An equivalent definition of *perfect* MV-algebra is the following: a non trivial MV-algebra A is *perfect* iff $A = Rad(A) \cup Rad(A)^*$, where $Rad(A)^* = \{x \in A : x^* \in Rad(A)\}$.
Let A be a proper MV-chain. If for some $n \geq 2$,

$$A/Rad(A) \cong \text{Ł}_n = \{0, 1/n, \ldots, (n-1)/n, 1\},$$

then we say that A is of rank n. If $A/Rad(A)$ is isomorphic to an infinite subalgebra of $[0,1]$, we say that A is of infinite rank. Totally ordered perfect algebras are then algebras of rank 1.

It is well known that the class $Loc(MV)$ of local MV-algebras intersects the variety $V(C)$ (that is the variety generated by Chang's algebra) just in the class of perfect MV-algebras, i.e.,:

$$Loc(MV) \cap V(C) = Perfect.$$

Let $\mathcal{MV}$ be the variety of MV-algebras. Note that proper subvarieties of $\mathcal{MV}$ generated by simple MV-chains (i.e, subalgebras of $[0,1]$) intersect $Loc(MV)$ just in their generators.

Then it is natural to ask where $Loc(MV)$ meets any other variety. For simplicity we limit ourselves to the cases when a given variety is generated by a single non simple chain of finite rank. Let us denote by $V(S_2^\omega)$ the variety generated by the Komori chain of rank 2. Then we set:

$$Loc(MV) \cap V(S_2^\omega) = Loc(V(S_2^\omega)).$$

By abuse of notations and terminology, for any set X and any MV-algebra B we say that the subalgebra of B^X of all functions f such that $f(X) \subseteq [s]_{Rad(B)}$ for some $s \in B$ is a full algebra of quasi constant B-functions and shall be denoted by $\mathbf{K}(B^X)$. Any subalgebra of $\mathbf{K}(B^X)$ shall be called an algebra of quasi constant B-functions. Then we have:

Proposition 1. *For any MV-algebra A, $A \in Loc(V(S_2^{\omega}))$ iff A is local and $A/Rad(A) \simeq Ł_2$ iff A is isomorphic to an algebra of quasi constant B-functions, where B is the greatest MV-algebra of rank* 2 *contained in an ultrapower of* $[0,1]$.

Note that the existence of the greatest MV-algebra of rank 2 in $[0,1]^*$ is ensured by results in [3].
More in general we can say that as perfect MV-algebras are not totally ordered generalization of Chang algebra C we can get non totally ordered generalizations of Komori algebras by considering the following class of local algebras:

$$Loc(MV) \cap V(S_n^{\omega}) = Loc(V(S_n^{\omega})) \qquad \forall n \in \mathbf{N}.$$

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On Symmetric MV-polynomials

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1 Introduction

An MV-algebra is an algebraic structure $A = (A, \oplus, ^*, 0)$ of type (2,1,0) satisfying the following axioms:

(1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(2) $x \oplus y = y \oplus x$;
(3) $x \oplus 0 = x$;
(4) $(x^*)^* = x$;
(5) $x \oplus 0^* = 0^*$;
(6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Therefore, if we define the costant 1 by $1 = 0^*$ and the operation $\odot$ by $x \odot y = (x^* \oplus y^*)^*$, then from (4), we obtain $1^* = 0$. Moreover, setting $y = 1$ in (6), it follows $x^* \oplus x = 1$. On A two new operations $\vee$ and $\wedge$ are defined as follows: $x \vee y = (x^* \oplus y)^* \oplus y$ and $x \wedge y = (x^* \odot y)^* \odot y$. The structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. We shall write $x \leq y$ iff $x \wedge y = x$. A remarkable example is the MV-algebra having, as support, the real interval $[0,1]$ and, as basic MV-algebraic operations on $[0,1]$,

$$x \oplus y = \min\,(1, x+y);$$
$$x^* = 1 - x.$$

We refer to this MV-algebra by $[0,1]$. For each positive integer n, let L_n be the set $\{0, \frac{1}{n}, \ldots, 1\}$ endowed with the following operations :

$x \oplus y = \min(x+y, 1)$,
$x^* = 1 - x$.

For each positive integer n, the algebra L_n is a finite totally ordered MV-algebra (MV-chain) and every non-trivial finite MV-chain is isomorphic to one of them.

Let A be an MV-algebra, $x \in A$ and n a nonegative integer. In the sequel we will denote by nx the element of A, inductively defined by $0x = 0$, $nx = (n-1)x \oplus x$. Analogously we will set $x^0 = 1$, $x^n = x \odot x^{n-1}$. Moreover we consider the $*$ operation more binding than any other operation, and $\odot$ operation more binding than $\oplus$.

Let $\mathcal{L}$ be the poset, under $\subseteq$, of subalgebras of the MV-algebra $[0,\ 1]$. $\mathcal{L}$ then has a unique minimal element, $\{0,\ 1\}$, and a unique maximal element, $[0,\ 1]$.

$\mathcal{L}$ also contains atoms, that is subalgebras $A \subseteq [0,\ 1]$ such that if $A^{'} \subseteq A$, then $A^{'} = \{0,\ 1\}$ or $A^{'} = A$. The algebra $\{0,\ 1/2,\ 1\}$ is such an atom.

Since, for a maximal ideal M of an MV-algebra A, $A/M \in \mathcal{L}$, we have a method to refine the structure of the maximal ideal space *MaxA*. Heuristically, *smaller* is the quotient A/M, *larger* is the maximal ideal M. In effect this provides a pre-order on the set of maximal ideals.

In this work we shall study these ideas for the set of maximal ideals of *finite type*, that is maximal ideals M with A/M finite. We shall first look at *super maximal* ideals M, that is, those maximal ideals M such that A/M is a *small* as possible, namely $A/M = \{0,\ 1\}$. Next we shall look at some classes of *big maximal* ideals M, that is, those maximal ideals M which if not super maximal are such that A/M is an atom of $\mathcal{L}$.

Our study will use a class of MV-polynomials, that we will call *symmetric* which shall permit us to construct the appropriate MV-algebras.

The first part of this work concerns supermaximal ideals. In a boolean algebra all maximal ideals are supermaximal (considering the boolean algebra as an MV-algebra).

Given an MV-algebra, its set of idempotents, $B(A)$, is a subalgebra which is a boolean algebra. We shall examine extensions of $B(A)$ in A, that is subalgebras A' of A such that $B(A) \subseteq A' \subseteq A$, that have supermaximal ideals. We shall also study properties of the set of supermaximal ideals.

The second part of this work will take up the case of certain extensions of $B(A)$ which may have big maximal ideals, and we shall study some properties of these algebras.

Both of these parts will be presented as a special case of subalgebras determined by certain symmetric MV-polynomials.

Definition 1 *Let A be an MV-algebra. An $M \in MaxA$ is called of* type n, *provided that $A/M \cong L_n$.*

Definition 2 *Let A be an MV-algebra. $M \in MaxA$ is called of* finite type, *if M is of type n for some integer n.*

Definition 3 *Given an MV-algebra A, an $M \in MaxA$ is called* supermaximal, *in symbols SMax provided $A/M \cong \{0,\ 1\}$.*

Definition 4 *Given an MV-algebra A, we shall call $M \in MaxA$,* big-maximal *iff $A/M \cong S$ where S has not nontrivial MV-algebras.*

We shall focus on big-maximal ideals of finite type.

Not every MV-algebra A has supermaximal ideals, for example $[0,\ 1]$, or less trivially, $[0,\ 1]^X$. We shall construct algebras which do have super maximal ideals, and some algebras where all maximal ideals are super maximal.

We shall look at the topological aspect of the subspace $SMaxA \subset MaxA$ of super maximal ideals.

We shall also look at some cases where A contains big maximals that are not super maximal.

2 Symmetric Polynomials

By an MV-polynomial (in one variable) we mean a polynomial $p(z)$ built from a symbol z and the symbols $\oplus, \odot, ^*, \vee,\ \wedge,\ 0,\ 1$. Given such a polynomial $p(z)$, we have an evident map on any given MV-algebra A, $p(z) : A \to A$, by evaluation, $p(a),\ a \in A$.

We shall call $p(z)$ *symmetric* if $p(z) = p(z^*)$. We shall call $p(z)$ *ideal-uniform* if for any MV-algebra A and every ideal $I \subseteq A$, we have $p(0) \in I$ and if $p(a),\ p(b) \in I$, then $p(a \oplus b) \in I$.

We immediately have:

Proposition 5 *If $p(z)$ is symmetric and ideal-uniform, then for any MV-algebra A and any ideal $I \subseteq A$ we have that $Sym(p,\ I) = \{a \in A \mid p(a) \in I\ \}$ is a subalgebra of A.*

Observe that, since every ideal in an MV-algebra is semi-prime, it suffices to check the ideal-uniform condition only on the prime ideals.

Proposition 6 *Let $p(z)$ be symmetric and ideal-uniform. Suppose if A is a linearly ordered MV-algebra and $p(z) = 0$ on A, then $A \cong L_n$, for some positive integer n. Then for any MV-algebra A and any ideal $I \subseteq A$, the subalgebra $Sym(p,I)$ satisfies the following:*

i) If Q is a prime ideal of $Sym(p,I)$ and $I \subseteq Q$, then $Sym(p,I)/Q \cong L_n$ or $Sym(p,I)/Q \cong \{0,\ 1\}$.
ii) If $I \subseteq J$, then $Sym(p,I) \subseteq S(p,J)$.

We will apply this proposition to several different symmetric MV-polynomials.

$Sym(p,\ I)$ (or just $Sym(I)$ if p is understood) will be called the *p-symmetric subalgebra over I*. A will be called a *p-symmetric algebra* if $A = Sym(p,\ I)$ for some $I \subseteq A$ where $p(z) \not\equiv 0,\ I \neq A$.

A simple example of a non-trivial (that is, non-constant) symmetric polynomial is $p_1(z) = z \wedge \overline{z}$. With above notations we get:

Lemma 7 *p_1 is symmetric and ideal-uniform.*

We shall examine some consequences of the above lemma.

Proposition 8 *Let A be an MV-algebra. $B(A)$ is a p_1-symmetric subalgebra of A.*

Proposition 9 *Let A be an MV-algebra, $I \subseteq A$ be an ideal and $Sym(I) = Sym(p_1,\ I)$. Then,*

i) I is an ideal in $Sym(I)$.
ii) $B(A) \subseteq Sym(I)$.
iii) $Sym(I)/I$ is a Boolean algebra.
iv) $B(A/I) \cong Sym(I)/I$.
v) $Sym(I)$ is the largest subalgebra R of A for which R/I is a Boolean algebra.
vi) If A is α-complete and I is an α-complete ideal, then $Sym(I)$ is α-complete.
vii) If J is an ideal and $I \subseteq J$, then $Sym(I) \subseteq Sym(J)$.

We can view, therefore, $Sym(I)$ as a generalization of $B(A)$. We shall compare some properties of $Sym(I)$ and $B(A)$.

Every ideal in a boolean algebra, if maximal, is supermaximal. On the other hand, $A = [0,\ 1]^X,\ X \neq \emptyset$ has no supermaximal ideals since the constant function $f(x) = 1/2$ satisfies $f \wedge \overline{f} = f$. As f has finite order it belongs to no ideal.

From Proposition 9 we have $B(A) \subseteq Sym(I) \subseteq A$ for any ideal I of A. Since every ideal I in an MV-algebra is contained in some prime ideal, it is evident that $Sym(I)$ always contains supermaximal ideals.

Let us now focus on an MV-algebra A that is p_1-symmetric over I for some ideal $I \subseteq A,\ I \neq A$.

Set $M^* = \{x \in A : x^* \in M\}$. Then we get:

Proposition 10 *Let $M \in$Max(A). The following are equivalent:*

i) $M \in$SMax(A).

ii) for all $x \in A,\ x \in M$ *or* $x^* \in M$.
iii) for all $x,\ y \in A,\ x \odot y \in M$ *implies* $x \in M$ *or* $y \in M$.
iv) $A = M \cup M^*$.

Proposition 11 *$SMax(A)$ is a closed subspace of $Spec(A)$.*

Proposition 12 *$SMax(A)$ is a closed boolean subspace of $Spec(A)$.*

Call an MV-algebra *A Boolean-mixed* , if A is not Boolean and $A = A' \times B$ where B is a Boolean algebra and A' is non-boolean MV-algebra. We shall prove that every p_1-symmetric algebra is a subdirect subalgebra of a Boolean-mixed algebra.

Denote by:

N_A the ideal of A, generated by the elements $\{x \wedge x^* : x \in A\}$ and
$N_A^{\perp}$, the set $\{x \in A : x \wedge a = 0,\ for\ every\ a \in N_A\}$.

With above notations we get:

Proposition 13 *Let A be an MV-algebra. Then following are equivalent:*

i) A is p_1-symmetric.
ii) A has a supermaximal ideal.
iii) A has a boolean homomorphic image.
iv) $N_A \neq A$.
*v) $A \in$ **BP**.*

Suppose then that A is a subdirect subalgebra of $A' \times B$ where A is non boolean and B is boolean MV-algebra respectively. Then A has B as a homomorphic image and by the above proposition, A is p_1-symmetric.

Suppose now that A is p_1-symmetric. Consider the ideal $(N_A)^{\perp}$. We note that $(N_A)^{\perp} \subseteq$ B(A). For if $x \in (N_A)^{\perp}$, then $x \wedge (x \wedge x^*) = 0$ and so $x \wedge x^* = 0$. Therefore $x \in B(A)$. Since $N_A \cap (N_A)^{\perp} = 0$, we have the condition for a subdirect representation of A with A/N_A and $A/(N_A)^{\perp}$. We consider two different representations.

Consider first the map $A \to A \times A/N_A$ given by $x \to (x,\ x/N_A)$. This map is an injective morphism, thus if A is non-Boolean we have A as a subdirect subalgebra of a Boolean-mixed algebra. Moreover we have $A \to A \times A/N_A \to A$ given by $x \to (x,\ x/N_A) \to x$; therefore A is a retract of a Boolean-mixed MV-algebra. We have,

Proposition 14 *A is a non-boolean p_1-symmetric MV-algebra iff A is a subdirect algebra of a boolean-mixed algebra.*

Theorem 15 *Suppose A is a retract of a boolean-mixed algebra. Then A is p_1-symmetric.*

3 Other Symmetric Functions

Here we want to consider other symmetric functions and the type of subalgebras of $[0,\ 1]$ they determine. These functions generalize $p_1(z)$ and are defined as follows:

Set:

$$q_0(z) = z \wedge z^*$$
$$p_2(z) = (z^2 \vee (z^*)^2) \wedge q_0(z)$$

and

$$p_3(z) = (z^3 \vee 2z^* \odot z^*) \wedge ((z^*)^3 \vee (2z \odot z) \wedge q_0(z).$$

Moreover, for $n = 5$ or $n = 7$,

$$p_n(z) = q_0(z) \wedge \bigwedge_{k=1}^{\frac{n-1}{2}} [z \odot (\frac{n-1}{k} z)^k \vee z^* \odot (k(z^*)^{\frac{n-1}{k}})] \wedge [z^* \odot (\frac{n-1}{k} z^*)^k \vee z \odot (kz^{\frac{n-1}{k}})].$$

With above notations we get:

Theorem 16 *Let A be an MV-algebra, I be an ideal of A and $n = 2,3,5,7$. Then we have:*

1. *$p_n(z)$ is symmetric and ideal-uniform.*
2. *$Sym(p_n, I)$ is a subalgebra of A.*
3. *I is an ideal of $Sym(p_n, I)$.*
4. *$B(A)$ is a subalgebra of $Sym(p_n, I)$*
5. *If Q is a prime ideal in $Sym(p_n, I)$ and $I \subseteq Q$, then Q is supermaximal or big ideal; thus $Sym(p_n, I)/Q \cong \{0,\ 1\}$ or $Sym(p_n, I)/Q \cong \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$.*

4 General Case p_n, n prime number and $n \geqslant 11$

Let n be a prime number and $n \geqslant 11$. Set:

$$q_{n,1}(z) = [z^n \vee z^* \odot ((n-1)z^*)] \wedge [(z^*)^n \vee z \odot ((n-1)z)]$$

$$q_{n,2}(z) = [z \odot (\frac{n-1}{2} z)^2 \vee z^* \odot (2(z^*)^{\frac{n-1}{2}})] \wedge [z^* \odot (\frac{n-1}{2} z^*)^2 \vee z \odot (2z^{\frac{n-1}{2}})]$$
$$q_{n,\frac{n-1}{2}}(z) = [z \odot (2z)^{\frac{n-1}{2}} \vee z^* \odot (\frac{n-1}{2} (z^*)^2)] \wedge [z^* \odot (2z^*)^{\frac{n-1}{2}} \vee z \odot (\frac{n-1}{2} z^2)]$$

To define $q_{n,k}(z)$, for $3 \leqslant k < \frac{n-1}{2}$, we need some preliminar considerations. Dividing the prime n by k yields a quotient d_0 and a remainder r_0, in symbols

$$n = kd_0 + r_0, 0 < r_0 < k. \tag{1}$$

If $d_0 < r_0$, you have to apply a similar process to n and r_0, obtaining

$$n = r_0 d_1 + r_1, 0 < r_1 < r_0. \tag{2}$$

If $d_1 < r_1$, you have to repeat the division algorithm to n and r_1 and so forth.

$$\ldots n = r_\ell d_{\ell+1} + r_{\ell 1}, 0 < r_{\ell+1} < r_\ell. \tag{3}$$

Since the finite sequences of positive intergers $(d_0, d_1, \ldots, d_{\ell+1})$ and $(r_0, r_1, \ldots, r_{\ell+1})$ are strictly increasing and decreasing respectively, there is $min\{i \in N : r_i < d_i\}$. Denote such a minimum by i_k.

$$n = r_{i_k - 1} d_{i_k} + r_{i_k}, 0 < r_{i_k} < r_{i_k - 1} \text{ and } 0 < r_{i_k} < d_{i_k}$$

Fuzzy intervals versus fuzzy numbers: is there a missing concept in fuzzy set theory?

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A fuzzy set [12] is a generalization of subset (at least in the naive sense). It is defined by a membership function from a basic set to the unit interval (or a suitable lattice) and its cuts are sets. Note that originally, the word "fuzzy" specifically refers to the introduction of shades or grades in all-or-nothing concepts.

However what is often called a fuzzy number is understood as a generalized interval, even if mathematicians of fuzzy sets in the past have proposed a different view of a fuzzy real number, starting with Hutton [5]. Often, it takes the form of a decreasing mapping from the reals to the unit interval or a suitable lattice (Grantner et al. [3]), or a probability distribution function (Lowen[6]); variants of such a fuzzy reals were also studied by Rodabaugh[10] and Hoehle [4]. To avoid confusion, we call a fuzzy set of numbers whose cuts are intervals a *fuzzy interval* (regardless of whether their cores are reduced to a point or not). Note that fuzzy intervals account for both imprecision and fuzziness. In contrast, we here take it for granted that a *fuzzy number* should be a fuzzy object of some kind each cut of which should be a number. This issue was a topic of (unresolved) debates in early Linz Seminars between pure mathematicians and applied ones (see Proc. of the 1st Linz Seminar, pp. 139-140, 1979).

Similarly there is a misunderstanding about the notion of defuzzification in engineering papers, whereby a fuzzy set of numbers is changed into a number. Yet, defuzzifying means removing gradedness, so that defuzzifying a fuzzy set should yield a set, not a point. And indeed in the past the notion of mean interval of a fuzzy interval was proposed as a natural way of extracting an interval from a fuzzy interval (Dubois & Prade [2], where the phrase "fuzzy number" was used in the sense of a fuzzy interval). See also recent works by Roventa and Spircu [10]. So, the defuzzification process in the engineering area can be split into two steps: removing fuzziness (thus getting an interval), and removing imprecision (by selecting a number in the interval). Randomly repeating this method yields a probability distribution (often the Shapley value).

One way of approaching the intuition of a (genuine) fuzzy number is to swap these two steps: given a fuzzy set of numbers, first remove imprecision, get a fuzzy number, and then defuzzify it. A fuzzy number is then supposed to express fuzziness only, WITHOUT imprecision. Mathematically, it can be modelled by a function from the unit interval to the real line (and not the converse). Note that we do not require monotonicity of the function so that some fuzzy numbers cannot be interpreted as a membership function (a number may then sometimes have more than one membership degree…). Algebraic structures of numbers (like groups) should be preserved for the most part when moving from numbers to fuzzy numbers (while fuzzy intervals just preserve algebraic properties of intervals). This view enables a fuzzy interval to be defined as a pair of particular (monotonic) fuzzy numbers, just as an interval is modelled by an ordered pair of numbers. Performing fuzzy interval analysis in the style of interval calculations, the combination of two fuzzy boundaries of fuzzy intervals may fail to be monotonic [1] (hence the necessity not to restrict to monotonic fuzzy numbers).

The introduction of “genuine” fuzzy numbers (to a wider audience, if we consider that similar considerations were more or less already discussed among pure mathematicians of fuzzy sets) may help clarify the situation in other problems. For instance, the fuzzy cardinality of fuzzy sets [11] has been a topic of debate and many proposals appeared in the 1980’s. It is clear that this notion has been more often than not envisaged as a fuzzy set of integers (hence involving some imprecision). The above discussion suggests it should not be so. Integers are defined as cardinalities of (finite) sets. Recently, Rocacher and Bosc [8] suggested to define fuzzy integers as (precise, but gradual) cardinalities of fuzzy sets. A fuzzy integer is then a (monotonic) mapping from the unit interval to the natural integers. They then define fuzzy negative integers [8] and fuzzy rationals [7] as equivalence classes of pairs of fuzzy integers, as in the classical setting. Fuzzy negative integers are no longer monotonic, generally. This view is totally along the line discussed above. A similar treatment applies to notions like probabilities of fuzzy events or distances between fuzzy sets, and more generally, fuzzy extensions of scalar evaluations of sets, where sets are mapped to numbers.

This discussion leads to introduce the notion of fuzzy element of a (fuzzy) set, a concept that was apparently missing in the theory. Topologists tried to introduce ideas of fuzzy points in the past, but this notion has often been controversial, and sterile in its applications. The aim of this talk is not to produce a full-fledged mathematical development. It seeks to informally introduce a natural notion of fuzzy number and fuzzy element, to outline elementary formal notions related to this notion and discuss its potential at shedding light on some yet ill-understood aspects of fuzzy set theory and its applications.

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From t-Norms to t-Norm Based Logics

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The aim of the talk is to give a general overview of some results of t-norms and t-norms based logics and to remark some of their relationship. Special mention will be due to the relation between t-norms results and completeness results of t-norm based logics and to the translation of some results on t-norm based logic to t-norm setting. At the end we will sketch some recent results about the most general t-norm based logic, the Monoidal t-norm based logic MTL and its involutive axiomatic extension IMTL.

From the definition of t-norms by Menger till the results contained in the book [23] of Klement, Mesiar and Pap there is a very long history where fuzzy sets and fuzzy systems play a very important role. One of the first important results about t-norms is the decomposition theorem of continuous t-norms as ordinal sum of copies of the three basic ones, i.e., Lukasiewicz, Product and Minimum. This result was firstly obtained by Moster and Shield in the setting of semigroups or monoids and after by Ling in the setting of probabilistic metric spaces. But here we are interested in the fuzzy set setting.

