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30th Linz Seminar on Fuzzy Set Theory

Abstracts

The Legacy of 30 Seminars – Where Do We Stand and Where Do We Go?

Bildungszentrum St. Magdalena, Linz, Austria
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Editors

LINZ 2009

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THE LEGACY OF 30 SEMINARS –
WHERE DO WE STAND AND WHERE DO WE GO?

Dedicated to the memory of Dan Butnariu

ABSTRACTS

Ulrich Bodenhofer, Bernard De Baets, Erich Peter Klement, Susanne Saminger-Platz
Editors

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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2009 is the 30th seminar in this series of meetings and is devoted to the theme "The Legacy of 30 Seminars - Where Do We Stand and Where Do We Go?". Different to previous years, the scope of the seminar does not restrict to a single sub-topic of fuzzy set theory. Instead, the goal is to view fuzzy set theory and the past and future contributions of the Linz seminars from additional perspectives. We want to determine the state of the art achieved within fuzzy set theory, to ask for the impacts on other fields (of mathematics and applications), and to discuss their future research directions and applications.

A large number of highly interesting contributions were submitted for possible presentation at LINZ 2009. In order to maintain the traditional spirit of the Linz Seminars — no parallel sessions and enough room for discussions — we selected those twenty-nine submissions which, in our opinion, fitted best to the focus of this very special seminar. This volume contains the abstracts of this impressive selection. These regular contributions are complemented by three reviews of renowned researchers and four invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

Ulrich Bodenhofer
Bernard De Baets
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Contents

Non-deterministic fuzzy semantics

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The idea behind the development of fuzzy logic is that in many cases propositions do not have a crisp truth value. Thus (to take a famous example) the question whether John is tall might not have a clear-cut, "yes" or "no" answer. Fuzzy logics try to solve this problem by allowing the whole range of numbers between 0 and 1 to serve as potential "truth values" for propositions. By this they strongly deviate from classical logic, which employs just the two extreme values: 0 and 1. However, current fuzzy logics (in the narrow sense) continue to closely follow classical logic in the way they treat complex formulas. All of them are based on the principle that the truth value assigned to a complex formula should be completely determined in a unique, crisp way by the truth values assigned to its components. In other words: basically, the phenomenon of fuzziness is limited to atomic formulas, but no fuzziness is allowed in the semantics of connectives. An interpretation of an n-ary connective is always a crisp n-ary function on the interval [0,1]. The various current fuzzy logics differ from one another in their choices of the specific interpretation functions used, but not in the basic underlying principle that the semantic interpretation of a connective should be a *function* from tuples of truth values to single truth values. Among other things, this leads to many counterintuitive results. Thus if p is assigned the value $1/2$ then $\neg p \vee p$ is also assigned the value $1/2$ in all the three basic fuzzy logics (according to the most common interpretation of $\vee$), even though 1 might be taken here as the intuitive value. One can try to remedy this by taking $A \vee B$ to mean $\neg A \rightarrow B$, where $\rightarrow$ is Łukasiewicz implication: then the value assigned to $\neg p \vee p$ would indeed be 1. Unfortunately, according to this interpretation of $\vee$, the value assigned to $p \vee p$ would again be 1 if p is assigned the value $1/2$, which is hardly intuitive or useful.

We believe that the truth functionality principle is in direct conflict with the idea and spirit of fuzzy logic. Its employment has been inherited from orthodox (deterministic) many-valued logics, and it makes fuzzy logic (in the narrow sense) nothing more than a branch of many-valued logic (in the narrow, deterministic sense). Instead, we suggest that in the framework of fuzzy logics the logical notions of "and", "or", "implies", etc should also get fuzzy interpretations.

Given the above observations, we propose a new general framework which suits the idea of fuzzy logic better than the conventional, but inflexible, deterministic many-valued logics: namely, the generalization of the latter given by *nondeterministic matrices*. This new framework was introduced in [5, 6, 4], where it was shown to have all the advantages of the framework of ordinary (deterministic, multi-valued) matrices. It was later successfully applied to a whole variety of families of logics (see e.g. [1–3]). The relevant definitions are as follows:

Definition 1. 1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, O \rangle$, where:

(a) $\mathcal{V}$ is a non-empty set of *truth values*.

(b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.

(c) For every n-ary connective $\diamond$ of $\mathcal{L}$, O includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

2. A *(legal) $\mathcal{M}$-assignment* in an Nmatrix $\mathcal{M}$ is a function from the set of formulas of $\mathcal{L}$ to $\mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \dots, \psi_n \in \mathcal{L}$:

$$v(\diamond(\psi_1, \dots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \dots, v(\psi_n))$$

3. An $\mathcal{M}$-assignment v (in an Nmatrix $\mathcal{M}$) is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$.

In applying the framework of Nmatrices to fuzzy logics, we should obviously take $\mathcal{V}$ to be the closed interval $[0,1]$, and $\mathcal{D} = \{1\}$. It also seems natural to demand that the sets assigned by the interpretations of the connectives be intervals. This leads to the following definition:

Definition 2. A *fuzzy Nmatrix* for a language $\mathcal{L}$ is an Nmatrix of the form $\langle [0,1], \{1\}, O \rangle$, where for every n-ary connective $\diamond$ of $\mathcal{L}$, O includes a corresponding n-ary function $\widetilde{\diamond}$ from $[0,1]^n$ to the set of nonempty subintervals (including singletons) of $[0,1]$.

Examples include some plausible interpretations of the basic connectives that allow in each case the freedom of choosing a value among those allowed by the interpretations of the basic deterministic fuzzy logics:

$$a \widetilde{\&} b = [max\{0, a+b-1\}, min\{a,b\}]$$

$$a \widetilde{\rightarrow} b = \begin{cases} \{1\} & a \leq b \\ [b, 1-a+b] & a > b \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \{1\} & a = 0 \\ [0, 1-a] & a > 0 \end{cases}$$

$$a \widetilde{\vee} b = [max\{a,b\}, min\{a+b, 1\}]$$

Note. It should be emphasized that although the values of the functions which interpret the connectives are now intervals rather than numbers, the values assigned to *formulas* by legal assignments are still single *numbers*.

The framework provided by fuzzy Nmatrices is only the first step in designing appropriate semantics on which truly fuzzy logics can be based. The reason is that the set of legal assignments allowed e.g. by the above nondeterministic connectives seems to be too large, so further constraints should be imposed on it. One source of such constraints might be the intent to validate certain axioms. Here are some examples of constraints of this type on an assignment v:

- $v(A \vee A) = v(A)$
- $v(A \vee B) = v(B \vee A)$
- $v(\neg A \vee A) = 1$

Another source of constraints might be the intent to preserve (at least to some extent) the connections between the connectives that are assumed in the usual, deterministic fuzzy logics (see e.g. [7]). Thus we may demand that every legal assignment v satisfy the following condition:

$$v(A \to B) = sup\{v(C) \mid v(A\&C) \leq v(B)\}$$

Note that if v satisfies this condition, $v(A) = 1$ for at least one formula A, and v respects the nondeterministic interpretation $\widetilde{\&}$ of & given in the foregoing, then v necessarily respects $\widetilde{\to}$ as well. Note also that every valuation which is legal according to one of the three basic deterministic fuzzy logics satisfies the above constraint, and respects the above nondeterministic interpretations of the standard connectives.

Accordingly, the obvious next step would be to explore the options of Nmatrices and constraints on the set of legal valuations best suited for the semantics of fuzzy logics, and find the logics generated by them. Obviously, such logics might prove more complicated than those based on deterministic semantics. However, they will surely represent a more general concept of fuzziness — and it would be an interesting task to compare them with the existing basic fuzzy logics.

References

1. A. Avron. Non-deterministic Matrices and Modular Semantics of Rules. In *Logica Universalis* (J.-Y. Beziau, ed.), 149–167, Birkhüser Verlag, 2005.
2. A. Avron. Logical Non-determinism as a Tool for Logical Modularity: An Introduction. In *We Will Show Them: Essays in Honor of Dov Gabbay*, (S. Artemov, H. Barringer, A. S. d'Avila Garcez, L. C. Lamb, and J. Woods, eds.), vol. 1, 105–124, College Publications, 2005.
3. A. Avron. Non-deterministic Semantics for Logics with a Consistency Operators. *International Journal of Approximate Reasoning*, 45:271–287, 2007.
4. A. Avron and B. Konikowska. Multi-valued Calculi for Logics Based on Non-determinism. *Journal of the Interest Group in Pure and Applied Logic*, 10:365–387, 2005.
5. A. Avron and I. Lev. Canonical Propositional Gentzen-Type Systems. In *Proceedings of the 1st International Joint Conference on Automated Reasoning (IJCAR 2001)* (R. Goré, A Leitsch, T. Nipkow, Eds), LNAI 2083, 529–544, Springer Verlag, 2001.
6. A. Avron and I. Lev. Non-deterministic Multiple-valued Structures. *Journal of Logic and Computation*, 15:241-261, 2005.
7. P. Hájek. *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, 1998.

Fuzzy Class Theory: a state of the art

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It is indisputable that mathematical structures arising around vague/fuzzy/non-bivalent concepts have a broad range of applications; therefore they have been intensively investigated during the last four decades. The discipline studying these structures is, maybe unfortunately, called *Fuzzy Mathematics*.

There is an ongoing project of the Prague research group in fuzzy logic, directed towards developing the *logic-based* fuzzy mathematics, i.e., an 'alternative' mathematics built in a formal analogy with classical mathematics, but using a suitable formal fuzzy logic instead of the classical logic. First steps in the development thereof were enabled by recent results in Mathematical Fuzzy Logic, especially by the emergence of higher-order fuzzy logics, proposed by Libor Běhounek and the present author, see [6]. This approach leads not only to an axiomatization, but also to a systematic study utilizing proof-theoretic and model-theoretic methods. Moreover, the unified formalism allows an interconnection of particular disciplines of fuzzy mathematics and provides the formal foundations of (part of) fuzzy mathematics.

The core of the project is a formulation of certain formalistic methodology (see [7]), proposing the foundational theory (see [6]), and studying the particular disciplines (see the list below) of fuzzy mathematics within this theory using our methodology. The proposed foundational theory is called Fuzzy Class Theory (FCT) and it is a first-order theory over multi-sorted predicate fuzzy logic, with a very natural axiomatic system which approximates nicely Zadeh's original notion of fuzzy set [19]. In paper [7] we claim that the whole enterprize of Fuzzy Mathematics can be formalized in FCT. This is still true as classical logic is formally interpretable inside formal fuzzy logics we use, however there are parts of fuzzy mathematics where our approach provides (very) little added value; see [3] for more details about relation of traditional and logic-based fuzzy mathematics.

An important feature of the theory is the gradedness of all defined concepts, which makes it more genuinely fuzzy than traditional approaches. Indeed, e.g. in the theory of fuzzy relations the majority of traditional characterizing properties, such as reflexivity, symmetry, transitivity, and so forth, are defined in a strictly crisp way,[1] i.e., as properties that either hold fully or do not hold at all. One may be tempted to argue that it is somewhat peculiar to fuzzify relations by allowing intermediate degrees of relationships, but, at the same time, to still enforce strictly crisp properties on fuzzy relations. This particularly implies that all results are effective only if some assumptions are fully satisfied, but say nothing at all if the assumptions are only fulfilled to a certain degree (even if they are *almost* fulfilled).

The papers written within the project so far can be divided into several groups (for more comprehensive list of papers together with their preprints and more details about the project in general see its webpage `www.cs.cas.cz/hp`):

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[1] The notion of fuzzy inclusion is a notable exception; graded properties of fuzzy relations were originally studied by Siegfried Gottwald in [15].

– Methodological issues: [3, 7, 9]
– Formalism of FCT: [6, 14] and freely available primer [8]
– Fuzzy relations: [4, 10]
– Fuzzy topology: [11–13]
– Fuzzy filters and measures: [17, 18]
– Fuzzy algebra and (interval) analysis: [1, 2, 5, 16]

In this talk we survey the basic logical prerequisites, formulate the methodological standpoint, put it in the context of other nonclassical-logic-based mathematics (intuitionistic, relevant, substructural, etc.), compare logic-based, categorial, and traditional fuzzy mathematics, sketch the formalism of FCT and illustrate it using simple examples from the theory of fuzzy relations. Finally, we address the possible outlooks of FCT and its role in future fuzzy mathematics.

References

1. Libor Běhounek. Towards a formal theory of fuzzy Dedekind reals. In Eduard Montseny and Pilar Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 946–954, Barcelona, 2005.
2. Libor Běhounek. Two notions of fuzzy lattice completion. In Vilém Novák and Martin Štěpnička, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 22–24, Ostrava, 2005. University of Ostrava.
3. Libor Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159(10):1153–1164, 2008.
4. Libor Běhounek, Ulrich Bodenhofer, and Petr Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159(14):1729–1772, 2008.
5. Libor Běhounek, Ulrich Bodenhofer, Petr Cintula, and Susanne Saminger-Platz. Graded dominance. In Erich Peter Klement, S. E. Rodabaugh, and Larry N. Stout, editors, *Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology. Abstracts of the 29th Linz Seminar on Fuzzy Set Theory*, pages 11–14, Linz, 2008.
6. Libor Běhounek and Petr Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.
7. Libor Běhounek and Petr Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.
8. Libor Běhounek and Petr Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.
9. Libor Běhounek and Petr Cintula. Features of mathematical theories in formal fuzzy logic. In Patricia Melin, Oscar Castillo, Lluis T. Aguilar, Janusz Kacprzyk, and Witold Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, Berlin etc., 2007.
10. Libor Běhounek and Martina Daňková. Relational compositions in Fuzzy Class Theory. To appear in Fuzzy Sets and Systems (doi:10.1016/j.fss.2008.06.013), 2008.
11. Libor Běhounek and Tomáš Kroupa. Interior-based topology in Fuzzy Class Theory. In Martin Štěpnička, Vilém Novák, and Ulrich Bodenhofer, editors, *New Dimensions in Fuzzy Logic and Related Technologies: Proceedings of the 5th Eusflat Conference*, volume I, pages 145–151. University of Ostrava, 2007.
12. Libor Běhounek and Tomáš Kroupa. Topology in Fuzzy Class Theory: Basic notions. In Patricia Melin, Oscar Castillo, Lluis T. Aguilar, Janusz Kacprzyk, and Witold Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 513–522. Springer, Berlin etc., 2007.
13. Libor Běhounek and Tomáš Kroupa. Continuous relations over topological spaces in Fuzzy Class Theory. In Erich Peter Klement, S. E. Rodabaugh, and Larry N. Stout, editors, *Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology. Abstracts of the 29th Linz Seminar on Fuzzy Set Theory*, pages 15–18, Linz, 2008.
14. Petr Cintula and Rostislav Horčík. Fuzzy Class Theory: Some advanced topics. In Martin Štěpnička, Vilém Novák, and Ulrich Bodenhofer, editors, *New Dimensions in Fuzzy Logic and Related Technologies. Proceedings of the 5th EUSFLAT Conference*, volume I, pages 137–144, Ostrava, 2007. University of Ostrava.
15. Siegfried Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View*. Vieweg, Wiesbaden, 1993.
16. Rostislav Horčík. Solution of a system of linear equations with fuzzy numbers. *Fuzzy Sets and Systems*, 159(14):1788–1810, 2008.

17. Tomáš Kroupa. Towards formal theory of measure on clans of fuzzy sets. In Eduard Montseny and Pilar Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 351–356, Barcelona, 2005.
18. Tomáš Kroupa. Filters in Fuzzy Class Theory. *Fuzzy Sets and Systems*, 159(14):1773–1787, 2008.
19. Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

Characterizations of discrete Sugeno integrals as lattice polynomial functions

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Abstract. We survey recent characterizations of the class of lattice polynomial functions and of the subclass of discrete Sugeno integrals defined on bounded chains.

1 Introduction

We are interested in the so-called discrete Sugeno integral, which was introduced by Sugeno [9, 10] and widely investigated in aggregation theory, due to the many applications in fuzzy set theory, data fusion, decision making, pattern recognition, image analysis, etc. For general background, see [1, 6] and for a recent reference, see [5].

A convenient way to introduce the discrete Sugeno integral is via the concept of (lattice) polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. More precisely, given a bounded chain L, by an n-ary polynomial function we simply mean a function $f: L^n \to L$ defined recursively as follows:

(i) For each $i \in [n] = \{1, \ldots, n\}$ and each $c \in L$, the projection $\mathbf{x} \mapsto x_i$ and the constant function $\mathbf{x} \mapsto c$ are polynomial functions from L^n to L.
(ii) If f and g are polynomial functions from L^n to L, then $f \vee g$ and $f \wedge g$ are polynomial functions from L^n to L.
(iii) Any polynomial function from L^n to L is obtained by finitely many applications of the rules (i) and (ii).

As shown by Marichal [7], the discrete Sugeno integrals are exactly those polynomial functions $f: L^n \to L$ which are idempotent, that is, satisfying $f(x, \ldots, x) = x$.

In this paper, we are interested in defining this particular class of lattice polynomial functions by means of properties which appear naturally in aggregation theory. We start in §2 by introducing the basic notions needed in this paper and presenting general characterizations of lattice polynomial functions as obtained in Couceiro and Marichal [2, 3]. In §3, we particularize these characterizations to axiomatize the subclass of discrete Sugeno integrals.

2 Characterizations of polynomial functions

Let L be a bounded chain and let S be a nonempty subset of L. A function $f\colon L^n \to L$ is said to be

- *S-idempotent* if for every $c \in S$, $f(c, \ldots, c) = c$.
- *S-min homogenous* if $f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c$ for all $\mathbf{x} \in L^n$ and $c \in S$.
- *S-max homogenous* if $f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c$ for all $\mathbf{x} \in L^n$ and $c \in S$.
- *horizontally S-minitive* if $f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c)$ for all $\mathbf{x} \in L^n$ and $c \in S$, where $[\mathbf{x}]^c$ is the n-tuple whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise.

- *horizontally S-maxitive* if $f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c)$ for all $\mathbf{x} \in L^n$ and $c \in S$, where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise.
- *median decomposable* if $f(\mathbf{x}) = \mathrm{median}\big(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\big)$ for all $\mathbf{x} \in L^n$ and $k \in [n]$, where $\mathbf{x}_k^c = (x_1, \ldots, x_{k-1}, c, x_{k+1}, \ldots, x_n)$ for all $c \in L$.
- *strongly idempotent* if $f(x_1, \ldots, x_{k-1}, f(\mathbf{x}), x_{k+1}, \ldots, x_n) = f(\mathbf{x})$ for all $\mathbf{x} \in L^n$ and $k \in [n]$.

Two vectors $\mathbf{x}$ and $\mathbf{x}'$ in L^n are said to be *comonotonic*, denoted $\mathbf{x} \sim \mathbf{x}'$, if $(x_i - x_j)(x'_i - x'_j) \geqslant 0$ for every $i, j \in [n]$. A function $f \colon L^n \to L$ is said to be

- *comonotonic minitive* if $f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}')$ whenever $\mathbf{x} \sim \mathbf{x}'$.
- *comonotonic maxitive* if $f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}')$ whenever $\mathbf{x} \sim \mathbf{x}'$.

For integers $0 \leqslant p \leqslant q \leqslant n$, define

$$L_n^{(p,q)} = \{\mathbf{x} \in L^n \colon |\{x_1, \ldots, x_n\} \cap \{0,1\}| \geqslant p \text{ and } |\{x_1, \ldots, x_n\}| \leqslant q\}.$$

For instance, $(c,d,c) \in L_n^{(0,2)}$, $(0,c,d), (1,c,d) \in L_n^{(1,3)}$, but $(0,1,c,d) \notin L_n^{(0,2)} \cup L_n^{(1,3)}$.

Let S be a nonempty subset of L. We say that a function $f \colon L^n \to L$ is

- *weakly S-min homogenous* if $f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$.
- *weakly S-max homogenous* if $f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$.
- *weakly horizontally S-minitive* if $f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c)$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$, where $[\mathbf{x}]^c$ is the n-tuple whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise.
- *weakly horizontally S-maxitive* if $f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c)$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$, where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise.
- *weakly median decomposable* if $f(\mathbf{x}) = \mathrm{median}\big(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\big)$ for all $\mathbf{x} \in L_n^{(0,2)} \cup L_n^{(1,3)}$ and $k \in [n]$.

A subset S of a lattice L is said to be *convex* if for every $a, b \in S$ and every $c \in L$ such that $a \leqslant c \leqslant b$, we have $c \in S$. For any subset $S \subseteq L$, we denote by $\overline{S}$ the convex hull of S, that is, the smallest convex subset of L containing S. The *range* of a function $f \colon L^n \to L$ is defined by $\mathcal{R}_f = \{f(\mathbf{x}) : \mathbf{x} \in L^n\}$.

A function $f \colon L^n \to L$ is said to be *nondecreasing* (*in each variable*) if, for every $\mathbf{a}, \mathbf{b} \in L^n$ such that $\mathbf{a} \leqslant \mathbf{b}$, we have $f(\mathbf{a}) \leqslant f(\mathbf{b})$. Note that if f is nondecreasing, then $\overline{\mathcal{R}}_f = [f(\mathbf{0}), f(\mathbf{1})]$. We say that a function $f \colon L^n \to L$ has a *componentwise convex range* if, for every $\mathbf{a} \in L^n$ and $k \in [n]$, the function $x \mapsto f_{\mathbf{a}}^k(x) = f(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n)$ has a convex range.

Theorem 1. *Let $f \colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) *f is a polynomial function.*
(ii) *f is median decomposable.*
(ii-w) *f is nondecreasing and weakly median decomposable.*
(iii) *f is nondecreasing, strongly idempotent, has a componentwise convex range.*
(iv) *f is nondecreasing, $\overline{\mathcal{R}}_f$-min homogeneous, and $\overline{\mathcal{R}}_f$-max homogeneous.*
(iv-w) *f is nondecreasing, weakly $\overline{\mathcal{R}}_f$-min homogeneous, and weakly $\overline{\mathcal{R}}_f$-max homogeneous.*
(v) *f is nondecreasing, $\overline{\mathcal{R}}_f$-min homogeneous, and horizontally $\overline{\mathcal{R}}_f$-maxitive.*
(v-w) *f is nondecreasing, weakly $\overline{\mathcal{R}}_f$-min homogeneous, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.*
(vi) *f is nondecreasing, horizontally $\overline{\mathcal{R}}_f$-minitive, and $\overline{\mathcal{R}}_f$-max homogeneous.*
(vi-w) *f is nondecreasing, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and weakly $\overline{\mathcal{R}}_f$-max homogeneous.*
(vii) *f is nondecreasing, $\overline{\mathcal{R}}_f$-idempotent, horizontally $\overline{\mathcal{R}}_f$-minitive, and horizontally $\overline{\mathcal{R}}_f$-maxitive.*

(vii-w) *f is nondecreasing, $\overline{\mathcal{R}}_f$-idempotent, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.*

(viii) *f is $\overline{\mathcal{R}}_f$-min homogeneous and comonotonic maxitive.*

(viii-w) *f is weakly $\overline{\mathcal{R}}_f$-min homogeneous and comonotonic maxitive.*

(ix) *f is comonotonic minitive and $\overline{\mathcal{R}}_f$-max homogeneous.*

(ix-w) *f is comonotonic minitive and weakly $\overline{\mathcal{R}}_f$-max homogeneous.*

(x) *f is $\overline{\mathcal{R}}_f$-idempotent, horizontally $\overline{\mathcal{R}}_f$-minitive, and comonotonic maxitive.*

(x-w) *f is $\overline{\mathcal{R}}_f$-idempotent, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and comonotonic maxitive.*

(xi) *f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and horizontally $\overline{\mathcal{R}}_f$-maxitive.*

(xi-w) *f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.*

(xii) *f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and comonotonic maxitive.*

Remark 1. In the special case when L is a bounded real interval $[a,b]$, by requiring continuity in each of the conditions of Theorem 1, we can replace $\overline{\mathcal{R}}_f$ with $\mathcal{R}_f$ and remove componentwise convexity in (iii) of Theorem 1.

3 Characterizations of discrete Sugeno integrals

Recall that discrete Sugeno integrals are exactly those lattice polynomial functions which are idempotent. In fact, $\{0,1\}$-idempotency suffices to completely characterize this subclass of polynomial functions.

We say that a function $f: L^n \to L$ is

- *Boolean min homogeneous* if $f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c$ for all $\mathbf{x} \in \{0,1\}^n$ and $c \in L$.
- *Boolean max homogeneous* if $f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c$ for all $\mathbf{x} \in \{0,1\}^n$ and $c \in L$.

Theorem 2. *Let $f: L^n \to L$ be a function. The following conditions are equivalent:*

(i) *f is a discrete Sugeno integral.*

(ii) *f is $\{0,1\}$-idempotent and median decomposable.*

(ii-w) *f is nondecreasing, $\{0,1\}$-idempotent, and weakly median decomposable.*

(iii) *f is nondecreasing, $\{0,1\}$-idempotent, strongly idempotent, has a componentwise convex range.*

(iv) *f is nondecreasing, Boolean min homogeneous, and Boolean max homogeneous.*

(v) *f is nondecreasing, $\{1\}$-idempotent, L-min homogeneous, and horizontally L-maxitive.*

(v-w) *f is nondecreasing, $\{1\}$-idempotent, weakly L-min homogeneous, and weakly horizontally L-maxitive.*

(vi) *f is nondecreasing, $\{0\}$-idempotent, horizontally L-minitive, and L-max homogeneous.*

(vi-w) *f is nondecreasing, $\{0\}$-idempotent, weakly horizontally L-minitive, and weakly L-max homogeneous.*

(vii) *f is nondecreasing, L-idempotent, horizontally L-minitive, and horizontally L-maxitive.*

(vii-w) *f is nondecreasing, L-idempotent, weakly horizontally L-minitive, and weakly horizontally L-maxitive.*

(viii) *f is $\{1\}$-idempotent, L-min homogeneous, and comonotonic maxitive.*

(viii-w) *f is $\{1\}$-idempotent, weakly L-min homogeneous, and comonotonic maxitive.*

(ix) *f is $\{0\}$-idempotent, comonotonic minitive, and L-max homogeneous.*

(ix-w) *f is $\{0\}$-idempotent, comonotonic minitive, and weakly L-max homogeneous.*

(x) *f is L-idempotent, horizontally L-minitive, and comonotonic maxitive.*

(x-w) *f is L-idempotent, weakly horizontally L-minitive, and comonotonic maxitive.*
(xi) *f is L-idempotent, comonotonic minitive, and horizontally L-maxitive.*
(xi-w) *f is L-idempotent, comonotonic minitive, and weakly horizontally L-maxitive.*
(xii) *f is L-idempotent, comonotonic minitive, and comonotonic maxitive.*

Remark 2. (i) As in Remark 1, when L is a bounded real interval $[a,b]$, componentwise convexity can be replaced with continuity in (iii) of Theorem 2.
(ii) The characterizations given in (iv) and (xii) of Theorem 2 were previously established, in the case of real variables, by Marichal [8, §4.3]. The one given in $(viii)$ was established, also in the case of real variables, by de Campos and Bolaños [4] with the redundant assumption of nondecreasing monotonicity.

References

1. G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*. Studies in Fuziness and Soft Computing. Springer, Berlin, 2007.
2. M. Couceiro and J.-L. Marichal. Representations and characterizations of lattice polynomial functions. (`http://arxiv.org/abs/0808.2619`).
3. M. Couceiro and J.-L. Marichal. Representations and characterizations of polynomial functions on chains. In preparation.
4. L. M. de Campos and M. J. Bolaños. Characterization and comparison of Sugeno and Choquet integrals. *Fuzzy Sets and Systems*, 52(1):61–67, 1992.
5. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, UK, to appear in 2009.
6. M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy measures and integrals*, volume 40 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg, 2000. Theory and applications.
7. J.-L. Marichal. Weighted lattice polynomials. *Discrete Mathematics*. to appear. (`http://arxiv.org/abs/0706.0570`).
8. J.-L. Marichal. *Aggregation operators for multicriteria decision aid*. PhD thesis, Institute of Mathematics, University of Liège, Liège, Belgium, December 1998.
9. M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, Tokyo, 1974.
10. M. Sugeno. Fuzzy measures and fuzzy integrals—a survey. In M. M. Gupta, G. N. Saridis, and B. R. Gaines, editors, *Fuzzy automata and decision processes*, pages 89–102. North-Holland, New York, 1977.

