LINZ 2008

29th Linz Seminar on Fuzzy Set Theory

Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology

Bildungszentrum St. Magdalena, Linz, Austria
February 12–16, 2008

Abstracts

Erich Peter Klement
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Editors

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Printed by: Universitätsdirektion, Johannes Kepler Universität, A-4040 Linz

Since their inception in 1979 the Linz Seminars on Fuzzy Sets have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work.

LINZ 2008 will be the 29th seminar carrying on this tradition and is devoted to the theme "Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology". The last decade has witnessed a significant development of the categorical, logical, and order-theoretic foundations of lattice-valued mathematics and their impact on algebra and topology. These developments have created or significantly strengthened bridges between lattice-valued mathematics, logic, sheaves, algebraic theories, quantales and order-theoretic structures, various subdisciplines of topology, and theoretical computer science. The purpose of the 29th Seminar is to discuss the synergy between these fields as well as identify important open questions.

Erich Peter Klement
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Contents

Graded dominance

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1 Introduction

The relation of dominance between aggregation operators has recently been studied quite intensively [9, 12, 10, 11, 13, 14]. We propose to study its 'graded' generalization in the foundational framework of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT) introduced in [1]. FCT is specially designed to allow a quick and sound development of graded, lattice-valued generalizations of the notions of traditional 'fuzzy mathematics' and is a backbone of a broader program of logic-based foundations for fuzzy mathematics, described in [2].

This short abstract is to be understood as just a 'teaser' of the broad and potentially very interesting area of graded dominance. We sketch basic definitions and properties related to this notion and present a few examples of results in the area of equivalence and order relations (in particular, we show interesting graded generalization of basic results from [6, 12]). Also some of our theorems are, for expository purposes, stated in a less general form here and can be further generalized substantively.

In this paper, we work in Fuzzy Class Theory over the logic MTL_Δ of all left-continuous t-norms [7]. The apparatus of FCT and its standard notation is explained in detail in the primer [3], which is freely available online. Furthermore we use $X \sqsubseteq Y$ for $\Delta(X \subseteq Y)$.

2 Inner truth values and truth-value operators

An important feature of FCT is the absence of variables for truth values. However, many theorems of traditional fuzzy mathematics do speak about truth values or quantify over operators on truth values like aggregation operators, copulas, t-norms, etc. In order to be able to speak of truth values within FCT, truth values need be *internalized* in the theory. This is done in [4] by a rather standard technique, by representing truth values by subclasses of a crisp singleton.[4] Thus we can assume that we do have variables $\alpha, \beta, \ldots$ for truth values in FCT; the class of the inner truth values is denoted by L.

Binary operators on truth values (including propositional connectives $\&, \neg, \ldots$) can then be regarded as functions $\mathbf{c}\colon \mathrm{L} \times \mathrm{L} \to \mathrm{L}$ or as fuzzy relations $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$. Consequently, graded class relations can be applied to such operators, e.g., fuzzy inclusion $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall \alpha, \beta)(\alpha\, \mathbf{c}\, \beta \to \alpha\, \mathbf{d}\, \beta)$. Many crisp classes of truth-value operators (e.g., t-norms, continuous t-norms, copulas, etc.) can be defined by formulae of FCT. The apparatus, however, enables also *partial* satisfaction of such conditions. In the

[4] Cf. [15] for an analogous construction in a set theory over a variant of Gödel logic. See [4] for details of the construction and certain metamathematical qualifications regarding the representation. Observe also a parallel with the power-object of 1 in topos theory.

following, we therefore give several *fuzzy* conditions on truth-value operators and use them as graded preconditions of theorems which need not be satisfied to the full degree. This yields a completely new *graded* theory of truth-value operators and allows non-trivial generalizations of well-known theorems on such operators, including their consequences for properties of fuzzy relations.

Definition 1. In FCT, we define the following graded properties of a truth-value operator $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$:

$$\begin{aligned}
\mathrm{Com}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha,\beta)(\alpha\,\mathbf{c}\,\beta \rightarrow \beta\,\mathbf{c}\,\alpha) \\
\mathrm{Ass}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha,\beta,\gamma)((\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma) \leftrightarrow (\alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\alpha)) \\
\mathrm{MonL}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha,\beta,\gamma)(\Delta(\alpha \rightarrow \beta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\gamma)) \\
\mathrm{MonR}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha,\beta,\gamma)(\Delta(\alpha \rightarrow \beta) \rightarrow (\gamma\,\mathbf{c}\,\alpha \rightarrow \gamma\,\mathbf{c}\,\beta)) \\
\mathrm{UnL}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha)(1\,\mathbf{c}\,\alpha \leftrightarrow \alpha) \\
\mathrm{UnR}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha)(\alpha\,\mathbf{c}\,1 \leftrightarrow \alpha)
\end{aligned}$$

For convenience, we also define

$$\begin{aligned}
\mathrm{Mon}(\mathbf{c}) &\equiv_{\mathrm{df}} \mathrm{MonL}(\mathbf{c}) \,\&\, \mathrm{MonR}(\mathbf{c}) \\
\mathrm{wMon}(\mathbf{c}) &\equiv_{\mathrm{df}} \mathrm{MonL}(\mathbf{c}) \wedge \mathrm{MonR}(\mathbf{c})
\end{aligned}$$

and analogously for Un.

The following theorem provides us with samples of basic graded results.

Theorem 1. *FCT proves the following graded properties of truth-value operators:*

1. $\mathrm{Mon}(\mathbf{c}) \,\&\, \mathrm{Un}(\mathbf{c}) \rightarrow (\mathbf{c} \subseteq \wedge)$
2. $\mathrm{wMon}(\mathbf{c}) \,\&\, (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \leftrightarrow \alpha) \rightarrow (\wedge \subseteq \mathbf{c})$
3. $\mathrm{Mon}(\mathbf{c}) \,\&\, \mathrm{Un}(\mathbf{c}) \rightarrow [(\alpha\,\mathbf{c}\,\alpha \leftrightarrow \alpha) \leftrightarrow (\forall\beta)((\alpha\,\mathbf{c}\,\beta) \leftrightarrow (\alpha \wedge \beta))]$

The three assertions above are generalizations of well-known basic properties of t-norms. Theorem 1.1 corresponds to the fact that the minimum is the greatest (so-called strongest) t-norm. Theorem 1.2 generalizes the basic fact that the minimum is the only idempotent t-norm, while 1.3 is a graded characterization of the idempotents of $\mathbf{c}$. [8].

3 Graded dominance

Definition 2. The graded relation $\ll$ of *dominance* between truth-value operators is defined as follows:

$$\mathbf{c} \ll \mathbf{d} \equiv_{\mathrm{df}} (\forall\alpha,\beta,\gamma,\delta)((\alpha\,\mathbf{d}\,\gamma)\,\mathbf{c}\,(\beta\,\mathbf{d}\,\delta) \rightarrow (\alpha\,\mathbf{c}\,\beta)\,\mathbf{d}\,(\gamma\,\mathbf{c}\,\delta))$$

Theorem 2. *FCT proves the following graded properties of dominance:*

1. $\Delta\mathrm{Com}(\mathbf{c}) \,\&\, \mathrm{Ass}^4(\mathbf{c}) \,\&\, \mathrm{Mon}(\mathbf{c}) \rightarrow (\mathbf{c} \ll \mathbf{c})$
2. $\mathrm{Un}(\mathbf{c}) \,\&\, \mathrm{Un}(\mathbf{d}) \,\&\, (\mathbf{c} \ll \mathbf{d}) \rightarrow (\mathbf{c} \subseteq \mathbf{d})$
3. $\Delta\mathrm{Com}(\mathbf{c}) \,\&\, \mathrm{Ass}^4(\mathbf{c}) \,\&\, \mathrm{Mon}^2(\mathbf{c}) \,\&\, (\mathbf{d} \sqsubseteq \mathbf{c}) \,\&\, (\mathbf{c} \subseteq \mathbf{d}) \rightarrow (\mathbf{c} \ll \mathbf{d})$
4. $\Delta\mathrm{Com}(\mathbf{d}) \,\&\, \mathrm{Ass}^4(\mathbf{d}) \,\&\, \mathrm{Mon}^2(\mathbf{d}) \,\&\, (\mathbf{d} \sqsubseteq \mathbf{c}) \,\&\, (\mathbf{c} \subseteq \mathbf{d}) \rightarrow (\mathbf{c} \ll \mathbf{d})$
5. $\mathrm{Mon}(\mathbf{c}) \,\&\, (\& \ll \mathbf{c}) \,\&\, ((\alpha \rightarrow \beta)\,\mathbf{c}\,(\gamma \rightarrow \delta)) \rightarrow ((\alpha\,\mathbf{c}\,\gamma) \rightarrow (\beta\,\mathbf{c}\,\delta))$
6. $\mathrm{Mon}(\mathbf{c}) \,\&\, (\& \ll \mathbf{c}) \,\&\, ((\alpha \leftrightarrow \beta)\,\mathbf{c}\,(\gamma \leftrightarrow \delta)) \rightarrow ((\alpha\,\mathbf{c}\,\gamma) \leftrightarrow (\beta\,\mathbf{c}\,\delta))$

Theorems 2.1 and 2.2 are generalizations of two basic facts, namely that every t-norm dominates itself and that dominance implies inclusion / pointwise order. Theorems 2.3 and 2.4 have no correspondences among known results; they provide us with bounds for the degree to which $(\mathbf{c} \ll \mathbf{d})$ holds, where the assumption $(\mathbf{d} \sqsubseteq \mathbf{c})$ & $(\mathbf{c} \subseteq \mathbf{d})$ would be obviously useless in the crisp non-graded framework (as it necessitates that $\mathbf{c}$ and $\mathbf{d}$ coincide anyway). Theorem 2.5 provides us with strengthened monotonicity of an aggregation operator $\mathbf{c}$ provided that $\mathbf{c}$ fulfills $\mathrm{Mon}(\mathbf{c})$ and dominates the conjunction of the underlying logic. Theorem 2.6 is then a kind of "Lipschitz property" of $\mathbf{c}$ (if we view $\leftrightarrow$ as a kind of generalized closeness measure).

Theorem 3. *FCT proves the following graded properties of dominance w.r.t.* $\wedge$*:*

1. $\mathrm{Mon}(\mathbf{c}) \rightarrow (\mathbf{c} \ll \wedge)$
2. $\Delta\mathrm{Mon}(\mathbf{c})\ \&\ \Delta\mathrm{Un}(\mathbf{c}) \rightarrow ((\wedge \ll \mathbf{c}) = (\wedge \subseteq \mathbf{c}))$
3. $\mathrm{wMon}^2(\mathbf{c}) \rightarrow ((\wedge \ll \mathbf{c}) \leftrightarrow (\forall \alpha, \beta)((\alpha\, \mathbf{c}\, 1) \wedge (1\, \mathbf{c}\, \beta) \leftrightarrow (\alpha\, \mathbf{c}\, \beta)))$

Theorem 3.1 is a graded generalization of the well-known fact that the minimum dominates any aggregation operator [12]. Theorem 3.2 demonstrates a rather surprising fact: that the degree to which a monotonic binary operation with neutral element 1 dominates the minimum is nothing else but the degree to which it is larger. Theorem 3.3 is an alternative characterization of operators dominating the minimum; for its non-graded version see [12, Prop. 5.1].

Example 1. Assertion 2. of Theorem 3 can easily be utilized to compute degrees to which standard t-norms on the unit interval dominate the minimum. It can be shown easily that

$$(\wedge \subseteq \mathbf{c}) = \inf_{x \in [0,1]} (x \Rightarrow \mathbf{c}(x,x))$$

holds, i.e. the largest "difference" of a t-norm $\mathbf{c}$ from the minimum can always be found on the diagonal. In standard Łukasiewicz logic, this is, for instance, 0.75 for the product t-norm and 0.5 for the Łukasiewicz t-norm itself. So we can infer that the product t-norm dominates the minimum with a degree of 0.75 (assuming that the underlying logic is standard Łukasiewicz!); with the same assumption, the Łukasiewicz t-norm dominates the minimum to a degree of 0.5.

4 Graded dominance and properties of fuzzy relations

The following theorems show the importance of graded dominance for graded properties of fuzzy relations. Theorem 4 is a graded generalization of the well-known theorem that uses dominance to characterize preservation of transitivity by aggregation [12, Th. 3.1] (compare also [6]).

Theorem 4. *FCT proves:*

$$\mathrm{Mon}(\mathbf{c}) \rightarrow ((\forall E, F)(\Delta\mathrm{Trans}(E)\ \&\ \Delta\mathrm{Trans}(F) \rightarrow \mathrm{Trans}(\mathrm{Op}_{\mathbf{c}}(E,F)) \leftrightarrow (\& \ll \mathbf{c})))$$

where $\mathrm{Op}_{\mathbf{c}}$ *is the class operation given by* $\mathbf{c}$*, i.e.,* $\langle x, y\rangle \in \mathrm{Op}_{\mathbf{c}}(E,F) \equiv Exy\ \mathbf{c}\ Fxy$.

The following theorem provides us with results on the preservation of various properties by symmetrizations of fuzzy relations.

Theorem 5. *FCT proves the following properties of the symmetrization of relations:*

1. $\mathrm{Com}(\mathbf{c}) \to (\mathrm{Sym}(\mathrm{Op}_{\mathbf{c}}(R, R^{-1})))$
2. $(\& \subseteq \mathbf{c})\ \&\ \mathrm{Refl}^2 R \to (\mathrm{Refl}(\mathrm{Op}_{\mathbf{c}}(R, R^{-1})))$
3. $(\& \subseteq \mathbf{c}) \to \mathrm{AntiSym}_{(\mathrm{Op}_{\mathbf{c}}(R, R^{-1}))} R$
4. $\mathrm{Mon}(\mathbf{c})\ \&\ (\& \ll \mathbf{c})\ \&\ \Delta \mathrm{Trans} R \to (\mathrm{Trans}(\mathrm{Op}_{\mathbf{c}}(R, R^{-1})))$

In the crisp case, the commutativity of an operator trivially implies the symmetry of symmetrizations by this operator. In the graded case, Theorem 5.1 above states that the degree to which a symmetrization is actually symmetric is bounded below by the degree to which the aggregation operator $\mathbf{c}$ is commutative. Theorems 5.2–4 are also well-known in the non-graded case [5, 6, 16]. Obviously, 5.4 is a simple corollary of Theorem 4.

Acknowledgments

Libor Běhounek was supported by the program Information Society project No. 1ET100300517. Petr Cintula was supported by grant No. A100300503 of GA AV ČR and Institutional Research Plan AV0Z10300504. The cooperation of the team was enabled by Program Kontakt/WTZ Czech Republic–Austria project No. 6–07–17 / 2–2007 "Formal foundations of preference modeling".

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Continuous relations over topological spaces in Fuzzy Class Theory

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Fuzzy topology has benefited from a considerable attention in the past (see, e.g., [12–14]). By now, there are several established approaches to fuzzy topology: lattice-theoretical, categorial, based on membership functions, etc. Recent advances in mathematical fuzzy logic, esp. after [10], enabled a new, *logic-based* approach to fuzzy topology. This approach is part of a broader program of logic-based fuzzy mathematics described in [3], which consists in the development of axiomatic theories over higher-order fuzzy logic. It extends and elaborates the methodology sketched by Höhle in [11, §5], namely a reinterpretation of classical definitions in a suitable calculus of fuzzy logic. A similar attempt to build fuzzy topology within a logical framework appeared also in [17]. We follow this line of research in the strictly formal framework of axiomatic theories over Hájek-style deductive fuzzy logic. The application of this kind of formalism leads to a universal gradedness of definitions and theorems which is not usual under traditional approaches; on the other hand it is limited to certain methodological presuppositions [1]: thus it complements (rather than competes with) the more traditional approaches. Due to the different strength of results, also some rather elementary theorems need to be proved anew in our setting, even though many results on related notions have already been obtained in the frameworks of more traditional approaches to fuzzy topology.

Initial results in our logic-based fuzzy topology have been presented in conference papers [6, 5], where the relationship between three notions of fuzzy topology (namely those based on open or closed sets, neighborhoods, and interior operators) were studied. In the present contribution we restrict our attention to fuzzy topology based on open sets and make first steps towards the notion of continuity in this setting. Instead of the more usual notion of continuous function, we study the notion of continuous *relation* between two fuzzy topologies. This enables us to avoid making such a basic concept as continuity depend on the notion of fuzzy function, which has many competing definitions.[3]

The notion of continuous relation between topological spaces has already appeared in several areas of mathematics. Continuous relations have been defined in general topology [16, 9] and also investigated by means of induced multifunctions in set-valued analysis [7]. In [15], their study was initiated in the purely formal framework of higher-order intuitionistic logic, as a part of formal (pointless) topology.

⋆ The work of the first author was supported by the program Information Society project No. 1ET100300517 and Institutional Research Plan AV0Z10300504.

⋆⋆ The work of the second author was supported by grant No. B100300502 of GA AV ČR and grant No. 1M0572 of MŠMT ČR.

[3] We are indebted for this idea to Rostislav Horčík.

In this paper we give initial observations on the concept of continuous fuzzy relation between a pair of fuzzy topologies. We work in the framework of Fuzzy Class Theory FCT (or higher-order fuzzy logic) introduced in [2]. For the sake of generality, here we use its variant over the logic MTL_Δ of all left-continuous t-norms [8]. Besides the original paper [2], the apparatus of FCT is described in detail in the primer [4], which is freely available online. Due to space restrictions, we do not repeat the definitions here. We use standard abbreviations and notions from [4, §1.6,1.7]; furthermore we use $X \sqsubseteq Y$ for $\Delta(X \subseteq Y)$ and Id_X for $\{\langle x,x\rangle \mid x \in X\}$.

In [6], the FCT notion of open fuzzy topology has been introduced. It uses the following predicates that express the (degree of) closedness of a fuzzy class of fuzzy classes τ under $\bigcup$ and $\cap$:

$$\mathsf{ic}(\tau) \equiv_{\mathrm{df}} (\forall A, B \in \tau)(A \cap B \in \tau)$$
$$\mathsf{Uc}(\tau) \equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\big(\bigcup \sigma \in \tau\big)$$

Definition 1. *In FCT, we define the predicate indicating the degree to which* $\tau \sqsubseteq \mathrm{Ker\,Pow}\, X$ *is an* open fuzzy topology *on a crisp class X as*

$$\mathsf{OTop}(X,\tau) \equiv_{\mathrm{df}} (\emptyset \in \tau)\ \&\ (X \in \tau)\ \&\ \mathsf{ic}(\tau)\ \&\ \mathsf{Uc}(\tau)$$

A predicate expressing that A such that $A \sqsubseteq X$ *is a* neighborhood *of x in* τ *is defined as*

$$\mathrm{Nb}_\tau(x,A) \equiv_{\mathrm{df}} (\exists B \in \tau)(B \subseteq A\ \&\ x \in B)$$

Given a class of classes τ*, we define the* interior *of a class A such that* $A \sqsubseteq X$ *as*

$$\mathrm{Int}_\tau(A) =_{\mathrm{df}} \bigcup \{B \in \tau \mid B \subseteq A\}$$

Models of the predicate OTop are closest to L-fuzzy topologies of Höhle-type studied in [13]. We assume the ground set X of an open fuzzy topology to be crisp, since quantification over fuzzy domains is not yet well understood in the fully graded setting of FCT. Even though there are no technical obstacles for using fuzzy X, graded definitions over fuzzy X would need a much more careful general discussion about their meaning and motivation. Thus in this contribution we stick to crisp ground sets of fuzzy topologies.

In the sequel we assume that $R \sqsubseteq X_1 \times X_2$ and $S \sqsubseteq X_2 \times X_3$, where each X_i is a crisp class. By τ_i we denote a fuzzy class of fuzzy classes such that $\tau_i \sqsubseteq \mathrm{Ker\,Pow}\, X_i$. In Definition 2 we introduce three predicates, each of them expressing a different definition of continuous relation (by open classes, by neighborhoods, and by the interior operator). It is worth mentioning that the definition of the predicate NCont resembles the one used by Sambin [15, §2.3] over intuitionistic logic.

Definition 2.

$$\mathrm{OCont}(R) \equiv_{\mathrm{df}} (\forall B \in \tau_2)(R^{\leftarrow} B \in \tau_1)$$
$$\mathrm{NCont}(R) \equiv_{\mathrm{df}} (\forall x \in X_1)(\forall B \in \tau_2)(R^{\rightarrow}\{x\} \| B \to (\exists A)(\mathrm{Nb}_{\tau_1}(x,A)\ \&\ A \subseteq R^{\leftarrow} B))$$
$$\mathrm{ICont}(R) \equiv_{\mathrm{df}} (\forall B)(R^{\leftarrow} \mathrm{Int}_{\tau_2}(B) \subseteq \mathrm{Int}_{\tau_1}(R^{\leftarrow} B))$$

The following proposition says that all of the above introduced predicates are fuzzily equivalent under rather general conditions: note that the second-order fuzzy classes τ_1 and τ_2 are only required to be closed under unions of fuzzy families of fuzzy classes. Also notice that the theorem is graded, i.e., the fuzzy equivalence holds at least to the degree of $\mathsf{Uc}(\tau_1)$ resp. $\mathsf{Uc}^2(\tau_2)$. We omit all proofs due to space restrictions.

Proposition 1. *It is provable in FCT:*

1. $\mathsf{Uc}(\tau_1) \to (\mathrm{NCont}(R) \leftrightarrow \mathrm{OCont}(R))$
2. $\mathsf{Uc}(\tau_1) \to (\mathrm{OCont}(R) \leftrightarrow \mathrm{ICont}(R))$
3. $\mathsf{Uc}^2(\tau_2) \to (\mathrm{ICont}(R) \leftrightarrow \mathrm{NCont}(R))$

As the properties OCont, ICont, and NCont are equivalent if τ_1 resp. τ_2 are sufficiently union-closed, we shall restrict our attention to the predicate OCont. The following proposition shows that continuous relations form a "fuzzy system of morphisms" between fuzzy topologies.

Proposition 2. *It is provable in FCT:*

1. $\mathrm{OCont}(\mathrm{Id}_X)$
2. $\mathrm{OCont}(R)\ \&\ \mathrm{OCont}(S) \to \mathrm{OCont}(R \circ S)$

Like in classical topology, when examining the continuity of a relation, it is sufficient to verify openness for preimages of open classes from a base.

Proposition 3. *FCT proves:*

$$\{\bigcup \nu \mid \nu \subseteq \sigma\} \subseteq \tau_2\ \&\ \mathsf{Uc}(\tau_1) \to [\mathrm{OCont}(R) \leftrightarrow (\forall B \in \sigma)(R^{\leftarrow} B \in \tau_1)]$$

A non-trivial example of continuous relations between fuzzy topological spaces is introduced in Example 1. The predicate OCont has crisp instances, too: in particular, continuous relations studied herein are special cases of so-called lower semicontinuous multifunctions investigated in set-valued analysis [7].

Example 1. In [6], an interval fuzzy topology on domains densely ordered by a crisp relation $\leq$ has been defined as the coarsest fully $\bigcup$-closed topology that fully contains the fuzzy subbase of fuzzily open fuzzy intervals $[A,B]$. It can be proved that $\leq$ is a continuous relation w.r.t. this topology: since the fuzzy family of open fuzzy intervals is closed under $\cap$, by Proposition 3 it is sufficient to prove that $\leq^{\leftarrow}[A,B]$ is open for any open fuzzy interval $[A,B]$. It can even be shown that $\leq^{\leftarrow}[A,B]$ is an open interval of the form $[-\infty, C]$ for a right-open C.[4]

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[4] Since the properties of openness and right-openness are fuzzy, the last two sentences should be read in the graded manner, i.e., as implications between the fuzzy conditions.

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Algebraic theory of lattice-valued fuzzy languages and automata

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In this paper we will give an overview of the most significant results concerning application of algebraic tools and concepts in study of fuzzy languages and automata. We will point to some older results, concerning classical fuzzy languages and automata taking membership values in the Gödel structure, but our attention will be aimed mostly to recent results on lattice-valued fuzzy languages and automata, especially to those concerning fuzzy languages and automata with membership values in lattice-ordered monoids and complete residuated lattices.

The central place in the algebraic theory of fuzzy languages and automata is held by various questions concerning recognition of fuzzy languages by fuzzy finite automata (FFA) and deterministic finite automata (DFA). In the classical theory of fuzzy languages and automata with membership values in the Gödel structure, the classes of all DFA-recognizable and all FFA-recognizable fuzzy languages coincide, but this does not necessary hold for other structures of membership values. Namely, Li and Pedrycz [19] studied fuzzy automata and languages over a lattice-ordered monoid $\mathscr{L}$, and they proved that the classes of all DFA-recognizable and all FFA-recognizable fuzzy languages over $\mathscr{L}$ coincide if and only if the reduct $\mathscr{L}^*$ of $\mathscr{L}$, with respect to join and multiplication, is a locally finite semiring. Bělohlávek [3] and Li and Pedrycz [19] developed a method for determinization of fuzzy automata, which results in a finite automaton if and only if $\mathscr{L}^*$ is locally finite. Ignjatović et all [11] developed another method, which can result in a finite automaton even if $\mathscr{L}^*$ is not locally finite, and always gives a smaller automaton than the mentioned method by Bělohlávek and Li and Pedrycz. Ignjatović et all [11, 12] also gave certain criterions for finiteness of the resulting deterministic automaton, and proved that this automaton is a minimal deterministic automaton recognizing all fuzzy languages which can be recognized by the original fuzzy automaton. Determinization of fuzzy automata was also studied by Li and Pedrycz [21], whereas Li [18] considered the problem of approximation of a non-deterministic fuzzy finite automaton by a deterministic one.

The Myhill-Nerode's type theory for fuzzy langauges and automata, in which fuzzy languages and automata are studied through right congruences and congruences on a free monoid, traces one's origin to the papers by Shen [37] and Malik et all [24] (see also [29]), and recently, it was further developed by Ignjatović et all [12]. Ignjatović et all [12] characterized DFA-recognizability of fuzzy languages through syntactic right congruences and syntactic congruences of a fuzzy language, and proved that for any fuzzy language there exists a minimal deterministic automaton recognizing it, which is unique up to an isomorphism. They also gave a construction of this automaton by means of the concept of a derivative of a fuzzy language, as well as by means of derivatives of certain crisp languages associated with a fuzzy language (kernel and cut languages), and they gave an algorithm for minimization of a deterministic automaton which recognizes a given fuzzy language. A similar algorithm, for deterministic automata recognizing fuzzy languages over a distributive lattice, was also given by Li and Pedrycz [21]. It is worth of mention that Ignjatović et all [12] studied fuzzy languages taking membership values in an arbitrary set having two distinguished elements 0 and 1, which are needed to take crisp languages into consideration. Recently, Bozapalidis and Louscou-Bozapalidou [4, 5], studied fuzzy languages recognized by finite monoids and established certain relationships between recognizability of fuzzy languages by finite monoids and fuzzy finite automata. Recognizability of fuzzy languages by monoids was also studied in [12], where it was proved that it is equivalent to DFA-recognizability.

Unlike deterministic automata, whose minimization is efficiently possible, it is well-known that the state minimization of non-deterministic automata is computationally hard. Fuzzy automata are generalizations of non-deterministic ones, and the mentioned problem also exist in work with fuzzy automata. For that reason, many researchers aimed their attention to efficient size reduction methods which do not necessarily give a minimal automaton. Size reduction algorithms for fuzzy automata given in [2, 6, 15, 26, 29, 32] are also based on the idea of computing and merging indistinguishable states, and the term minimization that we meet there does not mean the usual construction of the minimal one in the set of all fuzzy automata recognizing a given fuzzy language, but just the procedure of computing and merging indistinguishable states. M. Ćirić et all [7, 8] showed that the size reduction problem for fuzzy automata is related to the problem of solving a particular system of fuzzy relation equations. This system consists of infinitely many equations, and finding its general solution is a very difficult task, and M. Ćirić et all [7, 8] considered one of its special cases, a finite system whose solutions, called right invariant fuzzy equivalences, are common generalizations of right invariant or well-behaved equivalences used in reduction of non-deterministic automata, and congruences on fuzzy automata studied in [32]. They also gave a procedure for constructing the greatest right invariant fuzzy equivalence contained in a given fuzzy equivalence. It was shown that the method for reduction of fuzzy automata developed in [7, 8] gives better results than all other methods developed in [2, 6, 15, 26, 29, 32], and that these results can be even improved using fuzzy quasi-orders instead of fuzzy equivalences.

Finally, we will also talk about regular operations on fuzzy languages and related concepts, which have been considered in [1, 10, 14, 16, 19]. Li and Pedrycz [19] proved the Kleene's type theorem for fuzzy languages which asserts that a fuzzy language over a lattice-ordered monoid is FFA-recognizable if and only if it can be represented by a fuzzy regular expression, or equivalently, if it can be constructed from elementary languages using the regular operations on fuzzy languages – union, concatenation, Kleene star and scalar products. Certain related results were also obtained in [1, 16]. Ignjatović and Ćirić [10] studied fuzzy languages over a quantale $\mathscr{L}$, and proved that they can be represented by formal power series on $\mathscr{L}$ with coefficients which are crisp languages, and that regular operations on fuzzy languages can be represented by operations on power series which are defined by means of operations on crisp languages. They also proved that a fuzzy language is FFA-recognizable if and only if it can be represented by a rational power series, and that it is DFA-recognizable if and only if it can be represented by a polynomial whose coefficients are regular crisp languages.

Acknowledgments

Research supported by Ministry of Science, Republic of Serbia, Grant No. 144011

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Pointed semi-quantales and generalized lattice-valued quasi topological spaces

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Semi-quantales have been recently introduced by Rodabaugh [6] as a lattice-theoretic and an algebraic basis of powerset theories and lattice-valued topologies. In particular, lattice-valued quasi topology (**L**-quasi topology) referring to a semi-quantale **L** is defined in [6] as a generalization of many valued topology (**L**-topology) in Höhle's sense [2] by dropping the axiom "the characteristic function of the underlying set belongs to the **L**-topology". In a completely different context, Mulvey and Pelletier [5] proposed the quantal spaces as a generalization of topological spaces by means of quantales. A quantale space (X, τ_X), by definition, is a Gelfand quantale X together with an algebraically strong right embedding $\tau_X : X \to Q_X$ into a product Q_X of discrete Hilbert quantales. Motivating with the idea of quantal space, we define the notion of semi-quantale space: A semi-quantal space $(\mathbf{Q}, g)$ is a semi-quantale $\mathbf{Q}$ equipped with a semi-quantale morphism g from $\mathbf{Q}$ to the set-indexed product of some semi-quantales. If g is also an embedding, then the semi-quantal space $(\mathbf{Q}, g)$ is said to be T_0.

In this talk, our main problem is to find out whether there exists a categorical connection, possibly a categorical equivalence, between semi-quantal spaces and lattice-valued topological spaces. For this purpose, we first show that a (T_0) semi-quantal space $(\mathbf{Q}, g)$ can be identified with a (point-separating) small source $(\mathbf{Q}, \mathcal{H})$ [1] in the category **SQuant** of semi-quantales and semi-quantal morphisms [6], and call the associated (point-separating) small source $(\mathbf{Q}, \mathcal{H})$ a (spatially) pointed semi-quantale. Here a pointed semi-quantale $(\mathbf{Q}, \mathcal{H})$ is simply a semi-quantale $\mathbf{Q} = (Q, \leq, \otimes)$ together with a set-indexed family $\mathcal{H}$ of **SQuant**-morphisms from $\mathbf{Q}$ to some semi-quantales, and its spatiality means that for two distinct elements x, y of Q, there exists an element f of $\mathcal{H}$ such that $f(x) \neq f(y)$.

The identification of semi-quantal spaces with pointed semi-quantales basically results from the fact that the latter makes the formulation of results easier and simpler. In addition to this, spatially pointed semi-quantales can also be viewed as a generalization of spatial locales [3, 4]. In order to clarify this fact, let us first recall the category **Frm** of frames and frame morphisms [3]. **Frm** is obviously a non-full subcategory of **SQuant**. If we consider the non-full subcategory **SSQuant** of **SQuant** of semi-quantales and strong semi-quantale morphisms, i.e. semi-quantale morphisms preserving the top elements, then **Frm** will be full in **SSQuant**. In case **SQuant**-morphisms in the definitions of (T_0) semi-quantal spaces and (spatially) pointed semi-quantales preserve the top elements, we add the adjective "strong" in front of these concepts. Spatial locales can be defined as some special kinds of strong, spatially pointed semi-quantales: Indeed, if we particularly choose the semi-quantale $\mathbf{Q} = (Q, \leq, \otimes)$ as a locale (also known as a frame or a complete Heyting algebra), i.e. $\otimes = \wedge$ and $\wedge$ distributes over arbitrary joins, and if we consider the set $hom(\mathbf{Q}, \mathbf{2})$ of all frame morphisms from $\mathbf{Q}$ to the complete Boolean algebra $\mathbf{2} = (\{0, 1\}, \leq, \bigwedge)$, that are functions from Q to $\{0, 1\}$ preserving finite meets and arbitrary joins, then the spatiality of $\mathbf{Q}$ is obviously equivalent to that $(\mathbf{Q}, hom(\mathbf{Q}, \mathbf{2}))$ is a strong, spatially pointed semi-quantale. Consequently, in order to deal with our main problem, we will not operate with semi-quantal spaces directly, but their pointed semi-quantal identifications.

As a lattice-valued topological counterpart to (spatially) pointed semi-quantales, we will define a (T_0) generalized lattice-valued quasi topology to be a subproduct of a set-indexed family of semi-quantales. In order to justify the appropriateness of the term “generalized lattice-valued quasi topology”, it might be necessary to mention here that generalized lattice-valued quasi topology is a generalization of **L**-quasi topology. Thereafter, we will formulate the categories of pointed semi-quantales and of generalized lattice-valued quasi topological spaces that are denoted by **PSQuant** and **GQTop**. As an answer to our main, we will establish a functor Sp from the opposite category $\mathbf{PSQuant}^{op}$ of **PSQuant** to **GQTop**. Here the functor Sp :$\mathbf{PSQuant}^{op} \to$**GQTop** can be thought of as an extension of the functor $pt_{\mathcal{B}}$:$\mathbf{CGR}^{op} \to \mathcal{B}$-**TOP** in [2] to the present settings. We will conclude this talk with an important observation: If we denote the full subcategory of **PSQuant** of all spatially pointed semi-quantales and the full subcategory of **GQTop** of all T_0 generalized lattice-valued quasi topological spaces by **SPSQuant** and $\mathbf{GQTop}_0$, then the restriction of Sp to $\mathbf{SPSQuant}^{op}$ gives an equivalence from $\mathbf{SPSQuant}^{op}$ to $\mathbf{GQTop}_0$.

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What is fuzzification about?

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1 Introduction

The paper initiates a study of categorical approach to fuzzification processes. Traditionally fuzzification processes on a non-empty set X are performed by applying *Zadeh Extension Principle* (ZEP) [6, 9], and in this paper we are especially interested in arithmetic of fuzzy natural numbers. Linguistically, ZEP is usually intepreted such that any operation on X can be extended to an operation on L^X, where L is typically a completely distributive lattice and for $L = \{0,1\}$ we write $L = \mathbf{2}$. Is it possible to define arithmetical operations on $L^{\mathbb{N}}$ without basing them on the arithmetic of $\mathbb{N}$? Example 1 illustrates this situation.

Example 1. Consider there are some persons, each buying a couple of wine bottles. As a result, they have all together quite many wine bottles. Clearly, 'some persons', 'a couple of wine bottles' and 'quite many wine bottles' can be modelled as L-sets on the set of natural numbers $\mathbb{N}$. The hedge 'quite many' may be considered as some kind of aggregation of 'some' and 'a couple of'.

Example 1 suggests that the arithmetic of natural numbers is not really needed, while the extension principle says that the arithmetic on $\mathbb{N}$ is defined first and then these operations are extended. Now, we recall ZEP mathematically: Let X be a non-empty set. Then, for any $A_1, A_2, \ldots, A_n \in L^X$ and $x_1, x_2, \ldots, x_n \in X$ we have

$$f_L(A_1, A_2, \ldots, A_n)(z) = \bigvee_{f(x_1, x_2, \ldots, x_n) = z} \left(\bigwedge_{i=1}^{n} A_i(x_i) \right), \qquad \text{(MZEP)}$$

where f_L is an n-ary operation on L^X as an extension of $f \colon X^n \longrightarrow X$.

The formula (MZEP) is the traditional way to define, for example, arithmetic of fuzzy natural numbers as arithmetic with fuzzy'. On the other hand, the study in [3] suggests that approaching arithmetic of fuzzy natural numbers as 'arithmetic with fuzzy' is counter intuitive in monadic setting. In fact, the formation of $\mathbb{N}$ defines also the arithmetical operations. Indeed, $\mathbb{N}$ is formed by means of the successor (`succ`) operation, and other arithmetical operations on $\mathbb{N}$ are based on `succ`, that is, they are based on *enumeration*. In this paper we critically deliberate about extentions in the sense of Example 1 and extensions by means of (MZEP).

2 On monad compositions

In the sequel, let L be a completely distributive lattice. The covariant power-set functor L_{id} is obtained by $L_{id}X = L^X$, and for a morphism $X \xrightarrow{f} Y$ in Set we have ([6, 9])

$$\begin{aligned} L_{id}f(A)(y) &= \bigvee_{x\in X} A(x) \wedge f^{-1}(\{y\})(x) \\ &= \bigvee_{f(x)=y} A(x). \end{aligned} \tag{1}$$

Further, define $\eta_X : X \to L_{id}X$ by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and $\mu_X : L_{id}L_{id}X \to L_{id}X$ by

$$\mu_X(\mathcal{A})(x) = \bigvee_{A\in L_{id}X} A(x) \wedge \mathcal{A}(A). \tag{3}$$

We refer [1, 8] for more detailed discussion on power-set functors. Especially, L_{id} is categorically a correct choice to powerset operators in the sense of Rodabaugh ([8]). Moreover, it is clear that (MZEP) and (1) coincide when $n = 1$.

It is well known that the functor L_{id} can be extended to monad with η and μ defined in (2) and (3), respectively. Indeed, the following proposition can be presented:

Proposition 1 ([7]). $\mathbf{L}_{id} = (L_{id}, \eta, \mu)$ *is a monad.*

Note that $\mathbf{2}_{id}$ is the usual covariant power-set monad $\mathbf{P} = (P, \eta, \mu)$, where PX is the set of subsets of X, $\eta_X(x) = \{x\}$ and $\mu_X(\mathcal{B}) = \bigcup \mathcal{B}$, where $\mathcal{B} \in PPX$.

The problem of extending a functor to a monad is not a trivial one, and some strange situations may well arise as shown below. The id^2 functor can be extended to a monad with $\eta_X(x) = (x, x)$ and $\mu_X((x_1, x_2), (x_3, x_4)) = (x_1, x_4)$. Similarly, id^n can be extended to a monad. In addition, the proper power-set functor P_0, where $P_0X = PX \setminus \{\emptyset\}$, as well as $id^2 \circ P_0$ can, respectively, be extended to a monad in a unique way. However, $P_0 \circ id^2$ cannot be made to a monad [4].

Remark 1. Let $\mathbf{\Phi} = (\Phi, \eta^\Phi, \mu^\Phi)$ and $\mathbf{\Psi} = (\Psi, \eta^\Psi, \mu^\Psi)$ be monads over Set. The composition $\Phi \circ \Psi$ cannot always be extended to a monad as we see in the case of $P_0 \circ id^2$.

Especially, $L_0 \circ id^2$ cannot be extended to monad, where $L_0X = L_{id}X \setminus \{\emptyset\}$. One might now try the functor $id^2 \circ L_0$ to obtain (MZEP) as an approach to extend binary operations on X to binary operations on $L_{id}X$. However, the following discussion shows some problems.

Consider we have a binary operation $f\colon X \times X \to X$. It is clear that we can think f as a Set-morphism. Applying L_{id} we have then $L_{id}f\colon L_{id}(X \times X) \to L_{id}X$ such that for any $R \in L_{id}(X \times X)$ and $x, y \in X$,

$$L_{id}f(R)(z) = \bigvee_{z=f(x,y)} R(x, y).$$

Unfortunately, $L_{id}f$ is not a generalization of f in the sense that we should have an operation on $L_{id}X$. Indeed, we would like to have an operation $h\colon (id^2 \circ L_{id})X \to L_{id}X$, but it is clear that this is possible only if we have a natural transformation $\sigma\colon L_{id} \to id^2 \circ L_{id}$. Now, for $X \times X$ we have

$$\sigma_{X\times X}\colon L_{id}(X \times X) \to L_{id}(X \times X) \times L_{id}(X \times X).$$

As a conclusion of this discussion we can say that MZEP may be obtained directly by L_{id} for unary arithmetical operations only.

Concerning generalizations of terms, see [1], we adopt a more functorial presentation of the set of terms, as opposed to using the conventional inductive definition of terms, where we bind ourselves to certain styles of proofs. Even if a purely functorial presentation might seem complicated, there are advantages when we define corresponding monads, and, further, a functorial presentation simplifies efforts to prove results concerning compositions of monads.

For a set A, the constant set functor $A_{\mathtt{Set}}$ is the covariant set functor which assigns sets X to A, and mappings f to the identity map id_A. The sum $\sum_{i\in I}\varphi_i$ of covariant set functors φ_i assigns to each set X the disjoint union $\bigcup_{i\in I}(\{i\}\times\varphi_i X)$, and to each morphism $X\xrightarrow{f}Y$ in $\mathtt{Set}$ the mapping $(i,m)\mapsto(i,\varphi_i f(m))$, where $(i,m)\in(\sum_{i\in I}\varphi_i)X$.

Let k be a cardinal number and $(\Omega_n)_{n\leq k}$ be a family of sets. We will write $\Omega_n id^n$ instead of $(\Omega_n)_{\mathtt{Set}}\times id^n$. Note that $\sum_{n\leq k}\Omega_n id^n X$ is the set of all triples $(n,\omega,(x_i)_{i\leq n})$ with $n\leq k$, $\omega\in\Omega_n$ and $(x_i)_{i\leq n}\in X^n$.

A disjoint union $\Omega=\bigcup_{n\leq k}\{n\}\times\Omega_n$ is an operator domain, and an Ω-algebra is a pair $(X,(s_{n\omega})_{(n,\omega)\in\Omega})$ where $s_{n\omega}:X^n\to X$ are n-ary operations. The $\sum_{n\leq k}\Omega_n id^n$-morphisms between Ω-algebras are precisely the homomorphisms between the algebras.

The term functor can now be defined by transfinite induction. In fact, let $T_\Omega^0=id$ and define

$$T_\Omega^\alpha=(\sum_{n\leq k}\Omega_n id^n)\circ\bigcup_{\beta<\alpha}T_\Omega^\beta$$

for each positive ordinal α. Finally, let

$$T_\Omega=\bigcup_{\alpha<\bar{k}}T_\Omega^\alpha$$

where $\bar{k}$ is the least cardinal greater than k and $\aleph_0$. Clearly, $(n,\omega,(m_i)_{i\leq n})\in T_\Omega^\alpha X$, $\alpha\neq 0$, implies $m_i\in T_\Omega^{\beta_i}X$, $\beta_i<\alpha$.