After the definition of Fuzzy Sets by Zadeh in [31] the main work in what was called Fuzzy Logic was devoted in the seventies, to the definition of truth functions on the real unit interval corresponding to the usual connectives of propositional logic (and, or, negation and implication basically). These functions are also used to define the fuzzy set operations punctually and the inference in fuzzy rule based systems. In this setting t-norms plays a central place since they are the operation showsed to modelise conjunction operation and (with the exception of negation) to define other truth functions.

Before going into fuzzy logics in narrow sense, into the axiomatic multiple-valued systems corresponding to residuated logics which are standard complete with respect to interpretations over the real unit interval with truth functions given by a t-norm, its corresponding residuated implication and negation defined by “imply 0”, we want to survey some basic results obtained on the monoidal setting. Hölhe’s paper [19] is both a deep study of commutative monoids and its residuated implication and the first to give an axiomatic system related to them, the so-called monoidal logic. Even though this logic is complete with respect to resiuated lattices it is not properly a Fuzzy logic in the sense that it is not complete with respect to any algebra defined by a t-norm and its residuum over [0,1]. This logic is also found in Ono’s study of substructural logics (by the name Full Lambek with exchange and weakening, FLew. See [29]) and in Adillon-Verdú papers (By the name Intuitionistic logic without contraction, IPC/*. See [1]). This was the framework where t-norm based logic have been developped. An important result is that Monoidal logic and any of its axiomatic extensions are algebraizable in the sense of Block and Pigozzi and the equivalent semantic is the corresponding variety of algebras (residuated lattices for monoidal logic and the corresponding subvariety for its axiomatic extensions). This means that it is equivalent to study axiomatic extensions of Monoidal logic or to study the corresponding variety of algebras.

In the t-norm based logic setting properly defined, the known t-norm based logics before fuzzy sets have been defined were Lukasiewicz (infinite valued Lukasiewicz logic) and Gödel (which semantic is given by the minimum t-norm and its residuum). See, for example [4, 12] for a general overview of this logics. Hajek et alt. add to this two initial the study of Product logic in [17]. Afterwards, Hajek defined BL claiming that it is the logic of continuous t-norms. The method to prove this claim was to prove first that the BL-algebras are subdirect product of l.o. ones and second that the l.o. BL-algebras (BL-chains) are ordinal sums of the three basic ones like continuous t-norms (See [16, 5]). It is clear that the result about decomposition as ordinal sums of continuous t-norms have been the guide for the proof of standard completeness of BL. But in the logical setting we go further and Montagna et alt. proved in [2] that any BL-chain is an ordinal sum of Wajsberg hoops. Montagna also proved in [25] that there exist a t-norm generating the full variety of BL-algebras and Esteva, Godo and Montagna in [10] proved that the logic of any continuous t-norm is finitely axiomatizable and gave a method to find the axioms. Moreover in [25, 18] the authors proved that there are continuously many subvarieties of BL-algebras while there are only denumerable subvarieties generated by a continuous t-norm and its residuum. All these logical results give new results about t-norms and t-norms like over a chain and over a lattice. We will give some examples of them, some interesting results that goes from logic to t-norms place. Of course, there are many other works on particular t-norm-based logic corresponding to continuous t-norms like, for example, [24].

Moreover from the fact that a t-norm has a residuum if and only if it is left continuous, Esteva and Godo defined in [7] Monoidal t-norm based logic MTL claiming that it is the logic of left continuous t-norms and their residua (as BL was the logic of continuous t-norms). The claim was proved by Jenei and Montagna in [22] and generalized by Esteva et alt. in [8] to the involutive case (IMTL) and to weak contractive or pseudocomplemented case (SMTL). The method given by Jenei and Montagna to prove standard completeness is different from the one used for BL. It is clear that we can not use a similar way because of the lack of a structural theorems about left continuous t-norm (like decomposition theorem for continuous t-norm). This is a very important fact when we want to study subvarieties of MTL and IMTL. Some interesting results to study left continuous t-norms can be found in Jenei's papers [20, 21] where Jenei gave some methods to build some families of left continuous t-norms in the involutive case (IMTL-chains).

In the last part of the talk we will comment about the last results in the setting of MTL and IMTL logics and varieties. There are some varieties that are fully studied and axiomatized. The main ones are the nilpotent minimum ones NM (See [12]), the simple 4-contractive (or 4-potent), i.e. that satisfies the equation $x \vee \neg(x^3) = 1$ (See [14]) and the weak nilpotent minimum WNM (See [9]). Also some study was done in some particular t-norm-based logic (See for example [28] and [30]). This varieties are studied directly and the second one [14] is not t-norm based since there is no [0,1]-algebra belonging to this variety. In the talk we will give the results about WNM that are recently obtained and already not published. Finally we want to refer to the variety (and logic) studied following Jenei's method to built involutive left continuous t-norms. First we have proved in [26] that perfect IMTL algebras correspond to disconnected rotation of semihoops (generalizing the disconected rotation defined by Jenei in [0,1]). Second, in the same paper, we have proved that perfect IMTL algebras adding a fix point corresponds to connected rotation MTL without zero divisors and third we are working in the decomposable IMTL-chains (some genralization of ordinal sum of a perfect plus any IMTL) as the IMTL-chains obtained by the generalization of the rotation anihilation method of Jenei (See[21]). Moreover we have generalized in [27] the notion of perfect algebras to MTL algebras and have obtained a family of varieties that we will explain in the talk jointly with completeness results. This gave new methods to build left continuous t-norms. Finally some results about n-contractive (n-potent) IMTL algebras will be presented

The talk will finish with some ideas about future research in the study of subvarieties of MTL, SMTL and IMTL.

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Topological Categories of L-Sets and $(\mathbf{L},\mathbf{M})$-Topological Spaces Based on Structured Lattices

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In [3] the authors introduced the notion of *structured lattice* defined as a pair

$$(L,\Phi)$$

where L is a complete lattice and

$$\Phi = \{\phi_a | a \in L, a \neq \bot\}, \;\; \phi_a : L \longrightarrow [\bot, a]$$

is a family of $\bigwedge$-complete semilattice morphisms. Such a structure on L allows to build the *forward* and the *backward powerset operators* of any function between the underlying sets of any two L-sets, in such a way as to include in some sense both the situations considered in [1] and [5].

Thence, based on such a structured lattice, the notion of *ground category* as a concrete category of L-sets whose morphisms have "‘good"’ powerset operators has been given in [3].

Moreover, a large category including all possible ground categories on (L,Φ) has been considered. Such a category, denoted by **(L,Φ)-Set**, has all L-sets as objects, while morphisms are functions between the underlying sets of $Y \in L^X$ and $Z \in L^T$ that satisfy the condition

$$Y(x) \nearrow Z(f(x)), \; \forall x \in X$$

where $\nearrow$ is a pre-order relation on L induced by Φ.

Now we approach the problem of giving conditions on the structured lattice (L,Φ) that allow **(L,Φ)-Set** and the possible ground categories on (L,Φ) to be *topological* over **Set**. In case when those conditions are not satisfied we show how to get from (L,Φ), with weaker conditions, a new structured lattice $(\hat{L},\hat{\Phi})$ that gives topological ground categories.

Moreover we consider, under suitable conditions on (L,Φ), the *backward powerset functor* on every ground category **C** on (L,Φ)

$$\leftarrow_{(L,\Phi)} : \mathbf{C} \rightarrow \mathbf{CLat}^{\mathbf{op}}.$$

This functor can be used as a tool for the construction of categories of M-topological spaces on L-sets that are topological on their ground **C**, so extending results given in [4].

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An Abstract Approach Toward Evaluation of Fuzzy Rule Systems

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1 Introduction

There is the well known distinction between FATI and FITA strategies to evaluate systems of linguistic control rules w.r.t. arbitrary fuzzy inputs from $\mathcal{F}(\mathbf{X})$.

The core idea of a FITA strategy is that it is a strategy which **F**irst **I**nfers (by reference to the single rules) and **T**hen **A**ggregates starting from the actual input information A. Contrary to that, a FATI strategy is a strategy which **F**irst **A**ggregates (the information in all the rules into one fuzzy relation) and **T**hen **I**nfers starting from the actual input information A.

From the two standard interpolation strategies, the usual Mamdani/Assilian approach offers a FATI strategy, and the method of activation degrees provides a FITA strategy.

2 Some general evaluation strategies

Both these strategies use the set theoretic union as their aggregation operator. Furthermore, both of them refer to the compositional rule of inference (CRI) as their core tool of inference.

In general, however, the interpolation operators we intend to consider depend more generally upon some inference operator(s) as well as upon some aggregation operator.

By an *inference operator* we mean here simply a mapping from the fuzzy subsets of the input space to the fuzzy subsets of the output space.[1]

And an *aggregation operator* $\mathbf{A}$, as explained e.g. in [1, 2], is a family $(f^n)_{n\in\mathbb{N}}$ of operations, each f^n an n-ary one, over some partially ordered set $\mathbf{M}$ with a bottom element $\mathbf{0}$ and a top element $\mathbf{1}$, such that each operation f^n is non-decreasing, maps the bottom to the bottom: $f^n(\mathbf{0},\dots,\mathbf{0}) = \mathbf{0}$, and the top to the top: $f^n(\mathbf{1},\dots,\mathbf{1}) = \mathbf{1}$. Such an aggregation operator $\mathbf{A} = (f^n)_{n\in\mathbb{N}}$ is a *commutative* one iff each operation f^n is commutative. And $\mathbf{A}$ is an *associative* aggregation operator iff e.g. for $n = k+l$ one always has $f^n(a_1,\dots,a_n) = f^2(f^k(a_1,\dots,a_k), f^l(a_{k+1},\dots,a_n))$ and in general

$$f^n(a_1,\dots,a_n) = f^r(f^{k_1}(a_1,\dots,a_{k_1}),\dots,f^{k_r}(a_{m+1},\dots,a_n))$$

for $n = \sum_{i=1}^{r} k_i$ and $m = \sum_{i=1}^{r-1} k_i$.

Our aggregation operators further on are supposed to be commutative as well as associative ones.[2]

If we now consider interpolation operators Φ of FITA-type and interpolation operators Ψ of FATI-type then they have the abstract forms

$$\Psi_{\mathcal{D}}(A) = \mathbf{A}(\theta_1(A),\dots,\theta_n(A))\,, \tag{1}$$

$$\Xi_{\mathcal{D}}(A) = \widehat{\mathbf{A}}(\theta_1,\dots,\theta_n)(A)\,. \tag{2}$$

[1] This terminology has its historical roots in the fuzzy control community. There is no relationship at all with the logical notion of inference intended here.

[2] It seems that this is a rather restrictive choice from a theoretical point of view. However, in all the usual cases these restrictions are satisfied.

Here we assume that each one of the "local" inference operators θ_i is determined by the single input-output pair $\langle A_i, B_i \rangle$. Therefore we occasionally shall write $\theta_{\langle A_i, B_i \rangle}$ instead of θ_i only. And we have to assume that the aggregation operator $\mathbf{A}$ operates on fuzzy sets, and that the aggregation operator $\widehat{\mathbf{A}}$ operates on inference operators.

With this extended notation the formulas (1), (2) become

$$\Psi_{\mathcal{D}}(A) = \mathbf{A}(\theta_{\langle A_1,B_1 \rangle}(A), \ldots, \theta_{\langle A_n,B_n \rangle}(A)), \tag{3}$$

$$\Xi_{\mathcal{D}}(A) = \widehat{\mathbf{A}}(\theta_{\langle A_1,B_1 \rangle}, \ldots, \theta_{\langle A_n,B_n \rangle})(A). \tag{4}$$

Some particular cases of these interpolation procedures have been discussed in [6].

3 Stability conditions for the given data

If $\Theta_{\mathcal{D}}$ is a fuzzy inference operator of one of the types (3), (4), then the interpolation property one likes to have realized is that one has

$$\Theta_{\mathcal{D}}(A_i) = B_i \tag{5}$$

for all the data pairs $\langle A_i, B_i \rangle$. In the particular case that the operator $\Theta_{\mathcal{D}}$ is given via the CRI, this is just the problem to solve the system (5) of fuzzy relation equations.

Definition 1. *In the present generalized context let us call the property (5) the $\mathcal{D}$-*stability *of the fuzzy inference operator $\Theta_{\mathcal{D}}$.*

To find $\mathcal{D}$-stability conditions on this abstract level seems to be rather difficult in general. However, the restriction to fuzzy inference operators of FITA-type makes things easier.

It is necessary to have a closer look at the aggregation operator $\mathbf{A} = (f^n)_{n \in \mathbb{N}}$ involved in (1) which operates on $\mathcal{F}(\mathbf{Y})$, of course with inclusion as partial ordering.

Definition 2. *Having $B, C \in \mathcal{F}(\mathbf{Y})$ we say that C is* $\mathbf{A}$-negligible w.r.t. *B iff $f^2(B,C) = f^1(B)$ holds true.*

The core idea here is that in any aggregation by $\mathbf{A}$ the presence of the fuzzy set B among the aggregated fuzzy sets makes any presence of C superfluous.

Proposition 1. *Consider a fuzzy inference operator of FITA-type*

$$\Psi_{\mathcal{D}} = \mathbf{A}(\theta_{\langle A_1,B_1 \rangle}, \ldots, \theta_{\langle A_n,B_n \rangle}).$$

It is sufficient for the $\mathcal{D}$-stability of $\Psi_{\mathcal{D}}$, i.e. to have

$$\Phi_{\mathcal{D}}(A_k) = B_k \qquad \text{for all } k = 1, \ldots, n$$

that one always has $\theta_{\langle A_k,B_k \rangle}(A_k) = B_k$ and additionally that for each $i \neq k$ the fuzzy set $\theta_{\langle A_k,B_k \rangle}(A_i)$ is $\mathbf{A}$-negligible w.r.t. $\theta_{\langle A_k,B_k \rangle}(A_k)$.

The proof follows immediately from the corresponding definitions.

There is also a way to extend these considerations from inference operators (1) of the FITA type to those ones of the FATI type (2).

4 Stability conditions for modified data

The combined approximation and interpolation problem, as previously explained, sheds new light on the standard approaches toward fuzzy control via CRI-representable functions originating from the works of Mamdani/Assilian [5] and Sanchez [7] particularly for the case that neither the Mamdani/Assilian relation R_{MA}, determined by the membership degrees

$$R_{\mathrm{MA}}(x,y) = \bigvee_{i=1}^{n} A_i(x) * B_i(y)\,, \tag{6}$$

nor the Sanchez relation $\widehat{R}$, determined by the membership degrees

$$\widehat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \rightarrowtail B_i(y))\,, \tag{7}$$

offer a solution for the system of fuzzy relation equations. In any case both these fuzzy relations determine CRI-representable fuzzy functions which provide approximate solutions for the interpolation problem.

In other words, the consideration of CRI-representable functions determined by (6) as well as by (7) provides two methods for an approximate solution of the main interpolation problem. As is well known and explained e.g. in [3], the approximating interpolation function CRI-represented by $\widehat{R}$ always gives a lower approximation, and that one CRI-represented by R_{MA} gives an upper approximation for normal input data.

Extending these results, in [4] the iterative combination of these methods has been discussed to get better approximation results. For the iterations there, always the next iteration step consisted in an application of a predetermined one of the two approximation methods to the data family with the original input data and the real, approximating output data which resulted from the application of the former approximation method.

Therefore let us now, in the general context given earlier in this paper, discuss the problem of $\mathcal{D}$-stability for a modified operator $\Theta^*_{\mathcal{D}}$ which is determined by the kind of iteration of $\Theta_{\mathcal{D}}$ just explained.

Let us consider the $\Theta_{\mathcal{D}}$*-modified* data set $\mathcal{D}^*$ given as

$$\mathcal{D}^* = (\langle A_i, \Theta_{\mathcal{D}}(A_i)\rangle)_{1\le i\le n}\,, \tag{8}$$

and define from it the modified fuzzy inference operator $\Theta^*_{\mathcal{D}}$ as

$$\Theta^*_{\mathcal{D}} = \Theta_{\mathcal{D}^*}\,. \tag{9}$$

For these modifications, the problem of stability reappears. Of course, the new situation here is only a particular case of the former. And it becomes a simpler one in the sense that the stability criteria now refer only to the input data A_i of the data set $\mathcal{D} = (\langle A_i, B_i\rangle)_{1\le i\le n}$.

Proposition 2. *It is sufficient for the $\mathcal{D}^*$-stability of a fuzzy inference operator $\Psi^*_{\mathcal{D}}$ of FITA-type that one has*

$$\Psi^*_{\mathcal{D}}(A_i) = \Psi_{\mathcal{D}^*}(A_i) = \Psi_{\mathcal{D}}(A_i) \qquad \textit{for all}\ \ 1 \le i \le n$$

and that always $\theta_{\langle A_i, \Psi_{\mathcal{D}}(A_i)\rangle}(A_j)$ is **A***-negligible w.r.t. $\theta_{\langle A_i, \Psi_{\mathcal{D}}(A_i)\rangle}(A_i)$.*

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On Theories and Models in Fuzzy Predicate Logics

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In the last decades many formal systems of fuzzy logics were developed. Since the main differences between fuzzy and classical logics are grounded in the propositional level, the fuzzy predicate logics are still under-developed (compared to the propositional ones). After the monograph [6] only few new results have been achieved (notably results on arithmetical complexity of particular logics).

In this text we want to boost the interest in fuzzy predicate logics by contributing to the model theory of the fuzzy predicate logic. First, we generalize the completeness theorem, then we use it to get results on conservative extension of theories and on witnessed models.

1 Introduction and completeness theorem

We concentrate on basic predicate fuzzy logic BL∀ and stronger predicate calculi (see the monograph [6]). These logics have proved to be reasonably deep and well behaving as symbolic logical systems (see also e.g. [8, 9, 4, 11]). The reader familiar with the MTL hoop logic (HMTL, see [7]) and MTL delta logic ($\mathrm{MTL}_\triangle$, see [3]) will see that our results hold true also for such logics. Although they can be extended to even wider class of logics (see [2, 10]), we restrict ourselves to:

Convention 1 *By a propositional* fuzzy logic $\mathcal{L}$ *we understand here an axiomatic extension of either* HMTL *or* $\mathrm{MTL}_\triangle$*(a fortiori BL and its axiomatic extensions are included).*

We assume that the reader is familiar with the syntax and semantics of predicate fuzzy logics. We restrict ourselves to languages without functions, the generalization is almost straightforward. We recall that, for each $\mathcal{L}$-algebra $\mathbf{L}$, an $\mathbf{L}$-structure of a predicate language $\mathbf{M} = (M, (r_P)_{P\,predicate}, (m_c)_{c\,constant})$ where $M \neq \emptyset$, for each predicate P of arity n, r_P is an n-ary $\mathbf{L}$-fuzzy relation on M and for each constant c, $m_c \in M$. Having this, one defines for each formula φ (of the given language), the *truth value* $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ of φ in $\mathbf{M}$ determined by the $\mathcal{L}$-algebra $\mathbf{L}$ and evaluation v of free variables of φ in M in the usual (Tarskian) way. A Structure $\mathbf{M}$ is *safe* if this is defined for each φ and v.

By $(\mathbf{M},\mathbf{L}) \models \varphi$ we denote the fact $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = 1_{\mathbf{L}}$ for each $\mathbf{M}$-evaluation v. When $\mathbf{L}$ is known from the context we write $\mathbf{M} \models \varphi$ only. We say that $(\mathbf{M},\mathbf{L})$ is a model instead of saying that $\mathbf{L}$ is a $\mathcal{L}$-algebra and $\mathbf{M}$ is a safe $\mathbf{L}$-interpretation. Furthermore, we say that $(\mathbf{M},\mathbf{L})$ is a model of a theory T if $(\mathbf{M},\mathbf{L})$ is a model and all axioms of T are $\mathbf{L}$-true in $\mathbf{M}$ (i.e., $(\mathbf{M},\mathbf{L}) \models \alpha$ for each $\alpha \in T$).

Now we recall that for each propositional fuzzy logic we can define two distinct predicate logics. The first one is described in the monograph [6]. The second one results from this logic by omitting its last axiom (as described in [7]).

Definition 1. *Let* $\mathcal{L}$ *be a propositional fuzzy logic. The logic* $\mathcal{L}\forall^-$ *has axioms:*

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(P) the axioms resulting from the axioms of $\mathcal{L}$ by the substitution of the propositional variables by the formulae of Γ,

$(\forall 1)$ $(\forall x)\varphi(x) \to \varphi(t)$, *where t is substitutable for x in* φ,

$(\exists 1)$ $\varphi(t) \to (\exists x)\varphi(x)$, *where t is substitutable for x in* φ,

$(\forall 2)$ $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where x is not free in* χ,

$(\exists 2)$ $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where x is not free in* χ,

The deduction rules are those of the logic $\mathcal{L}$ *and generalization from* φ *infer* $(\forall x)\varphi$. *Furthermore, we define the logic* $\mathcal{L}\forall$ *as an extension of* $\mathcal{L}\forall^-$ *by the axiom:*

$(\forall 3)$ $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, *where x is not free in* χ.

The completeness theorem for Basic predicate logic was proven in [6] and the completeness theorems of other predicate fuzzy logic, defined in the literature were proven in the corresponding papers. We generalize these results in two aspects, first we prove it for each fuzzy logic (see Convention 1) and we prove it for arbitrary predicate languages (and not only countable ones as in the literature). Whereas the first generalization is rather trivial the second needs a new version of the proof of the fundamental lemma (about existence of Henkin extension). We also use this lemma and the proof in the next section. We also deal with both predicate logics $\mathcal{L}\forall$ and $\mathcal{L}\forall^-$.

Theorem 2. *Let* $\mathcal{L}$ *be a propositional fuzzy logic,* Γ *be a predicate language,* T *a theory and* φ *a formula. Then we have:*

- $T \vdash_{\mathcal{L}\forall^-} \varphi$ *iff* $\mathbf{M} \models \varphi$ *for each* $\mathbf{L}$*-model of theory* T *and each* $\mathcal{L}$*-algebra* $\mathbf{L}$.
- $T \vdash_{\mathcal{L}\forall} \varphi$ *iff* $\mathbf{M} \models \varphi$ *for each* $\mathbf{L}$*-model of theory* T *and each linearly ordered* $\mathcal{L}$*-algebra* $\mathbf{L}$.

2 Conservative extension

In classical logic, a theory T_2 is called an *extension* of a theory T_1 if the language of T_1 is a sublanguage of T_2 and each formula provable in T_1 is provable in T_2; T_2 is called a *conservative extension* if, in addition, each formula of the language of T_1 provable in T_2 is provable in T_1. A model-theoretic theorem then says that T_2 is a conservative extension of T_1 iff for each model $\mathbf{M}_1$ of T_1 there exists a model $\mathbf{M}_2$ of T_2 such that the restriction of $\mathbf{M}_2$ to the language of T_1 is elementarily equivalent to $\mathbf{M}_1$, i.e. for each sentence φ in the language of T_1, $\mathbf{M}_1 \models \varphi$ iff $\mathbf{M}_2 \models \varphi$. (This is an easy exercise of application of compactness and completeness.)