Subset systems in lattice-valued mathematics

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A subset system $\mathcal{Z}$ is a class-theoretic function from the category **Pos** of partially ordered sets (posets for short) and order-preserving functions to itself such that $\mathcal{Z}$ assigns to each poset P a subset $\mathcal{Z}(P)$ of the power set $\mathcal{P}(P)$ of P, and each order-preserving function $f : P \to Q$ maps each element of $\mathcal{Z}(P)$ (the so-called a $\mathcal{Z}$-set of P) to a $\mathcal{Z}$-set of Q. This formally means that $\mathcal{Z}$:**Pos**→**Pos** is a functor given by the rule

$$\mathcal{Z}\left(P \stackrel{f}{\to} Q\right) = \mathcal{Z}(P) \stackrel{\mathcal{Z}(f)}{\to} \mathcal{Z}(Q),$$

where $\mathcal{Z}(P) \subseteq \mathcal{P}(P)$ and $\mathcal{Z}(f)(A) = f(A)$ for all $A \in \mathcal{Z}(P)$. Subset systems motivated from the issues in computer science, were first introduced by Wright, Wagner and Thatcher [21]. Since then an increasing number of studies have been done on subset system-based generalizations of several order-theoretical structures, e.g. $\mathcal{Z}$-Semicontinuous posets [17], $\mathcal{Z}$-continuous posets [9], $\mathcal{Z}$-algebraic posets [5, 16, 20], $\mathcal{Z}$-continuous algebras [3, 4, 15] and $\mathcal{Z}$-frames [22], despite the fact that there is no known use of subset systems in lattice-valued mathematics. $\mathcal{Z}$-join-complete posets and $\mathcal{Z}$-join-continuous maps, also called $\mathcal{Z}$-complete posets and $\mathcal{Z}$-continuous maps, are basic ingredients of those studies. By definition, for a subset system $\mathcal{Z}$, a $\mathcal{Z}$-join-complete poset is a poset P provided with the additional property that each $\mathcal{Z}$-set of P has a join in P. And a $\mathcal{Z}$-join-continuous map is an order-preserving function $f : P \to Q$ such that for each $A \in \mathcal{Z}(P)$ having a join $\bigvee A$ in P, the join $\bigvee f(A)$ of $f(A)$ in Q exists and the equality $f(\bigvee A) = \bigvee f(A)$ holds. In a dual manner, $\mathcal{Z}$-meet-complete posets and $\mathcal{Z}$-meet-continuous maps can be defined by simply replacing joins in the preceding definition with meets.

Conjunction of $\mathcal{Z}$-join-completeness and $\mathcal{Z}$-meet-completeness of a poset gives rise to the notion of $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset providing an order-theoretical fundament for the application of subset systems in the theory of lattice-valued maps. In addition to this aspect of $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets, they also unify various kinds of enriched posets, e.g. posets themselves, bounded posets, semi-lattices, lattices, complete lattices, directed-complete posets [1] (also called up-complete posets [10]), chain-complete posets [14]. In other words, the distinction of one of such enriched posets from others is just the matter of the choice of the subset system pair $(\mathcal{Z}_1, \mathcal{Z}_2)$. For example; if we consider the subset systems $\mathcal{Z}_L$, $\mathcal{Z}_\emptyset$, $\mathcal{Z}_{one-two}$, $\mathcal{Z}_{directed}$, $\mathcal{Z}_{chain}$ and $\mathcal{Z}_G$ defined in such a way that for each poset P, the set $\mathcal{Z}(P)$ of all $\mathcal{Z}$-sets of P corresponding to each of such subset systems consists of no subsets of P, only the empty set $\emptyset$, all one-or two-element subsets of P, all directed subsets of P, all chains in P and all subsets of P, then $(\mathcal{Z}_L, \mathcal{Z}_L)((\mathcal{Z}_\emptyset, \mathcal{Z}_\emptyset)$, $(\mathcal{Z}_L, \mathcal{Z}_{one-two})$, $(\mathcal{Z}_{one-two}, \mathcal{Z}_{one-two})$, $(\mathcal{Z}_{directed}, \mathcal{Z}_L)$, $(\mathcal{Z}_{chain}, \mathcal{Z}_L)$ and $(\mathcal{Z}_G, \mathcal{Z}_G)$, resp.)-complete posets are exactly posets (bounded posets, meet-semi-lattices, lattices, directed-complete posets, chain-complete posets and complete lattices, resp.). $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets-natural generalizations of $\mathcal{Z}$-join-complete posets propose a new formalism providing a parametrization of lattice-theoretic concepts in terms of subset system pairs

$(\mathcal{Z}_1, \mathcal{Z}_2)$, and they might be used in many application fields of lattice theory, e.g. in the theory of continuous lattices [10], in domain theory [1], in programming language semantics, data types, flow diagrams and modelling λ-calculus in computer science (see [11, 13, 15], and references therein).

Categorical consideration of $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets, which will be a subject of this talk, requires $(\mathcal{Z}_1, \mathcal{Z}_2)$-continuity of a map-a new concept combining $\mathcal{Z}_1$-join-continuity and $\mathcal{Z}_2$-meet-continuity of a map. There are many particular cases of $(\mathcal{Z}_1, \mathcal{Z}_2)$-continuous maps playing a central role for the applications of lattice theory in various areas. For instance; $(\mathcal{Z}_{one-two}, \mathcal{Z}_{one-two})((\mathcal{Z}_{directed}, \mathcal{Z}_L)$, $(\mathcal{Z}_{chain}, \mathcal{Z}_L)$ and $(\mathcal{Z}_G, \mathcal{Z}_G)$, resp.)-continuous maps are known as lattice-morphisms [2, 7] (Scott-continuous functions [1], chain-continuous functions [13, 14] and complete lattice-morphisms [2], resp.), and Scott-continuous functions are an essential tool of domain theory [1] and the theory of continuous lattices [10].

$(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets and $(\mathcal{Z}_1, \mathcal{Z}_2)$-continuous maps constitute a category which we will denote by $(\mathcal{Z}_1, \mathcal{Z}_2)$**-CPos**. The categories $(\mathcal{Z}_1, \mathcal{Z}_2)$**-CPos** provide a useful parametrization of various common categories in terms of subset system pairs $(\mathcal{Z}_1, \mathcal{Z}_2)$. E.g. the categories of posets and order-preserving functions [2], of meet-semi-lattices and meet-semi-lattice-morphisms, of lattices and lattice-morphisms [2], of directed-complete posets and Scott-continuous maps [1], of chain-complete posets and chain-continuous maps [14] and of complete lattices and complete lattice-morphisms [2] can be written as $(\mathcal{Z}_L, \mathcal{Z}_L)$**-CPos**, $(\mathcal{Z}_L, \mathcal{Z}_{one-two})$**-CPos**, $(\mathcal{Z}_{one-two}, \mathcal{Z}_{one-two})$**-CPos**, $(\mathcal{Z}_{directed}, \mathcal{Z}_L)$**-CPos**, $(\mathcal{Z}_{chain}, \mathcal{Z}_L)$**-CPos** and $(\mathcal{Z}_G, \mathcal{Z}_G)$**-CPos**.

The comparison of the categories $(\mathcal{Z}_1, \mathcal{Z}_2)$**-CPos** and $(\mathcal{Z}_3, \mathcal{Z}_4)$**-CPos** as the former being a subcategory of the latter induces a preorder (i.e. a reflexive and a transitive relation) $\preccurlyeq$ on the conglomerate of all subset system pairs $(\mathcal{Z}_1, \mathcal{Z}_2)$. The preorder $\preccurlyeq$ enables us to enlarge the category $(\mathcal{Z}_1, \mathcal{Z}_2)$**-CPos** to the category $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$**-CPos** comprising all $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets (as objects) and $(\mathcal{Z}_3, \mathcal{Z}_4)$-continuous maps (as morphisms) where $(\mathcal{Z}_3, \mathcal{Z}_4) \preccurlyeq (\mathcal{Z}_1, \mathcal{Z}_2)$. Most of familiar constructs in lattice-theory are in the form of $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$**-CPos**. In other words, the constructs $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$**-CPos** form a subset system-based catalogue of order-theoretical constructs in which objects are posets having joins of certain subsets and meets of certain subsets and morphisms are order-preserving functions that maintain existing joins and existing meets.

Substructures of various kinds of order-theoretical notions, e.g. subsemi-lattices, sublattices, σ-sublattices and complete sublattices [7, 8] can be catalogued by $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$-subposets defined as follows: For subset systems $\mathcal{Z}_i$ $(i = 1, ..., 4)$ with $(\mathcal{Z}_3, \mathcal{Z}_4) \preccurlyeq (\mathcal{Z}_1, \mathcal{Z}_2)$, a subposet P of a $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset Q is a $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$-subposet of Q iff it is $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete and the inclusion map $i : P \hookrightarrow Q$ is a $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$**-CPos**-morphism, i.e. a $(\mathcal{Z}_3, \mathcal{Z}_4)$-continuous map. The main reason for considering $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets in the domain of lattice-valued mathematics is the fact that for some particular subset systems $\mathcal{Z}_i$ $(i = 1, ..., 4)$ with $(\mathcal{Z}_3, \mathcal{Z}_4) \preccurlyeq (\mathcal{Z}_1, \mathcal{Z}_2)$, a $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset L and a set X, the X-th power L^X of L, which is an object of $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$**-CPos**, and $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$-subposets of L^X (e.g. lattice-valued topologies [18], fuzzy closure systems [6] and fuzzily structured sets [19]) have a central place in the theory of lattice-valued maps. Although our primary objective in this talk is to introduce $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets and their elementary aspects, we will particularly show how fuzzy topologies in Hutton's sense [12], lattice-valued topologies in the sense of [18], fuzzy closure systems [6] and fuzzily structured sets [19] can be exemplified as some $(\mathcal{Z}_1, ..., \mathcal{Z}_4)$-subposets of some $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets.

References

1. S. Abramsky and A. Jung, Domain theory, in: S. Abramsky, D. Gabbay and T. S. E. Maibaum (Eds.), Handbook of Logic in Computer Science, Oxford University Press, 1994, pp. 1–168.

2. J. Adámek, H. Herrlich and G. E. Strecker, Abstract and concrete categories, (John Wiley & Sons, New York, 1990).
3. J. Adámek, V. Koubek, E. Nelson and J. Reiterman, Arbitrarily large continuous algebras on one generator, Trans. Amer. Math. Soc. 291 (1985) 681-699.
4. J. Adámek, E. Nelson and J. Reiterman, Continuous algebras revisited, J. Comput. Syst. Sci. 51 (3) (1995) 460-471.
5. H. -J. Bandelt and M. Erné, The category of Z-continuous posets, J. Pure Appl. Algebra 30 (1983) 219-226.
6. L. Biacino and G. Gerla, An extension principle for closure operators, J. Math. Anal. Appl. 198 (1) (1996) 1-24.
7. G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloquium Publications, Third Ed., Amer. Math. Soc., RI, 1967.
8. T.S. Blyth, Lattices and ordered algebraic structures, Springer-Verlag, London, 2005.
9. M. Erné, Z-continuous posets and their topological manifestation, Appl. Cat. Structures 7 (1999) 31-70.
10. G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A compendium of continuous lattices, Springer-Verlag, Berlin, 1980.
11. J.A. Goguen, J.W. Thatcher, E.G. Wagner and J.B. Wright, Initial algebra semantics and continuous algebras, J. Assoc. Comput. Mach. 24 (1977) 68-95.
12. B. Hutton, Products of fuzzy topological spaces, Topology Appl. 11 (1980) 59-67.
13. G. Markowsky and B. K. Rosen, Bases for chain-complete posets, IBM J. Res. Dev. 20 (1976) 138-147.
14. G. Markowsky, Categories of chain-complete posets, Theor. Comput. Sci. 4 (1977) 125-135.
15. E. Nelson, Z-continuous algebras, in : B. Banaschewski and R. E. Hoffmann (Eds.), Continuous Lattices, Proc. Conf., Bremen 1979, Lect. Notes Math. 871, Springer-Verlag, Berlin, 1981, pp. 315-334.
16. D. Novak, Generalization of continuous posets, Trans. Amer. Math. Soc. 272 (1982) 645-667.
17. R.C. Powers and T. Riedel, Z-Semicontinuous posets, Order 20 (2003) 365-371.
18. S. E. Rodabaugh, Point-set lattice-theoretic topology, Fuzzy Sets Syst. 40 (2) (1991) 297-345.
19. V. B. Shteinbuk and A. P. Šostak, On semigroups of continuous transformations of fuzzily structured sets, Semigroup Forum 43 (2) (1991) 135-145.
20. G. Venugopalan, Z-continuous posets, Houston J. Math. 12 (1986) 275-294.
21. J. B. Wright, E.G. Wagner and J.W. Thatcher, A uniform approach to inductive posets and inductive closure, Theoret. Comput. Sci. 7 (1978) 57-77.
22. D. Zhao, On projective Z-frames, Can. Math. Bull. 40 (1) (1997) 39-46.

Lattice-valued topological systems

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1 Overview

Program semantics can be done in a topological manner, and in order to capture this behavior of program semantics and make topological notions more applicable to computer science and, in particular, to program semantics, Vickers [8] introduced the notion of topological systems and the category **TopSys** using the concept of a satisfaction relation. The surpriing fact, despite the intention and the name, that topological systems represent a fundamentally different kind of mathematics from topology follows from an important observation of [3] that **TopSys** has ground category **Set** $\times$ **Loc**, which is also the same ground category for **Loc**-**Top**, one of the topological categories for variable-basis topology studied in various forms in [5] and its references. More precisely, from this observation emerge several remarkable facts, the first and third of which come from [3]:

1. **TopSys** is not topological over its ground **Set** $\times$ **Loc**, and therefore is fundamentally different mathematics from topology: there is a lack of initial and final structures in **TopSys** even for forgetful-functor structured *singleton* sources and sinks, respectively.
2. Topological systems are strikingly different from topology in the "algebraic" behavior of the **TopSys** isomorphisms: a continuous mapping in **TopSys** is a homeomorphism if and only if it is an isomorphism in the ground **Set** $\times$ **Loc**, behavior typical of group homomorphisms but not of space homeomorphisms.
3. Despite (1) and (2), two very different embeddings exist of **TopSys** into **Loc**-**Top**, which are potentially different ways of trying to mitigate the difference between **TopSys** and topology and open the possibility of applying variable basis topology to topological systems and program semantics.

Motivated both by the aesthetics of replacing satisfaction relations by frame-valued satisfaction relations, as well as by potential applications in Section 7 addressing database queries and (3) above, this note defines for each frame L, a category L-**TopSys** of L-valued topological systems. This construction has two important properties. First, for $L = \mathbf{2}$, L-**TopSys** is isomorphic to **TopSys**. Second, for each frame L, L-**Top** embeds fully into L-**TopSys**. The key to this embedding is the "fuzzifying" of the satisfaction relation discovered through the rewriting of topological systems as frame maps. This note also gives the supercategory **Loc**-**TopSys** of all L-**TopSys**'s, and it is obtained that **Loc**-**Top** embeds into **Loc**-**TopSys**, an embedding with potential applications for constructing the initial and final structures of systems as *systems*.

2 Category TopSys

Background for this section comes from [3] and [8], from which we summarize crucial information.

Definition 21 *The* ground category $\mathbf{Set} \times \mathbf{Loc}$ *comprises as objects all ordered pairs* (X,A) *with* X *a set and* A *a locale, together with all morphisms* (f,φ) *with* f *being a set mapping and* φ *being a localic mapping (which means the* φ^{op} *in the opposite direction is the concrete frame map preserving arbitrary joins and finite meets); compositions and identities are taken component-wise.*

Definition 22 *A topological system is an ordered triple* $(X,A,\models)$*, where* $(X,A) \in |\mathbf{Set} \times \mathbf{Loc}|$ *and* $\models$ *is a satisfaction relation on* (X,A)*, meaning that* $\models$ *is a relation from* X *to* A *such that both the following join and meet interchange laws hold:*

$$\text{if } S \text{ is a subset of } A\text{, then } x \models \bigvee S \text{ iff } \exists a \in S,\, x \models a,$$

$$\text{if } S \text{ is a finite subset of } A\text{, then } x \models \bigvee S \text{ iff } \forall a \in S,\, x \models a.$$

Definition 23 *The category **TopSys** has ground category* $\mathbf{Set} \times \mathbf{Loc}$ *and comprises the following data:*

(1) Objects are all topological systems $(X,A,\models)$*, where* $\models$ *is a satisfaction relation on* (X,A).
(2) Morphisms are all continuous maps $(f,\varphi) : (X,A,\models_1) \to (Y,B,\models_2)$*, where* $(f,\varphi) : (X,A) \to (Y,B)$ *is a ground morphism satisfying the continuity condition that for all* $x \in X$ *and all* $b \in B$*,*

$$f(x) \models_2 b \text{ if and only if } x \models_1 \varphi^{op}(b).$$

Associated with each topological system $(X,A,\models)$ is the important function $Ext : A \to \wp(X)$, where

$$Ext(a) = \{x \in X : x \models a\},$$

important in part for reasons that emerge below. The conditions on $\models$ allow *Ext* to functorially generate topological spaces from topological systems.

Example 24 *Let* $(X,\mathcal{T})$ *be a topological space.* $(\mathrm{X},\mathcal{T},\models)$ *is a topological system where* $x \models U$ *iff* $x \in U$.

Example 25 *Applicability of the above notions to program semantics can be seen by several examples in [8], the gist of which is the following: given two systems* $(X,A,\models_1),(Y,B,\models_2)$*, the carrier set in the first [second] system represents a set of input [output] bitstreams, the locale in each system represents a collection of "open" predicates or properties, and the satisfaction relation in each system determines which bitstream satisfies which properties. A program having X as the input bitstreams and Y as the output bitstreams could be a continuous map as in the definition above, several pseudocode examples of which are given in [8]—e.g., the interchanging of 0's and 1's is a continuous map between topological systems.*

3 Topological systems as frame maps

It is well-known that for all sets X, Y, we have the bijection

$$\wp(X \times Y) \cong (\wp(Y))^X$$

and the underlying correspondences

$$R \subset X \times Y \mapsto r_R : X \to \wp(Y) \text{ by } r_R(x) = \{y \in Y : xRy\}$$

$$r : X \to \wp(Y) \mapsto R_r \subset X \times Y \text{ by } xRy \Leftrightarrow y \in r(x).$$

Let a relation $\models \subset X \times A$ be given. Then the first correspondence above yields

$$(X, A, \models) \mapsto \models \subset X \times A \mapsto \models^{-1} \subset A \times X \mapsto r_{\models^{-1}} : A \to \wp(X).$$

Now let mapping $r : A \to \wp(X)$ be given. Then the second correspondence yields

$$r : A \to \wp(X) \mapsto \models_r \subset X \times A \text{ by } x \models_r a \Leftrightarrow x \in r(a).$$

Proposition 31 *If $(X, A, \models)$ is a topological system, then $r_{\models^{-1}} : A \to \wp(X)$ is a frame map which coincides with Ext; and if $r : A \to \wp(X)$ is a frame map, then $(X, A, \models_r)$ is a topological system. Hence for ground object $(X, A) \in |$**Set** $\times$ **Loc**$|$, the family $\mathcal{S}(X, A)$ of all satisfaction relations on (X, A) is bijective with Frm$(A, \wp(X))$.*

Proposition 32 *Let $(X, A, \models_1), (Y, B, \models_2)$ be topological systems and $(f, \varphi) : (X, A) \to (Y, B)$ be a ground morphism. Then $(f, \varphi) : (X, A, \models_1) \to (Y, B, \models_2)$ is a continuous map if and only if the square*

$$Ext_1 \circ \varphi^{op} = f^{\leftarrow} \circ Ext_2$$

commutes.

The above considerations allow us to give a definition of a category which is isomorphic to **TopSys**, and thus, this definition is an alternative definition of **TopSys**.

Definition 33 (Alternative) *The category **TopSys** has ground category* **Set** $\times$ **Loc** *and comprises the following data:*

(1) Objects are all topological systems (X, A, r), where $r : A \to \wp(X)$ is a frame map.
(2) Morphisms are all continuous maps $(f, \varphi) : (X, A, r_1) \to (Y, B, r_2)$, where $(f, \varphi) : (X, A) \to (Y, B)$ is a ground morphism satisfying the continuity condition

$$r_1 \circ \varphi^{op} = f^{\leftarrow} \circ r_2.$$

4 Two equivalent definitions of L-TopSys

Definition 41 (Alternative) *Let $L \in |\mathbf{Frm}|$. The category L-**TopSys** has ground category* $\mathbf{Set} \times \mathbf{Loc}$ *and comprises the following data:*

(1) Objects are all L-topological systems (X, A, r), where $r : A \to L^X$ is a frame map.
(2) Morphisms are all L-continuous maps $(f, \varphi) : (X, A, r_1) \to (Y, B, r_2)$, where $(f, \varphi) : (X, A) \to (Y, B)$ is a ground morphism satisfying the continuity condition

$$r_1 \circ \varphi^{op} = f_L^{\leftarrow} \circ r_2,$$

where $f_L^{\leftarrow}$ is the Zadeh preimage operator given by $f_L^{\leftarrow}(v) = v \circ f$.

We now indicate how the "alternative" definition yields a definition of L-**TopSys** expressed in terms of satisfactions relations. The bijection $\wp(X \times Y) \cong \wp(Y)^X$ can be trivially rewritten as $\mathbf{2}^{X \times Y} \cong (\mathbf{2}^Y)^X$, and thus, the underlying correspondences trivially become

$$R \subset X \times Y \mapsto r_R : X \to \wp(Y) \text{ by } \left[\forall x, \forall y,\ \chi_{r_R(x)}(y) = \chi_R(x,y)\right],$$

$$r : X \to \wp(Y) \mapsto R_r \subset X \times Y \text{ by } \left[\forall x, \forall y,\ \chi_{R_r}(x,y) = \chi_{r(x)}(y)\right].$$

Proposition 42 *Let $L \in |\mathbf{Frm}|$. The following hold:*

(1) For sets X, Y, we have the bijection $L^{X \times Y} \cong (L^Y)^X$ with underlying correspondences

$$R : X \times Y \to L \mapsto r_R : X \to L^Y \text{ by } r_R(x)(y) = R(x,y)$$

$$r : X \to L^Y \mapsto R_r : X \times Y \to L \text{ by } R_r(x,y) = r(x)(y).$$

(2) Given mapping $R : X \times A \to L$, the mapping $r_{R^{-1}} : A \to L^X$ is a frame map if and only if both the following hold:

$$\text{if } S \text{ is a subset of } A\text{, then } R\left(x, \bigvee_{a \in S} a\right) = \bigvee_{a \in S} R(x,a),$$

$$\text{if } S \text{ is a finite subset of } A\text{, then } R\left(x, \bigwedge_{a \in S} a\right) = \bigwedge_{a \in S} R(x,a).$$

*(3) Given ground morphism $(f, \varphi) : (X, A) \to (Y, B)$, $(f, \varphi) : (X, A, r_1) \to (Y, B, r_2)$ is a continuous map in L-**TopSys** if and only if $\forall x \in X$, $\forall b \in B$,*

$$R_{r_1}^{-1}(x, \varphi^{op}(b)) = R_{r_2}^{-1}(f(x), b).$$

The above proposition ensures that the alternative definition given above yields a category isomorphic to the one given in the following two definitions.

Definition 43 *Let L be a frame. An L-topological system is an ordered triple $(X, A, \models)$, where $(X, A) \in |\mathbf{Set} \times \mathbf{Loc}|$ and $\models$ is an L-satisfaction relation on (X, A), meaning that $\models\ : X \times A \to L$ is a mapping satisfying both of the following join and meet interchange laws:*

$$\text{if } S \text{ is a subset of } A\text{, then } \models\left(x, \bigvee_{a \in S} a\right) = \bigvee_{a \in S} \models(x,a),$$

$$\text{if } S \text{ is a finite subset of } A\text{, then } \models\left(x, \bigwedge_{a \in S} a\right) = \bigwedge_{a \in S} \models(x,a).$$

Definition 44 *Let $L \in |\mathbf{Frm}|$. The category L-**TopSys** has ground category* **Set** $\times$ **Loc** *and comprises the following data:*

(1) Objects are all L-topological systems $(X, A, \models)$, where $\models : X \times A \to L$ is an L-satisfaction relation.
(2) Morphisms are all L-continuous maps $(f, \varphi) : (X, A, \models_1) \to (Y, B, \models_2)$, where $(f, \varphi) : (X, A) \to (Y, B)$ is a ground morphism satisfying the continuity condition that for all $x \in X$ and all $b \in B$,

$$\models_1 (x, \varphi^{op}(b)) = \models_2 (f(x), b).$$

We record formally that whichever definition we choose for L-**TopSys**, we have:

Theorem 45 $\forall L \in |\mathbf{Frm}|$, L-**TopSys** *is a category.*

We may regard each L-**TopSys** as a category for fixed-basis topological systems, in contrast with **Loc**-**TopSys** introduced later in this note.

5 Relationships of L-TopSys to TopSys and L-Top

A topological system $(X, A, \models)$ can naturally be thought of as a **2**-topological system $(X, A, \models_{\mathbf{2}})$ where $\models_{\mathbf{2}} : X \times A \to \mathbf{2}$ is defined by

$$\models_{\mathbf{2}} (x, a) = \begin{cases} \top, & x \models a \\ \bot, & \text{otherwise} \end{cases}.$$

From this observation we have:

Proposition 51 **TopSys** *and* **2**-**TopSys** *are isomorphic categories; and hence $\forall L \in |\mathbf{Frm}|$, **TopSys** embeds into L-**TopSys**.*

Definition 52 *Let $L \in |\mathbf{Frm}|$. The category L-**Top** has ground category* **Set** *and comprises the following data:*

(1) Objects are all L-topological spaces (X, τ), where $\tau \subset L^X$ is closed under arbitrary $\bigvee$ and finite $\wedge$.
(2) Morphisms are all L-continuous maps $f : (X, \tau) \to (Y, \sigma)$, where $f : X \to Y$ is a mapping satisfying the conditions that

$$\forall v \in \sigma, f_L^{\leftarrow}(v) \in \tau.$$

It is shown in [3] that each L-**Top** (with L a frame) embeds into **TopSys**, but the proof is quite nontrivial. In marked contrast, each L-**Top** embeds very easily into the corresponding L-**TopSys**, as seen in this next theorem. This suggests that the relationship between L-topology and L-topological systems is perhaps the more natural one.

Theorem 53 *Define $E_L : L$-**Top** $\to L$-**TopSys** on spaces by*

$$E_L(X, \tau) = (X, \tau, \models),$$

where

$$\models (x, u) = u(x);$$

and for a morphism $f : (X, \tau) \to (Y, \sigma)$, put $E_L(f) : (X, \tau, \models) \to (Y, \sigma, \models)$ by

$$E_L(f) = \left(f, \left((f_L^{\leftarrow})_{|\sigma}\right)^{op}\right),$$

where $(f_L^{\leftarrow})_{|\sigma}$ is the restriction of the Zadeh preimage operator to the L-topology σ of the codomain. Then E_L is a full embedding.

6 Category Loc-TopSys

Definition 61 *The category **Loc-Top** for variable-basis topology has ground category* **Set** $\times$ **Loc** *and comprises the following data:*

1. *Objects are all topological spaces* (X, L, τ)*, where* $(X, \tau) \in |L\text{-}\textbf{Top}|$.
2. *Morphisms are all continuous maps* $(f, \varphi) : (X, L, \tau) \to (Y, M, \sigma)$*, where* $(f, \varphi) : (X, L) \to (Y, M)$ *is a ground morphism satisfying the continuity condition that*

$$\forall v \in \sigma,\ (f, \varphi)^{\leftarrow}(v) \in \tau,$$

where $(f, \varphi)^{\leftarrow}$ *is the standard variable-basis preimage operator given by*

$$(f, \varphi)^{\leftarrow}(v) = \varphi^{op} \circ v \circ f.$$

It is well-known that **Loc-Top** is a supercategory for all the L-**Top**'s, though there are far more morphisms in **Loc-Top** than are available in the mere union of the respective morphisms classes of the L-**Top**'s. In a similar fashion we can construct a supercategory of all the L-**TopSys**'s in an analogous way and with analogous enrichment of the morphism classes, namely by constructing the category **Loc-TopSys**. Further, the embedding of **Loc-Top** into **Loc-TopSys** not only lifts all the embeddings constructed in the previous section of each L-**Top** into the corresponding L-**TopSys**, but has other benefits as well, one of which is explored below.

The construction **Loc-TopSys** requires a larger ground category, namely **Set** $\times$ **Loc** $\times$ **Loc**.

Definition 62 *The ground category* **Set** $\times$ **Loc** $\times$ **Loc** *comprises as objects all ordered triples* (X, L, A) *with* X *a set and* L, A *locales, together with all morphisms* (f, φ, ψ) *with* f *being a set mapping and* φ, ψ *being localic mappings (with* φ^{op}, ψ^{op} *as the frame maps); compositions and identities are taken component-wise.*

Definition 63 *The category **Loc-TopSys** has ground category* **Set** $\times$ **Loc** $\times$ **Loc** *and comprises the following data:*

(1) Objects are all topological systems $(X, L, A, \models)$*, where* $(X, A, \models) \in |L\text{-}\textbf{TopSys}|$ *as defined above.*
(2) Morphisms are all continuous maps $(f, \varphi, \psi) : (X, L, A, \models) \to (Y, M, B, \models)$*, where* $(f, \varphi, \psi) : (X, L, A) \to (Y, M, B)$ *is a ground morphism satisfying the continuity condition that for all* $x \in X$ *and all* $b \in B$*,*

$$\models_1 (x, \psi^{op}(b)) = \varphi^{op}(\models_2 (f(x), b)).$$

To sum up, **Loc-TopSys** is the category of variable-basis topological systems as objects and continuous maps between them as morphisms.