A morphism $X\xrightarrow{f}Y$ in $\mathtt{Set}$ can also be extended to the corresponding Ω-homomorphism

$$(T_\Omega X,(\sigma_{n\omega})_{(n,\omega)\in\Omega})\xrightarrow{T_\Omega f}(T_\Omega Y,(\tau_{n\omega})_{(n,\omega)\in\Omega}),$$

where $T_\Omega f$ is defined to be the Ω-extension of $X\xrightarrow{f}Y\hookrightarrow T_\Omega Y$ associated to $(T_\Omega Y,(\tau_{n\omega})_{(n,\omega)\in\Omega})$.

We can now extend T_Ω to a monad. Define $\eta_X^{T_\Omega}(x)=x$. Further, let $\mu_X^{T_\Omega}=id_{T_\Omega X}^\star$ be the Ω-extension of $id_{T_\Omega X}$ with respect to $(T_\Omega X,(\sigma_{n\omega})_{(n,\omega)\in\Omega})$.

Proposition 2 ([7]). $\mathbf{T}_\Omega=(T_\Omega,\eta^{T_\Omega},\mu^{T_\Omega})$ *is a monad.*

Proposition 3 ([1]). $(L_{id}T_\Omega,\eta^{L_{id}T_\Omega},\mu^{L_{id}T_\Omega})$*, denoted* $\mathbf{L}_{id}\bullet\mathbf{T}_\Omega$*, is a monad.*

3 Fuzzification of logic - Where and how?

In [5] terms are described in a general setting in a substitution theory. This means essentially generalizing the underlying signature to involving usage of the composed monad $\mathbf{L}_{id}\bullet\mathbf{T}_\Omega$. An effort to generalize the notion of sentences can be found in [2].

The composed functor $T_\Omega L_{id}$ on the other hand is problematic as we are not able to extend it to a corresponding monad $\mathbf{T}_\Omega\bullet\mathbf{L}_{id}$. The distinction between $L_{id}T_\Omega$ and $T_\Omega L_{id}$ is important e.g. with respect to approaches to fuzzy arithmetic, as we need to understand if 'fuzzy arithmetic' produces terms in $L_{id}T_\Omega$ or $T_\Omega L_{id}$. In the latter a composition of substitutions is not possible as the underlying composed functor is not extendable to a monad. We are thus referred to staying within the set $L_{id}T_\Omega X$, and therefore we are NOT doing 'arithmetic with fuzzy' which has been the default approach for 'fuzzy arithmetic'. Especially, arithmetic need to be defined before fuzzification, thus, Example 1 is not an approach to 'fuzzy arithmetic'.

Fuzzy sets of arithmetic expressions, like `approximately` x, are then represented by mappings from $T_\Omega X$ to L. This is in our view intuitively more appealing.

Example 2. [3] Consider the element `approx(x+y)` of LTX, where $L = L_{id}$ and $T = T_\Omega$. With the substitution

```
x := approx 0
y := approx 60
```

applied to `approx(x+y)` we obtain the expression

```
approx(approx 0 + approx 60)
```

which is an element of $LTLTX$. However, applying μ_X^{LT} on `approx(approx 0 + approx 60)` brings $\mu_X^{LT}(\mathtt{approx}(\mathtt{approx0} + \mathtt{approx60}))$ to become an element of LTX.

Let us focus on semantics, fuzzy natural numbers again. It is clear that MZEP can be applied to unary operations, which can be seen as follows:

Example 3. Consider $(\mathbb{N}, \mathtt{succ})$, where $\mathtt{succ}\colon \mathbb{N} \to \mathbb{N}$, is the successor operation. It is clear that `succ` can be extended to $\mathtt{fsucc} = L_{id}\,\mathtt{succ}$ by means of (MZEP). In fact this is just a 'shift' by one unit to the right for $A \in L_{id}\mathbb{N}$. Notice that (MZEP) determines $\mathtt{fsucc}(A)(0) = 0$.

Fortunately, we can set $\mathtt{succ}^A(1) = \mathtt{fsucc}(A)$, where $\mathtt{succ}^A : \mathbb{N} \to L_{id}\mathbb{N}$. It may be also reasonable to extend $\mathtt{fsucc}^m(A)$ to an L-family of L-sets $\mathcal{A} \in L_{id}L_{id}\mathbb{N}$, which for all $k, m \in \mathbb{N}$ fulfills

$$\mathcal{A}(\mathtt{fsucc}^k(A)) = \eta_{\mathbb{N}}(m)(k),$$

where $\mathtt{fsucc}^k(A)$ means that the operation `fsucc` is applied for k times on A. Naturally, we identify $\mathtt{fsucc}^{\eta_{\mathbb{N}}(m)}(A)$ as $\mathcal{A}$, and it is clear that we have $\mathtt{fsucc}^m(A) = \mu_{\mathbb{N}}(\mathtt{fsucc}^{\eta_{\mathbb{N}}(m)}(A))$. Moreover, we can interpret this as an addition on $L_{id}\mathbb{N}$, thus we have

$$A + \eta_{\mathbb{N}}(m) = \mu_{\mathbb{N}}(\mathtt{fsucc}^{\eta_{\mathbb{N}}(m)}(A)).$$

Finally, an addition of fuzzy natural numbers $A, B \in L_{id}\mathbb{N}$ may be considered as

$$A + B = \mu_{\mathbb{N}}(\mathtt{fsucc}^B(A)). \tag{4}$$

Note that if $\mathcal{A} = \mathtt{fsucc}^B(A)$ we then have for all $k \in \mathbb{N}$, $\mathcal{A}(\mathtt{fsucc}^k(A)) = B(k)$. On syntactical point of view the authors think that (4) may be obtained by means of variable substitution described in [3].

4 Conclusion

In this paper we have critically deliberated about an extention of arithmetic in the sense of Example 1 as 'fuzzy arithmetic'. Moreover, there are doubts on extending the arithmetic of natural numbers by means of (MZEP). However, (MZEP) can be applied to unary operations and other arithmetical operations for fuzzy natural numbers may be produced applying the monad $\mathbf{L}_{id}$ as it was described in Section 3. It is not yet clear which kind of signature and equational logic would have the described semantics.

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L-fuzzy closure systems *

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1 Introduction and preliminaries

Closure systems (closure operators) are very useful tools in several areas of classical mathematics. In the framework of fuzzy set theory, fuzzy closure systems and fuzzy closure operators have been studied by Gerla etc., see e.g. [5–7], where a fuzzy set U is usually defined as a mapping from a universe X to the real interval [0,1] in the above mentioned works. R. Bělohlávek [2] introduced the notion of L_K-closure systems when L is a complete residuated lattice [8–10].

In this paper, we generalize Bělohlávek's L_K-closure system to a more general form, namely, (strongly) L-fuzzy closure system when L is a complete residuated lattice. In addition, we propose a new form of fuzzy closure operators, called strongly L-fuzzy closure operator, and show the strongly L-fuzzy closure operator is a suitable closure operator, which has close relationships with the strong L-fuzzy closure system. In categorical aspect, we establish a Galois correspondence between the category of (strongly) L-fuzzy closure system spaces and that of (strongly) L-fuzzy closure spaces. Finally, indeed, we point out that every Bělohlávek's L-closure system could be induced by a strongly L-fuzzy closure system.

Throughout this paper, a complete residuated lattice is a triple $(L,*,1)$ (denoted L, simply) such that (1) L is a complete lattice, (2) $(L,*,1)$ is a commutative monoid, (3) there exists a further binary operation $\rightarrow$ on L such that the condition $a*b \leq c \Longleftrightarrow a \leq b \rightarrow c$ holds for all a, b, $c \in L$. The greatest element of L is denoted by 1 and the least element of L is denoted by 0. Let X be a universe set and the family of all L-subsets on X will be denoted by L^X. By 0_X and 1_X, we denote the constant L-subset on X taking the value 0 and 1, respectively.

2 L-fuzzy closure systems and closure operator

Recall for a nonempty set X, a family Φ of subsets of X is a (classical) closure system on the set X if the intersection of any family of elements of Φ is an element of Φ. R.Bělohlávek [2] proposed a concept of L_K-closure system. Since an L_K-closure system is a classical family of L-subsets on X, as personal viewpoint, it may not be fuzzy closure system really. In the paper, we propose the following definition in L-fuzzy setting.

Definition 1. *A mapping φ: $L^X \rightarrow L$ is called an L-fuzzy closure system (L-fcs, in short) if and only if the following holds*

(S1) $\varphi(1_X) = 1$;

(S2) $\varphi\big(\bigwedge_{j\in J} U_j\big) \geq \bigwedge_{j\in J} \varphi(U_j)$ *for each family of* $\{U_j : j \in J\} \subseteq L^X$.

If φ satisfies (S1), (S2) and in addition,

(S3) $\varphi(a \rightarrow U) \geq \varphi(U)$ *for each* $U \in L^X$, $a \in L$,

then we say that φ is a strongly L-fuzzy closure system (SL−fcs, in short). A pair of (L^X, φ) is called a (strongly) L-fuzzy closure system space if φ is a (strongly) L-fuzzy closure system on X. □

* This work is supported by the National Natural Science Foundation of PR China (No. 10771200)

Obviously, every *SL*-fcs is an *L*-fcs, but we have examples to show the converse needn't be true.

The extension of (classical) closure operators in Birkhoff's sense to *L*-subsets, have been proposed by R. Bělohlávek, i.e., a fuzzy closure closure therein is a mapping $\mathcal{C} : L^X \to L^X$ satisfying (FC1), (FC2) and (FC3) in the definition below. The condition (FC4) seems to be new and readers should find its importance to the following contents.

Definition 2. *An L-fuzzy closure operator (L-fco, in short) on a universal set X is a mapping* $\mathcal{C} : L^X \to L^X$ *satisfying*

(FC1) $\forall U \in L^X$, $S(U, \mathcal{C}(U)) = 1$, *or equivalently,* $U \leq \mathcal{C}(U)$

(FC2) $\forall U, V \in L^X$, $S(U, V) \leq S(\mathcal{C}(U), \mathcal{C}(V))$;

(FC3) $\forall U \in L^X$, $\mathcal{C}(\mathcal{C}(U)) = \mathcal{C}(U)$;

If in addition $\mathcal{C}$ *satisfies*

(FC4) $\forall a \in L, U \in L^X$, $a * \mathcal{C}(U) \leq \mathcal{C}(a * U)$,

we say that $\mathcal{C}$ *is a strongly L-fuzzy closure operator (SL-fco, in short) on X. A pair of* $(L^X, \mathcal{C})$ *is called an (strongly) L-fuzzy closure space if* $\mathcal{C}$ *is an (strongly) L-fuzzy closure operator on X.* $\square$

For a given *L*-fcs (*SL*-fcs), there is a natural way to get an *L*-fco (*SL*-fco). In fact, for a given mapping $\varphi : L^X \to L$, we could define an operator $\mathcal{C}_\varphi : L^X \to L^X$ as follows:

$$\mathcal{C}_\varphi(U) = \bigwedge_{W \in L^X} \varphi(W) * S(U, W) \to W$$

for each $U \in L^X$. Then we have

Proposition 1. *(1) If* φ *is an L-fuzzy closure system, then* $\mathcal{C}_\varphi : L^X \to L^X$ *ia an L-fuzzy closure operator; (2) If* φ *is a strongly L-fuzzy closure system, then* $\mathcal{C}_\varphi : L^X \to L^X$ *ia a strongly L-fuzzy closure operator.* $\square$

Conversely, from a given *L*-fco (*SL*-fco), there is also a method to obtain an *L*-fcs (*SL*-fcs). Precisely, if $\mathcal{C} : L^X \to L^X$ be an operator, we can define $\varphi_{\mathcal{C}} : L^X \to L$ by

$$\varphi_{\mathcal{C}}(U) = S(\mathcal{C}(U), U)$$

for each $U \in L^X$, where $S(U, V) = \bigwedge_{x \in X} U(x) \to V(x)$ for each pair $(U, V) \in L^X \times L^X$ [3, 4, 8, 14]. Thus we have

Proposition 2. *If* $\mathcal{C} : L^X \to L^X$ *is an L-fuzzy closure operator, then* $\varphi_{\mathcal{C}} : L^X \to L$ *is an L-fuzzy closure system. In addition, if* $\mathcal{C} : L^X \to L^X$ *is a strongly L-fuzzy closure operator, then* $\varphi_{\mathcal{C}} : L^X \to L$ *is a strongly L-fuzzy closure system.* $\square$

By Propositions 1, 2, *L*-fcs (*SL*-fcs) and *L*-fco (*SL*-fco) could be induced one by another. These procedures have good representation as describing as follows.

Theorem 1. *Let* $\mathcal{C}$ *be a (strongly) L-fuzzy closure operator on X and* φ *a (strongly) L-fuzzy closure system on a set X, Then it hold*

$$\mathcal{C}_{\varphi_{\mathcal{C}}} = \mathcal{C} \quad \textit{and} \quad \varphi_{\mathcal{C}_\varphi} \geq \varphi. \quad \square$$

At the end of section, we offer examples to show how to get an *L*-closure system [2] from an *SL*-fcs and every *L*-closure system can be considered as a special *SL*-fcs.

Example 1. Suppose that $\varphi : L^X \to L$ is a strongly L-fuzzy closure system. Define $\mathcal{J} = \{W \in L^X \mid \varphi(W) = 1\}$. Then $\mathcal{J}$ is an L-closure system in the sense of [2]. In fact, by Corollary 3.1 [2] and (S3), it holds that

$$\forall W \in \mathcal{J},\, W \in \mathcal{J} \Rightarrow (a \to W) \in \mathcal{J}$$

for all $a \in L$.

Example 2. Suppose that $\mathcal{J} \subseteq L^X$ is an L-closure system. Define a mapping $\varphi : L^X \to L$ such that $\varphi(W) = 1$ when $W \in \mathcal{J}$ and =0 otherwise. Thus, it is easy to check that, by Definition 1, $\mathcal{J}$ is a strongly L-fuzzy closure system as desired.

3 Category of *L*-fuzzy closure system spaces

We devote the section to the categorical aspect of the relationship between L-fuzzy closure spaces and L-fuzzy closure system spaces. We refer to [1] for category theory.
Let $f : X \to Y$ be a mapping. f is called continuous from (L^X, φ) to (L^Y, ψ) if it holds $\varphi(f^{\leftarrow}(W)) \geq \psi(W)$ for all $W \in L^Y$, where and the following $f^{\leftarrow}(W) = W \circ f$ (using the notion of [12, 13]). The category of L-fuzzy closure system spaces with continuous mappings as morphisms is denoted by L-**FCss**. Write SL-**FCss** for the full subcategory of L-**FCss** composed of objects of all strongly L-fuzzy closure system spaces. And f is called continuous from $(L^X, \mathcal{C}_X)$ to $(L^Y, \mathcal{C}_Y)$ if it holds $\mathcal{C}_X(f^{\leftarrow}(W)) \leq f^{\leftarrow}(\mathcal{C}_Y(W))$ for all $W \in L^Y$. The category of L-fuzzy closure spaces with continuous mappings as morphisms is denoted by L-**FCos**. Write SL-**FCos** for the full subcategory of L-**FCos** composed of objects of all strongly L-fuzzy closure spaces. After the definitions and notions above, we want to show that there exists a Galois correspondence [1] between L-**FCss** (SL-**FCss**) and L-**FCos** (SL-**FCos**) indeed. Let us give two propositions

Proposition 3. *If a mapping $f : (L^X, \mathcal{C}_X) \to (L^Y, \mathcal{C}_Y)$ is continuous, then $f : (L^X, \varphi_{\mathcal{C}_X}) \to (L^Y, \varphi_{\mathcal{C}_Y})$ is continuous.* □

Proposition 4. *If a mapping $f : (L^X, \varphi_X) \to (L^Y, \varphi_Y)$ is continuous, then $f : (L^X, \mathcal{C}_{\varphi_X}) \to (L^Y, \mathcal{C}_{\varphi_Y})$ is continuous.* □

From Propositions 3 and 2(1), we obtain a concrete functor $\Xi : L$-**FCos**$\to L$-**FCss** defined by $\Xi : (L^X, \mathcal{C}) \longmapsto (L^X, \varphi_{\mathcal{C}})$ and $f \longmapsto f$. Note that we still write Ξ for the restriction of the functor $\Xi : L$-**FCos** $\to L$-**FCss** to the full subcategory SL-**FCos**, and by Proposition 2(2), $\Xi : SL$-**FCos** $\to SL$-**FCss** form a concrete functor also.

From Propositions 4 and 1(1) we obtain a concrete functor $\Upsilon : L$-**FCss**$\to L$-**FCos** defined by $\Upsilon : (L^X, \varphi) \longmapsto (L^X, \mathcal{C}_{\varphi})$ and $f \longmapsto f$. If we still write Υ for the restriction of the functor $\Upsilon : L$-**FCss**$\to L$-**FCos** to the full subcategory SL-**FCss**, then by Proposition 1(2), $\Upsilon : SL$-**FCss**$\to SL$-**FCos** form a concrete functor

By Theorem 1, if $\mathcal{C}$ is an L-fco (SL-fco) on a set X, then the identity map $id_X : (L^X, \mathcal{C}) \to (L^X, \Upsilon(\Xi(\mathcal{C}))) = (L^X, \mathcal{C}_{\varphi_{\mathcal{C}}})$ is continuous. Moreover, if φ is an L-fcs (SL-fcs) on a set Y, then the identity map $id_Y : (L^Y, \Xi(\Upsilon(\varphi))) = (L^Y, \varphi_{\mathcal{C}_{\varphi}}) \to (L^Y, \varphi))$ is continuous. Therefore, we obtain the following important theorem to reflects the close connection between the category of L-FCos (SL-FCos) and that of L-FCss (SL-FCss), categorically.

Theorem 2. *If an L is a complete residuated lattice, then (Ξ, Υ) is a Galois correspondence between the category of L-fuzzy closure spaces and that of L-fuzzy closure ststem spaces. Further, (Ξ, Υ) is a Galois correspondence between the category of strongly L-fuzzy closure spaces and that of strongly L-fuzzy closure system spaces.* □

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Topological duality methods for lattice ordered algebras

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Duality methods have been extremely useful in the study of various types of algebras and logics that have lattice reducts. This has been relatively less so for those based on chains. In recent years duality methods for lattices with additional operations have evolved significantly and the tools needed to apply duality to classes of algebras generated by chains are available. In this talk we give an introduction to duality theory with particular emphasis on 'double-quasioperator algebras', a class containing the lattice ordered algebras generated by chains.

Lattice-valued categories

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1 Introduction

The extension of the category notion to the lattice-valued case has been first considered in the seminal paper on L-fuzzy sets by J. A. Goguen in 1967: to that extent he used a quite general monoidal structure on the lattice L. In the last decade A. P. Sostak and others have considerably developed the original research line outlined by Goguen giving several examples of lattice-valued categories, mainly in the context of topological and algebraic structures: yet more they use an additional monoidal residuated structure on the lattice L.

Here we go further in the development of lattice-valued category theory with the purpose to link such a theory with the fundamental notions that a category comprises and reflects that in the many-valued context are L-sets and L- order relations. The main features of the approach we propose consist in the use of many-valued relations, instead of functions, to describe the structure of a category and to define functors and in the use of an implicative structure, instead of a monoidal one, in the involved lattice.

2 Basic notions and definitions

We assume L to be a complete distributive extended order (shortly cdeo) algebra (introduced in [2]), which is associative or commutative when needed, according to the following definitions.

Definition 1. *A complete, distributive extended-order algebra (cdeo algebra) is a (2.0)-type algebra* $(L, \rightarrow, \top)$ *such that*

- *the binary relation $\leq$ in L defined by*

$$x \leq y \text{ if and only if } x \rightarrow y = \top$$

  is a partial order;
- $(L, \leq)$ *is a complete lattice with maximum* $\top$ *and minimum, say,* $\bot$*;*
- $\bigvee A \rightarrow \bigwedge B = \bigwedge (A \rightarrow B)$*, for all* $A, B \in L$
  where $A \rightarrow B = \{a \rightarrow b \mid a \in A, b \in B\}$.

In a cdeo algebra $(L, \rightarrow, \top)$ the *adjoint product* $\otimes : L \times L \rightarrow L$ can be defined by

$$a \otimes b = \bigwedge \{x \in L \mid b \leq a \rightarrow x\}.$$

Definition 2. *A cdeo algebra* $(L, \rightarrow, \top)$ *is* commutative *if*

$$(c)\text{: } a \rightarrow (b \rightarrow c) = \top \Leftrightarrow b \rightarrow (a \rightarrow c) = \top,$$

for all $a, b, c \in L$*. L is* associative *if either of the following, equivalent conditions is satisfied*

(a_1) : $a \to (b \to c) = (\bigwedge\{x \mid a \to (b \to x) = \top\}) \to c$,

(a_2) : $a \to (b \to c) = (b \otimes a) \to c$,

Proposition 1. *L is commutative (associative, respectively) if and only if the adjoint product $\otimes$ is commutative (associative, respectively).*

A commutative and associative cdeo algebra is nothing but a complete residuated lattice or, with a different terminology, a strictly two sided commutative quantale. Our preference for the implicative approach has several motivation, including that we need all the properties of the implication operation but we do not need the commutativity or even the associativity of the product. Moreover a cdeo algebra suits well as a range for L-order relations and, if L is associative then the implication itself is an L-order on L. Eventually, we use just a cdeo algebra L to define L-categories and L-functors.

Definition 3. *An L-category $\mathcal{C}$ on a class of objects O and a class of morphisms $\mathcal{M}$ is a sextuple*

$$\mathcal{C} = (\omega, \mu, \delta, \gamma, *, \iota)$$

with

- *an L-class of objects, $\omega : O \to L$;*
- *an L-class of morphisms, $\mu : \mathcal{M} \to L$;*
- *a domain L-relation $\delta : O \times \mathcal{M} \to L, (X, f) \mapsto \delta_X(f)$ and a codomain L-relation $\gamma : O \times \mathcal{M} \to L, (Y, f) \mapsto \gamma_Y(f)$ such that for all $A, B \in O, f \in \mathcal{M}$*
$$\delta_A(f) \leq \omega(A), \;\; \gamma_B(f) \leq \omega(B) \tag{1}$$
- *an L-set $\mu_{XY} : \mathcal{M} \to L$, for every $(X, Y) \in O \times O$, such that*
$$\mu = \bigvee\{\mu_{XY} \mid X, Y \in O\} \tag{2}$$
and for all $X, Y \in O, f \in \mathcal{M}$
$$\mu_{XY}(f) \leq \delta_X(f) \otimes (\delta_X(f) \to \gamma_Y(f)); \tag{3}$$
- *a composition L-relation $* : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to L$, $(f, g, l) \mapsto (f * g)(l)$, such that for all $X, Y, Z \in O, f, g, l \in \mathcal{M}$*
$$\mu_{YZ}(g) \leq (f * g)(l) \to (\mu_{XY}(f) \to \mu_{XZ}(l)); \tag{4}$$
- *an identity L-relation $\iota : O \times \mathcal{M} \to L, (X, f) \mapsto \iota_X(f)$, such that for every $X \in O, \; f \in \mathcal{M}$*
$$\iota_X(f) \leq \mu_{XX}(f) \tag{5}$$
and for all $X \in O$,
$$\bigvee\{\iota_X(f) \mid f \in \mathcal{M}\} = \omega(X); \tag{6}$$
moreover the following conditions have to be satisfied for all $X, Y \in O$ and $f, g, h \in \mathcal{M}$
$$\mu_{XY}(g) \leq \iota_X(f) \to (f * g)(g) \tag{7}$$
$$\iota_Y(g) \leq \mu_{XY}(f) \to (f * g)(f) \tag{8}$$
$$(f * g) * h = f * (g * h) : \mathcal{M} \to L \tag{9}$$
where for all $f, g, h, l \in \mathcal{M}$
$((f * g) * h)(l) = \bigvee_p (f * g)(p) \otimes (p * h)(l)$
$(f * (g * h))(l) = \bigvee_q (f * q)(l) \otimes (g * h)(q)$.

Elementary examples show that L-sets, L-ordered sets and cdeo algebras can be viewed as L-categories. The above definition can be weakened, so allowing further examples of L-categories.

Definition 4. *An L-functor $\mathcal{F}$ from the L-category $\mathcal{C} = (\omega, \mu, \delta, \gamma, *, \iota)$ to the L-category $\mathcal{C}' = (\omega', \mu', \delta', \gamma', *', \iota')$ consists in a pair of L-relations, both denoted $\mathcal{F}$,*

$$\mathcal{F} : O_{\mathcal{C}} \times O_{\mathcal{C}'} \to L, \;\; \mathcal{F} : \mathcal{M}_{\mathcal{C}} \times \mathcal{M}_{\mathcal{C}'} \to L$$

such that the following conditions are satisfied, for all considered objects and morphisms

$$\mathcal{F}(X, X') \leq \omega(X) \to \omega'(X') \tag{10}$$

$$\mathcal{F}(f, f') \leq \mu(f) \to \mu'(f') \tag{11}$$

$$\mathcal{F}(f, f') \leq \mathcal{F}(A, A') \to (\delta_A(f) \to \delta'_{A'}(f')) \tag{12}$$

$$\mathcal{F}(f, f') \leq \mathcal{F}(B, B') \to (\gamma_B(f) \to \gamma'_{B'}(f')) \tag{13}$$

$$\mathcal{F}(f, f') \leq \mathcal{F}(Y, Y') \to (\mathcal{F}(X, X') \to (\mu_{XY}(f) \to \mu'_{X'Y'}(f'))) \tag{14}$$

$$\mathcal{F}(f, f') \leq \mathcal{F}(X, X') \to (\iota_X(f) \to \iota'_{X'}(f')) \tag{15}$$

$$\mathcal{F}(l, l') \leq \mathcal{F}(g, g') \to (\mathcal{F}(f, f') \to ((f * g)(l) \to (f' *' g')(l'))) \tag{16}$$

Under suitable conditions on L the composition of L-functors can be obtained by means of the composition of L-relations and the L-categories are so the objects of a category whose morphisms are L-functors.

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Localic real-valued functions: a general setting*

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By regarding the real numbers as a locale $\mathfrak{L}(\mathbb{R})$ rather than a space one arrives to the localic continuous real-valued functions [1]

$$h : \mathfrak{L}(\mathbb{R}) \to L$$

which essentially generalize classical continuous real-valued functions

$$f : X \to \mathbb{R}$$

(certainly a generalization for sober spaces).

It is then natural to consider lower and upper reals (like Dedekind sections but with only the left or right parts) in order to define upper semicontinuous and lower semicontinuous real-valued functions on locales. For upper semicontinuity, which is defined from below, one uses lower reals. The functions take their values as lower reals. For lower semicontinuity, which is defined from above, one uses upper reals.

Lower and upper reals are easily introduced in a pointfree way by defining suitable subframes of $\mathfrak{L}(\mathbb{R})$ ($\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$, respectively), independent of any notion of real number, using the fact that frames may be presented by generators and relations. This way, upper semicontinuous and lower semicontinuous real functions on a locale L are defined as, respectively, frame homomorphisms $\mathfrak{L}_l(\mathbb{R}) \to L$ and $\mathfrak{L}_u(\mathbb{R}) \to L$ satisfying some additional condition (see [2]), and have proved to be the right approach to develop semicontinuity in pointfree topology (see [2], [3], [4]).

Nevertheless, the fact that these three classes of morphisms have different domains is somewhat unsatisfactory: one does not see continuous maps inside upper and lower semicontinuous ones; in particular, one would always expect a map to be continuous if and only if it is both upper and lower semicontinuous. It is the main purpose of this paper to remedy this, by developing a general notion that encapsulates the three kinds of morphisms and plays the role of pointfree counterpart for the class $F(X,\mathbb{R})$ of all real-valued functions on a set X.

To put this in perspective, we recall that by the well-known adjunction between the categories of topological spaces and frames

$$\begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "\mathcal{O}"] & \mathsf{Frm} \arrow[l, shift left, "\Sigma"] \end{tikzcd}$$

there is a bijection

$$\mathsf{Top}(X,\mathbb{R}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)$$

* The first and the second named authors acknowledge financial support from the Ministry of Education and Science of Spain and FEDER under grant MTM2006-14925-C02-02. The third named author acknowledges financial support from the Centre of Mathematics of the University of Coimbra/FCT

between the set of all continuous real-valued functions from a space (X, OX) into $\mathbb{R}$ (endowed with the usual euclidean topology $\mathcal{T}_e$), and the set of all frame homomorphisms from $\mathfrak{L}(\mathbb{R})$ into the frame of open sets of X. Now, if we are interested in considering not only continuous but all real-valued functions from (X, OX) into $\mathbb{R}$, we use the fact that $F(X,\mathbb{R})$ is in an obvious bijection with $\mathsf{Top}((X,\mathcal{P}(X)),(\mathbb{R},\mathcal{T}))$ for any topology $\mathcal{T}$ on the reals. In particular, we have

$$F(X,\mathbb{R}) \simeq \mathsf{Top}((X,\mathcal{P}(X)),(\mathbb{R},\mathcal{T}_e)).$$

Therefore,

$$F(X,\mathbb{R}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}),\mathfrak{S}X)),$$

where $\mathfrak{S}X$ denotes the lattice of all subspaces of X.

One slogan of locale theory as generalized topology is that elements of the frame $\mathfrak{C}L$ of all sublocales of L are identified as *generalized subspaces*. Thus the above bijection justifies to think on the members of $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}),\mathfrak{C}L)$ as (generalized) real-valued functions on L and to adopt the definition that a *generalized real function* on a frame L is a homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$.

Since the real numbers are, essentially, the homomorphisms $\mathfrak{L}(\mathbb{R}) \to 2$, where 2 denotes the two element frame $\{0 < 1\}$, this also indicates, from the point of view of logic, that real-valued functions on L should be seen as the "$\mathfrak{C}(L)$-valued real numbers".

Our aim is to show that the real-valued functions on L play exactly the same role as they do in the classical framework, and that upper semicontinuous, lower semicontinuous and continuous real functions can be considered as particular cases of them, satisfying

$$\text{u.s.c.} \cap \text{l.s.c.} = \text{continuous}.$$

More specifically, we achieve the following:

- Upper and lower semicontinuous functions on a frame can now be identified with real-valued functions on the (dual) frame of sublocales such that the images of all elements of the form $(-,q)$ (resp. $(p,-)$) are closed sublocales.
- The identification above can only be stated for the semicontinuous functions that satisfy the additional condition considered in [2]. This clarifies the role of the mentioned condition.
- After having this bijection at hand one can see semicontinuous functions as a particular kind of real-valued functions on the (dual) frame of sublocales, with the same domain, namely $\mathfrak{L}(\mathbb{R})$.
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the (dual) frame of sublocales, we can compare them.
- Then by defining the algebraic operations for arbitrary real-valued functions that establish the function algebra $\mathcal{R}(\mathfrak{C}L)$, we obtain, in particular, a way of defining the sum of an upper semicontinuous function with a lower semicontinuous one.
- The class of continuous functions is precisely the intersection of the two classes of upper and lower ones.

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Uniform-type structures on lattice-valued spaces and frames

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The motivation for this work arose from an observation of Pultr and Rodabaugh in [4] that lattice-valued frames may be useful in the establishment of well-founded definitions of uniform-type structures.

For X a topological space, the lattice $O(X)$ of its open sets is a frame (i.e., a complete lattice in which finite meets distribute over arbitrary joins), and for any continuous map $f : X \to Y$, the map $O(f) : O(Y) \to O(X)$ defined by $O(f)(U) = f^{-1}(U)$ is a frame homomorphism (i.e., preserves finite meets and arbitrary joins). Thus we have a contravariant functor $O : \mathsf{Top} \to \mathsf{Frm}$ between the category of topological spaces and the category of frames. Besides, considering the standard spectrum construction $\Sigma(M) = \big(\{p : M \to \mathsf{2} \mid p \in \mathsf{Frm}\}, \{\Sigma_m \mid m \in M\}\big)$ for a frame M (where $\Sigma_m = \{p \mid p(m) = 1\}$) and defining, for each frame homomorphism $h : M \to N$, $\Sigma(h) : \Sigma(N) \to \Sigma(M)$ by $\Sigma(h)(p) = p \circ h$, a contravariant functor $\Sigma : \mathsf{Frm} \to \mathsf{Top}$ is obtained, which is a right adjoint for O.

The above adjunction $O \dashv \Sigma$ can be easily adapted to the uniform setting, giving an adjunction between the category UFrm of uniform frames (defined in [2] and studied in detail in [3] in terms of covers) and the category Unif of uniform spaces.

In [4], the authors introduced L-valued frames, which relate to frames in a way parallel to that in which L-valued topological spaces relate to topological spaces vía the ι_L functor. Moreover, when L is linearly ordered or, more generally, a spatial frame (see [1]), there is an adjunction between L-Top and L-Frm which shows that L-valued frames generalize L-valued topological spaces in a similar way to frames generalizing topological spaces.

Then, denoting by F_1 and F_2, respectively, the forgetful functors $\mathsf{Unif} \to \mathsf{Top}$ and $\mathsf{UFrm} \to \mathsf{Frm}$ forgetting the uniform structure, and denoting by χ_L^{T} and χ_L^{F} the (characteristic) functors embedding the categories of $\mathsf{2}$-valued objects in question in the corresponding categories of L-valued objects, we get the following incomplete diagram:

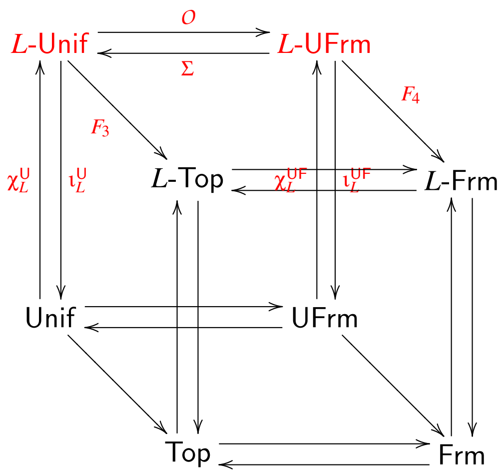
$$
\begin{tikzcd}
\textbf{?} \arrow[rr, shift left] \arrow[dr] & & \textbf{?} \arrow[ll, shift left] \arrow[dr] & \\
& L\text{-}\mathsf{Top} \arrow[rr, shift left, "O"] & & L\text{-}\mathsf{Frm} \arrow[ll, shift left, "\Sigma"] \\
\mathsf{Unif} \arrow[uu] \arrow[rr, shift left, "O"] \arrow[dr, swap, "F_1"] & & \mathsf{UFrm} \arrow[uu] \arrow[ll, shift left, "\Sigma"] \arrow[dr, "F_2"] & \\
& \mathsf{Top} \arrow[uu, "\chi_L^{\mathsf{T}}"] \arrow[rr, shift left, "O"] & & \mathsf{Frm} \arrow[uu, swap, "\chi_L^{\mathsf{F}}"] \arrow[ll, shift left, "\Sigma"]
\end{tikzcd}
$$

A natural question arises:

> *Does there exist two types of structure that would allow us to complete the cube (by filling in the two question marks) in such a way that the two new vertical arrows also represent embedding functors, that the two new diagonal arrows also represent forgetful functors, that the new horizontal arrows also establish an adjunction, and that the whole diagram commutes?*

It is our purpose in this work to show that all the points raised above can be addressed in a satisfactory way. We introduce categories L-Unif and L-UFrm (for strictly two-sided commutative quantales L) that fill in the two question marks: they are related respectively to the categories L-Top and L-Frm in a parallel way and again, when L is linearly ordered or a spatial frame, L-UFrm generalizes L-Unif in a similar way to uniform frames generalizing uniform spaces. We also present an equivalent description of L-Unif in terms of residuated mappings that will encompass Hutton's original definition whenever L is a Girard quantale.

We will therefore obtain the following commutative cube:



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Conditional probabilities on MV-algebras

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1 Introduction

In this talk we deal with a multivariate ”conditioning” as initiated by B.O. Koopman [7] for Boolean algebras, further developed by H.T. Nguyen et al. [8] and for MV-algebras and Girard algebras by U. Höhle and S. Weber [4],[5],[9],[10]. In the preparatory part, we consider ”reflectively radicable” residuated lattices, a certain class of residuated lattices having all n-fold roots (generalizing U. Höhle's square roots to an arbitrary order n). Next, we study ordered n-tuples (with $n \geq 2$) of elements of a residuated lattice L which with regard to possible applications to probability theory are interpret as *variables* in L. Then we introduce a lattice $L^{\mathbf{n}}$ of variables (”**n**-convolution” of L) equipped with a residuated structure generalizing the ”canonical extension” of L proposed by U. Höhle and S. Weber [4]. Further, we consider an associated to $L^{\mathbf{n}}$ lattice $\tilde{L^{\mathbf{n}}}$ of variables of a special form which we call *conditional events* w.r.t. variables in L generalizing the ”interval approach” of U. Höhle and S. Weber. In the second part, we describe ‘”mean values” of those variables and conditional events generalizing results of U. Höhle and S. Weber. Concrete examples are considered and the last step is to apply these constructions in a setting for measures in MV-algebras.

2 Residuated lattices

Definition 21 *A* (commutative, integral) residuated lattice *is a structure*

$$L = (L, \vee, \wedge, \times, \dot{\div}, \top, \bot)$$

with four binary operations and two constants such that

(i) $(L, \vee, \wedge, \top, \bot)$ *is a bounded lattice with the largest element* $\top$ *and the least element* $\bot$ *(with respect to the lattice ordering* $\leq$*);*

(ii) $(L, \times, \top)$ *is a commutative monoid with the unit element* $\top$*;*

(iii) there exists a further binary operation (in our notation) $\dot{\div} : L \times L \to L$*, called* residuum *(also known as* residuation *or* residual implication *with notations* $\to$ *or* $\to_\times$ *provided with the following property:*

$$a \times b \leq c \Leftrightarrow a \leq \frac{c}{b} \text{ (residuated pair).}$$

Definition 22 *A* Girard algebra $L = (L, \vee, \wedge, \times, \dot{\div}, \wp, \neg, \top, \bot)$ *is a residuated lattice* $(L, \vee, \wedge, \times, \dot{\div}, \top, \bot)$ *equipped with the additional two operations: the involution* $\neg : L \to L$ *(termed* negation *operation) and the second commutative semigroup operation* $\wp : L \times L \to L$ *with the unit element* $\bot$*, the dual operation associated with* $\times$*, by setting* $\neg a = \frac{\bot}{a}$ *and* $a \wp b = \neg(\neg a \times \neg b)$*, which in the case of* disjoint *pair* $a, b \in L$ *(i.e.,such that* $a \times b = \bot$*) will be denoted by* $a \dot{\wp} b$*.*

Definition 23 *(i) A Girard algebra L is called an MV-*algebra *if in it the* divisibility *property $a \times \frac{b}{a} = a \wedge b$ holds.*

(ii) An MV-algebra L is a Boolean algebra *if the semigroup operation $\times$ coincides with the lattice meet: $\times = \wedge$ (which implies that $\wp = \vee$).*

Example 24 *(i) The real unit interval $[0,1]$ equipped with the "nilpotent minimum" t-norm $a \times b$ equal to $\min(a,b)$ if $a+b > 1$ and equal to 0 if $a+b \leq 1$ (introduced by J. Fodor in 1995), with the residuum $\frac{a}{b}F$ equal to $\max(a, 1-b)$ if $b > a$ and equal to 1 if $b \leq a$, with the negation $\neg a = 1-a$ and with the "nilpotent maximum" t-conorm given by: $a \wp b$ is equal to $\max(a,b)$ if $a+b < 1$ and equal to 1 if $a+b \geq 1$ is a Girard algebra. This structure we calll Fodor algebra and denote it $[0,1]_F$.*

(ii) The interval $[0,1]$ equipped with the t-norm ("Łukasiewicz arithmetic conjunction") $a \times b = \max(a+b-1, 0)$, with the residuum (written as $\frac{\cdot}{\cdot}Ł$) $\frac{a}{b}Ł = \min(a+1-b, 1)$, with the negation $\neg a = 1-a$ and with the t-conorm ("Łukasiewicz arithmetic disjunction") $a \wp b = \min(a+b, 1)$ is an MV-algebra referred to as Łukasiewicz algebra. *This algebra will be denoted by $[0,1]_Ł$.*

Now we proceed to define "reflectively radicable" residuated lattices, a certain class of residuated lattices having all n-fold roots (generalizing U. Höhle's square roots to an arbitrary order n).

Definition 25 *Let L be a residuated lattice. If there exists a unary operation $\sqrt[n]{\cdot} : L \to L$ fulfilling the following two properties: $a^n \leq b \Rightarrow a \leq \sqrt[n]{b}$ and $(\sqrt[n]{b})^n \leq b$ for every natural number n and for every pair $a, b \in L$, then we say that L has all n-fold roots and will call $\sqrt[n]{a}$ n-fold root of a (noting that we use the notation c^n for the n-fold multiplication $c \times \ldots \times c$). Moreover, if every n-fold root $\sqrt[n]{b}$ of $b \in L$ satisfies the additional condition: $(\sqrt[n]{b})^n > b$ then we will say that L is* reflectively radicable.

We will say that a reflectively radicable MV-algebra L is strict *if for every natural number n the relation: $\neg\sqrt[n]{\perp} = (\sqrt[n]{\perp})^{n-1}$ holds.*

Example 26 *(i) Fodor algebra $[0,1]_F$ equipped with n-fold roots: $\sqrt{a} = \sqrt[3]{a} = \ldots$ equal to a if $a > \frac{1}{2}$ and equal to $\frac{1}{2}$ if $a \leq \frac{1}{2}$ obviously has all n-fold roots. But it is not reflectively radicable.*

(ii) Łukasiewicz algebra $[0,1]_Ł$ equipped with n-fold roots $\sqrt[n]{a} := \frac{a+n-1}{n}$ is reflectively radicable and even strict.

Proposition 27 *Let L be a strict MV-algebra. Then the identity $\neg(\sqrt[n]{a_1} \times \ldots \times \sqrt[n]{a_n}) = \sqrt[n]{\neg a_1} \times \ldots \times \sqrt[n]{\neg a_n}$ holds.*

3 Variables and conditional events in a residuated lattice

Let L be a residuated lattice and $\mathbf{n}$ be the chain of integers $\mathbf{n} = \{0, 1, \ldots, n-1\}$ with $n \geq 2$. Consider the set $L^{\mathbf{n}}$ of ordered n-tuples $f = \langle f_0, \ldots, f_{n-1} \rangle$ with $f_0 \leq \ldots \leq f_{n-1}$. We provide it with the pointwise partial ordering: $f \leq g \Leftrightarrow f_0 \leq g_0, \ldots, f_{n-1} \leq g_{n-1}$. Obviously these n-tuples (which will be referred to as *variables* in L) form a bounded lattice with $f \wedge g = \langle f_0 \wedge g_0, \ldots, f_{n-1} \wedge g_{n-1} \rangle$, $f \vee g = \langle f_0 \vee g_0, \ldots, f_{n-1} \vee g_{n-1} \rangle$, $\perp = \langle \perp, \ldots, \perp \rangle$ and $\top = \langle \top, \ldots, \top \rangle$.