The definitions of an extension and a conservative extension as formulated above are meaningful for theories over fuzzy logics. Our aim is to study a natural model-theoretic characterization, analogous to that for classical logic. Before we do so, we prepare few definitions.

Definition 2. *Let* $(\mathbf{M_1},\mathbf{L_1})$ *and* $(\mathbf{M_2},\mathbf{L_2})$ *be two models interpreting the same language. Say that* $(\mathbf{M_1},\mathbf{L_1})$ elementarily equivalent *to* $(\mathbf{M_2},\mathbf{L_2})$ *if for each sentence* φ *we have:* $(\mathbf{M_1},\mathbf{L_1}) \models \varphi$ *iff* $(\mathbf{M_1},\mathbf{L_1}) \models \varphi$.

Definition 3. *An elementary embedding of a model* $(\mathbf{M_1},\mathbf{L_1})$ *of a language* Γ_1 *into a model* $(\mathbf{M_2},\mathbf{L_2})$ *of a language* $\Gamma_2 \supseteq \Gamma_1$ *is a pair* (f,g) *such that* f *is an injection of the domain of* $\mathbf{M_1}$ *into the domain of* $\mathbf{M_2}$, g *is an isomorphism of* $\mathbf{L_1}$ *and a subalgebra of* $\mathbf{L_2}$ *such that for each* Γ_1*-formula* $\varphi(x_1,\dots,x_n)$ *and* $a_1,\dots,a_n \in \mathbf{M_1}$, $g(\|\varphi(a_1,\dots,a_n)\|^{(\mathbf{M_1},\mathbf{L_1})} = \|\varphi(f(a_1),\dots,f(a_n)\|^{(\mathbf{M_2},\mathbf{L_2})}$.

Here of course by $\|\varphi(a_1,\dots a_n)\|^{(\mathbf{M_1},\mathbf{L_1})}$ we mean $\|\varphi(x_1,\dots x_n\|^{\mathbf{L}_1}_{\mathbf{M}_1,v}$ for $v(x_i) = a_i, i = 1,\dots n$.

Definition 4. *For each model* $(\mathbf{M},\mathbf{L})$ *let* $Alg((\mathbf{M},\mathbf{L}))$ *be the subalgebra of* $\mathbf{L}$ *whose domain is the set* $\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}|\varphi,v\}$ *of truth degrees of all formulas* φ *under all* $\mathbf{M}$*-evaluations* v *of variables. Clearly, for each* φ *and* v, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{Alg((\mathbf{L},\mathbf{M}))}_{\mathbf{M},v}$. *Call* $(\mathbf{M},\mathbf{L})$ *exhaustive if* $\mathbf{L} = Alg((\mathbf{M},\mathbf{L}))$ *(i.e.* $\mathbf{L}$ *does not contain any unnecessary elements).*

We achieve the following characterizations analogous to the classical ones. However the proofs are much more complicated.

Theorem 3. *Let* T_1 *and* T_2 *be theories. Then the following claim are equivalent:*

1. T_2 *is a conservative extension of* T_1
2. *for each exhaustive model* $(\mathbf{M_1},\mathbf{L_1})$ *of* T_1 *there exists an elementary embedding of* $(\mathbf{M_1},\mathbf{L_1})$ *into a model* $(\mathbf{M_2},\mathbf{L_2})$ *of* T_2.
3. *for each model* $(\mathbf{M_1},\mathbf{L_1})$ *of* T_1 *there is a model* $(\mathbf{M_2},\mathbf{L_2})$ *of* T_2 *such that* $(\mathbf{M_1},\mathbf{L_1})$ *is elementarily equivalent to the restriction of* $(\mathbf{M_2},\mathbf{L_2})$ *to the language of* T_1.

3 Witnessed models and logics

Recall that the truth degree of a universally quantified formula is defined as the infimum of its instances and similarly for existentially quantified formula (supremum). The infimum may be smaller than the truth value of each instance (they do not have a minimum); dually for supremum (maximum).

Definition 5. *Call a formula* $(\exists x)\varphi$ *possibly containing free variables* $y_1,\dots,y_n$ *witnessed in* $(\mathbf{M},\mathbf{L})$ *if for each evaluation* $a_1,\dots a_n \in M$ *of* $y_1,\dots,y_n$ *there is a* $b \in M$ *such that* $\|(\exists x)\varphi(x,a_1,\dots,a_n)\|^{(\mathbf{M},\mathbf{L})} = \|\varphi(b,a_1,\dots,a_n)\|^{(\mathbf{M},\mathbf{L})}$; *similarly for* $(\forall x)\varphi$. *Call* $(\mathbf{M},\mathbf{L})$ *witnessed if each formula beginning by a quantifier its witnessed in* $(\mathbf{M},\mathbf{L})$.

The notion of a witnessed model was introduced in [5]. Consider the following two axiom schemas (cf. [1]).

$$\begin{array}{ll} (C\exists) & (\exists y)((\exists x)\varphi(x) \to \varphi(y)) \\ (C\forall) & (\exists y)(\varphi(y) \to (\forall x)\varphi(x)) \end{array}$$

Evidently, if $(\mathbf{M},\mathbf{L})$ is witnessed then all instances of $(C\exists),(C\forall)$ are true in $(\mathbf{M},\mathbf{L})$; but not necessarily conversely.

Definition 6. *Let* $\mathcal{L}$ *be a propositional fuzzy logic. We define the logic* $\mathcal{L}\forall^w$ *as an extension of* $\mathcal{L}\forall$ *by axioms* $(C\exists),(C\forall)$.

Lemma 1. *Łukasiewicz logic* Ł$\forall$ *proves* $(C\exists)$ *and* $(C\forall)$*, i.e.,* Ł$\forall$ = Ł$\forall^w$*; product logic* $\Pi\forall$ *proves* $(C\exists)$.

Remark 1. (1) The examples in [6] 5.3.6 can be used to show that $(C\exists)$ is unprovable in $G\forall$ and $(C\forall)$ is unprovable both in $G\forall$ and in $\Pi\forall$.

(2) To show that validity of $(C\forall),(C\exists)$ does not guarantee witnessedness it is enough take any non-witnessed model over standard Łukasiewicz, e.g. (N,r_P) where $r_P(n) = \frac{1}{n+1}$ (for $\forall$) of $r_P(n) = \frac{n}{n+1}$ (for $\exists$).

Theorem 4. *Let* $\mathcal{L}$ *be a propositional fuzzy logic and* $(\mathbf{M},\mathbf{L})$ *be an exhaustive model. Then* $(\mathbf{M},\mathbf{L})$ *is a model of* $\mathcal{L}\forall^w$ *iff it is an elementary submodel of a witnessed model.*

Theorem 5 (Witnessed completeness). *Let* $\mathcal{L}$ *be a propositional fuzzy logic,* Γ *be a predicate language,* T *a theory, and* φ *a formula. Then* $T \vdash_{\mathcal{L}\forall^w} \varphi$ *iff* $\mathbf{M} \models \varphi$ *for each witnessed* $\mathbf{L}$*-model of theory* T *and for each* $\mathcal{L}$*-chain* $\mathbf{L}$.

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Sheaves on Quantales

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Let Ω be a complete Heyting algebra , and X be an arbitrary set. One of the interesting contributions of sheaf theory consists in the identification of Ω-valued maps $X \xrightarrow{f} \Omega$ with subsheaves of the simple sheaf generated by X. This identification is based on the subobject classifier axiom — one of the most important axioms in topos theory — and makes heavily use of the distributivity of finite meets over arbitrary joins in the underlying lattice Ω.
In this talk we drop any distributivity condition between meets and joins and restate the previous result in the realm of complete lattices. In particular, this program implies an appropriate generalization of the concept of sheaves and the corresponding subobject classifier axiom. Since lattice-theoretically any complete lattice can be embedded into the lattice of selfadjoint elements of involutive, unital quantales (cf. Section 1), involutive and unital quantales (see Mulvey and Pellitier 2001) seem to represent the right level of generality for this kind of development. Thus this talk gives an introduction to the theory of *sheaves on involutive and unital quantales* Q and establishes the *classification* of certain subobjects by characteristic morphisms. As a special case of this situation a positive solution of the problem of identifying Q-valued maps with certain subsheaves on Q is specified.
As illustration of this train of thoughts the relevance of the previous results will be discussed in the special case of the involutive quantale determined by the non-commutative C^*-algebra of (2,2)-matrices.

Approximating Orders in Meet-Continuous Lattices and Subbasic Characterizations of Regularity Type Axioms

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1. Introduction

Some topological invariants, originally defined in terms of arbitrary open sets, have characterizations in terms of subbasic open sets. In general topology as well as in the theory of locales viewed as pointless topologies [5], the generation of an arbitrary open element is done by first constructing all the finite infs of subbasic elements, and subsequently by forming all the sups of those finite infs. It is the distributivity of finite infs over arbitrary sups (= frame law) which guarantees that such a construction yields the whole topology or the whole locale.

In the theory of L-valued topological spaces [2], a good many results are proved to hold for L a complete lattice. Under absence of the frame law, an L-topology $\tau \subset L^X$ is constructed from a subbase $\sigma \subset L^X$ in an external way:

$$\tau = \bigcap \{\rho : \sigma \subset \rho, \rho \text{ is an } L\text{-topology on } X\}.$$

For completely regular spaces of Hutton [4] a number of results hold for L a complete or a meet-continuous lattice. However, the subbasic characterization of complete regularity and, thus, the results depending on it have been proved under the assumption that L a frame (cf. [7]). Thus, the question naturally arises ([6]; also [8]) as to whether there is a subbasic characterization in a complete lattice setting. We shall show that this is the case provided L is meet-continuous.

2. Multiplicative auxiliary order on a meet-continuous lattice

There are many instances in which a complete lattice L carries a new binary relation which is stronger than the lattice order. A binary relation $\prec$ on a complete lattice L is called a *multiplicative auxiliary order* (cf. [1]) if it satisfies the following conditions:

$(1_\prec)$ $0 \prec \alpha$,

$(2_\prec)$ $\alpha \prec \beta$ implies $\alpha \leq \beta$,

$(3_\prec)$ $\alpha \leq \gamma \prec \delta \leq \beta$ implies $\alpha \prec \beta$,

$(4_\prec)$ $\alpha \prec \gamma$ and $\beta \prec \gamma$ imply $\alpha \vee \beta \prec \gamma$,

$(5_\prec)$ $\alpha \prec \beta$ and $\alpha \prec \gamma$ imply $\alpha \prec \beta \wedge \gamma$.

For each $\alpha \in L$ we write $\Downarrow \alpha = \{\beta \in L : \beta \prec \alpha\}$. An $\alpha \in L$ satisfies the *axiom of approximation* if $\alpha = \bigvee \Downarrow \alpha$. The order $\prec$ is called *approximating* if each member of L satisfies the axiom of approximation.

Examples of complete lattices with multiplicative auxiliary order can be found in [1], [5], [10], and [11].

A complete lattice L is called *meet-continuous* if $\alpha \wedge \bigvee D = \bigvee\{\alpha \wedge \delta : \delta \in D\}$ for every $\alpha \in L$ and every directed $D \subset L$.

2.1. Theorem. *Let L be a meet-continuous lattice with a multiplicative auxiliary order* $\prec$. *Then the set* $K = \{\alpha \in L : \alpha = \bigvee \Downarrow \alpha\}$ *is closed under non-empty finite infs and arbitrary sups* (*both formed in* L).

3. Subbasic characterizations of regularity axioms

Given a complete $(L, ')$, an L-topological space X and $a, b \in L^X$, we write

$$a \sqsubset b \Leftrightarrow \exists c \in L^X : a \leq c' \leq b$$

and

$$a \lhd b \Leftrightarrow a \leq L_1' \circ f \leq R_0 \circ f \leq b$$

for some $f \in C(X, \mathbb{I}(L))$, the family of all continuous maps from X to $\mathbb{I}(L)$.

3.1. Definition [4]. Let $(L,')$ be a complete lattice. An L-topological space (X, τ) is called:

(1) *regular* if $u = \bigvee\{v \in \tau : v \leq w' \leq u$ for some $w \in \tau\}$,

(2) completely regular if, given $u \in \tau$, there exist $\mathcal{A} \subset L^X$ and $\{f_a : a \in \mathcal{A}\} \subset C(X, \mathbb{I}(L))$ such that $u = \bigvee \mathcal{A}$ and $a \leq {L_1}' \circ f_a \leq R_0 \circ f_a \leq u$. [Without loss of generality the family $\mathcal{A}$ can be assumed to consist of open L-sets. Thus, complete regularity implies regularity.]

For $(L, ')$ a complete lattice and (X, τ) an L-topological space, the relation $\sqsubset$ is a multiplicative auxiliary order on τ. The space (X, τ) is L-regular iff $\sqsubset$ is approximating in τ.

3.2. Proposition ([6] or [7]). *For* $(L, ')$ *a complete lattice and* (X, τ) *an L-topological space, the following hold*:

(1) (X, τ) *is completely regular if and only iff* $u = \bigvee\{v \in \tau : v \lhd u\}$ *for each* $u \in \tau$;

(2) *If L is meet-continuous, then* $\lhd$ *is a multiplicative auxiliary order on* τ.

3.3. Definition [9]. Let L be a complete lattice and let $\prec$ be a multiplicative auxiliary order on the L-topology τ of an L-topological space X. Then:

(1) X is $\prec$*-regular* or τ is $\prec$*-regular* if $u = \bigvee \Downarrow u$ for every $u \in \tau$;

(2) A subbase σ of τ is $\prec$*-regular* if $v = \bigvee(\tau \cap \Downarrow v)$ for every $v \in \sigma$.

3.4. Theorem (Subbase characterization of $\prec$-regularity). *Let L be a complete lattice and* (X, τ) *be an L-topological space such that* τ *is meet-continuous*. *Let* σ *be a subbase of* τ *and* $\prec$ *be a multiplicative auxiliary order on* τ. *Then the following are equivalent*:

(1) τ *is* $\prec$*-regular,*

(2) σ *is* $\prec$*-regular and* $1_X = \bigvee \Downarrow 1_X$.

3.5. Theorem (Subbasic characterization of regularity axioms). *Let* $(L,')$ *be a meet-continuous lattice. Let* (X, τ) *be an L-topological space and let* $\sigma \subset L^X$ *be a subbase of* τ. *Then the following statements hold*:

(1) (X, τ) *is regular if and only if* $u = \bigvee\{v \in \tau : v \leq w' \leq u$ *for some* $w \in \tau\}$ *for every* $u \in \sigma$.

(2) (X, τ) *is completely regular if and only if, whenever* $u \in \sigma$, *there exist* $\mathcal{A} \subset L^X$ *and* $\{f_a : a \in \mathcal{A}\} \subset C(X, \mathbb{I}(L))$ *such that* $u = \bigvee \mathcal{A}$ *and* $a \leq L_1{}' \circ f_a \leq R_0 \circ f_a \leq u$. [Note that $\mathcal{A}$ can be assumed to be a subset of τ.]

4. Distributivity-free environment for regular and completely regular *L*-topological spaces

In the context of regularity type axioms, frames have been a lattice setting for those results whose proofs were based on the subbasic characterizations. An easy insepection of their proofs (see [6], [7], [9]) shows that the frame law can be eliminated, and all those results will continue to hold with unchanged proofs in the meet-continuous setting, if the appeal to the frame-like subbasic characterization is replaced by an application of Theorem 3.5. In particular, the following statements are now valid for L a meet-continuous lattice:

(1) (*Complete*) *regularity is inherited by initial structures; in particular by products.*

(2) (*Complete*) *regularity is preserved under stratification.*

(3) *The L-real real line, the unit L-interval, and L-cubes* (= *products of copies of the unit L-interval*) *are completely regular.*

A completely regular space X is called *L-Tychonoff* if X is a T_0-space, i.e., whenever $x \neq y$ in X, there exists an open L-set u such that $u(x) \neq u(y)$. Since already in a complete lattice setting the T_0-axiom is preserved under initial structures and stratification, the statements (1)–(3) continue to hold for L-Tychonoff spaces. If we replace frames by meet-continuous lattices, we can restate a result of [7] as follows:

4.1. Theorem (Tychonoff embedding theorem). *Let* $(L,')$ *be a meet-contin-uous lattice. A* [*stratified*] *space is L-Tychonoff if and only iff it is homeomorphic to a subspace of a* [*stratified*] *L-Tychonoff cube.*

We also note that with (1)–(3) at hand, virtually all the embbeding theorems of [7] become characterization theorems when the underlying lattice is meet-continuous.

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Geometry of Associativity ─ Theory and Application

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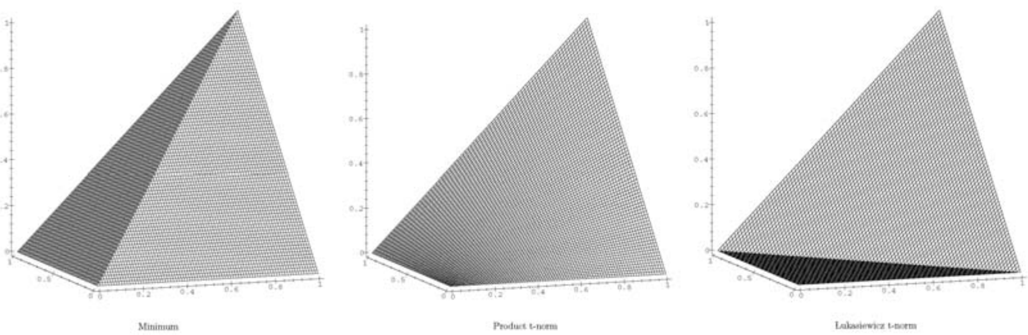
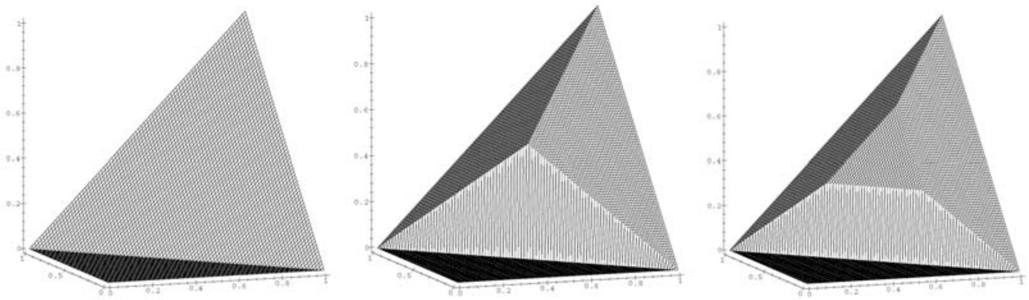
Commutativity of (binary) operations, that is the interchangeability of their arguments ($x*y=y*x$) is easily seen from the graph of the operations. The meaning of commutativity is just the invariance of the graph with respect to a reflection to the plane defined by $x=y$. Similar geometrical description for associativity is not known. That is, associativity of binary operations can not be seen simply by "looking at" their graphs. The following three operations are commutative and associative, their commutativity is readily seen from their graphs, but their associativity is not (see the figure below).



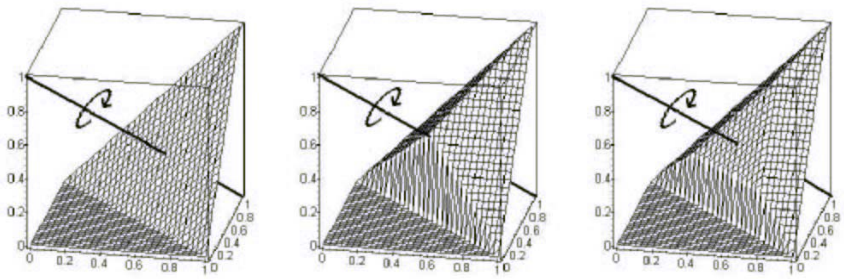
Investigation of associativity is one of the major problems in algebra. For example, semigroups, groups, rings and fields are all associative structures. In my opinion the reason of the difficulty of investigation of associativity is that we are able to "see things" in three dimensions only. In three dimensions the graph of an operation is defined as follows: There are two independent variables x and y, and the value $x*y$ is taken in the third axle. The meaning of associativity together with commutativity is that we can freely interchange the operands of the operation, that is, any two operands are interchangeable. We have seen above that interchangeability is just the invariance of the graph with respect to a reflection to a plane. Consider now the graph of an associative and commutative operation in four dimensions: There are three independent variables x, y, and z, and the value $x*y*z$ is taken in the fourth axle. It follows from the previous arguments that associativity and commutativity together are equivalent to the invariance of the four-dimensional graph with respect to three reflections to the "spaces" $x=y$, $x=z$, and $y=z$, respectively. That is, if we were able to "see things” in four dimension, then associativity together with commutativity were easily seen from the graph of the operation “for the first sight”.

Similar geometrical description of associativity is not known as of today.

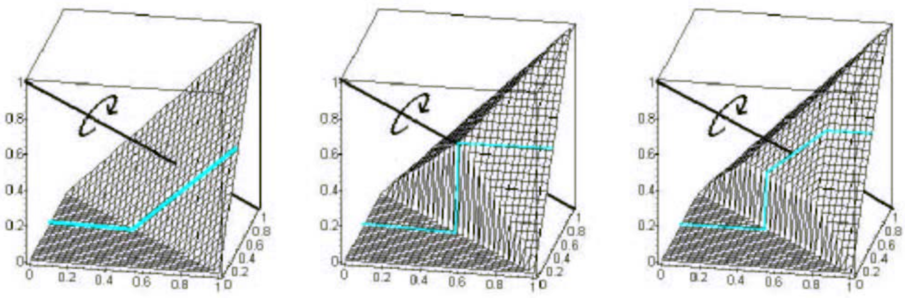
I have reported on a surprising geometrical property of a *special* class of associative functions in [8]. Namely, if we, in addition to commutativity and associativity, assume that the “border line” in between the 0 and the positive part of the graph is the function $y=1-x$, (three examples are plotted below)



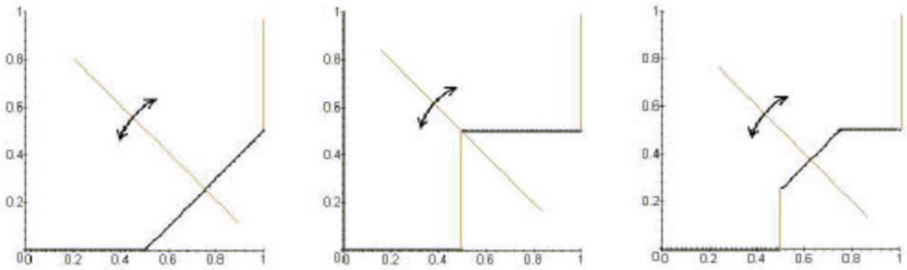
then the corresponding graphs are rotation-invariant with respect to a rotation with 120 degree (an illustration is in the following figure).



Moreover, vertical sections of graphs of such operations (see the yellow lines)



show as well a kind of symmetry.



The mentioned geometrical property *does not* characterize associativity. That is, there exist rotation-invariant functions which are not associative. The question suggests itself:

— Does there exist a geometrical characterization which does not assume the “border line” property, and which do characterize associativity.

In this talk we shall give a geometric characterization of commutative residuated semigroups (in particular, left-continuous t-norms) based on the notion of rotation-invariance and the notion of nuclei of quantale structures (see [10]).