Theorem 64 *Define* E : **Loc-Top** $\to$ **Loc-TopSys** *by*

$$E(X, L, \tau) = (X, L, \tau, \models),$$

$$E(f, \varphi) = (f, \varphi, (f, \varphi)^{\leftarrow})$$

where $\models (x, u) = u(x)$. *Then* E *is a full embedding.*

As mentioned previously, each L-**Top** embeds into **Loc**-**Top**, and this is done as follows:

$$E_{L\text{-}T}(X,\tau) = (X,L,\tau), \quad E_{L\text{-}T}(f) = (f,id_L),$$

where id_L is taken in Loc. Analogously, each L-**TopSys** embeds into **Loc**-**TopSys**, and all the embeddings of this and the previous secton mix appropriately, as seen in the next theorem:

Theorem 65 *Let $L \in |\mathbf{Frm}|$. The following hold:*

(1) $E_{L\text{-}TS} : L$-**TopSys** $\rightarrow$ **Loc**-**TopSys**, *defined by*

$$E_{L\text{-}TS}(X,A,\models) = (X,L,A,\models), \quad (f,\psi) = (f,id_L,\psi),$$

is an embedding.

(2) $E \circ E_{L\text{-}T} = E_{L\text{-}TS} \circ E_L$.

7 Potential applications

Example 71 (database queries) *Let X be a set of database queries, i.e., the set of statements which are generated from a standard database query language for a given database, and let Y be the entities in the given database. The following considerations indcate potential applications of L-satisfaction relations:*

1. *For each query, an entity is returned. Thus, this process may be modeled by a set function $f : X \rightarrow Y$.*
2. *One may think of the queries as satisfying properties, where each property may be identified with the set of queries for which the property is true. Likewise, there are properties associated with the database entities. M. Smyth has made the case that collections of logically related properties may be thought of as topologies [7]. Thus, from Example 24, we may think of X and Y as underlying topological systems $(X,A,\models_1)$, where $\models_1$ relates each property in A to the queries for which the property holds, and $(Y,B,\models_2)$, where $\models_2$ relates each property in B to the database entities for which the property holds. As stated above, when we query the database, we want the answers to our queries to be given by a set function $f : X \rightarrow Y$.*
3. *Further, there is a function from B to A which respects the query function f. This function maps a property of a query answer, i.e., a property of a database entity, to a corresponding property of the query. This function acts like Dijkstra's (weakest precondition) predicate transformer [4]. We call this function φ^{op}, and $(f,\varphi) : (X,A,\models_1) \rightarrow (Y,B,\models_2)$ is a continuous map between topological systems.*
4. *We can use the continuous map (f,φ) as follows. Suppose that we want to formulate a query that will give us an answer with a certain property b. Each element of $Ext_{\models_1}(\varphi^{op}(b))$ is such a query.*
5. *Further, let us assume, for example, that our query language is based on a natural language. Thus, we may naturally introduce fuzziness into our topological system $(X,A,\models_1)$, and $(X,A,\models_1)$ becomes an L-topological system for an appropriate frame L. The frame L need not be complicated mathematically to be useful in a finite database; L could, for example, be a small finite lattice. It could be appropriate to also consider $(Y,B,\models_2)$ as an L-topological system and to require (f,φ) to be an L-continuous function.*

6. *See [1] for an example of a fuzzy natural language query language, and see [2] and [6] for fuzziness in the set of database entities.*

Example 72 (initial and final structures) *One importance of the embedding* E : **Loc-Top** $\rightarrow$ **Loc-TopSys** *of the previous section stems from the fact that it gives us a different—and possibly better—answer than those given in [3] to the dilemma of **TopSys** failing to be a topological category over* **Set** $\times$ **Loc** *(Section 1 above), and this indicates another potential application of L-satisfaction relations. The following comments should be made:*

1. ***Loc-Top** is topological over* **Set** $\times$ **Loc** *[5]. From [3] comes two very different embeddings* $F_{\models}$ *and* F_k *of **TopSys** into **Loc-Top**, which means that forgetful functor structured sources [sinks] lacking unique initial [final] lifts in **TopSys** may be taken over to **Loc-Top** and lifted to that category—this is because **TopSys** and **Loc-Top** have the same ground—in each of two ways.*
2. *The problem now is getting these lifts "back" to a system setting. There is the "combining" functor* F_C *from **Loc-Top** back to **TopSys** given in [3], but it fails to be an embedding, and the question arises whether we can embed the solutions given by* $F_{\models}$ *and* F_k *into a systems setting. There is indeed an embedding available from [3], but it is the "empty" embedding* $E_{\varnothing}$ *which takes spaces to systems with empty carrier sets, and therefore* $E_{\varnothing}$ *does not seem very promising for applications.*
3. *The embedding* E : **Loc-Top** $\rightarrow$ **Loc-TopSys** *gives a "systems" solution to this dilemma: as before, move the problematic sources [sinks] over to **Loc-Top** using* $F_{\models}$ *or* F_k*, take the unique initial [final] lifts in **Loc-Top**, but now move the sources [sinks] with their lifts over to **Loc-TopSys** using the fact that the ground* **Set** $\times$ **Loc** *for* **Loc-Top** *embeds into the ground* **Set** $\times$ **Loc** $\times$ **Loc** *for* **Loc-TopSys**.
4. *Given what is available currently, we can find a systems solution via embedding to the lack of initial and final structures in **TopSys** only by means of* E : **Loc-Top** $\rightarrow$ **Loc-TopSys**, *which uses L-satisfaction relations.*

References

1. R. Bergmair, U. Bodenhofer, *Syntax-driven analysis of context-free languages with respect to fuzzy relational semantics*, Proceedings of the 2006 IEEE World Congress on Computational Intelligence (WCCI 2006).
2. B. P. Buckles, F. E. Petry, *A fuzzy representation of data for relational databases*, Fuzzy Sets and Systems, 7(3) (1982) 213-226.
3. J. T. Denniston, S. E. Rodabaugh, *Functorial relationships between lattice-valued topology and topological systems*, Quaestiones Mathematicae (to appear).
4. E. W. Dijkstra, *A Discipline of Programming*, Prentice-Hall: Engelwood Cliffs, New Jersey (1976).
5. S. E. Rodabaugh, *Categorical foundations of variable-basis fuzzy topology*, pp. 273–388 in U. Höhle, S. E. Rodabaugh, eds, *Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory, The Handbooks of Fuzzy Sets Series*, Volume **3**(1999), Kluwer Academic Publishers (Boston/Dordrecht/London).
6. S. Shenoi, A. Melton, L. T. Fan, *An equivalence classes model of fuzzy relational databases*, Fuzzy Sets and Systems, 38 (1990) 153-170.
7. M. Smyth, *Powerdomains and predicate transformers: a topological view*, Automata, Languages, and Programming, Lecture Notes in Computer Science, 154 (1983) 662-675.
8. S. J. Vickers, *Topology Via Logic*, Cambridge University Press (Cambridge, 1989).

Fuzziness, uncertainty and bipolarity: a critical review

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1 Introduction

Fuzzy set theory and its extensions seem to come to grips with three basic concepts that noticeably differ from each other but may appear conjointly in various circumstances:

1. **Gradualness**: the idea that many categories in natural language are a matter of degree, including truth. The extension of a *gradual* predicate is a fuzzy set, a set where the transition between membership and non-membership is gradual rather than abrupt. This is Zadeh's original intuition [29].
2. **Epistemic Uncertainty**: the idea of partial or *incomplete* information. In its most primitive form, it is often described by means of a set of possible values of some quantity of interest, *one and only one* of which is the right one. This is possibility theory [31].
3. **Bipolarity** : the idea that information can be described by distinguishing between *positive* and *negative* sides, possibly handled separately [16], as it seems to be the case in the human brain [2].

It is clear that the three notions interact closely with one another: truth may appear to be a bipolar notion as it goes along with falsity. A fuzzy set may account for epistemic uncertainty since it extends the notion of a set. Epistemic uncertainty is gradual since belief is often a matter of degree.

As a result the epistemological situation of fuzzy set theory and its extensions : interval-valued or type 2 fuzzy sets and variants, Atanassov orthopairs (membership-nonmembership), etc., is often unclear and leads to a lot of confusion in the literature:

- Controversies on the nature of vagueness between those who consider it basically as a matter of uncertainty of meaning, and those who incriminate the clash between the gradualness of categories and their Boolean-like use in natural language as the main source of vagueness (let alone the limited perception of shades by the human mind)[28]
- The existence of several views of membership functions: expressing similarity to prototypes, gradual incomplete information (possibility distributions), or utility functions in preference modeling [13].
- The temptation of interpreting any gradual notion as a membership function, even if *it does not refer to a set* (for instance the ambiguous notion of fuzzy number)[19].
- The confusion between gradual truth and gradual uncertainty (e.g. [18]), which turns out to be a variant of the former whereby sets (of truth-values, especially singletons) are confused with elements [14, 8].
- The confusion between membership functions and probability distributions [22]
- The confusion between a separate handling of positive and negative information, and uncertainty about truth-degrees [1].

The aim of this talk is to clarify some of these issues, by disentangling the three notions of gradualness, epistemic uncertainty and bipolarity, laying bare the precise role of fuzzy sets in connection with such notions, thus clarifying potential uses of membership functions, and existing extensions thereof, such as interval-valued fuzzy sets (IVFSs) , type 2 fuzzy sets and Atanassov orthopairs of fuzzy sets.

The following points are discussed in the talk:

1. The distinction between fuzzy sets representing some objective entity consisting of a weighted collection of items, and fuzzy sets that may model incomplete information and whose membership functions are interpreted as possibility distributions. The first kind is called *conjunctive fuzzy set* and the second one *disjunctive fuzzy set* [12]. This distinction leads to distinct extensions of mathematical notions to fuzzy sets: for instance, scalar distance between fuzzy sets vs. fuzzy-set-valued distance between underlying ill-known precise entities. When defining the variance of a fuzzy random variable, one may be interested in the variability of a membership function, or the range of variability of an ill-known random variable [3].
2. The distinction between a fuzzy set and a *gradual element* [17]. The latter is viewed as a selection function of the one-to-many mapping representing the nested set of cuts of a fuzzy set (i.e. picking one element in each cut). It does not represent a membership function, but a parameterized element. It seems that the notion of fuzzy real number (in contrast with fuzzy numbers understood as generalized real intervals) in lattice-valued topology as studied by Hutton [21] and other fuzzy topologists [23, 20, 27] is closer to the notion of gradual real number than to a fuzzy set. However operations between such fuzzy reals are based on the extension principle which is in agreement with interval computations. It explains why the fuzzy real line is not a group under addition. Using gradual reals allow to embed fuzzy real numbers into a group structure under addition.
3. The confusion between truth values and epistemic states in logic leads to an anomalous use of a truth-functional 3-valued logic as tool for handling uncertainty, by adding the ignorance state to the truth-values true and false [4, 14]. It comes down again to confusing elements and sets since epistemic states are disjunctive sets of truth-values, not truth-values [8]. Knowing that a proposition is true comes down to the singleton containing "true", while ignorance is modelled by the whole truth set, whose characteristic function is viewed as a possibility distribution. The same confusion can be observed with interval-valued fuzzy sets [30], their later variants (like Atanassov membership/non membership pairs [1]), and the type 2 fuzzy sets popular in engineering [24]. Namely, intervals of membership grades are interpreted as full-fledged truth-values to which a truth-functional calculus is applied, consisting of standard fuzzy set connectives extended with interval arithmetic [6]. This truth-functional calculus is a poor approximation of the right tool for reasoning about ill-known truth-values [4], which should be based on constraint propagation and respect the properties of the underlying fuzzy logic [9].
4. The notion of *bipolar fuzzy set* (distinguishing between and separately handling membership and non-membership degrees) is carefully distinguished from the idea of ill-known membership function. A possibility distribution encodes negative information (pointing out impossible values). Then positive information can be modelled by a second (lower) distribution (pointing out actually possible values). Such pairs of nested distributions formally come close to using interval-valued fuzzy sets[11]. They may also account for positive and negative preference, corresponding to goals and constraints respectively [15]. However while interval-valued fuzzy sets represent *less* information than a single membership function, the use of two membership functions in the bipolar possibilistic setting represents *more* information than a single one. The (upper) possibility distribution accounts for all epistemic states compatible with it (modelled by all included normal fuzzy

subsets), and the second (lower) distribution acts as a lower bound eliminating some of them. Similarly a *cloud* in the sense of Neumaier [25] can be viewed as two possibility distributions globally restricting a family of probability functions that is smaller than the family restricted by a single distribution [7].

5. It has been repeatedly said that the intuitionistic nature of the Atanassov system is dubious, because this structure is isomorphic to algebras of IVFSs [10]. Interpreting Atanassov orthopairs in terms of ill-known membership functions (as suggested by many authors) inherits the same flaw as the truth-functional calculus of interval-valued fuzzy sets. Taking Atanassov pairs of membership and non-membership functions as bipolar information leads to viewing such pairs as more informative than single membership functions, contrary to the IVFS-based ill-known fuzzy set view. The corresponding bipolar extensions of fuzzy set-theoretic operations on orthopairs are at odds from the IVFS-like connectives that Atanassov proposed, i.e. cannot be interpreted as interval arithmetics.

References

1. K. Atanassov, *Intuitionistic Fuzzy Sets*, Physica-Verlag, Heidelberg, 1999.
2. J. T. Cacioppo, W. L. Gardner, and G. G. Berntson. Beyond bipolar conceptualizations and measures: The case of attitudes and evaluative space. *Personality and Social Psychology Review*, 1:3– 25, 1997.
3. I. Couso, D. Dubois, S. Montes, L. Sanchez. On various definitions of the variance of a fuzzy random variable. *Proc.International Symposium on Imprecise Probability: Theories and Applications* (ISIPTA 2007), Prague (Czech Republic), 2007, Sipta, p. 135-144, 2007.
4. G. De Cooman From possibilistic information to Kleene's strong multi-valued logics. In *Fuzzy Sets, Logics and Reasoning about Knowledge* (D. Dubois et al., Eds.) Kluwer Academic Publishers (1999).
5. G. Deschrijver, E. Kerre, On the relationship between some extensions of fuzzy set theory, *Fuzzy Sets and Systems*, 133, 227-235, 2004.
6. G. Deschrijver, E. Kerre, Advances and challenges in interval-valued fuzzy logic, *Fuzzy Sets & Syst.*, 157, 622-627, 2006.
7. S. Destercke, D. Dubois, E. Chojnacki Unifying practical uncertainty representations. II: Clouds, *International Journal of Approximate Reasoning*, 49, 2008, 664-677
8. D. Dubois: On ignorance and contradiction considered as truth-values *Logic Journal of the IGPL* 16(2): 195-216 (2008)
9. D. Dubois. On degrees of truth, partial ignorance and contradiction. *Proc International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems* (IPMU 2008), Mlaga, Spain, p. 31-38, 2008.
10. D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, H. Prade. Terminological difficulties in fuzzy set theory - The case of intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 156, p. 485-491, 2005.
11. D. Dubois, P. Hajek, and H. Prade. Knowledge-driven versus data-driven logics. *J. of Logic, Language, and Information*, 9:65–89, 2000.
12. D. Dubois and H. Prade. On incomplete conjunctive information, *Computers and Mathematics with Applications*, 15(10), 797-810, 1988.
13. D. Dubois, H. Prade: The three semantics of fuzzy sets. *Fuzzy Sets and Systems*, 90, 141-150, 1997.
14. D. Dubois and H. Prade. Possibility theory, probability theory and multiple-valued logics: A clarification. *Ann. Math. and Artificial Intelligence*, 32:35–66, 2001.
15. D. Dubois and H. Prade. On the use of aggregation operations in information fusion processes. *Fuzzy Sets and Systems* 142,143-161, 2004.
16. D. Dubois and H. Prade. An introduction to bipolar representations of information and preference. *Int. J. Intell. Syst.* 23(8): 866-877, 2008.
17. D. Dubois and H. Prade. Gradual elements in a fuzzy set. *Soft Computing*, 12, 165-175, 2008.
18. Ch. Elkan (1993) The paradoxical success of fuzzy logic. In Proc. AAAI'93, Washington, DC, 698-703. Extended (with discussions), *IEEE Expert*, 9(4), 2-49 (1994).
19. J. Fortin D. Dubois H. Fargier, Gradual numbers and their application to fuzzy interval analysis, *IEEE trans. Fuzzy Systems*, 16, 388-402, 2008
20. U. Höhle, Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic, *Fuzzy Sets and Systems*, 24, 1987, 263-278.
21. B. Hutton, Normality in fuzzy topological spaces. *J. Math. Anal. Appl.* 50, 74-79, 1975.

22. B. Kosko, Fuzzy entropy and conditioning. *Information Sciences* 40(2): 165-174, 1986.
23. R. Lowen, *Fuzzy Set Theory*, Ch. 5 Fuzzy real numbers, Kluwer, Dordrecht, 1996.
24. J. M. Mendel: Advances in type-2 fuzzy sets and systems. *Information Sciences* 177(1): 84-110, 2007.
25. A. Neumaier, Clouds, fuzzy sets and probability intervals, *Reliable Comput.* 10, 249-272, 2004.
26. C. E. Osgood, G.J. Suci, and P. H. Tannenbaum. *The Measurement of Meaning*. Univ. of Illinois Press, Chicago, 1957.
27. S. E. Rodabaugh, Fuzzy addition in the L-fuzzy real line, *Fuzzy Sets and Systems*, 8, 39–51, 1982.
28. T. Williamson, *Vagueness* Routledge, London 1994.
29. L.A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.
30. L. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part I, *Information Sciences*, 8, 199-249, 1975.
31. L.A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1, 3–28, 1978.
32. L.A. Zadeh. L. Zadeh, PRUF: A meaning representation language for natural languages.*Int. J. of Man-Machine Studies*, 10, 395-460, 1978.

Introducing fuzziness in monads: cases of the powerobject monad and the term monad

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Fuzzy mathematics often starts by taking a piece of classical mathematics and introducing fuzziness to existing mathematical concepts, and then proceeds to a more essential adoption of a fuzzy perspective. Our paper explores this progression for two important monads: the powerset monad and its generalizations to fuzzy powerobjects, and the term monad and its generalization to using fuzzy sets of constants. This brings together two lines of research previously discussed at Linz.The powerset and its fuzzy analogs are important in the development of topology in a fuzzy world and the term monad and its fuzzy analogs are vital in understanding fuzzy computer science.

We work with fuzzy sets with values in a completely distributive lattice L equipped with a semigroup operator $\star$ which distributes over both $\bigwedge$ and $\bigvee$. Such a lattice will have residuation for both $\star$ and $\wedge$, we write $a \to -$ for the right adjoint to $a \star -$ and $a \Rightarrow -$ for the right adjoint to $a \wedge -$. This generalizes the setting of working over the unit interval with a continuous t-norm.

1 Categories of increasing fuzziness

Most of mathematics is done in the category Set whose objects are sets and morphisms are mappings (functions). One step in fuzzifying this is to replace subsets of A with functions from A to L. This still lives in the category Set: L^A is an object of Set and L-sets $\alpha\colon A \to L$ are just elements of L^A. To increase incorporate fuzziness from the start we can work in the category $\mathrm{Set}(L)$ introduced by Goguen in [6]. Others have worked in categories allowing multiple lattices or in categories generalizing the category of sets with relations instead of functions.

The category $\mathrm{Set}(L)$ has as objects pairs (A,α) where $\alpha\colon A \to L$ and as morphisms $f : (A,\alpha) \to (B,\beta)$ mappings $f\colon A \to B$ such that $\beta(f(a) \geq \alpha(a)$ for all $a \in A$.

It is known that $\mathrm{Set}(L)$ is topological over Set and has a monoidal structure using $(A,\alpha)\otimes(B,\beta) = (A \times B, \alpha \star \beta)$ and products using $(A,\alpha)\times(B,\beta) = (A\times B, \alpha \wedge \beta)$. Pultr [8] showed how to get exponentials for both. Because $\mathrm{Set}(L)$ is topological, it has all limits and colimits [1]. The category $\mathrm{Set}(L)$ is also a quasitopos, but not a topos. As pointed out in [9] the logic studied in fuzzy set theory is the logic of unbalanced subobjects (those with underlying map the identity) rather than the logic in the quasitopos structure. This observation informs our choice of fuzzy powerobject functor.

There is a lifting functor $\mathbf{C}\colon \mathrm{Set} \to \mathrm{Set}(L)$ taking A to the crisp fuzzy set $(A,\top)$. On functions this functor is the identity.

There are three natural functors from $\mathrm{Set}(L)$ to Set to consider:

1. the underlying set functor $\mathbf{U}$ taking (A,α) to A and $f : (A,\alpha) \to (B,\beta)$ to f

2. the full members functor **F** taking (A,α) to $\{a|\alpha(a)=\top\}$. A function $f:(A,\alpha)\to(B,\beta)$ takes full members to full members, so the action of this functor on maps is restriction to the full members.
3. the support functor **S** taking (A,α) to $\{a|\alpha(a)>\bot\}$. Again maps $f:(A,\alpha)\to(B,\beta)$ restrict to functions on the supports since we have $\beta(f(a))\geq\alpha(a)>\bot$.

We get $\mathbf{UC}=\mathbf{FC}=\mathbf{SC}=\text{id}$ and $\mathbf{U}\dashv\mathbf{C}\dashv\mathbf{F}$.

2 Monads with increasing fuzziness

2.1 Powerobject monads

As mentioned in Section 1, it is possible to replace a set A by L^A and still work in Set. The powerset monad $\mathbf{L}_{id}=(L_{id},\eta,\mu)$, from Manes [7], can be seen as the first step to introduce fuzziness in a categorical setting: The covariant powerset functor $L_{id}\colon\mathsf{Set}\to\mathsf{Set}$ is obtained by $L_{id}A=L^A$, and, following [5], for a morphism $f\colon A\to B$ in Set, by defining for all $y\in B$, $L_{id}f(\alpha)(y)=\bigvee_{f(x)=y}\alpha(x)$. The natural transformations $\eta_A\colon A\Longrightarrow L_{id}A$ by $\eta_A(x)(x')=\top$ if $x=x'$ and $\bot$ otherwise, and μ_A : $L_{id}L_{id}A\Longrightarrow L_{id}A$ by $\mu_A(\mathcal{A})(x)=\bigvee_{\beta\in L_{id}A}\alpha(x)\wedge\mathcal{A}(\beta)$. In [7] it was shown that $\mathbf{L}_{id}=(L_{id},\eta,\mu)$ is a monad.

Another variant of this monad uses a $\star$ in the definition of μ:

$$\mu^\star_X(\mathcal{A})(x)=\bigvee_{A\in L_{id}X}A(x)\star\mathcal{A}(A)$$

This can be made more fully fuzzy by considering the internal fuzzy powerobject functor from $\mathsf{Set}(L)$ to $\mathsf{Set}(L)$ which has action on objects given by

$$\mathcal{U}^\star(A,\alpha)=(L^A,\pi_{(A,\alpha)})\text{ where }\pi_{(A,\alpha)}(f)=\bigwedge_{a\in A}(f(a)\to\alpha(a))$$

Notice that one of the options for $\star$ is $\wedge$, in which case we write $\Rightarrow$ for the residuation and we get

$$\mathcal{U}^\wedge(A,\alpha)=(L^A,\xi_{(A,\alpha)})\text{ where }\xi_{(A,\alpha)}(f)=\bigwedge_{a\in A}(f(a)\Rightarrow\alpha(a))$$

There are three functors $\mathcal{U}^\star\colon\mathsf{Set}(L)\to\mathsf{Set}(L)$ giving unbalanced powerobjects as objects of $\mathsf{Set}(L)$: one contravariant (inverse image) and its covariant right adjoint and (corresponding to direct image) left adjoint taking $f:(A,\alpha)\to(B,\beta)$ to $\exists_f$ where

$$\exists_f(A,\alpha')(b)=\bigvee_{f(a)=b}\alpha'(a)$$

The covariant internal unbalanced powerobject monad uses the functor $\mathcal{U}^\star$ with $\exists_f$ for its action on maps.

The monadic structure comes from

$$\eta_{(A,\alpha)}:(A,\alpha)\Longrightarrow\mathcal{U}^\star(A,\alpha)$$

where

$$\eta_{(A,\alpha)}(a)(t)=\begin{cases}\top & \text{if } t=a\\ \bot & \text{otherwise}\end{cases}$$

Notice that the degree of membership of $\eta_{(A,\alpha)}(a)$ in $\mathcal{U}(A,\alpha)$ is

$$\bigwedge_t (\eta_{(A,\alpha)}(a)(t) \to \alpha(t)) = \top \to \alpha(a) = \alpha(a)$$

so $\eta_{(A,\alpha)}$ is a map in $\mathsf{Set}(L)$.

The union is given by the natural transformation

$$\mu^\star_{(A,\alpha)}(\mathcal{U}^\star)^2(A,\alpha) \Longrightarrow \mathcal{U}^\star(A,\alpha)$$

with

$$\mu^\star_A(L^A,\tau)(a) = \bigvee_f (\tau(f) \star f(a))$$

Proposition 1. *$(\mathcal{U}^\star, \eta, \mu^\star)$ is a monad.*

We can use the crisp functor to relate these monads on Set and $\mathsf{Set}(L)$. We get $\mathbf{C}(\nu_A) = \eta_{\mathbf{C}(A)}$ and $\mu^\star\colon\, \mathcal{U}^\star(\mathcal{U}^\star(\mathbf{C}(A)) = \mathbf{C}\left(L^{L^A}\right) \to \mathcal{U}^\star(\mathbf{C}(A)) = \mathbf{C}\left(L^A\right)$ is $\mathbf{C}(\mu^\star : L^{L^A} \to L^A)$.

2.2 Term monads

It is useful to adopt a more functorial presentation of the set of terms, as opposed to using the conventional inductive definition of terms, where we bind ourselves to certain styles of proofs. The term monad over Set is used, for example, in [4, 5]. In [7] it was shown that $\mathbf{T}_\Omega = (T_\Omega, \eta^{T_\Omega}, \mu^{T_\Omega})$ is a monad.

The first step to generalize term monad to fuzzy terms was to compose the monads $\mathbf{L}_{id}$ and $\mathbf{T}_\Omega$. This requires a distributive law in the form of a natural transformation $\sigma : \mathbf{T}_\Omega\mathbf{L}_{id} \Longrightarrow \mathbf{L}_{id}\mathbf{T}_\Omega$.

To introduce the term monad in $\mathsf{Set}(L)$ we set $id^0(A,\alpha) = (\{\varnothing\},\top)$ and $id^n(A,\alpha) = (id^nA, id^n(\alpha)$, where $id^n(\alpha)(a_1,\ldots,a_n) = \bigwedge_{i=1}^n \alpha(a_i)$. When we need the monoidal structure we replace $\wedge$ by $\star$. A constant $\mathsf{Set}(L)$- covariant functor $(A,\alpha)_{\mathsf{Set}(L)}$ assings any (X,ξ) to (A,α) and all morphisms $f\colon (X,\xi) \to (Y,\upsilon)$ to the identity morphism $id_{(A,\alpha)}$. If $\{(A_i,\alpha_i) \mid i \in I\}$ is a family of L-sets then the coproduct is $\bigsqcup_{i\in I}(A_i,\alpha_i)$.

Let k be a cardinal number and $\{(\Omega_n,\vartheta_n) \mid n \leq k\}$ be a family of L-sets. We have

$$\bigsqcup_{n\leq k}(\Omega_n,\vartheta_n)_{\mathsf{Set}(L)} \times id^n(X,\xi) = \Big(\bigcup_{n\leq k}\{n\} \times \Omega_n \times id^nX, \alpha\Big), \tag{1}$$

where $\alpha(n,\omega,(x)_{i\leq n}) = \vartheta_n(\omega) \wedge id^n(\xi)((x_i)_{i\leq n})$, $\omega \in \Omega_n$ and $(x_i)_{i\leq n} \in X^n$.