Theorem 31 *Let L be a residuated lattice. Then the bounded lattice $L^{\mathbf{n}}$ equipped with the convolution product (denoted by $*$) given by*

$$(f * g)_j = \bigvee_{i=0}^{j} f_i \times g_{j-i},\ j = 0, \ldots, n-1,$$

is a residuated lattice with convolution residuals in the following form:

$$(\frac{f}{g})_j = \bigwedge_{i=0}^{n-1-j} \frac{f_{i+j}}{g_i},\ j = 0,\ldots,n-1.$$

This residuated lattice will be referred to as the **n**-convolution *of L. Moreover, if L is a Girard algebra then so is its* **n**-*convolution* $L^{\mathbf{n}}$ *with the convolution negation* $\perp$ *and the dual operation # associated to* $*$ *given by:*

$$(f^{\perp})_j = \neg f_{n-1-j} \text{ and } (f\#g)_j = \bigwedge_{i=0}^{n-1-j} f_{n-1-i} \wp g_{i+j},\ j = 0,\ldots,n-1.$$

Corollary 32 *Let L be a Girard algebra and let* $L^{\mathbf{n}}$ *be its* **n**-*convolution. Then the following assertions are equivalent:*

(i) $L^{\mathbf{n}}$ *is an MV-algebra,*
(ii) L is a Boolean algebra.

Definition 33 *Let L be an MV-algebra and let* $a, f_0, \ldots, f_{n-2} \in L$ *be elements of L such that* $f_0 \leq \ldots \leq f_{n-2}$. *Then the* conditional event $(a \parallel f_0,\ldots,f_{n-2})$ *of* $a \in L$ *with respect to a "condition"* $\langle f_0,\ldots,f_{n-2},\top\rangle \in L^{\mathbf{n}}$ *is defined as the element of* $L^{\mathbf{n}}$ *(i.e., variable in L):*

$$(a \parallel f_0,\ldots,f_{n-2}) = \langle a\wedge f_0, \frac{a\wedge f_0}{f_{n-2}},\ldots,\frac{a\wedge f_0}{f_0}\rangle = \frac{\langle a\wedge f_0\rangle}{\langle f_0,\ldots,f_{n-2},\top\rangle},$$

which can be rewritten in the form:

$$(a \parallel f_0,\ldots,f_{n-2}) = \langle a\wedge f_0\rangle \dot{\#} \langle f_0,\ldots,f_{n-2},\top\rangle^{\perp}$$

in notations $\langle b\rangle$ *abbreviating the expression* $\langle b,\ldots,b\rangle$ *of a constant variable. (The next step would be to introduce conditional variables w.r.t. variables.)*

We denote by $\tilde{L^{\mathbf{n}}}$ the set of all conditional events in an *MV*-algebra *L*.

Lemma 34 *Variables in an MV-algebra L are in an one-to-one correspondence to conditional events in L via*

$$\langle g_0, g_1,\ldots,g_{n-1}\rangle = (g_0 \parallel \frac{g_0}{g_{n-1}},\ldots,\frac{g_0}{g_1}).$$

We introduce the bounded lattice and residuated structures in $\tilde{L^{\mathbf{n}}}$ carried to it by this correspondence.

Theorem 35 *Let L be an MV-algebra and let* $\tilde{L^{\mathbf{n}}}$ *be associated with it the lattice of conditional events. Then there exists a binary operation* $*$ *on* $\tilde{L^{\mathbf{n}}}$ *such that it is a Girard algebra. This operation is given by:*

$$(a \parallel f_0,\ldots,f_{n-2}) * (b \parallel g_0,\ldots,g_{n-2}) = (\alpha_0 \parallel \phi_0,\ldots,\phi_{n-2}),$$

where

$$\alpha_0 = (a\wedge f_0)\times(b\wedge g_0),$$

$$\phi_j = \frac{(a\wedge f_0)\times(b\wedge g_0)}{\frac{a\wedge f_0}{f_j}\times\frac{b\wedge g_0}{g_j}\times(f_j\vee g_j)} \wedge \frac{(a\wedge f_0)\times(b\wedge g_0)}{\bigvee_{i=1}^{n-j-2}\frac{a\wedge f_0}{f_{n-1-i}}\times\frac{b\wedge g_0}{g_{i+j}}},$$

$$j = 0,\ldots,n-3 \textit{ (with } n > 2\textit{)},$$
$$\phi_{n-2} = \frac{(a\wedge f_0)\times(b\wedge g_0)}{\frac{a\wedge f_0}{f_{n-2}}\times\frac{b\wedge g_0}{g_{n-2}}\times(f_{n-2}\vee g_{n-2})}.$$

Next, the residuals are given by:

$$\frac{(a\parallel f_0,\ldots,f_{n-2})}{(b\parallel g_0,\ldots,g_{n-2})} = (\beta_0\parallel \chi_0,\ldots,\chi_{n-2}),$$

where

$$\beta_0 = \frac{a\wedge f_0}{b\wedge g_0}\wedge\bigwedge_{i=0}^{n-2}\frac{\neg(b\wedge g_0)\times g_i}{\neg(a\wedge f_0)\times f_i},$$
$$\chi_0 = \beta\dot{\wp}(\neg(a\wedge f_0)\times f_0\times(b\wedge g_0)),$$
$$\chi_j = \beta\dot{\wp}[(\neg(a\wedge f_0)\times f_j\times(b\wedge g_0))\vee\bigvee_{i=0}^{j-1}(\frac{b\wedge g_0}{g_{n-1-j+i}}\times\neg(a\wedge f_0)\times f_i)],$$
$$j = 1,\ldots,n-2.$$

Further, the convolution negations are determined by:

$$(a\parallel f_0,\ldots,f_{n-2})^{\perp} = (\neg a\times f_0\parallel f_0,\frac{f_0}{f_{n-2}},\ldots,\frac{f_0}{f_1}).$$

Finally, the dual operation # associated with $*$ has the following form:

$$(a\parallel f_0,\ldots,f_{n-2})\#(b\parallel g_0,\ldots,g_{n-2}) = (\gamma_0\parallel\psi_0,\ldots,\psi_{n-2}),$$

where

$$\gamma_0 = \begin{cases}(a\wedge f_0)\wp(b\wedge g_0)\wp\neg(f_0\vee g_0) & \textit{if } n=2\\ (a\wedge f_0)\wp(b\wedge g_0)\wp\neg(f_0\vee g_0\vee\bigvee_{i=1}^{n-2}f_i\times g_{n-1-i}) & \textit{if } n>2,\end{cases}$$
$$\psi_j = \gamma\dot{\wp}(\neg(a\wedge f_0)\times\neg(b\wedge g_0)\times\bigvee_{i=0}^{j}f_i\times g_{j-i}),\ j=0,\ldots,n-2.$$

4 Mean values of variables and of conditional events

In this section we present a multivariate version of the U. Höhle and S. Weber "mean value approach" to conditioning in MV-algebras, in Girard algebras, and also in general residuated lattices described in [4],[5],[9],[10]. This approach of U. Höhle and S. Weber requires the existence of mean value functions with the crucial property of "compatibility with the complement". Here we bring forward a new property of "negative weak linearity" for mean value functions on Girard algebras.

Let $L = (L,\vee,\wedge,\times,\dot{:},\wp,\neg,\top,\perp)$ be a Girard algebra. In imitation of the J.-M. Andreoli and J.-Y. Girard's polarity (positive/negative) in linear logic ([1]) we name the semigroup operation $\times$ "positive multiplication", and the dual to it operation $\wp$ "negative multiplication". Moreover, we look at the lattice join $\vee$ as a "positive addition", and at the lattice meet $\wedge$ as a "negative addition". Thus, the reduct $L_{\vee\times} = (L,\vee,\perp,\times,\top)$ become a "positive" semiring and the reduct $L_{\wedge\wp} = (L,\wedge,\top,\wp,\perp)$ "negative" semiring. In the join-lattice reduct $L^{\mathbf{n}}_{\vee} = (L^{\mathbf{n}},\vee,\perp)$ we can introduce the structure of a $L_{\vee\times}$-semimodule, namely, the "positive" scalar multiplication by elements of the positive semiring $L_{\vee\times}$ (also denoted by $\times$) on $L^{\mathbf{n}}_{\vee}$ by the formula $a\times f = \langle a\times f_0,\ldots,a\times f_{n-1}\rangle$. Dually, starting from the negative semiring $L_{\wedge\wp}$ of a Girard algebra, in the meet-lattice reduct $L^{\mathbf{n}}_{\wedge} = (L^{\mathbf{n}},\wedge,\top)$ we can introduce the "negative" scalar multiplication (also denoted by $\wp$) by $a\wp f = \langle a\wp f_0,\ldots,a\wp f_{n-1}\rangle$.

Definition 41 *Let L be a residuated lattice, $\mathbf{n} = \{0, 1, \ldots, n-1\}$ be the finite chain of integers, $L^{\mathbf{n}}$ be the $\mathbf{n}$-convolution of it. A* mean value *function on L is a map $\mathbb{E}$ from $L^{\mathbf{n}}$ to L satisfying the following axioms:*

(i) $\mathbb{E}\langle a, \ldots, a\rangle = a$ *(idempotency),*

(ii) $f \leq g \Rightarrow \mathbb{E}f \leq \mathbb{E}g$ *(isotonocity).*

(iii) In the case when L is a Girard algebra a mean value function $\mathbb{E}$ on L will be said to be negatively weakly linear *provided that*

$$a \times f = \bot \Rightarrow \mathbb{E}(a \wp f) = a \wp \mathbb{E}f \text{ (negative weak linearity)}$$

for each scalar $a \in L$ and each $f \in L^{\mathbf{n}}$, where the last two multipliers a and $\mathbb{E}f$ among themselves are always disjoint, i.e., $a \times \mathbb{E}f = \bot$. This is because

$$a \times f = \bot \Rightarrow f \leq \langle a\rangle^{\perp} = \langle \neg a\rangle \Rightarrow \mathbb{E}f \leq \mathbb{E}\langle \neg a\rangle = \neg a \Rightarrow a \times \mathbb{E}f = \bot.$$

Thus, in notation for products of disjoint elements, we can reformulate the axiom as

$$\mathbb{E}(a \dot{\wp} f) = a \dot{\wp} \mathbb{E}f.$$

(iv) A mean value function $\mathbb{E}$ on a Girard algebra L is said to be compatible *with the negation of L if it satisfies the following additional condition:*

$$\neg(\mathbb{E}f) = \mathbb{E}(f^{\perp}), \text{ i.e., } \neg(\mathbb{E}\langle f_0, \ldots, f_{n-1}\rangle) = \mathbb{E}\langle \neg f_{n-1}, \ldots, \neg f_0\rangle.$$

(v) A mean value function $\mathbb{E}$ on a general residuated lattice L is said to be bisymmetric *if it satisfies the relation:*

$$\begin{aligned} &\mathbb{E}\langle \mathbb{E}\langle f_{0,0}, \ldots, f_{0,n-1}\rangle, \ldots, \mathbb{E}\langle f_{n-1,0}, \ldots, f_{n-1,n-1}\rangle\rangle \\ &= \mathbb{E}\langle \mathbb{E}\langle f_{0,0}, \ldots, f_{n-1,0}\rangle, \ldots, \mathbb{E}\langle f_{0,n-1}, \ldots, f_{n-1,n-1}\rangle\rangle \text{ (bisymmetry)} \end{aligned}$$

for all matrices

$$\begin{pmatrix} f_{0,0} & \cdots & f_{0,n-1} \\ \vdots & \ddots & \vdots \\ f_{n-1,0} & \cdots & f_{n-1,n-1} \end{pmatrix}$$

such that all rows and all columns constitute elements of $L^{\mathbf{n}}$, i.e., it satisfies: $f_{i,j} \leq f_{k,l}$ whenever $i \leq k$ and $j \leq l$.

Example 42 *To each $c \in L$ there exist bisymmetric mean value functions $\mathbb{E}_c$ and $\mathbb{E}^c$ on L (not necessarily compatible with $\bot$) given by*

$$\mathbb{E}_c f = f_0 \vee f_1 \times c \vee \ldots \vee f_{n-1} \times c^{n-1},$$

$$\mathbb{E}^c f = \frac{f_0}{c^{m-1}} \wedge \ldots \wedge \frac{f_{m-2}}{c} \wedge f_{m-1}.$$

Moreover, the latter function is negatively weakly linear.

Proposition 43 *Let L be a reflectively radicable MV-algebra and let $L^{\mathbf{n}}$ be the $\mathbf{n}$-convolution of it (with $n \geq 2$). Then*

$$\mathbb{E}\langle f_0, f_1, \ldots, f_{n-1}\rangle = \sqrt[n]{f_0} \times \sqrt[n]{f_1} \times \ldots \times \sqrt[n]{f_{n-1}}$$

gives a negatively weakly linear and bisymmetric mean value function $\mathbb{E} : L^{\mathbf{n}} \to L$ on L. If L is strict then $\mathbb{E}$ is still compatible with the negation.

Example 44 *Let $[0,1]_{Ł}$ be Łukasiewicz algebra (which is a strict MV-algebra). Then mean values of variables and of conditional events of the preceding proposition take the form:*

$$\mathbb{E}_{Ł}\langle r_0, r_1, \ldots, r_{n-1}\rangle = \frac{1}{n}(r_0 + r_1 + \ldots + r_{n-1})$$

and

$$(a \mid r_0, r_1, \ldots, r_{n-2}) = \min(a, r_0) + 1 - \frac{1}{n}(1 + r_0 + r_1 + \ldots + r_{n-2})$$

(with $a, r_0, \ldots, r_{n-1} \in [0,1]$).

Example 45 *Let $L = [0,1]$ be Goguen and Łukasiewicz algebras simultaneously (equipped with residums $\dot{-}G$ and $\dot{-}Ł$, respectively). For arbitrary $\langle r_0, r_1, \ldots, r_{n-1}\rangle \in [0,1]^{\mathbf{n}}$ (with $n \geq 2$) and $k \in [0,1]$, consider*

$$\frac{r_0}{\frac{r_0}{\frac{r_1}{\vdots_{r_{n-1}}}G}Ł}G = \begin{cases} \dfrac{r_0}{r_0+1-\dfrac{r_1}{\cdots-\dfrac{r_{n-2}}{r_{n-2}+1-r_{n-1}}}} & \text{if } \langle r_0, r_{n-1}\rangle \neq \langle 0,1\rangle \\ k & \text{if } \langle r_0, r_{n-1}\rangle = \langle 0,1\rangle. \end{cases}$$

This quantity defines a mean value function,denoted by $\mathbb{E}_{G,Ł}$ from $[0,1]^{\mathbf{n}}$ to $[0,1]$. In particular, its mean values are the following: in the case when $n = 2$,

$$\mathbb{E}_{G,Ł,k}\langle r_0, r_1\rangle = \begin{cases} \frac{r_0}{r_0+1-r_1} & \text{if } \langle r_0, r_1\rangle \neq \langle 0,1\rangle \\ k & \text{if } \langle r_0, r_1\rangle = \langle 0,1\rangle, \end{cases}$$

in the case when $n = 3$,

$$\mathbb{E}_{G,Ł,k}\langle r_0, r_1, r_2\rangle = \begin{cases} \dfrac{r_0}{r_0+1-\frac{r_1}{r_1+1-r_2}} & \text{if } \langle r_0, r_2\rangle \neq \langle 0,1\rangle \\ k & \text{if } \langle r_0, r_2\rangle = \langle 0,1\rangle, \end{cases}$$

in the case when $n = 4$,

$$\mathbb{E}_{G,Ł,k}\langle r_0, r_1, r_2, r_3\rangle = \begin{cases} \dfrac{r_0}{r_0+1-\dfrac{r_1}{r_1+1-\frac{r_2}{r_2+1-r_3}}} & \text{if } \langle r_0, r_3\rangle \neq \langle 0,1\rangle \\ k & \text{if } \langle r_0, r_3\rangle = \langle 0,1\rangle. \end{cases}$$

5 Conditional probabilities on *MV*-algebras

Definition 51 ([2]) *Let L be a Girard algebra. A map $\mathbb{P} : L \to [0,1]$ is called an* uncertainty measure *if $\mathbb{P}$ satisfies the following conditions:*

(i) $\mathbb{P}\bot = 0$, $\mathbb{P}\top = 1$ *(boundary conditions),*
(ii) $a \leq b \Rightarrow \mathbb{P}a \leq \mathbb{P}b$ *(isotonicity.*

In the case when L is an MV-algebra, an uncertainty measure $\mathbb{P}$ is called *additive* (see, e.g.,[2]) (or *weakly additive* [10]) if it satisfies the axiom: $a \times b = \bot \Rightarrow \mathbb{P}(a \wp b) = \mathbb{P}a + \mathbb{P}b$. We take note of the popular "additivity" of the operation $\wp$. But the author will stick to his guns.

Definition 52 *We say that an uncertainty measure $\mathbb{P}$ on an MV-algebra L is* negatively weakly multiplicative *(instead of its usual name "additive") if it satisfies the axiom:*

(i) $a \times b = \bot \Rightarrow \mathbb{P}(a \wp b) = \mathbb{P}a + \mathbb{P}b$ *(negative weak multiplicativity),*

which, viewing $[0,1]$ *as the Łukasiewicz algebra* $[0,1]_Ł$ *(see [?]), can be rewritten in the form:*

(i)' $\mathbb{P}(a\dot{\wp}b) = \mathbb{P}a\dot{\wp}\mathbb{P}b$,

since $\mathbb{P}a$ *and* $\mathbb{P}b$ *are disjoint in* $[0,1]_Ł$*, because*

$$\mathbb{P}a \times \mathbb{P}b = \max(\mathbb{P}a + \mathbb{P}b - 1, 0) = \max(\mathbb{P}(a\dot{\wp}b) - 1, 0) = 0.$$

A negatively weakly multiplicative measure $\mathbb{P}$ *on an MV-algebra L will be shortly referred to as* probability *on L.*

Note that, replacing $[0,1]_Ł$ by an arbitrary *MV*-algebra *M*, we can talk about *M*-valued probabilities.

Definition 53 *Let L and M be two MV-algebras. A map* $\mathbb{P} : L \to M$ *is said to be an* uncertainty *M*-valued *measure if it satisfies the conditions*

(i) $\mathbb{P}\bot = \bot$, $\mathbb{P}\top = \top$,

(ii) $a \leq b \Rightarrow \mathbb{P}a \leq \mathbb{P}b$.

An uncertainty M-valued measure $\mathbb{P}$ *is said to be* negatively weakly multiplicative *if it satisfies the additional axiom:*

(iii) $\mathbb{P}(a\dot{\wp}b) = \mathbb{P}a\dot{\wp}\mathbb{P}b$ *for every disjoint pair* a, b *of elements of L.*

A negatively weakly multiplicative M-valued measure $\mathbb{P}$ *on an MV-algebra L will be shortly referred to as M-*valued probability *on L.*

We will also consider uncertainty measures on the lattice $L^{\mathbf{n}}$ of variables in an *MV*-algebra *L* (the **n**-convolution of *L*) and on the associated to it lattice $\tilde{L^{\mathbf{n}}}$ of conditional events. Observe that these are *MV*-algebras only in the case when *L* is a Boolean algebra. In the case when *L* is an *MV*-algebra, $L^{\mathbf{n}}$ and $\tilde{L^{\mathbf{n}}}$ unfortunate become Girard algebras.

In analogy to the classical situation (when *L* is a Boolean algebra) we introduce the concepts of a simple random variable and its expectation (see, e.g., [6]). We also associate a simple random variable with a chain of $L^{\mathbf{n}}$.

Definition 54 *Let L be an MV-algebra. Let* **a** *be a finite partition of* $\top$ *in L, i.e., a finite sequence* $\{a_0, a_1, \ldots, a_k\}$ *of pairwise disjoint and different from* $\bot$ *elements of L such that* $a_0\dot{\wp}a_1\dot{\wp}\ldots\dot{\wp}a_k = \top$. *By a simple random variable in L will be meant a real valued function* ξ *determined on* **a** *by setting:*

$$\xi = r_0 I_{a_0} + r_1 I_{a_1} + \ldots + r_k I_{a_k},$$

where $r_0, r_1, \ldots, r_k$ *are pairwise different real numbers and* I_{a_i} *denotes the "indicator" of* a_i*:*

$$I_{a_i}(a_j) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j. \end{cases}$$

Definition 55 *Let L be an MV-algebra, let* $\mathbb{P}$ *be a probability on it, and let* ξ *be a simple random variable defined on a finite partition* $\mathbf{a} = \{a_0, a_1, \ldots, a_k\}$ *of* $\top$ *in L:*

$$\xi = r_0 I_{a_0} + r_1 I_{a_1} + \ldots + r_k I_{a_k}.$$

Then the expectation $\mathbf{E}\xi$ *of* ξ *is the sum:*

$$\mathbf{E}\xi = r_0\mathbb{P}a_0 + r_1\mathbb{P}a_1 + \ldots + r_k\mathbb{P}a_k.$$

Let $f = \langle f_0, f_1, \ldots, f_{n-1}\rangle$ be a variable of the $\mathbf{n}$-convolution $L^{\mathbf{n}}$ of L having $k(\leq n-1)$ "covers" in it, i.e., a finite sequence $f_{i_0}, f_{i_1}, \ldots, f_{i_k}$ of components of f exists such that $f_0 = f_{i_0}$ and f_{i_j} covers $f_{i_{j-1}}$ in f for $j = 1, \ldots, k$, $f_{i_j} > f_{i_{j-1}}$, but that $f_{i_j} > f_l > f_{i_{j-1}}$ for no component f_l of f). We shall proceed to associate a simple random variable with f. For, define $a_0, a_1, \ldots, a_{k+1}$ to be the sequence of elements of L given by setting:

$$a_0 = f_0,\, a_1 = \neg f_0 \times f_{i_1}, \ldots, a_k = \neg f_{i_{k-1}} \times f_{i_k}$$

and else

$$a_{k+1} = \neg(((a_0 \dot{\wp} a_1)\dot{\wp} \ldots)\dot{\wp} a_k) = \neg f_{i_k}.$$

From the disjoint decomposition property of L it follows that in the case when $f_0 \neq \bot$ and $f_{n-1} \neq \top$ the sequence $a_0, a_1, \ldots, a_{k+1}$ obviously forms a partition of $\top$ in L. The other possible cases to consider are as follows: the case when $f_0 = \bot$ and $f_{n-1} = \top$, we have that the shorter sequence $a_1, \ldots, a_k$ forms a partition of $\top$; in the case when $f_0 = \bot$ but $f_{n-1} \neq \top$ then the sequence $a_1, \ldots, a_{k+1}$ makes a partition of $\top$, and in the case when $f_0 \neq \bot$ but $f_{n-1} = \top$ then the partition of $\top$ is formed by the sequence $a_0, \ldots, a_k$. Now a general component f_i of f can be written in the next form:

$$f_i = \begin{cases} a_0 & \text{if } i < i_1 \\ a_0 \dot{\wp} a_1 & \text{if } i_1 \leq i < i_2 \\ \vdots & \vdots \\ ((a_0 \dot{\wp} a_1)\dot{\wp} \ldots)\dot{\wp} a_k & \text{if } i_k \leq i \leq n-1. \end{cases}$$

With this in mind, we arrive at the following:

Definition 56 *The simple random variable ξ_f associated with the variable $f = \langle f_0, f_1, \ldots, f_{n-1}\rangle$ (as above) is given by setting:*

$$\xi_f = I_{a_0} + (1 - \frac{i_1}{n})I_{a_1} + \ldots + (1 - \frac{i_k}{n})I_{a_k}$$

(considering $I_1 = 1$ and $I_0 = 0$).

We shall need the next observation.

Proposition 57 *For the expectation $\mathbf{E}\xi_f$ of ξ_f there are two expressions:*

$$\mathbf{E}\xi_f = \mathbb{P}a_0 + \sum_{j=1}^{k}(1 - \frac{i_j}{n})\mathbb{P}a_j = \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{P}f_i.$$

Now we shall proceed to generalize Theorem 6.5 [2] establishing the existence of the extension of uncertainty measures from a given Boolean algebra to its MV-algebra extension (with $n = 2$ in op. cit.).

Theorem 58 *Let L be a Boolean algebra, and $\mathbb{P}$ be a probability on L. Let $L^{\mathbf{n}}$ be the MV-algebra $\mathbf{n}$-convolution of L. Then $\mathbb{P}$ has a unique extension to a probability $\mathbf{E}$ on $L^{\mathbf{n}}$, i.e., there exists a unique probability $\mathbf{E}$ on $L^{\mathbf{n}}$ such that the restriction of $\mathbf{E}$ to L coincides with $\mathbb{P}$. In particular, $\mathbf{E}$ is given by*

$$\mathbf{E}\langle f_0, \ldots, f_{n-1}\rangle = \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{P}f_i,$$

which, by Proposition 5.7, can be understood as the expectation $\mathbf{E}\xi_f$ of the simple random variable ξ_f associated with f.

Corollary 59 *In the setting of preceding theorem replacing $L^{\mathbf{n}}$ with $\tilde{L}^{\mathbf{n}}$, a probability $\mathbb{P}$ on a Boolean algebra L can be extended to a probability $\mathbf{E}$ on $\tilde{L}^{\mathbf{n}}$ via*

$$\mathbf{E}(a \parallel f_0,\ldots,f_{n-2}) = \mathbb{P}(a \wedge f_0) + \frac{1}{n}(\mathbb{P}(\neg f_0) + \ldots + \mathbb{P}(\neg f_{n-2})),$$

which can be understood as the sum of $\mathbb{P}(a \wedge f_0)$ and the expectation $\mathbf{E}\xi_g$ of the simple random variable ξ_g associated with $g = \langle \bot, \neg f_{n-2}, \ldots, \neg f_0 \rangle$.

Note that the constructions of Theorem 5.8 are special cases of the following propositions (generalizing respective propositions in [2], [9]).

Proposition 510 *Let L be an MV-algebra and $L^{\mathbf{n}}$ be its $\mathbf{n}$-convolution. Let $\mathbb{E} : [0,1]^n \to [0,1]$ be a mean value function. Then each probability $\mathbb{P}$ on L can be extended to an uncertainty measure $\mathbf{E}$ on $L^{\mathbf{n}}$ via*

$$\mathbf{E}\langle f_0, f_1, \ldots, f_{n-1}\rangle = \mathbb{E}\langle \mathbb{P}f_0, \mathbb{P}f_1, \ldots, \mathbb{P}f_{n-1}\rangle.$$

For the set $\tilde{L}^{\mathbf{n}}$ of conditional events, this result becomes

Proposition 511 *In the setting of the preceding proposition replacing $L^{\mathbf{n}}$ with $\tilde{L}^{\mathbf{n}}$, each probability $\mathbb{P}$ on L can be extended to an uncertainty measure $\mathbf{E}$ on $\tilde{L}^{\mathbf{n}}$, the "mean value extension" of $\mathbb{P}$ (as called in [9] in the case when $n = 2$), via*

$$\mathbf{E}(a \parallel f_0,\ldots,f_{n-2}) = \mathbb{E}\langle \mathbb{P}(a \wedge f_0), \mathbb{P}((a \wedge f_0)\dot{\wp}\neg f_{n-2}), \ldots, \mathbb{P}((a \wedge f_0)\dot{\wp}\neg f_0)\rangle.$$

Definition 512 *In the setting of Proposition 5.10 the uncertainty measure $\mathbf{E}$ on $L^{\mathbf{n}}$ will be referred to as the* expectation *on $L^{\mathbf{n}}$ and its values as* expected *values.*

In the setting of Proposition 5.11 expected values $\mathbf{E}(a \parallel f_0,\ldots,f_{n-2})$ will be referred to as conditional probabilities *on L and denoted by $\mathbb{P}(a \mid f_0,\ldots,f_{n-2})$. Thus, in the new notation, we have that*

$$\mathbb{P}(a \mid f_0,\ldots,f_{n-2}) = \mathbb{E}\langle \mathbb{P}(a \wedge f_0), \mathbb{P}((a \wedge f_0)\dot{\wp}\neg f_{n-2}), \ldots, \mathbb{P}((a \wedge f_0)\dot{\wp}\neg f_0)\rangle. \tag{1}$$

Proposition 513 *Let L be an MV-algebra and $L^{\mathbf{2}}$ be its $\mathbf{2}$-convolution. Let $\mathbb{P}$ be a probability on L and let $\mathbb{E} : [0,1]^{\mathbf{2}} \to [0,1]$ be the usual "arithmetic mean": $\mathbb{E}\langle r_0, r_1\rangle = \frac{1}{2}(r_0 + r_1)$. Then, according Proposition 5.10, $\mathbb{P}$ can be extended to an expectation $\mathbf{E}$ on $L^{\mathbf{2}}$ via*

$$\mathbf{E}\langle f_0, f_1\rangle = \frac{1}{2}(\mathbb{P}f_0 + \mathbb{P}f_1).$$

Moreover, this expectation on $L^{\mathbf{2}}$ has the following property:

$$f * g = \bot \Rightarrow \mathbf{E}(f \ddot{\#} g) = \mathbf{E}f + \mathbf{E}g$$

for every disjoint pair $f = \langle f_0, f_1\rangle$, $g = \langle g_0, g_1\rangle \in L^{\mathbf{2}}$ such that

$$(f_1 \times \neg f_0) \wedge (g_1 \times \neg g_0) \leq f_1 \times g_1$$

(which obviously holds if L is a Boolean algebra).

Proposition 514 *In the setting of the preceding proposition replacing $L^{\mathbf{2}}$ with $\tilde{L^{\mathbf{2}}}$, according Proposition 5.11, $\mathbb{P}$ can be extended to a conditional probability $\mathbb{P}$ on L via*

$$\mathbb{P}(a \mid f_0) = \mathbb{P}(a \wedge f_0) + \frac{1}{2}\mathbb{P}(\neg f_0)$$

$$(= \frac{1}{2}(\mathbb{P}(a \wedge f_0) + \mathbb{P}((a \wedge f_0)\dot{\wp}\neg f_0)).$$

Moreover, this conditional probability on L has the following property:

$$(a \parallel f_0) * (b \parallel g_0) = (\perp \parallel \top) \Rightarrow \mathbb{P}(c \mid h_0)$$

$$= \mathbb{P}(a \mid f_0) + \mathbb{P}(b \mid g_0) \textit{ (with } (c \parallel h_0) := (a \parallel f_0)\dot{\#}(b \parallel g_0))$$

for every disjoint pair $(a \parallel f_0)$, $(b \parallel g_0)$ of conditional events such that

$$\neg f_0 \wedge \neg g_0 \leq \frac{a}{f_0} \times \frac{b}{g_0}$$

(which always holds if L is a Boolean algebra).

To motivate the choice of the notation $\mathbb{P}(a \mid f_0, \ldots, f_{n-2})$ and the name "conditional probability", consider the case when $\mathbb{P}$ is a probability measure on a Boolean algebra L. Then (1) takes the form:

$$\mathbb{P}(a \mid f_0, \ldots, f_{n-2}) = \mathbb{E}\langle \mathbb{P}(a \wedge f_0), \mathbb{P}(a \wedge f_0) + 1 - \mathbb{P}f_{n-2}, \ldots, \mathbb{P}(a \wedge f_0) + 1 - \mathbb{P}f_0 \rangle. \tag{2}$$

Consider a mean value function on the real unit interval $[0,1]$ defined by

$$\mathbb{E}^2_{G,Ł,k}\langle r_0, r_1 \rangle = \begin{cases} \frac{r_0}{r_0+1-r_1} & \text{if } \langle r_0, r_1 \rangle \neq \langle 0, 1 \rangle \\ k & \text{if } \langle r_0, r_1 \rangle = \langle 0, 1 \rangle, \end{cases}$$

where k is an arbitrary number in $[0,1]$ (see Example 4.5). From (2) (with $n = 2$) it follows that

$$\mathbb{P}(a \mid f_0) = \begin{cases} \frac{\mathbb{P}(a \wedge f_0)}{\mathbb{P}(a \wedge f_0) + 1 - (\mathbb{P}(a \wedge f_0) + 1 - \mathbb{P}f_0)} & \text{if } \mathbb{P}f_0 \neq 0 \\ k & \text{if } \mathbb{P}f_0 = 0 \end{cases}$$

$$= \begin{cases} \frac{\mathbb{P}(a \wedge f_0)}{\mathbb{P}f_0} & \text{if } \mathbb{P}f_0 \neq 0 \\ k & \text{if } \mathbb{P}f_0 = 0, \end{cases}$$

which is the usual definition of (Kolmogorovian) conditional probability (with the usual notation).

For $\langle r_0, r_1, r_2 \rangle \in [0,1]^{\mathbf{3}}$ (with $r_0 \leq r_1, \leq r_2$) and $k \in [0,1]$,consider

$$\mathbb{E}^3_{G,Ł,k}\langle r_0, r_1, r_2 \rangle = \begin{cases} \frac{r_0}{r_0 + 1 - \frac{r_1}{r_1+1-r_2}} & \text{if } \langle r_0, r_2 \rangle \neq \langle 0, 1 \rangle \\ k & \text{if } \langle r_0, r_2 \rangle, \end{cases}$$

which defines a mean value function from $[0,1]^{\mathbf{3}}$ to $[0,1]$ (see Example 4.5 again). From (2) (with $n = 3$) it follows that

$$\mathbb{P}(a \mid f_0, f_1) = \begin{cases} 1 - \frac{\mathbb{P}f_0 - \mathbb{P}(a \wedge f_0)}{\mathbb{P}f_0 - \mathbb{P}(a \wedge f_0)(\mathbb{P}f_1 - \mathbb{P}f_0)} & \text{if } \mathbb{P}f_0 \neq 0 \\ k & \text{if } \mathbb{P}f_0 = 0 \end{cases}$$

(not forgetting the condition: $f_0 \leq f_1$). This formula can be considered as a generalization of the usual conditional probability.

Next, for $\langle r_0, r_1, r_2, r_3 \rangle \in [0,1]^{\mathbf{4}}$ (with $r_0 \leq \ldots \leq r_3$) and $k \in [0,1]$, consider

$$\mathbb{E}^4_{G,Ł,k}\langle r_0, r_1, r_2, r_3 \rangle = \begin{cases} \frac{r_0}{r_0+1-\frac{r_1}{r_1+1-\frac{r_2}{r_2+1-r_3}}} & \text{if } \langle r_0, r_3 \rangle \neq \langle 0, 1 \rangle \\ k & \text{if } \langle r_0, r_3 \rangle = \langle , 1 \rangle, \end{cases}$$

which defines a mean value function from $[0,1]^{\mathbf{4}}$ to $[0,1]$. From this we obtain that

$$\mathbb{P}(a \mid f_0, f_1, f_2)$$
$$= \begin{cases} 1 - \frac{\mathbb{P}f_0 - \mathbb{P}(a \wedge f_0)}{\mathbb{P}f_0 - \mathbb{P}(a \wedge f_0)((\mathbb{P}f_1 - \mathbb{P}f_0)(1 - \mathbb{P}f_2 + \mathbb{P}(a \wedge f_0)) + \mathbb{P}f_2 - \mathbb{P}f_0)} & \text{if } \mathbb{P}f_0 \neq 0 \\ k & \text{if } \mathbb{P}f_0 = 0 \end{cases}$$

(together with the condition that $f_0 \leq f_1 \leq f_2$) which can be considered as an another (more high level) generalization of the traditional conditional probability.

Similarly, one can consider the case $n = 5$ etc.

Finally, we arrive at

Definition 515 *Let L and M be two MV-algebras. Let $\mathbb{P} : L \to M$ be a M-valued probability on L and let $\mathbb{E} : M^{\mathbf{n}} \to M$ be a mean value function. Then the expression $\mathbb{P}(a \mid f_0, \ldots, f_{n-2})$ defined by the formula*

$$\mathbb{P}(a \mid f_0, \ldots, f_{n-2}) := \mathbb{E}\langle \mathbb{P}(a \wedge f_0), \mathbb{P}((a \wedge f_0) \dot{\wp} \neg f_{n-2}), \ldots, \mathbb{P}((a \wedge f_0) \dot{\wp} \neg f_0) \rangle \tag{3}$$

*will be called M-*valued conditional probability *on L.*

Note that in the case when $n = 2$ and $\mathbb{P}$ is the identity operator on L our L-valued conditional probability $\mathbb{P}(a \mid f_0)$ becomes the conditional operator in the sense of U. Höhle and S. Weber.

Proposition 516 *Let L and M be two MV-algebras. Let $\mathbb{P} : L \to M$ be a M-valued probability and $\mathbb{E} : M^{\mathbf{n}} \to M$ be a negatively weakly linear mean value function. Then the following decomposition*

$$\mathbb{P}(a \mid f_0, \ldots, f_{n-2}) = \mathbb{P}(a \wedge f_0) \dot{\wp} \mathbb{P}(\perp \mid f_0, \ldots, f_{n-2})$$

holds for all $a, f_0, \ldots, f_{n-2} \in L$.

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On some topological properties of the subalgebra of normal convex functions of the algebra of truth values of type-2 fuzzy sets

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1 Introduction

The principle subject of this paper is a subalgebra $\mathbb{L}_U$ of the algebra $\mathbb{L}$ of normal, convex functions in the algebra of truth values of type-2 fuzzy sets. The algebra $\mathbb{L}$ is defined and some basic properties are given in the abstract "Automorphisms of subalgebras of the algebra of truth values of type-2 fuzzy sets," for this meeting. We have shown that $\mathbb{L}$ is complete as a lattice, and that the subalgebra $\mathbb{L}_U$ of upper semicontinuous functions is a complete continuous lattice [4]. For any complete continuous lattice, the Lawson topology is a compact Hausdorff topology [3]. We have shown that the Lawson topology on this complete continuous lattice is countably based, so by [3] it is a separable topological space which is metrizable by a complete metric. We are interested in finding a metric for this topological space as well as a simple description of this topology.

2 The lattice $\mathbb{D}_U$

It is convenient to use a representation of the lattice $\mathbb{L}_U$ that is much more intuitive that the usual type-2 representation; for example, one in which join and meet of functions are pointwise.

Definition 1. *Let $D = \{f : [0,1] \to [0,2] : f$ is decreasing and 1 is an accumulation point of the values$\}$. Let $\underline{1}$ be the constant function in D with value 1. We say $f \in D$ is **band semicontinuous** (BSC) if $f \vee \underline{1}$ is lower semicontinuous and $f \wedge \underline{1}$ is upper semicontinuous. Equivalently, $f^{-1}([\alpha, 2-\alpha])$ is closed for every $0 \leq \alpha \leq 1$. Let $D_U = \{f \in D : f$ is band semicontinuous$\}$.*

As the union and intersection of two closed sets is closed, it follows that the pointwise join and meet of two LSC functions is LSC and the pointwise join and meet of two USC functions is USC.

Theorem 1. [4] *The algebra $\mathbb{D} = \left(D, \wedge, \vee, \overline{0}, \overline{1}\right)$ is a complete lattice, where $\wedge$ and $\vee$ are ordinary pointwise minimum and maximum, $\overline{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x > 0 \end{cases}$ and $\overline{1}(x) = \begin{cases} 2 \text{ if } x < 1 \\ 1 \text{ if } x = 1 \end{cases}$. With the operation $f'(x) = 2 - f(1-x)$, $\mathbb{D}$ becomes a De Morgan algebra. The algebra $\mathbb{D}_U = \left(D_U, \wedge, \vee, \overline{0}, \overline{1}\right)$ is a complete, continuous lattice, and $(\mathbb{D}_U, ')$ is also a De Morgan algebra.*

A function in $\mathbb{L}$ can be characterized as those functions $[0,1] \to [0,1]$ satisfying the two properties

$$f = f^L \wedge f^R$$
$$f^{LR} = \underline{1}$$

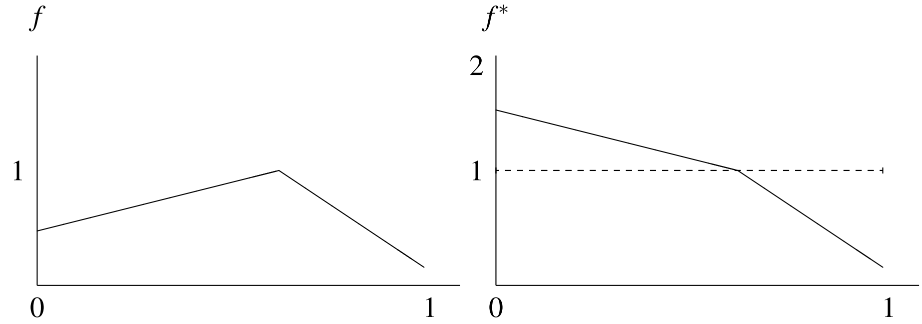
where $f^L(x) = \sup\{f(y) : 0 \leq y \leq x\}$ and $f^R(x) = \sup\{f(y) : x \leq y \leq 1\}$. Note that f^L is monotone increasing and f^R is monotone decreasing.

Theorem 2. [4] *The algebra $\mathbb{L}$ is isomorphic to $\mathbb{D}$ via the map $\Phi : \mathbb{L} \to \mathbb{D}$ where*

$$\Phi(f)(x) = \begin{cases} 2 - f(x) & \text{if } f(x) = f^L(x) \\ f(x) & \text{otherwise} \end{cases}$$

The restriction of Φ to $\mathbb{L}_U$ gives an isomorphism $\mathbb{L}_U \approx \mathbb{D}_U$. The map Φ also gives isomorphisms between the respective De Morgan algebras.

Roughly, $\Phi(f)$ is produced by taking the mirror image of the increasing portion of f about the line $y = 1$, and leaving the remainder of f alone. The following diagram illustrates the situation.



As the intersection of any family of closed sets is closed, the pointwise join of any family of LSC functions is LSC, and the pointwise meet of any family of USC functions is USC. It follows that for any function f there is a largest LSC function f^- beneath f, the pointwise join of all LSC functions beneath f, and there is a smallest USC function f^+ above f. Define $* : \mathbb{D} \to \mathbb{D}_U : f \mapsto f^*$ by

$$f^*(x) = \begin{cases} f^-(x) \text{ if } f(x) \geq 1 \\ f^+(1) \text{ if } f(x) \leq 1 \end{cases}$$

Then f and f^* agree a.e. and $f^{**} = f^*$.

We show that the set D is closed in the product topology τ_p on $[0,2]^{[0,1]}$ and that the function $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$ is a metric on D. This metric induces a topology on D that we will denote by τ_d.

The function $* : (D, \tau_p) \to (D_U, \tau_d)$ is continuous, and we conclude that (D_U, τ_d) is a compact Hausdorff space. We will describe the relationship between these topologies and the Lawson topology [2, 3].

3 Metric lattices

Another way of looking at this lattice is as a metric lattice.

Definition 2. *A **valuation** on a lattice L is a map $v : L \to R$ such that*

$$v(x) + v(y) = v(x \vee y) - v(x \wedge y)$$

*A valuation is **isotone** if $x \leq y$ implies $v(x) \leq v(y)$. It is **positive** if $x < y$ implies $v(x) < v(y)$. A lattice with a positive valuation is called a **metric lattice**.*

On any metric lattice, the function $d(f,g) = v(f \vee g) - v(f \wedge g)$ is a metric.