As a consequence, associativity can be “seen” even from the three-dimensional graph. This geometrical understanding of associativity has already led to an elegant solution of a long-standing open problem of C. Alsina, M. J. Frank and B. Schweizer concerning the convex combination of t-norms [9]. Namely, at the end of the talk we shall present an answer to the question whether the convex combination of two left-continuous t-norms can ever be a t-norm.

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Sections of Triangular Norms

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1 Introduction

Two types of sections of triangular norms are of special interest for the characterization and description of t-norms.

The fist type is the intersection of the graph of a t-norm $T\colon [0,1]^2 \to [0,1]$ with a plane which is normal to the domain $[0,1]^2$ of T and whose intersection with the domain is given by some linear constraint $ax+by+c=0$, i.e., we consider the function $T|_{\{(x,y)\in[0,1]^2 | ax+by+c=0\}}$ which always can be written as a function in one variable.

We are concentrating on the following important special cases:

(i) *diagonal section* $\delta_T\colon [0,1] \to [0,1]$, i.e., $x-y=0$ given by $\delta_T(x) = T(x,x)$;
(ii) *horizontal* and *vertical sections* $h_c, v_c\colon [0,1] \to [0,1]$ with $c \in [0,1]$, i.e., $y-c=0$ or $x-c=0$, given by $h_c(x) = T(x,c)$ and $v_c(y) = T(c,y)$;
(iii) *sections parallel to the opposite diagonal* $s_c\colon [\max(c-1,0), \min(c,1)] \to [0,1]$ with $c \in [0,2]$, i.e., $x+y-c=0$ given by $s_c(x) = T(x, c-x)$.

Note that, because of the continuity of t-norms we have $h_c = v_c$ and, for all $x \in [\max(c-1,0), \min(c,1)]$, $s_c(x) = s_c(c-x)$.

The second type of sections is the intersection of the graph of the t-norm with a plane which is parallel to its domain. For the sake of simplicity, we restrict ourselves to the case of continuous Archimedean t-norms, although also more general t-norms can be investigated from this point of view (see, e.g., [9]). For a continuous Archimedean t-norm and each $\alpha \in]0,1]$ the *level function* $l_\alpha\colon [\alpha,1] \to [0,1]$ is given by

$$l_\alpha(x) = \sup\{y \in [0,1] \mid T(x,y) = \alpha\}.$$

Observe that, for each $x \in [\alpha,1]$, we have $l_\alpha(x) = R_T(x,\alpha)$, where $R_T\colon [0,1]^2 \to [0,1]$ is the residual implication [8] induced by T, i.e., the level functions of T coincide are just the horizontal sections of R_T. Moreover, if $t\colon [0,1] \to [0,\infty]$ is an additive generator of T then for each $x \in [\alpha,1]$

$$l_\alpha(x) = t^{-1}(t(\alpha) - t(x)). \tag{1}$$

From (1) we immediately see that l_α is a decreasing involution on $[\alpha,1]$ (compare with strong negations in fuzzy logics), implying $l_\alpha \circ l_\alpha = \mathrm{id}_{[\alpha,1]}$.

In Section 2 we briefly discuss diagonal, horizontal and vertical sections, Section 3 deals with sections parallel to the opposite diagonal, and in Section 4 level functions are considered. Finally we shall relate these sections to different types of continuity.

2 Diagonal, horizontal and vertical sections

Several algebraic properties of a t-norm T are, in fact, properties of its diagonal section δ_T: the Archimedean property, the existence of zero divisors, the structure of the set of idempotent elements, etc.

For each t-norm T, its diagonal section δ_T is a non-decreasing function such that $\delta_T \leq \mathrm{id}_{[0,1]}$ and $\delta_T(1) = 1$. Clearly, if T is continuous so is δ_T. The converse of this is not true in general (a counterexample is provided by the *Krause t-norm* [12, Appendix B.1]).

Moreover, two different continuous t-norms with the same diagonal section are necessarily incomparable [12, Corollary 7.18] (this is not true for non-continuous t-norms, see [12, Example 7.19]). Examples of incomparable continuous t-norms with the same diagonal section can be found in [12, Example 6.1].

The diagonal sections of continuous t-norms can be completely characterized [10, 14, 20, 21]. If $\delta\colon [0,1] \to [0,1]$ is a non-decreasing function satisfying $\delta_T \leq \mathrm{id}_{[0,1]}$ and $\delta_T(1) = 1$ then the following are equivalent:

(i) there exists a continuous t-norm T with $\delta_T = \delta$.
(ii) δ is continuous and the restriction $\delta|_{[0,1]\setminus\delta^{-1}(\{x\in[0,1[\,\mid\delta(x)=x\})}$ is strictly increasing.

As shown in [2, 3], a strict t-norm is uniquely determined by its diagonal section and by its values on some small parts of the opposite diagonal sector or of a vertical section (also the knowledge of some vertical section and a small part of the diagonal section is sufficient). In [6] it was proved that a strict 1-Lipschitz t-norm T is uniquely determined by its diagonal section δ_T if $(\delta_T)'(1^-) = 2$.

There are several open problems in the context of diagonal sections of t-norms:

Open Problem 1. Characterize the set of diagonal sections of t-norms.
Open Problem 2. Characterize the set of continuous diagonal sections of t-norms (observe that in [15] it was shown that the function $\delta\colon [0,1] \to [0,1]$ given by

$$\delta(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0,0.5]\,, \\ 0.25 & \text{if } x \in\,]0.5,0.75]\,, \\ 3x-2 & \text{otherwise.} \end{cases}$$

is not the diagonal section of a t-norm).
Open Problem 3. Characterize the set of diagonal sections of 1-Lipschitz t-norms.

Finally, we mention that the associativity of a binary operation is closely related to its horizontal and vertical sections [18, Theorem 5.1.1]: given an arbitrary set X, then a binary operation $T\colon X^2 \to X$ is associative if and only if for all $a,b \in X$ we have $h_a \circ v_b = v_b \circ h_a$.

3 Sections parallel to the opposite diagonal

Our attention was drawn to this type of sections of t-norms by the Open Problem 5 in [13] posed by J. C. Fodor, looking for a characterization of continuous Archimedean t-norms satisfying

$$T(x,y) \leq \delta_T\left(\tfrac{x+y}{2}\right) \qquad (2)$$

for all $(x,y) \in [0,1]^2$. Note that this means that each section s_c attains its maximal value at the point $\frac{c}{2}$.

Recently, in [7] this problem was solved (even for continuous t-norms). A continuous t-norm is a solution of (2) if and only if it is an ordinal sum of continuous Archimedean t-norms solving (2). A continuous Archimedean t-norm is a solution of (2) whenever its additive generator $t\colon [0,1] \to [0,\infty]$ fulfills the following two conditions:

(i) t is convex on the set $\left[t^{-1}(\frac{t(0)}{2}), 1\right]$,

(ii) for all $u \in \left[0, t^{-1}(\frac{t(0)}{2})\right]$ we have $t(u) + t\big(2t^{-1}(\frac{t(0)}{2})\big) \geq t(0)$.

However, there are also non-continuous solutions of (2) (see [11]).

Inspired by [4], we recall an interesting property of t-norms which is related to its sections parallel to the opposite diagonal, the Schur concavity (which was already studied in the context of continuous t-norms in [1]): a t-norm T is said to be *Schur concave* if for all $\lambda, x, y \in [0,1]$ we have

$$T(x,y) \leq T(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y). \tag{3}$$

The Schur concavity of T means that each section s_c is non-decreasing on the interval $\left[\max(c-1, 0, \frac{c}{2}\right]$.

From [1] we know the following. A continuous t-norm is Schur concave if and only if it is an ordinal sum of Schur concave continuous Archimedean t-norms. A continuous Archimedean t-norm is Schur concave if and only if it has an additive generator $t\colon [0,1] \to [0,\infty]$ such that for all $\lambda \in [0,1]$ and for all $(x,y) \in [0,1]^2$ with $t(x) + t(y) < t(0)$ we have

$$t(x) + t(y) \geq t(\lambda x + (1-\lambda)y) + t((1-\lambda)x + \lambda y).$$

There are also non-continuous t-norms which are Schur concave, e.g., the nilpotent minimum [5, 17].

Evidently, the Schur concavity of a t-norm T implies that T satisfies (2).

An even stronger property of t-norms is the 1-Lipschitz property which, in the case of continuous Archimedean t-norms, is equivalent with the convexity of the additive generators. Each continuous 1-Lipschitz t-norm is an ordinal sum of continuous Archimedean t-norms with convex additive generators.

In general, the implications mentioned so far cannot be reversed. However, for weakly cancellative t-norms, i.e., continuous t-norms with strictly increasing diagonal section, we have:

Proposition 1. *If a continuous t-norm T is weakly cancellative then the following are equivalent:*

(i) *T satisfies* (2)*;*

(ii) *T is Schur concave;*

(iii) *T is* 1*-Lipschitz;*

(iv) *T is a copula.*

Open Problem 4. Characterize the set of of all t-norms satisfying (2).

Open Problem 5. Prove or disprove: 1-Lipschitz t-norms have only concave sections s_c.

4 Level functions

Several properties of level functions were discussed in [9], we only mention two of them:

If, for some left-continuous t-norm T, there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $]0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $l_{\alpha_n}(x) = 1 + \alpha_n - x$ holds for all $x \in [\alpha_n, 1]$, then T is the Łukasiewicz t-norm.

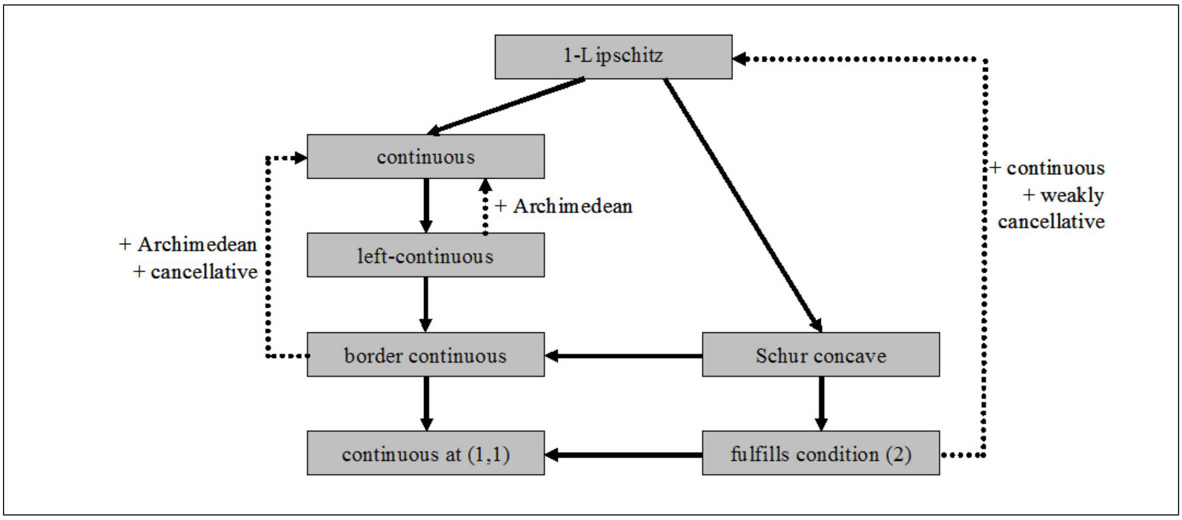

Fig. 1. Several implications

If, for some left-continuous t-norm T, there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $]0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 0$ and $l_{\alpha_n}(x) = \frac{\alpha_n}{x}$ holds for all $x \in [\alpha_n, 1]$, then T is the product t-norm.

As shown in [16], each level function of a 1-Lipschitz Archimedean t-norm is convex. A counterexample for the converse is given by the t-norm generated by the additive generator $t\colon [0,1] \to [0,\infty]$ given by $t(x) = \frac{1+\cos\pi x}{2}$ (originally introduced in [1] showing that a continuous Schur concave t-norm need not be 1-Lipschitz).

Note that the convexity of all level functions implies the Schur concavity. Again the converse is not true in general, as can be seen from the t-norm generated by the additive generator t given by $t(x) = \min(1-\frac{x}{2}, \max(\frac{3}{2}-2x, \frac{1-x}{2}))$.

Recall that the Schur concavity was defined by means of sections parallel to the opposite diagonal. However, there is also an equivalent characterization in terms of level functions:

Proposition 2. *A continuous Archimedean t-norm T is Schur concave if and only if for all $z \in]0,1[$ with $\delta_T(z) = \alpha > 0$ the function $l_\alpha + \mathrm{id}_{[\alpha,1]}\colon [\alpha,1] \to [0,2]$ is non-increasing on $[\alpha, z]$.*

Similarly, it is possible to characterize solutions of (2) by means of level functions [19]:

Proposition 3. *A continuous Archimedean t-norm T satisfies (2) if and only if for all $z \in]0,1[$ with $\delta_T(z) = \alpha > 0$ we have $l_\alpha(x) + x \geq 2z$ for each $x \in [\alpha,1]$.*

Observe that, similarly as the 1-Lipschitz property of a t-norm implies its continuity, the Schur concavity implies its border continuity and (2) implies the continuity at the point $(1,1)$. Figure 1 visualizes the implications mentioned in this paper.

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Inconsistency and Refutation in Type-2 Fuzzy Logics

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Definition 1 ($\mathfrak{T}$–$\mathfrak{D}$ Type-2 Fuzzy Logic)
A tuple $\Lambda =_{\text{def}} \langle \text{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models \rangle$ *shall be called a* $\mathfrak{T}$–$\mathfrak{D}$ ***Type-2 Fuzzy Logic***

- *with **logical language*** Frm,
- *with **truth value lattice*** $\mathfrak{T}$,
- *with **semantics*** $\mathfrak{S}$,
- *with **validity degree lattice*** $\mathfrak{D}$,
- *with **label lattice*** $\mathfrak{L}$,
- *and with **model relation*** $\models$,

$=_{\text{def}}$ *i.* Frm *is a nonempty set,*

ii. $\mathfrak{T} = \langle T, \sqcap_T, \sqcup_T \rangle$ *is a complete chain and* $\mathfrak{D} = \langle D, \sqcap_D, \sqcup_D \rangle$, $\mathfrak{L} = \langle L, \sqcap_L, \sqcup_L \rangle$ *are complete lattices with at least two elements each, with induced partial orders* $\sqsubseteq_T$, $\sqsubseteq_D$, $\sqsubseteq_L$, *respectively,*

iii. $\mathfrak{S} \subseteqq T^{\text{Frm}}$,

iv. For every $t \in T$ *there exists* $x_t \in \text{Frm}$ *and* $\text{Val}_t \in \mathfrak{S}$ *such that* $\text{Val}_t(x_t) = t$,

v. $\models$ *is a ternary relation on* $\mathfrak{S} \times \mathfrak{L}^{\text{Frm}} \times D$ *such that for every* $\text{Val} \in \mathfrak{S}$, $x \in \text{Frm}$, *and* $\ell \in L$ *there exists a **unique*** $d \in D$ *such that* $\text{Val} \models_{d} \langle x, \ell \rangle$,

vi. if $x, y \in \text{Frm}$ *and* $\text{Val}, \text{Val}' \in \mathfrak{S}$ *such that* $\text{Val}(x) = \text{Val}'(y)$, *then for all* $\ell \in L$ *and* $d \in D$,

$$\text{Val} \models_{d} \langle x, \ell \rangle \text{ iff } \text{Val}' \models_{d} \langle y, \ell \rangle \,, \tag{1}$$

vii. if $\ell, \ell' \in L$ *such that* $\ell \neq \ell'$, *then there exists* $t \in T$ *such that for* $d, d' \in D$,

$$\text{if } \text{Val}_t \models_{d} \langle x_t, \ell \rangle \text{ and } \text{Val}_t \models_{d'} \langle x_t, \ell' \rangle \text{ then } d \neq d', \tag{2}$$

viii. for all $\ell \in L$,

$$\text{Val}_1 \models_{1} \langle x_1, \ell \rangle \,, \tag{3}$$

ix. for every $t \in T$ *and* $d \in D$, *there exists* $\ell_d^t \in L$ *such that for* $t' \in T$ *and* $d' \in D$,

$$\text{if } \text{Val}_{t'} \models_{d'} \langle x_{t'}, \ell_d^t \rangle \text{ then } d' = \begin{cases} 1, & \text{if } t' = 1 \\ d, & \text{if } t' \neq 1 \text{ and } t \sqsubseteq_T t' \\ 0, & \text{if } \textit{\textbf{not}}\ t \sqsubseteq_T t' \end{cases} \tag{4}$$

x. for $s, t \in T$, $\ell \in L$, *and* $c, d \in D$ *such that*

$$\text{Val}_s \models_{c} \langle x_s, \ell \rangle \text{ and } \text{Val}_t \models_{d} \langle x_t, \ell \rangle \,,$$

it holds that

$$\text{if } s \sqsubseteq_T t \text{ then } c \sqsubseteq_D d. \tag{5}$$

xi. for $t \in T$, $\ell, \ell' \in L$, and $c, d \in D$ such that

$$\mathrm{Val}_t \models_c \langle x_t, \ell' \rangle \text{ and } \mathrm{Val}_t \models_d \langle x_t, \ell \rangle,$$

it holds that

$$\mathrm{Val}_t \models_{c \sqcap_D d} \langle x_t, \ell' \sqcup_L \ell \rangle, \tag{6}$$

xii. for $t \in T$, $\ell, \ell' \in L$, and $c, d \in D$ such that

$$\mathrm{Val}_t \models_c \langle x_t, \ell' \rangle \text{ and } \mathrm{Val}_t \models_d \langle x_t, \ell \rangle,$$

it holds that

$$\mathrm{Val}_t \models_{c \sqcup_D d} \langle x_t, \ell' \sqcap_L \ell \rangle, \tag{7}$$

The case that $\mathfrak{T}$ is not a chain, but more generally an arbitrary complete lattice is treated in [2]. It leads to significant complications without yielding many more interesting results.

It has been shown in [2, Observation 4.1.2 and Theorem 4.1.3] that

$$\Lambda = \langle \mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models \rangle$$

is a $\mathfrak{T}$–$\mathfrak{D}$ Type-2 Fuzzy Logic if and only if $\mathfrak{L}$ is isomorphic with a complete lattice $\mathfrak{L}' = \langle L', \cup, \cap \rangle$ where $L' \subseteq D^T$ is a set of **non-decreasing** (wrt. $\sqsubseteq_T, \sqsubseteq_D$) mappings such that for every $\ell \in L'$, $\ell(1) = 1$, and where $\cup, \cap$ are the fuzzy set union and intersection, respectively, induced by $\sqcup_D, \sqcap_D$, and where $\mathrm{Val} \models_d \langle x, \ell \rangle$ is defined by $d =_{\mathrm{def}} \ell(\mathrm{Val}(x))$.

Hence, here we identify labels $\ell \in L$ with non-decreasing mappings from D^T ($\mathfrak{D}$-fuzzy sets on T) such that $\ell(1) = 1$, and we identify $\sqcap_L$ with the **fuzzy set union**, $\sqcup_L$ with the **fuzzy set intersection** and $\sqsubseteq_L$ with the **superset relation** for fuzzy sets induced by $\sqcup_D, \sqcap_D, \sqsubseteq_D$, respectively. The graded model relation $\mathrm{Val} \models_d \langle x, \ell \rangle$ is defined by $d =_{\mathrm{def}} \ell(\mathrm{Val}(x))$.

We consider $\mathfrak{L}$-*fuzzy sets of formulae* from L^{Frm}.

We define the **model $\mathfrak{D}$-fuzzy set** of a labelled formula $\langle x, \ell \rangle$ for every $\mathrm{Val} \in \mathfrak{S}$ by

$$\mathrm{Mod}(\langle x, \ell \rangle)(\mathrm{Val}) =_{\mathrm{def}} \ell(\mathrm{Val}(x))$$

and for $\mathcal{X} : \mathrm{Frm} \to L$:

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) =_{\mathrm{def}} \bigsqcap_{D,\, x \in \mathrm{Frm}} \mathrm{Mod}(\langle x, \mathcal{X}(x) \rangle)(\mathrm{Val}).$$

The **semantic consequence relation** is then defined straightforwardly for $\mathcal{X} : \mathrm{Frm} \to L$ and $\langle x, \ell \rangle$ by

$$\mathcal{X} \Vdash \langle x, \ell \rangle =_{\mathrm{def}} \mathrm{Mod}(\mathcal{X}) \subseteq \mathrm{Mod}(\langle x, \ell \rangle)$$

and the respective $\mathfrak{L}$-fuzzy set of consequences by

$$\mathrm{Cons}(\mathcal{X})(x) =_{\mathrm{def}} \bigsqcup_L \{ \ell \mid \ell \in L \text{ and } \mathcal{X} \Vdash \langle x, \ell \rangle \}.$$

Let $\mathfrak{U} =_{\mathrm{def}} \langle [0,1], \min, \max \rangle$ be the **real unit interval**. A $\mathfrak{U}$–$\mathfrak{U}$ Type-2 Fuzzy Logic is called simply **Type-2 Fuzzy Logic**. This simpler class and its applications will be studied in further publications; here a little more variation is needed for characterization results.

It has been proved in [2, Observation 5.2.4 and Corollary 5.3.2] that choosing $\mathfrak{D}$ to be *two-valued* leads to **fuzzy logic in narrow sense** [4,3] and choosing $\mathfrak{T}$ to be *two-valued* leads to **possibilistic logic**

with neccessity-valued formulae [1]. In this sense, a $\mathfrak{T}$–$\mathfrak{D}$ Type-2 Fuzzy Logic could also be called **possibilistic fuzzy logic**, but this name has already been used for several different logical systems.

The purpose of this talk is to demonstrate how the concepts of **inconsistency** and **refutation** can be investigated in this setting.

We recall the following two classical definitions of the concepts:

1. A set of logical formulae is **inconsistent** if it has no model.
2. The concept of **refutation** refers to the fact that classically, $X \Vdash x$ holds for a logical formula x and a set of logical formulae X if and only if $X \cup \{\neg x\}$ has no model, so proving the latter means proving the former *by refutation*.
   It is pointed out that refutation is an essential part of some proof systems like *semantic tableaux* and *resolution*, which are especially well suited for automated proving.

It should be clear that both concepts become significantly more complicated in the logical setting presented here:

1. By the gradedness of the model relation, the concept of inconsistency is neccessarily graded as well.
   Furthermore, we are considering $\mathfrak{L}$*-fuzzy sets* of formulae here, so the labels have to be taken into account as well.
2. When considering refutation wrt. the relationship $\mathcal{X} \Vdash \langle x, \ell \rangle$, it is not enough to negate the formula x. Furthermore, a 'dual' label $\tilde{\ell}$ has to be constructed which can be attached to $\neg x$.

Definition 2 (Inconsistency distribution) *For this definition, assume that* Frm *contains a formula* $\bot$ *such that for all* $\mathrm{Val} \in \mathfrak{S}$, $\mathrm{Val}(\bot) = 0$.