Consider $(\Omega,\vartheta) = \bigsqcup_{n\leq k}(\Omega_n,\vartheta_n)$ as a fuzzy operator domain. The term functor over $\mathsf{Set}(L)$ can now be defined by transfinite induction. Let $T^0_{(\Omega,\vartheta)} = id$ and $T^1(X,\xi)$ be the right side of the equation 1. Define

$$T^\iota_{(\Omega,\vartheta)}(X,\xi) = \bigsqcup_{n\leq k}(\Omega_n,\vartheta_n)_{\mathsf{Set}(L)} \times id^n \bigvee_{0<\kappa<\iota} T^\kappa_{(\Omega,\vartheta)}(X,\xi)$$

for each positive ordinal ι. Finally, let $T_{(\Omega,\vartheta)}(X,\xi) = \bigsqcup\{T^0(X,\xi), \bigvee_{0<\iota<\bar{k}} T^\iota_{(\Omega,\vartheta)}(X,\xi)\}$, where $\bar{k}$ is the least cardinal greater than k and $\aleph_0$. Notice that $T^\iota_{(\Omega,\vartheta)}, T_{(\Omega,\vartheta)}\colon \mathsf{Set}(L) \to \mathsf{Set}(L)$ and $\bigvee_{0<\iota<\bar{k}} T^\iota_{(\Omega,\vartheta)}(X,\xi)$ denotes the colimit for the family $\{T^\iota_{(\Omega,\vartheta)}(X,\xi) \mid 0 < \iota < \bar{k}\}$.

Lemma 1. *For each positive ordinal there exists a unique α_ι, such that $T^\iota_{(\Omega,\vartheta)}(X,\xi) = (T^\iota_\Omega X, \alpha_\iota)$, and there exists a unique α such that $T_{(\Omega,\vartheta)}(X,\xi) = (T_\Omega X, \alpha)$.*

Using Lemma 1 we can easily see that $T_{(\Omega,\vartheta)}$ indeed is a functor. Further, we can extend the functor to a monad, since $T_{(\Omega,\vartheta)}$ can be shown to be idempotent. Once we have the fuzzy term monad on $\mathsf{Set}(L)$ we consider the composition with the unbalanced powerobject monad by constructing a natural transformation $\sigma^\star : \mathbf{T}_{(\Omega,\vartheta)}\mathcal{U}^\star \Longrightarrow \mathcal{U}^\star\mathbf{T}_{(\Omega,\vartheta)}$ as a prerequisite for the monad composition of $\mathcal{U}^\star$ and $\mathbf{T}_{(\Omega,\vartheta)}$ using distributive laws [2].

This understanding of the term monad in a fuzzy setting gives a start to a well founded non-classical logic programming. What remains is a similar understanding of unification.

References

1. J. Ádamek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, John Wiley & Sons, Inc., 1990.
2. J. Beck. Distributive laws. *Seminars on Triples and Categorical Homology Theory, 1966/67*. Lecture Notes in Mathematics 80, Springer-Verlag, 1969, pp. 119 – 140.
3. P. Eklund, W. Gähler. Fuzzy filter functors and convergence. In S. E. Rodabaugh, E. P. Klement, U. Höhle (eds.) *Applications of category theory to fuzzy subsets*. Kluwer Academic, 1992, 109 – 136.
4. P. Eklund, M. A. Galán, J. Medina, M. Ojeda Aciego, A. Valverde, *Powersets of terms and composite monads*, Fuzzy Sets and Systems, **158** (2007), 2552-2574.
5. W. Gähler. Monads and convergence. *Proc. Conference Generalized Functions, Convergences Structures, and Their Applications*, Dubrovnik (Yugoslavia) 1987. Plenum Press, New York, 1988, pp. 29 – 46.
6. J. A. Goguen. *L*-fuzzy sets. *Journal of Mathematical Analysis and Applications*, 18:145–174, 1967.
7. E. Manes. *Algebraic Theories*. Springer, Berlin, 1976.
8. A. Pultr. Closed categories and *L*-fuzzy sets. In *Vortrage zur Automaten und Algoritmentheorie*. Technische Universität Dresden, 1975.
9. L. N. Stout The logic of unbalanced subobject in a category of two closed sturctures. In S. E. Rodabaugh, E. P. Klement, U. Höhle (eds.) *Applications of Category Theory to Fuzzy Subsets*. Kluwer Academic, 1992, pp. 74 – 106.

On generalized continuous and left-continuous t-norms over chains: a survey

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Continuous t-norms, as operations over [0,1], have been deeply studied (see [16, 3] for general surveys). The prominent examples are Łukasiewicz ($x *_L y = \max(0, x+y-1)$), Minimum and product t-norms and the decomposition theorem for continuous t-norm states, roughly speaking, that any continuous t-norm is an ordinal sum of these three basic t-norms [20, 17]. In the last decade, the case of left-continuous t-norms (very relevant from the logical point of view) has also deserved increasing attention, see for example [6, 9–14, 18, 19, 22]. Unfortunately, in this latter case we are still lacking a decomposition theorem as in the case for continuous t-norms which make things much harder and only partial results are known.

From a logical point of view, there have also been many interesting advances in the recent past regarding fuzzy logical systems based on t-norms. Namely, the logic of continuous t-norms and their resiuda, called BL, introduced in [8], and the logic of left continuous t-norm and their residua, called MTL (Monoidal t-norm based logic) [7], have been introduced and deeply studied. Both logics, as well as all their axiomatic extensions, are complete with respect to the class of linearly ordered algebras (chains) of their corresponding varieties and, in many cases, also with respect to the subclass of chains over the real unit interval, the so-called *standard* chains, which are defined by a t-norm and its residuum.

In parallel, varieties of MTL-algebras (the algebraic counterpart of the logics BL, MTL and other related t-norm based fuzzy logics) and their classes of chains have been also deeply studied, see for instance [8, 5] for BL-chains and [21] for MTL-chains. Along this line, a new decomposition theorem of BL-chains (and in particular of continuous t-norms) as ordinal sums of Wajsberg hoops has been proved, where the notion of ordinal sum is slightly different than the usual one for t-norms (see [4, 2, 1] for basic notions and results). For MTL chains it has also been proved a decomposition theorem as ordinal sum of prelinear semihoops [21]. Nevertheles, this decomposition does not help much since so far it is unknown a characterization of the indecomposable linearly ordered semihoops. This is the main reason why in this paper we restrict ourselves mainly to continuous t-norms and, their generalizations, BL-chains.

The aims of this paper are:

- To provide a summary of the recent results on BL and MTL chains and its potential interest for researchers on t-norms. We will give the definition of hoop and its basic types, the notion of ordinal sum of hoops and the decomposition theorem of any BL-chains as ordinal sum of Wajsberg hoops.
- To motivate that a proper generalization of continuous and left-continuous t-norms over other domains than the unit interval is the one provided by BL and MTL chains (or even by BL and MTL algebras).

– To study and characterize these “generalized” continuous and some left-continuous t-norms over general chains, specially over the rational unit interval.

We believe that the results surveyed in this paper may serve for a better understanding of Zadeh's fuzzy sets [23] when taking values over linearly ordered scales other than the real unit interval

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References

1. P. Aglianò, I. M. A. Ferreirim and F. Montagna. Basic hoops: an algebraic study of continuous t-norms. *Studia Logica* 87 (2007), 73-98.
2. P. Aglianò and F. Montagna. Varieties of BL-algebras I: general properties, *Journal of Pure and Applied Algebra*, 181 (2003), 105–129.
3. C. Alsina and M. J. Frank and B. Schweizer. *Associative Functions: Triangular Norms and Copulas*, World Scientific Publishing Company (2006).
4. W. J. Blok and I. M. A. Ferreirim. *Hoops and their implicational reducts*, Algebraic Logic, Banach Center pub. 28, Warsaw (1993).
5. R. Cignoli, F. Esteva, L. Godo and A. Torrens. Basic Fuzzy Logic is the logic of continuous t-norms and their residua. *Soft Computing* 4 (2000),106-112.
6. R. Cignoli, F. Esteva, L. Godo and F. Montagna. On a class of left-continuous t-norms. *Fuzzy Sets and Systems* 131 (2002), 283-296.
7. F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems* 124 (2001), 271-288.
8. P. Hájek. *Metamathematics of fuzzy logic*, Trends in Logic vol. 4, Kluwer, Dordrecht (1998).
9. S. Jenei. On the convex combination of left-continuous t-norms. *Aequationes Mathematicae* 72 (2006), 47-59.
10. S. Jenei. How to construct left-continuous triangular norms? State of the Art. *Fuzzy Sets and Systems* 143 (2004), 27-45.
11. S. Jenei. Structure of left-continuous triangular norms with strong induced negations (III): Construction and decomposition. *Fuzzy Sets and Systems* 128 (2002), 197-208.
12. S. Jenei. Structure of left-continuous triangular norms with strong induced negations, (I) Rotation construction. *J. Appl. Non-Classical Logics* 10 (2000), 83-92.
13. S. Jenei. New family of triangular norms via contrapositive symmetrization of residuated implications. *Fuzzy Sets and Systems* 110 (2000), 157-174.
14. S. Jenei and F. Montagna. On the continuity points of left-continuous t-norms. *Archive for Mathematical Logic* 42 (2003), 797-810.
15. S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica* 70 (2002), 183–192.
16. E. P. Klement, R. Mesiar and E. Pap. *Triangular Norms*, Kluwer Academic Publishers, Dordrecht (2000).
17. C. H. Ling. Representation of associative functions. *Publ. Math. Debrecen* 12 (1965), 189-212.
18. K. Maes and B. De Baets. On the structure of left-continuous t-norms that have a continuous contour line. *Fuzzy Sets and Systems* 158 (2007), 843-860.
19. K. Maes and B. De Baets. The triple rotation method for constructing t-norms. *Fuzzy Sets and Systems* 158 (2007), 1652-1674.
20. P. S. Mostert and A. L. Shields. On the structure of semigroups on a compact manifold with boundary. *Annals of Math.* 65 (1957), 117–143.
21. C. Noguera. Algebraic study of Axiomatic Extensions of Triangular Norm Based Fuzzy Logics, Monografies de l'Institut d'Investigació en Intel·ligència Artificial (CSIC) n. 32 (2007).
22. T. Vetterlein. Regular left-continuous t-norms. *Semigroup Forum* 77 (2008), 339 - 379.
23. L. A. Zadeh. Fuzzy sets. *Information Control* 8 (1965), 338–353.

Dialogue games and the proof theory of fuzzy logics — a review and outlook

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Already in the 1970s Robin Giles [17, 16] presented a characterization of infinite-valued Łukasiewicz logic **Ł** that combines a Lorenzen-style dialogue game [20] with a simple scheme for betting on the correctness of atomic statements. A central feature of Giles's model of reasoning is to separate (1) the analysis of logical connectives from (2) the interpretation of 'fuzzy' atomic assertions. To this aim the stepwise reduction of logically complex assertions to their atomic components (1) is guided by Lorenzen-style dialogue rules that regulate idealized debates between a proponent and an opponent of an assertion. As for (2), the two players agree to pay a fixed amount of money to the opposing player for each incorrect statement that they make. The (in)correctness of stating an atomic sentence p is decided by an elementary (yes/no) experiment E_p associated with p. 'Fuzziness' arises from the stipulation that the experiments may be dispersive, i.e., yield different results upon repetition; only a fixed success probability is known for E_p. Giles demonstrated that an initial statement F can be asserted by the proponent without having to expect a loss of money, independently of the probabilities assigned to the elementary experiments, if and only if F is valid in **Ł**.

The purpose of this contribution is to connect Giles's dialogue game theoretic characterization of Łukasiewicz logic with more recent developments in the proof theory of fuzzy logics. We want to demonstrate that dialogue games not only provide an alternative semantic foundation for different t-norm based fuzzy logics, but also relate in a natural way to appropriate Gentzen-type proof systems. In fact already Giles himself, in a paper with A. Adamson [1], presented a sequent calculus for **Ł** that is based on the search for winning strategies in his dialogue game. However, a version of the cut rule is needed for the completeness of this system. This not only renders the calculus non-analytic (i.e., the subformula property cannot be maintained) but also spoils the direct connection with winning strategies. In contrast, as indicated in [9], the logical rules of the *hypersequent* system **HŁ** of [21] can be viewed as rules for the systematic construction of generic winning strategies in Giles's game and thus connect strategies with *analytic*, i.e., cut-free derivations. Hypersequents were introduced by Avron [2, 3] as a generalization of Gentzen sequents that takes account of disjunctive or parallel forms of reasoning. In our context, hypersequents correspond to multisets of possible dialogue states, joined disjunctively at the meta-level. Making the relation between winning strategies and analytic proofs precise is somewhat tricky and involves interesting design choices for a formal presentation of the game that was described rather informally by Giles. (This is the subject of current work pursued in [11].)

Giles's game for **Ł** can be generalized to Gödel logic **G** and Product logic **P** by using alternative evaluations of atomic states and an extended dialogue rule for implication. The logical rules of the so-called r-hypersequent system of [6] that are uniform for **Ł**, **G**, and **P** correspond to the dialogue rule of the generalized game, as indicated in [9]. This observation constitutes no means the final result of analyzing the correspondence between winning strategies and analytic proofs. To the contrary, it triggers various further investigations into the foundations of reasoning with fuzzy logics. In this context we mention the fact, hinted at in [9, 11], that Cancellative Hoop Logic **CHL** [8] naturally emerges from Giles's game for **Ł** and a corresponding hypersequent system if we disallow experiments that can never succeed, i.e., if only non-zero probabilities of positive results are assigned to elementary experiments. (As a consequence, $\bot$ is removed from the language.) A further example is the dialogue game based characterization of the logic **SŁ** in [14] that connects the concept of supervaluation with degree based evaluation by extending Łukasiewicz logic **Ł** with an appropriate modality for 'supertruth'. Yet another recent example is the dialogue game based analysis of interval based fuzzy logics that attempts to solve puzzles about the truth functionality of logical connectives in the context of 'imprecise knowledge' [12].

There exist also other types of dialogue games, not based on Giles's approach, that are also closely related to specific proof systems. We will review two cases that are of interest for fuzzy logic: (1) Truth comparison games for finite and infinite valued Gödel logics [15] that correspond to sequent-of-relations systems as developed in [4, 5] and (2) parallel versions of Lorenzen's original dialogue game for intuitionistic logic that are adequate for intermediate logics, in particular for Gödel logic **G** [10, 13].

On a more general level, we argue that there is a conspicuous discrepancy between the by now vast amount of results in the algebraic, model theoretic approach to fuzzy logics (in Zadeh's narrow sense) and the still rather preliminary state of knowledge in proof theory for the same realm of logics. Indeed, relevant books like [19, 7, 23] restrict attention to axiomatic, Hilbert-style calculi for the syntactic presentation of logics and do not address the challenge to find adequate analytic (cut-free) proof systems. We emphasize that only calculi of the latter type provide a suitable base for automated deduction, but also for extracting information (e.g., about counter models) from failed proof search. (The forthcoming book [22] will be the first monograph focusing on the proof theory of fuzzy logics.) It is of central significance in this context that *both*, the dialogue theoretic analysis and the proof theoretic approach to fuzzy logics, aim at an analysis of *reasoning* with fuzzy propositions and predicates that goes beyond the mere characterization of validity and entailment relations. This fact alone provides ample ground for future research into the foundations of deductive fuzzy logics.

References

1. A. Adamson and R. Giles. A game-based formal system for $Ł_\infty$. *Studia Logica*, 1(38), 49-73, 1979.
2. A. Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52(4):939–951, 1987.
3. A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4(3–4):225–248, 1991.
4. M. Baaz and C. Fermüller. Analytic calculi for projective logics. TABLEAUX 1999, LNAI 1617, 36–50, Springer 1999.
5. M. Baaz, A. Ciabattoni, and C. Fermüller. Cut-elimination in a sequents-of-relations calculus for Gödel logic. 31st International Symposium on Multiple Valued Logic (ISMVL'2001), 181–186. IEEE, 2001.
6. A. Ciabattoni, C.G. Fermüller, and G. Metcalfe: Uniform Rules and Dialogue Games for Fuzzy Logics. In: *LPAR 2004*, F. Baader, A. Voronkov (eds.), Springer LNAI 3452 (2005), 496-510.
7. R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordrecht, 1999.
8. F. Esteva, L. Godo, P. Hájek, and F. Montagna. Hoops and fuzzy logic. *Journal of Logic and Computation*, 13(4):532–555, 2003.
9. C.G. Fermüller. Revisiting Giles - connecting bets, dialogue games, and fuzzy logics. In O. Majer, A.V. Pietarinen, and T. Tulenheimo, editors, *Games: Unifying Logic, Language, and Philosophy*. Springer, to appear.
10. C.G. Fermüller. Parallel dialogue games and hypersequents for intermediate logics. In M. Cialdea Mayer and F. Pirri, editors, *Proceedings of TABLEAUX 2003*, volume 2796 of *LNCS*. Springer, 2003.
11. C.G. Fermüller and G. Metcalfe. Giles's Game and the Proof Theory of Łukasiewicz Logic. In preparation.

12. C.G. Fermüller. Truth Value Intervals, Bets, and Dialogue Games. Submitted.
13. C.G. Fermüller and A. Ciabattoni. From intuitionistic logic to Gödel-Dummett logic via parallel dialogue games. In *Proceedings of the 33rd IEEE International Symposium on Multiple-Valued Logic*, Tokyo, May 2003.
14. C.G. Fermüller and R. Kosik: Combining supervaluation and degree based reasoning under vagueness. In: *LPAR 2006*, M. Hermann, A. Voronkov (eds.), Springer LNAI 4246 (2006), 212-226.
15. C.G. Fermüller and N. Preining. A dialogue game for intuitionistic fuzzy logic based on comparison of degrees of truth. In *Proceedings of InTech'03*, 2003.
16. R. Giles: A non-classical logic for physics. In: R. Wojcicki, G. Malinkowski (Eds.) *Selected Papers on Łukasiewicz Sentential Calculi*. Polish Academy of Sciences, 1977, 13-51.
17. R. Giles: A non-classical logic for physics. *Studia Logica* 33(4), 399-417, 1974.
18. R. Giles: Semantics for Fuzzy Reasoning. *Int. J. Man-Machine Studies* 17 (1982), 401-415.
19. P. Hájek: *Metamathematics of Fuzzy Logic*. Kluwer, 1998.
20. P. Lorenzen: Logik und Agon. In: *Atti Congr. Internat. di Filosofia*, Vol. 4, Florence 1960, 187-194.
21. G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.
22. G. Metcalfe, N. Olivetti, and D. Gabbay. Proof Theory for Fuzzy Logics. Submitted.
23. V. Novák, I. Perfilieva, J. Močkoř. *Mathematical Principles of Fuzzy Logic*, Kluwer, 1999.

Fuzzy relations and preference modelling

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This talk contains an overview on the development of binary fuzzy relations and preference modelling. It also reflects my personal reminiscences on particular outcomes of Linz Seminars in the field.

Fuzzy relations in general, and preferences in particular, have played an important role at some Linz Seminars. The very first lecture linked to fuzzy relations was delivered by Llorenc Valverde in 1982, entitled "Indistinguishability relations with implicative part". Then the Twelfth Linz Seminar in 1990 was devoted to "Applications of Logical and Algebraic Aspects of Fuzzy Relations", and it was the first time I participated in. The chair of the PC was Llorenc Valverde. In 2001, the subject of the 22nd Linz Seminar was "Valued Relations and Capacities in Decision Theory", chaired by Marc Roubens. The 27th Linz Seminar in 2006 was concentrated on "Preferences, Games and Decisions", co-chaired by Marc Roubens and myself.

In the overview we consider traces of binary fuzzy relations and emphasize their role in the characterization of some fundamental properties of fuzzy relations. We deal also with the transitivity property of binary fuzzy relations. We show a general representation theorem for any T-transitive binary fuzzy relation, where T is a left-continuous triangular norm. The study is carried out by using traces.

We consider two frameworks in which preferences can be expressed in a gradual way. The first framework is that of fuzzy preference structures as a generalization of Boolean (2-valued) preference structures. A fuzzy preference structure is a triplet of fuzzy relations expressing strict preference, indifference and incomparability in terms of truth degrees. An important issue is the decomposition of a fuzzy preference relation into such a structure. The second framework is that of reciprocal relations as a generalization of the 3-valued representation of complete Boolean preference relations. Reciprocal relations, also known as probabilistic relations, leave no room for incomparability, express indifference in a Boolean way and express strict preference in terms of intensities. We describe properties of fuzzy preference relations in both frameworks, focusing on transitivity-related properties.

After that we give a state-of-the-art overview of representation and construction results for fuzzy weak orders, concentrating on results that hold in the most general case when the underlying set is possibly infinite.

In the last part we introduce quaternary fuzzy relations in order to describe difference structures. Three models are developed and studied, based on three different interpretations of a fuzzy implication. Functional forms of the quaternary fuzzy relation are determined by solutions of functional equations of the same type. These quaternary fuzzy relations turn out to be representable fuzzy weak orders on the set of pairs of alternatives.

Aggregation on bipolar scales

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1 Introduction

Most of the time aggregation functions are defined on $[0,1]$, where 0 and 1 represent the lowest and highest scores along each dimension. We may desire to consider a third particular point e of the interval, which will play a particular role, for example a neutral value or an annihilator value (this is the case with uninorms). For convenience, we may always consider that we work on $[-1,1]$, and 0 corresponds to our particular point e.

The motivation for such a study has its roots in psychology. In many cases, scores or utilities manipulated by humans lie on a *bipolar scale*, that is, a scale with a neutral value making the frontier between good or satisfactory scores, and bad or unsatisfactory scores. With our convention, good scores are positive ones, while negative scores reflect bad scores. Very often our behaviour with positive scores is not the same as with negative ones, hence it becomes important to define aggregation functions that are able to reflect the variety of aggregation behaviours on bipolar scales.

In the sequel, we will consider several ways to define bipolar aggregation functions, starting from some aggregation funnction defined on $[0,1]$. We first consider associative aggregation functions, and treat separetely the case of minimum, and maximum, then we will turn to nonassociative aggregation functions.

This work is closely related and brings new insights to the following mathemetical and applied fields:

(i) algebraic structures, such as rings, groups and monoids, ordered Abelian groups. In particular, Section 3 offers a incursion into nonassociative algebra, a domain which has been scarcely investigated. Many-valued logics dealing with bipolar notions is also concerned.
(ii) integration, measure theory by providing a new type of integral (Choquet integral w.r.t. a bi-capacity). In the finite case, the notion bi-capacity is related to bi-set functions, which are known in some domains of discrete mathematics and combinatorial optimization (bisubmodular base polyhedron, see, e.g., Fujishige [1]).
(iii) decision making and mathematical economics, since the motivation of this work is rooted there. This work offers a generalization of the well-known Cumulative Prospect Theory (see Section 5).

The material presented here is drawn from Chapter 9 of [7], a forthcoming monograph on aggregation functions written by the authors.

We introduce first the fundamental concept of pseudo-difference.

Definition 1. *Let* S *be a t-conorm (see [8] for details on t-norms and t-conorms).*

(i) The S*-difference is defined for any* (a,b) *in* $[0,1]^2$ *by* $a \overset{\mathsf{S}}{-} b := \inf\{c \in [0,1] \mid \mathsf{S}(b,c) \geqslant a\}$.

(ii) The pseudo-difference associated to S *is defined for any* (a,b) *in* $[0,1]^2$ *by*

$$a \ominus_{\mathsf{S}} b := \begin{cases} a \overset{\mathsf{S}}{-} b & \text{if } a \geqslant b \\ -(b \overset{\mathsf{S}}{-} a) & \text{if } a < b \\ 0, & \text{if } a = b, \end{cases}$$

Proposition 1. *If* S *is a continuous Archimedean t-conorm with additive generator s, then* $a \overset{\mathsf{S}}{-} b = s^{-1}(0 \vee (s(a) - s(b)))$, *and* $a \ominus_{\mathsf{S}} b = g^{-1}(g(a) - g(b))$, *with* $g(x) := s(x)$ *for* $x \geqslant 0$, *and* $g(x) := -s(-x)$ *for* $x \leqslant 0$.

2 Associative bipolar operators

In this section, we want to define associative and commutative operators where 0 is either a neutral element or an annihilator element, which we call respectively *(symmetric) pseudo-addition* and *(symmetric) pseudo-multiplication*. This section is mainly based on [3].

We denote respectively by $\oplus, \otimes : [-1,1]^2 \to [-1,1]$ these operators.

2.1 Pseudo-additions

Our basic requirements are the following, for any $x,y,z \in [-1,1]$:

A1 Commutativity: $x \oplus y = y \oplus x$
A2 Associativity: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
A3 Neutral element $x \oplus 0 = 0 \oplus x = x$.
A4 Nondecreasing monotonicity: $x \oplus y \leqslant x' \oplus y'$, for any $x \leqslant x'$, $y \leqslant y'$.

The above requirements mean that we recognize $\oplus$ as a t-conorm when restricted to $[0,1]$, which we denote by S. Since $[-1,1]$ is a symmetric interval, and if 0 plays the role of a neutral element, then we should have

A5 Symmetry: $x \oplus (-x) = 0$, for all $x \in]-1,1[$.

From **A1**, **A2**, and **A5** we easily deduce $(-x) \oplus (-y) = -(x \oplus y), \quad \forall (x,y) \in]-1,0]^2 \cup [0,1[^2$. Then, **A3** and **A4** permit to define $\oplus$ on $[-1,1]$:

$$x \oplus y = \begin{cases} \mathsf{S}(x,y) & \text{if } x,y \in [0,1] \\ -\mathsf{S}(-x,-y) & \text{if } x,y \in [-1,0] \\ x \ominus_{\mathsf{S}} (-y) & \text{if } x \in [0,1[, y \in]-1,0] \\ 1 \text{ or } -1 & \text{if } x = 1, y = -1, \end{cases} \tag{1}$$

with the remaining cases being determined by commutativity. We distinguish several cases for S. We write for convenience $x \oplus (-y) = x \ominus y$ for any $x, y \in [-1,1]^2$.

S *is a strict t-conorm* with additive generator s. Let us rescale $\oplus$ on $[0,1]^2$, calling U the result:

$$\mathsf{U}(z,t) := \frac{((2z-1)\oplus(2t-1))+1}{2}. \tag{2}$$

We introduce $g : [-1,1] \to [-\infty,\infty]$ by $g(x) := s(x)$ for positive x, $g(x) := -s(-x)$ for negative x, i.e., g is a symmetrization of s. Then $x \oplus y = g^{-1}(g(x)+g(y))$ for any $x,y \in [-1,1]$, with the convention $\infty - \infty = \infty$ or $-\infty$. Also, U is a generated uninorm that is continuous (except at $(0,1)$ and $(1,0)$), strictly increasing on $]0,1[^2$, has neutral element $\frac{1}{2}$, and is conjunctive (respectively, disjunctive) when the convention $\infty - \infty = -\infty$ (respectively, $\infty - \infty = \infty$). Finally, we have:

Theorem 1. *Let* S *be a strict t-conorm with additive generator* s *and* $\oplus$ *the corresponding pseudo-addition. Then* $(]-1,1[,\oplus)$ *is an Abelian group.*

S *is a nilpotent t-conorm* with additive generator s. It is easy to see that the construction does not lead to an associative operator.

S *is the maximum operator.* This case will be treated in Section 3.

S *is an ordinal sum of continuous Archimedean t-conorms.* In this case too, associativity cannot hold everywhere.

2.2 Pseudo-multiplications

Our first requirements are, for any $x,y,z \in [-1,1]$

M0 0 is an annihilator element: $x \otimes 0 = 0 \otimes x = 0$
M1 Commutativity: $x \otimes y = y \otimes x$
M2 Associativity: $x \otimes (y \otimes z) = (x \otimes y) \otimes z$.

Let us adopt for the moment the following.

M3 Nondecreasing monotonicity on $[0,1]^2$: $x \otimes y \leqslant x' \otimes y'$, for any $0 \leqslant x \leqslant x' \leqslant 1$, $0 \leqslant y \leqslant y' \leqslant 1$.
M4 Neutral element for positive elements: $x \otimes 1 = 1 \otimes x = x$, for all $x \in [0,1]$,

then axioms **M1** to **M4** make $\otimes$ a t-norm on $[0,1]^2$, and **M0** is deduced from them. If pseudo-addition and pseudo-multiplication are used conjointly, a natural requirement is then distributivity.

M5 Distributivity of $\otimes$ with respect to $\oplus$: $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ and $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ for all $x,y,z \in [-1,1]$.

Then under **A1** to **A4**, and **M1** to **M4**, axiom **M5** can be satisfied on $[0,1]^2$ if and only if $\oplus = \vee$. Finally we can show:

Proposition 2. *Under **M1** to **M5** and **A3**, **A5**,* $\otimes$ *has the form* $x \otimes y = \mathrm{sign}(x \cdot y)\mathsf{T}(|x|,|y|)$, *for some t-norm* T.

If distributivity is not needed, we can impose monotonicity of $\otimes$ on the whole domain $[-1,1]^2$:

M3' Nondecreasing monotonicity for $\otimes$: $x \otimes y \leqslant x' \otimes y'$, $-1 \leqslant x \leqslant x' \leqslant 1$, $-1 \leqslant y \leqslant y' \leqslant 1$.

Then, if we impose in addition

M4' Neutral element for negative numbers: $(-1) \otimes x = x$ for all $x \leqslant 0$, up to a rescaling in $[0,1]^2$, $\otimes$ is a nullnorm with $a = 1/2$. In summary, we have shown the following.