Theorem 3. [1] *A metric lattice is a metric space in which* $\vee$ *and* $\wedge$ *are uniformly continuous.*

Theorem 4. *For* $f \in D_U$*, let*

$$v(f) = \int_0^1 f(x)\,dx$$

Then $\mathbb{D}_U$ *is a metric lattice.*

Thus the theory of metric lattices as given in [1] applies to $\mathbb{D}_U$.

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Lower separation axioms for many valued topological spaces

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Let $(L,\leq)$ be a frame. Following Fourman and Scott [1], an L-valued set is a pair (X,E) consisting of a set X and an *L-valued equality* $E : X\times X \to L$ satisfying the following conditions:

(E1) $E(x,y) = E(y,x)$, (Symmetry)

(E2) $E(x,y)\wedge E(y,z) \leq E(x,z)$ (Transitivity)

for all $x,y,z \in X$.

An $f \in L^X$ is called *E-strict* and *E-extensional* iff for all $x,y \in X$ it satisfies the following two conditions:

(S0) $f(x) \leq E(x,x)$, (E-strictness)

(S1) $f(x)\wedge E(x,y) \leq f(y)$. ($E$-extensionality)

Given an L-valued set (X,E), the collection

$$\mathcal{P}(X,E) = \{f \in L^X : f \text{ has (S0) and (S1)}\}$$

is a frame under pointwise ordering: $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$, the top being $\mathbb{E} \in L^X$ defined by $\mathbb{E}(x) = E(x,x)$.

Following Höhle [4, Def. 2.3.2(a)] and [3, p. 351] (also cf. [5]), a *topology* on (X,E) is a subframe $\tau \subseteq \mathcal{P}(X,E)$ such that $a\wedge\mathbb{E} \in \tau$ for all $a \in L$.

Then (X,E,τ) is called an *L-valued topological space* and is an object of the category L-**TOP** in which $\psi : (X,E,\tau) \to (Y,F,\sigma)$ is a morphism provided that the following two conditions are satisfied:

(1) ψ is a morphism in the category of L-valued sets, i.e., $\psi : X \to Y$ is a map such that $E \leq F\circ(\psi,\psi)$ and $E_{|\Delta} = (F\circ(\psi,\psi))_{|\Delta}$ where $\Delta = \{(x,x) : x \in X\}$ and $(\psi,\psi)(x_1,x_2) = (\psi(x_1),\psi(x_2))$ for all $(x_1,x_2) \in X\times X$,

(2) $g\circ\psi \in \tau$ whenever $g \in \sigma$.

In this talk we shall discuss a system of lower separation axioms T_0, T_1 and T_2 in the category L-**TOP**. These axioms generalize, respectively, the T_0-axiom proposed independently by Rodabaugh [8], Šostak [9], and Wuyts and Lowen [10] (and by Liu [7] in quite a different yet equivalent way), as well as the T_1-axiom of [6]. The T_2-axiom is formulated in terms of the limit map of an L-valued filter. A link between the T_2-axiom and sobriety is provided by the following: if L is a complete Boolean algebra and (X,E) is L-valued set which is *complete*, then every T_2-separated L-valued topological space is sober. Examples of T_2-separated L-valued topological spaces include L-probabilistic metric spaces in the sense of [2].

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Topologies on quantale sets and topological representations of spectra of C^*-algebras

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The attempt to understand the spectrum of non commutative C^*-algebras as topological structure led to the idea to develop a concept of non commutative topological spaces (cf. [1,4]). From the very beginning to was evident that irreducible representations of C^*-algebras should intrinsically relate to «points» of these non commutative spaces — an idea which can also be found in various textbooks in the form that irreducible respresentations of non commutative C^*-algebras are viewed as filling a role similar that of complex homomorphisms of commutative C^*-algebras. In contrary to the commutative setting the problem arises here that the Hilbert space is varying and depends on the respective irreducible representation. In order to overcome these difficulties F. Borceaux introduced the concept of quantic spaces (cf. pp. 124 in [7]), and more recently C.J. Mulvey and J.W. Pelletier formulated their concept of quantale spaces (cf. [6]). We do not follow here these approaches, but lean heavily on the Gelfand-Naimark-Segal construction from which we can conclude that for every C^*-algebra there exists a respentation having sufficiently many irreducible subrepresentations. Hence in contrast to C.J Mulvey and J.W. Pelletier we fix the underlying Hilbert space and understand irreducible representations as *global* points and irreducible subrepresentations as *local* points of the quantised universe. This approach leads to *quantale sets* (cf. [2]) or more precisely to Q-valued sets where Q is the Hilbert quantale associated with the underlying Hilbert space (cf. [5]). On the basis of quantale sets we introduce a concept of *non commutative topological spaces* (cf. [3]) in such a way that closed left-ideals of C^*-algebras can be identified with strict and extensional Q-valued maps forming the «open subsets » of these structures. In particular, the multiplication of closed left-ideals corresponds to the multiplication of the respective Q-valued maps.
Moreover, partially defined quantale homomorphisms having the subquantale $I(Q)$ of all two-sided elements as their domain form a further source for non commutative topological spaces. This construction can be viewed as a non idempotent *and* non commutative generalization of space representations of Ω-valued locales where Ω is an arbitrary frame.

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Interval analysis done in Fuzzy Class Theory

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Interval analysis is a well established area of mathematics which has been investigating since Moore published his thesis [4] in 1960s. One of the important parts of interval analysis focuses on systems of linear equations whose parameters are only known to belong to given intervals. There are plenty of deep results dealing with solution sets of such systems (see e.g. [5, 6]). One of them for instance tells us that the solution set in each orthant forms a convex polyhedral set, see [6, Theorem 3.6]. In this paper we are going to show that we can derive analogous results also if we formally develop interval analysis within fuzzy logic. Instead of classical set theory we use Fuzzy Class Theory (FCT) which was introduced in [1] as a ground theory for fuzzy mathematics. Originally it was built over the logic ŁΠ. However, the definitions and basic results from [1] work in any logic extending MTLΔ, where MTLΔ is Monoidal T-norm Based Logic expanded by Baaz's operator Δ. MTL is the logic of left-continuous t-norms and their residua introduced in [2]. Also our results in this paper can be proved in any fuzzy logic that is at least as strong and expressive as MTLΔ.

Before we start with a formulation of the results, we present a motivational example. Let $[a_i^{\uparrow}, a_i^{\downarrow}], [b_i^{\uparrow}, b_i^{\downarrow}]$ be crisp intervals. Then we can ask whether there is a line p which "goes through" all the Cartesian products $[a_i^{\uparrow}, a_i^{\downarrow}] \times [b_i^{\uparrow}, b_i^{\downarrow}]$. Formally, this means that we ask if p has a non-empty intersection with each Cartesian product $[a_i^{\uparrow}, a_i^{\downarrow}] \times [b_i^{\uparrow}, b_i^{\downarrow}]$. This question leads to a typical task from classical interval analysis (for details see [6]). The answer is affirmative iff the following interval system of linear equations has a solution:

$$\begin{aligned} [a_1^{\uparrow}, a_1^{\downarrow}]k + q &= [b_1^{\uparrow}, b_1^{\downarrow}], \\ &\vdots \\ [a_n^{\uparrow}, a_n^{\downarrow}]k + q &= [b_n^{\uparrow}, b_n^{\downarrow}], \end{aligned}$$

where k is the slope of p and q is its offset. A solution of such an interval system is usually defined as follows: we say that a pair (k,q) describing the line p is a solution iff the following classical first-order formula holds:

$$\bigwedge_{i=1}^{n} \left(\exists(\alpha_i, \beta_i) \in [a_i^{\uparrow}, a_i^{\downarrow}] \times [b_i^{\uparrow}, b_i^{\downarrow}]\right) (\alpha_i k + q = \beta_i),$$

where $\wedge$ is the classical conjunction. Observe that this formula is true if there exists a point in each box $[a_i^{\uparrow}, a_i^{\downarrow}] \times [b_i^{\uparrow}, b_i^{\downarrow}]$ which lies on p. In order to solve this problem, classical interval analysis gives us methods how to find the set of all solutions.

The purpose of this paper is to discuss what happens if we change the crisp intervals to fuzzy intervals (i.e. normal convex fuzzy classes). Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be fuzzy intervals. Analogously to the previous task we would like to find a line p which goes through all fuzzy points $A_i \times B_i$.

⋆ The author was partly supported by the Program "Information Society" under project 1ET100300517 and partly by the Institutional Research Plan AV0Z10300504.

Clearly, as the sets $A_i \times B_i$ are not crisp, some of the lines satisfy this condition better than the others. The corresponding system of equations which we have to solve is the following one:

$$\begin{gathered} A_1 k + q = B_1 \,, \\ \vdots \\ A_n k + q = B_n \,. \end{gathered}$$

A solution to this system of equations can be defined in the same way as in the case of classical interval analysis, only interpreted in fuzzy logic. We say that a pair (k,q) describing a line p is a solution to such an extent to which the following first-order formula holds:

$$\mathop{\&}_{i=1}^{n} (\exists(\alpha_i, \beta_i) \in A_i \times B_i)(\alpha_i k + q = \beta_i) \,,$$

where & is the strong conjunction in our fuzzy logic and $\exists$ is interpreted by supremum as usual in first-order fuzzy logic. The corresponding solution set is a fuzzy set of pairs (k,q) where a pair (k,q) belongs to this set to a degree to which there are points in each fuzzy point $A_i \times B_i$ lying on the corresponding line.

Such a definition has also a reasonable interpretation. The truth degree to which a crisp point (x,y) belongs to a fuzzy point $A_i \times B_i$ can be understood as a penalty which we have to pay if the line in demand goes through this point. The greatest truth degree 1 represents no penalty and the lowest truth degree 0 the unacceptable penalty (i.e. a line going through this point cannot be a solution by no means). Then the truth degree to which a pair (k,q) describing a line p belongs to the solution set can be interpreted as follows: in each fuzzy point $A_i \times B_i$ we find the "best" point lying on p (i.e. the point with the greatest truth degree), the truth degree of this point tells us how "good" this point is, and then we compute the conjunction of the truth degrees of all these points (i.e. we have to sum all the penalties we receive in each fuzzy point $A_i \times B_i$). The way the penalties are summed together depends on the chosen fuzzy logic. For instance, in the standard semantics of Łukasiewicz logic, where the conjunction is interpreted by the Łukasiewicz t-norm, the penalties are summed by the usual addition and truncated at a maximum penalty.

Now we define our task formally. Let $\mathbf{A} = (A_{ij})$ be an $m \times n$ matrix of fuzzy intervals and $B = (B_1, \ldots, B_n)$ be an n-tuple of fuzzy intervals. The system $\mathbf{A}x = B$ is called a fuzzy interval linear system (FILS). Although the definition of solution set in the above-mentioned example involves only existential quantifier, it is reasonable to define the solution set more generally using also universal quantifier (see [6]). In order to define the formula describing the solution set, we split the matrix $\mathbf{A}$ and the tuple B into two disjoint parts according to the quantifiers. We define $\mathbf{A}^{\forall} = (A^{\forall}_{ij})$, $\mathbf{A}^{\exists} = (A^{\exists}_{ij})$, $B^{\forall} = (B^{\forall}_i)$, and $B^{\exists} = (B^{\exists}_i)$, where

$$A^{\forall}_{ij} = \begin{cases} A_{ij} & \text{if } A_{ij} \text{ should be} \\ & \text{quantified by } \forall, \\ \{0\} & \text{otherwise,} \end{cases} \qquad A^{\exists}_{ij} = \begin{cases} A_{ij} & \text{if } A_{ij} \text{ should be} \\ & \text{quantified by } \exists, \\ \{0\} & \text{otherwise.} \end{cases}$$

Analogously for $B^{\forall}$ and $B^{\exists}$. Then we have $\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}$, $B = B^{\forall} + B^{\exists}$. Now we can write down the formal definition of the solution set.

Definition 1. *Let $(\mathbf{A}^{\forall} + \mathbf{A}^{\exists})x = B^{\forall} + B^{\exists}$ be a FILS. Then its solution set is the following fuzzy class:*

$$\begin{aligned} \Xi(\mathbf{A}^{\forall}, \mathbf{A}^{\exists}, B^{\forall}, B^{\exists}) =_{\mathrm{df}} \{x \mid ((\forall \mathbb{U} \in \mathbf{A}^{\forall})(\forall u \in B^{\forall}) \\ (\exists \mathbb{E} \in \mathbf{A}^{\exists})(\exists e \in B^{\exists})((\mathbb{U} + \mathbb{E})x = u + e)\} \,. \end{aligned}$$

The first important results which can be generalized to fuzzy logic is [6, Theorem 3.4]. The theorem characterizes the solutions by means of the arithmetic defined on fuzzy intervals by means of Zadeh's extension principle.

Theorem 1. *Let $(\mathbf{A}^{\forall}+\mathbf{A}^{\exists})x = B^{\forall}+B^{\exists}$ be a fuzzy linear system. Then a vector x belongs to the solution set $\Xi(\mathbf{A}^{\forall},\mathbf{A}^{\exists},B^{\forall},B^{\exists})$ to the same degree as the formula $\mathbf{A}^{\forall}x - B^{\forall} \subseteq B^{\exists} - \mathbf{A}^{\exists}x$ holds, i.e.,*

$$x \in \Xi(\mathbf{A}^{\forall},\mathbf{A}^{\exists},B^{\forall},B^{\exists}) \leftrightarrow \mathbf{A}^{\forall}x - B^{\forall} \subseteq B^{\exists} - \mathbf{A}^{\exists}x\,.$$

The operations in $\mathbf{A}^{\forall}x - B^{\forall}$ and $B^{\exists} - \mathbf{A}^{\exists}x$ are the arithmetic operations defined by means of Zadeh's extension principle.

The second important result which we can derive in FCT is an analogous version of [6, Theorem 3.6]. Let $(\mathbf{A}^{\forall}+\mathbf{A}^{\exists})x = B^{\forall}+B^{\exists}$ be a classical interval linear system. The theorem states that the solution set $\Xi(\mathbf{A}^{\forall},\mathbf{A}^{\exists},B^{\forall},B^{\exists})$ in each orthant forms a convex polyhedron defined by a usual system of linear inequalities $\mathbb{C}x \leq d$ where entries in $\mathbb{C}$ lie among the bounds of intervals appearing in $\mathbf{A}^{\forall}+\mathbf{A}^{\exists}$, and entries of d among the bounds of intervals appearing in $B^{\forall}+B^{\exists}$.

So far we are not able to generalize completely this theorem to the fuzzy case. We can do it only for FILS where $\mathbf{A}^{\forall}$ and $B^{\forall}$ consist of the crisp singletons $\{0\}$. Thus assume that all quantifiers in Definition 1 of the solution set are existential. We will denote the solution set in this case by $\Xi(\mathbf{A}^{\exists},B^{\exists})$ and call it the united solution set like in the classical interval analysis

It is clear that unlike the crisp intervals, a fuzzy interval need not have a crisp lower-bound and upper-bound. Nevertheless, we can replace them respectively by a down-class and up-class. Let A be a fuzzy interval. A down-class and an up-class generated by A are defined respectively as follows:

$$A^{\downarrow} =_{\mathrm{df}} \{x \mid (\exists a \in A)(x \leq a)\}\,, \quad A^{\uparrow} =_{\mathrm{df}} \{x \mid (\exists a \in A)(x \geq a)\}\,.$$

Let $K = \{\uparrow,\downarrow\}^n$ be the set of all sequences of symbols $\uparrow$, $\downarrow$ whose length is n. The j-th component of $k \in K$ will be denoted by k_j. Further, we define $\varepsilon_{jk} = 1$ if $k_j =\uparrow$ and -1 otherwise. Let Q_k, $k \in K$, be the family of all orthants of $\mathbb{R}^n$, i.e., we have for each Q_k:

$$Q_k = \{x \in \mathbb{R}^n \mid \varepsilon_{1k}x_1 \geq 0 \ \&\ \cdots\ \&\ \varepsilon_{nk}x_n \geq 0\}\,,$$

where x_j stands for the j-th component of x. Each Q_k is obviously crisp. Finally, we define $-k_j =\downarrow$ if $k_j =\uparrow$ and $\uparrow$ otherwise.

Theorem 2. *Let $\mathbf{A}^{\exists}x = B^{\exists}$ be a FILS, Q_k an orthant, and $x \in Q_k$. Then*

$$x \in \Xi(\mathbf{A}^{\exists},B^{\exists}) \ \leftrightarrow\ \mathop{\&}_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}^{k_j} x_j \leq (B_i^{\exists})^{\downarrow} \ \&\ \sum_{j=1}^{n} A_{ij}^{-k_j} x_j \geq (B_i^{\exists})^{\uparrow} \right)\,,$$

where $A \leq B \equiv_{\mathrm{df}} (\exists x \in A)(\exists y \in B)(x \leq y)$.

Note that this result is in fact of the same shape as [6, Theorem 3.6] which was mentioned above. It tells us that a solution set in a given orthant Q_k is determined by a system of linear inequalities. The only difference is that we have to use the down-classes and up-classes instead of the bounds of crisp intervals.

It turns out that Theorem 2 is quite useful if we want to describe the united solution set for a specific fuzzy logic and a specific shape of fuzzy intervals. For instance, we are able to describe explicitly the solution set for the logic of Łukasiewicz t-norm if the fuzzy intervals are of the trapeziodal shape (see [3]).

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A Dempster-Shafer theory inspired logic

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1 Introduction

In this work we aim to introduce a procedure that allows one to translate the familiar Dempster-Shafer evidential universe to sets of formulas and then analyze the semantics and properties of the corresponding logic. The philosophical inspiration behind this work follows Brouwer's idea of intuitionism that does not allow non-constructive proofs [7]. Earlier research findings were also reported in [1], [2], similar representations that favored modal rather superintuitionistic logics can be found in [3] [4].

We represent the Dempster-Shafer theory frames of discernment as Kripke models, where a frame of discernment Θ is defined as a collection of sets and their corresponding beliefs, which in their own turn defined through the mass assignments which are not necessarily additive or monotone [5]. By a Kripke model we understand a triple $\mathfrak{F} = \langle W, R, V \rangle$ where W is some set, relation R is a partial order on W and valuation $V : \mathbf{Var}\mathcal{L} \to 2^W$ is a multivalued map [6]. The proposed procedure allows one to induce a Kripke model given a frame of discernment and the probability mass assignments of its elements. The idea behind the procedure is to represent different elements of the frame of discernment as worlds in a Kripke model and to deduce the relation R by using the knowledge about the set order on set Θ.

Definition 1 *Relation R. Assume that*

$$p = \text{“}x \in A\text{”};\ q = \text{“}x \in B\text{”};\ p, q \in \mathcal{L};\ A, B \in \Theta$$

and let

$$V(p) = v,\ V(q) = w;\ \textit{where } v, w \in W$$

then

1. *If $A \cap B \neq \emptyset$, then $V(p \wedge q) \neq \emptyset$.*
2. *If $A \cap B = \emptyset$, then $v\not R\, w$ and $w \not R\, v$, where $a \not R\, b$ means that a cannot see b.*
3. *There is no $w_{nil} \in W$ such that $V(n) = w_{nil}$, where $n =$“$x \in \emptyset$”, i.e., there is no separate point corresponding to the empty set.*
4. *$B \subset A$ requires vRw and $w \not R\, v$.*
5. *If there is $w \in W$ such that $w \in V(p)$, then there must exist $v \in W$ such that $v \in V(r)$, where $r =$ “$x \notin A$”*

When a model is built according to the definition above the worlds that correspond to the core elements of the frame of discernment 'inherit' their mass assignments. Whenever the evidence is updated two or more frames of discernment are combined together. In terms of the models it means combining two models together which sometimes involves creating new nodes, i.e. nodes that were not present in the original constructions. If a new node is formed then we assume that if $m_1(A) = s_1$ and $m_2(B) = s_2$ the updated belief in $A \wedge B$ is given by $m(A \wedge B) = s_1 s_2$: Shafer's postulate used for developing the evidence combination rule.

The proposed procedure for translating frames of discernment to Kripke models also works when one needs to update the evidence. In other words given two Kripke models representing two different frames of discernment they can be combined in a single model that represents updated evidence. The probability mass assignment for the worlds of the new model is the same as the one produced by taking an orthogonal sum of the probability mass assignments of the underlying frames of discernment.

2 Logic L_{DS}

The procedure of representing Dempster-Shafer frames of discernment by Kripke frames, gives rise to a si-logic[1], L_{DS}.

A logic can be defined in different ways: through a set of first-order conditions on Kripke frames as in the case with L_{DS}, or through a finite set of axioms as in the case with **Int** and **Cl**. A finite set of axioms that describe a logic gives one a *calculus*. Having a calculus is extremely convenient for both validating or refuting formulas and for answering fundamental questions of completeness and soundness.

Logic L_{DS} is defined as the set of formulas in $\mathbf{For}\mathcal{L}$ validated in every Kripke frame $\mathfrak{F} = \langle W, R \rangle$ that satisfies

$$\forall x, y, z, u \in W((xRy \wedge xRz \wedge y\not R\, z) \rightarrow (y\not R\, u \vee z\not R\, u)) \quad (1)$$

Equation 1 cannot be directly translated to formulas in $\mathbf{For}\mathcal{L}$ and thus that there is no finite axiomatization for L_{DS}. However, logic L_{DS} is sound and complete.

Theorem 1 *There is no finite set of axioms describing L_{DS}.*

The fact above is proven by considering the truth-preserving operations on the Kripke model and demonstrating that the models that violate the conditions of equation 1 can be mapped into trees and quoting the fact that any Kripke model may be transformed into a disjoint union of trees, thus making the task of searching for the formulas provable in the frames that satisfy equation 1 but refuted in the ones that don't impossible.

3 Completeness of L_{DS}

To show that logic L_{DS} is complete and sound regardless of the fact that it does not form a calculus, we use the parallelism between Heyting algebras and Kripke models. To generate the corresponding class of algebras we use the fact that for every finite Heyting algebra $\mathfrak{U}$ there exists a Kripke frame $\mathfrak{F}$ such that $\mathfrak{U}$ is isomorphic to $Up\,(\mathfrak{F})$ [6]. We call these isomorphic algebras *duals* $\mathfrak{F}^+$ of Kripke frames and take advantage of semantic equivalence between them and Kripke frames: $\mathfrak{F} \models \phi$ iff $\mathfrak{F}^+ \models \phi$.

When the connection between algebras and Kripke models is established the semantic analysis of L_{DS} is reduced to analyzing the corresponding class of algebras $\mathcal{C}_{DS}$. The class of algebras can be conveniently defined through the propositions below.

Proposition 1 *All duals of rooted frames have the second greatest element.*

[1] si-logic stands for superintuitionistic logic, hence 'a' rather than 'an' article

Proposition 2 *Duals of Kripke frames that do not satisfy equation 1 have the second least element.*

A Kripke model violates equation 1 only if it is both rooted and has the greatest element. We thus can think of $\mathcal{C}_{DS}$ as a class of algebras that have either second largest or second least element but not both. The task of showing the completeness and soundness of L_{DS} is now reduced to showing that the algebras in $\mathcal{C}_{DS}$ form a variety [8].

A set of algebras forms a variety if it is closed under taking subalgebras, products and homomorphic images. A fairly straightforward check reveals that class $\mathcal{C}_{DS}$ is indeed closed under the three operations mentioned above and thus forms a variety $\mathrm{Var}\mathcal{C}_{DS}$ as desired.

4 Limits of representation

The next step of our analysis involved trying to find the 'reasonable limits' for the proposed parallelism. The Dempster-Shafer theory is the approach that possesses a fairly rich formal set of instruments that enable one to update and transform known evidence. The evidence update should not be confused with evidence combination: the first deals with transforming separate frames of discernment, the second with combining two different frames. There are also virtually no limitations imposed on the belief functions (except for normalization).

In the present inquiry we tried to look at the possibility to develop a formal apparatus that helps to translate the operations on frames of discernment (different from taking the orthogonal sum) to the operations on Kripke models as well as seeing the effects of such operations on the corresponding logic.

It was shown that operations of frame refinement and frame coarsening can be translated to the language of Kripke models and lead to the same relation between the cores (sets of elements with non-zero probability assignments) of the respective structures. However, imposing a new non-monotonous belief function on a refined frame, i.e. redefining the core using arbitrary mass assignments, may lead to producing a Kripke model that will ultimately violate the conditions of equation 1. On the bright side, we have also demonstrated how this problem can be resolved by building a new Kripke model whose nodes correspond to the core elements of the *refined* frame.

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Many-valued logic for modifiers of fuzzy sets

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Abstract. Modifiers are self-maps on the set of all fuzzy sets. Their idea is to modify each fuzzy set to another fuzzy set. In this paper, we introduce a logic for substantiating fuzzy-set modifiers. Its language, axiomatization, and semantics is presented. Furthermore, completeness of the logic is proved.

1 Fuzzy sets and their modifiers

We begin with recalling some notions and notation. Let us denote by $\mathbb{I}$ the unit interval $[0,1]$. For any nonempty set X, we denote by $\mathbb{I}^X$ the set of all mappings $X \to \mathbb{I}$. The elements of $\mathbb{I}^X$ are called *fuzzy sets* on X.

Because the interval $\mathbb{I}$ may be ordered with its usual order, also the set $\mathbb{I}^X$ can be ordered *pointwise* by setting

$$\mu \leq \nu \iff (\forall x \in X)\, \mu(x) \leq \nu(x)$$

for all $\mu, \nu \in \mathbb{I}^X$. The set $\mathbb{I}^X$ has the greatest element $\mathbf{1}\colon x \mapsto 1$ and the least element $\mathbf{0}\colon x \mapsto 0$. It is easy to observe that with respect to the pointwise order, $\mathbb{I}^X$ is a distributive lattice such that for all $\mu, \nu \in \mathbb{I}^X$ and $x \in X$,

$$(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\} \text{ and } (\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}.$$

Any mapping $\mathcal{M}\colon \mathbb{I}^X \to \mathbb{I}^X$ is called a *modifier* $\mathcal{M}$. A modifier $\mathcal{M}$ is

(i) a *substantiating modifier* if $\mathcal{M}(\mu) \leq \mu$ and
(ii) a *weakening modifier* if $\mu \leq \mathcal{M}(\mu)$

for all $\mu \in \mathbb{I}^X$. Our next example is modified from an example appearing in [1].

Example 1. In Fig. 1 is presented a subjective assignment of degrees of tallness of men. For instance, if someone is smaller than 165 cm, then he is not tall to any degree. If his height is 180 cm, we might say that he is tall to, say, with degree 0.7. If he is over 190 cm, then he is tall, period.

In Fig. 1 one can also find the membership function of 'very tall'. The fuzzy set 'very tall' is *modified* from the fuzzy set 'tall'. It can be easily observed that the membership function values of 'very tall' are below the values of the fuzzy set 'tall'. Therefore, 'very' is a substantiating modifier.

We could also define a fuzzy set 'more or less tall' and notice that its membership function values are always above the values of 'tall'. Hence, 'more or less' can be interpreted as a weakening modifier.

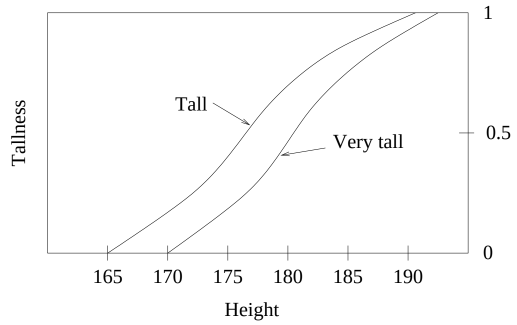

Fig. 1. Subjective degrees of tallness of men

Remark 1. Example 1 deals with linguistic expressions. Such modifiers are called *hedges* by Lakoff [1]. Note also that the study of Lakoff is done more or less from the viewpoint of 'philosophical logic' and he only suggested some possible axiomatizations. Mattila has presented several axiom systems and defined Kripke-style semantics for modifier logics; see [2], for instance. He has also outlined a many-valued modifier logic in [3]. However, the current work is the first study in which algebraic semantics is presented to valuate modifier logic formulas.

We may also define an implication operation on $\mathbb{I}^X$ in various ways. For instance,

$$(\mu \to \nu)(x) := \min\{1, 1 - \mu(x) + \nu(x)\}.$$

In the next section we will study the logic of *normal substantiating modifiers*. A modifier $\mathcal{M}$ is *normal* if it satisfies the following conditions for all $\mu, \nu \in \mathbb{I}^X$:

(i) $\mathcal{M}(\mathbf{1}) = \mathbf{1}$
(ii) $\mathcal{M}(\mu \to \nu) \to (\mathcal{M}(\mu) \to \mathcal{M}(\nu))$

2 Syntax for modifier logic

The propositional modifier language consists of an enumerable set P of *propositional variables*, propositional *constants* $\perp$ and $\top$, logical connective of *implication* $\to$, *modifier operators* $\mathcal{M}_0$, $\mathcal{M}_1$, $\ldots$, $\mathcal{M}_n$, and *parenthesis* (and). *Well-formed formulas* are defined inductively as follows:

(i) Every propositional variable and propositional constant is a formula.
(ii) If A and B are formulas, then so are $A \to B$ and $\mathcal{M}_i(A)$ for all $0 \leq i \leq n$.

Let us denote by Φ the set of all formulas.

Before we present our axiom system, we introduce for the sake of simplicity the following abbreviations for *negation*, *disjunction*, *conjunction*, and *equivalence*:

$$\begin{aligned} \neg A &:= A \to \perp \\ A \vee B &:= (A \to B) \to B \\ A \wedge B &:= \neg(\neg A \vee \neg B) \\ A \leftrightarrow B &:= (A \to B) \wedge (B \to A) \end{aligned}$$

The logical system has the following nine *axioms*:

(Ax1) $A \to (B \to A)$
(Ax2) $(A \to B) \to ((B \to C) \to (A \to C))$
(Ax3) $(A \vee B) \to (B \vee A)$
(Ax4) $(\top \to A) \to A$
(Ax5) $\neg\neg A \to A$
(Ax6) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(Ax7) $\mathcal{M}_0 A \leftrightarrow A$
(Ax8) $\mathcal{M}_l A \to \mathcal{M}_k A$ for all $l \geq k$
(Ax9) $\mathcal{M}_i(A \to B) \to (\mathcal{M}_i A \to \mathcal{M}_i B)$ for all $0 \leq i \leq n$.

The following are the *rules of inference*:

$$\text{(MP)}\quad \frac{A \qquad A \to B}{B} \qquad\qquad (\mathrm{N}_i)\quad \frac{A}{\mathcal{M}_i(A)} \quad \text{for all } 0 \leq i \leq n$$

The first rule is the classical *modus ponens* and the other says that the so-called *necessitation* analogy applies to all modifiers. We may call it the *rule of substantiation*.

A formula A is said to be *provable*, denoted by $\vdash A$, if there is a finite sequence $A_1, A_2, \ldots, A_n$ of formulas such that $A = A_n$ and for every $1 \leq i \leq n$: either A_i is an axiom or A_i is the conclusion of some inference rules, whose premises are in the set $\{A_1, \ldots, A_{i-1}\}$.

Proposition 1. *For all formulas A and B, we have for* $0 \leq i \leq n$*:*

(i) $\vdash \mathcal{M}_i(A) \to A$ (reflexivity rule)

(ii) $\dfrac{\mathcal{M}_i(A) \qquad A \to B}{B}$ (modified modus ponens)

3 Modifier algebras and semantics for logic of fuzzy set modifiers

Here we define semantics for our logic. A *modifier algebra* is an algebra

$$(L, \to, m_0, \ldots, m_n, \mathbf{0}, \mathbf{1}),$$

where $\to$ is a binary operation, each m_i is a unary operation, $\mathbf{0}, \mathbf{1} \in L$, and the following conditions hold for all $a, b, c \in L$:

(m1) $a \to (b \to a) = \mathbf{1}$
(m2) $(a \to b) \to ((b \to c) \to (a \to c)) = \mathbf{1}$
(m3) $a \vee b = b \vee a$
(m4) $\mathbf{1} \to a = a$
(m5) $a'' \to a = \mathbf{1}$
(m6) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(m7) $m_0\, a = a$
(m8) $m_l\, a \to m_k\, a = \mathbf{1}$ for all $l \geq k$
(m9) $m_i(a \to b) \to (m_i a \to m_i b) = \mathbf{1}$ for all $0 \leq i \leq n$
(m10) $m_i\, \mathbf{1} = \mathbf{1}$ for all $0 \leq i \leq n$,

where the following abbreviations are employed:

$$a' := a \to \mathbf{0}, \quad a \vee b := (a \to b) \to b, \quad a \wedge b := (a' \vee b')'$$

For a modifier algebra L, we may define a binary relation $\leq$ by setting

$$a \leq b \iff a \to b = \mathbf{1}.$$

Proposition 2. *If L is a modifier algebra, then the partially ordered set $(L, \leq)$ is a bounded distributive lattice such that* $\sup_{\leq}\{a,b\} = a \vee b$ *and* $\inf_{\leq}\{a,b\} = a \wedge b$*; the least element is* $\mathbf{0}$ *and* $\mathbf{1}$ *is the greatest element.*

The formulas of the logic can be now valuated canonically as presented in the following. Let L be a modifier algebra. Any map $v\colon P \to L$ is called a *valuation* on L. The valuation v can be extended canonically to the set of all formulas Φ as follows:

$$
\begin{aligned}
v(A \to B) &= v(A) \to v(B) \\
v(\bot) &= \mathbf{0} \\
v(\top) &= \mathbf{1} \\
v\,(\mathcal{M}_i(A)\,) &= m_i(v(A))
\end{aligned}
$$

Completeness of the logic can be proved by the 'standard' method of applying Lindenbaum-Tarski algebras.

Theorem 1 (Completeness Theorem). *Let $A \in \Phi$. Then $\vdash A$ if and only if $v(A) = \mathbf{1}$ for every valuation v on any modifier algebra.*

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Semilinear space and a compact $\mathbf{T_1}$-space SpecA

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1 Introduction

Let A be an MV-algebra with the underlying lattice L. Denote by $W(A)$ the set consisting of sets $W(a)$, $a \in A$ which form a base of closed sets in $SpecA$. It is known [4] that $\widehat{L} = (L, \vee, 0)$ is a semilinear space over A with the external multiplication $pa = p \odot a$. Then an isomorphism $A \longrightarrow W(A)$, $a \mapsto W(a)$, carries $\widehat{L}$ onto a semilinear space $W(\widehat{L})$ over $W(A)$. If $SpecA$ is a compact T_1-space, in fact, Hausdorff, then both A and $W(A)$ are Boolean algebras. Moreover, $W(A)$ is totally disconnected precisely when the sets $W(a)$, $a \neq 1$, constitute a decomposition of $SpecA$. Our goal is to find a linear combination in $W(\widehat{L})$ which characterizes the case. We also form the corresponding connection with level sets in fuzzy mathematics.

Lattices, MV-algebra and their ideals are referenced as [1], [2] and [3]. For the spectral space $SpecA$, see [1]. Its open base consists of sets $V(a) = \{p \in SpecA \mid a \notin p\}$ and closed base of $W(a) = \{p \in SpecA \mid a \in p\}$ where p is a prime ideal of the MV-algebra A and $a \in A$. For definitions of ring and field of sets we refer to[2], and semilinear space with needed consepts to [4]. The two fuzzy mathematics concepts, fuzzy sets and level sets, can be found from [5].

A topological space is called a T_1-space iff its singles are closed. The following two definitions can be found from [1] or [5]:

Let A be an MV-algebra. A nonempty set $S \subseteq A$ is an orthogonal set if $0 \notin S$ and $a, b \in S$, $a \neq b$ implies $a \wedge b = 0$.

A decomposition of a topological space T is an union $T = \cup T_i$, where each T_i is a nonempty subset of T and $T_i \cap T_j = \emptyset$, $i \neq j$.

2 SpecA as a $\mathbf{T_1}$-space

Proposition 1. *[5] Let A be an MV-algebra. Then the sets $W(a_i)$, $a_i \in A$, $a_i \neq 1$ constitute a decomposition of a topological space $SpecA$ iff the set $S^* = \{a_i{}^* \mid a_i{}^* \in A\}$ is an orthogonal subset of A.*

Proposition 2. *Let $A = (L, \oplus, \odot, {}^*, 0, 1)$ be an MV-algebra with the underlying lattice $(L, \wedge, \vee, 0, 1)$, and the set $W(A) = \{W(a) \mid a \in A\}$ a base of closed sets of $SpecA$. Then a mapping*

$$W : L \longrightarrow W(L), \;\; a \longmapsto W(a)$$

is a lattice isomorphism.

Proof. By [5], for any $b, c \in L$, $b = c$ iff $W(b) = W(c)$ implying that $W : L \longrightarrow W(L)$ is a bijection. Moreover,

$$W(b \vee c) = W(b) \cap W(c) \;\text{ and }\; W(b \wedge c) = W(b) \cup W(c) \tag{1}$$

Since p is an ideal, $b \vee c \in p$ iff $b \in p$ and $c \in p$ and the first equation is proved. To show the second one we suppose that $p \in W(b \wedge c)$, equivalently, $b \wedge c \in p$. Since p is a prime ideal, then $b \in p$ or $c \in p$ implying $p \in W(b) \cup W(c)$. Conversely, let $p \in W(b) \cup W(c)$, that is, $p \in W(b)$ or $p \in W(c)$. If $p \in W(b)$, then $b \in p$. Since $b \wedge c \leq b$ and p is an ideal, $b \wedge c \in p$ meaning that $p \in W(b \wedge c)$. It is proved that $W : L \longrightarrow W(L)$ is a lattice isomorphism.

By [2], any compact T_1-space X is determined up to homeomorphism by any ring of closed sets of X which constitutes a base of closed sets. This base is a distributive lattice. Moreover, X has a field of closed sets which form a base iff it is totally disconnected.

Corollary 1. *The following holds:*

(i) $W(A)$ *is a field of closed sets of SpecA.*

(ii) SpecA is a compact T_1*-space iff it is totally disconnected. In this case, SpecA is homeomorphic with* $W(A)$ *which constitutes a base of closed sets of SpecA.*

(iii) If SpecA is a T_1*-space, then* A *and* $W(A)$ *are Boolean algebras.*

Proof. According to (1), $W(b) \cup W(c)$ and $W(b) \cap W(c)$ belong to $W(A)$ for any pair $W(b)$, $W(c) \in W(A)$. The set-complement of $W(a)$ is $V(a) = W(a^*) = SpecA \setminus W(a)$, and so $W(A)$ is a field of closed sets. Let *SpecA* be a compact T_1-space. Since $W(A)$ forms a base of closed sets in *SpecA*, then *SpecA* is homeomorphic with $W(A)$ and precisely, in this case, it is totally disconnected. Finally, any field of sets is a Boolean algebra [2].

Proposition 3. *Suppose SpecA is a compact* T_1*-space. Then* $W(A)$ *is totally disconnected iff the sets* $W(a)$*, for every* $a \in A$*,* $a \neq 1$*, constitute a decomposition of SpecA.*

Proof. By Corollary 1, $W(A)$ is a T_1-space having then closed singles. Suppose $W(A)$ is totally disconnected and let a, b be any pair in A, $a \neq b$ iff $W(a) \neq W(b)$ [5]. Choose a disconnection $W(A) = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$ such that $C_1 = \{W(a)\}$, $W(b) \in C_2 = W(A) \setminus \{W(a)\} = W(\cup c)$,$c \neq a$. Therefore, $W(b) \subseteq W(\cup c)$, $W(a) \cap W(\cup c) = \emptyset$ which implies $W(a) \cap W(b) = \emptyset$ leading to $W(a \vee b) = W(a) \cap W(b) = \emptyset = W(1)$, that is,by [5], $a \vee b = 1$ being equivalent with $a^* \wedge b^* = 0$ (also for $a, b \neq 1$). Proposition 1 implies that the sets $W(a)$ form a decomposition of *SpecA*. The converse holds trivially because, by Corollary 1, $W(A)$ is always totally disconnected.

3 W($\widehat{L}$) as a semilinear space

By [4], we conclude: Let $A = (L, \oplus, \odot, ^*, 0, 1)$ be an MV-algebra and $L = (L, \vee, \wedge, 0, 1)$ its underlying lattice. Define operations in L as follows: $a + b = a \vee b$ and $pa = p \odot a$. Then $\widehat{L} = (L, +, 0) = (L, \vee, 0)$ is a semilinear space over A with the external multiplication $pa = p \odot a$. If A is a Boolean algebra, $pa = p \wedge a$.

Proposition 4. *Let* A *be a* MV*-algebra with an underlying lattice* L*. Then the isomorphism* $W : L \longrightarrow W(L)$*,* $a \mapsto W(a)$ *maps the semilinear space* $\widehat{L}$ *onto the semilinear space* $W(\widehat{L})$*.*

Proof. We show that

$$W(\widehat{L}) = (W(L), \vee_W, W(0)) = (W(L), \cap, W(0))$$

is a semilinear space over $W(A)$.

Let $\vee$ and $\wedge$ be two binary operations in the lattice L. Suppose that $\vee_W$ and $\wedge_W$ are the corresponding operations in $W(L)$. Put the binary operations in $W(L)$:

$$W(a) \vee_W W(b) = W(a) \cap W(b) \text{ and } W(a) \wedge_W W(b) = W(a) \cup W(b)$$

for all $W(a), W(b) \in W(A)$. Then, by the equations (1)

$$W(a \vee b) = W(a) \cap W(b) = W(a) \vee_W W(b)$$
$$W(a \wedge b) = W(a) \cup W(b) = W(a) \wedge_W W(b)$$

For a set complement in $W(A)$ we use the same notation * as for the complement in A. Moreover, $W(a^*) = W(a)^*$.

Hence, W maps the complement of $a \in A$ to the complement of $W(a)$. Also, $W : L \longrightarrow W(L)$ carries both the lattice - and MV-operations of L, resp. A, to the corresponding operations of $W(A)$. In A, we have the lattice order but the dual order in $W(A)$: $W(a) \subseteq W(b)$ iff $W(b) \leq W(a)$. It follows that $W(0) = SpecA$ and $W(1) = \emptyset$ are the least and greatest elements in $W(A)$.