Let $\mathcal{X} \in L^{\mathrm{Frm}}$*. The* ***inconsistency distribution*** *of* $\mathcal{X}$ *is defined by*

$$\mathrm{inc}(\mathcal{X}) =_{\mathrm{def}} \mathrm{Cons}(\mathcal{X})(\bot). \tag{8}$$

Definition 3 (Refutation) *Assume to be given two unary mappings* $\nu_{\mathfrak{D}} : D \to D$, $\nu_{\mathfrak{T}} : T \to T$ *with the following properties, for* $c, d \in D$ *and* $s, t \in T$*:*

$$\begin{array}{lll} \textit{(order reversion)} & s \sqsubseteq_{\mathbb{T}} t \text{ iff } \nu_{\mathfrak{T}}(t) \sqsubseteq_{\mathbb{T}} \nu_{\mathfrak{T}}(s) & c \sqsubseteq_{\mathbb{D}} d \text{ iff } \nu_{\mathfrak{D}}(d) \sqsubseteq_{\mathbb{D}} \nu_{\mathfrak{D}}(c) \quad (9) \\ \textit{(involution)} & \nu_{\mathfrak{T}}(\nu_{\mathfrak{T}}(t)) = t & \nu_{\mathfrak{D}}(\nu_{\mathfrak{D}}(d)) = d \quad (10) \end{array}$$

and assume further that Frm *contains a unary operator symbol* $\neg$ *interpreted by* $\nu_{\mathfrak{T}}$.

Let $\mathcal{X} \in L^{\mathrm{Frm}}$ *and* $\langle x, \ell \rangle \in \mathfrak{L}^{\mathrm{Frm}}$ *be given.*

ℓ *is said to* ***admit refutation***

$=_{\mathrm{def}}$ *the mapping* $\tilde{\ell} : T \to D$ *defined for* $t \in T$ *by*

$$\tilde{\ell}(t) =_{\mathrm{def}} \begin{cases} 1 & \text{if } t = 1 \\ \nu_{\mathfrak{D}}(\ell(\nu_{\mathfrak{T}}(t))) & \text{if } t \neq 1 \end{cases} \tag{11}$$

is in L.

If ℓ *admits refutation, then* $\mathcal{X} \Vdash \langle x, \ell \rangle$ *is said to be* ***characterised by refutation***

$=_{\mathrm{def}}$ $\mathcal{X} \Vdash \langle x, \ell \rangle$ *iff* $\ell \sqsubseteq_{\mathbb{L}} \mathrm{inc}\left(\mathcal{X} \cup \left\langle \neg x, \tilde{\ell} \right\rangle\right)$.

During the talk, sufficient and neccessary criteria for a label to admit refutation, and for a relationship $\mathcal{X} \Vdash \langle x, \ell \rangle$ to be characterised by refutation, shall be given.

It will turn out that in fact the cases where one of $\mathfrak{T}$ or $\mathfrak{D}$ is two-valued are the only ones where refutation can safely be applied without further preparation (as is well-known for fuzzy logic in narrow sense and possibilistic logic) while for arbitrary Type-2 Fuzzy Logics, care has to be taken about the range of labels which are admitted in a refutation-based derivation system.

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Omitting Types in Fuzzy Predicate Logics

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Abstract. The paper describes in detail the omitting types theory in two predicate fuzzy logic. The first part of this paper studies the omitting types theory in predicate first-order fuzzy logic in narrow sense with evaluated syntax [4] which is based on Łukasiewicz MV-algebra. The second part describes this theory in basic fuzzy predicate logic [3] which is based on linearly ordered BL-algebra. The way of presentation is heavily influenced by the classical book on mathematical logic, especially by C.C. Chang and H.J. Keisler [1].

Keywords: Łukasiewicz MV-algebra, BL-algebra, predicate fuzzy logic with evaluated syntax, basic fuzzy predicate logic, model theory, completeness in predicate fuzzy logic with evaluated syntax and in basic fuzzy predicate logic.

The starting point of our discussion is the notation of a set of formulas in BL-logic and a fuzzy set of formulas in fuzzy logic with evaluated syntax. The following is a precise definitions:

Definition 1. By $\Sigma_L(x_1,\dots,x_n)$ we denote a set of evaluated formulas (a fuzzy set of formulas) from the language J such that each formula
$A(x_1,\dots,x_n) \in \Sigma_L(x_1,\dots,x_n)$ has all its free variables among $x_1,\dots,x_n$.

By $\Sigma_{BL}(x_1,\dots,x_n)$ we denote a set of formulas of the language J such that each formula has all its free variables among $x_1,\dots,x_n$.

Realization and omitting of Σ_L

Definition 2. (Realization of Σ_L) Let Σ_L be a fuzzy set introduced above and let $\mathcal{V}$ be a structure for the language J. We say that Σ_L is *realized* in the structure $\mathcal{V}$ *in s degree* $c > 0$ if there is n-tuple $v_1,\dots,v_n \in V$ of elements such that

$$\mathcal{V}(A(^{v_1}/x_1,\dots,^{v_n}/x_n)) \geq c \vee \Sigma_L(A) \tag{1}$$

holds for each evaluated formula $A \in_a \Sigma_L$.

When is a fuzzy set of formulas realized in some non-zero degree by some model of a fuzzy theory T?

Lemma 1. *Let T be a fuzzy theory and let $\Sigma_L(x_1,\dots,x_n)$ be as above. The following are equivalent:*

- *T has a model which realizes Σ_L in a degree $c > 0$, $c \geq \check{a}_\Sigma$.*
- *Every finite subset of Σ_L is realized in some model of T in a degree $d > 0$, $c \geq d \geq \check{a}_\Sigma$.*

Definition 3. (Omitting Σ_L) Let Σ_L be a fuzzy set introduced above and $\mathcal{V}$ be a structure for the language J. We say that Σ_L is *b-omitted* if to *each* n-tuple $v_1,\dots v_n \in V$ of elements there is a formula $A \in_a \Sigma_L$ such that $b \leq a$ and

$$\mathcal{V}(A(^{v_1}/x_1,\dots,^{v_n}/x_n)) < b. \tag{2}$$

We now take up the question: When is a set Σ_L *omitted* in some degree in some model of a fuzzy theory T? This is a more difficult question, and we need more than the compactness theorem to answer it.

Theorem 1. *Let T be a consistent fuzzy theory with the language J and $\Sigma_L(x_1,\dots,x_n)$ be a set of evaluated formulas which is not isolated in T. Then there exists a model of T which omits $\Sigma_L(x_1,\dots,x_n)$ in some non-zero degree.*

Realization and omitting of Σ_{BL}

Definition 4. (Realization of Σ_{BL}) Let Σ_{BL} be a set introduced
above and $\mathcal{V}$ be a structure for the language J. We say that Σ_{BL} is a *realized* in the safe L-structure $\mathcal{V}$ denoted by $\mathcal{V} \models \Sigma_{BL}(v_1,\dots,v_n)$ if there is n-tuple $v_1,\dots,v_n \in V$ of elements such that for each formula $A \in \Sigma_{BL}$

$$\mathcal{V}(A(v_1/x_1,\dots,v_n/x_n)) = \mathbf{1}. \tag{3}$$

The next lemma answers the question: When is a set of formulas realized by some model of a theory T? Its proof is a simple application of the compactness theorem.

Lemma 2. *Let T be a theory and let $\Sigma_{BL}(x_1,\dots,x_n)$ be as above. The following are equivalent:*

- *T has a model which realizes Σ_{BL}.*
- *Every finite subset of Σ_{BL} is realized in some model of T .*

Definition 5. (Omitting Σ_{BL}) Let Σ_{BL} be a set introduced above and $\mathcal{V}$ be a safe L-structure for the language J. We say that $\mathcal{V}$ *omits* Σ_{BL} if to each n-tuple $v_1,\dots,v_n \in V$ of elements there is a formula $A \in \Sigma_{BL}$ such that

$$\mathcal{V}(A(v_1/x_1,\dots,v_n/x_n)) < \mathbf{1}.$$

We now take up the question: When is a set Σ_{BL} *omitted* in some safe L-model of a theory T?

Theorem 2. (Omitting Types Theorem) *Let T be a consistent theory with countable language J and $\Sigma_{BL}(x_1,\dots,x_n)$ be a set of formulas which is not isolated in T. Then there exists a countable model of T which omits $\Sigma_{BL}(x_1,\dots,x_n)$.*

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A Comprehensive Theory of Evaluating Linguistic Expressions in Fuzzy Logic

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Most applications of fuzzy logic explicitly consider linguistic expressions such as "small, very warm, medium age, more or less high pressure" etc. This is the case especially in fuzzy control, but also in fuzzy decision-making, classification and many other applications which are usually based on fuzzy IF-THEN rules.

All the considered linguistic expressions fall into the class of the, so called, *evaluating linguistic expressions*. This class, though linguistically narrow, is rich enough and encompasses also expressions such as "deep, very intelligent, rather narrow, medium important, very tall, extremely nice, about 100, not too expensive" and many others. Note that we use them very often in natural language since they serve us as an essential tool for evaluation of a performance, quality, satisfaction, etc.

Most of the proposed theories of fuzzy IF-THEN rules do not care too much about the linguistic aspect of these expressions and their semantics is only roughly outlined without too much care about how they are indeed understood by people. Namely, the meaning of evaluating expressions is often characterized only by simple triangular fuzzy sets in the universe of real numbers and most care is focused on the way how is the universe covered by them. For, the rules are used in engineering applications, where the goal is to describe imprecisely a certain function and so, the above linguistic expressions are, in fact, used only as auxiliary labels which help to get orientation in the given task.

However, we are convinced that a careful study of the real semantics of the evaluating expressions is important and can be very helpful also in the purely technical applications because the expert knowledge that is usually an initial source of information uses them. Applications can be expected also in robotics — imagine a robot obeying instructions in natural language. We argue that evaluating linguistic expressions form an essential constituent of the special agenda of fuzzy logic as discussed, e.g., in the books [4, 14].

Our goal in this paper is to analyze structure of the evaluating expressions and especially, provide a formal theory of their semantics. We will demonstrate that they are inherently vague and that their vagueness is always a manifestation of the, more or less hidden, sorites paradox. Hence, the means of formal fuzzy logic seem to be suitable for capturing their semantics.

A further question arises, which kind of a formal logical system should be used. A lot for characterization of the meaning of evaluating expressions has been done in the predicate first-order fuzzy logic with evaluated syntax (see [14]). Let us stress that when modeling semantics of words, it is indispensable to distinguish between their intension and extension of expressions (cf. [3, 7]). Because of the simplicity of evaluating expressions, the concepts of intension and extension can be somehow captured using the above logic. However, we want our theory to provide a potential to be included in a theory of a wider part of natural language semantics. Therefore, we prefer the means of *fuzzy type theory* [13]. Besides other advantages, it enables us to formulate explicitly behavior of the evaluating expressions in various contexts (in predicate logic, this is only implicit) and has a potential for further development including the generalized (fuzzy) quantifiers.

A concept of great significance in our theory is that of *fuzzy equality* (fuzzy equivalence; fuzzy similarity). This is an imprecise equality using which we may characterize various degrees of similarity between objects. Note that the role of such relations in modeling of the linguistic semantics has been raised already in [11] where a related concept of the, so called, *indiscernibility relation* has been employed.

We will give reasons and develop a formal theory in which the semantics of *all* evaluating expressions can be *uniquely* characterized using the fuzzy equality. This makes the theory transparent and elegant. Let us remark that the theory of evaluating linguistic expressions has been also supported by the psycholinguistic investigation [8].

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Fuzzy Logics and Substructural Logics

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1 Introduction

Substructural logics are originally defined as logics which lack some of structural rules if we formulate them in Gentzen's sequent systems. Now, substructural logics are regarded as logics of residuated structures, as the recent development of algebraic study of substructural logics shows (see [12]).

In the following, we will focus on either extensions of the substructural logic $\mathbf{FL_e}$ or on those of $\mathbf{FL_{ew}}$. The logic $\mathbf{FL_e}$ is formalized in a sequent system obtained from the sequent system **LJ** for intuitionistic logic by deleting both *weakening rules* and *contraction rule* and adding rules for *fusion*. It is sometimes called intuitionistic l inear logic (without exponentials). Also, the logic $\mathbf{FL_{ew}}$ is the sequent system obtained from $\mathbf{FL_e}$ by adding weakening rules. The logic $\mathbf{FL_{ew}}$ is introduced and its syntactic and algebraic properties are studied in Ono-Komori [13]. We note that $\mathbf{FL_{ew}}$ is equivalent to *monoidal logic* introduced by U. Höhle, which is characterized semantically by the class of all *commutative, integral residuated lattices*. Therefore, fuzzy logics can be regarded as a particular class of substructural logics. Here, by *fuzzy logics* we mean extensions of Hájek's basic logic **BL**, or sometimes those of Esteva-Godo's monoidal t-norm logic **MTL**.

Our purpose of the present paper is to select several topics of substructural logics which are relevant to fuzzy logics, and to give a brief survey of them from both proof-theoretic and algebraic point of view. For general information on fuzzy logics and many-valued logics, see [5] and [2].

2 Sequent systems for substructural logics

Let $\mathbf{FL_e}$ be the sequent system obtained from **LJ** by deleting contraction rule and weakening rules:

$$\frac{\alpha,\alpha,\Gamma\Rightarrow\theta}{\alpha,\Gamma\Rightarrow\theta}\ (\text{contraction})$$

$$\frac{\Gamma\Rightarrow\theta}{\alpha,\Gamma\Rightarrow\theta}\ (\text{left-weakening}) \qquad \frac{\Gamma\Rightarrow}{\Gamma\Rightarrow\alpha}\ (\text{right-weakening})$$

and then adding the following rules for *fusion*:

$$\frac{\Gamma\Rightarrow\alpha \quad \Delta\Rightarrow\beta}{\Gamma,\Delta\Rightarrow\alpha\cdot\beta}\ (\Rightarrow\cdot) \qquad \frac{\alpha,\beta,\Gamma\Rightarrow\theta}{\alpha\cdot\beta,\Gamma\Rightarrow\theta}\ (\cdot\Rightarrow)$$

It is convenient to introduce also two constants 1 and 0. The constant 1 will behave as the unit for fusion, and 0 is used for defining the negation $\neg\alpha$ of a formula α by $\alpha\to 0$. We assume the following initial sequents and rules for them:

$$\Rightarrow 1 \qquad 0\Rightarrow$$

$$\frac{\Gamma \Rightarrow \theta}{1, \Gamma \Rightarrow \theta} \ (1 \text{ weakening}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0} \ (0 \text{ weakening})$$

We can introduce also constants for the top $\top$ and the bottom $\bot$ by taking the following initial sequents:

$$\Gamma \Rightarrow \top \qquad \bot, \Gamma \Rightarrow \gamma$$

The logic $\mathbf{FL_{ew}}$ is defined by the sequent system obtained from $\mathbf{FL_e}$ by adding the above left and right weakening rules. Clearly, in $\mathbf{FL_{ew}}$ both 1- and 0-weakening rules become redundant, and moreover 1 and 0 become provably equivalent to $\top$ and $\bot$, respectively. By abuse of language, we identify these logics with corresponding sequent systems.

In the same way as the above, we can introduce a sequent system $\mathbf{CFL_e}$ ($\mathbf{CFL_{ew}}$) which is obtained form the sequent system **LK** for classical logic by deleting both contraction and wekening rules (only contraction rules, respectively). The former is equal to linear logic **MALL** by J.-Y. Girard and the latter is studied by V. Grishin in 70s. They are defined also by adding the law of double negation $\neg\neg\alpha \Rightarrow \alpha$ as an initial sequent to $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$, respectively.

We note here that 0 must be distinguished from $\bot$ in $\mathbf{FL_e}$. To show this, let us define a new negation $\sim \alpha$ by $\alpha \to \bot$, and consider the logic obtained from $\mathbf{FL_e}$ by adding $\sim\sim \alpha \Rightarrow \alpha$ as an initial sequent. Then, left-weakening rule is derivable in it and in fact, the logic is equivalent to $\mathbf{CFL_{ew}}$, not to $\mathbf{CFL_e}$.

It will be necessary here to give a precise definition of *extensions* of $\mathbf{FL_e}$, or *logics over* $\mathbf{FL_e}$. A set of formula **L** is a logic over $\mathbf{FL_e}$ if

1. every formula provable in $\mathbf{FL_e}$ belongs to **L**,
2. if both α and $\alpha \to \beta$ are in **L** then β is also in **L**,
3. if both α and β are in **L** then $\alpha \wedge \beta$ is also in **L**,
4. if α is in **L** then every substitution instance of α is also in **L**.

3 Cut elimination theorem and its consequences

We have the following.

Theorem 1. *Cut elimination theorem holds for any of* $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$.

In the usual proof of cut elimination theorem for **LK** and **LJ**, we need to replace cut rules by *mix rules*. On the other hand, this is not necessary for the above four systems, and thus the proof becomes much easier than that for **LK** or **LJ**, since they lack contraction rules (see [13, 11] for the detail).

We say that a logic **L** has *Craig's interpolation property* if for all formulas α and β, if $\alpha \to \beta$ is provable in **L** then there exists a formula γ such that

1. both $\alpha \to \gamma$ and $\gamma \to \beta$ are provable in **L**,
2. any propositional variable in γ appears in both α and β.

Also, we say that a logic **L** has the *disjuntion property* if for all formulas α and β, if $\alpha \vee \beta$ is provable in **L** then either α or β is provable in **L**. By using the standard proof-theoretic argument, we can show the following results as consequences of cut elimination theorem. The disjunction property of these logics comes from the fact that none of them have right-contraction rule (see [11] for the details).

Theorem 2. *All of logics* $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$ *are decidable, and also have both Craig's interpolation property and the disjunction property.*

Proof-theoretic methods are quite powerful in deriving various logical properties, once a given logic is formulated in a sequent system for which cut elimination theorem holds. But this means at the same time that they can be applied only to a limited class of logics. For instance, we have difficulties of formulating fuzzy logics in sequent systems in general, though hypersequent systems can be introduced in particular case. Thus, semantical approach will be more appropriate to fuzzy logics in studying their logical properties.

4 Algebraic structures for $\mathbf{FL_e}$

We introduce here algebraic structures for extensions of $\mathbf{FL_e}$. A *commutative residuated lattice* (CRL) is an algebra of the form $\langle P; \vee, \wedge, \cdot, 1, \rightarrow, \rangle$ such that

1. $\langle P; \vee, \wedge \rangle$ is a lattice,
2. $\langle P; \cdot, 1 \rangle$ is a commutative monoid,
3. $xy \leq z \Longleftrightarrow x \leq y \rightarrow z$ for all $x, y, z \in P$.

For general information on residuated lattices, see e.g. [6]. By an $\mathbf{FL_e}$-*algebra* we mean a CRL $\mathbf{P}$ with an (*arbitrary*) element $0 \in P$. A CRL $\mathbf{P}$ is *integral* if the unit 1 is at the same time the greatest element of P. An integral CRL is called an $\mathbf{FL_{ew}}$-*algebra* if 0 is moreover the least element. Thus, $\mathbf{FL_{ew}}$-algebras are nothing but *bounded* CRLs having 0 as the least element. From the syntactical point of view, the integrality corresponds to *left-weakening rule* and the asumption that 0 is equal to the least corresponds to *right-weakening rule*.

Suppose that a given algebra $\mathbf{P}$ with the greatest element 1 satisfies the above first two conditions and is complete as a lattice. Then, $\mathbf{P}$ satisfies also the above third condition (*the law of residuation*) iff the monoid operation $\cdot$ is a *left-continuous* t-norm.

A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is *valid* in an $\mathbf{FL_e}$-algebra $\mathbf{P}$ if $v(\alpha_1) \cdots v(\alpha_m) \leq v(\beta)$ holds for every assignment v on $\mathbf{P}$. Then by the standard argument, we can show that for any sequent S, S is provable in $\mathbf{FL_e}$ ($\mathbf{FL_{ew}}$) iff it is valid in all $\mathbf{FL_e}$-algebras ($\mathbf{FL_{ew}}$-algebras, respectively). The completeness result of this kind holds always between every logic over $\mathbf{FL_e}$ and a corresponding variety of $\mathbf{FL_e}$-algebras.

Since the logic $\mathbf{MTL}$ is obtained from $\mathbf{FL_{ew}}$ by adding the prelinearity axiom $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$ as an axiom, none of extensions of $\mathbf{MTL}$ has the disjunction property. Let us introduce here a property, called *Halldén completeness*, which is weaker than the disjunction property. We say that a logic $\mathbf{L}$ is Halldén complete, if for all formulas α and β which have no variables in common, if $\alpha \vee \beta$ is provable in $\mathbf{L}$ then either α or β is provable in $\mathbf{L}$.

H. Kihara has showed recently that a characterization of Halldén complete superintuitionistic logics given by A. Wroński [14] holds also for logics over $\mathbf{FL_{ew}}$. That is, the following holds.

Theorem 3. *The following three conditions are mutually equivalent for any logic* $\mathbf{L}$ *over* $\mathbf{FL_{ew}}$.

1. $\mathbf{L}$ *is Halldén complete,*
2. $\mathbf{L}$ *cannot be represented as the intersection of two incomparable logics,*
3. $\mathbf{L}$ *is characterized by a single well-connected* $\mathbf{FL_{ew}}$-*algebra.*

Here, an $\mathbf{FL_{ew}}$-algebra $\mathbf{P}$ is *well-connected* if for all $x, y \in P$ if $x \vee y = 1$ then either $x = 1$ or $y = 1$. It is clear that every linearly-ordered $\mathbf{FL_{ew}}$-algebra is well-connected. As a corollary, we have the following.

Theorem 4. *All of basic logic* **BL**, *Łukasiewicz logic Ł, Gödel logic* **G** *and product logic* Π *are Halldén complete, while the logic* ŁΠ**G** *which is the intersection of Ł,* Π *and* **G**, *is not Halldén complete.*

For each logic **L** over $\mathbf{FL_e}$, we introduce a relation $\vdash_{\mathbf{L}}$, called the *deducibility relation* of **L** by

$\alpha_1, \ldots, \alpha_m \vdash_{\mathbf{L}} \beta \iff v(\alpha_i) \geq 1$ for each i implies $v(\beta) \geq 1$ holds for every assignment v on every $\mathbf{FL_e}$-algebra **P** which validates **L**.

Each deducibility relation is a consequence relation in the sense of abstract algebraic logic. We can show the *algebraization* of the deducibility relation and also the following *local deduction theorem* (see [4]).

Theorem 5. *Let* **L** *be any logic over* $\mathbf{FL_e}$. *For all* Γ, α *and* β, $\Gamma, \alpha \vdash_{\mathbf{L}} \beta$ *iff* $\Gamma \vdash_{\mathbf{L}} (\alpha \wedge 1)^n \to \beta$ *for some* $n \geq 0$.

We note that while the provability in $\mathbf{FL_e}$ is decidable as shown in Theorem 2, the deducibility in $\mathbf{FL_e}$ is undecidable.

5 Finite model property and finite embeddability property

A useful semantical method of showing the decidability of a logic **L** is to prove the *finite model property* (FMP), i.e. to prove that **L** is characterized by a class of *finite* algebras. In other words, **L** has the FMP iff the variety $V(\mathbf{L})$ of $\mathbf{FL_e}$-algebras determined by **L** is generated by its finite members. By Harrop's result, **L** is decidable if it is finitely axiomatizable and has the FMP. In the study of modal logic, to show the FMP is the most powerful and successful method in proving the decidability. On the other hand, it is not easy to show the FMP of a given substructural logic. For instance, whether $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ have the FMP or not remained open until the middle of 90s while their decidability is shown already in 80s as an easy consequence of cut elimination. Strangely enough, cut elimination results are used in showing the FMP in proofs by Lafont [9] and Okada-Terui [10], who have solved problems affirmatively.