Proposition 3. *Under **M1**, **M2**, **M3'**, **M4** and **M4'**, $\otimes$ has the following form:*

$$x \otimes y = \begin{cases} \mathsf{T}(x,y) & \text{if } x,y \geqslant 0 \\ \mathsf{S}(x+1,y+1)-1 & \text{if } x,y \leqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

for some t-norm T *and t-conorm* S.

3 Symmetric minimum and maximum

The previous section has shown that except for strict t-conorms, there was no way to build a pseudo-addition fulfilling requirements **A1** to **A5**. Hence extending the maximum on $[-1,1]^2$ in this way is not possible. However, we will show that this is in fact almost possible (see [2] for details).

3.1 The symmetric maximum

Our basic requirements are:

SM1 $\varovee$ coincide with $\vee$ on $(L^+)^2$.
SM2 Commutativity
SM3 Associativity
SM4 0 is a neutral element
SM5 $-x$ is the symmetric image of x, i.e. $x \varovee (-x) = 0$.

These requirements are already contradictory. In fact, **SM1** and **SM5** imply that associativity (**SM3**) cannot hold. The following can be shown.

Proposition 4. *Under conditions (**SM1**), (**SM5**) and (**SM6**), no operation is associative on a larger domain than* $\varovee$ *defined by:*

$$x \varovee y = \begin{cases} -(|x| \vee |y|) & \text{if } y \neq -x \text{ and } |x| \vee |y| = -x \text{ or } = -y \\ 0 & \text{if } y = -x \\ |x| \vee |y| & \text{otherwise.} \end{cases} \quad (3)$$

Except for the case $y = -x$, $x \varovee y$ equals the absolutely larger one of the two elements x and y.

3.2 Symmetric minimum

The case of the symmetric minimum is less problematic. The following requirements determine it uniquely.

Sm1 $\varowedge$ coincides with $\wedge$ on $(L^+)^2$
Sm2 Rule of signs: $-(x \varowedge y) = (-x) \varowedge y = x \varowedge (-y)$, for all $x, y \in L$.

Under **Sm1** and **Sm2**, we get

$$x \varowedge y := \begin{cases} -(|x| \wedge |y|) & \text{if } \operatorname{sign}(x) \neq \operatorname{sign}(y) \\ |x| \wedge |y| & \text{otherwise.} \end{cases} \quad (4)$$

As for pseudo-multiplications, we could as well impose a different rule of signs, namely $-(x \varowedge y) = (-x) \varowedge (-y)$, and impose monotonicity on the whole domain. This would give, up to a rescaling, a nullnorm, namely $\mathsf{Med}_{0.5}(x,y) := \mathsf{Med}(x,y,0.5)$.

4 Separable aggregation functions

We consider here not necessarily associative functions A. A simple way to build bipolar aggregation functions is the following. Let $\mathsf{A}^+, \mathsf{A}^-$ be given aggregation functions on $[0,1]^n$. We define A on $[-1,1]^n$ by $\mathsf{A}(\mathbf{x}) := \psi(\mathsf{A}^+(\mathbf{x}^+), \mathsf{A}^-(\mathbf{x}^-)), \quad \forall \mathbf{x} \in [-1,1]^n$, where $\mathbf{x}^+ := \mathbf{x} \vee \mathbf{0}$, $\mathbf{x}^- := (-\mathbf{x})^+$, and ψ is a pseudo-difference (Definition 1). A bipolar aggregation function defined as above is called *separable*.

We give as illustration three cases of interest.

$\mathsf{A}^+ = \mathsf{A}^-$ *is a strict t-conorm* S. If $\mathsf{A}^+ = \mathsf{A}^-$ is a strict t-conorm S with generator s, and $\ominus_{\mathsf{S}}$ is taken as pseudo-difference, we recover the construction of Section 2.

$\mathsf{A}^+ = \mathsf{A}^-$ *is a continuous t-conorm* S. We know by Section 2 that associativity is lost if S is not strict. Restricting to the binary case, it is always possible to apply the definition of $\oplus$ given by (1), taking the associated pseudo-difference operator $\ominus_{\mathsf{S}}$. For example, considering $\mathsf{S} = \mathsf{S}_{\mathbf{L}}$, we easily obtain $\mathsf{A}(x,y) = ((x+y) \wedge 1) \vee (-1)$.

$\mathsf{A}^+, \mathsf{A}^-$ *are integral-based aggregation functions.* An interesting case is when $\mathsf{A}^+, \mathsf{A}^-$ are integral-based aggregation functions, such as the Choquet or Sugeno integrals. Then we recover various definitions of integrals for real-valued functions. Specifically, let us take $\mathsf{A}^+, \mathsf{A}^-$ to be Choquet integrals with respect to capacities μ^+, μ^-, and ψ is the usual difference $\ominus_{\mathbf{L}}$. Then:

- Taking $\mu^+ = \mu^-$ we obtain the *symmetric Choquet integral* or Šipoš integral $\check{\mathcal{C}}_\mu(\mathbf{x}) := \mathcal{C}_\mu(\mathbf{x}^+) - \mathcal{C}_\mu(\mathbf{x}^-)$.
- Taking $\mu^- = \overline{\mu^+}$ we obtain the *asymmetric Choquet integral* $\mathcal{C}_\mu(\mathbf{x}) := \mathcal{C}_\mu(\mathbf{x}^+) - \mathcal{C}_{\overline{\mu}}(\mathbf{x}^-)$.
- For the general case, we obtain what is called in decision making theory the *Cumulative Prospect Theory (CPT)* model [9] $\mathrm{CPT}_{\mu^+,\mu^-}(\mathbf{x}) := \mathcal{C}_{\mu^+}(\mathbf{x}^+) - \mathcal{C}_{\mu^-}(\mathbf{x}^-)$.

5 Integral-based aggregation functions

It is possible to generalize the above definitions based on the Choquet integral to a much wider model called the Choquet integral w.r.t bicapacities (see [4, 5] and [6] for a general construction).

We introduce $\mathcal{Q}(N) := \{(A,B) \mid A,B \subseteq N, A \cap B = \varnothing\}$. A *bicapacity* w on N is a function $w : \mathcal{Q}(N) \to \mathbb{R}$ satisfying $w(\varnothing,\varnothing) = 0$, and $w(A,B) \leqslant w(C,D)$ whenever $A \subseteq C$ and $B \supseteq D$ (monotonicity).

Definition 2. *Let w be a bicapacity and $\mathbf{x} \in \mathbb{R}^n$. The Choquet integral of $\mathbf{x}$ with respect to w is given by $\mathcal{C}_w(\mathbf{x}) := \mathcal{C}_{\nu_{N^+_{\mathbf{x}}}}(|\mathbf{x}|)$, where $\nu_{N^+_{\mathbf{x}}}$ is a game on N defined by $\nu_{N^+_{\mathbf{x}}}(C) := w(C \cap N^+_{\mathbf{x}}, C \cap N^-_{\mathbf{x}})$, and $N^+_{\mathbf{x}} := \{i \in N \mid x_i \geqslant 0\}$, $N^-_{\mathbf{x}} = N \setminus N^+_{\mathbf{x}}$.*

The CPT model (and hence the asymmetric and symmetric Choquet integrals) are recovered taking a bicapacity of the form $w(A,B) = \mu_+(A) - \mu_-(B)$, for all $(A,B) \in \mathcal{Q}(N)$.

References

1. S. Fujishige. *Submodular functions and optimization*. Elsevier, 2nd edition edition, 2005.
2. M. Grabisch. The Möbius function on symmetric ordered structures and its application to capacities on finite sets. *Discrete Mathematics*, 287(1-3):17–34, 2004.
3. M. Grabisch, B. De Baets, and J. Fodor. The quest for rings on bipolar scales. *Int. J. of Uncertainty, Fuzziness, and Knowledge-Based Systems*, 12(4):499–512, 2004.
4. M. Grabisch and Ch. Labreuche. Bi-capacities. Part I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151:211–236, 2005.

5. M. Grabisch and Ch. Labreuche. Bi-capacities. Part II: the Choquet integral. *Fuzzy Sets and Systems*, 151:237–259, 2005.
6. M. Grabisch and Ch. Labreuche. Bipolarization of posets and natural interpolation. *J. of Mathematical Analysis and Applications*, 343:1080–1097, 2008. doi: 10.1016/j.jmaa.2008.02.008.
7. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Number 127 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009.
8. E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Kluwer Academic Publishers, Dordrecht, 2000.
9. A. Tversky and D. Kahneman. Advances in prospect theory: cumulative representation of uncertainty. *J. of Risk and Uncertainty*, 5:297–323, 1992.

Attachment between fuzzy points and fuzzy sets

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In the context of point-set lattice-theoretic approach to fuzzy topological spaces fuzzy points and local properties play an important role either from a theoretical point of view or for application to topology and topological systems. In this respect, several kinds of relations involving fuzzy points and fuzzy sets have been considered.
In [3] a so called attachment relation between L-sets is proposed, depending on a suitable family of filters in L. This allows to state functorial relationship between the category of L-topological spaces, **L-Top**, and either the category **Top** of topological spaces or the category **TopSys** of topological systems, as these are defined in [6]. In particular, under the assumption of spatiality there is an embedding of **L-Top** into **TopSys** and an embedding of **L-Top** into **Top**. The latter one maps (X,τ) to a topological space (S_X,τ^*), where S_X is the set of all L-points on X and τ^* is frame-isomorphic to τ.

The basic definitions and results can be summarized as follows.

Definition 01 *An attachment family, or more simply an attachment in a complete lattice L is a family* $\mathcal{F} = \{F_\lambda | \lambda \in L\}$ *of subsets of L with*

- $F_\perp = \emptyset$
- F_λ *completely coprime filter* $\forall \lambda \neq \perp$.

Definition 02 *The attachment is said to be*

- *spatial if*
  (sp) $\alpha \not\leq \beta \Rightarrow \exists \lambda : \alpha \in F_\lambda, \beta \notin F_\lambda$
- *isotonic if*
  (i) $\lambda \leq \mu \Rightarrow F_\lambda \subseteq F_\mu$
- *symmetrical if*
  (s) $\mu \in F_\lambda \Rightarrow \lambda \in F_\mu$
- *inverse isotonic if*
  (ii) $F_\lambda \subseteq F_\mu \Rightarrow \lambda \leq \mu$

The following relationship between the conditions listed in the above definition holds.

Proposition 03 *For any attachment* $\mathcal{F}$ *in L one has*

- $(s) \implies (i)$
- $(s),(ii) \implies (sp)$.

If $\mathcal{F}$ is an attachment in L, a binary relation $\mathcal{A}$, called *attachment relation*, is defined between L-sets on any set X

$$A \mathcal{A} B \text{ if } B(x) \in F_{A(x)} \text{ for some } x \in X.$$

For any $A \in L^X$, denote $A^* = \{\lambda_x \in S_X | \ \lambda_x \mathcal{A} A\}$
and consider the function

$$* : L^X \longrightarrow \mathcal{P}(S_X), \ \ A \mapsto A^*.$$

Then

Proposition 04 *Whatever is the fixed attachment*

(1) $*$ *is a frame map;*

under (sp)

(2) $*$ *is an embedding of posets.*

Let (X,τ) be any L-topological space and denote $\tau^* = \{A^* | \ A \in \tau\}$.

Then clearly (S_X, τ^*) is a topological space; moreover, τ and τ^* are isomorphic frames if the attachment is spatial

For any function $f : X \to Y$, consider

$$f^* : S_X \longrightarrow S_Y, \quad \lambda_x \mapsto \lambda_{f(x)}.$$

Then the correspondences

$$\begin{array}{rcl} (X,\tau) & \longmapsto & (S_X, \tau^*) \\ f \in \mathbf{L\text{-}Top}((X,\tau),(Y,\sigma)) & \longmapsto & f^* \in \mathbf{Top}((S_X,\tau^*),(S_Y,\sigma^*)) \end{array}$$

define a functor

$$* \ : \ \mathbf{L\text{-}Top} \longrightarrow \mathbf{Top}.$$

and the correspondences

$$(X,\tau) \mapsto (S_X, \tau, \mathcal{A}), \quad f \mapsto (f^*, (f_L^{\leftarrow})^{op})$$

define a functor

$$*_{Sys} : \mathbf{L\text{-}Top} \longrightarrow \mathbf{TopSys}.$$

Moreover the following holds.

Proposition 05 *If the attachment is spatial, then the functors* $* : \mathbf{L\text{-}Top} \to \mathbf{Top}$ *and* $*_{Sys} :$ ***L-Top*** $\longrightarrow$ ***TopSys*** *are embeddings.*

Interesting examples of attachment come from lattice-ordered implicative and multiplicative structures related to (non-classical) logics with primes (see [3]): roughly speaking, if L is such a kind of structure, an L-point λ_x is attached to an L-set A if the product (i.e. conjunction) of the existing value λ of x and the belongness degree $A(x)$ of x to A does not go below some prescribed prime. The negation operation in L may be involved, too.

We notice the following results relating special attachment relations with spatial frames and with order reversing involutions and primes in L.

Proposition 06 *For any complete lattice L the following are equivalent*

1. *A spatial attachment exists in L*

2. *L is a spatial frame*
3. *$\forall \lambda \not\leq \mu$ a prime $\alpha \in L$ exists s.t. $\mu \leq \alpha$, $\lambda \not\leq \alpha$.*

Proposition 07 *Let L be a complete lattice. An attachment $\mathcal{F}$ exists in L that satisfies (ii) and (s) if and only if an order reversing involution exists in L and all the elements of L other than $\top$ are prime.*

This latter result shows that thirty years of applications of quasi-coincidence relation to fuzzy topological spaces, usually obtained by specific and long computations, get a motivation and a simplification from the above embedding and isomorphisms. In fact the quasi-coincidence relation introduced in [5] and widely considered in Chinese schools since 1980 comes from the *Łukasiewicz algebra* $([0,1], \vee, \wedge, \otimes_L, \rightarrow_L, 0, 1)$ by the attachment $\mathcal{F} = \{F_\lambda | \lambda \in L\}$, where $F_\lambda = \{\mu | \lambda \otimes_L \mu > 0\}$, which satisfies (ii) and (s).
Most applications can be considered in L-topological spaces, for any spatial frame L: as an example, in [4] some basic topological notions such as the neighborhood structure, interior and exterior of L-sets, separation axioms and compactness are introduced in the L-topological space (X, τ), by means of a spatial attachment in L, not just rephrasing those concepts as in classical topology but concretely transporting the corresponding notions in the space $(S_X, \tau^\star)$ by the functor $\star$: **L-Top** $\rightarrow$ **Top**.
The q-neighborhood structure of an L-topological space (X, τ) is nothing but the neighborhood structure in the space of L-points $(\mathcal{S}_X, \tau^*)$ which explains why it works fine to approach most L-topological properties and in the applications to discrete fuzzy topological dynamical systems (see e.g. [1]), whose points are, in fact, fuzzy points.
Eventually we remark that the functor described above from **L-Top** to **TopSys** gives a useful contribution to the detailed analysis of the relationship between lattice-valued topology and topological systems exploited in [2].

References

1. L.Chen, H. Kou, M.-K. Luo, W. Zhang, Discrete Dynamical Systems in *L*-topological spaces, Fuzzy Sets and Systems 156 (2005), 25-42.
2. J. T. Denniston, S. E. Rodabaugh, Functorial relationship between lattice-valued topology and topological systems, (Preprint).
3. C. Guido, Fuzzy points and attachment, (Preprint).
4. C. Guido, V. Scarciglia, *L*-topological spaces as spaces of points, (Preprint).
5. B.-M. Pu, Y.-M. Liu, Fuzzy topology I. Neighborhood structure of fuzzy points and More-Smith convergence, J. Math. Anal. Appl. 76 (1980), 571-599.
6. S. Vickers, Topology Via Logic, Cambridge Tracts in Theoretical Computer Science 5 (1989), Cambridge University Press.

An observation on (un)decidable theories in fuzzy logic

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Mathematical fuzzy logic (fuzzy logic as a kind of mathematical logic) has been presented at most Linz meetings (the present author lectured about it in years 1996, 1997, 2000, 2003, 2004, 2005, 2007). The present contribution continues this by presenting a (surprising) fact on decidability and undecidability of fuzzy theories. In classical logic the extension of a decidable theory T by a single axiom φ is again a decidable theory thanks to the deduction theorem: $T,\varphi \vdash \psi$ iff $T \vdash \varphi \to \psi$. (It follows that a decidable theory has a decidable complete extension, which is useful e.g. for the proof of essential undecidability some theories.) Also for some fuzzy logics (Gödel logic, logics with Baaz's Delta) provability in T,φ recursively reduces to provability in T and decidability of T implies decidability of (T,φ) due to specific deduction theorem of these logics. But in general the answer is negative. We are going to present a decidable theory T over Łukasiewicz logic and its extension (T,φ) which is undecidable (but of course recursively axiomatizable). We shall construct T and φ in Łukasiewicz *propositional* logic but it gives trivially a example in predicate logic (propositional variables understood as nullary predicates, then each formula is logically equivalent to a quantifier free formula).

Let $R(n,m)$ be a recursive relation on natural numbers such that its existential projection $(\exists n)R(n,m)$ is not recursive. Let T be a theory over Łukasiewicz propositional logic whose language consists of propositional variables q, p_n (n positive natural) and whose axioms are $q^n \to p_m$ for all n,m with $R(n,m)$. (q^n is $q\&\ldots\&q$, n conjuncts, as usual.)

The theory has a trivial crisp model evaluating q by 1 and evaluating p_m by 1 iff $(\exists n)R(n,m)$, otherwise evaluating p_m by 0.

Theorem. $(T,q) \vdash p_m$ iff $(\exists n)R(n,m)$, hence (T,q) is undecidable.
(Easy.)

Theorem. The theory T is decidable.

Surprisingly difficult. Hint: The set of all formulas φ provable in T is of course Σ_1 (recursively enumerable). One can show that also the set of formulas unprovable in T is Σ_1: one can recursively reduce the problem of T-unprovability of a formula to the satisfiability problem of open formulas in the ordered field of reals, the latter being decidable (even PSPACE, [1]).

Remark. (1) The theorem trivially holds for BL; simply add the schema $\neg\neg\alpha \to \alpha$ for each α to the axioms of T.

(2) In my paper [3] a weak arithmetic is defined over the fuzzy logic BL$\forall$ and Gödel's incompleteness theorem for it is proved (each axiomatizable extension of this arithmetic consistent over BL$\forall$ is incomplete in the sense of fuzzy logic) and it is claimed that essential undecidability follows. But the

problem of the existence of a decidable complete extension of a decidable theory over the logic BL∀ seems to remain open as well as the problem whether the weak arithmetic over BL∀ is essentially undecidable.

References

1. Canny J: Some algebraic and geometric computations in PSPACE. In Proc. 20th Ann. ACM Symp. Theory of Computing, pages 460–467, 1988.
2. Hájek P.: Metamathematics of fuzzy logic. Kluwer 1998.
3. Hájek P.: Mathematical Fuzzy Logic and Natural Numbers. Fundamenta Informaticae 81 (2007) p. 155-163.

The variety generated by the truth value algebra of type-2 fuzzy sets

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1 Introduction

The algebra of truth values of type-2 fuzzy sets was introduced by Zadeh in 1975, generalizing the truth value algebras of ordinary fuzzy sets, and of interval-valued fuzzy sets. This algebra is quite a complicated object. There is an extensive literature on it. Many fundamental mathematical properties of this algebra have been developed, but many basic questions remain open. This paper addresses some questions about the variety generated by this algebra of truth values, and our principal result is that it is generated by a finite algebra. In particular, this algebra of fuzzy type-2 truth values is locally finite. Many natural questions remain; for example, finding an equational base for the variety.

We begin by giving the definition of our algebra, and listing some of its known properties.

2 The algebra of fuzzy truth values of type-2 fuzzy sets

The underlying set of the algebra of truth values of type-2 fuzzy sets is $M = Map([0,1],[0,1])$, the set of all functions from the unit interval into itself. The operations imposed are certain convolutions of operations on $[0,1]$. These are the binary operations $\sqcup$ and $\sqcap$, the unary operation *, and the nullary operations $\bar{1}$ and $\bar{0}$ as spelled out below.

$$\begin{aligned}(f \sqcup g)(x) &= \sup\{f(y) \wedge g(z) : y \vee z = x\} \\ (f \sqcap g)(x) &= \sup\{f(y) \wedge g(z) : y \wedge z = x\} \\ f^*(x) &= \sup\{f(y) : 1-y = x\} = f(1-y) \\ \bar{1}(x) = \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1 \end{cases} &\text{ and } \bar{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}\end{aligned} \tag{1}$$

These yield the algebra $\mathbb{M} = (M, \sqcup, \sqcap, ^*, \bar{1}, \bar{0})$, *the algebra of truth values for fuzzy sets of type-2*.

Using some auxiliary operations, it is fairly routine to verify the following properties of the algebra $\mathbb{M}$. The details may be found in [1].

Corollary 1. *Let $f,\ g,\ h \in M$. The following equations hold in $\mathbb{M}$.*

1. $f \sqcup f = f$; $f \sqcap f = f$
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
4. $\bar{1} \sqcap f = f$; $\bar{0} \sqcup f = f$
5. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
6. $f^{**} = f$
7. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$

Easy examples show that $\mathbb{M}$ is not a lattice. However, $\mathbb{M}$ is a *De Morgan bisemilattice*—a general algebra with two binary operations, both of which are idempotent, commutative, and associative, and a unary operation satisfying (6) and (7). There is a fairly extensive literature on distributive bisemilattices, but $\mathbb{M}$ is not distributive.

Notice that the algebra $\mathbb{M}$ has an additional equation connecting directly the operations $\sqcup$, and $\sqcap$, namely equation (5).

3 The main results

We begin with a bit of notation, and then list our main results.

- $\mathbb{M} = (M, \sqcap, \sqcup, {}^*, 0, 1)$.
- $\mathcal{V}(\mathbb{M})$ denotes the variety generated by the algebra $\mathbb{M}$.
- $\mathcal{V}(\mathbb{EQ})$ denotes the variety generated by the equations in (1)–(7) above.

Definition 1. *An algebra is **locally finite** if every subalgebra generated by a finite subset is finite. A variety is locally finite if every member of it is locally finite.*

Since $\mathcal{V}(\mathbb{EQ})$ contains all ortholattices, it is not locally finite, as it is well known that ortholattices are not locally finite.

Theorem 1. $\mathbb{M} = (M, \sqcap, \sqcup, {}^*, 0, 1)$ *is locally finite with a uniform upper bound on the size of a subalgebra in terms of the size of a generating set.*

Corollary 2. *The variety* $\mathcal{V}(\mathbb{M})$ *generated by* $\mathbb{M}$ *is locally finite.*

Corollary 3. $\mathbb{M}$ *satisfies an equation not a consequence of the equations in (1)–(7) above. Thus this set of equations is not an equational base for the variety generated by* $\mathbb{M}$.

We have not yet found such an equation. Actually, we have a stronger result than Theorem 1.

Theorem 2. $\mathcal{V}(\mathbb{M})$ *is generated by an algebra with 32 elements. In particular, this variety is locally finite.*

The proof of this theorem is effected by showing that $\mathcal{V}(\mathbb{M})$ is generated by the subalgebra $\mathbb{E} = (\{0,1\}^{[0,1]}, \sqcap, \sqcup, {}^*, 0, 1)$ of $\mathbb{M}$, and then constructing homomorphisms of $\mathbb{E}$ into an appropriate algebra with 32 elements.

Analogous results hold for the algebra $\mathbb{M}$ without the unary operation *, except that the variety that algebra generates is generated by an algebra with 8 elements.

We have not found an equational base for the variety $\mathcal{V}(\mathbb{M})$, with or without the unary operation *.

References

1. C. Walker and E. Walker. The algebra of fuzzy truth values, *Fuzzy Sets and Systems*, 149:309-347, 2005.
2. C. Walker and E. Walker. Sets with type-2 operations, *International Journal of Approximate Reasoning*, in press.
3. L. Zadeh. The concept of a linguistic variable and its application to approximate reasoning, *Inform Sci.*, 8:199-249, 1975.

Fuzzy logic in machine learning

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The purpose of this talk is threefold. First, it is intended to convey an idea of the state-of-the-art in the field of fuzzy logic-based machine learning, to be understood as the application of theoretical concepts, methods, and techniques from fuzzy set theory and fuzzy logic in the field of machine learning and related research areas, such as data mining and knowledge discovery. Second, the potential contributions that fuzzy logic can make to machine learning shall be assessed in a somewhat systematic and critical way, highlighting potential advantages of fuzzy extensions and recent advances in combining machine learning methods with fuzzy modeling and inference techniques, but also pointing to some deficiencies and pitfalls of this line of research. Finally, some promising directions of future research shall be sketched and promoted, including problems of ranking and preference learning, the representation of uncertainty in model induction and prediction, and the use of fuzzy modeling techniques for feature generation.

An overview on the algebraic aspects of residuated monoids on $[0,1]$ with outlooks

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Abstract. First we shall give a historical overview in which we shall confine our attention to the *algebraic* development of left-continuous t-norms only. It will be followed by recent results of the speaker: The main *geometric aspects* of the study of residuated lattices will be summarized together with a new finding in this direction. Then the structural description of both e-involutive uninorms on $[0,1]$ and e-involutive finite involutive uninorm chains will be presented. This description involves a striking new construction, called *skew-symmetrization*, in which one *has* to leave the accustomed residuated setting, and has to enter a *co-residuated* setting too. The main "philosophical" contribution of this result says: for the description of residuated structures one needs as well co-residuation; it is a surprising observation in the theory of residuated lattices, a theory which goes back to 70 years.

1 Historical overview, geometric aspects

Many mathematical theories dealing with t-norms have been using only the *left*-continuity assumption for the t-norm. Despite a complete structural description of continuous t-norms has been available since 1957 [12] not even an example left-continuous t-norms was known till a good decade ago. Then the first example, the nilpotent minimum was introduced in [2]. Even after this paper a conjecture went to print saying that the nilpotent minimum (up to isomorphism) may be the only left-continuous (but not continuous) t-norm. Next a sequence of papers was published, mainly by the author of the present abstract, in which several methods constructing left-continuous t-norms have been introduced. For some details, see [4] and the references therein. The sudden abundance of example left-continuous t-norms has called for structural description of any kind. In the opinion of the author such a characterization for the *whole* class of left-continuous t-norms does not exist. However, this does not exclude the possibility of characterizing *sub*classes of them. In an attempt to attach this problem decomposition theorems have been established for the rotation, the rotation-annihilation and the triple rotation theorems. An important structural description has been given for the weakly cancellative class in [3] by relating them to full Hahn groups. Another important result is that the rotation construction is descriptive enough to characterize the structure of perfect IMTL-algebras [10].

The extreme structural complexity of the class of left-continuous t-norms has called for a better insight. In response to this requirement a novel geometric approach to understand the structure of residuated lattices has been introduced [7, 6]. The main focus has been to determine how the associativity property can be seen from the surface of the graph of a commutative associative operation and from the sections thereof. This geometric understanding of associativity has several immediate algebraic applications including the introduction of the rotation construction. Next, motivated by the mail result of [11] another geometric result, the so-called reflection-invariance lemma has been established

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in [5]. It is this lemma that has lead the author to observe that co-residuated operations play a crucial role in the algebraic study of residuated lattices, a topic which is in the focus of the recent talk. As a demonstration the structural description of e-involutive uninorms will be presented here. The structural description inevitably relies on co-residuated operations which are derived from the uninorm to be described, thus leading to the notion of skew pairs and skew duals.

2 Structural description of e-involutive uninorms

Residuated lattices and substructural logics are subjects of intense investigation. Substructural logics encompass among many others, classical logic, intuitionistic logic, relevance logics, many-valued logics, fuzzy logics, linear logic and their non-commutative versions. Equivalent algebraic counterparts of substructural logics are classes of residuated lattices. Below we give a structural description for both e-involutive uninorms on the real unit interval $[0,1]$ and finite e-involutive uninorm chains.

For any binary operation $\circledast$ (on a poset) which is commutative and non-decreasing one can define its *residuum* $\to_\circledast$ by $x \circledast y \le z \iff x \to_\circledast z \ge y$. The displayed equivalence is often referred to as *adjointness conditions*. If $\to_\circledast$ exists, it has an equivalent description, namely, $\to_\circledast$ is the unique binary operation on the poset such that we have $x \to_\circledast y = \max\{z \mid x \circledast z \le y\}$.