$\widehat{L} = (L, \vee, 0)$ is a semilinear space over A with the external multiplication $pa = p \odot a$. This means that $W(\widehat{L}) = (W(L), \vee_W, W(0))$ is a semilinear space over $W(A)$ with the $\vee_W$-operation in $W(\widehat{L})$: Again,

$$W(a \vee b) = W(a) \cap W(b) = W(a) \vee_W W(b).$$

If b is a linear combination of a_i, then, by [4], the coefficients $p = a_i \to b$ leading to

$$p \odot a_i = (a_i \to b) \odot a_i = ((a_i \to b) \to a_i^*)^* = ((b^* \to a_i^*) \to a_i^*)^* = (b^* \vee a_i^*)^* = b \wedge a_i$$

For the external multiplication $W(p)W(a)$ in $W(\widehat{L})$ we obtain

$$W(pa) = W(p \odot a) = W(b \wedge a_i) = W(p) \cup W(a) = W(p)W(a) \text{ in the general case.}$$
$$W(pa) = W(p \wedge a) = W(p) \cup W(a) = W(p)W(a) \text{ in the Boolean case.}$$

Proposition 5. *Let A be a complete MV-algebra and $W(\widehat{L})$ a semilinear space over $W(A)$ where $\widehat{L} = (L, \vee, 0)$ is a semilinear space over A. Then, for a pseudo-complement $z^{\circledast}$ of any $z \in A$, $W(z^{\circledast})$ can be represented as a linear combination of elements $W(a)$ in $W(\widehat{L})$, the number of which is finite or infinitely denumerable:*

$$W(z^{\circledast}) = \cap W(a) \quad a \in (z^{\circledast}) \; = \; \{a \mid a \wedge z = 0\}$$

If A is a complete Boolean algebra, the linear combination consists of only one element, itself $V(z) = W(z^*)$, z^* is a complement of z. For level sets

$$\mathcal{A}_{W(z^{\circledast})} = \cap \mathcal{A}_{W(a)} = \cap W(a), \text{ and in the Boolean case, } \mathcal{A}_{V(z)} = V(z)$$

Let $SpecA$ be a compact T_1-space. Then the following holds:

(i) A and $W(A)$ are Boolean algebras,
(ii) $SpecA$ and $W(A)$ are totally disconnected and compact Hausdorff,
(iii) The sets $W(a)$ $a \in A$, $a \neq 1$, constitute a decomposition of $SpecA$, i.e., $\{a^* \mid a \in A, a \neq 1\}$ is a maximal orthogonal set.

(iv) The disjoint sets $W(a) = \mathcal{A}_{W(a)}$ form a topological base for SpecA,
(v) The sets $W(a)$ form a base of closed sets in SpecA.

Proof. Since A is a complete MV-algebra, then for every $z \in A$ there exists a pseudo-complement $z^{\circledast}$ satisfying $a \wedge z = 0$ iff $a \leq z^{\circledast}$ for any $a \in A$, and therefore $z^{\circledast} = \vee\{a \mid a \wedge z = 0\} = \vee\{a \mid a \in (z^{\circledast})\}$ [3].

Suppose the above a-sets are finite or infinitely denumerable. Because p is an ideal, $a \vee b \in p$ iff $a \in p$ and $b \in p$, i.e., $W(a \vee b) = W(a) \cap W(b)$. By induction, $W(\vee a) = \cap W(a)$. Consequently, $W(z^{\circledast}) = W(\vee a) = \cap W(a)$. Let A be a complete Boolean algebra. Then pseudo-complements $z^{\circledast}$ and complements z^* coinside [3]. Also, the equation $a \wedge z = 0$ has a unique solution $a = z^*$. This means that $V(z) = W(z)^* = W(z^*) = W(a)$ for any $z \in A$, denoted by $z = a^*$. The assertion for level sets follows from the fact $p = \mathcal{A}_p$, where p is any lattice element [5] and $\cap \mathcal{A}_p = \mathcal{A}_{\cap p}$.

Let *SpecA* be a compact T_1-space, equivalently, it is compact Hausdorff. According to Proposition 3, the sets $W(a)$ for every $a \in A$, $a \neq 1$ constitute a decomposition of *SpecA* iff $W(A)$ is totally disconnected. But, by Corollary 1, *SpecA* and $W(A)$ are totally disconnected and $W(A)$ forms a base of closed sets in *SpecA*. On the other hand, $W(A) = \mathcal{A}_{W(A)}$ is a topological base for *SpecA* and all the sets $W(a)$ are disjoint.

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Abstract. Adjunctions between categories are common and important. The existence of an adjunction tells us much about the corresponding categories, including that there are equivalent subcategories. If we replace categories by concrete categories and functors by concrete functors, we can generalize adjunctions to correspondences. This generalization allows us not only to retain some important adjunction properties and results; it also allows us to introduce order theoretic concepts into the concrete categorical setting.

1 From polarities to correspondences

Correspondences are not only generalizations of adjuctions, they are also generalizations of connections and Galois connections. In fact, there is an interesting history of development of generalizations to correrspondences. In 1940, G. Birkhoff introduced polarities [2]. These were generalized to (order-reversing) Galois connections by O. Ore in 1944 [9]. The next step in the development was not a generalization, but in 1953, J. Schmidt modified Galois connections so that the maps were order-preserving [10]. In 1982, H. Crapo generalized Galois connections to connections [3].

By this time, adjunctions were well known and often used in category theory and in applications of category theory. It was commonly accepted that adjunctions were the natural categorical generalization of Galois connections, and indeed adjunctions are a natural categorical generalization of Galois connnections. However, they are not the only generalization, and in fact, the generalization to Galois correspondences introduced by Herrlich and Hušek in 1990 allows for a more natural generalization of Galois connections in that some of the lattice theoretical properties of Galois connections are preserved in this generalization [4]. See also [1].

There was also another line of development from connections to correspondences. It is well known that order-preserving Galois connections are better suited for applications in computer science than are order-reversing Galois connections. However, even order-preserving Galois connections have limited applicability in computer science. These limitations were the reason why the applications of Galois connections to computer science in [7] were restricted to Galois connections with one map being an injection. These limitations were overcome by Melton, Schröder, and Strecker with the introduction of Lagois connections in [8]. By using Lagois connections, the computer science examples which were modeled with injective Galois connections in [7] could be given in full generality in [8]. In 2004, Melton introduced Lagois correspondences [6].

For the sake of completeness, the next section begins with the definition and some properties of connections. Then the necessary background for defining correspondences is given; correspondences are defined; and a theorem detailing some of their properties is given.

2 Correspondences

Definition 2.01 Let $(P,\leq)$ and $(Q,\leq)$ be partially ordered sets, and let $f : P \to Q$ and $g : Q \to P$ be order-preserving maps. (f,P,Q,g) or simply (f,g) is said to be a *connection* if $fgf = f$ and $gfg = g$.

When (f,P,Q,g) is a connection, we denote the image of Q under g by P^*, i.e., $P^* = g[Q]$. Likewise, $Q^* = f[P]$. Further, f^* denotes the restriction of f from P^* to Q^*, and g^* denotes the restriction of g from Q^* to P^*.

Theorem 2.02 Let (f,P,Q,g) be a connection.

1. $\forall p \in P$, $p \in P^*$ iff $p = gf(p)$ and $\forall q \in Q$, $q \in Q^*$ iff $q = fg(q)$.
2. f is injective iff g is surjective iff $gf = 1_P$ and if f is surjective iff g is injective iff $fg = 1_Q$.
3. P^* and Q^* are isomorphic posets with f^* and g^* being isomorphisms.
4. If P and Q are (complete) lattices, then so are P^* and Q^*, though they need not be sublattices.

Theorem 2.02 is well known; for a proof, see, for example, [8].

Definition 2.03 Let $\mathbf{A}$ and $\mathbf{B}$ be categories and $G : \mathbf{A} \to \mathbf{B}$ a functor. G is said to be a *faithful* functor if whenever $f,g : A \to A'$ are $\mathbf{A}$-morphisms with $G(f) = G(g)$, then $f = g$.

Definition 2.04 Let $\mathbf{A}$ and $\mathbf{X}$ be categories and $U : \mathbf{A} \to \mathbf{X}$ a functor. $(\mathbf{A},U)$ or simply $\mathbf{A}$ is said to be a *concrete category over* $\mathbf{X}$ if U is a faithful functor.

Definition 2.05 Let $\mathbf{X}$ be a category, and let $(\mathbf{A},U)$ and $(\mathbf{B},V)$ be concrete categories over $\mathbf{X}$. A functor $G : \mathbf{A} \to \mathbf{B}$ is said to be a *concrete functor* from $(\mathbf{A},U)$ to $(\mathbf{B},V)$ if $U = VG$.

Let $(\mathbf{A},U)$ be a concrete category over $\mathbf{X}$. The fibers determined by U are pre-ordered classes where the pre-order is defined as follows. If A and B are $\mathbf{A}$-objects such that $U(A) = U(B) = X$, then $A \leq B$ if and only if there exists an $\mathbf{A}$-morphism $f : A \to B$ such that $U(f) = id_X$. Further, if $G_1, G_2 : \mathbf{A} \to \mathbf{B}$ are concrete functors, then $G_1 \leq G_2$ if and only if $G_1(A) \leq G_2(A)$ for each $\mathbf{A}$-object A. [1]

For the sake of simplicity, we assume that our concrete categories are amnestic; that is, we assume that all pre-ordered fibers in our concrete categories are partially ordered classes.

Definition 2.06 Let $(\mathbf{A},U)$ and $(\mathbf{B},V)$ be concrete categories over $\mathbf{X}$, and let $G : A \to B$ and $F : B \to A$ be concrete functors over $\mathbf{X}$. $(G,(\mathbf{A},U),(\mathbf{B},V),F)$ or simply (G,F) is said to be a *correspondence* if G and F are quasi-inverses, i.e., if $GFG = G$ and $FGF = F$.

Theorem 2.07 Let $(G,(\mathbf{A},U),(\mathbf{B},V),F)$ be a correspondence.

1. $\forall g \in Mor(\mathbf{A})$, if $g \in Mor(\mathbf{A}^*)$ then $g = FG(g)$ and $\forall f \in Mor(\mathbf{B})$, if $f \in Mor(\mathbf{B}^*)$ then $f = GF(f)$[1].
2. If G is an embedding, then $FG = 1_{\mathbf{A}}$; if F is surjective, then $FG = 1_{\mathbf{A}}$; if G is surjective, then $GF = 1_{\mathbf{B}}$; and if F is an embedding, then $GF = 1_{\mathbf{B}}$. Further, G is an embedding (respectfully, surjective) iff F is surjective (respectively, an embedding).
3. $(\mathbf{A}^*,U)$ and $(\mathbf{B}^*,V)$ are isomorphic concrete categories with restrictions to G^* and F^* being inverse functors.

[1] This statement implies a similar statement for objects in $\mathbf{A}^*$ and $\mathbf{B}^*$ since a functor's images of objects must correspond to its images of identity morphisms.

4. Whatever limits or colimits exist in **A**, also exist in $\mathbf{A}^*$, and whatever limits or colimits exist in **B**, also exist in $\mathbf{B}^*$. Further, whatever lattice structure exists in **A** or **B** also exists in $\mathbf{A}^*$ and in $\mathbf{B}^*$. Thus, if **A** or **B** is complete or co-complete, then so are $\mathbf{A}^*$ and $\mathbf{B}^*$.

3 Example

Let $\mathbf{Top_c}$ be the category of topological spaces and convergence related functions. ("Convergence related functions" are defined below.) Let **Rel** be the category of sets with assoicated relations and relation preserving functions.

Given a topological space (X,τ), one can define a relation ρ_τ on X by $x_1\rho_\tau x_2$ if and only if whenever $U \in \tau$ and $x_1 \in U$, then $x_2 \in U$.

Definition 3.08 Let (X,ρ) be a relation. (X,ρ) is said to be a *complete quasi order* (cqo) if ρ is a quasi order (i.e., if ρ is reflexive and transitive) and if whenever D is a directed subset of X, then a least upper bound of D ($lubD$) exists in X.

Definition 3.09 Let (X,ρ) be a relation, and let $C \subseteq X$. C is said to have the *convergence related property* if whenever D be a directed subset of X with $lubD \in C$, then D is eventually in C.

Definition 3.010 Let $f : (X,\sigma) \to (Y,\tau)$ be a continuous function. f is said to be *convergence related* if whenever $V \in \tau$ has the convergence-related property with respect to ρ_τ then $f^{-1}(V)$ has the convergence-related property with respect to ρ_σ.

A topological space (X,τ) is said to be a Scott topological space if the topology is the set of all ρ_τ-up-closed sets with the convergence related property.

Define $G : \mathbf{Top_c} \to \mathbf{Rel}$ such that $G(X,\tau) = (X,\rho_\tau)$. Define $F : \mathbf{Rel} \to \mathbf{Top_c}$ such that $F(X,\rho)$ is the Scott topology determined by ρ.

Theorem 3.011 $(G,(\mathbf{Top_c},U),(\mathbf{Rel},V),F)$ is a correspondence where U and V are forgetful functors to **Set**.

This example generalizes the main result in [5].

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Categories of fuzzy topological spaces with localization at points

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We study *weakly stratified* (all constants are open) *L–topological spaces* with respect to a continuous frame $(L, \leq)$ (with top element 1). Their category is denoted by WS-L-TOP.

Revised axioms of EQ-algebras *

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1 Introduction

Every many-valued logic is uniquely determined by the algebraic properties of the structure of its truth values. It is generally accepted that in fuzzy logic, it should be a residuated lattice, possibly fulfilling some additional properties. A higher-order fuzzy logic — the *fuzzy type theory* (FTT) — has been introduced in [10]. The algebra of truth values considered there is the IMTL$_\Delta$-algebra. In [9], other kinds of FTT have been introduced, namely those where the algebra of truth values is one of Łukasiewicz$_\Delta$, BL$_\Delta$ or ŁΠ-algebra (see [4, 3, 6, 2] and elsewhere).

The basic connective in FTT, however, is a fuzzy equality since it is developed as a generalization of the elegant classical formal system originated by A. Church and L. Henkin (see [1, 7]). In the algebras considered in FTT so far, however, the basic connective is implication and the equivalence is derived on the basis of it. An extension of MV-algebra by similarity (fuzzy equality) has been presented in [5]. An algebra of truth values specific for FTT that is called *EQ-algebra* has been introduced in [11, 12]. In this paper, we continue the work on these algebras with the goal to provide grounds for new establishment of FTT. For this reason, we have slightly revised the axioms of EQ-algebras so that the algebras introduced in [11, 12] will now be referred to as weak EQ-algebras.

Let us also remark that a concept related to EQ-algebras are *equivalential algebras* introduced in [8]. It turns out, however, that they are very special algebras that are of little interest for fuzzy logics since they are strongly related only to Heyting algebras.

Recall that a fuzzy relation $E : U \times U \to L$ is called a *fuzzy equality* on U if it is *reflexive* ($E(u,u) = \mathbf{1}$), *symmetric* ($E(u,v) = E(v,u)$) and *transitive* ($E(u,v) \otimes E(v,w) \leq E(u,w)$).

2 EQ-algebras

Definition 1. *EQ-algebra is the algebra*

$$\mathcal{L} = \langle L, \wedge, \otimes, \sim, \mathbf{1} \rangle \tag{1}$$

of type (2, 2, 2, 0) where for all $a, b, c \in L$:

- *(E1)* $\langle L, \wedge, \mathbf{1} \rangle$ *is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element $\mathbf{1}$). We put $a \leq b$ iff $a \wedge b = a$, as usual.*
- *(E2)* $\langle L, \otimes, \mathbf{1} \rangle$ *is a (commutative) monoid and $\otimes$ is isotone w.r.t. $\leq$.*
- *(E3)* $a \sim a = \mathbf{1}$,
- *(E4)* $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$,
- *(E5)* $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$,

* The research was supported by project MSM 6198898701.

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$,
(E7) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$,
(E8) $a \otimes b \leq a \sim b$.

The operation "$\wedge$" is called meet (infimum), "$\otimes$" is called product and "$\sim$" is a fuzzy equality.

Axiom (E3) is axiom of reflexivity, (E4) is substitution axiom, (E5) is congruence axiom, (E6)–(E7) are monotonicity axioms and (E8) is axiom of boundedness.

Clearly, $\leq$ is the classical partial order. We will also put

$$a \to b = (a \wedge b) \sim a, \tag{2}$$

and

$$\tilde{a} = a \sim \mathbf{1}, \tag{3}$$

$a, b \in L$. The derived operation (2) will be called implication.

EQ-algebra is *separated* if for all $a, b \in L$

(E9) $a \sim b = \mathbf{1}$ implies $a = b$.

The following properties hold in EQ-algebras:

(a) $a \sim b = b \sim a$, (symmetry)
(b) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$, (transitivity)
(c) $(a \to b) \otimes (b \to c) \leq a \to c$. (transitivity of implication)

Let $a \leq b$. Then

(a) $a \to b = \mathbf{1}$. If $\mathcal{L}$ is separated then $a \to b = \mathbf{1}$ implies that $a \leq b$.
(b) $c \to a \leq c \to b, \qquad b \to c \leq a \to c$.
(c) $a \sim b = b \to a$.
(d) $\tilde{a} \leq \tilde{b}$.

Let us introduce the following induced operations:

$$a \leftrightarrow b = (a \to b) \wedge (b \to a), \tag{4}$$
$$a \mathrel{\hat{\Leftrightarrow}} b = (a \to b) \otimes (b \to a). \tag{5}$$

Lemma 1. *The following holds in every EQ-algebra $\mathcal{L}$:*

(a) $(a \wedge b) \leftrightarrow a = (a \wedge b) \mathrel{\hat{\Leftrightarrow}} a = a \to b$.
(b) $a \mathrel{\hat{\Leftrightarrow}} b \leq a \sim b \leq a \leftrightarrow b$.
(c) Both $\leftrightarrow$ as well as $\mathrel{\hat{\Leftrightarrow}}$ fulfil axioms (E3), (E4), (E6)–(E8).
(d) If $\mathcal{L}$ is linearly ordered then $a \leftrightarrow b = a \mathrel{\hat{\Leftrightarrow}} b = a \sim b$.

Lemma 2. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$ be a residuated lattice and $f : L \to L$ be a $\wedge$-homomorphism such that $a \Leftrightarrow b \leq f(a) \Leftrightarrow f(b)$ holds for all $a \in L$ where $a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a)$. Put*

$$a \sim b = f(a) \Leftrightarrow f(b).$$

Then $\langle L, \wedge, \otimes, \sim, \mathbf{1} \rangle$ is an EQ-algebra.

Example 1. Let $L = [0,1]$, $x \otimes y = 0 \vee (x+y-1)$ be the Łukasiewicz conjunction and define $f : [0,1] \to [0,1]$ by $f(x) = 1 \wedge (x+k)$ for some $k \in [0,1]$ and put

$$x \sim y = 1 - |f(x) - f(y)|, \qquad x, y \in [0,1]. \tag{6}$$

Then

$$\mathcal{L} = \langle [0,1], \wedge, \otimes, \sim, 1 \rangle$$

is an EQ-algebra in which $\tilde{x} = f(x)$. This algebra is separated.

Example 2. Let $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$ be a residuated lattice. Furthermore, let $*$ be a new monoidal operation such that $* \leq \otimes$. Then $\mathcal{L} = \langle L, \wedge, *, \Leftrightarrow, \mathbf{1} \rangle$ is a separated EQ-algebra. If $\mathcal{L}$ is linearly ordered then also $\mathcal{L} = \langle L, \wedge, *, \diamondsuit, \mathbf{1} \rangle$ is a separated EQ-algebra where $a \diamondsuit b = (a \Rightarrow b) \otimes (b \Rightarrow a)$.

There are also examples of finite non-trivial (i.e. non-residuated) algebras.

Lemma 3. *If $a \to b = \mathbf{1}$ then $a \leq b$ or $a \sim b = \mathbf{1}$ or $a \| b$.*

By this lemma, we can have comparable elements a, b such that $a > b$, $a \sim b = \mathbf{1}$, and $a \to b = \mathbf{1}$. Such an ordered couple $\langle a, b \rangle$ will be called *pathological*.

A filter is a subset $F \subset L$ such that the following is fulfilled:

(i) If $a \in F$ and $a \leq b$ then $b \in L$.
(ii) If $a, b \in F$ then $a \otimes b \in F$.
(iii) If $a \to b \in F$ for a non-pathological couple $\langle a, b \rangle \in L^2$ and $c \in L$ then $a \otimes c \to b \otimes c \in F$.

Put

$$D = \{\mathbf{1}\} \cup \{u \mid (\forall a, b, c \in L)(\exists n)((a \to b = \mathbf{1} \text{ and } a \| b) \text{ implies } u \geq ((a \otimes c) \to (b \otimes c))^n)\}. \tag{7}$$

Lemma 4. *The set D defined in (7) is a filter and $D \subset L$.*

Put

$$a \approx_F b \text{ iff } a \sim b \in F. \tag{8}$$

The relation $\approx_F$ is the equivalence relation on L. The equivalence class w.r.t. $\approx_F$ will be denoted by $[a]$. Furthermore, we will define a factor-algebra

$$\mathcal{L}|F = \langle L|F, \wedge, \otimes, \sim_F, \mathbf{1} \rangle \tag{9}$$

in the standard way where $[a] \sim_F [b] = [a \sim b]$. The top element is $[\mathbf{1}]$.

Theorem 1. *Let an EQ-algebra $\mathcal{L}$ does not contain pathological couples. Then*

(a) To every $a \neq \mathbf{1}$ there is a maximal filter $F \subset L$ such that $a \notin F$.
(b) The relation $\approx_F$ is a congruence in it.
(c) The algebra (9) is a separated EQ-algebra and $f : a \mapsto [a]$ is a homomorphism of $\mathcal{L}$ onto $\mathcal{L}|F$.

Let $\mathcal{L}$ contain also the bottom element $\mathbf{0}$. Then we put $\neg a = a \sim \mathbf{0}$.

(i) EQ-algebra is *spanned* if

(E10) $\tilde{\mathbf{0}} = \mathbf{0}$.

(ii) *semiseparated* if for all $a \in E$,

(E11) $a \sim \mathbf{1} = \mathbf{1}$ implies $a = \mathbf{1}$.

(iii) EQ-algebra is *good* if for all $a \in L$,

(E12) $a \sim \mathbf{1} = a$.

(iv) EQ-algebra is *involutive* (IEQ-algebra) if for all $a \in L$,

(E13) $\neg\neg a = a$.

(v) EQ-algebra is *residuated* if for all $a, b, c \in L$,

(E14) $(a \otimes b) \wedge c = a \otimes b$ iff $a \wedge ((b \wedge c) \sim b) = a$.

If the EQ-algebra is good then it is spanned but not vice-versa. Each residuated EQ-algebra is good and separated. Many properties of good EQ-algebras become the standard properties known from the theory of residuated lattices. E.g., in every good EQ-algebra $a \leftrightarrow \mathbf{1} = a \Leftrightarrow \mathbf{1} = a$. An EQ-algebra $\mathcal{L}$ is good iff $a \otimes (a \sim b) \leq b$ for all $a, b \in L$. IEQ-algebra is good, spanned and separated.

An EQ-algebra is *complete* if it is a complete $\wedge$-semilattice. A lattice ordered EQ-algebra is an EQ-algebra that is at the same time also a lattice. It is a *lattice EQ-algebra* (ℓEQ-algebra) if it is lattice ordered and, moreover, the following additional substitution axiom holds:

(E15) $((a' \vee b) \sim c) \otimes (a' \sim a) \leq ((a \vee b) \sim c)$.

A complete EQ-algebra is a complete lattice ordered EQ-algebra. A finite EQ-algebra is latice ordered. A complete residuated EQ-algebra is a complete residuated lattice.

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Discrete duality and its application to bounded lattices with operators

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1 Introduction

Duality theory emerged from the work by Marshall Stone [18] on Boolean algebras and distributive lattices in the 1930s. Later in the early 1970s Larisa Maksimova [10, 11] and Hilary Priestley [15, 16] developed analogous results for Heyting algebras, topological Boolean algebras, and distributive lattices. Duality for bounded, not necessarily distributive lattices, was developed by Alstir Urquhart [19]. Since the 1970s establishing a duality between and class of algebras and a class of (ultrafilter) frames has become an important methodological problem both in algebra and in logic.

All the abovementioned classical duality results are developed using topological spaces as dual structures of algebras. Imposing a topology on the (ultrafilter) frame allows the original algebra to be recovered - in particular, as the algebra of basic open sets of the topology. Recent developments building on Priestley duality include canonical extensions of Gehrke[3, 5] and the coalgebraic approach of Venema[20]. The relationship between these developments and dualities for Boolean algebras is given [2, 20].

Here we consider dualities from a different angle with the aim of presenting a framework that serves as a tool for developing Kripke-style semantics for logics whose algebraic semantics is given by a class of algebras (see, for example, [17, 13, 7]).

In an attempt to arrive at this framework we use two ideas. The first, from non-classical logic, is that, given a formal language, its Kripke semantics can be derived from its algebraic semantics, and vice versa. The second is that underlying Urquhart's [19] duality is a relational representation of bounded lattices in terms of abstract relational structures, namely doubly ordered sets $(X, \leq_1, \leq_2)$. Urquhart's duality then shows that any bounded lattice can be represented as the lattice of Galois closed sets of a Galois connection between the lattice of $\leq_1$-increasing subsets of X and the lattice of $\leq_2$-increasing subsets of X. The relational structures involved may be referred to as frames following the terminology of non-classical logics. A topology is not involved in the construction of these frames and hence they may be thought of as having a discrete topology. The relational representation may be referred to as a discrete duality.

Manifestations of discrete duality for classes of lattices with operators can be traced back to Jónsson and Tarski [8, 9], Goldblatt [6], Allwein and Dunn [1]. Essentially the idea is that for lattices with operators (such as, for example, negation, fusion, implication, necessity operators, sufficiency operators, monotone operators) the frames require an additional binary relation satisfying certain conditions derived from the properties of the particular operator. (See, for example, [4, 12].) In each case these conditions are sufficient to ensure a discrete duality for the particular class of lattices with operators.

Establishing a discrete duality therefore involves the following steps. Given a class Alg of algebras (resp. a class Frm of frames) we define a class Frm of frames (resp. a class Alg of algebras). Next, for an algebra $W \in \mathsf{Alg}$ we define its canonical frame $\mathcal{X}(W)$ and for each frame $X \in \mathsf{Frm}$ we define its complex algebra $\mathcal{C}(X)$. Then we prove that $\mathcal{X}(W) \in \mathsf{Frm}$ and $\mathcal{C}(X) \in \mathsf{Alg}$. A discrete duality between Alg and Frm holds provided that the following facts are proved:

(D1) Every algebra $W \in \mathsf{Alg}$ is embeddable into the complex algebra $\mathcal{C}(\mathcal{X}(W))$ of its canonical frame.
(D2) Every frame $X \in \mathsf{Frm}$ is embeddable into the canonical frame $\mathcal{X}(\mathcal{C}(X))$ of its complex algebra.

The representation theorems (D1) and (D2) play an important role in proving completeness of logics with respect to the Kripke-style semantics determined by the class of frames associates with a given class of algebras [13].

In this talk we attempt to arrive at a general framework for various bounded lattices with operators, including Heyting algebras with operators [14], lattices with modal-type operators, lattices with relation algebra operators, and lattice with negations [4].

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Semilinear spaces, systems of linear-like equations, fixed points of contraction and dilatation operators

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1 Introduction

Linear behavior of fuzzy systems has been recently discovered in [5]. Two cases may occur: a behavior of a system is characterized by fuzzy IF-THEN rules (expert knowledge and similar) or it is characterized by a set of input-output pairs of fuzzy sets (monitoring, collecting knowledge, etc.). In the second case, the problem of solving a respective system of fuzzy relation equations arises [6, 11]. However, it has not yet been observed (besides, probably in [3]) that the mentioned problem is similar to the problem of solving systems of linear equations. In this contribution, we show that a system of fuzzy relation equations can be considered as a system of linear-like equations in a semilinear space over a residuated lattice. We will concentrate on systems with $\sup -*$ composition and $\inf \rightarrow$ composition because they are more popular in practical applications. Both systems are considered on finite universes.

In this contribution we change an angle under which this problem is usually considered (see, e.g., [1, 4, 6, 10]). We concentrate on a characterization of possible right-hand sides which make a system solvable with a given left-hand side. We prove that the right-hand side of a systems with $\sup -*$ composition ($\inf \rightarrow$ composition) must be a fixed point of a special contraction (dilatation) operator. Moreover, we show that a set of fixed points of a contraction operator is a linear subspace of an underlying vector space. We characterize each fixed point of a contraction operator as an eigenvector of the respective similarity matrix. Last, but not least, we show that the class of equivalence of each fixed point of a contraction operator is set of solutions of the system with $\inf \rightarrow$ composition.

2 Semi-linear spaces

In this section, a *semilinear space* is taken as a semimodule over a $\vee$-reduct of a residuated lattice supplied with an additional scalar operation. The latter is introduced on the basis of a new idea (different from "natural" definitions used in [5, 7, 9]), which makes utilizing knowledge about residual mappings possible.

Definition 1. *Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a residuated lattice ([8]) and $\mathcal{L}_\vee$ its semiring reduct. Let $\langle A, \vee, \mathbf{0}\rangle$ be a $\vee$-semilattice with the least element.*

We say that $\mathcal{A} = \langle A, \vee, \bar{*}, \mathbf{0}\rangle$ is an idempotent semilinear space over a residuated lattice (shortly, a semilinear space) if the scalar multiplication $\bar{*} : L \times A \rightarrow A$ is defined so that

- *$\langle A, \vee, \mathbf{0}\rangle$ is a semimodule over $\mathcal{L}_\vee$ (The scalar multiplication is $\bar{*}$),*
- *for each $\lambda \in L$ the mapping $h_\lambda : A \rightarrow A$, defined by $h_\lambda(\mathbf{a}) = \lambda \bar{*} \mathbf{a}$, has a residual, i.e. the isotone mapping $g_\lambda : A \rightarrow A$ such that*

$$(g_\lambda \circ h_\lambda)(\mathbf{a}) \geq \mathbf{a}, \tag{1}$$

$$(h_\lambda \circ g_\lambda)(\mathbf{a}) \leq \mathbf{a}. \tag{2}$$

We say that a semilinear space $\mathcal{A}$ is *lattice-ordered* if its carrier is a lattice with respect to the order given above. The elements of a semilinear space are called *vectors* and denoted by bold characters, and elements of L are called *scalars* and denoted by Greek characters.

Based on the fact (see e.g. [2]) that if a residual mapping exists then it is unique, we define another scalar operation $\bar{\rightarrow}$ on A:

$$\lambda \bar{\rightarrow} \mathbf{a} = g_\lambda(\mathbf{a}). \tag{3}$$

It is true that for any $\mathbf{a} \in A$, $\lambda \bar{\rightarrow} \mathbf{a} = \max\{\mathbf{b} \in A \mid \lambda \bar{*} \mathbf{b} \leq \mathbf{a}\}$ holds true if and only if the right-hand side exists. Therefore, if $\mathcal{A}$ has the greatest element $\mathbf{1}$ then for any $\mathbf{a} \in A$, $0 \bar{\rightarrow} \mathbf{a} = \mathbf{1}$.

The following is the example of a semilinear space which we will use in the sequel.

Example 1. Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a residuated lattice on L. The set of n-dimensional vectors L^n, $n \geq 1$, such that

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \iff a_1 \leq b_1, \ldots, a_n \leq b_n$$

is a lattice ordered semilinear space over $\mathcal{L}$ where for arbitrary $\lambda \in L$

$$\lambda \bar{*} (a_1, \ldots, a_n) = (\lambda * a_1, \ldots, \lambda * a_n),$$
$$\lambda \bar{\rightarrow} (a_1, \ldots, a_n) = (\lambda \rightarrow a_1, \ldots, \lambda \rightarrow a_n).$$

3 Systems of fuzzy relation equations under $\circ$ and $\triangleright$ compositions

In what follows, we fix a residuated lattice $\mathcal{L}$ with a support L and consider L^m and L^n, $m, n \geq 1$, as semilinear spaces over $\mathcal{L}$. Throughout this section, let $A = (a_{ij})$ be a fixed $n \times m$ matrix and $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{d} = (d_1, \ldots, d_m)$ vectors, all have components from L. The following two systems of linear-like equations

$$\bigvee_{j=1}^{m} (a_{ij} * x_j) = b_i, \quad i = 1, \ldots n, \tag{4}$$

$$\bigwedge_{i=1}^{n} (a_{ij} \rightarrow y_i) = d_j, \quad j = 1, \ldots m, \tag{5}$$

are considered with respect to unknown vectors $\mathbf{x} = (x_1 \ldots, x_m)$ and $\mathbf{y} = (y_1 \ldots, y_n)$.

In the literature, which is related to fuzzy sets and systems, the above considered systems are known as systems of fuzzy relation equations:

$$A \circ \mathbf{x} = \mathbf{b}$$
$$A \triangleright \mathbf{y} = \mathbf{d}$$

where $\circ$ is the so called sup $-*$ composition and $\triangleright$ is the inf $\rightarrow$ composition.

The problem of solvability of systems (4) or (5) will be considered in the following formulation:

Given $n \times m$ matrix A, characterize all vectors $\mathbf{b} \in L^n$ (all vectors $\mathbf{d} \in L^m$) such that (4) (respectively, (5)) is solvable.

4 Fixed points of the contraction and dilatation operators

In this section, we will show that problems (4) and (5) are equivalent with the respective fixed point problems of contraction and dilatation operators.

Definition 2. *Let $A = (a_{ij})$ be a $n \times m$ matrix with components from L. We say that*

- *$AA^{\rightarrow} : L^n \mapsto L^n$ is a contraction operator if for any $\mathbf{b} \in L^n$ such that $\mathbf{b} = (b_1, \ldots, b_n)$, $(AA^{\rightarrow})(\mathbf{b}) = ((AA^{\rightarrow})\mathbf{b}_1, \ldots, (AA^{\rightarrow})\mathbf{b}_n)$ where*

$$(AA^{\rightarrow})\mathbf{b}_i = \bigvee_{j=1}^{m} (a_{ij} * \bigwedge_{l=1}^{n} (a_{lj} \rightarrow b_l)), \quad i = 1, \ldots, n. \tag{6}$$

- *$A^{\rightarrow}A : L^m \mapsto L^m$ is a dilatation operator if for any $\mathbf{d} \in L^m$ such that $\mathbf{d} = (d_1, \ldots, d_m)$, $(A^{\rightarrow}A)(\mathbf{d}) = ((A^{\rightarrow}A)\mathbf{d}_1, \ldots, (A^{\rightarrow}A)\mathbf{d}_m) \in L^m$ where*

$$(A^{\rightarrow}A)\mathbf{d}_j = \bigwedge_{i=1}^{n} (a_{ij} \rightarrow \bigvee_{l=1}^{m} (a_{il} * d_l)), \quad j = 1, \ldots, m. \tag{7}$$

Due to the limitation of space, we will give only some results related to the contraction operator $AA^{\rightarrow}$ and its fixed points.

Proposition 1. *Let $A = (a_{ij})$ be a $n \times m$ matrix with components from L, $AA^{\rightarrow} : L^n \mapsto L^n$ the corresponding contraction operator. Then the following holds true:*

- *for each $\mathbf{b} \in L^n$, $(AA^{\rightarrow})\mathbf{b} \leq \mathbf{b}$,*
- *for each $\mathbf{b} \in L^n$, $\mathbf{b}_0 = (AA^{\rightarrow})\mathbf{b}$ is a fixed point of of $AA^{\rightarrow}$,*
- *for each $\mathbf{b}_1, \mathbf{b}_2 \in L^n$, $\mathbf{b}_1 \leq \mathbf{b}_2$ implies $(AA^{\rightarrow})(\mathbf{b}_1) \leq (AA^{\rightarrow})(\mathbf{b}_2)$,*
- *$\mathbf{b}_0 \in L^n$ is a fixed point of $AA^{\rightarrow}$ if and only if there exists $\mathbf{x} \in L^m$ such that $A \circ \mathbf{x} = \mathbf{b}_0$,*
- *if $\mathbf{b}_0 \in L^n$ is a fixed point of $AA^{\rightarrow}$ such that $\mathbf{b}_0 = (AA^{\rightarrow})\mathbf{b}$ then $A^{\rightarrow}\mathbf{b} \leq A^{\rightarrow}\mathbf{b}_0$,*
- *if $\mathbf{b}_1, \mathbf{b}_2 \in L^n$ are fixed points of $AA^{\rightarrow}$ then $\mathbf{b}_1 \vee \mathbf{b}_2$ is a fixed point too.*

Theorem 1. *Let $A = (a_{ij})$ be a $n \times m-$ matrix with components from L, $AA^{\rightarrow} : L^n \mapsto L^n$ the corresponding contraction operator. Then the set of fixed points of $AA^{\rightarrow}$ is a linear subspace of L^n generated by vector-columns $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of A.*

5 Theorem of isomorphisms

In this section, we show that the contraction operator $AA^{\rightarrow}$ establishes an equivalence $\equiv_{(AA^{\rightarrow})}$ on L^n such that each class $[\cdot]_{\equiv_{(AA^{\rightarrow})}}$ is a set of solutions of the system with inf $\rightarrow$ composition.

Lemma 1. *Let $A = (a_{ij})$ be a $n \times m$ matrix with components from L, $AA^{\rightarrow} : L^n \mapsto L^n$ the corresponding contraction operator. Let $\mathbf{b}_1, \mathbf{b}_2 \in L^n$, $\mathbf{d}_1, \mathbf{d}_2 \in L^m$. The following are equivalences on L^n and L^m respectively:*

- *$\mathbf{b}_1 \equiv_{(AA^{\rightarrow})} \mathbf{b}_2 \iff (AA^{\rightarrow})(\mathbf{b}_1) = (AA^{\rightarrow})(\mathbf{b}_2)$;*
- *$\mathbf{d}_1 \equiv_A \mathbf{d}_2 \iff (A \circ \mathbf{d}_1) = (A \circ \mathbf{d}_2)$.*

Theorem 2. *Let $L(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ be the linear subspace of L^n generated by vector-columns of A. The following isomorphisms between factor spaces can be established:*

1. $L^n/_{\equiv_{(AA^{\rightarrow})}} \cong L(\mathbf{a}_1,\dots,\mathbf{a}_m)$,
2. $L^m/_{\equiv_A} \cong L(\mathbf{a}_1,\dots,\mathbf{a}_m)$,
3. $L^n/_{\equiv_{(AA^{\rightarrow})}} \cong L^m/_{\equiv_A}$.

Theorem 3. *Let $\mathbf{b}_0$ be a fixed point of $(AA^{\rightarrow})$. The class of equivalence $[\mathbf{b}_0]_{\equiv_{(AA^{\rightarrow})}}$ is a set of solutions of (5) with the righthand side given by $A^{\rightarrow}\mathbf{b}_0$, i.e.*

$$A \triangleright \mathbf{y} = A^{\rightarrow}\mathbf{b}_0.$$

6 Similarity matrices in a semi-linear space

Definition 3. *An $n \times n$ matrix E is a similarity matrix if for all $i,j,k = 1,\dots,n$, the following holds:*

- $S_{ii} = 1$,
- $S_{ij} = S_{ji}$,
- $S_{ij} * S_{jk} \leq S_{ik}$.

We say that vector $(b_1,\dots,b_n) \in L^n$ is *extensional* with respect to E if for all $l,k = 1,\dots,n$,

$$E_{lk} \leq b_l \leftrightarrow b_k.$$

Let $\mathbf{a}^1,\dots,\mathbf{a}^m \in L^n$ be vectors and $\mathbf{a}^j = (a_1^j,\dots,a_n^j)$, $j = 1,\dots,n$. We consider the following $n \times n$ similarity matrix E^A:

$$E^A_{lk} = \bigwedge_{j=1}^{m} (a_l^j \leftrightarrow a_k^j). \tag{8}$$

We say that vectors $\mathbf{a}^1,\dots,\mathbf{a}^m$ *generate* E^A or that they are *generating vectors*.

We will show that a fixed point of the operator $AA^{\rightarrow}$ is an eigenvector of a similarity matrix generated by the column vectors $\mathbf{a}^1,\dots,\mathbf{a}^m$ of A.

Theorem 4. *Let $\mathbf{b} \in L^n$ be a fixed point of the operator $AA^{\rightarrow}$ where $A = (a_{ij})$ is an $n \times m$ matrix with components from L. Let E^A be a similarity matrix generated by the column vectors $\mathbf{a}^1,\dots,\mathbf{a}^m$ of A. Then $\mathbf{b}$ is extensional with respect to E^A, and it is a fixed point of the operator $(E^A)(E^A)^{\rightarrow}$.*

Corollary 1. *Let the assumptions of Theorem 4 be fulfilled. Then any fixed point $\mathbf{b} \in L^n$ of the operator $AA^{\rightarrow}$ is a fixed point of both operators $(E^A) : L^n \mapsto L^n$ and $(E^A)^{\rightarrow} : L^n \mapsto L^n$, i.e.*

- $(E^A\mathbf{b})_i = \bigvee_{j=1}^n (e_j^i * b_j) = b_i, \quad i = 1,\dots n,$
- $((E^A)^{\rightarrow}\mathbf{b})_i = \bigwedge_{j=1}^n (e_j^i \rightarrow b_j) = b_i, \quad i = 1,\dots n.$

Acknowledgement. The research has been supported by the project MSM 6198898701 of the MŠMT ČR.

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Functorial generation of non-stratified, anti-stratified, and normalized spaces

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1 Preliminaries and purpose

All lattices L of membership values are frames, i.e., complete lattices satisfying the first infinite distributive law. The inconsistent frame with $\perp$ equal $\top$ is allowed.

The fixed-basis category L-**Top** comprises all *L-topological spaces* (X,τ), where τ is a subframe of L^X, together with all *L-continuous morphisms* $f:(X,\tau)\to(Y,\sigma)$, where $f:X\to Y$ is a function and $(f_L^{\leftarrow})^{\rightarrow}(\sigma)\subset\tau$, the notation $(f_L^{\leftarrow})^{\rightarrow}$ indicating the traditional image operator of the Zadeh preimage operator defined by $f_L^{\leftarrow}(v)=v\circ f$. Note **Top** embeds into each L-**Top** and is isomorphic to **2**-**Top** via $G_\chi:\mathbf{Top}\to L\text{-}\mathbf{Top}$ given by

$$G_\chi(\mathfrak{T})=\{\chi_U:U\in\mathfrak{T}\},\quad G_\chi(X,\mathfrak{T})=\big(X,G_\chi(\mathfrak{T})\big),\quad G_\chi(f)=f.$$

The variable-basis category **Loc**-**Top**, cf. [3], [4], [5], [20], [21], [22], [23], [25], [27], [30], [31], comprises all *topological spaces* (X,L,τ) such that $(X,\tau)\in|L\text{-}\mathbf{Top}|$, together with *continuous morphisms* $(f,\varphi):(X,L,\tau)\to(Y,M,\sigma)$ such that $(f,\varphi):(X,L)\to(Y,M)$ in the *ground category* **Set** $\times$ **Loc** and $((f,\varphi)^{\leftarrow})^{\rightarrow}(\sigma)\subset\tau$, where $(f,\varphi)^{\leftarrow}(v)=\varphi^{op}\circ v\circ f$ (φ^{op} being a frame map). Each L-**Top** embeds into **Loc**-**Top** via

$$(X,\tau)\mapsto(X,L,\tau),\quad f\mapsto(f,id_L),$$

and **Top** embeds into **Loc**-**Top** via the composition of this embedding with G_χ. Further, **Loc** embeds into **Loc**-**Top** via both the singleton functor S and the empty functor $\varnothing$:

$$S(A)=(\mathbf{1},A,A^{\mathbf{1}}),\quad S(\varphi)=(id_{\mathbf{1}},\varphi),$$
$$\varnothing(A)=(\varnothing,A,A^{\varnothing}),\quad \varnothing(\varphi)=(id_{\varnothing},\varphi).$$

Thus **Loc**-**Top** is, up to functorial isomorphism, a supercategory of both **Top** and **Loc**, apparently the first known such category.