Then, Blok-van Alten [1] introduced a purely algebraic method of proving the FMP. We say that a class K of $\mathbf{FL_e}$-algebras has the *finite embeddability property* (FEP), if every finite *partial* algebra of some member of K can be embedded into some finite member of K. The FEP induces a stronger consequence than the FMP. In fact, if a class K has the FEP then every universal sentence that fails in K will fail in a finite member of K. Therefore, if K is moreover finitely axiomatizable then the universal theory is decidable. In [1], the following is shown.

Theorem 6. *The variety* FL_{ew} *of all* $\mathbf{FL_{ew}}$*-algebras has the FEP, while the variety* FL_e *of all* $\mathbf{FL_e}$*-algebras doesn't.*

Let us apply their proof to subvarieties of FL_{ew} satisfying equations for the prelinearity $(x \to y) \vee (y \to x) = 1$, the pseudo-complementation $x \wedge \neg x = 1$ and the involution $\neg\neg x = x$. Then, we can show the FEP not only of each of these three varieties but also of a variety satisfying any combination of these three equations (in Kowalski-Ono in an unpublished note, 2001). Thus, we have the following.

Theorem 7. *Every extension of* $\mathbf{FL_{ew}}$ *obtained by adding any combination of the prelinearity axiom, the pseudo-complementation axiom* $\neg(\alpha \wedge \neg\alpha)$ *and the involution axiom (i.e. the law of double negation) is decidable.*

6 Almost maximal logics — logics just below classical logic

It is easy to see that among consistent logics over $\mathbf{FL_{ew}}$ classical logic **CL** is the greatest logic. Then what logics (over $\mathbf{FL_{ew}}$) will come just below **CL**? To see this, let us say that a logic **L** over $\mathbf{FL_{ew}}$ is *almost maximal* if **L** is strictly weaker than **CL** and moreover there exists no consistent logics except **CL** that are strictly stronger than **L**. Among logics over intuitionistic logic there exists a single almost maximal logic $\mathbf{H_3}$ which is characterized by the 3-valued Heyting algebra. Also, Y. Komori [7] gave a complete list of almost maximal logics over Ł that are countably many. Another interesting example of almost maximal logics is product logic Π [3].

Let us call $\alpha^n \to \alpha^{n+1}$, the *n-potent* axiom. The 1-potent axiom is no other than the axiom of contraction. M. Ueda (2000) with T. Kowalski showed the following [8].

Theorem 8. *There exist exactly six almost maximal logics over* **MTL** *with the 2-potent axiom, and there exist uncountably many almost maximal logics over* **MTL** *with the 3-potent axiom.*

In contrast with this, Y. Katou (2001) proved the following.

Theorem 9. *Almost maximal logics over* **BL** *consist of* $\mathbf{H_3}$, Π *and almost maximal logics over Ł.*

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Semi-Linear Spaces and Their Bases

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Introduction

The notion of a linear space is one of the central notions in mathematics and its applications. Therefore, generalization of the linear space to a weaker structure, such as commutative monoid or semiring, is of certain interest. From the application point of view, these spaces may be suitable for solving semi-linear equations and systems of semi-linear equations with fuzzy coefficients.

Main Definitions

We will use semirings, BL-algebras ([3]) and dual BL-algebras ([5]) as underlying structures. Generally, a semiring is a set with two associative operations $+$ and $\cdot$ which fulfil the distributive laws. For our purposes, we will require some additional properties.

Definition 1
A semiring $\mathcal{R} = \langle R, +, \cdot, 0_R, 1_R \rangle$ *is an algebraic structure ([1, 2]) such that:*

(i) $\langle R, +, 0_R \rangle$ *is a commutative monoid.*
(ii) $\langle R, \cdot, 1_R \rangle$ *is a monoid.*
(iii) $r \cdot (s + t) = r \cdot s + r \cdot t$ *holds for all* $r, s, t \in R$.
(iv) $0_R \cdot r = r \cdot 0_R = 0_R$ *holds for all* $r \in R$.

A semiring is called *commutative* if $\langle R, \cdot, 1_R \rangle$ is a commutative monoid.

A typical example of a commutative semiring is a set $\mathbb{N}$ of non-negative integers with addition and multiplication. Below, we will use semirings which can be taken as reducts of BL-algebras or MV-algebras (see [1, 4]).

The following definition of a semimodule is taken from J. S. Golan [2].

Definition 2
Let $\mathcal{R} = \langle R, +, \cdot, 0_R, 1_R \rangle$ *be a semiring. A left* $\mathcal{R}$*-semimodule is a commutative monoid* $\mathcal{A} = \langle A, +_A, 0_A \rangle$ *for which there is defined an external multiplication* $R \times A \longrightarrow A$ *denoted by* ra, *which for all* $r, r' \in R$ *and* $a, a' \in A$ *satisfies the following equalities:*

(i) $(r \cdot r')a = r(r'a)$,
(ii) $r(a +_A a') = ra +_A ra'$,
(iii) $(r + r')a = ra +_A r'a$,
(iv) $1_R\, a = a$,
(v) $0_R\, a = r\, 0_A = 0_A$.

The definition of a *right* $\mathcal{R}$-semimodule is analogous, where the external multiplication is defined as a function $A \times R \longrightarrow A$. An $\mathcal{R}$*-bisemimodule* is a both right and left $\mathcal{R}$-semimodule, i.e. it satisfies the equality $(ra)r' = r(ar')$.

Definition 3

Let semiring $\mathcal{R}$ be a reduct of a BL-algebra $\mathcal{L}$ or dual BL-algebra $\mathcal{L}_d$. Then a semimodule over $\mathcal{L}$ ($\mathcal{L}_d$) is called a semilinear space.

The following are examples of semimodules and semilinear spaces over respective algebras and their reducts.

Example 1

1. *Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a BL-algebra on L, $\mathcal{L}^{\vee} = \langle L, \vee, *, 0, 1\rangle$ its semiring reduct. Let us consider the set of all n-dimensional vectors $A = L^n$, $n \geq 1$, and define*

$$(a_1,\dots,a_n) + (b_1,\dots,b_n) = (a_1 \vee b_1, \dots a_n \vee b_n),$$
$$p \cdot (a_1,\dots,a_n) = (p * a_1,\dots,p * a_n)$$

where $p \in L$. The neutral element in A is the vector $(0,\dots,0)$.

2. *Let $\mathcal{L} = \langle L, \vee, \wedge, \oplus, \ominus, 0, 1\rangle$ be a dual BL-algebra on L, $\mathcal{L}^{\wedge} = \langle L, \wedge, \oplus, 1, 0\rangle$ its semiring reduct. Let us take the set of all n-dimensional vectors $A = L^n$, $n \geq 1$, and define*

$$(a_1,\dots,a_n) + (b_1,\dots,b_n) = (a_1 \wedge b_1, \dots a_n \wedge b_n),$$
$$p \cdot (a_1,\dots,a_n) = (p \oplus a_1,\dots,p \oplus a_n)$$

where $p \in L$. The neutral element in A is the vector $(1,\dots,1)$.

Linear Dependence and Independence

Let A be some left semi-linear space over a BL-algebra $\mathcal{L}$ or a dual BL-algebra $\mathcal{L}_d$. By a linear combination of vectors $a_1,\dots,a_n \in A$ we mean the following expression

$$\alpha_1 a_1 + \cdots + \alpha_n a_n$$

where $\alpha_1,\dots,\alpha_n \in R$ are scalars called also coefficients. This linear combination uniquely determines a certain vector from A.

Definition 4

By the definition, a single vector a is linearly independent. Vectors $a_1,\dots,a_n$, $n \geq 2$, are linearly independent if none of them can be represented by a linear combination of the others.

Otherwise, we say that vectors $a_1,\dots,a_n$ are linearly dependent.

An infinite set of vectors is linear independent if any finite subset of it is linear independent.

Definition 5

A linear independent set of generators of a semi-linear space A is called a basis of A.

The following theorem describes coefficients of a linear combination of vectors $\mathbf{a}_1,\dots,\mathbf{a}_m \in L^n$ which expresses a vector $\mathbf{b}$ provided that the latter is expressible by at least one of such combinations.

Theorem 1

Let $A = L^n$ be the semi-linear space of n-dimensional vectors over $\mathcal{L}^{\vee}$ where $\mathcal{L}$ is a BL-algebra. Let vector $\mathbf{b} \in L^n$ be represented by a linear combination of vectors $\mathbf{a}_1,\dots,\mathbf{a}_m \in L^n$. Then $\mathbf{b}$ can be represented by the linear combination of $\mathbf{a}_1,\dots,\mathbf{a}_m$ with coefficients

$$\hat{x}_i = \bigwedge_{j=1}^{n} (a_{ij} \rightarrow b_j), \quad i = 1,\dots,m. \tag{1}$$

It is worth noticing that if a vector $\mathbf{b} \in L^n$ can be represented by a linear combination of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ then the representation is not necessarily unique.

Corollary 1
Let $A = L^n$ be the semi-linear space of n-dimensional vectors over $\mathcal{L}^\vee$ where $\mathcal{L}$ is a BL-algebra. Then the zero vector $\mathbf{0} = (0, \dots, 0) \in L^n$ is representable by the linear combination of arbitrary vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ with the respective coefficients

$$\hat{x}_i = \bigwedge_{j=1}^{n} \neg a_{ij}, \quad i = 1, \dots, m. \tag{2}$$

By the criterion, suggested below, it is possible to investigate whether the given system of vectors is linear independent.

Theorem 2
Let $A = L^n$ be the semi-linear space of n-dimensional vectors over $\mathcal{L}^\vee$ where $\mathcal{L}$ is a BL-algebra. Vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$, $m \geq 2$, are linearly independent if and only if

$$(\forall l \in \{1, \dots, m\})(\exists i \in \{1, \dots, n\}) \left(a_{li} \not\leq \bigvee_{j=1, j \neq l}^{m} a_{ji} * \left(\bigwedge_{k=1}^{n} a_{jk} \to a_{lk} \right) \right). \tag{3}$$

Corollary 2
Let $A = L^n$ be the semi-linear space of n-dimensional vectors over $\mathcal{L}^\vee$ and $\mathcal{L}$ be a linearly ordered BL-algebra. Vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ are linearly independent if and only if

$$(\forall l \in \{1, \dots, m\})(\exists i \in \{1, \dots, n\}) \left(a_{li} > \bigvee_{j=1, j \neq l}^{m} a_{ji} * \left(\bigwedge_{k=1}^{n} a_{jk} \to a_{lk} \right) \right). \tag{4}$$

Let us remind that in the case of a linear space, we distinguish linearly dependent and linearly independent vectors by analyzing coefficients of their linear combinations leading to zero vectors. As we will see below, this characterization is unhelpful in the case of semi-linear spaces where we care about the expressibility property. To exemplify this claim, let us take the reduct $\mathcal{L}^\vee$ of Łukasiewicz algebra on $[0,1]$ and for $a \in (0,1)$ consider the following set of linearly independent vectors from L^n :

$$\begin{aligned} \mathbf{a}_1 &= (a, 0, 0, \dots, 0) \\ \mathbf{a}_2 &= (0, a, 0, \dots, 0) \\ &\dots\dots\dots\dots \\ \mathbf{a}_n &= (0, 0, 0, \dots, a). \end{aligned} \tag{5}$$

It is easy to see that the linear combination

$$\neg a\mathbf{a}_1 \vee \dots \vee \neg a\mathbf{a}_n = \mathbf{0}$$

with non-zero coefficients $\neg a$ gives the zero vector.

On the other hand, the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{a}_1 + \mathbf{a}_2$ are linearly dependent, and again, their linear combination with all coefficients equal to $\neg a$ gives the zero vector. Therefore, independently on the fact whether the vectors are linearly dependent or not (in the sense of our definition), their linear combination with non-zero coefficients may be *equal to the zero vector*. Note that this may happen if at least one of the coefficients given by (2) is non-zero.

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Necessary and Sufficient Conditions for Powersets in Set and Set $\times$ C to Form Algebraic Theories

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1 Motivation

Definition 1. *An* ***algebraic theory*** *[9]* ***(in clone form)*** *in a* ***ground*** *category $\mathcal{C}$ is an ordered triple $\mathbf{T} = (T, \eta, \diamond)$ specified by the following data and axioms:*

D1. $T : |\mathcal{C}| \to |\mathcal{C}|$ *is an object function on* $\mathcal{C}$.
D2. η *assigns to each* $A \in |\mathcal{C}|$ *a* $\mathcal{C}$ *morphism* $\eta_A : A \to T(A)$.
D3. $\diamond$ *assigns to each pair of* $\mathcal{C}$ *morphisms,* $f : A \to T(B), g : B \to T(C)$, *a* $\mathcal{C}$ *morphism* $g \diamond f : A \to T(C)$.
A1. $\diamond$ *is associative, i.e. for each* $f : A \to T(B), g : B \to T(C), h : C \to T(D)$,

$$h \diamond (g \diamond f) = (h \diamond g) \diamond f$$

A2. η *furnishes identities, i.e. for each* $f : A \to T(B)$,

$$\eta_B \diamond f = f$$

A3. $\diamond$ *is compatible with the composition* $\circ$ *of* $\mathcal{C}$ *morphisms, i.e. given* $f : A \to B, g : B \to T(C)$, *and setting* $f^\Delta : A \to T(B)$ *by*

$$f^\Delta = \eta_B \circ f$$

then it is the case that

$$g \diamond f^\Delta = g \circ f$$

Remark 1. [9]. The following hold:

1. η furnishes two-sided identities in (A2).
2. $\mathbf{T}$ induces a new category $\mathcal{C}_\mathbf{T}$, the **Kleisli category of T**, with $|\mathcal{C}_\mathbf{T}| = |\mathcal{C}|$, the morphisms are $\mathcal{C}$ morphisms of the form $f : A \to T(B)$, the composition is $\diamond$, and the identities are the components of η.
3. Each $\mathcal{C}$ morphism $f : A \to B$ lifts to a $\mathcal{C}$ morphism $T(f) : T(A) \to T(B)$ by $T(f) = f^\Delta \diamond id_{T(A)}$. In fact, $T : \mathcal{C} \to \mathcal{C}$ is a functor and η is a natural transformation from $id_\mathcal{C}$ to T. In the sequel, we also write $f_\mathbf{T}^\to = T(f) = f^\Delta \diamond id_{T(A)}$, since in many applications this lifting is acting as an image operator between powersets.

Example 1. [9]. Each semigroup S induces an algebraic theory $\mathbf{T} = (T, \eta, \diamond)$ in **Set** as follows: $T : |\mathbf{Set}| \to |\mathbf{Set}|$ by $A \mapsto T(A)$, where $T(A)$ is the family of all finite, ordered, grouping-symbol-free strings of variables from A taking values in S; $\eta_A : A \to T(A)$ by insertion of variables $a \mapsto a$; and given $f : A \to T(B), g : B \to T(C)$, we set $g \diamond f : A \to T(C)$ by $g \diamond f = g^\# \circ f$, where $g^\# : T(B) \to T(C)$ by the concatenation $b_1 \ldots b_n \mapsto g(b_1) \ldots g(b_n)$.

Example 2. [9]. Traditional powersets collectively form an algebraic theory $\mathbf{T} = (T, \eta, \diamond)$ in $\mathcal{C} = \mathbf{Set}$ as follows: $T : |\mathbf{Set}| \to |\mathbf{Set}|$ by $T(X) = \wp(x)$; $\eta_X : X \to \wp(X)$ by $\eta(x) = \{x\}$; and given $f : X \to \wp(Y)$, $g : Y \to \wp(Z)$, put $g \diamond f : X \to \wp(Z)$ by $(g \diamond f)(x) = \bigcup_{y \in f(x)} g(y)$. The image powerset operator $f^{\to} : \wp(X) \to \wp(Y)$ of a function $f : X \to Y$ is generated as $f^{\Delta} \diamond id_{T(A)} : \wp(X) \to \wp(Y)$ (see Remark 1.2(3))—i.e. $f^{\to} = f^{\to}_{\mathbf{T}} = T(f)$, from which image operator the Adjoint Functor Theorem (AFT) for the partially-ordered case then gives the all-important pre-image operator $f^{\leftarrow} : \wp(X) \leftarrow \wp(Y)$. Since $\mathbf{T}$ generates the image powerset operator of a function via the construction $T(f) = f^{\Delta} \diamond id_{T(A)}$, we say that $\mathbf{T}$ generates the traditional "powerset theory" or that the traditional "powerset theory" is algebraicly generated. In the sequel, the language *an algebraic theory generates a "powerset theory"* or *a "powerset theory" is algebraicly generated* means that the lifting of $\mathcal{C}$ morphisms f by $f^{\to}_{\mathbf{T}} = T(f) = f^{\Delta} \diamond id_{T(A)}$ in Remark 1.2(3) coincides with the image operator of that "powerset theory".

Question 1. Are there conditions under which the powersets occuring in fuzzy sets form an algebraic theory in **Set** in the fixed-basis case or an algebraic theory in $\mathbf{Set} \times \mathbf{C}$ in the variable-basis case, and are these conditiions both necessary and sufficient?

2 Fixed-Basis Powersets

Definition 2. *A **complete quasi-monoidal lattice (cqml)** $(L, \leq, \otimes)$ is a complete lattice $(L, \leq)$ equipped with a **tensor product** $\otimes : L \times L \to L$ isotone in both variables and with $\top$ idempotent; and the category **Cqml** comprises all cqml's together with mappings preserving arbitrary $\bigvee$, $\otimes$, and $\top$ [7], [16]. If additionally $\otimes$ is associative and distributes across arbitrary $\bigvee$ from both sides (implying $\bot$ is a two-sided zero), then $(L, \leq, \otimes)$ is a **quantale** [10], [19]. Finally, $(L, \leq, \otimes)$ is a **strictly two-sided (sts)** quantale if it is a quantale for which $\top$ is a two-sided identity [7].*

Lemma 1. ***(characterization lemma)**. Let $(L, \leq, \otimes) \in |\mathbf{Cqml}|$ and let $\mathbf{T} = (T, \eta, \diamond)$ be given by the following data:*

D1. $T : |\mathbf{Set}| \to |\mathbf{Set}|$ *by* $T(X) = L^X$.
D2. $\forall X \in |\mathbf{Set}|$, η *determines* $\eta_X : X \to L^X$ *given by*

$$\eta_X(x) = \chi_{\{x\}}$$

D3. $\forall f : X \to L^Y, g : Y \to L^Z$ *in **Set**, put* $g \diamond f : X \to L^Z$ *by*

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) \otimes (g(y))(z)]$$

*Then **T** is an algebraic theory in **Set** if and only if $(L, \leq, \otimes)$ is a sts-quantale.*

Remark 2. (**doubling theories**). Since $\otimes$ is generally not commutative, it follows that the tensor products appearing in the definition of the clone composition in (D3) of the Characterization Lemma are ordered according to our choice. Restated, the clone composition

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) \otimes (g(y))(z)]$$

could also be chosen as

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(g(y))(z) \otimes (f(x))(y)]$$

This yields an alternative clone composition and therefore an alternative theory **T**. Let us denote the theory presented in the Characterization 5.1.1 by $\mathbf{T}_1 = (T, \eta, \diamond_1)$ and the alternative theory by $\mathbf{T}_2 = (T, \eta, \diamond_2)$. We have then the following corollary:

Corollary 1. *The following are equivalent:*

1. $\mathbf{T}_1 = (T, \eta, \diamond_1)$ *is an algebraic theory in* ***Set****.*
2. $(L, \leq, \otimes)$ *is a sts-quantale.*
3. $\mathbf{T}_2 = (T, \eta, \diamond_2)$ *is an algebraic theory in* ***Set****.*

Theorem 1. ***(algebraic generation of fixed-basis "powerset theories")****. Let* $(L, \leq, \otimes)$ *be a sts-quantale. Then the algbraic theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *each lift* $f : X \to Y$ *in* ***Set*** *to the same* $T(f) : L^X \to L^Y$ *via* $f^{\Delta} \diamond id_{T(A)}$ *(Remark 1.2(3); and further,* $f^{\rightarrow}_{\mathbf{T}_1} = f^{\rightarrow}_{\mathbf{T}_2} = T(f) = f^{\rightarrow}_L$ *(the standard Zadeh image operator).*

Remark 3. This theorem shows that two different algebraic theories in **Set** can "generate" the same "powerset" theory in **Set** for lattice-valued mathematics, and in particular, they generate the standard image operator of Zadeh. It should be noted that a special case of sufficiency of the Characterization Lemma—when L is a locale with $\otimes = \wedge$—appears in [9] along with the corresponding special case of the above theorem.

3 Variable-Basis Powersets

Finding necessary and sufficient conditions under which the powersets in lattice-valued mathematics form an algebraic theory in a variable-basis ground category of the form $\mathcal{C} = \mathbf{Set} \times \mathbf{C}$—$\mathbf{C}$ a subcategory of $\mathbf{Loqml} \equiv \mathbf{Cqml}^{op}$—is more delicate than for the fixed-basis case. The previous section may be viewed as a special case of this section by setting $\mathbf{C} = \{L\}$ (with the identity morphism), where $L \in |\mathbf{Loqml}|$.

We fix some notation. Recall $\mathbf{Set} \times \mathbf{C}$ has: objects (X, L), with $X \in |\mathbf{Set}|$ and $L \in |\mathbf{C}|$; morphisms $(f, \phi) : (X, L) \to (Y, M)$, with $f : X \to Y$ in **Set** and $\phi : L \to M$ in **C**, i.e. $\phi^{op} : L \leftarrow M$ a concrete morphism in $\mathbf{C}^{op} \subset \mathbf{Cqml}$; and the product composition and identities.

Initially, we shall give necessary and sufficient conditions on $\mathbf{C} \subset \mathbf{Loqml}$ for "right-adjoint" theories $\mathbf{T}(\vdash)$ and "left-adjoint" theories $\mathbf{T}(\dashv)$ (both defined below) to be algebraic in $\mathbf{Set} \times \mathbf{C}$; and we also give sufficient conditions on $\mathbf{C} \subset \mathbf{Loqml}$ for "adjoint" theories $\mathbf{T}(*)$ (also defined below) to be algebraic in $\mathbf{Set} \times \mathbf{C}$. Then, analogous to the fixed-basis case, we shall "double" each of these theories to obtain four theories $\mathbf{T}_{1,2}(\vdash)$ and $\mathbf{T}_{1,2}(\dashv)$, for each of which there are necessary and sufficient conditions on $\mathbf{C} \subset \mathbf{Loqml}$ making that theory algebraic in $\mathbf{Set} \times \mathbf{C}$, as well as two theories $\mathbf{T}_{1,2}(*)$, for each of which there are sufficient conditions on $\mathbf{C} \subset \mathbf{Loqml}$ making that theory algebraic in $\mathbf{Set} \times \mathbf{C}$.