Let $\mathcal{C} = \langle X, \le, \bot, \top \rangle$ be a bounded poset. A *involution* over $\mathcal{C}$ is an order reversing bijection on X such that its composition by itself is the identity map of X. Involutions are continuous in the order topology of $\mathcal{C}$. *T-conorms* (resp. *t-norms*) over $\mathcal{C}$ are commutative monoids on X with unit element $\bot$ (resp. $\top$). T-conorms and t-norms are *duals* of one another. That is, for any involution $'$ and t-conorm $\oplus$ over $\mathcal{C}$, the operation $\odot$ on X defined by $x \odot y = (x' \oplus y')'$ is a t-norm over $\mathcal{C}$. Vice versa, for any involution $'$ and t-norm $\odot$ over $\mathcal{C}$, the operation $\oplus$ on X defined by $x \oplus y = (x' \odot y')'$ is a t-conorm over $\mathcal{C}$. *Uninorms* over $\mathcal{C}$ [13, 1] are commutative monoids on X with unit element e (which may be different from $\bot$ and $\top$). Every uninorm over $\mathcal{C}$ has an *underlying t-norm* $\odot$ and *t-conorm* $\oplus$ acting on the subdomains $[\bot, e]$ and $[e, \top]$, respectively. That is, for any uninorm $\circledast$ over $\mathcal{C}$, its restriction to $[\bot, e]$ is a t-norm over $[\bot, e]$, and its restriction to $[e, \top]$ is a t-conorm over $[e, \top]$.

Definition 1. *$\langle X, \le, \bot, \top, e, f, \circledast \rangle$ is called an* involutive uninorm algebra *if $\mathcal{C} = \langle X, \le, \bot, \top \rangle$ is a bounded poset, $\circledast$ is a uninorm over $\mathcal{C}$ with unit element e, for every $x \in X$, $x \to_\circledast f = \max\{z \in X \mid x \circledast z \le f\}$ exists, and for every $x \in X$, we have $(x \to_\circledast f) \to_\circledast f = x$. If $\mathcal{C}$ is a chain, we call $\langle X, \le, \bot, \top, e, f, \circledast \rangle$ an involutive uninorm chain. In an involutive uninorm algebra one can define an order-reversing involution by $x' = x \to_\circledast f$.*

Definition 2. *$\langle X, \le, \bot, \top, e, \circledast \rangle$ is called an e-*involutive uninorm algebra *if $\langle X, \le, \bot, \top, e, e, \circledast \rangle$ is an involutive uninorm algebra.*

Definition 3. *For any binary operation $\bullet$ (on a poset) which is commutative and non-decreasing one can define its* co-residuum *$\leftarrow_\bullet$ by $x \bullet y \ge z \iff x \leftarrow_\bullet z \le y$. If $\leftarrow_\bullet$ exists it has an equivalent description: $\leftarrow_\bullet$ is the unique binary operation on the poset such that $x \leftarrow_\bullet y = \min\{z \mid x \circledast z \le y\}$.*

Definition 4. *For any commutative residuated chain $\langle X, \le, \oplus, \to_\oplus, 1 \rangle$, define $\circledbullet : X \times X \to X$ by $x \circledbullet y = \inf\{u \oplus v \mid u > x, v > y\}$, and call it the* skewed pair *of $\oplus$. For any commutative co-residuated chain $\langle X, \le, \circledbullet, \to_\circledbullet, 1 \rangle$, define $\oplus : X \times X \to X$ by $x \oplus y = \sup\{u \circledbullet v \mid u < x, v < y\}$, and call it the* skewed pair *of $\oplus$. Call $(\oplus, \circledbullet)$ a* skew pair.

Definition 5. *Let $(L_2, \leq)$ be a chain and $L_1 \subseteq L_2$. Let $(L_1, \oplus, \to_\oplus, \leq, \top)$ be a commutative residuated chain and $'$ be an order reversing involution on L_2. The operation $\odot$ is said to be* dual *to $\oplus$ with respect to $'$ if $\odot$ is a binary operation on $(L_1)' = \{x' \mid x \in L_1\}$ given by $x \odot y = (x' \oplus y')'$. We say that the operation $\circledbullet$ is* skew dual *to $\oplus$ with respect to $'$ if $\circledbullet$ is the skewed pair of $\odot$.*

Definition 6. *Let $\mathcal{C} = \langle X, \leq, \perp, \top \rangle$ be a bounded chain, and $'$ be an involution on X with fixed point $e \in X$. For any left-continuous t-conorm $\oplus$ on $[e, \top]$, define its skew symmetrization $\oplus_{\underline{\mathbf{s}}} : X \to X$ as follows.*

$$x \oplus_{\underline{\mathbf{s}}} y = \begin{cases} x \oplus y & \text{if } x, y \in [e, \top] \\ (x \to_\oplus y')' & \text{if } x \in [e, \top] \text{ and } y \in [\perp, e] \text{ and } x \leq y' \\ (y \to_\oplus x')' & \text{if } x \in [\perp, e] \text{ and } y \in [e, \top] \text{ and } x \leq y' \\ (x' \circledast y')' & \text{if } x, y \in [\perp, e] \\ x' \leftarrow_{\circledast} y & \text{if } x \in [\perp, e] \text{ and } y \in [e, \top] \text{ and } x \geq y' \\ y' \leftarrow_{\circledast} x & \text{if } x \in [e, \top] \text{ and } y \in [\perp, e] \text{ and } x \geq y' \end{cases}, \tag{1}$$

where $\circledast$ denotes the skewed pair of $\oplus$.
For any left-continuous t-norm $\odot$ on $[\perp, e]$, define its skew symmetrization $\odot_{\bar{\mathbf{s}}} : X \to X$ as follows.

$$x \odot_{\bar{\mathbf{s}}} y = \begin{cases} (x' \circledbullet y')' & \text{if } x, y \in [e, \top] \\ x' \leftarrow_{\circledbullet} y & \text{if } x \in [e, \top] \text{ and } y \in [\perp, e] \text{ and } x \leq y' \\ y' \leftarrow_{\circledbullet} x & \text{if } x \in [\perp, e] \text{ and } y \in [e, \top] \text{ and } x \leq y' \\ (y \to_\odot x')' & \text{if } x \in [e, \top] \text{ and } y \in [\perp, e] \text{ and } x \geq y' \\ (x \to_\odot y')' & \text{if } x \in [\perp, e] \text{ and } y \in [e, \top] \text{ and } x \geq y' \\ x \odot y & \text{if } x, y \in [\perp, e] \end{cases}, \tag{2}$$

where $\circledbullet$ denotes the skewed pair of $\odot$.

Theorem 1. *Any e-involutive uninorm on $[0,1]$ and any finite e-involutive uninorm chain can be represented as the skew symmetrization of its underlying t-conorm or t-norm.*

Corollary 1. *For any e-involutive uninorm chain on $[0,1]$ and for any finite e-involutive uninorm chain, its underlying t-norm and t-conorm form a skew dual pair with respect to $'$. Furthermore, $\circledast$ is self skew dual with respect to $'$.*

References

1. J. C. Fodor, R. R. Yager, A. Rybalov, *Structure of uninorms*, International Journal Of Uncertainty Fuzziness And Knowledge-Based Systems, 5 (1997), 411–427.
2. J.C. Fodor, *Contrapositive symmetry of fuzzy implications*, *Fuzzy Sets and Systems*, 69 (1995) 141–156.
3. R. Horčik, *Structure of commutative cancellative integral residuated lattices on (0, 1]*, Algebra universales 57 (2007) 303–332.
4. S. Jenei, *How to construct left-continuous triangular norms state of the art*, Fuzzy Sets and Systems, 143 (2004), 27-45.
5. S. Jenei, *On reflection invariance of residuated chains*, Annals of Pure and Applied Logic, (to appear)
6. S. Jenei, *On the geometry of associativity*, Semigroup Forum, 74 (3): 439-466 (2007)
7. S. Jenei, *On the structure of rotation-invariant semigroups*, Archive for Mathematical Logic, 42 (2003), 489–514.
8. S. Jenei, *Structural description of a class of involutive uninorms via skew symmetrization* (submitted)
9. S. Jenei, F. Montagna, *On the continuity points of left-continuous t-norms*, Archive for Mathematical Logic, 42 (2003), 797–810.
10. C. Noguera, F. Esteva, J Gispert, *Perfect and bipartite IMTL-algebras and disconnected rotations of prelinear semihoops*, Archive for Mathematical Logic 44 (7): 869–886 (2005)

11. K.C. Maes, B. De Baets, *On the structure of left-continuous t-norms that have a continuous contour line*, Fuzzy Sets and Systems 158 (2007), 843–860.
12. P.S. Mostert, A.L. Shields, *On the structure of semigroups on a compact manifold with boundary*. Ann. Math., 65 (1957), 117–143.
13. R. R. Yager, A. Rybalov, *Uninorm aggregation operators*, Fuzzy Sets and Systems, Vol. 80 (1996), 111–120.

Lipschitzian De Morgan triplets with strong negations

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Abstract. We study Lipschitzian strong negations and Lipschitzian De Morgan triplets (T,S,n), where T is a t-norm, S is a t-conorm, and n is a strong negation. We also investigate the relationships between the best Lipschitzian constants of connectives T, S and n. Several examples are included.

1 Introduction

In this contribution we study Lipschitzian strong negations and Lipschitzian De Morgan triplets (T,S,n) of fuzzy connectives, where T is a t-norm, S is a t-conorm, and n is a strong negation [1, 4–6], especially the relationships between the best Lipschitzian constants of connectives T, S and n. In the paper all functions T, S and n are considered to be Lipschitzian with respect to the L_1-norm. The Lipschitzian property of connectives T, S, n is very desirable, because it guarantees the stability of these functions with respect to input errors [2, 3]. Simply said, k-Lipschitzian functions do not increase the changes of inputs by more than a multiplicative factor of k. Therefore this property is of great importance in fuzzy sets and fuzzy logic applications where possible errors in input values of membership functions cannot be avoided.

2 Lipschitzian strong negations

Definition 1. (i) *A decreasing function* $n\colon [0,1] \to [0,1]$ *is called a negation if*

$$n(0) = 1 \quad \textit{and} \quad n(1) = 0.$$

(ii) *A negation n is a strong negation if it is an involution, i.e.,* $n \circ n = id_{[0,1]}$.

Note that each strong negation is a strict negation, i.e., it is continuous and strictly decreasing.

We will be interested in Lipschitzian (strong) negations. Lipschitzian properties of one place $[0,1] \to [0,1]$ functions are defined as follows.

Definition 2. *Let* $c \in [0,\infty[$. *A function* $f\colon [0,1] \to [0,1]$ *is c-Lipschitzian if for all* $x_1, x_2 \in [0,1]$,

$$|f(x_1) - f(x_2)| \le c\,|x_1 - x_2|. \tag{1}$$

Evidently, if f is k-Lipschitzian, then it is also p-Lipschitzian for any $p > k$.

Definition 3. (i) *A function $f\colon [0,1] \to [0,1]$ is Lipschitzian if*

$$\sup\left\{ \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|} \,|\, x_1,x_2 \in [0,1], x_1 \neq x_2 \right\} \tag{2}$$

is finite.

(ii) *If f is Lipschitzian then the number c^* which is equal to the supremum in (2), is called the best Lipschitzian constant of f.*

Evidently, if $n\colon [0,1] \to [0,1]$ is a c-Lipschitzian negation then necessarily $c \geq 1$.

Proposition 1. *Fix a constant $c \in [1,\infty[$. Then for each c-Lipschitzian strong negation n it holds $\underline{n}_c \leq n \leq \overline{n}_c$, where the functions $\underline{n}_c, \overline{n}_c\colon [0,1] \to [0,1]$ are c-Lipschitzian strong negations given by*

$$\underline{n}_c(x) = \max\left(1-cx, \frac{1-x}{c}\right), \tag{3}$$

$$\overline{n}_c(x) = \min\left(c\,(1-x), 1-\frac{x}{c}\right). \tag{4}$$

The function $\underline{n}_c$ is the smallest c-Lipschitzian strong negation and $\overline{n}_c$ is the greatest one.

Example 1. Let $c = 1$. Then for each $x \in [0,1]$, $\underline{n}_1(x) = \overline{n}_1(x) = 1-x$, which means that the only 1-Lipschitzian strong negation is the standard negation $n_s\colon [0,1] \to [0,1]$, $n_s(x) = 1-x$.

Proposition 2. *For the best Lipschitzian constant c^* of a Lipschitzian strong negation n it holds*

(i) $c^* = |n'(0^+)|$ *if n is a convex function,*
(ii) $c^* = |n'(1^-)|$ *if n is a concave function.*

(*Here n' denotes the derivative of n.*)

Example 2. Consider the Sugeno negations $\{n_\lambda\}_{\lambda\in]-1,\infty[}$ given by

$$n_\lambda(x) = \frac{1-x}{1+\lambda x}, \quad x \in [0,1].$$

All functions n_λ are strong negations. Moreover, the functions n_λ are concave for $\lambda \in]-1,0]$, and convex for $\lambda \in [0,\infty[$. As the derivative is given by $n'_\lambda(x) = -\frac{1+\lambda}{(1+\lambda x)^2}$, $x \in [0,1]$, by Proposition 2, the best Lipschitzian constant of n_λ is

$$c^*_\lambda = \begin{cases} \frac{1}{1+\lambda} & if \;\; \lambda \in]-1,0[, \\ 1+\lambda & if \;\; \lambda \in [0,\infty[. \end{cases}$$

Example 3. The function $n\colon [0,1] \to [0,1]$, $n(x) = \sqrt{1-x^2}$, is a strong negation, but it is not Lipschitzian. The function n is concave and $|n'(1^-)| = +\infty$.

The following claim is immediate due to the involutivity of strong negations.

Proposition 3. *If n is a c-Lipschitzian strong negation then at all points, where the derivative exists, it holds*

$$|n'(x)| \in \left[\frac{1}{c}, c\right].$$

Note that the derivatives of the strong negations $\underline{n}_c, \overline{n}_c$ defined by (3), (4), have minimal ranges, $Ran\,\underline{n}'_c = Ran\,\overline{n}'_c = \left\{-\frac{1}{c}, -c\right\}$.

Trillas [13] characterized all decreasing involutions on the interval $[0,1]$. According to this characterization, see also Th. 2.38 in the monograph by Klement et al. [6], each strong negation n can be represented in the form

$$n(x) = g^{-1}(1 - g(x)),$$

where $g\colon [0,1] \to [0,1]$ is a monotone bijection. The function g is called a generator of n.

Proposition 4. *Let $g\colon [0,1] \to [0,1]$ be a monotone p-Lipschitzian bijection, whose inverse function g^{-1} is r-Lipschitzian. Then the strong negation n generated by g is c-Lipschitzian where $c = p\,.\,r$.*

Remark 1. By Proposition 4, if p^* is the best Lipschitzian constant of the function g and r^* is the best Lipschitzian of g^{-1}, then the strong negation n generated by g has the best Lipschitzian constant $c^* \le p^*.\,r^*$.

Proposition 5. *For each Lipschitzian strong negation n – with the best Lipschitzian constant c^* – there is a monotone bijection $g\colon [0,1] \to [0,1]$ generating n, such that g is p^*-Lipschitzian, g^{-1} is r^*-Lipschitzian and $c^* = p^*.\,r^*$, where p^*, r^* are the best Lipschitzian constants of g and g^{-1}, respectively.*

Note that if n is a c-Lipschitzian strong negation, then the function

$$g(x) = \frac{1 - n(x) + x}{2}, \quad x \in [0,1], \tag{5}$$

is an increasing $[0,1] \to [0,1]$ bijection generating n. It can be shown that the function g is $p^* = \frac{c^*+1}{2}$-Lipschitzian and g^{-1} is $r^* = \frac{2c^*}{c^*+1}$-Lipschitzian, and these constants are the best Lipschitzian constants for g and g^{-1}. Evidently, $c^* = p^*.\,r^*$.

The following example illustrates Proposition 5.

Example 4. Consider again the Sugeno negations $\{n_\lambda\}_{\lambda \in]-1,\infty[}$, see Example 2. Given a $\lambda \in]-1,\infty[$, the function $g_\lambda\colon [0,1] \to [0,1]$,

$$g_\lambda(x) = \frac{(2+\lambda)x + \lambda x^2}{2(1+\lambda x)},$$

constructed by (5), generates n_λ. All functions g_λ are differentiable, with the derivative

$$g'_\lambda(x) = \frac{1}{2} + \frac{1+\lambda}{2(1+\lambda x)^2}.$$

If $\lambda \ge 0$,

$$\frac{\lambda+2}{2(1+\lambda)} \le g'_\lambda(x) \le \frac{\lambda+2}{2}.$$

The best Lipschitzian constant for g_λ is $p^*_\lambda = \frac{\lambda+2}{2}$ and for g_λ^{-1} is $r^*_\lambda = \frac{2(1+\lambda)}{\lambda+2}$. By Proposition 5, for the best Lipschitzian constant c^*_λ of n_λ we have $c^*_\lambda = p^*_\lambda.\,r^*_\lambda = \lambda + 1$, which agrees with the result of Example 2. Similarly, for $\lambda \in]-1,0]$, $p^*_\lambda = \frac{\lambda+2}{2(1+\lambda)}$, $r^*_\lambda = \frac{2}{\lambda+2}$, thus $c^*_\lambda = p^*_\lambda.\,r^*_\lambda = \frac{1}{\lambda+1}$.

3 Lipschitzian De Morgan triplets

Assume that T is a t-norm, S is a t-conorm and n is a strong negation. A triplet (T,S,n) is a *De Morgan triplet*, if for all $(x,y) \in [0,1]^2$ it holds

$$S(x,y) = n(T(n(x),n(y))),$$

or, equivalently,

$$T(x,y) = n(S(n(x),n(y))).$$

More details can be found, e.g., in the monographs [4–6].

In the next parts of the paper we will be interested in Lipschitzian De Morgan triplets. A De Morgan triplet (T,S,n) is called a *Lipschitzian De Morgan triplet* if all functions T, S and n are Lipschitzian.

Definition 4. *Let $k \in [0,\infty[$. A function $F\colon [0,1]^2 \to [0,1]$ is k-Lipschitzian if for all $x_1,x_2,y_1,y_2 \in [0,1]$,*

$$|F(x_1,y_1) - F(x_2,y_2)| \le k\,(|x_1-x_2| + |y_1-y_2|). \tag{6}$$

Definition 5. (i) *A function $F\colon [0,1]^2 \to [0,1]$ is Lipschitzian if*

$$\sup\left\{\frac{|F(x_1,y_1)-F(x_2,y_2)|}{|x_1-x_2|+|y_1-y_2|} \mid (x_1,y_1),\,(x_2,y_2) \in [0,1]^2,\,(x_1,y_1) \neq (x_2,y_2)\right\} \tag{7}$$

is finite.

(ii) *If F is Lipschitzian then the number k^* which is equal to the supremum in (7), is called the best Lipschitzian constant of F.*

Note that for each Lipschitzian function F, $k^* = \inf\{k \in [0,\infty[\mid F \text{ is } k-\text{Lipschitzian}\}$.

From the existence of neutral element, it follows that neither t-norms nor t-conorms can be k-Lipschitzian with $k < 1$. Thus the smallest best Lipschitzian constant k^* in the framework of t-norms and t-conorms is $k^* = 1$. The class of 1-Lipschitzian t-norms has been characterized by Moynihan [10]. According to this characterization a continuous Archimedean t-norm T is 1-Lipschitzian if and only if it has a convex additive generator. Note that 1-Lipschitzian t-norms are copulas. More details on t-norms can be found, e.g., in monographs [1, 6, 11].

A partial characterization of k-Lipschitzian t-norms for $k > 1$ has been given by Shyu [12]. A complete characterization of Archimedean k-Lipschitzian t-norms based on the k-convexity of their additive generators has been given by Mesiarová [7–9]. In general, a t-norm T is k-Lipschitzian if and only if it is an ordinal sum of t-norms with k-Lipschitzian summands [7].

Theorem 1. *Let (T,S,n) be a De Morgan triplet with a strong negation n. If the t-norm T is a-Lipschitzian and the negation n is c-Lipschitzian then the t-conorm S is b-Lipschitzian, where $b = a c^2$.*

Corollary 1. *A De Morgan triplet (T,S,n) with a strong negation n is Lipschitzian if and only if the functions T and n (S and n) are Lipschitzian.*

Remark 2. (i) Obviously, if the t-conorm S is b-Lipschitzian and the negation n is c-Lipschitzian then T is a a-Lipschitzian t-norm with $a = b c^2$.

(ii) Though the constants $b = a c^2$ in Theorem 1 and $a = b c^2$ in item (i) of this remark, need not be the best Lipschitzian constants for S and T, respectively, Theorem 1 and Remark 2(i) cannot be strengthen in general.

(iii) For the best Lipschitzian constants a^*, b^* and c^* of functions T, S and n, respectively, it holds $a^* \leq b^*.(c^*)^2$ and $b^* \leq a^*.(c^*)^2$. Evidently, both inequalities can be turned into equalities iff $c^* = 1$, i.e., iff $n = n_s$, and then $b^* = a^*$.

Example 5. Let $(T, S_L, \overline{n}_2)$ be the De Morgan triplet with the Łukasiewicz t-conorm S_L, $S_L\colon [0,1]^2 \to [0,1]$, $S_L(x,y) = \min(x+y, 1)$, which is 1-Lipschitzian, and with the negation $\overline{n}_2$ – the greatest 2-Lipschitzian strong negation, $\overline{n}_2(x) = \min\left(2(1-x), 1-\frac{x}{2}\right)$, introduced in Theorem 1. Using the notation from Remark 2(i), we have $b = 1$ and $c = 2$. As the t-norm T is given by

$$T(x,y) = \overline{n}_2\left(\min(\overline{n}_2(x) + \overline{n}_2(y), 1)\right),$$

it holds $T\left(\frac{5}{6}, \frac{5}{6}\right) = \frac{2}{3}$, $T\left(\frac{3}{4}, \frac{3}{4}\right) = 0$, i.e., for couples $\left(\frac{5}{6}, \frac{5}{6}\right)$ and $\left(\frac{3}{4}, \frac{3}{4}\right)$ we obtain

$$\frac{\left|T\left(\frac{5}{6}, \frac{5}{6}\right) - T\left(\frac{3}{4}, \frac{3}{4}\right)\right|}{\left|\frac{5}{6} - \frac{3}{4}\right| + \left|\frac{5}{6} - \frac{3}{4}\right|} = 4 = b\,c^2.$$

Hence, the best Lipschitzian constant for T is $a^* = 4$.

In BL-algebras of Hájek [5], T is a continuous t-norm and R_T is the residual implication related to T, i.e., $R_T\colon [0,1]^2 \to [0,1]$, $R_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}$. Then the negation $n_T\colon [0,1] \to [0,1]$ given by $n_T(x) = R_T(x,0)$, is a strong negation if and only if T is a nilpotent t-norm generated by a normed additive generator t ($t\colon [0,1] \to [0,1]$ is a decreasing bijection), i.e.,

$$T(x,y) = t^{-1}\left(\min(1, t(x) + t(y))\right),$$

and then

$$n_T(x) = t^{-1}(1 - t(x)).$$

A triplet (T, S, n_T) is a De Morgan triplet iff $S = S_T$, where

$$S_T(x,y) = t^{-1}\left(\max(0, t(x) + t(y) - 1)\right), \quad (x,y) \in [0,1]^2.$$

More details can also be found in the monograph by Klement et al. [6].

Theorem 2. *Let T be a nilpotent t-norm generated by a normed additive generator t. Let the function t be p-Lipschitzian and let its inverse t^{-1} be r-Lipschitzian. Then the De Morgan triplet (T, S_T, n_T) is a Lipschitzian De Morgan triplet whose components T, S_T and n_T are $p\,.\,r$-Lipschitzian functions.*

Note that in general the constant $p\,.\,r$ in Theorem 2 cannot be improved. However, if p^* and r^* are the best Lipschitzian constants of t and t^{-1}, respectively, then for the best Lipschitzian constants a^*, b^*, c^* of T, S_T and n_T, respectively, it holds $a^* \leq p^*.\,r^*$, $b^* \leq p^*.\,r^*$ and $c^* \leq p^*.\,r^*$.

Example 6. Put $t = \overline{n}_2$, i.e. $t(x) = \min\left(2(1-x), 1-\frac{x}{2}\right)$, $x \in [0,1]$. The function t and also its inverse are 2-Lipschitzian functions ($p^* = r^* = 2$). The t-norm T generated by t is given by

$$T(x,y) = \begin{cases} x+y-1 & \text{if } x+y \geq \frac{5}{3} \\ 4x+4y-6 & \text{if } \frac{3}{2} \leq x+y < \frac{5}{3} \\ \max(0, x+4y-4, 4x+y-4) & \text{otherwise,} \end{cases}$$

and the negation n_T generated by t is given by

$$n_T(x) = \min\left(1 - \frac{x}{4}, \frac{3}{2} - x, 4(1-x)\right).$$

Both functions T and n_T are 4-Lipschitzian ($a^* = c^* = 4 = p^*.r^*$).
However, the t-conorm S_T is 1-Lipschitzian ($b^* = 1$), because of the fact that its additive generator $s = 1 - t$, $s(x) = \max\left(2x - 1, \frac{x}{2}\right)$, is convex (compare with the result of Moynihan [10] for t-norms). Note that S is given by

$$S_T(x,y) = \begin{cases} \min\left(x+y, \frac{x+y+2}{4}\right) & \text{if } (x,y) \in \left[0, \frac{2}{3}\right]^2, \\ \min\left(1, \max\left(\frac{x}{4}+y, x+\frac{y}{4}, x+y-\frac{1}{2}\right)\right) & \text{otherwise.} \end{cases}$$

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References

1. C. Alsina, M.J. Frank, and B. Schweizer, *Associative Functions: Triangular Norms and Copulas*. World Scientific, Singapore, 2006.
2. G. Beliakov, A. Pradera, and T. Calvo, *Aggregation Functions: A Guide for Practitioners.* Springer, Heidelberg, Berlin, New York, 2007.
3. T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar, Aggregation operators: Properties, classes and construction methods. In T. Calvo, G. Mayor, R. Mesiar, eds., Aggregation Operators. New Trends and Applications, Physica Verlag, Heidelberg, New York, 2002, pp. 3-104.
4. J. Fodor, M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.
5. P. Hájek, *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht, 1998.
6. E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms*. Kluwer Academic Publishers, Dordrecht, 2000.
7. A. Mesiarová, A note on two open problems of Alsina, Frank and Schweizer. *Aequationes Math.* **72** (2006) 41–46.
8. A. Mesiarová, Extremal k-Lipschitz t-conorms. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems* **14** (2006) 247–257.
9. A. Mesiarová, Approximation of k-Lipschitz t-norms. *Int. J. General Systems* **36** (2007) 205–218.
10. R. Moynihan, On τ_T-semigroups of probability distribution functions II. *Aequationes Math.* **17** (1978) 19–40.
11. R.B. Nelsen, *An Introduction to Copulas*. Lecture Notes in Statistics 139, Springer Verlag, New York, 1999.
12. Y.-H. Shyu, *Absolute continuity in the* τ_T*-operations*. PhD Thesis, Illinois Institute of Technology, Chicago, 1984.
13. E. Trillas, Sobre funciones de nagación en la teoría de conjuntas difusos. *Stochastica* **3** (1979) 47–60.

Foundations of the theory of (L,M)-fuzzy topological spaces

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1 Introduction and motivation

To explain the motivation of our work we give a short glimpse into the history of *Fuzzy Topology* called more recently *Lattice-Valued Topology* or *Many Valued Topology*. In 1968, C. L. Chang [1] introduced the notion of a fuzzy topology on a set X as a subset $\tau \subseteq [0,1]^X$ satisfying the natural counterparts of the axioms of topology: (1) $0_X, 1_X \in \tau$; (2) $U,V \in \tau \Rightarrow U \wedge V \in \tau$; (3) $\mathcal{U} \subseteq \tau \Rightarrow \bigvee \mathcal{U} \in \tau$. Five years later, J. A. Goguen [2] replaced the interval $[0,1]$ with an arbitrary complete infinitely distributive lattice L thus obtaining the concept of an L-fuzzy topology or just an *L-topology*.[3] In 1980, U. Höhle [3] came to the concept of an L-fuzzy topology being an L-subset T of the powerset $\mathcal{P}(X) \approx 2^X$, that is a map $T : \mathcal{P}(X) \to L$ such that: (1) $T(\emptyset) = T(X) = 1$, (2) $T(U \cap V) \geq T(U) \wedge T(V)$ for any U, $V \in \mathcal{P}(X)$, and (3) $T(\bigvee \mathcal{U}) \geq \bigwedge T(\mathcal{U})$ for all $\mathcal{U} \subseteq \mathcal{P}(X)$. The latter concept is now called a *fuzzifying topology*, after the 1991 paper by Mingsheng Ying [4].

Finally, in 1985, the authors of this abstract independently introduced the concept of an L-fuzzy topology X as a map $T : L^X \to L$ such that: (1) $T(0_X) = T(1_X) = 1$; (2) $T(U \wedge V) \geq T(U) \wedge T(V)$ $\forall U,V \in L^X$; (3) $T(\bigvee \mathcal{U}) \geq \bigwedge T(\mathcal{U})$ $\forall \mathcal{U} \subseteq L^X$; see [5], [10], [11]. For historical reason we note that in [10] the case $L = [0,1]$ was considered and developed, while [5] merely introduces (without further developing) fuzzy topologies of the form $T : L^X \to M$ with L and M being complete lattices in a variable-basis setting á la S. E. Rodabaugh [8] (see next section).