The full subcategory SL-**Top** [S**Loc**-**Top**] [9], [15] of all stratified L-topological [topological] spaces has objects (X,τ) [(X,L,τ)] such that $\{\underline{\alpha}:\alpha\in L\}\subset\tau$, where $\underline{\alpha}:X\to L$ is the constant map with value α.

The category SL-**Top** is fundamentally related to **Top** via the adjunction $\omega\dashv\iota$ [14], [15], where $\omega:\mathbf{Top}\to SL\text{-}\mathbf{Top}$ and $\iota:\mathbf{Top}\leftarrow L\text{-}\mathbf{Top}$ as follows, using $\langle\langle\ \rangle\rangle$ to indicate the L-topology or topology generated by a subbasis:

$$\omega(\mathfrak{T})=\mathbf{Top}\left[(X,\mathfrak{T}),(L,\langle\langle\{L-\downarrow(\alpha)\}:\alpha\in L\}\rangle\rangle)\right],$$

$$\omega(X,\mathfrak{T})=(X,\omega(\mathfrak{T})),\quad \omega(f)=f,$$

$$\iota(\tau) = \langle\langle\{[u \not\leq \alpha] : u \in \tau, \alpha \in L\}\rangle\rangle, \quad \iota(X,\tau) = (X, \iota(\tau)), \quad \iota(f) = f.$$

And SL-**Top** is fundamentally related to **Top** via the adjunction $V \dashv G_k$, where $V : SL\text{-}\mathbf{Top} \to L\text{-}\mathbf{Top}$ is the forgetful functor and the *stratification functor* $G_k : SL\text{-}\mathbf{Top} \leftarrow L\text{-}\mathbf{Top}$ is given by

$$G_k(\tau) = \tau \vee \{\underline{\alpha} : \alpha \in L\} = \langle\langle \tau \vee \{\underline{\alpha} : \alpha \in L\}\rangle\rangle,$$

$$G_k(X,\tau) = (X, G_k(\tau)), \quad G_k(f) = f.$$

The category S**Loc**-**Top** has similar relationships to both **Top** and **Loc**-**Top** which can be constructed by the reader.

The necessity of stratified spaces in lattice-valued topology is well known, in part because of their natural generation by ω. The necessity of non-stratified spaces was recognized as early as 1986 [7], [21] from the standpoint of the L-spectrum and L-soberification functors. This paper surveys recent and ongoing work in L-frames and topological systems which provides a more complete picture of the necessity of non-stratified spaces, their natural generation by a variety of functors, and the necessity of both stratified and non-stratified spaces in the L-**Top**'s and **Loc**-**Top**.

2 Levels of non-stratification

Since all L-topological spaces are stratified for $|L| \leq 2$, sequens it is assumed that $|L| \geq 3$ unless stated otherwise.

Definition 1. *Let* $(X,\tau) \in |L\text{-}\boldsymbol{Top}|$.

1. (X,τ) *is* non-stratified *if* $\exists \alpha \in L - \{\bot, \top\}$, $\underline{\alpha} \notin \tau$.
2. (X,τ) *is* anti-stratified [2], [19] *if* $\forall \alpha \in L - \{\bot, \top\}$, $\underline{\alpha} \notin \tau$.
3. (X,τ) *is* normalized [1] *if* $\forall u \in \tau - \{\underline{\bot}\}$, $||u|| = \top$, *where* $||u|| = \bigvee_{x \in X} u(x)$.

Proposition 1. *The following hold* [1]*:*

1. *Normalized* $\Rightarrow$ *anti-stratified* $\Rightarrow$ *non-stratified; and none of these implications reverses.*
2. *L-homeomorphisms preserve each of these conditions: stratified, non-stratified, anti-stratified, normalized.*
3. $\forall (X,\mathfrak{T}) \in |L\text{-}\boldsymbol{Top}|$, $\forall L \in |\mathbf{Frm}|$, *each* $G_\chi(X,\mathfrak{T})$ *is normalized.*

3 Non-stratification and L-spectra

Fix frame L, let $A \in |\mathbf{Loc}|$, and recall from [6], [7], [12], [13], [16], [17], [21], [22], [23], [24], [26] that

$$Lpt(A) = \mathbf{Frm}(A,L), \quad \Phi_L : A \to L^{Lpt(A)} \text{ by } \Phi_L(a)(p) = p(a),$$

$$LPT(A) = (Lpt(A), (\Phi_L)^{\rightarrow}(A)).$$

Then the L-topological space $LPT(A)$ is the *L-spectrum of A*. It is known that the *L-sober* spaces are precisely the L-spectra. Recall that A is *L-spatial* if $\Phi_L : A \to (\Phi_L)^{\rightarrow}(A)$ is an order-isomorphism.

Theorem 1. *If* $|\mathbf{Frm}(L,L)| > 1$, *then* $LPT(A)$ *is non-stratified, in which case each L-topological space has a non-stratified Čech-Stone compactification; and if* $\forall \alpha \in L - \{\bot, \top\}$, $\exists \psi \in \mathbf{Frm}(L,L)$, $\psi(\alpha) \neq \alpha$, *then* $LPT(A)$ *is anti-stratified, in which case each L-topological space has a anti-stratified Čech-Stone compactification.*

Given $(X,\tau) \in |L\text{-}\mathbf{Top}|$, put $L\Omega(X,\tau) = \tau$. With appropriate actions on morphisms, $L\Omega$ and LPT are functors, $L\Omega \dashv LPT$, and this adjunction sets up the equivalence between L-sobriety and L-spatiality as well as various classes of Stone representations and Čech-Stone compactifications. The composition $LPT \circ L\Omega$ is called (the) *L-soberification* (*functor*); and the composition $LPT \circ \Omega$ is called (the) *L-*$\mathbf{2}$ *soberification* (*functor*), where $\Omega : \mathbf{Top} \to \mathbf{Loc}$ is the classical functor [11] equivalent to $\mathbf{2}\Omega$ via G_χ.

Theorem 2. *The following hold:*

1. *If* $(X,\tau) \in |L\text{-}\mathbf{Top}|$ *is non-stratified [anti-stratified, normalized], then so is* $LPT\,(L\Omega(X,\tau))$.
2. *If* $(X,\mathfrak{T}) \in |\mathbf{Top}|$, *then* $LPT\,(\Omega(X,\mathfrak{T}))$ *is normalized.*
3. $\forall L \in |\mathbf{Frm}|$, $\mathbb{R}^*(L)$ *and* $\mathbb{I}^*(L)$ [26] *are normalized.*
4. $\forall L \in |\mathbf{CBool}|$, $\mathbb{R}(L)$ *and* $\mathbb{I}(L)$ *are normalized.*

The relevance of being normalized is that a collection $\{(X_\gamma,\tau_\gamma) : \gamma \in \Gamma\}$ of normalized spaces necessarily has a product-separated product topology [1]; localic products also have a form of product-separation; so this property could be related to how $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$ compares with $\bigoplus_{\gamma\in\Gamma}\tau_\gamma$.

4 Anti-stratification and *L*-frames

For this section L is restricted to be a complete chain; $L^\bullet \equiv \{t \in L : t < \top\}$. An *L-frame* A [10], [16], [17], [18], [19] is a pair $(A^{\mathrm{u}}, A^{\mathrm{l}})$—$A^{\mathrm{u}}$ is the "upper" frame and A^{l} is the "lower" frame—together with a family $\{\varphi_t^A : A^{\mathrm{u}} \to A^{\mathrm{l}} \mid t \in L^\bullet\}$ of frame morphisms satisfying the following axioms:

(F0) For $\varnothing \neq S \subset L^\bullet$, $\varphi^A_{\bigwedge S} = \bigvee_{t\in S}\varphi_t^A$.
(F1) $\{\varphi_t^A : A^{\mathrm{u}} \to A^{\mathrm{l}} \mid t \in L^\bullet\}$ is an extremal epi-sink.
(F2) $\{\varphi_t^A : A^{\mathrm{u}} \to A^{\mathrm{l}} \mid t \in L^\bullet\}$ is a mono-source.

An *L-frame morphism* is a pair $(h^{\mathrm{u}}, h^{\mathrm{l}}) : A \to B$ such that $h^{\mathrm{u}} : A^{\mathrm{u}} \to B^{\mathrm{u}}$, $h^{\mathrm{l}} : A^{\mathrm{l}} \to B^{\mathrm{l}}$ are frame morphisms and for each $t \in L^\bullet$, $h^{\mathrm{l}} \circ \varphi_t^A = \varphi_t^B \circ h^{\mathrm{u}}$.

The motivation for L-frames (particularly (F0)) stems from both presheaves [18] and the ι functor of Section 1 above [16], [18]. The role of ι is especially critical since—letting L-**Frm** be the category of all L-frames and L-frame morphisms and L-**Loc** be its dual—the spectrum adjunction $\text{ſ}L\Omega \dashv \blacksquare$ between L-**Top** and L-**Loc** lives on ι-*sobriety* (a fundamentally different sobriety than L-sobriety) defined by saying (X,τ) is ι-sober iff $(X,\iota(\tau))$ is traditionally sober.

It is known that L-**Frm** is complete and cocomplete and has epi-mono decomposition with diagonalization property [18]. Intrinsic to these proofs are the two forgetful functors $\mathcal{U}^{\mathrm{u}}$, $\mathcal{U}^{\mathrm{l}} : L$-**Frm** $\to$ **Frm** given by

$$\mathcal{U}^{\mathrm{u}}(A) = A^{\mathrm{u}},\ \ \mathcal{U}^{\mathrm{u}}(h) = h^{\mathrm{u}};\quad \mathcal{U}^{\mathrm{l}}(A) = A^{\mathrm{l}},\ \ \mathcal{U}^{\mathrm{l}}(h) = h^{\mathrm{l}}.$$

The left adjoint $\mathcal{L} : L$-**Frm** $\leftarrow$ **Frm** of $\mathcal{U}^{\mathrm{u}}$ is constrcuted using quotients of frame coproducts; the right adjoint $\mathcal{R} : L$-**Frm** $\leftarrow$ **Frm** of $\mathcal{U}^{\mathrm{l}}$ is constructed using quotients of frame products; and $\mathcal{L}$ and certain modifications $\mathcal{R}^*$ of $\mathcal{R}$, Ω^* of Ω, and $\blacksquare^*$ of $\blacksquare$ yield the following factorizations:

Theorem 3. *For each complete chain L, the following hold* [19]*:*

1. $LPT \cong \blacksquare \circ \mathcal{L}^{op}$;
2. $\omega = \blacksquare^* \circ \mathcal{R}^* \circ \Omega^*$.

Every L-sober space is ι-sober, but not conversely (for L a complete chain, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are ι-sober but not L-sober); and, using Section 2 above, upper frames of L-frames always produce L-sober and hence anti-stratified L-topological spaces. On the other hand, lower frames of an L-frames resulting from ordinary topological spaces (via Ω^*) always produce (via $\blacksquare^*$) stratified L-topological spaces. Thus the single category L-**Frm** (or L-**Loc**) argues for including in L-**Top** both stratified and non-stratified spaces.

5 Non-stratification and topological systems

Given $(X,A) \in |\mathbf{Set} \times \mathbf{Loc}|$, a binary relation $\models \subset X \times A$ is a *satisfaction* relation [32] if the following *join* and *meet interchange laws* hold:

$$\forall x \in X, \forall \{a_i\}_{i \in I} \subset A,\ x \models \bigvee_{i \in I} a_i \Leftrightarrow x \models a_i \text{ for some } i \in I,$$

$$\forall x \in X, \forall \text{ finite } \{a_i\}_{i \in I} \subset A,\ x \models \bigwedge_{i \in I} a_i \Leftrightarrow x \models a_i \text{ for each } i \in I,$$

in which case $(X,A,\models)$ is a *topological system* [32]. The relation $\models$ is *maximal* [2] if it is $X \times (A - \{\bot\})$. Having ground category $\mathbf{Set} \times \mathbf{Loc}$, **TopSys** [32] comprises all topological systems together with ground morphisms $(f,\varphi) : (X,A,\models) \to (Y,B,\models)$ satisfying

$$\forall x \in X, \forall b \in B,\ f(x) \models_2 b \Leftrightarrow x \models_1 \varphi^{op}(f(b)),$$

such ground morphisms being called *continuous maps*.

Given $(X,\mathfrak{T}) \in |\mathbf{Top}|$, $(X,\mathfrak{T},\in)$ is a topological system; and given $A \in |\mathbf{Loc}|$, $(Pt(A),A,\models)$ is a topological system, where $Pt(A) = \mathbf{2}pt(A)$ and $p \models a$ iff $p(a) = \top$. The first [second] correspondence is part of an embedding E_V [$E_{\mathbf{Loc}}$] of **Top** [**Loc**] into **TopSys** [32], making **TopSys** the second known category that is, up to functorial isomorphism, a supercategory of both **Top** and **Loc**. How are the two supercategories **Loc**-**Top** and **TopSys** related? The most important relationships include the following three concrete functors from **TopSys** to **Loc**-**Top** [2], the first two of which are embeddings and the second and third of which use the notion of *extent*—for $a \in A$, $Ext(a) = \{x \in X : x \models a\}$:

$$\tau_{\models} = \left\{u \in A^X : (\forall x \in X, x \models u(x)) \text{ or } u = \underline{\bot}\right\},$$

$$F_{\models}(X,A,\models) = (X,A,\tau_{\models}),$$

$$\tau_k = \left\langle \left\{\underline{a} \wedge \chi_{Ext(a)} : a \in A\right\}\right\rangle,$$

$$F_k(X,A,\models) = (X,A,\tau_k),$$

$$\tau^k = \left\langle \left\{\underline{a} \wedge \chi_{Ext(b)} : a,b \in A\right\}\right\rangle,$$

$$F^k(X,A,\models) = \left(X,A,\tau^k\right).$$

Theorem 4. *The following hold:*

1. $\forall (X,A,\models) \in |\mathbf{TopSys}|$, $F_{\models}(X,A,\models)$ *is non-stratified if and only if* $(X,A,\models)$ *is non-maximal* [2].
2. $\forall (X,A,\models) \in |\mathbf{TopSys}|$, $F_k(X,A,\models)$ *is non-stratified if and only if* $(X,A,\models)$ *is non-maximal* [2].

3. $\forall(X,A,\models) \in |\mathbf{TopSys}|$, $F^k(X,A,\models)$ *is stratified* [2].
4. $\forall(X,\mathfrak{T}) \in |\mathbf{Top}|$, $F_{\models}(X,\mathfrak{T},\in)$ *is normalized* [1] *and* $F_k(X,\mathfrak{T},\in)$ *is anti-stratified* [2].
5. $\forall A \in |\mathbf{SpatLoc}|$, $F_{\models}(Pt(A),A,\models)$ *is normalized* [1] *and* $F_k(Pt(A),A,\models)$ *is anti-stratified* [2].
6. $\forall A \in |\mathbf{Loc}|$ *with* $\mathrm{Pr}(A^\bullet) - \{\perp\} \neq \varnothing$, $F_{\models}(Pt(A),A,\models)$ *and* $F_k(Pt(A),A,\models)$ *are non-stratified* [2].
7. $\forall A \in |\mathbf{Loc}|$ *with* $|A| \geq 3$, $F_{\models}(Pt(A),A,\models)$ *and* $F_k(Pt(A),A,\models)$ *are non-stratified* [2].

Remark 1. **TopSys** provides both stratified and non-stratified spaces to **Loc**-**Top**, arguing for **Loc**-**Top** including both types of spaces. On the other hand, **Loc**-**Top** may provide information for **TopSys**: **TopSys** is not topological over $\mathbf{Set} \times \mathbf{Loc}$ [2]; **Loc**-**Top** is topological over $\mathbf{Set} \times \mathbf{Loc}$ [9]; so initial and final structures lacking in **TopSys** are provided in **Loc**-**Top**. This latter benefit points up the need for functors from **Loc**-**Top** to **TopSys**, the most important of which, so far, is a functor which combines the embeddings of all the L-**Top**'s into **TopSys**, each of which generalizes E_V and provides anti-stratified spaces via $F_{\models}$ and F_k even when starting with stratified spaces [2].

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Bounded lattices, complete sublattices, and extensions of t-norms

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Lattices and, in particular, bounded lattices are used in any many-valued logic as sets of truth values [4–7, 11, 12]. Such bounded lattices need not necessarily form a chain (a first attempt in this direction is described in [4, Section 15.2], compare [1, 3] and also the paraconsistent logic in [2]). Conjunction and implication form the basic constituents w.r.t. logical operations on such lattices of truth values — the conjunction is interpreted by some triangular norm on L and the implication, most often, by its related residuum/residual implication.

In this contribution we focus on triangular norms on some bounded lattice L of truth values in particular on its construction as extensions of a triangular norm acting on a (complete) sublattice S of L. We investigate the strongest as well as the weakest extension. It is interesting to see that in case of the strongest extension the sublattice S has to fulfill some conditions, and can, therefore, not necessarily be chosen arbitrarily. In case of the weakest extension the (complete) sublattice S can be chosen without any restrictions. Ensuring that the (complete) sublattice S can be chosen arbitrarily in both cases implies that L be a horizontal sum of chains (compare also [8–10]).

We further investigate the case of strongest and weakest extensions of families of t-norms on corresponding families of sublattices and particular cases of bounded lattices of truth values such as horizontal sums, ordinal sums, and Cartesian products of bounded lattices. We further relate our results to properties of triangular norms such as, e.g., the intermediate value property.

Acknowledgments

Susanne Saminger-Platz has been on a sabbatical year at the Dipartimento di Matematica "Ennio De Giorgi", Università del Salento (Italy), when writing and investigating parts of this paper. She therefore gratefully acknowledges the support by the Austrian Science Fund (FWF) in the framework of the Erwin Schrödinger Fellowship J 2636-N15. Radko Mesiar was also supported by the grants MSM6198898701 and VEGA 1/3006/06.

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From quantale algebroids to topological spaces

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1 Introduction

The notion of fuzzy set introduced by Zadeh in [13] and generalized by Goguen in [2] induced many lines of research to study different fuzzy structures. In particular, [3, 8] consider fixed-basis as well as variable-basis fuzzy topologies and its properties. On the other hand, localic theory was being developed in 1960's and 1970's and was given a coherent statement by Johnstone in [5]. One of the milestones was the result of Papert and Papert [6] (described more succinctly by Isbell [4]) stating that the (obvious) functor $\mathbf{Top} \xrightarrow{O} \mathbf{Loc}$ has a right adjoint. As a result one got an appropriate environment in which to develop (pointless) topology.

A fuzzy analogue of the aforesaid adjunction was considered in, e.g., [8] (replacing **Loc** by the dual of the category **CQML** of complete quasi-monoidal lattices) as well as in [9] (in a more general way for the category L-**Loc** of L-locales).

Motivated by the aforesaid results we considered in [11] an adjunction between the dual of the category Q-**Alg** of algebras over a given unital commutative quantale Q [12] and the category Q-**Top** of stratified Q-topological spaces. With appropriate changes in the definitions of sobriety and spatiality we got an equivalence of the categories of Q-sober spaces and Q-spatial Q-algebras (already included in the L-sobriety of [9]).

By analogy with the notion of quantaloid of Rosenthal [10] we introduced in our talk during "The 45th Summer School on Algebra and Ordered Sets" in Tále (Slovakia) the notion of quantale algebroid. It is our purpose now to generalize the results of [11] to the new setting. The necessary categorical background can be found in [1].

2 Algebraic preliminaries

This section introduces the category Q-**Abrds** of Q-algebroids over a given unital commutative quantale Q. Start by recalling the notion of quantale from [10].

Definition 1. *A* quantale *is a $\bigvee$-lattice Q together with an associative binary operation $\otimes$ satisfying: $q \otimes (\bigvee S) = \bigvee_{s \in S} (q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s \in S} (s \otimes q)$ for $q \in Q$ and $S \subseteq Q$. A quantale Q is* unital *provided that there exists an element $1 \in Q$ with $1 \otimes q = q = q \otimes 1$ for $q \in Q$. Q is* commutative *provided that $q \otimes s = s \otimes q$ for $q, s \in Q$.*

Every frame is a unital commutative quantale, e.g., the chain $\mathbf{2} = \{0, 1\}$ is a quantale.

Definition 2. *A map $Q \xrightarrow{f} S$ between quantales is a* quantale homomorphism *provided that f preserves $\otimes$ and $\bigvee$.*

Definitions 1 and 2 give the category **Quant** of quantales and quantale homomorphisms.

On the next step we define the category Q-**Mod** of left Q-modules over a given unital quantale Q [7, 10] motivated by the category of left modules over a ring.

Definition 3. *Given a unital quantale Q, Q-**Mod** is the category, the objects of which (called* unital left Q-modules*) are pairs $(A,*)$, where A is a $\bigvee$-lattice and $Q\times A \xrightarrow{*} A$ is a map such that for $q,q' \in Q$, $a \in A$ and $T \subseteq Q$, $S \subseteq A$:*

- $q*(\bigvee S) = \bigvee\limits_{s\in S}(q*s)$ *and* $(\bigvee T)*a = \bigvee\limits_{t\in T}(t*a)$;
- $q*(q'*a) = (q\otimes q')*a$;
- $1*a = a$.

Morphisms $(A,*) \xrightarrow{f} (B,*)$ (called unital left Q-module homomorphisms*) are $\bigvee$-preserving maps $A \xrightarrow{f} B$ such that $f(q*a) = q*f(a)$ for $a\in A$ and $q\in Q$.*

It is easy to see that the category **2**-**Mod** is isomorphic to the category **CSLat**$(\bigvee)$ of $\bigvee$-lattices and $\bigvee$-preserving maps.

Now we are ready to define the category Q-**Alg** of Q-algebras over a given unital commutative quantale Q motivated by the category of algebras over a commutative ring.

Definition 4. *Given a unital commutative quantale Q, Q-**Alg** is the category, the objects of which (called* Q-algebras*) are Q-modules $(A,*)$ such that A is a quantale with the property that: $q*(a\otimes b) = (q*a)\otimes b = a\otimes(q*b)$ for $a,b\in A$ and $q\in Q$. Morphisms $(A,*) \xrightarrow{f} (B,*)$ (called* Q-algebra homomorphisms*) are quantale homomorphisms $A \xrightarrow{f} B$ which are also Q-module homomorphisms.*

One can easily see that the category **2**-**Alg** is isomorphic to the category **Quant**.

The next definition is motivated by the concept of quantaloid of Rosenthal [10]. Recall that *quantaloids* are categories, whose hom-sets are $\bigvee$-lattices, with composition in the category preserving $\bigvee$ in both variables.

Definition 5. *Given a unital commutative quantale Q, a* Q-algebroid *is a category, whose hom-sets are Q-modules, with composition of morphisms preserving $\bigvee$ and $*$ in both variables.*

If $\mathbf{A}$ is a Q-algebroid, then for every $\mathbf{A}$-morphisms f,g with $f\circ g$ defined, $q*(f\circ g) = (q*f)\circ g = f\circ(q*g)$ for $q\in Q$, and for every $\mathbf{A}$-object A, the hom-set $\mathbf{A}(A,A)$ is a unital Q-algebra (from here the terminolgy "quantale algebroid"). Notice that quantale algebroids can be thought of as quantale algebras "with many objects".

Definition 6. *A functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ between Q-algebroids is a* Q-algebroid homomorphism *provided that on hom-sets it induces a Q-module morphism $\mathbf{A}(A,A') \to \mathbf{B}(F(A),F(A'))$.*

Definitions 5 and 6 give the (quasi)category Q-**Abrds** of Q-algebroids and Q-algebroid homomorphisms. The category **2**-**Abrds** is isomorphic to the (quasi)category **Qtlds** of quantaloids of [10]. There exists the (obvious) forgetful functor Q-**Abrds** $\xrightarrow{|-|}$ **CAT** with **CAT** being the (quasi)category of categories and functors.

Example 1. A quantale algebroid with one object is just a unital quantale algebra. Given a unital commutative quantale Q, the categories Q-**Mod** and Q-**Alg** are Q-algebroids.

From now on we do not distinguish between (quasi)categories and categories.

3 Topological preliminaries

This section introduces the category $\mathbf{S}(\mathbf{Q}\text{-}\mathbf{Top})$ of stratified $\mathbf{Q}$-topological spaces for a particular Q-algebroid $\mathbf{Q}$. Start by fixing a unital commutative quantale Q. Since every unital commutative quantale is an algebra over itself (with action given by multiplication), Example 1 provides a Q-algebroid $\mathbf{Q}$. Moreover, every category $\mathbf{X}$ provides the category $|\mathbf{Q}|^{\mathbf{X}}$ of all functors $\mathbf{X} \xrightarrow{F} \mathbf{Q}$ which is a Q-algebroid as well.

Definition 7. *Given a category* $\mathbf{X}$*,* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$ *is a* (quasi, –, stratified) $\mathbf{Q}$-topology *on* $\mathbf{X}$ *provided that* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$ *is a (subcategory, subquantaloid, sub(Q-algebroid)) of* $|\mathbf{Q}|^{\mathbf{X}}$*. The pair* $(\mathbf{X}, \mathcal{O}_{\mathbf{Q}}(\mathbf{X}))$ *is a* (quasi, –, stratified) $\mathbf{Q}$-topological space *provided that* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$ *is a (quasi, –, stratified)* $\mathbf{Q}$*-topology on* $\mathbf{X}$*.*

For shortness sake we will denote a topological space $(\mathbf{X}, \mathcal{O}_{\mathbf{Q}}(\mathbf{X}))$ by $\mathbf{X}$ alone. The following lemma motivates the term "stratified".

Lemma 1. *A* $\mathbf{Q}$*-topological space* $\mathbf{X}$ *is stratified iff for every* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$*-object* F *and every* $q \in Q$*,* $F \xrightarrow{\alpha^q} F$ *with* $\alpha^q_X = q$ *is a* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$*-morphism.*

Proof. $\alpha^q = q * \alpha^1 = q * 1_F$ and $q * \beta = \alpha^q \circ \beta$ for $q \in Q$ and β in $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$. □

Definition 8. *A functor* $\mathbf{X} \xrightarrow{H} \mathbf{Y}$ *between (quasi, –, stratified)* $\mathbf{Q}$*-topological spaces is* $\mathbf{Q}$-continuous *provided that* $FH \xrightarrow{\alpha H} F'H$ *is a* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$*-morphism for every* $\mathcal{O}_{\mathbf{Q}}(\mathbf{Y})$*-morphism* $F \xrightarrow{\alpha} F'$*.*

Definitions 7 and 8 provide the categories $\mathbf{QU}(\mathbf{Q}\text{-}\mathbf{Top})$, $\mathbf{Q}\text{-}\mathbf{Top}$ and $\mathbf{S}(\mathbf{Q}\text{-}\mathbf{Top})$ of (quasi, –, stratified) $\mathbf{Q}$-topological spaces and $\mathbf{Q}$-continuous functors. The objects of $\mathbf{S}(\mathbf{Q}\text{-}\mathbf{Top})$ will be referred to as $\mathbf{Q}$-spaces. Notice that all notions generalize the existing ones used in the fuzzy community (see, e.g., [3, 8]).

4 Q-algebroids versus Q-topological spaces

This section generalizes for the category Q-$\mathbf{Abrds}$ the aforesaid adjunction of [4, 6]. Start by fixing a unital commutative quantale Q. According to the notation used by many authors, the dual of the category Q-$\mathbf{Abrds}$ is denoted by $\mathbf{Lo}(Q\text{-}\mathbf{Abrds})$ (the "$\mathbf{Lo}$" stands for *localic*). If H is a $\mathbf{Lo}(Q\text{-}\mathbf{Abrds})$-morphism, the respective Q-$\mathbf{Abrds}$-morphism is denoted by H^*.

There exists a functor $\mathbf{S}(\mathbf{Q}\text{-}\mathbf{Top}) \xrightarrow{\mathcal{O}_{\mathbf{Q}}} \mathbf{Lo}(Q\text{-}\mathbf{Abrds})$ defined by: $\mathcal{O}_{\mathbf{Q}}(\mathbf{X} \xrightarrow{H} \mathbf{Y}) = \mathcal{O}_{\mathbf{Q}}(\mathbf{Y}) \xrightarrow{\mathcal{O}^*_{\mathbf{Q}}(H)} \mathcal{O}_{\mathbf{Q}}(\mathbf{X})$ with $\mathcal{O}^*_{\mathbf{Q}}(H)(F \xrightarrow{\alpha} F') = FH \xrightarrow{\alpha F} F'H$.

Theorem 1. *The functor* $\mathbf{S}(\mathbf{Q}\text{-}\mathbf{Top}) \xrightarrow{\mathcal{O}_{\mathbf{Q}}} \mathbf{Lo}(Q\text{-}\mathbf{Abrds})$ *has a right adjoint.*

Proof. Given a Q-algebroid $\mathbf{A}$, construct the universal arrow as follows. Let $\mathrm{Pt}_{\mathbf{Q}}(\mathbf{A})$ be the full subcategory of $|\mathbf{Q}|^{|\mathbf{A}|}$ with objects Q-$\mathbf{Abrds}(\mathbf{A}, \mathbf{Q})$. For every $\mathbf{A}$-object A define a Q-algebroid morphism $\mathrm{Pt}_{\mathbf{Q}}(\mathbf{A}) \xrightarrow{P_A} \mathbf{Q}$ by: $P_A(F \xrightarrow{\alpha} F') = F(A) \xrightarrow{\alpha_A} F'(A)$. For every $\mathbf{A}$-morphism $A \xrightarrow{f} A'$ define a natural transformation $P_A \xrightarrow{\Omega^f} P_{A'}$ by: $\Omega^f_F = F(f)$. Let $\mathcal{O}_{\mathbf{Q}}(\mathrm{Pt}_{\mathbf{Q}}(\mathbf{A}))$ be the category with objects all P_A and morphisms all Ω^f. The required universal arrow $\mathbf{A} \xrightarrow{\varepsilon^*_{\mathbf{A}}} \mathcal{O}_{\mathbf{Q}}(\mathrm{Pt}_{\mathbf{Q}}(\mathbf{A}))$ is then given by: $\varepsilon^*_{\mathbf{A}}(A \xrightarrow{f} A') = P_A \xrightarrow{\Omega^f} P_{A'}$. □

Theorem 1 gives the adjunction $(\eta, \varepsilon) : \mathcal{O}_{\mathbf{Q}} \dashv \mathrm{Pt}_{\mathbf{Q}} : \mathbf{Lo}(Q\text{-}\mathbf{Abrds}) \to \mathbf{S}(\mathbf{Q}\text{-}\mathbf{Top})$ where: given a Q-$\mathbf{Abrds}$-morphism $\mathbf{A} \xrightarrow{H^*} \mathbf{B}$, $\mathrm{Pt}_{\mathbf{Q}}(H)(F \xrightarrow{\alpha} F') = FH^* \xrightarrow{\alpha H^*} F'H^*$ and given a $\mathbf{Q}$-space $\mathbf{X}$, $\mathbf{X} \xrightarrow{\eta_{\mathbf{X}}} \mathrm{Pt}_{\mathbf{Q}}(\mathcal{O}_{\mathbf{Q}}(\mathbf{X}))$ is defined by: $\eta_{\mathbf{X}}(X)(F \xrightarrow{\alpha} F') = F(X) \xrightarrow{\alpha_X} F'(X)$ and $\eta_{\mathbf{X}}(f)_F = F(f)$.

5 Q-sobriety versus Q-spatiality

This section considers some consequences of the adjunction of Theorem 1.

Definition 9. *A* $\mathbf{Q}$*-space* $\mathbf{X}$ *is* $\mathbf{Q}$-T_0 ($\mathbf{Q}$-sober) *provided that* $\eta_{\mathbf{X}}$ *is a monomorphism (isomorphism) in* $\mathbf{CAT}$.

One can easily see that every $\mathbf{Q}$-sober $\mathbf{Q}$-space is $\mathbf{Q}$-T_0 but not vice versa.

Lemma 2. *A* $\mathbf{Q}$*-space* $\mathbf{X}$ *is* $\mathbf{Q}$-T_0 *(*$\mathbf{Q}$*-sober) iff for every* $\mathbf{X}$*-morphisms* f,g *with* $f \neq g$ *there exists a* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$*-object* F *such that* $F(f) \neq F(g)$ *(*$\eta_{\mathbf{X}}$ *is a* $\mathbf{Q}$*-homeomorphism). The* $\mathbf{Q}$*-space* $\mathrm{Pt}_{\mathbf{Q}}(\mathbf{A})$ *is* $\mathbf{Q}$*-sober for every Q-algebroid* $\mathbf{A}$.

The next definition is induced by the notion of spatial locale of [5].

Definition 10. *A Q-algebroid* $\mathbf{A}$ *is* $\mathbf{Q}$-spatial *provided that for every* $\mathbf{A}$*-morphisms* f,g *with* $f \neq g$ *there exists a* $\mathrm{Pt}_{\mathbf{Q}}(\mathbf{A})$*-object* F *such that* $Ff \neq Fg$.

Lemma 3. *A Q-algebroid* $\mathbf{A}$ *is* $\mathbf{Q}$*-spatial iff* $\varepsilon^*_{\mathbf{A}}$ *is an isomorphism. The Q-algebroid* $\mathcal{O}_{\mathbf{Q}}(\mathbf{X})$ *is* $\mathbf{Q}$*-spatial for every* $\mathbf{Q}$*-space* $\mathbf{X}$.

Let $\mathbf{Q}$-$\mathbf{Sob}$ be the full subcategory of $\mathbf{S}(\mathbf{Q}\text{-}\mathbf{Top})$ with objects all $\mathbf{Q}$-sober $\mathbf{Q}$-spaces and let $\mathbf{Q}$-$\mathbf{Spat}$ be the full subcategory of $\mathbf{Lo}(Q\text{-}\mathbf{Abrds})$ with objects all $\mathbf{Q}$-spatial Q-algebroids.

Theorem 2. *The restriction of the adjunction of Theorem 1 to the categories* $\mathbf{Q}$-$\mathbf{Sob}$ *and* $\mathbf{Q}$-$\mathbf{Spat}$ *gives an equivalence. The inclusion* $\mathbf{Q}\text{-}\mathbf{Sob} \xrightarrow{E} \mathbf{Q}\text{-}\mathbf{Top}$ *has a left adjoint* $\mathrm{Pt}_{\mathbf{Q}} \circ \mathcal{O}_{\mathbf{Q}}$*. The inclusion* $\mathbf{Q}\text{-}\mathbf{Spat} \xrightarrow{E} \mathbf{Lo}(Q\text{-}\mathbf{Abrds})$ *has a right adjoint* $\mathcal{O}_{\mathbf{Q}} \circ \mathrm{Pt}_{\mathbf{Q}}$.

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When does a category built on a lattice with a monoidal structure have a monoidal structure?

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Those of us working in Fuzzy sets regularly do mathematics with reasoning taking its truth values in a lattice. The simplest case is when that lattice has only two elements, giving classical logic. The next level of complexity happens when we let the lattice of truth values be [0,1] and consider logical operations resulting from connectives defined there. The connectives $\wedge$ and $\vee$ come from the lattice structure, but further logical connectives typically need further structure on the lattice. A DeMorgan negation gives a self adjoint contravariant functor from the lattice to itself. An implication can be gotten as a right adjoint $a \Rightarrow -$ to the functor $- \wedge a$. A t-norm gives a monoidal structure on the lattice of truth values. Generalizing to an arbitrary lattice with a monoidal structure leads to considerations of quantale valued logics, linear logics, and other variants (many of which have been studied by Hájek and others in the Prague school [5, 8, 7, 6],[2],[1]. One can go even further with the generalization and ask for truth values in a category with a monoidal (or perhaps monoidal closed) structure rather than just a lattice.

A number of different categories have been proposed which can be thought of as places for mathematics with truth values in one of these rich structures to live. Topoi provide one model, since they have truth values forming a Heyting algebra and have a rich higher order internal logic in which mathematics can be carried out. The Higgs construction of the topos of sheaves on a complete Heyting algebra ([9], [3]) has provided a pattern for several attempts to produce a notion of fuzziness in which equality as well as existence is fuzzy and the maps are strict extensional relations. See, for instance [13], [11], [10], [12]. Some of these constructions have used the monoidal structure on the lattice in an attempt to capture fuzzy logic (perhaps from a t-norm) rather than the intuitionistic logic which results from using $\wedge$ and $\Rightarrow$.

The Goguen category **Set**(L) of L valued fuzzy sets with (crisp) functions which do not reduce degree of membership has also been carefully studied. It is not a topos, but has a rich internal logic just the same. Its structure was described in [15], [17], [4], and [14] and generalizations to a non-commutative $\star$ were in [16].

Interpretation of predicate logic in any of these categorical settings gives rise to consideration of categories of predicates on an object and also to category objects internalizing them. One would hope that these would have the algebra structures which arise for the various logics studied.

This paper investigates when these various constructions lead to monoidal, symmetric monoidal, monoidal closed, and locally monoidal closed categories.

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Towards the development of the basic picture on sets evaluated over an overlap algebra

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Sambin in his forthcoming book [6] introduced the theory of "The Basic Picture", that generalizes both the notion of topological space and its point-free version. He also introduced the concept of overlap algebra to the aim of putting in algebraic form those properties that are needed to define such topological structures. The goal of our work is to generalize such topological notions in the context of many-valued sets. In particular, we want to test whether Sambin's original algebrization of his new topological notions can be considered also the algebrization of their many-valued version.

Since all his work was developed in a predicative constructive set theory [5], then it comes natural to place our project in the context of sets evaluated on a complete Heyting algebra H. The reason is that a complete Heyting algebra can be taken as an algebraic counterpart of the logic underlying the set theoretic foundation adopted by Sambin, which is intuitionistic logic. When trying to do this, we realized that the power collection $\mathcal{P}_H(X)$ of a H-valued set does not in general enjoy all the algebraic properties of $\mathcal{P}(X)$ that Sambin expressed via the notion of overlap algebra. In more precise terms, the power collection of a H-set is not in general an overlap algebra, if we consider an overlap relation as exactly the many-valued version of that defined in $\mathcal{P}(X)$. Since H itself, seen as $\mathcal{P}(\mathtt{1})$, does not come necessarily equipped with a structure of an overlap algebra, we then assumed that H is an overlap algebra as a starting point. In fact it turned out that if we take H to be a complete Heyting algebra equipped with a primitive notion of overlap, then $\mathcal{P}_H(X)$ inherits a structure of overlap algebra, where the overlap relation is the pointwise one induced by H. This led us to place our project in the context of sets evaluated on an overlap algebra.

Then, after defining the category $\mathbf{Rel}(H)$ of sets and relations evaluated on a overlap algebra H (H-sets and H-relations for short) by adopting the definitions in [1], we started to analyze the relationships between this category and the category **OA** of overlap algebras as defined in [6]. In particular, we considered the functor $\mathcal{P} : \mathbf{Rel}(H) \to \mathbf{OA}$ associating the power collection of all H-valued subsets to each H-set. We then investigated whether this is full and faithful as the analogous functor defined in [6] from the category **Rel** of sets and relations (in the foundation adopted by Sambin) to **OA**. It turned out that $\mathcal{P}$ is faithful but not generally full. We also characterized the morphisms under which $\mathcal{P}$ is full, by applying analogous results of fullness in [2, 4]. Such conditions seem to be better understood if we consider the representation of sets evaluated on a complete Heyting algebra in terms of sheaves as shown in [1, 7, 3]. Hence, before carrying on our general project, we want to investigate this and to characterize the role of the overlap relation in the context of sheaves.

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Fuzzy sets and geometric logic

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Abstract. Höhle has shown that fuzzy sets, valued in a frame (complete Heyting algebra) Ω, can be identified with certain sheaves over Ω: they are the subsheaves of constant sheaves, and more general sheaves can be got as quotients of the fuzzy sets.
The technical development is complicated by the fact that there are four different, but equivalent, technical expressions of the notion of sheaf: local homeomorphisms, presheaves with the sheaf pasting condition, and Ω-valued sets with or without a completeness condition. Fuzzy sets relate most naturally to the Ω-valued sets, and in fact they first lead one to incomplete Ω-valued sets. However for technical reasons Höhle describes the sheaf constructions known from topos theory in terms of complete Ω-valued sets.
In my talk I shall present the geometric fragment of those topos theoretic constructions, namely those constructions that are preserved by inverse image functors of geometric morphisms. I shall describe how (i) the geometric constructions correspond to a natural idea of sheaf as continuous set-valued map on a space, (ii) they enable pointwise reasoning even for non-spatial frames (without enough points) and (iii) they have simple descriptions in terms of incomplete Ω-valued sets.
My aim is to give a simpler picture of sheaves and how fuzzy sets lie within them.

1 Introduction

In his two papers [2] and [3], Ulrich Höhle has shown how fuzzy sets, valued in a frame, may be considered as particular kinds of sheaves over the corresponding locale. There are two particular insights that underly this treatment. The first is that, of the various equivalent different ways of expressing the notion of sheaf, the one most relevant to fuzzy sets is that of the "Ω-valued set". The second is that fuzzy sets are then seen as subsheaves of the "constant sheaves" that correspond to standard sets, and that the general sheaf can then be got as a quotient of a fuzzy set. From this point of view, fuzzy set theory is mathematically deficient in that it does not include quotienting, and when quotienting is added one obtains sheaf theory.

As is well known, the category of sheaves over a locale is a topos and so supports categorical operations corresponding to those of (intuitionistic) set theory. These include products (cartesian products), pullbacks (fibred products), coproducts (disjoint unions), coequalizers (quotients), exponentiation (function sets), the subobject classifier (set of logical truth values) and power objects (powersets). In [2] these are described concretely in terms of the Ω-valued set structure of the sheaves.

The purpose of my talk is to publicize a certain class of operations that are particularly well behaved. These are the *geometric* operations, and are known from topos theory as those operations that are preserved by the inverse image functors of geometric morphisms between toposes. Although they omit some of the topos-valid (intuitionistic) operations, they have an inherent continuity that makes it useful to restrict oneself to the geometric operations where possible. When sheaves are viewed as local homeomorphisms, the geometric operations have the important property that they can be calculated stalkwise. At first sight this approach would seem to be useful only when the locale is spatial (so that are enough points and hence enough stalks), but in fact it can be made sense of for general locales.

⋆ I am very grateful to Prof. Steve Rodabaugh, Prof. Larry Stout and the organizers of the 29th Linz Seminar on Fuzzy Set Theory (2008) for inviting me to present this material there.

The geometric constructions provide a key to treating locales as spaces of points, and their sheaves as continuous assignments of stalks to points.

A technical point that arises is in regard to the *completeness* of the Ω-valued sets. In general, different Ω-valued sets can present the same sheaf. However, any Ω-valued set can be completed to give a canonical representative. There is then an isomorphism (not just an equivalence) between the categories of sheaves and of complete Ω-valued sets. Höhle describes his constructions in terms of the complete Ω-valued sets, but the geometric operations can be described in particularly simple ways as constructions on the uncompleted Ω-valued sets and we shall describe examples of these.