With regard to variable-basis "powerset theories"—underlying lattices *and* underlying sets both change from powerset to powerset, together with morphisms from $\mathbf{Set} \times \mathbf{C}$ *and any possible (forward) image operators between powersets*—we are surprised to find that the left-adjoint theories $\mathbf{T}_{1,2}(\dashv)$ generate the "powerset theory" first presented in [11] and extensively studied in [12], [13], [14], [15]. On the other hand, the right-adjoint theories $\mathbf{T}_{1,2}(\vdash)$ algebraicly generate a *new* variable-basis "powerset theory" in $\mathbf{Set} \times \mathbf{C}$ with a *new* image operator for variable-basis fuzzy sets.

3.1 Necessary and sufficient conditions for algebraic theories $\mathbf{T}_{1,2}(\vdash)$ and $\mathbf{T}_{1,2}(\dashv)$ in $\mathbf{Set} \times \mathbf{C}$

Definition 3. *Let $\phi : L \to M$ in **Loqml**. Then we define $\phi^{\dashv} \equiv (\phi^{op})^{\dashv} : L \to M$ in **Set**, $\phi^{\vdash} \equiv (\phi^{op})^{\vdash} : L \to M$ in **Set**, where*

$$\phi^{\dashv}(a) = \bigwedge_{a \leq \phi^{op}(b)} b, \quad \phi^{\vdash}(a) = \bigvee_{\phi^{op}(b) \leq a} b$$

*Note that $\phi^{op} : L \leftarrow M$ is in **Cqml**—this implies $\phi^{\vdash}$ is isotone and*

$$\phi^{op} \dashv \phi^{\vdash}$$

—and note $\phi^{\vdash}$ is denoted ϕ^{*} in [16]. Also note that $\phi^{\dashv}$ is isotone and preserves $\bot$; note that if $\phi^{\dashv}$ preserves arbitrary $\bigvee$—equivalently, ϕ^{op} preserves arbitrary $\bigwedge$, then

$$\phi^{\dashv} \dashv \phi^{op}$$

and note that $\phi^{\dashv}$ is denoted $[\phi]$ in [14], [15]. Observe that given $\phi : L \to M$ and $\psi : M \to N$, we have $(\psi \circ \phi)^{\dashv} = \psi^{\dashv} \circ \phi^{\dashv}$ and $(\psi \circ \phi)^{\vdash} = \psi^{\vdash} \circ \phi^{\vdash}$. Finally, given a set X, $\langle \phi^{\vdash} \rangle : L^X \to M^X$ and $\langle \phi^{\dashv} \rangle : L^X \to M^X$ by $\langle \phi^{\vdash} \rangle (a) = \phi^{\vdash} \circ a$ and $\langle \phi^{\dashv} \rangle (a) = \phi^{\dashv} \circ a$.

Definition 4. *Let $\mathbf{Quant}^*$ [$\mathbf{Quant}^*(\vdash)$, $\mathbf{Quant}^*(\dashv)$] denote the following data:*

Q1. Each object of $\mathbf{Quant}^*$ [$\mathbf{Quant}^*(\vdash)$, $\mathbf{Quant}^*(\dashv)$] is a stsa-quantale.
Q2. Given two objects L, M in $\mathbf{Quant}^*$ [$\mathbf{Quant}(\vdash)$, $\mathbf{Quant}(\dashv)$], $\phi : L \to M$ is a morphism in $\mathbf{Quant}^*$ [$\mathbf{Quant}(\vdash)$, $\mathbf{Quant}(\dashv)$] if the following hold:
   *(a) $\phi : L \to M$ is a morphism in **Loqml**.*
   *(b) $\exists \phi^* : L \to M$ in **Cqml** (this mapping need not be unique) [$\phi^{\vdash} : L \to M$ is in $\mathbf{Cqml}$, $\phi^{\dashv} : L \to M$ is in $\mathbf{Cqml}$].*

Proposition 1. *Each of $\mathbf{Quant}^*$, $\mathbf{Quant}(\vdash)$, $\mathbf{Quant}(\dashv)$ with the composition and identities of **Loqml** becomes a subcategory of **Loqml**.*

Example 3. $\mathbf{Quant}^*$, $\mathbf{Quant}(\vdash)$, $\mathbf{Quant}(\dashv)$ are non-trivial with respect to morphisms: there are morphisms in these categories in addition to identities and isomorphisms.

1. Let $L = \{\bot, a, \top\}$ and $M = \{\bot, \top\}$, and let $\phi^{op} : L \leftarrow M$ be given by $\phi^{op}(\bot) = \bot$, $\phi^{op}(\top) = \top$. Then both $\phi^{\vdash}, \phi^{\dashv}$ preserve each of arbitrary $\bigvee$, arbitrary $\bigwedge$, and $\top$.
   (a) Let L be equipped with $\otimes$ equal to the binary meet. Then both $\phi^{\vdash}, \phi^{\dashv} : L \to M$ are in **Cqml**.
   (b) Let L be equipped with $\otimes$ equal to the binary join. Then both $\phi^{\vdash}, \phi^{\dashv} : L \to M$ are in **Cqml**.
2. Let $M = \{\bot, \alpha, \top\}$ and let $L = \{\bot, a, b, c, \top\}$ be the locale representing the product topology of the Šierpinski topology with itself—$\bot$ is meet-irreducible (i.e. prime) and $\{a, b, c, \top\}$ is the four-point diamond with $\bot \leq a \leq b \leq \top$, $\bot \leq a \leq c \leq \top$, and b, c unrelated. Now let $\phi^{op} : L \leftarrow M$ be given by $\phi^{op}(\bot) = \bot$, $\phi^{op}(\top) = \top$, $\phi^{op}(\alpha) = c$.
   (a) Let L, M be equipped with $\otimes$ equal to the binary meet. Then $\phi^{\dashv} : L \to M$ is in **Cqml**.
   (b) Let L, M be equipped with $\otimes$ equal to the binary join. Then $\phi^{\dashv} : L \to M$ is in **Cqml**.
3. Let M be as in the previous example, and let L be the dual of the L of the previous example. Let ϕ be as given in the previous example.
   (a) Let L, M be equipped with $\otimes$ equal to the binary meet. Then $\phi^{\vdash} : L \to M$ is in **Cqml**.
   (b) Let L, M be equipped with $\otimes$ equal to the binary join. Then $\phi^{\vdash} : L \to M$ is in **Cqml**.

4. Let $L = M = \mathbb{I} \equiv [0,1]$ and let $\phi^{op} : L \leftarrow M$ as follows: $[0,1/4]$ maps homeomorphically to $[0,1/2]$ with $0 \mapsto 0$ and $1/4 \mapsto 1/2$; $[1/4,3/4]$ maps to $\{1/2\}$; and $[3/4,1]$ maps homeomorphically to $[1/2,1]$ with $3/4 \mapsto 1/2$ and $1 \mapsto 1$.
   (a) Let L,M be equipped with $\otimes$ equal to the binary meet. Then $\phi^{\dashv} : L \to M$ is in **Cqml**.
   (b) Let L,M be equipped with $\otimes$ equal to the binary join. Then $\phi^{\dashv} : L \to M$ is in **Cqml**.
   These examples are actually typical of two entire classes of examples.
5. Let X,Y be sets, let $f : X \to Y$ be a surjective function, let N be a complete lattice, put $L = N^X$, $M = N^Y$, $\phi^{op} : L \leftarrow M$ by $\phi^{op} = f_N^{\leftarrow}$. Now let L,M be equipped with $\otimes$ equal to the binary join. Then $\phi^{\dashv} \equiv f_N^{\rightarrow} : L \to M$ is in **Cqml**.

Lemma 2. ***(characterization of right-adjoint theories)****. Let* $\mathbf{C} \subset \mathbf{Loqml}$ *and let* $\mathbf{T}(\vdash) = (T,\eta,\diamond)$ *be the structure given by the following data:*

D1. $T : |\mathbf{Set} \times \mathbf{C}| \to |\mathbf{Set} \times \mathbf{C}|$ *by* $T(X,L) = \left(L^X, L\right)$.
D2. $\forall (X,L) \in |\mathbf{Set} \times \mathbf{C}|$, η *determines* $\eta_{(X,L)} : (X,L) \to \left(L^X, L\right)$ *given by*

$$\eta_{(X,L)} = (\eta_X, id_L)$$

where $\eta_X(x) = \chi_{\{x\}}$, $id_L : L \to L$ *in* $\mathbf{C}$
D3. $\forall (f,\phi) : (X,L) \to \left(M^Y, M\right)$, $\forall (g,\psi) : (Y,M) \to \left(N^Z, N\right)$ *in* $\mathbf{Set} \times \mathbf{C}$*, put* $(g,\psi) \diamond (f,\phi) : (X,L) \to \left(N^Z, N\right)$ *by*

$$(g,\psi) \diamond (f,\phi) = (g \diamond f, \phi \diamond \psi)$$

where

$$\phi \diamond \psi = \phi \circ \psi$$

and

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} \left[\psi^{\vdash}\left(\left(f(x)\right)(y)\right) \otimes \left(g(y)\right)(z)\right]$$

Then $\mathbf{T}(\vdash) = (T,\eta,\diamond)$ is an algebraic theory in $\mathbf{Set} \times \mathbf{C}$ if and only if

$$\mathbf{C} \subset \mathbf{Quant}(\vdash)$$

Lemma 3. ***(characterization of left-adjoint theories)****. Let* $\mathbf{C} \subset \mathbf{Loqml}$ *and let* $\mathbf{T}(\dashv) = (T,\eta,\diamond)$ *be the structure given by the following data:*

D1. $T : |\mathbf{Set} \times \mathbf{C}| \to |\mathbf{Set} \times \mathbf{C}|$ *by* $T(X,L) = \left(L^X, L\right)$.
D2. $\forall (X,L) \in |\mathbf{Set} \times \mathbf{C}|$, η *determines* $\eta_{(X,L)} : (X,L) \to \left(L^X, L\right)$ *given by*

$$\eta_{(X,L)} = (\eta_X, id_L)$$

where

$$\eta_X(x) = \chi_{\{x\}},\ id_L : L \to L \text{ in } \mathbf{C}$$

D3. $\forall (f,\phi) : (X,L) \to \left(M^Y, M\right)$, $\forall (g,\psi) : (Y,M) \to \left(N^Z, N\right)$ *in* $\mathbf{Set} \times \mathbf{C}$*, put* $(g,\psi) \diamond (f,\phi) : (X,L) \to \left(N^Z, N\right)$ *by*

$$(g,\psi) \diamond (f,\phi) = (g \diamond f, \phi \diamond \psi)$$

where

$$\phi \diamond \psi = \phi \circ \psi$$

and

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} \left[\psi^{\dashv}\left(\left(f(x)\right)(y)\right) \otimes \left(g(y)\right)(z)\right]$$

Then $\mathbf{T}(\dashv) = (T, \eta, \diamond)$ is an algebraic theory in $\mathbf{Set} \times \mathbf{C}$ if and only if

$$\mathbf{C} \subset \mathbf{Quant}(\dashv)$$

Lemma 4. ***(existence of adjoint-like theories)***. *Let $\mathbf{C} \subset \mathbf{Loqml}$ and let $\mathbf{T}(*) = (T, \eta, \diamond)$ be the structure given by the following data:*

D1. $T : |\mathbf{Set} \times \mathbf{C}| \to |\mathbf{Set} \times \mathbf{C}|$ *by* $T(X, L) = \left(L^X, L\right)$.
D2. $\forall (X, L) \in |\mathbf{Set} \times \mathbf{C}|$, η *determines* $\eta_{(X,L)} : (X, L) \to \left(L^X, L\right)$ *given by*

$$\eta_{(X,L)} = (\eta_X, id_L)$$

where

$$\eta_X(x) = \chi_{\{x\}},\ id_L : L \to L \text{ in } \mathbf{C}$$

D3. $\forall (f, \phi) : (X, L) \to \left(M^Y, M\right), \forall (g, \psi) : (Y, M) \to \left(N^Z, N\right)$ *in* $\mathbf{Set} \times \mathbf{C}$, *put* $(g, \psi) \diamond (f, \phi) : (X, L) \to \left(N^Z, N\right)$ *by*

$$(g, \psi) \diamond (f, \phi) = (g \diamond f, \phi \diamond \psi)$$

where

$$\phi \diamond \psi = \phi \circ \psi$$

and

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [\psi^*((f(x))(y)) \otimes (g(y))(z)]$$

providing a function $\psi^* : L \to M$ exists and assuming that $(\ )^*$ chooses a unique such morphism.

Then $\mathbf{T} = (T, \eta, \diamond)$ is an algebraic theory in $\mathbf{Set} \times \mathbf{C}$ if $\mathbf{C} \subset \mathbf{Quant}^*$.

Remark 4. (**doubling theories**). Since $\otimes$ is generally not commutative, it follows that the tensor products appearing in the definition of the clone compositions in (D3) of Lemmas 6.1.5.1, 6.1.5.2, 6.1.5.3 are ordered according to our choice. As in Remark 5.1.2, different clone compositions could be chosen by reversing these tensor products; e.g., in the case of Lemma 6.1.5.1,

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} \left[\psi^{\vdash}((f(x))(y)) \otimes (g(y))(z)\right]$$

could also be chosen as

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} \left[(g(y))(z) \otimes \psi^{\vdash}((f(x))(y))\right]$$

This yields for each of $\mathbf{T}(\vdash), \mathbf{T}(\dashv), \mathbf{T}(*)$ an alternative clone composition and therefore an alternative theory. Let us denote the theory presented in Lemma 6.1.5.1 by $\mathbf{T}_1(\vdash) = (T, \eta, \diamond_1)$ and the alternative theory by $\mathbf{T}_2(\vdash) = (T, \eta, \diamond_2)$, the theory presented in Lemma 6.1.5.2 by $\mathbf{T}_1(\dashv) = (T, \eta, \diamond_1)$ and the alternative theory by $\mathbf{T}_2(\dashv) = (T, \eta, \diamond_2)$, and the theory presented in Lemma 6.1.5.3 by $\mathbf{T}_1(*) = (T, \eta, \diamond_1)$ and the alternative theory by $\mathbf{T}_2(*) = (T, \eta, \diamond_2)$. We have then the following corollary:

Corollary 2. *The following statements hold:*

I. The following statements are equivalent:

(a) $\mathbf{T}_1(\vdash)$ [$\mathbf{T}_1(\dashv)$] *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$.

(b) $\mathbf{C}\subset\mathbf{Quant}(\vdash)$ [$\mathbf{Quant}(\dashv)$].

(c) $\mathbf{T}_2(\vdash)$ [$\mathbf{T}_2(\dashv)$] *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$.

II. *The condition* $\mathbf{C}\subset\mathbf{Quant}^*$ *is sufficient for each of the following statements:*

(a) $\mathbf{T}_1(*)$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$.

(b) $\mathbf{T}_2(*)$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$.

3.2 Algebraic generation of "powerset theories" in $\mathbf{Set}\times\mathbf{C}$ from left-adjoint theories

Theorem 2. *Let* $\mathbf{C}\subset\mathbf{Quant}(\dashv)$. *Then the algebraic theories* $\mathbf{T}_1(\dashv)$ *and* $\mathbf{T}_2(\dashv)$ *in* $\mathbf{Set}\times\mathbf{C}$ *both lift* $(f,\phi):(X,L)\to(Y,M)$ *to the same* $T(f,\phi):\left(L^X,L\right)\to\left(M^Y,M\right)$ *and*

$$(f,\phi)^{\to}_{\mathbf{T}_1(\dashv)}=(f,\phi)^{\to}_{\mathbf{T}_2(\dashv)}=T(f,\phi)=\langle\phi^{\dashv}\rangle\circ f^{\to}_L=(f,\phi)^{\to}_{\dashv}$$

where $(f,\phi)^{\to}_{\dashv}$ *is the image operator introduced in [11] and further studied in [12], [13], [14], [15]. Hence each of* $\mathbf{T}_1(\dashv)$ *and* $\mathbf{T}_2(\dashv)$ *algebraicly generates this "powerset theory" in* $\mathbf{Set}\times\mathbf{C}$.

Remark 5. Because of this theorem, the "powerset theory" of [11], [12], [13], [14], [15] is called a *left-adjoint* "powerset theory", and it is algebraicly generated by two different algebraic theories if $\mathbf{C}\subset\mathbf{Quant}(\dashv)$.

Remark 6. Letting $(f,\phi)^{\leftarrow}_{\dashv}$ be the preimage operator of a left-adjoint "powerset theory", the proof that $(f,\phi)^{\to}_{\dashv}\dashv(f,\phi)^{\leftarrow}_{\dashv}$ requires only that the lattices be cqml's and that ϕ^{op} additionally preserve arbitrary $\bigwedge$; indeed, given cqml's, the adjunction is logically equivalent to ϕ being in **Loqml** such that ϕ^{op} preserve arbitrary $\bigwedge$—see [14], [15]. Thus, the powerset theories constructed in the above theorem account for a significant part of left-adjoint "powerset theories", but not for all of them; i.e. there are significant left-adjoint "powerset theories" in $\mathbf{Set}\times\mathbf{C}$ which do not arise from left-adjoint algebraic theories constructed in the previous Section even though they behave like powerset theories algebraicly generated from such algebraic theories. Restated, for each $\mathbf{C}\subset\mathbf{Loqml}$, there is an appropriate preimage operator making **C-Top** and **C-FTop** topological over $\mathbf{Set}\times\mathbf{C}$: some of these preimage operators come from algebraicly generated left-adjoint powerset theories, but most do not; however, the syntax of all these preimage operators is the same, namely $(f,\phi)^{\leftarrow}_{\dashv}(b)=\phi^{op}\circ b\circ f$.

3.3 Algebraic generation of new "powerset theories" in $\mathbf{Set}\times\mathbf{C}$ from right-adjoint theories

This subsection creates new variable-basis "powerset theories" in $\mathbf{Set}\times\mathbf{C}$—dubbed *right-adjoint* "powerset theories" from right-adjoint algebraic theories in $\mathbf{Set}\times\mathbf{C}$. The significance of these new "powerset theories" will be developed in future work.

Theorem 3. *Let* $\mathbf{C}\subset\mathbf{Quant}(\dashv)$. *Then the algebraic theories* $\mathbf{T}_1(\vdash)$ *and* $\mathbf{T}_2(\vdash)$ *in* $\mathbf{Set}\times\mathbf{C}$ *both lift* $(f,\phi):(X,L)\to(Y,M)$ *to the same* $T(f,\phi):\left(L^X,L\right)\to\left(M^Y,M\right)$ *and*

$$(f,\phi)^{\to}_{\mathbf{T}_1(\vdash)}=T(f,\phi)=\langle\phi^{\vdash}\rangle\circ f^{\to}_L=(f,\phi)^{\to}_{\mathbf{T}_2(\vdash)}$$

The image operator induced by these algebraic theories is new and leads to the following definition.

Definition 5. ***(right-adjoint forward/image operators)****. Let* $(f,\phi):(X,L)\to(Y,M)\in\mathbf{Set}\times\mathbf{Loqml}$. *Then* $(f,\phi)^{\to}_{\vdash}:L^X\to M^Y$ *is defined by*

$$(f,\phi)^{\to}_{\vdash}=\left\langle\phi^{\vdash}\right\rangle\circ f^{\to}_{L}$$

i.e. $\forall a\in L^X,\forall y\in Y,[(f,\phi)^{\to}_{\vdash}(a)](y)=\phi^{\vdash}[(f^{\to}_{L}(a))(y)]$.

Lemma 5. *If* $\mathbf{C}\subset\mathbf{Quant}(\vdash)$ *and* $(f,\phi):(X,L)\to(Y,M)\in\mathbf{Set}\times\mathbf{C}$, *then* $(f,\phi)^{\to}_{\vdash}:L^X\to M^Y$ *preserves arbitrary* $\bigvee$.

Proposition 2. *Let* $\mathbf{C}\subset\mathbf{Quant}(\vdash)$ *and* $(f,\phi):(X,L)\to(Y,M)\in\mathbf{Set}\times\mathbf{C}$. *Then* $\exists!\,(f,\phi)^{\leftarrow}_{\vdash}:L^X\leftarrow M^Y$ *such that* $(f,\phi)^{\to}_{\vdash}\dashv(f,\phi)^{\leftarrow}_{\vdash}$.

Remark 7. The previous proposition says that there is a new preimage operator to go with our new image operator. A rigorous definition of redundacy of powerset operators can be given, and if $\mathbf{C}\subset\mathbf{Quant}(\vdash)$, then the right-adjoint powerset operators are not redundant, i.e. they really are new. In future work we shall try to characterize when the new preimage operator preserves arbitrary $\bigvee$ and binary $\otimes$ and thereby serve to make new topological categories over $\mathbf{Set}\times\mathbf{C}$.

Remark 8. It is an open question as to how the work summarized in this abstract relates to recent work on powerset operators in [2], [3], [4].

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A Natural Fuzzy Equivalence Relation

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1 Introduction

We present a natural interpretation of fuzzy equivalence relations as a part of the natural interpretation of fuzzy set theory in a cumulative Heyting valued model for intuitionistic set theory.

With the natural interpretation we can deduce most of the standard equations or inequalities of the basic concepts of fuzzy sets or fuzzy relations ([3]), and we can make clear the meaning of the extension principle by Zadeh ([5]). We can also consider notions such as operations of fuzzy subsets of different universes, fuzzy relations and mappings between fuzzy subsets ([1]).

Here we present a natural interpretation of fuzzy equivalence relations and fuzzy partitions. In the model equivalence relations together with quotient sets and corresponding partitions can be treated in a similar way as in the usual set theory. We can naturally consider fuzzy equivalence relations and corresponding fuzzy partitions, which are mutually one-to-one correspondent (up to similarity).

This paper is an improved version of our previous works [2] [4], in which some technical errors were found recently. By modifying the definition of partition in the model and checking the arguments, we have overcome the defects and verified the main results again. We assume the reader is familiar with the basic notions appeared in either one of our previous papers [1] [3] [5].

2 Equivalence relations in the Heyting valued model

Let H be a complete Heyting algebra with standard operations and constants, V be the class of all crisp sets, and On be the class of all ordinals.

Definition 1 *The H-valued model V^H is constructed as follows.*
For every ordinal α, V^H_α is defined by induction:

$$V^H_0 = \phi, \qquad V^H_\alpha = \bigcup_{\beta<\alpha} V^H_\beta \quad (\textit{if } \alpha \textit{ is a limit ordinal}),$$

$$V^H_{\alpha+1} = \big\{ u = \langle |u|, Eu\rangle \,;\, |u| : \mathcal{D}u \longrightarrow H,\, \mathcal{D}u \subseteq V^H_\alpha,\ Eu \in H,$$
$$|u|(x) \leq Eu \wedge Ex \ \ (\forall x \in \mathcal{D}u)\big\}.$$

Then $V^H = \bigcup_{\alpha\in On} V^H_\alpha$.