Note also that some authors (J.A. Goguen [2] was the first) consider topological-type structures in the context of L-fuzzy sets in case when L is endowed with an additional binary operation $*$ which allows to introduce residuation in L. However, we will not scare this direction since our interest here concerns mainly the role of the lattice properties of L in the research of topological-type structures in the context of fuzzy sets.

2 (L,M)-fuzzy topologies and (L,M)-fuzzy topological spaces: Basic notions

Already in 1993, the authors of this talk started with their project to extend the well-established second author's theory from $L = M = [0,1]$ to a general theory of (L,M)-fuzzy topological spaces (so far, the only published outcome is our papers [6], [7]) because it was clear for us that the role of the lattice for fuzzy sets is quite different than the role of the range lattice for the fuzzy topology. Thus we came to the agreement to consider an (L,M)-fuzzy topology on a set X being a map $T : L^X \to M$ satisfying the properties analogous to the properties of an L-fuzzy topology defined in the previous paragraph. Since M usually necessitates more lattice properties than L we have chosen to assume M being completely

[3] In actual fact, Goguen replaced $[0,1]$ by – what is now called – a *quantale* with unit being the top element of L.

distributive [4] and L being a complete lattice. The pair (X,T) is called an (L,M)-fuzzy topological space. The aim of this talk is to present main ideas and discuss principal concepts and results of the theory of (L,M)-fuzzy topological spaces so far developed. In the following we list the main items to be discussed in the talk.

3 The tools

When developing the theory of (L,M)-fuzzy topological spaces we essentially rely on the auxiliary concepts of a $\bigvee$-map and a $\wedge$-map, "extracting" separate conditions of an (L,M)-fuzzy topology. By a $\bigvee$-map we call a mapping $S : L^X \to M$ such that $S(\bigvee \mathcal{A}) \geq \bigwedge S(\mathcal{A})$ for each $\mathcal{A} \subseteq L^X$; a mapping $S : L^X \to M$ is called a $\wedge$-map if $S(A \wedge B) \geq S(A) \wedge S(B)$ for any $A, B \in L^X$. In particular, we define an operator $T : M^{L^X} \to M^{L^X}$, assigning to a map $S : L^X \to M$ a $\bigvee$-map $T_S : L^X \to M$, which moreover is a $\wedge$-map (and hence essentially an (L,M)-fuzzy topology) whenever S itself is a $\wedge$-map. A pair (X,S) will be refereed to as a $\bigvee$- space ($\wedge$-space) if S is a $\bigvee$-map (a $\wedge$-map) resp.

Another tool essentially used and investigated in the work are powerset operators introduced by S.E. Rodabaugh [9]. It is shown that powerset operators $(f_L^{\leftarrow})_M^{\leftarrow} : M^{L^X} \to M^{L^X}$ and $(f_L^{\leftarrow})_M^{\rightarrow} : M^{L^X} \to M^{L^X}$ are both $\wedge$-map preserving and $\bigvee$-map preserving.

4 Lattice properties of (L,M)-fuzzy topologies

Let $\mathbb{T}(X,L,M)$ stand for the family of all (L,M)-fuzzy topologies on a set X. Then

- $\mathbb{T}(X,L,M)$ is a complete lattice in which sups and infs are defined as follows:

$$\left(\bigwedge \mathcal{T}\right)(A) = \bigwedge_{T \in \mathcal{T}} T(A) \quad \text{and} \quad \left(\bigvee \mathcal{T}\right)(A) = \bigvee\{\alpha \in M : A \in \bigvee_{T \in \mathcal{T}} T_\alpha\}$$

  where $A \in L^X$, $\mathcal{T} \subseteq \mathbb{T}(X,L,M)$ and $T_\alpha = \{U \in L^X \mid T(U) \geq \alpha\}$;
- $(\bigwedge \mathcal{T})_\alpha = \bigcap_{T \in \mathcal{T}} T_\alpha$.
- $(\bigvee \mathcal{T})_\alpha = \bigcap\{\bigvee_{T \in \mathcal{T}} T_\gamma : \gamma \in \mathrm{COPRIME}(M) \text{ and } \gamma \ll \alpha\}$
  where $\ll$ is the way-below relation in M.

5 Category **TOP**(L,M)

A mapping $f : X \to Y$ where (X,T_X) and (Y,T_Y) are (L,M)-fuzzy topological spaces is called continuous if $T_X(f^{-1}(V) \geq T_Y(V)$ for each $V \in L^Y$. Let **TOP**(L,M) be the category with (L,M)-topological spaces as objects and continuous mappings as morphisms. Some properties of this category are discussed. In particular, basing on the results of the previous paragraph we show the existence of final and initial structures in **TOP**(L,M). This allows to describe basic operations and constructions such as products, co-products, etc., in this category. Its full subcategories **TOP**(2,2), **TOP**(L,2), **TOP**(2,L) and **TOP**(L,L) with appropriately chosen lattice L are, respectively, the categories of topological spaces, of Chang-Goguen L-topological spaces, of fuzzifying topological spaces and of L-fuzzy topological spaces.

[4] The condition of complete distributivity in some cases can be weakened, however we assume it here in order to make exposition more homogeneous

6 Lower set valued L-topological spaces

An important special case of (L,M)-fuzzy topological spaces are $(L,\mathbb{L}(M))$-fuzzy topological spaces where $\mathbb{L}(M)$ is the complete lattice of lower subsets of the lattice M. Such spaces were first considered in our paper [6] under the name *lower set valued L-topological spaces*.

Given an arbitrary map $T : L^X \to \mathbb{L}(M)$ let $T^\vee : L^X \to M$ be defined by $T^\vee(A) = \bigvee T(A)\ \forall A \in L^X$. Further, given an arbitrary map $\mathcal{S} : L^X \to M$ let $\mathcal{S}^\downarrow : L^X \to \mathbb{L}(M)$ be defined by $\mathcal{S}^\downarrow(A) = \downarrow \mathcal{S}(A)$. We show that the operations $(\cdot)^\vee$ and $(\cdot)^\downarrow$ give rise to an adjoint situation $(\cdot)^\vee \dashv (\cdot)^\downarrow$ from the category $\mathbf{TOP}(L,\mathbb{L}(M))$ to the category $\mathbf{TOP}(L,M)$.

7 Interior operators and (L,M)-fuzzy topologies

Let (X,T) be an (L,M)-$\bigvee$-space, in particular, an (L,M)-fuzzy topological space. By setting

$$\mathrm{Int}_T(A,\alpha) = \bigvee\{U \in L^X \mid U \le A, T(U) \ge \alpha\},$$

we define a monotone operator $\mathrm{Int}_T : L^X \times M \to L^X$ such that for all $A \in L^X$, $\alpha \in M$:
(1) $\mathrm{Int}(1_X,\alpha) = 1_X$; (2) $\mathrm{Int}(A,\alpha) \le A$; (3) $\mathrm{Int}(\mathrm{Int}(A,\alpha),\alpha) = \mathrm{Int}(A,\alpha)$; (4) if $\mathrm{Int}(A,\alpha) = A^0\ \forall \alpha \in M^0 \subseteq M$, then $\mathrm{Int}(A, \vee M^0) = A^0$. Conversely given a monotone operator $\mathrm{Int} : L^X \times M \to L^X$ satisfying properties (1) - (4), by setting

$$T_{\mathrm{Int}}(U) = \bigvee\{\alpha \mid \mathrm{Int}(U,\alpha) \ge U\}$$

an (L,M)-$\bigvee$-space (X,T_{Int}) is obtained. Besides $T_{\mathrm{Int}_T} = T$ and $\mathrm{Int}_{T_{\mathrm{Int}}} = \mathrm{Int}$. Other properties of interior operators and their relations with (L,M)-fuzzy topologies are discussed.

8 Neighbourhood and q-neighbourhood operators in (L,M)-fuzzy topological spaces

Let (X,T) be an (L,M)-$\wedge$-space, in particular an (L,M)-fuzzy topological space. By setting

$$\mathcal{N}(x,\alpha,U) = \sup\{T(V) \mid V \le U, V(x) \ge \alpha\}$$

we define an operator $\mathcal{N} : X \times L \times L^X \to M$. This operator has the following properties:

(1N) if $\mathcal{N}(x,\alpha,U) > 0$, then $U(x) \ge \alpha$;
(2N) $\bigvee_{U \in L^X} \mathcal{N}(x,\alpha,U) = 1$;
(3N) $\mathcal{N}(x,\alpha,U_1 \wedge U_2) \ge \mathcal{N}(x,\alpha,U_1) \wedge \mathcal{N}(x,\alpha,U_2)$;
(4N) if $U \le U', \alpha' \le \alpha$, then $\mathcal{N}(x,\alpha',U') \ge \mathcal{N}(x,\alpha,U)$;
(5N) $\mathcal{N}(x,\alpha,U) \le \bigvee_{V \le U} \Big(\mathcal{N}(x,\alpha,V) \wedge (\Big(\bigwedge_{y \in X, W(y) \ge \beta} \mathcal{N}(y,\beta,W)\Big)\Big)$.

Conversely, starting with an operator $\mathcal{N} : X \times L \times L^X \to M$ satisfying (1N) - (5N), by setting $T_{\mathcal{N}}(U) = \inf_{x,\alpha} \mathcal{N}(x,\alpha,U)$ we obtain an (L,M)-$\wedge$-map $T : L^X \to M$. Besides $\mathcal{N}_{T_{\mathcal{N}}} = \mathcal{N}$ and $T_{\mathcal{N}_T} \ge T$. In case when T is an (L,M)-fuzzy topology the equality $T_{\mathcal{N}_T} = T$ holds.

Neighbourhood operators are used to characterize local structure of (L,M)-fuzzy topological spaces. In particular, local description of continuity for mappings of (L,M)-fuzzy topological spaces is given in terms of neghbourhood operators.

An alternative tool to describe local structure of (L,M)-fuzzy topological spaces and continuity of mappings is given by means of q-neighbourhood operators.

References

1. C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* **24** (1968), 182-190.
2. J. A. Goguen, The fuzzy Tychonoff theorem, *J. Math. Anal. Appl.* **43** (1973), 737–742.
3. U. Höhle, Upper semicontinuous fuzzy sets and applications, *J. Math. Anal. Appl.* **78** (1980), 659–673.
4. Ying Mingsheng, A new approach to fuzzy topology (I), *Fuzzy Sets and Syst.* **39** (1991), 303–321.
5. T. Kubiak, On Fuzzy Topologies, Ph. D. Thesis, Adam Mickiewicz Univ., Poznań, Poland (1985).
6. T. Kubiak, A. Šostak, Lower set valued fuzzy topologies, *Queast. Math.* **20** (1997), 423–429.
7. T. Kubiak, A. Šostak, A fuzzification of the category of M-valued L-topological spaces. *Applied General Topology* **5** (2004), 137–154.
8. S. E. Rodabaugh, A categorical accommodation of various notions of fuzzy topology, *Fuzzy Sets and Syst.* 9 (1983), 241–265.
9. S. E. Rodabaugh, Powerset operator foundations for poslat fuzzy set theories and topologies, [Chapter 2 in:] *Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory*, U. Höhle, S.E. Rodabaugh (Eds.), The Handbook of Fuzzy Sets Series, Vol.3, Kluwer Academic Publishers, Dordrecht, Boston, 1999, pp. 91–116.
10. A. Šostak, On a fuzzy topological structure, *Suppl. Rend. Circ. Matem. Palermo,* Ser II, **11** (1985), 89-103.
11. A. Šostak, Two decades of fuzzy topology: basic ideas, notions and results, *Russian Math. Surveys* **44:6** (1989), 125-186.

Behavioral analysis of aggregation functions

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1 Introduction

Aggregation functions arise wherever aggregating information is important: applied and pure mathematics (probability, statistics, decision theory, functional equations), operations research, computer science, and many applied fields (economics and finance, pattern recognition and image processing, data fusion, etc.). For recent references, see Beliakov et al. [2] and Grabisch et al. [9].

Let $\mathbb{I}$ be a real interval, bounded or not. Given an aggregation function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$, it is often useful to define values or indices that offer a better understanding of the general behavior of F with respect to its variables. These indices may constitute a kind of identity card of F and enable one to classify the aggregation functions according to their behavioral properties.

For example, given an internal aggregation function A (“internal” means $\mathsf{Min} \leqslant \mathsf{A} \leqslant \mathsf{Max}$, where Min and Max are the minimum and maximum functions, respectively), it might be convenient to appraise the degree to which A is conjunctive, that is, close to Min. Similarly, it might be very instructive to know which variables, among $x_1, \ldots, x_n$, have the greatest influence on the output value $\mathsf{A}(\mathbf{x})$.

In this note, we present various indices, such as: andness and orness degrees of internal functions, idempotency degrees of conjunctive and disjunctive functions, importance and interaction indices, tolerance indices, and dispersion indices.

Sometimes different indices can be considered to measure the same behavior. In that case it is often needed to choose an appropriate index according to the nature of the underlying aggregation problem.

The material presented here is a summary of [9, Chapter 10], a forthcoming monograph on aggregation functions written by the authors.

2 Expected values and distribution functions

A very informative treatment of a given function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ consists in applying it to a random input vector and examining the behavior of the output signal by computing its distribution function. However, determining an explicit form of the distribution remains very difficult in general.

Instead, we can calculate the expected value or, more generally, the moments of the output variable and derive indices that would provide information on the location of the output values within the range of the function.

3 Importance indices

When using a given aggregation function A of n variables, one may wonder which are the most influential variables in the computation of $\mathsf{A}(\mathbf{x})$, if any. We may say that no such variable exists if A is symmetric. In case symmetry does not hold, e.g., for weighted aggregation functions and for integral-based ones, it is very instructive to know the level or percentage of contribution of each variable in the computation of the result. We call this level of contribution or influential power the *importance index*.

For weighted aggregation functions, a naive answer to the above question is to take as importance index simply the weight of each variable. A first simple reason to discard this idea is that the definition, meaning and normalization of weights differ from one aggregation function to another: just consider the weighted arithmetic mean, with weights in $[0,1]$ summing up to 1, and the weighted maximum, whose weights are in $[0,1]$ with no summation condition, but whose maximum value is 1. A second reason is that intuitively, a weight value, say 0.5, does not have the same effect in a weighted arithmetic mean as in a weighted geometric mean.

A natural approach when $\mathbb{I}$ is a bounded closed interval $[a,b]$ is to compute an average of the marginal contribution of variable x_i.

4 Interaction indices

Although the notion of importance index is useful to analyze a given aggregation function, the description it provides is still very primitive. Take for example the arithmetic mean, the minimum, and the maximum. Since they are symmetric, they have the same importance index for all coordinates, yet they are extremely different aggregation functions, because the minimum operator is a conjunctive aggregation function, the maximum operator is disjunctive, and the arithmetic mean is neither conjunctive nor disjunctive.

The question is how to quantify or describe the difference between these aggregation functions, which are indistinguishable by the importance index. Since andness and orness are reserved to internal aggregation functions, other indices have to be found.

The key of the problem lies in the interrelation between variables. The notion of importance index is based on the variation of the aggregated value vs. the total variation of a given variable, the others being fixed. We may consider the variation induced by the mixed variation of two variables, or more. This is expressed by the *second order (total) variation of* A *with respect to coordinates i and j*.

5 Tolerance indices

Some internal aggregation functions are more or less intolerant (respectively, tolerant) in the sense that they are bounded from above (respectively, below) by one of the input values or by a function of these values.

Here, we mainly deal with internal aggregation functions having veto and/or favor coordinates as well as k-conjunctive and k-disjunctive internal aggregation functions. Starting from the properties of these functions, we define indices that provide degrees to which an internal aggregation function is intolerant or tolerant.

6 Measures of arguments contribution and involvement

Given an aggregation function A, we consider in this final section the following two indices:

1. *The index of uniformity of arguments contribution*, which measures the uniformity of contribution of the n components of $\mathbf{x} \in \mathbb{I}^n$ in the computation of the aggregated value $\mathsf{A}(\mathbf{x})$.
2. *The index of arguments involvement*, which measures the proportion of arguments among $x_1, \ldots, x_n$ that are involved in the computation of the aggregated value $\mathsf{A}(\mathbf{x})$.

References

1. J. F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.
2. G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*. Studies in Fuziness and Soft Computing. Springer, Berlin, 2007.
3. J. J. Dujmović. Seven flavors of andness/orness. In B. De Baets, J. Fodor, and D. Radojević, editors, *Proc. Eurofuse 2005*, pages 81–92, Belgrade, 2005.
4. J. M. Fernández Salido and S. Murakami. Extending Yager's orness concept for the OWA aggregators to other mean operators. *Fuzzy Sets and Systems*, 139(3):515–542, 2003.
5. K. Fujimoto, I. Kojadinovic, and J.-L. Marichal. Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. *Games Econom. Behav.*, 55(1):72–99, 2006.
6. M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 5(5):587–607, 1997.
7. M. Grabisch and C. Labreuche. How to improve acts: an alternative representation of the importance of criteria in MCDM. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 9(2):145–157, 2001.
8. M. Grabisch, C. Labreuche, and J.-C. Vansnick. On the extension of pseudo-Boolean functions for the aggregation of interacting criteria. *European J. Oper. Res.*, 148(1):28–47, 2003.
9. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, UK, to appear in 2009.
10. M. Grabisch, J.-L. Marichal, and M. Roubens. Equivalent representations of set functions. *Mathematics of Operations Research*, 25(2):157–178, 2000.
11. M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Internat. J. Game Theory*, 28(4):547–565, 1999.
12. I. Kojadinovic, J.-L. Marichal, and M. Roubens. An axiomatic approach to the definition of the entropy of a discrete Choquet capacity. *Information Sciences*, 172:131–153, 2005.
13. A. Kolesárová. Revision of parametric evaluation of aggregation functions. In *Proc. 13th Zittau Fuzzy Colloquium*, pages 202–211, Zittau, Germany, September 2006.
14. C. Labreuche. Determination of the criteria to be improved first in order to improve as much as possible the overall evaluation. In *10th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2004)*, pages 609–616, Perugia, Italy, July 2004.
15. J.-L. Marichal. Behavioral analysis of aggregation in multicriteria decision aid. In *Preferences and decisions under incomplete knowledge*, pages 153–178. Physica, Heidelberg, 2000.
16. J.-L. Marichal. The influence of variables on pseudo-Boolean functions with applications to game theory and multicriteria decision making. *Discrete Appl. Math.*, 107(1-3):139–164, 2000.
17. J.-L. Marichal. Entropy of discrete Choquet capacities. *European J. Oper. Res.*, 137(3):612–624, 2002.
18. J.-L. Marichal. Tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral. *European J. Oper. Res.*, 155(3):771–791, 2004.
19. J.-L. Marichal. Cumulative distribution functions and moments of lattice polynomials. *Statistics & Probability Letters*, 76(12):1273–1279, 2006.
20. J.-L. Marichal. k-intolerant capacities and Choquet integrals. *European J. Oper. Res.*, 177(3):1453–1468, 2007.
21. J.-L. Marichal. Multivariate integration of functions depending explicitly on the minimum and the maximum of the variables. *J. Math. Anal. Appl.*, 341(1):200–210, 2008.
22. J.-L. Marichal. Weighted lattice polynomials of independent random variables. *Discrete Applied Mathematics*, 156(5):685–694, 2008.
23. J.-L. Marichal and I. Kojadinovic. Distribution functions of linear combinations of lattice polynomials from the uniform distribution. *Statistics & Probability Letters*, 78(8):985–991, 2008.
24. J.-L. Marichal, I. Kojadinovic, and K. Fujimoto. Axiomatic characterizations of generalized values. *Discrete Applied Mathematics*, 155(1):26–43, 2007.
25. J.-L. Marichal and P. Mathonet. Approximations of Lovász extensions and their induced interaction index. *Discrete Appl. Math.*, 156(1):11–24, 2008.
26. J.-L. Marichal and M. Roubens. Entropy of discrete fuzzy measures. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(6):625–640, 2000.

27. T. Murofushi. A technique for reading fuzzy measures (I): the Shapley value with respect to a fuzzy measure. In *2nd Fuzzy Workshop*, pages 39–48, Nagaoka, Japan, October 1992. In Japanese.
28. T. Murofushi and M. Sugeno. Some quantities represented by the Choquet integral. *Fuzzy Sets and Systems*, 56(2):229–235, 1993.
29. G. Owen. Multilinear extensions of games. *Management Sci.*, 18:P64–P79, 1972.
30. L. S. Shapley. A value for *n*-person games. In *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pages 307–317. Princeton University Press, Princeton, N. J., 1953.
31. R. J. Weber. Probabilistic values for games. In A. E. Roth, editor, *The Shapley value*, pages 101–119. Cambridge Univ. Press, Cambridge, 1988.
32. R. R. Yager. A class of fuzzy measures generated from a Dempster-Shafer belief structure. *Int. J. Intell. Syst.*, 14(12):1239–1247, 1999.
33. R. R. Yager. On the entropy of fuzzy measures. *IEEE Trans. on Fuzzy Systems*, 8:453–461, 2000.

Contribution on some construction methods for aggregation functions

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Abstract. In this paper, based on [14], we present some well established construction methods for aggregation functions as well as some new ones.

There is a well-known demand for an ample variety of aggregation functions having predictable and tailored properties to be used in modelling processes. The need for bigger flexibility and ability of fitting more accurate aggregation functions requires the extension of aggregation functions buffer, and one of approaches how to reach this is just based on construction methods. Several construction methods have been introduced and developed for extending the known classes of aggregation functions (defined either on $[0,1]$ or, possibly, on some other domains). There are several construction methods, introduced in many fields [1–5, 7, 9, 14, 15, 24, 25]. Obviously, new construction methods should be a central issue in the rapidly developing field of aggregation functions. In this paper we present some well established construction methods as well as some new ones.

The first group of construction methods can be characterized "from simple to complex". They are based on standard arithmetical operations on the real line and fixed real functions. The second group of construction methods starts from given aggregation functions to construct new ones. Here we can start either from aggregation functions with a fixed number of inputs (e.g., from binary functions only) or from extended aggregation functions. Observe that some methods presented are applicable to all aggregation functions (for example, transformation), while some of them can be applied only to specific cases. Finally, there are construction methods allowing us to find aggregation functions when only some partial knowledge about them is available. For more details on this topic we recommend [14], Chapter 6. In our presentation we will discuss these items:

- transformation of aggregation functions (recall the classical transformation of the sum into the product),
- composed aggregation (recall recursive aggregation functions, convex sums, etc.),
- weighted aggregation functions (quantitative and qualitative approaches),
- aggregation based on optimalisation (mixture operators, for example),
- ordinal sums of aggregation functions (covering in one formula well-known ordinal sums of t-norms and t-conorms).

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References

1. Aczél, J.: Lectures on functional equations and their applications, Mathematics in Science and Engineering, Vol. 19, Academic Press, New York, 1966.
2. Aczél, J., Dhombres, J.: Functional equations in several variables, Encyclopedia of Mathematics and its Applications 31, With applications to mathematics, information theory and to the natural and social sciences, Cambridge University Press, Cambridge, 1989.
3. Alsina, C., Frank, M. J. and Schweizer, B.: Associative functions, Triangular norms and copulas, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
4. Bajraktarević, M.: Sur une équation fonctionnelle aux valeurs moyennes, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske. Ser. II 13 (1958), 243–248.
5. Beliakov, G.,Pradera, A., Calvo, T.: Aggregation Functions: A Guide for Practitioners, Studies in Fuziness and Soft Computing, Springer, Berlin, 2007.
6. Birkhoff, G.: Lattice theory, Third edition. American Mathematical Society Colloquium Publication, Vol. XXV. American Mathematical Society, Providence, R. I., 1967.
7. Bouchon-Meunier, B. (ed.): Aggregation and fusion of imperfect information, Studies in Fuzziness and Soft Computing. 12. Heidelberg: Physica-Verlag, 1998.
8. Bullen, P. S.: Handbook of means and their inequalities, Mathematics and its Applications 560, Kluwer Academic Publishers Group, Dordrecht, 2003.
9. Calvo, T. and Mayor, G. and Mesiar, R. (eds.), Aggregation operators, Studies Fuzziness Soft Computing 97, Physica, Heidelberg, 2002.
10. Calvo, T., Mesiar, R., Yager, R.R.: Quantitative Weights and Aggregation, IEEE Trans. Fuzzy Syst. 12 (2004), 62-69.
11. Calvo, T., Pradera, A.: Double aggregation operators, Fuzzy Sets and Systems 142(1) (2004), 15-33.
12. Cutello, V., Montero, J.: Recursive connective rules, Int. J. Intelligent Systems 14 (1999), 3-20.
13. Fujimoto, K., Murofushi, T., Sugeno, M.: Canonical hierarchical decomposition of Choquet integral over finite set with respect to null additive fuzzy measure, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 6 (1998), 345-363.
14. Grabisch, M., Marichal, J. -L., Mesiar, R., Pap, E.: Aggregation Functions, Cambridge University Press (in press).
15. Klement, E. P., Mesiar, R., Pap, E.: Triangular norms, Trends in Logic—Studia Logica Library 8, Kluwer Academic Publishers, Dordrecht, 2000.
16. Luo, X., Jennings, N. R.: A spectrum of compromise aggregation operators for multi-attribute decision making, Artificial Intelligence 171 (2007), 161-184.
17. Ovchinikov, S., Dukhovny: Integral representation of invariant functionals, J. Math. Anal. Appl. 244(1) (2000), 228-232.
18. Marques, P., Ricardo, A., Ribeiro, R. A.: Aggregation with generalized mixture operators using weighting functions, Fuzzy Sets and Systems 137 (2003), 43-58.
19. Mesiar, R., Špirková, J.: Weighted means and weighting functions, Kybernetika (Prague) 42 (2006), 151-160.
20. Mesiar, R., Špirková, J., Vavríková, L.: Weighted aggregation operators based on minimization, Inform. Sci. 178 (2008), 1133–1140.
21. Mesiar, R., Rückschlossová, T.: Characterization of invariant aggregation operators, Fuzzy Sets and Systems 142(1) (2004), 63-73.
22. Mesiar, R., Vivona, D.: Two-step integral with respect to fuzzy measure, Tatra Mountains Mathematical Publications 16 (1999), 359–368.
23. Nelsen, R. B.: An introduction to copulas, Lecture Notes in Statistics 139, Springer-Verlag, New York, 1999.
24. Torra, V., Narukawa, Y.: Modeling decisions: Information Fusion and Aggregation Operators, Cognitive Technologies, Springer, 2007.
25. Yager, R. R.: Aggregation operators and fuzzy systems modeling, Fuzzy Sets and Systems 67 (1994), 129-145.

Probability in many-valued logics

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The relationship between probability and many-valued logics is a main theme in the tradition of Linz Seminars. A great number of papers, including several chapters of the Elsevier Handbook of Measure Theory, and various monographs owe their existence to what might be called the Linz Seminar cross-fertilization. A wealth of interesting results have been obtained in probability theory for t-norm logics, featuring various integral representations. In particular, the concept of state s in an MV-algebra A (i.e., a unit preserving map $s\colon A \to [0,1]$ which is additive on $\odot$-incompatible elements of A) has many characterizations, including:

1. s is the restriction to A of a positive normalized real homomorphism of the unital ℓ-group corresponding to A via the Γ functor,
2. s arises via Riesz representation as the integral on the maximal ideal space of A, relative to some regular Borel probability measure (this is the Kroupa-Panti theorem, showing that states are a finitely additive algebraization of σ-additive probability measures),
3. s is a coherent probability assessment (in the sense of De Finetti) for A with respect to the set of possible worlds given by all homomorphisms of A into $[0,1]$,
4. s is a limit, in the Tychonov cube $[0,1]^A$, of convex combinations of homomorphisms of A into $[0,1]$ (because extremal states coincide with such homomorphisms, by a result of Goodearl and Handelman). In other words, letting A be the Lindenbaum algebra of some theory Θ in Łukasiewicz logic, s is a limit of convex combinations of valuations satisfying all formulas of Θ.

In view of these deep relations between logic and probability—generalizing Carathéodory time-honored boolean algebraic probability theory—one may naturally ask if a further generalization is possible in other $[0,1]$-logics L. One finds in the literature various definitions of "state", capturing some of the above features (1)-(5), which in L need no longer be equivalent. Whatever L is, one can anyway define the benchmark notion of "coherent" probability assessment for any finite set E of formulas in L, and any set W of truth-valuations in L for the formulas of E. This is so because, independently of any logic context, for any finite set $E = \{X_1,\dots,X_n\}$ whose elements are called "events" and closed set $W \subseteq [0,1]^E$ of "possible worlds", a bookmaker's map $b\colon E \to [0,1]$ is said to be incoherent (in the sense of De Finetti) if a bettor can fix stakes $s_1,\dots,s_n \in \mathbb{R}$ such that the bookmaker looses at least 1 (million euro) in any possible world $V \in W$. Whether or not E and W arise from some logic L, it turns out that there is a theory Θ in Łukasiewicz logic such that coherent maps coincide with restrictions to E of states in the Lindenbaum algebra of Θ. Thus coherent probability assessments in L can always be interpreted in Łukasiewicz logic. We will exemplify this result to the Aguzzoli-Gerla-Marra theory of coherent probability assessments for Gödel logic.