2 Background

2.1 Locales

Standard references for frames and locales are [4] and [9]. For the topos-theoretic account of constructive locales see [7].

Definition 1. *A* frame *is a complete lattice in which binary meet distributes over arbitrary joins. A* frame homomorphism *is a function between frames that preserves finite meets and arbitrary joins. We write* **Fr** *for the category of frames and frame homomorphisms.*

Frames embody the idea of "point-free topology". A frame is intended to be a "lattice of opens", except that these opens are not specified as subsets of a given set of points. Points of frames are nonetheless defined, but for some frames there are not enough of them to distinguish between all the opens – frames need not be *spatial*. One might wonder therefore what virtue there is in the point-free approach to topology: not only does it obfuscate the topology by converting it to lattice theory, it does not even capture the established theory. However, it turns out that in constructive mathematics (for example, in the mathematics one can obtain by replacing sets by sheaves) it gives a theory that is better behaved than point-set topology, retaining classical theorems that otherwise are lost.

We shall use the language of *locales*. For present purposes, we may think of a locale as "a frame pretending to be a topological space" and define the category **Loc** of locales to be the opposite of the category of frames. That is to say, the objects are the same, but a morphism (a *continuous map*, or just *map*) $f : X \to Y$ between locales is a frame homomorphism (the *inverse image function*) in the opposite direction. We shall write ΩX for the frame corresponding to X, and $\Omega f : \Omega Y \to \Omega X$ for the frame homomorphism. The purpose of this duplication of notation is to allow us to use a language that supports spatial intutions in point-free topology.

A *point* of a locale X is a map $1 \to X$, where $\Omega 1$ is the frame of truth-values. A *generalized point at stage* W is a map $W \to X$. Then the ordinary points are often called *global* points.

Composition with a map $f : X \to Y$ transforms points of X to points of Y, just as with an ordinary continuous map. If one just considers global points here, then there is a problems, because a locale need not have"enough" points. That is to say, the action on global points does not in general define the locale map. This is clearest in those locales that are non-trivial, but have no global points at all.

However, composition with f also transforms points at any given stage W. Now, we do have enough points. In fact, consider the *generic* point, the identity map $\mathrm{Id} : X \to X$, which is a point at stage X. Transforming this immediately gives f, as a point of Y at stage X. An additional property of this point transformation is that it respects *change of stage*. Suppose we have $\alpha : W_1 \to W_2$. Then composition with α transforms points at stage W_2 to points at stage W_1. Associativity of composition, i.e. $f \circ (x \circ \alpha) = (f \circ x) \circ \alpha$, says that the point transformer f commutes with change of stage.

In general, suppose we have a point transformer that commutes with change of stage. More precisely, –

1. For each stage W, we have a function F_W that transforms points $x : W \to X$ of X at stage W to points $F_W(x) : W \to Y$ of Y at stage W.
2. If $\alpha : W_1 \to W_2$, then $F_{W_2}(x) \circ \alpha = F_{W_1}(x \circ \alpha)$.

Let $f = F_X(\mathrm{Id}) : X \to Y$. Then if $x : W \to X$ we have

$$F_W(x) = F_W(\mathrm{Id} \circ x) = F_W(\mathrm{Id}) \circ x = f \circ x.$$

From this it follows that morphisms are equivalent to "generalized point transformers that commute with change of stage". This is a completely general argument that applies in any category. We shall see later for locales how to use logical techniques to guarantee commuting with change of stage.

2.2 Sheaves

Fundamental though sheaves are, a big conceptual difficulty is that they have a variety of different but equivalent technical expressions. The two best known definitions are as *local homeomorphisms* and as *pasting presheaves*. In addition, Höhle's connection with fuzzy sets is via a lesser known, and quite different, notion, that of Ω-*valued set*; and this comes in two different flavours, complete and incomplete.

I'll summarize the four kinds here, partly to show how bad the problem is. The four really are different, and choosing one rather than another can make a big difference to ease of calculation – but different calculations can require different choice of sheaf style. A key message of my talk is to illustrate a more fundamental conceptual base: a sheaf over a space X is a *continuous set-valued map* $X \to \mathbf{Set}$. The problem with this, unfortunately, is we cannot make it precise by topologizing the class of sets in the ordinary way. (In fact, it is an example of the *generalized topological spaces*, i.e. *toposes*, of Grothendieck. Topos theory says it has opens, but only three: the empty space, the entire space, and the space of non-empty sets. The usual definition of continuity, applied with this topology, will not give us the notion of sheaf. The space of sets is one for which there are too few opens to define the topology, so a more elaborate description is needed, using sheaves. A sheaf over **Set** will be a functor F from $\mathbf{Set}_f$, the category of finite sets, to **Set**. This can then be extended to a functor defined for every set S, by taking a colimit of sets $F(S_0)$ where S_0 is a finite family of elements of S. This extended functor is the "continuous set-valued map from **Set** to **Set**." The topos described here as "the space of sets" is normally called the *object classifier*.)

The various definitions of sheaf are technical expressions of this idea of "continuous set-valued map". This comes out most clearly in the first definition, that a sheaf over X is a *local homeomorphism* $p : Y \to X$, i.e. a (continuous) map such that each y in Y has an open neighbourhood V for which the image $p(V)$ is open and p homeomorphically maps V to $p(V)$. Y is the *display space* for the sheaf. The subspace topology on each *stalk* $p^{-1}(x)$ is discrete, and the assignment of stalks to points of X is the "continuous set-valued map" on X. Intuitively, the definition says that if x is varied slightly, then the stalk $p^{-1}(x)$ does not make discontinuous changes. We also need to consider the sheaf morphisms. If $p : Y \to X$ and $q : Z \to X$ are two local homomorphisms, then a morphism between them is a map $f : Y \to Z$ such that $q \circ f = p$. For each x, f will map $p^{-1}(x)$ into $q^{-1}(x)$.

The second technical expression is via the notion of presheaf. If p is a local homeomorphism, then for each open U in X one can consider the set $\mathrm{Sect}_p(U)$ of *local sections* of p over U, maps $\sigma : U \to Y$ such that $p \circ \sigma$ is the identity on U, so for x in U, σ continuously selects $\sigma(x) \in p^{-1}(x)$. If $U \subseteq U'$ then there is a restriction map from $\mathrm{Sect}_p(U')$ to $\mathrm{Sect}_p(U)$, and this makes Sect_p a contravariant functor (presheaf) from the topology ΩX to the category of sets. Moreover, it has the following "sheaf pasting" condition. Suppose U_i ($i \in I$) is a family of opens in X, and suppose we have a family of sections

$\sigma_i \in \mathrm{Sect}_p(U_i)$ such that for each pair (i,j), σ_i and σ_j have the same restriction to $\mathrm{Sect}_p(U_i \cap U_j)$. There is a unique $\sigma \in \mathrm{Sect}_p(\bigcup_i U_i)$ that restricts to every σ_i.

Presheaves $F : (\Omega X)^{op} \to \mathbf{Set}$ satisfying the sheaf pasting condition are equivalent to local homeomorphisms. A sheaf morphism from F to G is a natural transformation – that is to say, for each $U \in \Omega X$ a function $\theta_U : F(U) \to G(U)$ such that if $V \subseteq U$ and $a \in F(U)$ then $\theta_V(a|V) = \theta_U(a)|V$. (The symbol $|$ denotes restriction, arising from the functoriality of presheaves. For instance, $a|V$ is $F(V \subseteq U)(a)$.)

The definition of sheaf over X that was exploited by Höhle is that of *ΩX-valued set*. (To speak of these in generality, without specifying a particular X, we refer to "Ω-valued sets".) Let A be a set. An *ΩX-valuation* on A is a function $E : A \times A \to \Omega X$ satisfying $E(a,b) = E(b,a)$ and $E(a,b) \wedge E(b,c) \leq E(a,c)$. This is the simplest of all the definitions of sheaf, but the definition of morphism is more complicated. If A and B are two ΩX-valued sets then a morphism from A to B is a function $\theta : A \times B \to \Omega X$ such that

$$\begin{aligned} \theta(a,b) &\leq E(a,a) \wedge E(b,b) \\ E(a',a) \wedge \theta(a,b) \wedge E(b,b') &\leq \theta(a',b') \\ \theta(a,b) \wedge \theta(a,b') &\leq E(b,b') \\ E(a,a) &\leq \bigvee_{b \in B} \theta(a,b) \end{aligned}$$

Finally, for an ΩX-valued set A we say that a function $s : A \to \Omega X$ is a *singleton* if it satisfies

$$\begin{aligned} s(a) &\leq E(a,a) \ (s \text{ is } \textit{strict}) \\ s(a) \wedge E(a,b) &\leq s(b) \ (s \text{ is } \textit{extensional}) \\ s(a) \wedge s(b) &\leq E(a,b). \end{aligned}$$

(Actually, the third condition implies the first. But we separate them out in order to make explicit the properties of strictness and extensionality.) Then A is *complete* if for every singleton s there is a unique $a \in A$ such that for all b we have $s(b) = E(a,b)$. For complete ΩX-valued sets, morphisms can be defined more simply: this is because a morphism $\theta : A \to B$ defines, for each $a \in A$, a singleton $\theta(a,-)$ and hence an element of B. In fact, the morphisms are equivalent to *functions* $\psi : A \to B$ such that $E(a,a') \leq E(\psi(a),\psi(a'))$ and $E(a,a) = E(\psi(a),\psi(a))$.

With four different notions of sheaf, there is a complex web of equivalences between them. We have already seen how, from the local homeomorphism, one gets the pasting presheaf of local sections. The equivalence between pasting presheaves F and complete ΩX-valued sets A is relatively straightforward: A is the disjoint union of the sets $F(U)$. If $a_i \in F(U_i)$ ($i = 1,2$), then

$$E(a_1,a_2) = \bigvee \{V \in \Omega X \mid a_1|V = a_2|V\}.$$

Every ΩX-valued set can be completed by taking the set of all singletons, and this respects the morphisms. There are some advantages in completing. The morphisms are simpler, and in addition it gives a canonical representation of the sheaf: two complete ΩX-valued sets are isomorphic as sheaves iff they are structurally isomorphic as ΩX-sets, whereas incomplete ΩX-valued sets can be structurally quite different but still give isomorphic sheaves. However, the completion process itself is non-trivial.

One connection we shall particularly focus on is the transformation from an ΩX-valued set A to a local homeomorphism. If x is a point of X, then we can define a partial equivalence relation (symmetric and transitive, but not necessarily reflexive) $\sim_x$ on A by $a \sim_x b$ if x in $E(a,b)$, and so we get a set $A/\sim_x$. Taking these as stalks, their disjoint union can be topologized so as to make the projection map a local homeomorphism. $E(a,a)$ describes the region on which a is defined, and $E(a,b)$ the region on which a and b are defined and equal. We shall describe this more carefully in Section 4. Note that each element $a \in A$ provides a local section $\tilde{a}$ over $E(a,a)$ of the corresponding sheaf: if x is in $E(a,a)$ then $\tilde{a}(x)$ is the $\sim_x$-equivalence class $[a]$ of a. The sets $\{\tilde{a}(x) \mid x \text{ in } E(a,a)\}$ $(a \in A)$ form a base of opens for the display space.

2.3 Sheaves over locales

Now let X be a locale. Clearly the presheaf and ΩX-valued set definitions transfer directly, since they are expressed in terms of the frame ΩX and do not mention points. Less clearly, the local homeomorphism definition also transfers: a locale map p can be defined to be a local homeomorphism if the unique map $!: X \to 1$ and the diagonal map $\Delta: X \to X \times X$ are both open ([7]; see also [10]). This is again equivalent to the pasting presheaf notion, using the same idea of defining the set of sections $\mathrm{Sect}_p(U)$ for each $U \in \Omega X$, but now using locale maps $\sigma: U \to Y$.

If $p: Y \to X$ is a local homeomorphism (between locales) and $x: 1 \to X$ is a global point of X, then we can construct the stalk $p^{-1}(x)$ using a pullback (or fibred product)

$$\begin{array}{ccc} p^{-1}(x) & \longrightarrow & Y \\ x^*p \downarrow & & \downarrow p \\ 1 & \underset{x}{\longrightarrow} & X \end{array}$$

The universal characterization of pullback is equivalent to defining the generalized points of $p^{-1}(x)$ at any stage W: they are the points y of Y such that $p(y) = x$ (where $p(y)$ is by definition the composite $p \circ y$). It can be proved that $p^{-1}(y)$ is a discrete locale, i.e. one whose frame is a powerset, so the locale corresponds to a set. Thus again the stalks provide an assignment of sets to points of X.

Since X might not have enough global points, it by now seems less plausible that a sheaf can be sensibly viewed as a continuous set-valued map; nonetheless this works. The above argument for a global point x also works for generalized points, due to the fact that pullback preserves local homeomorphisms. Hence if we think of "sets at stage W" as the sheaves over W, each point $x: W \to X$ gives us by pullback a "generalized stalk" x^*p.

2.4 Direct and inverse image functors

We shall write $\mathcal{S}X$ for the category of sheaves over X. This is ambiguous, since we have four different definitions of sheaf, and we get four equivalent but non-isomorphic categories. Nonetheless, we shall work with the ambiguous notation, leaving it to be interpreted according to one's current favourite definition.

If $f: X \to Y$ is a map, then we get from it *two* functors between $\mathcal{S}X$ and $\mathcal{S}Y$, forming an adjoint pair. The left adjoint is $f^*: \mathcal{S}Y \to \mathcal{S}X$ is the *inverse image functor*, and the right adjoint $f_*: \mathcal{S}X \to \mathcal{S}Y$ is the *direct image functor*. An important property of f^* is that it preserves not only colimits (as does any left adjoint) but also finite limits. (Such an adjoint pair is a *geometric morphism* from X to Y, and these are in fact equivalent to the locale maps.) A key part of our discussion here is of the "geometric" constructions, those that are preserved by every f^*.

An interesting fact is that the ease of constructing f^* and f_* depends on what definition of sheaf one is using.

For pasting presheaves, f_* is easy. If $F : (\Omega X)^{op} \to \mathbf{Set}$ is a pasting presheaf, then $f_*(F)$ is got by composing with Ωf. Explicitly, $f_*(F)(\beta) = F(\Omega f(\beta))$. Because complete Ω-valued sets A are closely related to the pasting presheaves, f_* can also be easily calculated for them as a pullback along Ωf.

$$f_*(A) = \{(a,\beta) \in A \times \Omega Y \mid E(a,a) = \Omega f(\beta)\}$$
$$E((a_1,\beta_1),(a_2,\beta_2)) = \bigvee\{\beta \leq \beta_1 \wedge \beta_2 \mid \Omega f(\beta) \leq E(a_1,a_2)\}.$$

On the other hand, f^* is harder for these, as it involves a completion step (or "sheafification" for the presheaves).

For local homeomorphisms, as we saw in Section 2.3, f^* is easy, constructed as a pullback

$$\begin{array}{ccc} f^*(Z) & \longrightarrow & Z \\ f^*p \downarrow & & \downarrow p \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

Finally, f^* is easy for Ω-valued sets (non necessarily complete). If (B,E_X) is an ΩY-valued set, then $f^*(B)$ is B again, with ΩX-valuation $E_X(b_1,b_2) = \Omega f(E_Y(b_1,b_2))$. Note how this matches the stalks. The stalk of (B,E_X) over x is isomorphic to the stalk of (B,E_Y) over $f(x)$, and this shows the pullback construction of the local homeomorphism.

We shll be particularly interested in how the geometric constructions, the ones preserved by f^*, are done for Ω-valued sets.

3 Geometric logic

Geometric logic is a positive logic matched to topological structure. For example, the logical connectives in its propositional fragment are finite conjunction and arbitrary disjunction, matching the finite intersections and arbitrary unions with which one can combine open sets. It goes along with a certain notion of "geometric construction", and our claim here is to show that this is the essence of continuity. For locales at least, a continuous map $X \to Y$ is a geometric construction of points of Y out of points of X. This is surprising for two reasons. First, no continuity proof is needed. The geometricity constraints mean that geometric constructions are intrinsically continuous. In a sense, obeying those constraints means foregoing the ability to construct discontinuous functions. Second, it applies even to locales, where there might not be enough points. We then generalize this to the situation where we might be constructing not points of a locale but more elaborate set-based structures, and this will include the notion of sheaf as continuous set-valued map.

Compared with ordinary classical logic, geometric logic has an added layer of structure. An axiom in a geometric theory is not simply a set of sentences (formulae with no free variables), as in classical logic, but a set of *sequents.*

3.1 Propositional geometric logic

Let Σ be a propositional signature, i.e. a set of propositional symbols. A geometric *formula* is built out of them using finitary conjunction ($\wedge$) and arbitrary disjunction ($\bigvee$). We shall not go into the logical rules here, but there are enough to ensure that each formula is equivalent to one expressed as a disjunction of finite conjunctions of propositional symbols. A geometric *sequent* is of the form $\phi \to \psi$, where ϕ and ψ are geometric formulae. A geometric theory over Σ is a set of sequents.

In fact a geometric theory (Σ, T) is structurally the same as a presentation of a frame using generators (the propositional symbols in the signature) and relations (the sequents in theory). A model of the theory (Σ, T) is exactly the same as a point of the corresponding locale, which we shall write $[\Sigma, T]$. Hence one can think of a locale as "the space of models for a propositional theory". The geometric formulae give the opens of the locale.

This can be extended. The usual logical notion of model requires each propositional symbol to be interpreted as a truth value, so that interpreting them all is the same as designating a subset of the symbols (those that are interpreted as **true**). But this still makes sense if the symbols are interpreted in any other frame ΩW, and then the models of (Σ, T) in ΩW are the same as frame homomorphisms $\Omega[\Sigma, T] \to W$, i.e. locale maps $W \to [\Sigma, T]$, i.e. generalized points of $[\Sigma, T]$ at stage W. Hence we can say in generality that the points of the locale $[\Sigma, T]$ are the models of (Σ, T).

Now consider maps $[\Sigma_1, T_1] \to [\Sigma_2, T_2]$. These are models of (Σ_2, T_2) in $\Omega[\Sigma_1, T_1]$. But all the ingredients of such a model are made geometrically (i.e. using $\wedge$ and $\bigvee$) from the symbols of Σ_1 and one deduces that maps are equivalent to geometrically defined transformations of models of (Σ_1, T_1) into models of (Σ_2, T_2). To define a map of locales $f : X \to Y$ one says "let x be a point of X" (technically this is then going to be the "generic" point of X in ΩX, corresponding to the identity map $X \to X$) and then, geometrically, defines a point $f(x)$ of Y. Maps can be defined pointwise, even if X does not have enough global points, and no continuity proof is needed!

Conceptually, therefore, –

- a locale is the "space of models" of a propositional geometric theory, and
- a map is a geometric transformation of models.

To summarize it as a slogan, *continuity is geometricity*.

However, there is a subtle question here: Why geometricity? A frame is a complete Heyting algebra, and has non-geometric structure such as the Heyting arrow and negation. Suppose we say "let x be a point of X", and then, *non*-geometrically, define a point $f(x)$ of Y. We can apply this to the generic point, the identity map on X, and thus get a point of Y at stage X, in other words a map $X \to Y$. However, in terms of the discussion in Section 2.1, the non-geometricity means this does not commute with change of stage. This is because change of stage for α is achieved by applying the frame homomorphism $\Omega\alpha$, and non-geometric operations are not preserved by frame homomorphisms.

Example 1. Let $\mathbb{S}$ be the *Sierpinski* locale, given by the geometric theory with one generator P and no relations. It has (in classical mathematics) two points, got by interpreting P as either **true** or **false**. It has three opens, 0 (bottom), P and 1 (top). A point of $\mathbb{S}$ at stage W, in other words a model of the theory in ΩW, is just an open of W. Now consider the point transformer F_W got by applying Heyting negation $\neg$ in ΩW. The generic point, as element of $\Omega\mathbb{S}$, is P, and in $\Omega\mathbb{S}$ we have $\neg P = 0$. The corresponding frame homomorphism takes 1 to 1, and P and 0 both to 0. When we use composition with this to give a point transformer, we find it always takes any open U of W to 0, and not to $\neg U$ as intended. It just happens that for the generic point P, we have $\neg P$ and 0 are equal.

3.2 Predicate geometric logic

There is also *predicate* geometric logic. For this, we allow the signature Σ to include sorts, and function symbols and predicates together with their arities (including the sorts of the arguments and results). Then terms can be built from sorted variables and the function symbols in the usual way, and geometric formulae are built from terms and predicate symbols using not only $\wedge$ and $\bigvee$, but also equality $=$ and existential quantification $\exists$. Then a geometric sequent is of the form $(\forall xyz \cdots)(\phi \to \psi)$, where

"$xyz\cdots$" is a finite list of sorted variables, and ϕ and ψ are geometric formulae in which every free variable is in the list $xyz\cdots$. A geometric theory is again a set of sequents.

The infinitary disjunctions make this an unusual logic, with a natural type theory (sort constructions) associated with it. They give us the power within geometric theories to characterize certain sorts up to isomorphism. Suppose, for example, we want to characterize a sort N as the natural numbers. We can do this with a constant 0, a successor map s, and sequents

$$(\forall n)(s(n) = 0 \to \mathbf{false})$$
$$(\forall mn)(s(m) = s(n) \to m = n)$$
$$(\forall n)(\mathbf{true} \to \bigvee_{i\in\mathbb{N}} n = s^i(0))$$

(Here, the exponent i in $s^i(0)$ is not part of the logical syntax, but is meant to suggest an inductive definition of formulae ϕ_i in which ϕ_0 is the formula $n = 0$, ϕ_1 is $n = s(0)$, ϕ_2 is $n = s(s(0))$ and so on.)

Because of this ability, which is impossible in finitary logic, geometric logic embodies a "geometric type theory". The geometric type constructions include finite limits (products, pullbacks, equalizers, ...), arbitrary colimits (coproducts, quotients, ...) and also all free algebra constructions.

3.3 Geometric logic of sheaves

The usual interpretation of predicate logic in sets can be generalized to sheaves. Sorts are interpreted as sheaves, function symbols as sheaf morphisms from a sheaf product to another sheaf, and predicates as subsheaves of sheaf products. Once that is done, terms can be interpreted as sheaf morphisms and formulae as subsheaves. Specific categorical structure in the category of sheaves is needed for this; for instance, equalizers are needed in order to interpret $=$. In fact, all can be done using finite limits and arbitrary colimits. Moreover, particular properties of the way these limits and colimits interact with each other ensure that the rules of geometric logic are valid in the sheaf interpretations.

An important fact is that all this structure is preserved by inverse image functors α^*, and it follows that the α^*s also preserve the geometric type constructions.

Now recall the argument in Section 3.1 that "continuity is geometricity". It said that to define a locale map $f : X \to Y$ it sufficed to provide a geometric transformation of points of X into points of Y, even though (i) there may not be enough global points of X, and (ii) no explicit continuity proof is given. We now show how sheaves can be defined the same way: it suffices to provide a geometric transformation of points of X into sets. In other words, to describe a sheaf, it suffices to describe the stalks, as long as the description is geometric. The same also goes for sheaf morphisms. This, then, is the technical content of our claim that sheaves can be thought of as continuous set-valued maps.

Let X be a locale, and suppose $F(x)$ is a set, described geometrically in terms of its parameter x, a point of X. The category $\mathcal{S}X$ of sheaves over X includes the opens, for the opens can be identified with the subsheaves of the constant sheaf 1. Hence the generic point of X in ΩX can also be found in $\mathcal{S}X$. Applying F to it gives an on object of $\mathcal{S}X$, a sheaf S. For any generalized point $x : W \to X$, we know that $x^*(S)$ is got by pullback, and so is the stalk of S over x. But x^* preserves geometric constructions, so $x^*(S)$ is constructed from x in the same way as S is constructed from the generic point.

The same argument also applies to sheaf morphisms: to describe one, it suffices to give a geometric description of its action on the stalks.

4 The local homeomorphism of an ΩX-valued set

Let (A,E) be an ΩX-valued set. Höhle defines the frame $P(A,E)$ to have as its elements the strict, extensional maps. (Let us here use the notation $P(A,E)$ for the locale rather than the frame. It is the display locale for the sheaf.) He shows that if X is spatial then so is $P(A,E)$, and the display map $\mathrm{pt}(P(A,E)) \to \mathrm{pt}X$ is a local homeomorphism corresponding to the sheaf for (A,E). Actually, we can do better than that, for we can define the local homeomorphism even in the non-spatial case.

Lemma 1. *$\Omega P(A,E)$ can be presented by generators and relations as*

$$\begin{aligned} \mathrm{Fr}\langle \Omega X \textit{ (qua frame), } \tilde{a}\ (a \in A) \mid 1 &\leq \bigvee_{a\in A} \tilde{a} \\ \tilde{a} \wedge \tilde{b} &\leq E(a,b) \\ E(a,c) \wedge \tilde{c} &\leq \tilde{a}\rangle. \end{aligned}$$

Proof. Let us write F for the frame presented as stated. To define a homomorphism $\theta : F \to \Omega P(A,E)$ we must describe its action on the generators and show that it respects the relations (including that it preserves the frame structure on ΩX). It will map $\alpha \in \Omega X$ to the strict, extensional map $\theta(\alpha)(a) = \alpha \wedge E(a,a)$, and it will map the formal generator $\tilde{a}$ to the strict, extensional map that Höhle already calls $\tilde{a}$, defined by $\tilde{a}(b) = E(a,b)$. This respects the relations. Next we define $\phi : \Omega P(A,E) \to F$ by $\phi(f) = \bigvee_{a\in A} f(a) \wedge \tilde{a}$. This is easily seen to be a homomorphism, using the fact that the joins and binary meets on the strict, extensional maps are defined argumentwise. Finally, it remains to show that θ and ϕ are mutually inverse. The more interesting part is that $\phi \circ \theta$ is the identity on F, and it suffices to check its action on the generators. If $\alpha \in \Omega X$ then $\phi \circ \theta(\alpha) = \bigvee_{a\in A} \alpha \wedge E(a,a) \wedge \tilde{a}$. From the second relation (and putting $b = a$) we see $\tilde{a} \leq E(a,a)$, so it suffices to show $\alpha = \bigvee_{a\in A} \alpha \wedge \tilde{a}$, which follows from the first relation. For $\tilde{a}$ we have $\phi \circ \theta(\tilde{a}) = \bigvee_{b\in A} E(a,b) \wedge \tilde{b}$ and this equals $\tilde{a}$ by the third relation.

The advantage of this is that the generators and relations give us a direct description of the points that applies even to generalized points: a point is a function from the generators to Ω that respects the relations. On the generators α this gives us a point x of X. On the generators $\tilde{a}$ we find a subset $S \subseteq A$ such that (i) S is non-empty, (ii) if $a,b \in S$ then x satisfies $E(a,b)$, and (iii) if x satisfies $E(a,c)$ and $c \in S$ then $a \in S$: in other words, S is an equivalence class for $\sim_x$. Hence a point of $P(A,E)$ can be described geometrically as a pair (x,u) where x is a point of X and u is an element of its stalk.

(Actually, the use of the frame ΩX is non-geometric. However, this can be circumvented by using a presentation of it by generators and relations.)

This is the standard description that applies to any sheaf. The projection map $p : P(A,E) \to X$ is defined on points (x,u) by forgetting u. However, its inverse image function Ωp is also clear enough; it is the inclusion of generators α.

5 Geometric constructions on ΩX-valued sets

[2] describes a full range of topos-theoretic constructions on complete ΩX-valued sets. For some of them, completeness is a very convenient part of the construction. However, our contention here is that for the geometric constructions (which are performed stalkwise on the local homeomorphisms) there are simpler and more natural constructions that work directly on the uncompleted ΩX-valued sets and avoid the need to complete.

As the most fundamental construction, let us look at morphisms. Suppose $\theta : A \to B$ is a morphism of ΩX-valued sets. Given a point x, let us write $[a]$ for the equivalence class of a under $\sim_x$, assuming

x in $E(a,a)$ (i.e. $a \sim_x a$, which is required in order for a to have an equivalence class – remember that $\sim_x$ is only a *partial* equivalence relation). We can define $\theta_x : A/\sim_x \to B/\sim_x$, by $\theta_x([a]) = [b]$, where x in $\theta(a,b)$. To show that there is such a b, we have x in $E(a,a) \leq \bigvee_b \theta(a,b)$. If we have another candidate b' then x is in $\theta(a,b) \wedge \theta(a,b') \leq E(b,b')$ and so $b \sim_x b'$ and $[b] = [b']$. To show that $[b]$ is independent of choice of a, if we also have $a \sim_x a'$ then x is in $E(a',b) \wedge \theta(a,b) \leq \theta(a',b)$.

We give an illustration of the geometric reasoning.

Proposition 1. *Let $\theta : A \to B$ be a morphism of ΩX-valued sets.*

1. *θ is monic (all stalk functions θ_x are 1-1) iff $\theta(a,b) \wedge \theta(a',b) \leq E(a,a')$ for all a,a',b.*
2. *θ is epi (all stalk functions θ_x are onto) iff $E(b,b) \leq \bigvee_a \theta(a,b)$.*
3. *θ is an isomorphism iff both the above conditions hold.*

Proof. (1) Monicness is characterized geometrically, so θ is monic if we can prove geometrically that every stalk function is monic (1-1). Let x be a point of X. θ_x is monic iff $\theta_x([a]) = \theta_x([a']) \Rightarrow a \sim_x a'$, i.e. if x in $\theta(a,b) \wedge \theta(a',b)$ then x in $E(a,a')$. The result follows.

(2) is a similar argument, and (3) combines the first two.

Now recall the motivation for the link with fuzzy sets: fuzzy sets are *subsheaves of constant sheaves*, and to get more general sheaves one needs a notion of quotienting.

5.1 Constant sheaves

As a local homeomorphism over X, the constant sheaf $!^*(A)$ for a set A is just the projection map $X \times A \to X$. (Note that we are treating A as a sheaf over the one-point locale 1, and the notation $!^*(A)$ is referring to the unique map $! : X \to 1$.) The stalk at any point x is clearly A, so it is the *stalks* that are constant. The pasting presheaf is certainly not a constant functor.

This can be presented as an ΩX-valued set A, with $E(a,b) = \bigvee\{\mathbf{true} \mid a = b\}$. In other words, $E(a,b)$ is the top element **true** of the frame iff $a = b$. It is easy to calculate that $a \sim_x b$ iff $a = b$, so the stalk $A/\sim_x$ is A, as expected. Note that this argument is geometric, so we do not have to worry about whether there are enough points. We know that this ΩX-valued set presents the right sheaf.

5.2 Subsheaves

Suppose A is an ΩX-valued set. How can we describe its subsheaves A'? A partial equivalence relation can express not only quotients but also subsets, by restricting the elements that are self-equivalent, and in the context of a ΩX-valued set A this would be done by making each $E(a,a)$ possibly smaller. This suggests taking $A' = A$, but using a smaller ΩX-valuation E'. If $\phi : A \to \Omega X$ is a strict, extensional map then $E'(a,b) = E(a,b) \wedge \phi(a)$ is also an ΩX-valuation. (This is also clear from Section 4, since each strict, extensional map is equivalent to an open of the display locale.) The relation E' is also a monic morphism from (A',E') to (A,E).

Now suppose $\theta : B \to A$ is a morphism of ΩX-valued sets. We can define a strict, extensional map ϕ on A by $\phi(a) = \bigvee_b \theta(b,a)$ that defines the image of θ. If (A',E') is the induced subsheaf, then we find that θ factors through (A',E') as an epi followed by a monic. This is the image factorization of θ. If θ itself is monic, then the epi part is an isomorphism and one sees that the monic is equivalent to the subsheaf (A',E'). Thus every monic can be presented (up to isomorphism) using a strict, extensional map, and subsheaves are equivalent to strict, extensional maps.

Let us apply this now to subsheaves of constant sheaves. Let A be a set. Then the subsheaves of $!^*(A)$ are equivalent to strict, extensional maps on A. But these are just arbitrary functions $\phi : A \to \Omega X$, in other words fuzzy sets on A.

5.3 Quotients

A quotient of (A,E) is defined by an an ΩX-valuation E' that is smaller than E, but does not affect the definedness of any a: we have $E'(a,a) = E(a,a)$. Now consider an arbitrary (A,E). The function $\phi(a) = E(a,a)$ gives a fuzzy set on A and hence a subsheaf of the constant sheaf $!^*A$, and (A,E) is a quotient of it. Hence when one allows quotients as well as subsheaves one can move beyond fuzzy sets to arbitrary sheaves.

5.4 Other geometric constructions

We give some more examples to illustrate the techniques. Note that in each case the construction is close to what is familiar from set theory. This depends on the fact that we are happy to work with incomplete ΩX-valued sets. If we had to complete, the constructions would be made much more complicated.

Products: Let A and B be ΩX-valued sets. How can we define the product as ΩX-set? It would seem natural use the set product $A \times B$, with equality defined componentwise – $(a_1,b_1) \sim_x (a_2,b_2) = a_1 \sim_x a_2$ and $b_1 \sim_x b_2$. This gives us the definition of E on $A \times B$, namely

$$E((a_1,b_1),(a_2,b_2)) = E(a_1,a_2) \wedge E(b_1,b_2).$$

To check that that does indeed define the sheaf product, it suffices to check the stalks. This is because binary product is a geometric construction, so sheaf product exists and is calculated stalkwise. In other words, we must check $(A \times B)/\sim_x \cong A/\sim_x \times B/\sim_x$, which is clear. We also need the product projections $p : A \times B \to A$ and $q : A \times B \to B$. Clearly we want $p_x([a,b]) = [a']$ iff $a \sim_x a'$ (and $b \sim_x b$), which translates into

$$p((a,b),a') = E(a,a') \wedge E(b,b).$$

One can then check that this is a morphism and gives the correct stalk functions.

Equalizers: Let A and B be ΩX-valued sets and let $\theta, \phi : A \to B$ be morphisms. The equalizer is a subsheaf of A, given by the same set A and a strict extensional map ψ. The elements of the corresponding substalk are those $[a]$ such that $\theta_x([a]) = \phi_x([a])$, so we get

$$\psi(a) = \bigvee_b \theta(a,b) \wedge \phi(a,b).$$

Again, one must check that this gives the right stalks.

List sets: If A is a set, we write A^* for the set of finite lists of elements of A. How can we make the analogous construction for ΩX-valued sets A? It would seem natural to use the set A^*, with $(a_i)_{i=0}^{m-1} \sim_x (b_i)_{i=0}^{n-1}$ if $m = n$ and for each index i we have $a_i \sim_x b_i$. This translates into

$$E((a_i)_{i=0}^{m-1},(b_i)_{i=0}^{n-1}) = \bigvee \{ \bigwedge_{i=0}^{m-1} E(a_i,b_i) \mid m = n \}.$$

(Note that the expression on the right evaluates to 0 if $m \neq n$, since the set of disjuncts is then empty.) This gives the correct stalks. Associated structure, such as the concatenation operation that, with the emepty list as unit, makes the list set into a monoid, can also be checked stalkwise.

6 Conclusions

Their range of different technical expressions can make sheaves daunting to the newcomer. However, there is a simple unifying intuition: a sheaf over a locale X is a continuous set-valued function, the value at a point x being the stalk. We have described why continuity may be thought of as geometricity of the construction. Geometric constructions on sheaves, such as finite limits, arbitrary colimits and free algebra constructions, can be performed stalkwise. When carried out on ΩX-valued sets, they can often be formulated simply if one does not require the ΩX-valued sets to be complete, and can be verified by checking the actions on stalks.

7 Further reading

The recomended introduction to sheaves from a topos point of view is [8]. [1] is less deep, but shows well how logic and set theory translate into category theory. The ultimate comprehensive reference is [5], [6], but is not for the beginner. [10] develops in more detail the relationship between continuity and geometricity, and sheaves as continuous set-valued maps.

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Automorphisms of subalgebras of the algebra of truth values of type-2 fuzzy sets

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1 Introduction

Fuzzy sets with fuzzy subsets of $[0,1]$ as truth values were introduced by Zadeh [6]. This notion extends the notion of both ordinary fuzzy sets and interval-valued fuzzy sets. By now, there is a large literature, both theoretical and applied, on the topic. In [1], there is a treatment of the mathematical basics of type-2 fuzzy sets, that is, of the algebra, as defined by Zadeh, of these truth values. Let **M** denote this algebra of fuzzy truth values. In [2], a study was begun of automorphisms of **M**. This study was continued in [3], where the automorphism group of the algebra was explicitly determined in terms of the automorphism group of the unit interval with its usual lattice structure. This theorem has several corollaries concerning characteristic subalgebras of **M** and their automorphism groups.

The algebra **M** has many subalgebras, and their study is relevant because each subalgebra could serve as the basis of a fuzzy theory, where a fuzzy set in this theory is a mapping of a universal set into this subalgebra. The subalgebras considered are typically characteristic. That is, automorphisms of the algebra of truth values induce automorphisms of these subalgebras. Characteristic subalgebras are of special interest because they are "canonical". If an algebra is characteristic, then there is no subalgebra isomorphic to it sitting in the containing algebra in the same way. A number of subalgebras were proved characteristic in the papers [2–5]. But characteristic subalgebras can have automorphisms other than those induced by automorphisms of the containing algebra. This paper addresses that issue. A subalgebra of special interest is the subalgebra of convex normal functions. It is a De Morgan algebra, and in particular, a lattice. Though characteristic, it has automorphisms not induced by those of **M**, and those automorphisms are the principal focus of this paper.

2 The algebra of fuzzy truth values

The algebra of truth values for fuzzy sets of type-2 is the set of all mappings of $[0,1]$ into $[0,1]$ with operations certain convolutions of operations on $[0,1]$, as follows.

Definition 1. *On* $[0,1]^{[0,1]}$*, let*

$$(f \sqcup g)(x) = \bigvee_{y \vee z = x} (f(y) \wedge g(z))$$

$$(f \sqcap g)(x) = \bigvee_{y \wedge z = x} (f(y) \wedge g(z))$$

$$f^*(x) = \bigvee_{y' = x} f(y) = f(x')$$

$$\bar{1}(x) = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } x \neq 1 \end{cases}$$

$$\bar{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}$$

The algebra $\mathbf{M} = \left([0,1]^{[0,1]}, \sqcup, \sqcap, ^*, \bar{0}, \bar{1}\right)$ is the basic algebra of truth values for type-2 fuzzy sets, and is analogous to the algebra $([0,1], \vee, \wedge, ', 0, 1)$, which is basic for type-1 or ordinary fuzzy set theory.

Determining the properties of the algebra **M** is helped by introducing the following auxiliary operations.

Definition 2. *For $f \in \mathbf{M}$, let f^L and f^R be the elements of* $\mathbf{M}$ *defined by*

$$f^L(x) = \vee_{y \le x} f(y)$$
$$f^R(x) = \vee_{y \ge x} f(y)$$

The point of this definition is that the operations $\sqcup$ and $\sqcap$ in $\mathbf{M}$ can be expressed in terms of the pointwise max and min of functions, as follows.

Theorem 1. *The following hold for all $f, g \in \mathbf{M}$.*

$$\begin{aligned} f \sqcup g &= \left(f \wedge g^L\right) \vee \left(f^L \wedge g\right) \\ &= (f \vee g) \wedge \left(f^L \wedge g^L\right) \end{aligned}$$

$$\begin{aligned} f \sqcap g &= \left(f \wedge g^R\right) \vee \left(f^R \wedge g\right) \\ &= (f \vee g) \wedge \left(f^R \wedge g^R\right) \end{aligned}$$

Using these auxiliary operations, it is fairly routine to verify the following properties of the algebra $\mathbf{M}$. The details may be found in [1].

Corollary 1. *Let f, g, $h \in \mathbf{M}$. The basic properties of $\mathbb{M}$ follow.*

1. $f \sqcup f = f$; $f \sqcap f = f$
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
3. $\bar{1} \sqcap f = f$; $\bar{0} \sqcup f = f$
4. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
5. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
6. $f^{**} = f$
7. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$

3 Automorphisms

In our study of automorphisms, we limit ourselves initially to the algebra $\mathbb{M} = \left([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{0}, \bar{1}\right)$, that is, the algebra $\mathbf{M}$ without its negation *, and similarly for subalgebras of $\mathbf{M}$. This allows more automorphisms, avoids certain technicalities, and it turns out that the results can be specialized to $\mathbf{M}$. For an automorphism α of $\mathbb{I} = ([0,1], \vee, \wedge, 0, 1)$, α_L and α_R defined by $\alpha_L(f) = \alpha f$ and $\alpha_R(f) = f\alpha$ are automorphisms of $\mathbb{M}$. The principle result for $\mathbb{M}$ is that every automorphism is of the form $\alpha_L \beta_R$, and uniquely so [3]. Thus $Aut(\mathbb{M}) \approx Aut(\mathbb{I}) \times Aut(\mathbb{I})$. This has many corollaries. But a characteristic subalgebra of $\mathbb{M}$ may have automorphisms not induced by those of $\mathbb{M}$.

One characteristic subalgebra of special interest is the subalgebra of normal convex functions. And element f of $\mathbb{M}$ is normal if $\vee_{x \in [0,1]} f(x) = 1$, and is convex if whenever $x \le y \le z$, then $f(y \ge f(x) \wedge f(z)$. Equivalently, f is convex if $f = f^L \wedge f^R$. The normal functions form a characteristic subalgebra, and so do the convex functions. Their intersection is thus characteristic. It is a De Morgan algebra, and as a lattice, it is maximal among the subalgebras of $\mathbb{M}$ that are lattices. We denote this subalgebra by $\mathbb{L}$.

Theorem 2. *Let α, β, and γ be automorphisms of $([0,1], \vee, \wedge, 0, 1)$. Then Φ defined by*

$$\Phi(f) = \alpha\,(f\gamma)^L \wedge \beta\,(f\gamma)^R$$

is an automorphism of $\mathbb{L}$. Distinct triples (α, β, γ) of automorphisms of $([0,1], \vee, \wedge, 0, 1)$ yield distinct automorphisms of $\mathbb{L}$.

The upshot of this theorem is that $\mathbb{L}$ has many more automorphisms than those induced by automorphisms of $\mathbb{M}$. We do not know if $\mathbb{L}$ has automorphisms other than those specified in the theorem above. A basic tool in our investigation of automorphisms of subalgebras of $\mathbb{M}$ is the determination of the irreducible elements of a subalgebra. An element f is *join irreducible* if $f = g \sqcup h$ implies $f = g$ or $f = h$. *Meet irreducible* is defined similarly, and an element is *irreducible* if it is both join and meet irreducible. Irreducible elements are carried to irreducible elements by automorphisms. We have determined the irreducible elements of $\mathbb{L}$ [2]. One irreducible element of $\mathbb{L}$ is the constant function 1. It would be important to know whether or not this irreducible element is carried to itself by automorphisms of $\mathbb{L}$, but we have not been able to determine this. We do conjecture that this is so, and that the group of automorphisms of $\mathbb{L}$ consists of those automorphisms specified in the theorem above.