Definition 2 *Let $u, v \in V^H$ and $\varphi, \psi, \varphi(a)$ be formulas of V^H.*
For atomic formulae,

$$\|\mathrm{E}u\| = Eu, \qquad \|u \in v\| = \bigvee_{y\in\mathcal{D}v} \big(v(y) \wedge \|u = y\|\big),$$

$$\|u = v\| = \bigwedge_{x\in\mathcal{D}u} \big(u(x) \to \|x \in v\|\big) \wedge \bigwedge_{y\in\mathcal{D}v} \big(v(y) \to \|y \in u\|\big) \wedge Eu \wedge Ev.$$

For compound formulae,

$$\|\varphi \wedge \psi\| = \|\varphi\| \wedge \|\psi\|, \qquad \|\varphi \vee \psi\| = \|\varphi\| \vee \|\psi\|,$$
$$\|\varphi \to \psi\| = \|\varphi\| \to \|\psi\|, \qquad \|\neg\varphi\| = \neg\|\varphi\|,$$
$$\|\forall x\varphi(x)\| = \bigwedge_{u\in V^H} \big(Eu \to \|\varphi(x)\|\big), \qquad \|\exists x\varphi(x)\| = \bigvee_{u\in V^H} \big(Eu \wedge \|\varphi(x)\|\big).$$

A sentence φ of V^H is *valid in* V^H ($\models \varphi$) if $\|\varphi\| = \mathbf{1}$. We say u is a *subset of* v *in* V^H ($u \sqsubseteq v$) if $\|u \subseteq v\| = \|\forall x(x \in u \to x \in v)\| = \mathbf{1}$. If $u \sqsubseteq v$ and $v \sqsubseteq u$, we say u and v are *similar* ($u \sim v$). For every crisp set x in V, the *check set* $\check{x} \in V^H$ is defined recursively by: $\mathcal{D}\check{x} = \{\check{y}; y \in x\}$, $E\check{x} = \mathbf{1}$, $\check{x} : \check{y} \longmapsto \mathbf{1}$.

Basic notions such as pair, ordered pair, and cartesian product etc. are naturally defined in V^H. So all axioms of intuitionistic set theory are valid in V^H.

A *relation in* V^H is a subset of a cartesian product in V^H. For $R, u, v \in V^H$, R is a *relation from* u *to* v *in* V^H if R is a subset of $u \times v$ in V^H. The identity relation, compositions of relations, and inverse relations are naturally defined.

Definition 3 *For* $u, R \in V^H$, *R is an* equivalence relation on u *if R is a reflexive, symmetric, and transitive relation, that is, the followings hold.*

(1) $\|(\forall x \in u)(xRx)\| = \mathbf{1}$.
(2) $\|\forall x\forall y(xRy \to yRx)\| = \mathbf{1}$.
(3) $\|\forall x\forall y\forall z(xRy \wedge yRz \to xRz) = \mathbf{1}$.

Let R be an equivalence relation on u in V^H. For every $x \in V^H$, define the *equivalence class* $[x]$ by:

$$\mathcal{D}[x] = \mathcal{D}(u), \;\; E[x] = Ex, \;\; [x] : y \longmapsto \|xRy\|.$$

Then for all $x, y \in V^H$, $\|xRy\| = \|\exists z(z \in [x] \cap [y])\| \leq \|[x] = [y]\|$.

For an equivalence relation R on u, define the *quotient set* $Q = u/R$ by:

$$\mathcal{D}Q = \{[x]; x \in \mathcal{D}u\}, \;\; EQ = Eu, \;\; Q : [x] \longmapsto \|x \in u\|.$$

Then for all $x, y \in V^H$, $\|xRy\| = \|\exists p \in Q(x \in p \wedge y \in p)\|$.

Definition 4 *Let* $u, P \in V^H$. *P is a* partition of u *if the followings hold.*

(1) $\|\forall p \in P\,(\exists x(x \in p))\| = \mathbf{1}$.
(2) $\|\forall p \in P\,(p \subseteq u)\| = \mathbf{1}$.
(3) $\|\forall x \in u\,\exists p \in P\,(x \in p)\| = \mathbf{1}$.
(4) $\|\forall p, q \in P\,(\exists x(x \in p \cap q) \to p = q)\| = \mathbf{1}$.

If R is an equivalence relation on u and $Q = u/R$ is its quotient set, then Q becomes a partition of u.

Conversely for a partition P of u, define $R = R_P \in V^H$ by:

$$\mathcal{D}R = \mathcal{D}u \times \mathcal{D}u, \;\; ER = Eu \wedge EP, \;\; R : \langle xy\rangle \longmapsto \|\exists p \in P\,(x \in p \wedge y \in p)\|.$$

Then R becomes an equivalence relation on u, and is called the *equivalence relation induced from P*.

Theorem 1 *Let $u, R, Q, P \in V^H$.*
(1) *If R is an equivalence relation on u and Q is its quotient set, then the equivalence relation induced from Q is similar to R.*
(2) *If P is a partition of u and R is the equivalence relation induced from P, then the quotient set u/R is similar to P.*

Therefore there is a natural one-to-one correspondence (up to similarity) between equivalence relations on u and partitions of u for every $u \in V^H$.

3 Fuzzy equivalence relations and fuzzy partitions

We briefly recall the most basic notions of the natural interpretation of fuzzy sets and fuzzy relations (Cf. [1] [3] [5]).

Let X be a crisp set. Every set in V^H is called an *H-fuzzy set*, and every subset of $\check{X}$ in V^H is called an *H-fuzzy subset of X*. For every H-fuzzy set A, the *membership function of A on X* is the mapping $\mu_A : X \longrightarrow H; x \longmapsto \| \check{x} \in A \|$. An H-fuzzy set A is called *normal on X* if $\mu_A(x) = \mathbf{1}$ for some $x \in X$.

Let X, Y be crisp sets. If R is an H-fuzzy subset of $X \times Y$, R is called an *H-fuzzy relation from X to Y*. In case $X = Y$, R becomes a relation on $\check{X}$, and it is called an *H-fuzzy relation on X*.

Theorem 2 *Let R be an H-fuzzy relation on X. Then R is an equivalence relation iff for all $x, y, z \in X$ the followings hold.*

(1) $\mu_R\langle xx \rangle = \mathbf{1}$.
(2) $\mu_R\langle xy \rangle = \mu_R\langle yx \rangle$.
(3) $\mu_R\langle xy \rangle \wedge \mu_R\langle yz \rangle \leq \mu_R\langle xz \rangle$.

Let R be an H-fuzzy equivalence relation on X. For each $x \in X$, the equivalence class $[\check{x}]$ is defined by:

$$\mathcal{D}[\check{x}] = \{\check{y}; y \in X\},\ E[\check{x}] = \mathbf{1},\ [\check{x}] : y \longmapsto \|xRy\|.$$

Obviously $[\check{x}]$ is an H-fuzzy subset of X for every $x \in X$. For all $x, y \in X$,

$$\mu_R\langle xy \rangle = \|\exists u(u \in [\check{x}] \cap [\check{y}])\| = \|[\check{x}] = [\check{y}]\|, \text{ and } [\check{x}] = [\check{y}] \text{ iff } \mu_R\langle xy \rangle = \mathbf{1}.$$

Definition 5 *Let $\mathcal{P}$ be a family of H-fuzzy subsets of X. $\mathcal{P}$ is an H-*fuzzy partition family of X *if it satisfies the following three conditions.*

(1) *Every $A \in \mathcal{P}$ is normal on X.*
(2) *For every $x \in X$ there is a unique $A \in \mathcal{P}$ such that $\mu_A(x) = \mathbf{1}$.*
(3) *For all $A, B \in \mathcal{P}$ and all $x, y \in X$, $\mu_A(x) \wedge \mu_A(y) \wedge \mu_B(x) \leq \mu_B(y)$.*

Let R be an H-fuzzy equivalence relation on X and $\mathcal{P} = \mathcal{P}_R = \{[\check{x}]; x \in X\}$. Then $\mathcal{P}$ becomes an H-fuzzy partition family of X, and is called *the H-fuzzy partition family induced from R*.

Proposition 1 *Let $\mathcal{P}$ be an H-fuzzy partition family of X and $A, B \in \mathcal{P}$.*

(1) $\|A = B\| = \|\exists u(u \in A \cap B)\|$.
(2) *For all $x, y \in X$, $\mu_A(x) \wedge \mu_A(y) \wedge \mu_B(x) \wedge \neg\mu_B(y) = \mathbf{0}$.*

Conversely for an H-fuzzy partition family $\mathcal{P}$ of X, define $R = R_{\mathcal{P}} \in V^H$ by:

$$\mathcal{D}R = \{\langle \check{x}\check{y} \rangle;\ x, y \in X\},\ \ ER = \mathbf{1},\ \ R : \langle \check{x}\check{y} \rangle \longmapsto \bigvee_{A \in \mathcal{P}} (\mu_A(x) \wedge \mu_A(y)).$$

Then R becomes an H-fuzzy equivalence relation on X, and is called the *H-fuzzy equivalence relation induced from* $\mathcal{P}$.

Two families $\mathcal{P}$ and $\mathcal{Q}$ of H-fuzzy sets are said to be *similar* if for every $A \in \mathcal{P}$ there is a unique $B \in \mathcal{Q}$ similar to A, and vice versa.

Theorem 3 *Let X be a crisp set.*
(1) *If R is an H-fuzzy equivalence relation on X and $\mathcal{P} = \mathcal{P}_R$ is the H-fuzzy partition family induced from R, then the H-fuzzy equivalence relation induced from $\mathcal{P}$ is similar to R.*
(2) *If $\mathcal{P}$ is an H-fuzzy partition family of X and $R = R_{\mathcal{P}}$ is the H-fuzzy equivalence relation induced from $\mathcal{P}$, then the H-fuzzy partition family induced from R is similar to $\mathcal{P}$.*

Therefore there is a natural correspondence between H-fuzzy equivalence relations on X and H-fuzzy partition families of X.

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A Categorical Semantics for Fuzzy Predicate Logics

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The object of this study is to look at categorical approaches to many valued logic, both propositional and predicate, to see how different logical properties result from different parts of the situation. In particular, the relationship between the *Categorical Fabric* I introduced at Linz in 2004 and the Fuzzy Logics studied by Hájek and others in [2, 1, 3] comes from restricting the kind of structures used for truth values. We see how the structure of the various kinds of algebras shows up in the categorical logic. Since categorical logic gives one form for axioms (coming from identity morphisms) and many rules arising from structures in the category (products, coproducts, terminal objects, initial objects, tensors, right adjoints and the like) this gives a form of logic similar to natural deduction or a sequent calculus rather than the systems with few rules and several axioms previously proposed.

The form of categorical logic I have been working with recently provides a semantic setting in which types are not determined by their individuals, so a predicate logic based on variables and constants needs to be replaced with a predicate logic using predicates as properties with types. Quantification (following Lawvere [8]) becomes the action of right and left adjoints to the functor which allows us to transport a predicate of one type back along a morphism in the base category of types. Inference rules for quantification come from the adjointness property. Substitution, change of type, and quantification are actions taken along morphisms. This gives a logic with no variables. Rules which usually are stated with constraints on substitutability and freeness of variables are instead stated in terms of predicates transported using inverse image functors.

Since the semantics of fuzzy sets is usually done in a fabric over **Sets**, where the terminal is a generator, we can use properties of global sections to get a form of the rule of generalization. Since the full understanding of fuzzy sets comes from looking at fuzzy points with less than full membership it behooves us to consider semantics at all levels, and not just at the level of full truth. This is particularly true in systems like those of Höhle in [6, 5, 4] or Stout [9, 10] where the logic is internalized through a form of subobject representation. The resulting semantics is similar to the Kripke-Joyal-Beth semantics for topos logic given in Lambek and Scott [7, p.164].

The categorical setting gives a predicate logic without variables and constants. The language in the more traditional sense comes from a structure built on a particular freely generated cartesian category. Formulas involving n-ary predicates, variables, and constants have a clear meaning in that more restricted context. Interpretation of the language of a fuzzy theory in other categorical fabrics is given by application of a product preserving functor. Completeness results to date have addressed how well the predicate logic induced by various kinds of algebras has captured semantics in terms of this kind of interpretation.

Another interpretation of completeness results comes from possible worlds semantics for modal logic. A fabric gives a family of possible worlds and the transition maps allowing for transworld identity and accessibility. A completeness result then gives a model for the necessary truths in a theory, those which hold in all accessible worlds. In some sense they also give a preferred world, the natural home for a particular kind of fuzzy reasoning.

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BL- and MTL-algebras as Partial Structures

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1 Introduction

It is a well-known fact that MV-algebras are intervals of abelian ℓ-groups. The probably most intutive way to see this is to associate with an MV-algebra the corresponding effect algebra, a structure which is based on a partial addition. Also BL-algebras may be identified with certain partial structures, which in turn generate representing ℓ-groups. We shall review these facts, and we shall see that even the structure of certain MTL-algebras (which are not BL-algebras) may be enlightened in this way.

2 MV-algebras, effect algebras, and ℓ-groups

There is a one-to-one correspondence between MV-algebras, which are the Lindenbaum algebras of theories of Łukasiewicz logic, and lattice-ordered effect algebras fulfilling the Riesz decomposition property (see e.g. [DvPu]). Namely, let $(L;\leq,\oplus,\sim,0,1)$ be an MV-algebra; define $a+b = a\oplus b$ if a is the smallest element x such that $x\oplus b = a\oplus b$, and if b is the smallest element y such that $a\oplus y = a\oplus b$; else let $a+b$ undefined. Then $(L;\leq,+,0,1)$ is the corresponding effect algebra. That is, (E1) $(L;\leq,0,1)$ is a bounded poset, (E2) $+$ is an associative, commutative, and cancellative binary operation w.r.t. which 0 is a neutral element, and (E3) $a\leq b$ iff $a+x=b$ for some x. That L is lattice-ordered simply means that $(E;\leq)$ is a lattice; that L fulfils the Riesz decomposition property means that for any a,b,c,d such that $a+b=c+d$ there are e_1,e_2,e_3,e_4 such ! that $e_1+e_2=a,\ e_3+e_4=b,\ e_1+e_3=c,\ e_2+e_4=d$.

Apart from technical difficulties caused by the fact that the addition is only partially defined, it is a clear advantage of effect algebras that representing ℓ-groups are easily constructible. Namely, we may associate with any effect algebra $(L;\leq,+,0,1)$ the group $\mathcal{G}(L)$ freely generated by the elements of L and subject to the conditions $a+b=c$ whenever this equation holds among elements a,b,c in L. Assuming then that $(L;\leq,+,0,1)$ has the Riesz decomposition property, the canoncial embedding $L\to\mathcal{G}(L)$ is injective, and $\mathcal{G}(L)$ may be partially ordered in a way that L is isomorphic to $\mathcal{G}(L)[0,1] = \{a\in\mathcal{G}(L)\colon\ 0\leq a\leq 1\}$, where the partial operation on $\mathcal{G}(L)[0,1]$ is the group addition restricted to those pairs of elements the sum of which is below 1 [Rav]. We note that this construction works without assuming a lattice order and moreover ev! en in the non-commutative case [DvVe].

3 BL-algebras, weak effect algebras, and po-groups

There is a similar correspondence between BL-algebras, the Lindenbaum algebras of theories of Basic fuzzy logic [Haj1], and a certain generalization of effect algebras [Vet1]. Let a BL-algebra L be given. For what follows, it is advantageous to invert the partial order of L and to change the notation accordingly: Call $(L;\leq,\oplus,\ominus,0,1)$ a dual BL-algebra if $(L;\leq_{\text{BL}},\odot_{\text{BL}},\to_{\text{BL}},0_{\text{BL}},1_{\text{BL}})$ is a BL-algebra, where $a\leq b$ if $b\leq_{\text{BL}} a,\ a\oplus b = a\odot_{\text{BL}} b,\ a\ominus b = b\to_{\text{BL}} a,\ 0=1_{\text{BL}}$, and $1=0_{\text{BL}}$. We now may restrict the total addition $\oplus$ to a partial one just like in the case of MV-algebras. Then $(L;\leq,+,0,1)$ is a weak effect algebra [Vet1]. That is, the axioms (E1) and! (E2) hold as well as the following ones: (E3') if $a \leq b$, then there is largest $\bar{a} \leq a$ such that $\bar{a} + x = b$ for some x; (E4) if $a + c$ and $b + c$ are defined, then $a \leq b$ iff $a + c \leq b + c$.

Obviously, every effect algebra is a weak effect algebra: (E3') is a weakened form of (E3); and (E4) is implied by (E3) and cancellativity of $+$. Furthermore, the above construction defines a one-to-one correspondence between BL-algebras and a certain subclass of the weak effect algebras. To characterize this subclass algebraically is difficult. Among the properties that are fulfilled, we mention only one [Vet2]: the Riesz decomposition property, defined as above.

Now, although weak effect algebras are even much more difficult to handle than effect algebras, we again have the advantage that a representing po-group can easily be constructed. Let $(L; \leq, +, 0, 1)$ be a weak effect algebra. Let $(L_0; \preceq, +, 0)$ be the structure resulting from L by removing the 1 element and by replacing the partial order $\leq$ by a new one: $a \preceq b$ if $a + x = b$ for some x. Then we have that L_0 is a generalized effect algebra [HePu]; generalized effect algebras may be considered as effect algebras with the largest element removed. Because $a \preceq b$ implies $a \leq b$, we actually may say that the weak effect algebra L is a generalized effect algebra whose partial order is extended and to which a largest element 1 is added.

When L fulfils the Riesz decomposition property, then so does the generalized effect algebra L_0. It follows that there is an abelian po-group $\mathcal{G}(L_0)$ into which L_0 can be isomorphically embedded; the construction works exactly as in the case of effect algebras.

If L even arises from some BL-algebra, these facts are in nice accordance with the well-known structure theorem for BL-algebras (see e.g. [AgMo]). Namely, if L is totally ordered, L is the disjoint union of convex sets $L_\iota, \iota \in I$, such that $a \preceq b$ iff $a \leq b$ and $a, b \in L_\iota$ for some ι or $a = 0$. It follows that $\mathcal{G}(L_0)$ is the direct sum of totally ordered abelian groups and finally that L is the ordinal sum of intervals of the positive cone of totally ordered abelian groups.

4 MTL-algebras, weak effect algebras, and po-groups

We have seen that in order to represent a BL-algebra by means of an abelian po-group, we take the corresponding weak effect algebra and then generate from it a representing po-group. Now, weak effect algebras corresponding to BL-algebras fulfil several special properties among which only one was used to make sure that the po-group exists: the Riesz decomposition property. So we wonder if not a larger class of algebras than only BL-algebras may undergo an analysis via po-groups.

What we have in mind are naturally those algebras which generalize BL-algebras and which have been resisting a detailed analysis until today: the MTL-algebras, the algebras corresponding to the equally named logic. Only special cases were clarified, and what we offer here is to bring some of them on a common line. We call $(L; \leq, \oplus, 0, 1)$ a dual MTL-algebra if (M1) $(L; \leq, 0, 1)$ is a bounded lattice, (M2) $(L; \oplus, 0)$ is a commutative semigroup with neutral element 0, (M3) $\cdot \oplus a$ is monotone for all fixed a, (M4) for all a, b, there is a smallest x such that $a \oplus x \geq b$, (M5) $(a \ominus b) \wedge (b \ominus a) = 0$.

From a (dual) MTL-algebra L, we may form a partial algebra $(L; \leq, +, 0, 1)$ just like in the case of BL-algebras. The original addition is reobtained by $a \oplus b = \max\{a' + b' \colon \; a' \leq a, b' \leq b\}$. However, we must say that under no additional assumptions, not much can be proved about this algebra; we were not even able to verify that $+$ is in general associative.

On the other hand, we may selectively consider those algebras for which analogous constructions to those mentioned above, are possible. Let us form $(L_0; \preceq, +, 0)$ as above: $L_0 = L \backslash \{1\}$ and $a \preceq b$ if $a + x = b$ for some x. Let us assume that this is a generalized effect algebra. This does not seem to be a restrictive condition; we do not know counterexamples. We are then interested in the case that $(L_0; \preceq, +, 0)$ is embeddable into the positive cone of some po-group. Unfortunatley, there is basically only one algebraic condition known which implies the group embeddability: the Riesz decomposition property. Assuming this condition, however, is rather restrictive; we still would cover certain non-BL MTL-algebras, but not as many as desired.

What remains is to proceed in the opposite direction: exploring MTL-algebras arising from po-groups not by searching hopelessly algebraic properties making such an analysis possible, but by studying the po-groups themselves. In this way, we might still not get all MTL-algebras, but at least a satisfactorily wide class of them. We restrict ourselves to totally ordered abelian groups, whose structure is perfectly known (cf. e.g. [Fuc]), and to the following construction.

Let $(R;+,0)$ be a subgroup of $(\mathbb{R};+,0)$ and call a subset I of R (temporarily) a domain of R if I is either $[0,a]$ for some $a \in R^+$ or R^+ or $[a,0]$ for some $a \in R^-$ or R^- or R. I is understood to be endowed with the partial operation $+$, which is the restriction of the group addition to I.

Now let R and P be two subgroups of $(\mathbb{R};+,0)$. Let I be a domain of R such that $I \subseteq R^+$; and for every $r \in I$, let I_r be a domain of P such that the following condition are fulfilled. (i) $I_0 \subseteq R^+$, and if I has a greatest element m, then I_m is bounded from above; (ii) for $r,s \in I$ such that $r \leq s$, and $a \in I_r, b \in I_s$ such that $a \leq b$, there is a $c \in I_{s-r}$ such that $c \geq b-a$, (iii) for any $r \in I$, let $C_r^+ = \{a+b\colon a \in I_s,\, b \in I_t,\, s+t = r,\, a+b \geq \sup I_r\}$ and similarly $C_r^- = \{a+b\colon a \in I_s,\, b \in I_t,\, s+t = r,\, a+b \leq \inf I_r\}$; then (sums of sets being understood elementwise) $C_s^+ + I_t \subseteq C_{s+t}^+$ and $C_s^- + I_t \subseteq C_{s+t}^-$ if $s+t$ exists.

Let now $L_0 = \{(r,a)\colon\ r \in I,\, a \in I_r\}$; let $\leq$ be the lexicographical order; and define the partial addition $+$ componentwise whenever this is (for both components) possible. Moreover, extend $+$ in the following way. For some $a \in I_r$, let $a^\sim = (a \vee \inf I_r) \wedge \sup I_r$; for $(r,a),(s,b) \in L_0$, let $(r,a) \oplus (s,b) = (r+s,(a+b)^\sim)$ if $r+s$ exists and else $= (\sup I, \sup I_r)$. Letting $L = L_0 \cup \{1\}$, where 1 is a new greatest element, $(L;\leq,\oplus,0,1)$ becomes a dual MTL-algebra.

We give two examples. (i) Consider $L = \mathbb{N} \times \mathbb{R}^+$. This leads to an example of Hájek [Haj2] of a left-continuous, non-continuous t-norm. (ii) Let $L = \{(a,b)\colon\ \ a = 0, b \in \mathbb{R}^+$ or $a = \frac{1}{2}, b = 0$ or $a = 1, b \in \mathbb{R}^-\}$. This corresponds to Jenei's rotated product t-norm [Jen].

Our construction allows generalizations into several directions. First of all, the number of subgroups of $\mathbb{R}$ from which our universe is formed from, may certainly be chosen greater than 2. Second, what is called a domain here can be defined more flexible; the possibility could be allowed that the addition is further restricted. To illustrate this, consider the real unit interval, endowed with the addition only in case the result is 1; when extending $+$ as above, we are led to what corresponds to the annihilated minimum t-norm (see e.g. [Jen]). Finally, we treat here only totally ordered abelian groups, and then use this total order for the constructed MTL-algebra. We could actually also start from any abelian po-group and then extend the given order to a total one. – Whereas the first two aims seem to be easily achievable, the third one is probably difficult.

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