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A complete characterization of all weakly additive measures and of all valuations on the canonical extension of any finite chain equipped with its unique MV-algebra structure

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Let $\mathbb{L}$ be an MV-algebra, where Chang's [1] original operations $\cdot$, $+$ and $^-$, resp. will be interpreted as and denoted by "intersection" $\sqcap$, "union" $\sqcup$ and "complementation" $'$, resp. Let m be an uncertainty measure on $\mathbb{L}$ in the sense of [6] as an isotonic mapping from $\mathbb{L}$ to the real interval $[0,1]$ with boundary conditions $m(0) = 0$, $m(1) = 1$ and compatibility $m(a') = 1 - m(a)$ with respect to complementation, where the bottom and top elements in $\mathbb{L}$ and in $[0,1]$ will be denoted by the same symbols. Then the additivity of m has the clear and well known meaning as

$$m(a\dot{\sqcup}b) = m(a) + m(b) \quad \text{for all "disjoint unions"},$$

denoted as $\dot{\sqcup}$ and given by $a \sqcap b = 0$. It is clear that additive measure are also valuations on the underlying lattice $\mathbb{L}$.

But if $\mathbb{L}$ has a structure poorer than an MV-algebra, it turns out to be a non-trivial problem to find a reasonable notion of additivity for uncertainty measures m on $\mathbb{L}$. In our joint papers [2] resp. [3] with U. Höhle, we proposed for quantum De Morgan algebras resp. Girard algebras $\mathbb{L}$ to define additivity of m by

$$(SA) \quad m(a\dot{\sqcup}b) = m(a) + m(b) \quad \text{only for "divisible unions"},$$

i. e. only for those disjoint unions which are divisible in the sense of

$$a' \sqcap (a \sqcup b) = b \quad \text{and} \quad b' \sqcap (a \sqcup b) = a.$$

Particularly, the structure of Girard algebras seems to be the natural one in the problem of conditioning, because in [3] we have proved that any Girard algebra $\mathbb{L}$ has a unique "canonical Girard algebra extension" $\tilde{\mathbb{L}} := \{(a,b) \in \mathbb{L} \times \mathbb{L} : a \leq b\}$. Furthermore, we proved that the canonical extension $\tilde{\mathbb{L}}$ of a Boolean algebra $\mathbb{L}$ is an MV-algebra and any additive measure m on $\mathbb{L}$ has a unique extension to an additive measure $\tilde{m}$ on $\tilde{\mathbb{L}}$, given by

$$\tilde{m}(a,b) = \frac{m(a) + m(b)}{2}.$$

For a not Boolean MV-algebra $\mathbb{L}$, the canonical extension $\tilde{\mathbb{L}}$ is not an MV- but only a Girard algebra.

My talk during the Linz Seminar 2007 was dedicated to discuss in this situation the problem of extending the additivity of m on $\mathbb{L}$ to $\tilde{m}$ on $\tilde{\mathbb{L}}$. There we presented examples where a unique extension $\tilde{m}$ exists which is additive in the sense of (SA), but also examples where such additive extensions do not exist. Therefore, we called (SA) "strong additivity", and we proposed as a second reasonable notion the "weak additivity" of an uncertainty measure m on a Girard algebra $\mathbb{L}$, which is not MV-algebra, by

$$(WA) \quad m(a \dot{\sqcup} b) = m(a) + m(b) \quad \text{only on all sub-MV-algebras of } \mathbb{L}.$$

In this talk, we start with a finite chain $\mathbb{L} = \mathbb{C}_n$ with exactly $n-1$ different non-trivial elements, say $0 < a_1 < \ldots < a_{n-1} < 1$. In the extended paper [6] of my above mentioned talk Linz 2007, it was proved that $\mathbb{L}$ has a unique MV-algebra structure given by

$$\begin{aligned} & a_j \sqcap a_k = 0,\ a_j \dot{\sqcup} a_k = a_{j+k} \quad \text{for} \quad j+k \leq n, \\ & a_k' = a_{n-k} \quad \text{for} \quad 0 \leq k \leq n, \quad \text{including} \quad a_0 = 0, a_1 = 1, \\ & a_j \sqcap a_k = a_{j+k-n} \quad \text{for} \quad j+k > n. \end{aligned}$$

Therefore, $\mathbb{C}_n$ is an MV-algebra "generated by" a_1, written for short as $\mathbb{C}_n = \mathbb{M}(a_1)$, in the sense that any element $a_k \neq 0$ of $\mathbb{C}_n$ is the disjoint union of k terms of a_1. But the main result in [6] for $\mathbb{L} = \mathbb{C}_n$ was that for $n \geq 4$ there does not exist a strongly additive measure $\tilde{m}$ on $\tilde{\mathbb{C}}_n$. This was the motivation to look for (all) weakly additive measures $\tilde{m}$, which requires to look for all sub-MV-algebras of $\tilde{\mathbb{C}}_n$. Now, we can give a complete solution to this problem, which will be outlined in the following.

For any $0 \leq t \leq n$, $\mathbb{M}_t := \{(a_{k-t}, a_k) : t \leq k \leq n\} \cup \{(0,0),(1,1)\}$ is a sub-chain of $\tilde{\mathbb{C}}_n$. Clearly, $\mathbb{M}_0 = \mathbb{M}(a_1, a_1)$ can be identified with $\mathbb{C}_n$. Furthermore, $\mathbb{M}_1 = \mathbb{M}(0, a_1)$ has n non-trivial elements and $\mathbb{M}_n = \mathbb{M}(0,1)$ has 1 non-trivial element. For $n \geq 3$, it will be proved that any other sub-MV-algebra of $\tilde{\mathbb{C}}_n$ is a proper subset of some $\mathbb{M}_t$ and, therefore, is generated by an element $(a_{x-t}, a_x) \in \mathbb{M}_t$ for some $x \geq t$. Furthermore, it will be given a procedure with several steps which permits to decide whether or not a given $(a_j, a_k) \in \tilde{\mathbb{C}}_n$ with $t := k - j \geq 2$ and $k \leq \frac{n}{2}$ is contained in some sub-MV-algebra of $\mathbb{M}_t$. In a first step, $L \geq 1$ and $0 \leq l < k$ can be uniquely determined by $\frac{n-j}{k} = L + \frac{l}{k}$. For $l = 0$ the answer is positive, for $l = 1$ negative. For any other $l \geq 2$ the answer is:

(a_j, a_k) is contained in some sub-MV-algebra of $\tilde{\mathbb{C}}_n$ if and only if
$t \leq l$ and there exists x with $t \leq x \leq l$ such that $k = Q \cdot x,\ l = R \cdot x$ with some R, Q.

Trivially, for any $k \leq \frac{n}{2}$, $\mathbb{M}(a_k, a_k')$ is a sub-MV-algebra with 1 non-trivial element. Moreover, it will be proved that for all $1 \leq j+1 \leq k < \frac{n}{2}$, none of the elements (a_j, a_k') are in some sub-MV-algebra.

As a simple consequence of the above briefly outlined analysis we obtain the characterization of all weakly additive measures $\tilde{m}$ on $\tilde{\mathbb{C}}_n$ by the values

$$\tilde{m}(a_j, a_k) = \frac{k}{n+k-j}$$

of $\tilde{m}$ on all sub-MV-algebras of $\tilde{\mathbb{C}}_n$, i. e. which is based on the "mean value function"

$$M(x,y) = \frac{y}{1+y-x},$$

a notion introduced and discussed in [4,5] in the context of conditioning.

On the other hand, also all valuations $\tilde{m}$ on $\tilde{\mathbb{C}}_n$ will be characterized by a completely different form. Finally, it will be proved that an uncertainty measure $\tilde{m}$ on $\tilde{\mathbb{C}}_n$ is both weakly additive and valuation, and then is unique, if and only if $n = 3$ or $n = 5$.

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Implication structures, fuzzy subsets, and enriched categories

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1 Introduction

In this talk, we present an analysis of some basic notions in fuzzy subset theory form the point of view of enriched category theory. The results demonstrate that ideas and methods from enriched category theory are very useful in the study of fuzzy subsets; and that a remarkable part of the mathematical theory of fuzzy subsets is an important and interesting application and/or an indispensable part of the general theory of enriched categories.

The following table shows the correspondence between some basic notions in the theory of fuzzy subsets and those in that of enriched categories.

Fuzzy subsets	**Enriched categories**
implication structure	closed category
monoidal implication structure	monoidal closed category
commutative, unital quantale	symmetric, monoidal closed category
fuzzy preorder	enriched category
many valued set	symmetric enriched category
fuzzy subset	functor from a discrete category
fuzzy powerset	functor category
fuzzy relation	distributer
fuzzy function	left adjoint in a 2-category

2 Implication structure

Implication is one of the most discussed concept in fuzzy logic. There is a vast literature on this concept, to name a few, [1, 3, 11, 15, 17, 19, 26, 49, 54], etc. Implication structures are meant to capture the basic features of ”implication” in lattice valued logic (or, fuzzy logic).

Definition 1. *An implication structure is a triple $(L, \to, D)$ where L is a complete lattice; $\to: L \times L \longrightarrow L$ is a binary function, called the implication; and D is a subset of L, called the set of designated values. These data satisfy that for all $a, b, c \in L$:*

(I1) $a \to b \leq a \to c$ if $b \leq c$;

(I2) $a \to c \leq b \to c$ if $a \geq b$;

(I3) $a \to b \leq (c \to a) \to (c \to b)$;

(I4) $a \to b \in D \iff a \leq b$;

(I5) $e \to a \leq a$ for all $e \in D$.

An implication structure $(L, \to, D)$ is complete if D has a smallest element.

Definition 2. *If $\to_1$ and $\to_2$ are implications on a complete lattice L with the same set of designated values, we say that $\to_1$ is conjugate to $\to_2$ if $a \leq (b \to_1 c) \iff b \leq (a \to_2 c)$ for all $a, b, c \in L$.*

Clearly, if an implication $\rightarrow$ has a conjugate, its conjugate must be unique. Therefore, we shall write $\rightarrow_{\mathrm{c}}$ for the conjugate of an implication $\rightarrow$ in the sequel.

Suppose that P, Q are partially ordered sets. A pair of non-increasing maps $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ is said to be a Galois connection if $q \leq f(p) \iff p \leq g(q)$ for all $p \in P, q \in Q$. Suppose that $\rightarrow_1$ and $\rightarrow_2$ are implications on a complete lattice L with the same set of designated values. Then $\rightarrow_1$ is conjugate to $\rightarrow_2$ if and only if for all $c \in L$, the pair of functions $(-) \rightarrow_1 c : L \longrightarrow L$ and $(-) \rightarrow_2 c : L \longrightarrow L$ is a Galois connection.

Proposition 1. *Suppose that $(L, \rightarrow, D)$ is an implication structure and that $\rightarrow$ has a conjugate $\rightarrow_{\mathrm{c}}$. Then the followings hold.*

(1) For all $a, b, c \in L$, $a \rightarrow_{\mathrm{c}} (b \rightarrow c) = b \rightarrow (a \rightarrow_{\mathrm{c}} c)$.

(2) For all $a, b \in L$, $a \rightarrow b = \bigwedge_{c \in L}[(b \rightarrow c) \rightarrow_{\mathrm{c}} (a \rightarrow c)]$.

(3) $(\bigvee a_t) \rightarrow b = \bigwedge(a_t \rightarrow b)$ for all $b \in L$ and all subset $\{a_t\}$ of L.

Definition 3. *Let $(L, \rightarrow, D)$ be an implication structure. Then*

(1) $(L, \rightarrow, D)$ is symmetric if $\rightarrow$ is conjugate to itself, i.e., it satisfies the exchange principle

(S) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$ for all $a, b, c \in L$.

(2) $(L, \rightarrow, D)$ has a neutral element if there is some $I \in L$ such that

(N) $I \rightarrow a = a$ for all $a \in L$.

(3) $(L, \rightarrow, D)$ is inf-preserving on the second argument if it satisfies

(IPS) $a \rightarrow (\bigwedge b_i) = \bigwedge(a \rightarrow b_i)$.

Proposition 2. *Suppose that $(\rightarrow, \rightarrow_{\mathrm{c}})$ is a conjugate pair of implications on L.*

(1) I is a neutral element for $\rightarrow$ if and only if I is a neutral element for $\rightarrow_{\mathrm{c}}$.

(2) $\rightarrow$ is inf-preserving on the second argument if and only if $\rightarrow_{\mathrm{c}}$ is inf-preserving on the second argument.

It is easy to check that the set D of designated values in an implication structure $(L, \rightarrow, D)$ is an upper set in L; and that if $(L, \rightarrow, D)$ has a neutral element I, this element must be the smallest element in D. However, a complete implication structure does not necessarily have a neutral element.

Example 1. ([48]) By a unital quantale is meant a triple $(L, *, I)$, where L is a complete lattice; I is an element in L; and $*$ is a binary operation $* : L \times L \longrightarrow L$, subject to the following conditions:

(1) $(L, *, I)$ is a monoid;

(2) $a * (\bigvee b_i) = \bigvee(a * b_i)$ and $(\bigvee b_i) * a = \bigvee(b_i * a)$ for all $a, b_i \in L$.

If $(L, *, I)$ is a unital quantale. Let $\rightarrow_r, \rightarrow_l$ be defined by

$$a \leq b \rightarrow_r c \iff b * a \leq c \iff b \leq a \rightarrow_l c.$$

Then $(\rightarrow_r, \rightarrow_l)$ is a conjugate pair of implications.

Example 2. (Gaines-Rescher, [21, 47]) Let L be a complete lattice. This example presents two implication structures on L with $D = \{1\}$. The first is given by:

$$a \rightarrow b = \begin{cases} 1, \ a \leq b; \\ 0, \ \text{otherwise}. \end{cases}$$

This is the smallest implication (under pointwise order) on L. Clearly, $(L, \rightarrow, \{1\})$ is a complete implication structure. $(L, \rightarrow, \{1\})$ is inf-preserving on the second argument but has no neutral element in general. This implication is called the Gaines-Rescher implication in the literature. The Gaines-Rescher implication on a complete lattice L has a conjugate if and only if $L = \{0,1\}$.

The second is given by

$$a \to b = \begin{cases} 1, & a \le b; \\ b, & \text{otherwise.} \end{cases}$$

This is the smallest implication on L with 1 as a neutral element. Clearly, $(L, \to, \{1\})$ is symmetric and complete, but not inf-preserving on the second argument in general. It is easy to check that $\to$ is inf-preserving on the second argument if and only if L is linearly ordered. In this case, $\to$ is the Gödel implication, and it is generated by the commutative, unital quantale $(L, \wedge, 1)$.

Definition 4. *An implication structure $(L, \to, D)$ is monoidal if there exist an element $I \in L$ and a binary operation $* : L \times L \longrightarrow L$ such that*

(1) $*$ is order-preserving;

(2) $(L, *, I)$ is a monoid;

(3) $a * b \le c \iff b \le a \to c$.

In this case, $(L, \to, D)$ is determined by the monoid $(L, *, I)$. We write $(L, \to, D, *, I)$ for a monoidal implication structure.

Proposition 3. *Suppose that $(L, \to, D, *, I)$ is a monoidal implication structure. Then, for all $a, b, c \in L$,*

(1) I is a neutral element;

(2) $a \to (b \to c) = b * a \to c, a * (a \to b) \le b, (a \to b) * (b \to c) \le a \to c$;

(3) $a \to (\bigwedge b_i) = \bigwedge (a \to b_i)$, $a * (\bigvee b_i) = \bigvee (a * b_i)$;

(4) $a * b = \bigwedge \{c : b \le a \to c\}$, $b \to c = \bigvee \{a : b * a \le c\}$;

Theorem 1. *Suppose that $(\to, \to_{\mathrm{c}})$ is a conjugate pair of implications on a complete lattice L with $D \subseteq L$ as the set of designated values. Then the following conditions are equivalent.*

(1) $(L, \to, D)$ is quantale-based, i.e., there exist a binary operation $*$ on L and an element $I \in L$ such that $(L, *, I)$ is a unital quantale and that $(\to, \to_{\mathrm{c}}) = (\to_r, \to_l)$.

(2) $(L, \to, D)$ is monoidal.

(3) $(L, \to, D)$ is complete and is inf-preserving on the second argument.

Corollary 1. *Suppose that $*$ is a binary operation on a complete lattice L such that $a * (\bigvee b_i) = \bigvee (a * b_i)$ and $(\bigvee b_i) * a = \bigvee (b_i * a)$ for all $a, b_i \in L$. Then $*$ is associative.*

Definition 5. *A closed map $f : (L_1, \to_1, D_1) \longrightarrow (L_2, \to_2, D_2)$ between implication structures is a map $f : L_1 \longrightarrow L_2$ such that*

(1) $f : L_1 \longrightarrow L_2$ preserves order;

(2) $f(D_1) \subseteq D_2$;

(3) $f(a \to_1 b) \le f(a) \to_2 f(b)$.

In the following we shall omit the index i in $\to_i$ if no confusion would arise.

Proposition 4. *Suppose that $(L_1, \to_1, D_1, *, I_1), (L_2, \to_2, D_2, \odot, I_2)$ are monoidal implication structures and that $f : L_1 \longrightarrow L_2$ is an order-preserving function. Then the following are equivalent:*

(1) $f : (L_1, \to, D_1) \longrightarrow (L_2, \to, D_2)$ is a closed map;

(2) $f(a) \odot f(b) \le f(a * b)$, for all $a, b \in L_1$ and $I_2 \le f(I_1)$.

If we regard the two points lattice $\{0,1\}$, with ordering $0<1$, as a category then $\mathbf{2}=\{\{0,1\},\wedge,1\}$ is a symmetric, monoidal closed category. Thus, we can define a closed category over $\mathbf{2}$, see [16], p. 549. It is easy to check that (1) if $(L,\rightarrow,D)$ is an implication structure with a neutral element I, then $(L,\rightarrow,I)$ is a closed category over $\mathbf{2}$, and vice versa; (2) a closed map between implication structures with neutral elements is exactly a closed functor in the sense of [16]; and (3) $(L,\rightarrow,D)$ is a (symmetric) monoidal implication structure (with I as neutral element) if and only if $(L,\rightarrow,I)$ is a (symmetric) monoidal closed category over $\mathbf{2}$. Therefore, implication structures are a generalization of closed categories over $\mathbf{2}$, and they can be regarded as closed categories without, possibly, neutral elements.

3 $\mathcal{L}$-categories and $\mathcal{L}$-sets

In this section, $(L,\rightarrow,D)$ is always assumed to be an implication structure. Sometimes, we write simply $\mathcal{L}$ for the triple $(L,\rightarrow,D)$.

Definition 6. *Let $\mathcal{L}=(L,\rightarrow,D)$ be an implication structure. An $\mathcal{L}$-category is a set X together with a function $R: X\times X\longrightarrow L$ subject to the conditions:*
(1) $R(x,x)\in D$ for all $x\in X$;
(2) $R(y,z)\leq R(x,y)\rightarrow R(x,z)$ for all $x,y,z\in X$.

X is called the underlying set of $(X,R(-,-))$, and $R(-,-)$ is called the hom-functor. We often write X for (A,R) and $X(-,-)$ for $R(-,-)$, if the hom-functor is clear from the context. In this case, write $|X|$ for the underlying set of X.

Two elements x and y in X are said to be *isomorphic* if $X(x,y)\in D$ and $X(y,x)\in D$. X is called *antisymmetric* if different elements in X are always non-isomorphic.

Given an $\mathcal{L}$-category A, if we interpret $A(a,b)$ as the degree to which b precedes a, then the condition $A(a,a)\in D$ is to require that A be reflexive and the condition $X(a,b)\leq A(c,a)\rightarrow A(c,b)$ is a variation of transitivity of order relation. For this reason, $\mathcal{L}$-categories can be studied as fuzzy preorders. When $\mathcal{L}$ is a (symmetric) quantale-based implication structure, this has been done in [4, 5, 7, 33, 36, 40, 42, 43, 52, 53, 46, 55, 56, 59, 60].

Definition 7. *An $\mathcal{L}$-functor $f: X\longrightarrow Y$ between $\mathcal{L}$-categories is a function $f:|X|\longrightarrow|Y|$ such that $X(x,y)\leq Y(f(x),f(y))$.*

If $\mathcal{L}=(L,\rightarrow,D)$ is an implication structure with a neutral element, i.e., $\mathcal{L}$ is a closed category over $\mathbf{2}$, then an L-category is exactly a category over $\mathcal{L}$ and an $\mathcal{L}$-functor is exactly an $\mathcal{L}$-functor in the sense of Eilenberg and Kelly [16]. The category of $\mathcal{L}$-categories and $\mathcal{L}$-functors is denoted by $\mathcal{L}$-**Cat**.

Example 3. (1) ([16]) Let $\mathcal{L}=(L,\rightarrow,D)$ be an implication structure. For any $a,b\in L$, let $R(a,b)=a\rightarrow b$. Then (L,R) is an antisymmetric $\mathcal{L}$-category by (I3) and (I4). The $\mathcal{L}$-category structure on L is called the canonical $\mathcal{L}$-category structure on L. In the remainder of this article, we shall write $(\mathcal{L},\rightarrow)$ for this $\mathcal{L}$-category.

(2) ([55]) Let X be a set and $\lambda: X\longrightarrow L$ a function. For any $x,y\in X$, let $V_\lambda(x,y)=\lambda(x)\rightarrow\lambda(y)$. Then (X,V_λ) is an $\mathcal{L}$-category.

Example 4. (Discrete $\mathcal{L}$-categories) Let $\mathcal{L}=(L,\rightarrow,D)$ be an implication structure such that D has a least element e. Given a set X and $x,y\in X$, let $X(x,y)=e$ if $x=y$ and $X(x,y)=0$ if $x\neq y$. Then X becomes an $\mathcal{L}$-category, called a discrete $\mathcal{L}$-category. In this case, every function $X\longrightarrow L$ is an $\mathcal{L}$-functor.

Let $\mathcal{L} = (L, \to, D)$ be an implication structure such that $\to$ has a conjugate $\to_{\mathrm{c}}$. Let X be an $\mathcal{L}$-category. For all $x, y \in X$, let $X^{\mathrm{op}}(x,y) = X(y,x)$. Then X^{op} is an $\mathcal{L}_c$-category, where $\mathcal{L}_c = (L, \to_{\mathrm{c}}, D)$. Indeed, for all $x,y,z \in X$, because $X(x,z) \leq X(y,x) \to X(y,z)$, we obtain that $X(y,z) \leq X(x,z) \to_{\mathrm{c}} X(y,z)$, therefore $X^{\mathrm{op}}(x,y) \leq X^{\mathrm{op}}(z,x) \to_{\mathrm{c}} X^{\mathrm{op}}(z,y)$.

In particular, if $\mathcal{L} = (L, \to, D)$ is symmetric, we can talk about symmetric L-categories. An $\mathcal{L}$-category X is said to be symmetric if $X(x,y) = X(y,x)$ for any $x, y \in X$. A symmetric $\mathcal{L}$-category shall also be called an $\mathcal{L}$-set. The full subcategory consisting of $\mathcal{L}$-sets is denoted by $\mathcal{L}$-**Set**.

Example 5. ([52]) Let $\mathcal{L} = (L, \to, D)$ be an implication structure, X an $\mathcal{L}$-category, and $x \in X$. Then the function $X(x,-) : X \longrightarrow (\mathcal{L}, \to)$ is an $\mathcal{L}$-functor. If $\to$ has a conjugate $\to_{\mathrm{c}}$, then $X(-,x) : X^{\mathrm{op}} \longrightarrow (\mathcal{L}, \to_{\mathrm{c}})$ is an $\mathcal{L}_{\mathrm{c}}$-functor.

Proposition 5. *(c.f. I.8.6 in [16]) Suppose that $\mathcal{L} = (L, \to, D)$ is a complete implication structure; X be an $\mathcal{L}$-category and $a \in X$. Then, for any $\mathcal{L}$-functor $f : X \longrightarrow \mathcal{L}$,*

$$\bigwedge_{x \in X} (X(a,x) \to f(x)) \in D \iff f(a) \in D.$$

Lemma 1. *(Yoneda lemma) Suppose that $\mathcal{L} = (L, \to, D)$ is an implication structure such that $\to$ has a conjugate $\to_{\mathrm{c}}$. Let X be an $\mathcal{L}$-category and $a \in X$. Then for any $\mathcal{L}$-functor $f : X \longrightarrow \mathcal{L}$,*

$$\bigwedge_{x \in X} (X(a,x) \to_{\mathrm{c}} f(x)) = f(a).$$

Example 6. (Functor category) Let $\mathcal{L}$ be a complete implication structure which is inf-preserving on the second argument. For any $\mathcal{L}$-categories X, Y, let $[X,Y]$ be the set of all $\mathcal{L}$-functors $X \longrightarrow Y$. For any $f, g \in [X,Y]$, let

$$[X,Y](f,g) = \bigwedge_{x \in X} Y(f(x), g(x)).$$

Clearly, $[X,Y](f,f) \in D$ for any $f \in [X,Y]$. Because

$$\begin{aligned}
[X,Y](f,g) &= \bigwedge_{x \in X} Y(f(x), g(x)) \\
&\leq \bigwedge_{x \in X} \Big(Y(h(x), f(x)) \to Y(h(x), g(x)) \Big) \\
&\leq \bigwedge_{x \in X} \Big(\bigwedge_{y \in X} Y(h(y), f(y)) \to Y(h(x), g(x)) \Big) \\
&= [X,Y](h,f) \to [X,Y](h,g)
\end{aligned}$$

for any $f, g, h \in [X,Y]$, $[X,Y]$ becomes an $\mathcal{L}$-category. This $\mathcal{L}$-category is called the functor category from X to Y.

In particular, if X is a set, considered as a discrete $\mathcal{L}$-category, then $[X, \mathcal{L}] = L^X$ and $[X, \mathcal{L}](\lambda, \mu)$ is known as the degree that λ is a subset of μ [3, 9].

Example 7. Let $\mathcal{L} = (L, \to, \{1\})$, where L is a complete lattice, $\to$ is the Gaines-Rescher implication on L. Then for any set X, considered as a discrete $\mathcal{L}$-category, the functor category is essentially the complete lattice L^X under pointwise order.

Suppose that $(L,\to,D)$ is complete and inf-preserving on the second argument and that $\to$ has a conjugate $\to_{\mathrm{c}}$; or equivalently, $(L,\to,D)$ is quantale-based. Given an $\mathcal{L}$-category X and two $\mathcal{L}$-functors $f,g : X \longrightarrow \mathcal{L}$, let

$$[X,\mathcal{L}]_{\mathrm{c}}(f,g) = \bigwedge_{x\in X} (f(x) \to_{\mathrm{c}} g(x)).$$

Then it is easy to check that $[X,\mathcal{L}]_{\mathrm{c}}$ becomes an $\mathcal{L}_{\mathrm{c}}$-category. Thus, $[X,\mathcal{L}]_{\mathrm{c}}^{\mathrm{op}}$ is an $\mathcal{L}$-category.

Proposition 6. *(co-Yoneda embedding, [52]) Let $\mathcal{L} = (L,\to,D)$ be a quantale-based implication structure; X an $\mathcal{L}$-category, and $a \in X$. Then, for any $\mathcal{L}$-functor $f : X \longrightarrow \mathcal{L}$,*

$$[X,\mathcal{L}]_{\mathrm{c}}(X(a,-), f) = f(a).$$

Therefore,

$$\mathbf{y}' : X \longrightarrow [X,\mathcal{L}]_{\mathrm{c}}^{\mathrm{op}},\ a \mapsto X(a,-),$$

is a fully faithful $\mathcal{L}$-functor in the sense that $X(a,b) = [X,\mathcal{L}]_{\mathrm{c}}^{\mathrm{op}}(\mathbf{y}'(a),\mathbf{y}'(b))$ for all $a,b \in X$.

$\mathbf{y}'$ in the above lemma is called the co-Yoneda embedding. Similarly, let $[X^{\mathrm{op}},\mathcal{L}_{\mathrm{c}}]$ denote the set of all $\mathcal{L}_{\mathrm{c}}$-functors $X^{\mathrm{op}} \longrightarrow \mathcal{L}_{\mathrm{c}}$ and for all $f,g \in [X^{\mathrm{op}},\mathcal{L}_{\mathrm{c}}]$, let

$$[X^{\mathrm{op}},\mathcal{L}_{\mathrm{c}}](f,g) = \bigwedge_{x\in X} (f(x) \to g(x).$$

Then $[X^{\mathrm{op}},\mathcal{L}_{\mathrm{c}}]$ becomes an $\mathcal{L}$-category. It is routine to check that the correspondence

$$\mathbf{y} : X \longrightarrow [X^{\mathrm{op}},\mathcal{L}_{\mathrm{c}}],\ a \mapsto X(-,a),$$

defines a fully faithful $\mathcal{L}$-functor, $\mathbf{y}$ is called the Yoneda embedding [52].

4 Two categories of $\mathcal{L}$-subsets

In this section, $(L,\to,D)$ is always assumed to be a complete implication structure which is inf-preserving on the second argument, if not otherwise specified.

The category $\mathcal{L}$-**FSet** consists of the following data:

- Objects: pairs (X,λ), where X is a set, $\lambda : X \longrightarrow L$ is a function;
- Morphisms: a morphism $f : (X,\lambda) \longrightarrow (Y,\mu)$ is a function $f : X \longrightarrow Y$ such that $\lambda(x_1) \to \lambda(x_2) \leq \mu(f(x_1)) \to \mu(f(x_2))$ for all $x_1,x_2 \in X$;
- Composition: the usual composition of functions.

Intuitively, a morphism in $\mathcal{L}$-**FSet** is a function which makes the membership degrees less dramatically changed.

Before elaborating on the category $\mathcal{L}$-**FSet**, we recall some basic notions of concrete categories from [2]. By a concrete category over the category **Set** of sets is meant a pair $(\mathbf{A},U)$, where $\mathbf{A}$ is a category and $U : \mathbf{A} \longrightarrow \mathbf{Set}$ is a faithful functor. A concrete category $(\mathbf{A},U)$ is often abbreviated to $\mathbf{A}$ if the functor U is obvious. Let $(\mathbf{A},U)$ be a concrete category. Given a set X, the fibre of X is the preordered class consisting of all $\mathbf{A}$-objects A with $U(A) = X$ ordered by: $A \leq B$ if and only if $\mathrm{id}_X : UA \longrightarrow UB$ is an $\mathbf{A}$-morphism. $\mathbf{A}$-objects A and B are said to be equivalent provided that $A \leq B$ and $B \leq A$. $(\mathbf{A},U)$ is said to be amnestic provided that the fibre of any set X is a partially ordered class, that is, no two different $\mathbf{A}$-objects are equivalent. A concrete category $(\mathbf{A},U)$ is said to be transportable provided that for every $\mathbf{A}$-object A and every bijection $UA \xrightarrow{k} X$ there exists a $\mathbf{A}$-object B with $UB = X$ such that $UA \xrightarrow{k} X$ is an $\mathbf{A}$-isomorphism.

Proposition 7. *$\mathcal{L}$-**FSet** is a concrete category over **Set**. It is transportable but not amnestic.*

By aid of the category $\mathcal{L}$-**FSet**, we can talk about the *shape* of $\mathcal{L}$-subsets.

Definition 8. *(X,λ) and (X,μ) are of the same shape if they are equivalent in $\mathcal{L}$-**FSet**, i.e., $\lambda(x) \to \lambda(y) = \mu(x) \to \mu(y)$ for all $x,y \in X$.*

Example 8. Let L be a complete lattice equipped with the Gaines-Rescher implication. Then (X,λ) and (X,μ) are of the same shape if and only if $\lambda(x) \leq \lambda(y) \iff \mu(x) \leq \mu(y)$ for all $x,y \in X$.

An $\mathcal{L}$-subset $\lambda : X \longrightarrow L$ can be regarded as an expression of a property of elements in X. The above example says that, with respect to the Gaines-Rescher implication, two function $\lambda,\mu : X \longrightarrow L$ express the same property if they define the same order on X in the way that $x \leq y$ if $\lambda(x) \leq \lambda(y)$. It is the thus-defined order, not the particular value $\lambda(x)$, that is most important.

Example 9. Let $\mathcal{L} = (L,\to,\{1\})$, where $L = [0,1]$ and $\to$ is the implication corresponding to the Łukasiewicz t-norm on $[0,1]$. Then $\lambda,\mu : X \longrightarrow L$ are of the same shape if and only if there is some real number a such that $\lambda(x) = \mu(x) + a$ for all $x \in X$. Hence, λ and μ are of the same shape if and only if the difference between $\lambda(x)$ and $\mu(x)$ is a constant.

Example 10. Let $\mathcal{L} = (L,\to,\{1\})$, where $L = [0,1]$ and $\to$ is the implication corresponding to the product t-norm. If $\mathcal{L} = (L,\to,\{1\})$. Then $\lambda,\mu : X \longrightarrow L$ are of the same shape if and only if there is some real number a such that $\lambda(x) = a \cdot \mu(x)$ for all $x \in X$. Hence, λ and μ are of the same shape if and only if λ equals the product of μ with a real number.

Example 11. Let $\mathcal{L} = (L,\to,\{1\})$, where $L = [0,1]$ and $\to$ is the implication corresponding to the t-norm min. A $\mathcal{L}$-subset $\lambda : X \longrightarrow L$ is said to be regular if either (i) $\max\{\lambda(x) : x \in X\} = 1$; or (ii) $\{\lambda(x) : x \in X\}$ has no maximal element. The constant function $\underline{1}$ is regular, but, for every $a \neq 1$, the constant function $\underline{a}$ is not regular. Given a fuzzy set $\lambda : X \longrightarrow L$, let $\lambda_{\#}(x) = 1$ if $\lambda(x)$ is maximal in $\{\lambda(z) : z \in X\}$, otherwise let $\lambda_{\#}(x) = \lambda(x)$. Then $\lambda_{\#}$ is regular, called the regularization of λ. Then $\lambda,\mu : X \longrightarrow L$ are of the same shape if and only if $\lambda_{\#} = \mu_{\#}$. That is, λ and μ are of the same shape if and only if they have the same regularization.

For each morphism $f : (X,\lambda) \longrightarrow (Y,\mu)$ in $V : \mathcal{L}$-**FSet**, it is easy to see that $f : (X,V_\lambda) \longrightarrow (Y,V_\mu)$ is a morphism in $\mathcal{L}$-**Cat**. Thus, the correspondence $(X,\lambda) \mapsto (X,V_\lambda)$ defines a concrete functor $V : \mathcal{L}$-**FSet**$\longrightarrow \mathcal{L}$-**Cat**.

Theorem 2. *If $\mathcal{L}$ has a neutral element, then the functor $V : \mathcal{L}$-**FSet**$\longrightarrow \mathcal{L}$-**Cat** satisfies the following conditions:*

(1) V is full and faithful.

*(2) $\mathcal{L}$-**Cat** is amnestic, transportable, and concretely complete.*

If L is linearly ordered with $1 \in L$ as neutral element, then

*(3) V is concrete limit-dense, i.e., for every $\mathcal{L}$-category A there exists a small diagram $E : J \longrightarrow \mathcal{L}$-**FSet** such that A is a concrete limit of $V \circ E$.*

(4) V preserves limits of small diagrams.

*(5) For every amnestic, transportable, and concretely complete category **C** and every concrete functor $F : \mathcal{L}$-**FSet**$\longrightarrow$ **C** that preserves concrete limits of small diagrams, there exists a unique concrete functor $\overline{F} : \mathcal{L}$-**Cat**$\longrightarrow$ **C** that preserves concrete limits of small diagrams, with $F = \overline{F} \circ V$.*

Thus, if L is linearly ordered with $1 \in L$ as neutral element, $\mathcal{L}$-**Cat** is the universal completion of the amnestic modification [2] of $\mathcal{L}$-**FSet** in the sense of Herrlich [27].

When $(L, \to, D)$ is generated by a commutative, unital quantale structure on L, we can introduce another category of $\mathcal{L}$-subsets, the category $\mathcal{L}$-**CSet**. $\mathcal{L}$-**CSet** consisting of the following data:

- Objects: pairs (X, λ), where X is a set, $\lambda : X \longrightarrow L$ is a function;
- Morphisms: a morphism $f : (X, \lambda) \longrightarrow (Y, \mu)$ is a function $f : X \longrightarrow Y$ such that $\lambda(x_1) \leftrightarrow \lambda(x_2) \leq \mu(f(x_1)) \leftrightarrow \mu(f(x_2))$ for all $x_1, x_2 \in X$;
- Composition: the usual composition of functions.

Parallel to the definition of the functor $V : \mathcal{L}$-**FSet**$\longrightarrow \mathcal{L}$-**Cat** to $\mathcal{L}$-**CSet**, we can define a functor $H : \mathcal{L}$-**CSet**$\longrightarrow \mathcal{L}$-**Set**.

Theorem 3. *If $(L, ra, \{1\})$ is a commutative, unital quantale based and linearly ordered implication structure, then $\mathcal{L}$-*__Set__* is the universal completion of $\mathcal{L}$-*__CSet__.

5 Fuzzy functions as left adjoints

$\mathcal{L} = (L, \to, D, *, I)$ is assumed to be a symmetric, quantale-based implication structure in this section. We show that left adjoints in the 2-category of $\mathcal{L}$-sets can be regarded as the fuzzy functions between $\mathcal{L}$-sets.

Let X, Y be $\mathcal{L}$-sets. An $\mathcal{L}$-relation from X to Y, in symbols $R : X \rightharpoonup Y$, is an $\mathcal{L}$-functor $R : X \otimes Y \longrightarrow (\mathcal{L}, \to)$, that is, for all $x_1, x_2 \in X, y_1, y_2 \in Y$,

$$X(x_2, x_1) * R(x_1, y_1) * Y(y_1, y_2) \leq R(x_2, y_2).$$

The composition of two fuzzy relations $R : X \rightharpoonup Y$ and $S : Y \rightharpoonup Z$ is given by

$$S \circ R(x, z) = \bigvee_{y \in Y} R(x, y) * S(y, z)$$

for all $x \in X, z \in Z$.

An $\mathcal{L}$-relation $R : X \rightharpoonup Y$ is exactly a *distributer* from X to Y in [52].

For each $\mathcal{L}$-set X, let $\mathrm{id}_X : X \rightharpoonup X$ be the $\mathcal{L}$-relation given by $\mathrm{id}_X(x_1, x_2) = X(x_1, x_2)$. Then it is easy to check that $R \circ \mathrm{id}_X = R = \mathrm{id}_Y \circ R$ and that $(R \circ S) \circ T = R \circ (S \circ T)$ for any $\mathcal{L}$-relations $R : X \rightharpoonup Y, S : Y \rightharpoonup Z$ and $T : Z \rightharpoonup W$. Thus, we obtain a category $\mathcal{L}$-**Rel** of which the objects are $\mathcal{L}$-sets and the morphisms are $\mathcal{L}$-relations, the category of $\mathcal{L}$-relations. It is an interesting fact that for any $\mathcal{L}$-sets X and Y, the homset $\mathcal{L}$-**Rel**(X, Y) is itself an $\mathcal{L}$-category (as the functor category from $X \otimes Y$ to $\mathcal{L}$). We write $[X \rightharpoonup Y]$ for the functor category $[X \otimes Y, \mathcal{L}]$.

Theorem 4. *$\mathcal{L}$-*__Rel__* is a category enriched over the monoidal closed category $\mathcal{L}$-*__Cat__.

Therefore, the category of $\mathcal{L}$-relations is a 2-category, indeed, a locally partially ordered 2-category with:

- Objects: $\mathcal{L}$-sets;
- 1-morphisms: $\mathcal{L}$-relations $R : X \rightharpoonup Y$;
- 2-morphisms: there is exactly one morphism from 1-morphism $R : X \rightharpoonup Y$ to 1-morphism $R' : X \rightharpoonup Y$ if $R \leq R'$ under the pointwise order.

Definition 9. *An $\mathcal{L}$-relation $R : X \rightharpoonup Y$ is said to be functional if it is a left adjoint in the 2-category $\mathcal{L}$-**Rel**, that is, there exists an $\mathcal{L}$-relation $S : Y \rightharpoonup X$ such that $\mathrm{id}_X \leq S \circ R$ and $R \circ S \leq \mathrm{id}_Y$. In this case, S is called a right adjoint of R.*

Example 12. ([44]) Suppose $f : X \longrightarrow Y$ is an $\mathcal{L}$-functor. Then f generates an $\mathcal{L}$-relation $f_\natural : X \rightharpoonup Y, f_\natural(x,y) = Y(f(x),y)$. It is easy to verify that $f_\natural$ is functional and that a right adjoint of $f_\natural$ is given by $f^\natural : Y \rightharpoonup X, f^\natural(y,x) = Y(y,f(x))$.

In order to show that functional $\mathcal{L}$-relations can be regarded as fuzzy functions, we introduce the following

Definition 10. *(C.f. [12, 35]) A singleton of an $\mathcal{L}$-set X is an $\mathcal{L}$-functor $s : X \longrightarrow (\mathcal{L}, \rightarrow)$ such that*
(s1) $s(x) * s(y) \leq X(x,y)$; and
(s2) $\bigvee_{x \in X} s(x) * s(x) \geq I$.

Put differently, a singleton of X is an $\mathcal{L}$-relation $s : \mathbf{1} \rightharpoonup X$ such that $s^{\mathrm{op}} : X \rightharpoonup \mathbf{1}$ is a right adjoint to s; hence, s is a functional $\mathcal{L}$-relation from the terminal $\mathcal{L}$-set $\mathbf{1}$ to X.

We say that the unit element I in a quantale $(L, *, I)$ is *square accessible* if $\bigvee A \geq I \Rightarrow \bigvee\{a * a : a \in A\} \geq I$ for any subset $A \subseteq L$. This condition first appeared in [35] (for idempotents). It is easy to check that every BL-algebra [26] satisfies this condition.

Lemma 2. *Suppose the unit element I is square accessible and X is an $\mathcal{L}$-set. Then every functional $\mathcal{L}$-relation $\mathbf{1} \rightharpoonup X$ is a singleton.*

Lemma 3. *If the unit element I is square accessible, then for every functional $\mathcal{L}$-relation $F : X \rightharpoonup Y$, the right adjoint of F is given by $F^{\mathrm{op}} : Y \rightharpoonup X, F^{\mathrm{op}}(y,x) = F(x,y)$.*

Theorem 5. *Suppose the unit element I is square accessible and X, Y are $\mathcal{L}$-sets. Then, an $\mathcal{L}$-relation $F : X \rightharpoonup Y$ is functional if and only if it satisfies*
(F1) $F \circ F^{\mathrm{op}} \leq \mathrm{id}_Y$, that is, $F(x,y_1) * F(x,y_2) \leq Y(y_1,y_2)$ for all $x \in X$ and $y_1, y_2 \in Y$;
(F2) $F^{\mathrm{op}} \circ F \geq \mathrm{id}_X$, that is, $\bigvee_{y \in Y} F(x_1,y) * F(x_2,y) \geq X(x_1,x_2)$ for all $x_1, x_2 \in X$.

The condition (F1) says that F is "single-valued" and (F2) says that F is "totally defined" on X. Thus, F is a *fuzzy function*.

Acknowledgement: This work is supported by NCET (05-0779) and Natural Science Foundation of China (10771147).

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