Tribes revisited

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Abstract. Tribes became a successful basis of measure theory on fuzzy sets. During their investigation, new requirements occurred. We discuss possible modifications of the original definition of a tribe.

1 Original definitions

Classical measure theory is based on the notion of an algebra (=field) of subsets of a set (in case of finitely additive measures), resp. a σ-algebra of subsets (in case of countably additive measures). It is the basis of probability theory on classical events whose occurrence can be described in yes-no terms.

Fuzzy measure theory tries to extend this approach to events whose satisfaction is gradual, described by a many-valued scale of truth degrees, usually (a subset of) the real interval $[0,1]$. Such events are naturally represented by fuzzy sets. In order to define measures and probabilities, we need to restrict ourselves to adequate collections of fuzzy subsets of some universal set X. This idea gave origin to the notion of clan, resp. tribe, as a fuzzification of an algebra, resp. σ-algebra of subsets. Clans were studied already by O. Wyler in [24], then used by D. Butnariu and E.P. Klement in [3–5, 9], where tribes were introduced as a σ-complete analogue. This approach assumes a fixed continuous triangular norm (t-norm) T and a strong fuzzy negation $'$. These operations are naturally extended to pointwise operations on fuzzy subsets of X and T can be generalized to any countable arity. A T*-clan* on X is a collection $\mathcal{T} \subseteq [0,1]^X$ such that

(T1) $0 \in \mathcal{T}$,
(T2) $f \in \mathcal{T} \implies 1-f \in \mathcal{T}$,
(T3) $f,g \in \mathcal{T} \implies T(f,g) \in \mathcal{T}$.

A T-clan $\mathcal{T}$ is called a T*-tribe* on X if it satisfies a stronger version of the latter condition:

(T3+) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^N \implies T_{n\in\mathbb{N}} f_n \in \mathcal{T}$.

A functional $\mu\colon \mathcal{T} \to [0,\infty[$ is called a T*-measure* if it satisfies the following conditions:

(M1) $\mu(0) = 0$,
(M2) $f,g \in \mathcal{T} \implies \mu(T(f,g)) + \mu(S(f,g)) = \mu(f) + \mu(g)$, where S is the triangular conorm dual to T,
(M3) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^N,\ f_n \nearrow f,\ f \in \mathcal{T} \implies \mu(f_n) \nearrow \mu(f)$.

The notion of T-measure is a special case of a *stochastic measure* introduced by Höhle [8].

These definitions follow the classical approach. Many interesting properties of measures on these structures have been derived in [4] and subsequent papers (e.g. [1, 2, 11–15, 17, 18], see an overview by S. Weber and E.P. Klement in [23]). The set $\mathcal{B} = \mathcal{T} \cap \{0,1\}^X$ of *Boolean elements* of $\mathcal{T}$ forms a σ-algebra of subsets of X. For Archimedean Frank (and some other, so-called *sufficient* [6]) t-norms, all elements of $\mathcal{T}$ are $\mathcal{B}$-measurable.

The consequences of the line initiated by Butnariu and Klement have led also to proposals of revisions. In the sequel, we present some of these arguments.

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2 The choice of negation

The original definition of a T-tribe works only with the standard fuzzy negation $'\colon x \mapsto 1-x$. This can be replaced by an arbitrary fuzzy negation which is *strong* (=involutive). Condition (T2) then reads:

(T2) $f \in \mathcal{T} \implies f' \in \mathcal{T}$.

In condition (M2), S is the triangular conorm dual to T with respect to $'$.

This change is not essential. Each strong negation has a generator. Applying this generator pointwise to all fuzzy sets in the T-tribe, we obtain a new collection in which the negation is standard. The t-norm T and its dual t-conorm S are modified to some T^*, S^*. We obtain a T^*-tribe which differs only by the change of scale of membership degrees.

3 Tribes as σ-lattices

Fuzzy sets are ordered by the usual ordering of functions, $f \leq g \iff \forall x \in X : f(x) \leq g(x)$. Let T_{M} be the minimum t-norm. In a T_{M}-clan $\mathcal{T}$, $T_{\mathrm{M}}(f,g) = f \wedge g$ is the infimum (with respect to the above ordering) and a $\mathcal{T}$ is a lattice. If $\mathcal{T}$ is a T_{M}-tribe, it is a σ-complete lattice.

Now let us consider a T-tribe $\mathcal{T}$, where T is an arbitrary continuous t-norm. Condition (M3) applies only to the case when the countable supremum $f = \bigvee_{n\in\mathbb{N}} f_n$ is in $\mathcal{T}$. If we want to omit the assumption $f \in \mathcal{T}$, we need $\mathcal{T}$ to be a σ-complete lattice, equivalently a T_{M}-tribe (and at the same time a T-tribe). This observation led to a modification of the definition of a T-tribe – condition (T3) is replaced by the following:

(T4) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^N,\ f_n \nearrow f \implies f \in \mathcal{T}$.

Together with (T3), it implies (T3+), thus it is a strengthening. This definition occurred first in [16] and was used later in [19, 20].

It was proved in [4] that if T is an Archimedean Frank t-norm, then every T-tribe is also a T_{L}-tribe (where T_{L} is the Łukasiewicz t-norm) and a T_{M}-tribe. Thus it satisfies also (T4). For non-Frank t-norms, the two definitions differ. However, the basic result of [19] is that T-tribes which admit reasonably rich collections of T-measures are essentially those where T is an Archimedean Frank t-norm. Therefore the difference between these definitions is not much important. Condition (T4) seems to fit better to the definition of a T-measure.

There is a non-symmetry in the definition of a T-measure: It preserves limits of *increasing* sequences, but not of *decreasing* ones. Therefore the following requirement was added in [19]:

(M4) $(f_n)_{n\in\mathbb{N}} \in \mathcal{T}^N,\ f_n \searrow f,\ f \in \mathcal{T} \implies \mu(f_n) \searrow \mu(f)$.

In σ-algebras, this dual requirement was unnecessary, but in tribes it excludes T-measures which seem to have low applicability because they depend only on the *supports* of fuzzy sets,

$$\operatorname{Supp} f = \{x \in X \mid f(x) > 0\} \subseteq X\,,$$

not on a finer scale of their membership degrees.

4 Clans with underlying σ-algebras

Closedeness under countable operations (condition (T3+) or (T4)) seems to be a natural requirement, following the analogy with σ-algebras. However, fuzzy sets combine two "directions": The first – *horizontal* or *qualitative* – is determined by the support of the fuzzy set. The second – *vertical* or *quantitative* – is determined by the membership degrees of elements of the support; these are numbers from $[0,1]$. The above definitions imply that a T-tribe $\mathcal{T}$ is closed under countable operations in both directions.

As pointed out by D. Mundici and T. Kroupa (personal communication, 2008, cf. also [10]), the theory of σ-*additive* measures requires only the *horizontal* closedness under countable operations, i.e., the set $\mathcal{B}$ of Boolean elements of $\mathcal{T}$ must form a σ-algebra of subsets of X. (We identify subsets of X with their characteristic functions.) However, there is no need to require such a strong restriction for the *vertical* direction. The set of membership values at $x \in X$,

$$\mathcal{T}(x) = \{f(x) \mid f \in \mathcal{T}\} \subseteq [0,1] \, .$$

must be closed under $'$ and T with *finite* arity. If T is a strict t-norm and $\mathcal{T}(x)$ is closed under T with *countable* arity, it can be only $\{0,1\}$ or the whole interval $[0,1]$. This restriction is unnecessary, we may admit, e.g., a collection of fuzzy sets whose membership degrees are rational. Then we have only a T-clan. However, we admit only such a T-clan whose elements are $\mathcal{B}$-measurable.

5 Algebraization of tribes

This approach is inspired by Boolean algebras. These are algebras defined by equations, without a reference to any set representation. In fact, in Boolean algebras this makes no difference; due to Stone theorem, every (abstract) Boolean algebra can be represented as an algebra (field) of subsets of some set. The difference becomes important if we require σ-completeness. There are σ-complete Boolean algebras which cannot be represented as σ-algebras of subsets. (There is still a weaker representation obtained by the Loomis–Sikorski theorem, see [22].)

Another analogy can be found in MV-algebras [7, 21]: σ-complete MV-algebras are a generalization of T_{L}-tribes and T_{L}-tribes are exactly those σ-complete MV-algebras which can be represented by fuzzy sets. We want to find a common generalization of (abstract) σ-complete MV-algebras and T-tribes.

With H. Weber (personal communication), we suggested such an algebra based on a σ-complete lattice. The set of Boolean elements is expected to form an (abstract) Boolean σ-algebra. Monotonicity of the t-norm implies that the corresponding operation should distribute over countable suprema. A proposed definition might be the following:

A *fuzzy* σ-*algebra* is an algebra $(\mathcal{A}, T, {}', \wedge, 0, 1)$ of type $(2,1,2,0,0)$, where $(\mathcal{A}, \wedge, 0, 1)$ is a bounded σ-complete lattice, $' \colon \mathcal{A} \to \mathcal{A}$ is an antitone involutive mapping, and $T \colon \mathcal{A}^2 \to \mathcal{A}$ is an operation which is commutative, associative, has a neutral element 1, and satisfies

$$T\Big(a, \bigwedge_{n\in\mathbb{N}} b_n\Big) = \bigwedge_{n\in\mathbb{N}} T(a, b_n) \, , \qquad T\Big(a, \bigvee_{n\in\mathbb{N}} b_n\Big) = \bigvee_{n\in\mathbb{N}} T(a, b_n)$$

for all $a \in \mathcal{A}$, $(b_n)_{n\in\mathbb{N}} \in: \mathcal{A}^{\mathbb{N}}$. This notion admits only those T-tribes which are also T_{M}-tribes. On the other hand, it is more general in the sense that it admits a collection of fuzzy sets with operations defined pointwise, but using *different* fuzzy negations and t-norms at different points. From the algebraic point of view, there is no way how to distinguish and exclude such cases. A fuzzy σ-algebra may allow to apply a rich collection of algebraic methods (developed for varieties).

References

1. G. Barbieri, M. Navara, and H. Weber. Characterization of T-measures. *Soft Computing*, 8:44—50, 2003.
2. G. Barbieri and H. Weber. Measures on clans and on MV-algebras. In E. Pap, editor, *Handbook of Measure Theory*. Elsevier Science, Amsterdam, Chapter 22, pages 911–946, 2002.
3. D. Butnariu and E. P. Klement. Triangular norm-based measures and their Markov kernel representation. *J. Math. Anal. Appl.*, 162:111–143, 1991.
4. D. Butnariu and E. P. Klement. *Triangular Norm-Based Measures and Games with Fuzzy Coalitions*. Kluwer Academic Publishers, Dordrecht, 1993.
5. D. Butnariu and E. P. Klement. Triangular norm-based measures. In E. Pap, editor, *Handbook of Measure Theory*. Elsevier Science, Amsterdam, Chapter 23, pages 947–1010, 2002.
6. D. Butnariu, E.P. Klement, R. Mesiar, M. Navara. Sufficient triangular norms in many-valued logics with standard negation. *Arch. Math. Logic* 44:829–849, 2005.
7. R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*. Kluwer Academic Publishers, Dordrecht, 1999.
8. U. Höhle. Fuzzy measures as extensions of stochastic measures. *J. Math. Anal. Appl.* 92:372–380, 1983.
9. E. P. Klement. Construction of fuzzy σ-algebras using triangular norms. *J. Math. Anal. Appl.*, 85:543–565, 1982.
10. T. Kroupa. Every state on semisimple MV-algebra is integral. *Fuzzy Sets and Systems*, 157(20):2771–2782, 2006.
11. R. Mesiar. Fundamental triangular norm based tribes and measures. *J. Math. Anal. Appl.*, 177:633–640, 1993.
12. R. Mesiar. On the structure of T_s-tribes. *Tatra Mt. Math. Publ.*, 3:167–172, 1993.
13. R. Mesiar and M. Navara. T_s-tribes and T_s-measures. *J. Math. Anal. Appl.*, 201:91–102, 1996.
14. M. Navara. A characterization of triangular norm based tribes. *Tatra Mt. Math. Publ.*, 3:161–166, 1993.
15. M. Navara. Nearly Frank t-norms and the characterization of T-measures. In D. Butnariu and E. P. Klement, editors, *Proceedings of the 19th Linz Seminar on Fuzzy Set Theory*, pages 9–16, Linz, 1998.
16. M. Navara. Characterization of measures based on strict triangular norms. *J. Math. Anal. Appl.*, 236:370–383, 1999.
17. M. Navara, P. Pták. Convex structure of the space of fuzzy measures. In: *Proc. 24th Linz Seminar on Fuzzy Set Theory,* Univ. Linz, Austria, pages 149–151, 2003.
18. M. Navara, P. Pták. Regular measures on tribes of fuzzy sets. In: *Mathematics of Fuzzy Systems*, E.P. Klement and E. Pap (eds.), Univ. Linz, Austria, pages 153–161, 2004.
19. M. Navara. Triangular norms and measures of fuzzy sets. In: E.P. Klement and R. Mesiar (eds.), *Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms*, Elsevier, pages 345–390, 2005.
20. M. Navara. Probability and conditional probability on tribes of fuzzy sets. In: D. Dubois, E.P. Klement, R. Mesiar: *Fuzzy Sets, Probability, and Statistics — Gaps And Bridges*, Johannes Kepler University, Linz, Austria, pages 84–88, 2007.
21. B. Riečan and D. Mundici. Probability on MV-algebras. In E. Pap, editor, *Handbook of Measure Theory*. Elsevier Science, Amsterdam, Chapter 21, pages 869–910, 2002.
22. R. Sikorski. *Boolean Algebras*. 3rd edition, Springer, Berlin, 1969.
23. S. Weber and E. P. Klement. Fundamentals of a generalized measure theory, In U. Höhle et al., editors, *Mathematics of Fuzzy Sets. Logic, Topology, and Measure Theory*, pages 633–651, Kluwer Academic Publishers, Dordrecht, 1999.
24. O. Wyler. Clans. *Comp. Math.*, 17:172–189, 1966.

Convex combinations of triangular norms

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In the list of open problems by Alsina, Frank, and Schweizer [2], the following two problems have been stated:

Problem 1. Is the arithmetic mean, or for that matter any convex combination, of two distinct t-norms ever a t-norm?

Problem 2. More specifically, can T_P be expressed as a convex combination of two associative copulas?

We recall that a convex combination of two t-norms T_1, T_2 is a function $F = \alpha T_1 + (1-\alpha) T_2$ where $\alpha \in [0,1]$. It is immediate that for trivial convex combinations, i.e. for $\alpha \in \{0,1\}$ or for $T_1 = T_2$, the answer is positive. A positive example can be given even for non-trivial convex combinations of non-continuous t-norms [3, 10, 12]. For example, let T_1 be an ordinal sum of the product t-norm T_P on the carrier $[0, \frac{1}{2}]$. Let T_2 be a binary operation on $[0,1]$ such that $T_2(x,y) = 0$ for $x, y \in [0, \frac{1}{2}]$ and $T_2(x,y) = \min\{x,y\}$ otherwise. It is easy to check that T_2 is a left-continuous t-norm. Observe now that any convex combination of T_1 and T_2 is a left-continuous t-norm. However, for continuous t-norms the problem still has not been answered completely although it is conjectured that the answer is negative [2].

Thus, in order to exclude the trivial cases mentioned above, whenever we write "convex combination" we mean a function $\alpha T_1 + (1-\alpha) T_2$ where $\alpha \in]0,1[$, $T_1 \neq T_2$, and both t-norms are continuous.

Here we briefly outline the results related to convex combinations of t-norms which have been done so far. In the historically first paper dealing with this problem, Tomás [11] has given the following result:

Proposition 1. *[11] Let T_1, T_2, and T_3 be strict t-norms with additive generators t_1, t_2, and t_3, respectively, derivable on $]0,1]$, with derivatives distinct from zero, continuous on $]0,1]$, and*

$$\lim_{x \to 0_+} \frac{t_1'(x)}{t_3'(x)} = \frac{t_1'(1)}{t_3'(1)}\,.$$

If $T_3 = \frac{T_1+T_2}{2}$ for all $x, y \in [0,1]$, then $T_1 = T_2 = T_3$.

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However, the smoothness assumption [1] and the constraint are rather restrictive. In the papers by Ouyang, Fang, and Li [7, 8], the whole class of continuous t-norms is treated under no additional assumptions. For example, they prove [7, Theorems 2.1 and 2.2] that a convex combination of a continuous Archimedean t-norm and a continuous non-Archimedean t-norm is never a t-norm. In other words, if a convex combination of two continuous t-norms is a t-norm again, then both combined t-norms are ordinal sums with the same structure of summand carriers and with continuous Archimedean summands. By this result, in order to clarify the convex structure of the class of continuous t-norms it is sufficient to clarify the convex structure of the class of continuous Archimedean t-norms. By another result of theirs [7, Theorem 3.1], a convex combination of a strict and a nilpotent t-norm is never a t-norm. Thus even the latter task can be subdivided into solving the convex structure of the nilpotent class and of the strict class separately. Another type of results are no-way theorems for pairs of t-norms satisfying an additional property. One of them is due to Jenei [3] and applies to all pairs of *left-continuous* t-norms with an additional property that both t-norms share a *level set* of special properties (which, on the other hand, significantly reduces the generality of the result). An immediate consequence of this result is that for two nilpotent t-norms T_1, T_2 which share a z-level set for $z \in [0,1[$, none of their convex combinations is a t-norm. Let us mention also the recent result by Mesiar and Mesiarová-Zemánková [4] where it is stated that a convex combination of two continuous t-norms with the same diagonal section is never a t-norm. (We recall that a *diagonal section* of a t-norm T is the function $x \mapsto T(x,x)$.)

In the recent work [5, 6, 9, 10], the authors have presented several new findings on continuous Archimedean t-norms. First, it is the characterization of the associativity of t-norms by means of the web geometry and the Reidemeister condition which leads to proofs of the following two theorems:

Theorem 1. *A non-trivial convex combination of two distinct nilpotent t-norms is never a t-norm.*

Theorem 2. *A non-trivial convex combination of a strict and a nilpotent t-norm is never a t-norm.*

The proof of the second theorem is an alternative proof of the result given earlier by Ouyang, Fang, and Li [7, Theorem 3.1].

Further results have been obtained by the study of relations between the shape of a t-norm and the shape of its (multiplicative and additive) generator. This enabled the proof of the following theorem:

Theorem 3. *Let T be a strict t-norm. Define*

$$b_T\colon\ ]0,1[\ \to\ ]0,1[\ : y \mapsto T'(0,y).$$

Suppose that the value $b_T(y)$ is defined for every $y \in\ ]0,1[$. Then there are three mutually exclusive possibilities:

(i) the function b_T is the constant function 1,
(ii) the function b_T is an order isomorphism of the interval $]0,1[$,
(iii) the function b_T is the constant function 0.

Moreover, in the case (ii) the unique continuous extension of b_T to the whole unit interval is a multiplicative generator of T.

Definition 1. *Let T be a strict t-norm such that the function $b_T\colon\ ]0,1[\ \to\ ]0,1[$, given by $b_T(y) = T'(0,y)$, is well defined. According to Theorem 3, the following four classes of strict t-norms are defined:*

1. *if b_T is a bijection then $T \in \mathcal{T}_{\mathrm{R0}}$,*
2. *if $b_T(y) = 0$ then $T \in \mathcal{T}_0$,*
3. *if $b_T(y) = 1$ then $T \in \mathcal{T}_1$,*
4. *if $T \notin \mathcal{T}_{\mathrm{R0}} \cup \mathcal{T}_0 \cup \mathcal{T}_1$ then $T \in \mathcal{T}_{\mathrm{N}}$.*

Note that these four classes form a partition of the class of all strict t-norms.

Theorem 3 allows an immediate proof of the following proposition:

Proposition 2. *Let T_1 and T_2 belong to two distinct classes from $\mathcal{T}_{\mathrm{R0}}$, $\mathcal{T}_0$, $\mathcal{T}_1$. Then no non-trivial convex combination of T_1 and T_2 is a t-norm.*

Definition 2. *By $\mathcal{T}_{\mathrm{R1}}$ we denote the set of all strict t-norms whose multiplicative generators θ satisfy $\theta'(1) \in \,]0,\infty[$. (If one of multiplicative generators of a t-norm satisfies this condition, all its multiplicative generators satisfy it, too.)*

Finally, for the intersection of $\mathcal{T}_{\mathrm{R0}}$ and the newly defined class $\mathcal{T}_{\mathrm{R1}}$, the following condition can be given:

Theorem 4. *Let $T_1, T_2 \in \mathcal{T}_{\mathrm{R0}} \cap \mathcal{T}_{\mathrm{R1}}$. Let θ_1 and θ_2 be multiplicative generators of T_1 and T_2 given by $\theta_1(y) = T'(0,y)$ and $\theta_2(y) = T'(0,y)$, $y \in [0,1]$, respectively. Let $t_1 = -\ln\theta_1$, resp. $t_2 = -\ln\theta_2$, be the corresponding additive generators of T_1, resp. T_2.*

If a non-trivial convex combination of T_1 and T_2 is a t-norm then, necessarily, for each $y \in [0,1]$ at least one of the following conditions is satisfied:

$$\theta_2'(y) = \frac{\theta_2'(1)}{\theta_1'(1)}\,\theta_1'(y)\,, \tag{1}$$

$$t_1'(y) = t_2'(y)\,. \tag{2}$$

This directly leads to the following corollary:

Corollary 1. *Let $T_1, T_2 \in \mathcal{T}_{\mathrm{R0}} \cap \mathcal{T}_{\mathrm{R1}}$ be two distinct strict t-norms. If the (multiplicative and additive) generators of both T_1 and T_2 are absolutely continuous, then no non-trivial convex combination of T_1 and T_2 is a t-norm.*

References

1. C. Alsina and M. S. Tomás: Smooth convex t-norms do not exist. *Proceedings of the American Mathematical Society*, 102(2):317-320, 1988.
2. C. Alsina, M. J. Frank, and B. Schweizer. Problems on associative functions. *Aequationes Mathematicae*, 66(1–2):128–140, 2003.
3. S. Jenei. On the convex combination of left-continuous t-norms. *Aequationes Mathematicae*, 72(1–2):47–59, 2006.
4. R. Mesiar and A. Mesiarová-Zemánková. Convex combinations of continuous t-norms with the same diagonal function. *Nonlinear Analysis*, To appear.
5. M. Navara and M. Petrík. Explicit formulas for generators of triangular norms. *Semigroup Forum*, 2008. Submitted.
6. M. Navara, M. Petrík. Two methods of reconstruction of generators of continuous t-norms. *12th International Conference Information Processing and Management of Uncertainty in Knowledge-Based Systems*, Málaga, Spain, 2008.
7. Y. Ouyang and J. Fang. Some observations about the convex combinations of continuous triangular norms. *Nonlinear Analysis*, 2007.
8. Y. Ouyang, J. Fang, and G. Li. On the convex combination of T_D and continuous triangular norms. *Information Sciences*, 177(14):2945–2953, 2007.
9. M. Petrík. Convex combinations of strict t-norms. *Soft Computing - A Fusion of Foundations, Methodologies and Applications*, 2008. Submitted.
10. M. Petrík, P. Sarkoci. Convex combinations of nilpotent triangular norms. *Journal of Mathematical Analysis and Applications*, 350:271–275, 2009. DOI: 10.1016/j.jmaa.2008.09.060
11. M. S. Tomás. Sobre algunas medias de funciones asociativas. *Stochastica*, XI(1):25–34, 1987.
12. T. Vetterlein. Regular left-continuous t-norms. *Semigroup Forum*. To appear.

Fuzzy logic in broader sense: its current state and future

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The so called *Fuzzy Logic in Narrow Sense* (FLn) is the basic constituent of mathematical fuzzy logic. Its development has been initiated by the paper of J. A Goguen [7] and the first, highly sophisticated paper is that of J. Pavelka [24]. His work has been continued by the author of this paper in [14] and especially in the book [13]. Fuzzy logic has been established as a generalization of classical mathematical logic with clearly distinguished syntax and semantics. The syntax consists of precise definitions of the formula, proof, formal theory, model, provability, etc. and the semantics is formed by a special residuated lattice. The latter in the above mentioned works is an MV-algebra and, namely, the standard Łukasiewicz algebra since Pavelka proved that completeness of the syntax w.r.t. semantics can be kept providing that the corresponding algebra fulfils the following four equations:

$$\bigvee_{i\in I}(a \to b_i) = a \to \bigvee_{i\in I} b_i, \qquad \bigwedge_{i\in I}(a \to b_i) = a \to \bigwedge_{i\in I}(b_i) \tag{1}$$

$$\bigvee_{i\in I}(a_i \to b) = \bigwedge_{i\in I} a_i \to b, \qquad \bigwedge_{i\in I}(a_i \to b) = \bigvee_{i\in I} a_i \to b \tag{2}$$

These equations are in $[0,1]$ equivalent to continuity of $\to$, which is fulfilled only by Łukasiewicz implication and its isomorphs. The resulting logic is quite strong since it is complete with respect to the generalized syntax, in which all formulas are evaluated by elements from the underlying algebra and the completeness thus takes the form of generalization of the Gödel completeness:

$$T \vdash_a A \quad \text{iff} \quad T \models_a A, \qquad a \in L,$$

for all all formulas $A \in F_{J(T)}$ and *fuzzy theories* T where the latter are determined by *fuzzy sets* of axioms. This means that the situation when an axiom needs not be fully true is acceptable in this logic. For this reason, its preferred name is *fuzzy logic with evaluated syntax* ($\text{Ev}_{\text{Ł}}$).

The major turning point in the development of FLn is represented by the book of P. Hájek [8]. He relaxed the requirement that completeness should be fulfilled for every truth value and considers only traditional syntax, so that the resulting logic fulfils a weaker form of completeness:

$$T \vdash A \quad \text{iff} \quad T \models A$$

where $T \models A$ means that $\mathcal{M}(A) = 1$ for every model $\mathcal{M}$ of the (classical) theory T. This approach opened the door to introducing various other formal systems which are now taken as constituents of FLn. They usually differ from each other on the basis of the assumed algebra of truth values, which then determines all of the properties of the given calculus. It is argued in [19] that the most distinguished calculi are MTL-, IMTL-, BL-, Łukasiewicz- and, perhaps, ŁΠ-fuzzy logic. The MTL-logic is the basic kind of the, so called, *core fuzzy logic* [9]. The discussion, however, is not yet

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finished. FLn is in detail studied especially from the algebraic point of view. One can hardly estimate, how far this work can still continue. It seems that FLn is in the position of a theory promising a deeply justified technique for modeling various manifestations of the vagueness phenomenon and for various other applications. There are not, however, so many results in the latter.

One of exceptions is the program announced by P. Cintula and L. Běhounek [1, 2]. Several paper have already been published and the program seems to be very promising.

Nevertheless, we think that we should move on further in the development of the logical part so that the power of FLn were fully acknowledged. A possible direction is *Fuzzy Logic in Broader Sense* (FLb) which was established by V. Novák in 1995 in [15] as a program for extension of FLn, which aims at developing of a *formal theory of natural human reasoning that would include mathematical models of special expressions in natural language with regard to their vagueness*. There is an overlap with two other paradigms proposed in the literature, namely *commonsense reasoning* and *precisiated natural language* (PNL).

The main drawback of the up-to-date formalizations of commonsense reasoning, in our opinion, is that it neglects the vagueness present in the meaning of natural language expressions (cf. [3] and the citations therein).

PNL, on the other hand, is based on two main premises:

(a) Much of the world knowledge is perception based,
(b) perception based information is intrinsically fuzzy.

It is important to stress that the term *precisiated natural language* means "a reasonable working formalization of semantics of natural language without pretension to capture it in detail and fineness". Its goal is to provide an acceptable and applicable technical solution. It should also be noted that the term *perception* is not considered here as a psychological term but rather as a result of human, intrinsically imprecise measurement.

PNL methodology requires the presence of *World Knowledge Database* and *Multiagent, Modular Deduction Database*. The former contains all the necessary information including perception based propositions describing the knowledge acquired by direct human experience, which can be used in the deduction process. The latter contains various rules of deduction. No exact formalization of PNL, however, has been developed until recently, so it should be taken mainly as a reasonable methodology.

Thus, our concept of FLb is a glue between both paradigms that should consider the best of each. During the years, it has been slowly developed and so far, it consists of the following theories:

(a) Formal theory of evaluative linguistic expressions [20],
(b) formal theory of fuzzy IF-THEN rules [5, 22],
(c) formal theory of perception-based logical deduction [4, 23, 17, 16],
(d) formal theory of intermediate quantifiers [18, 21].

Let us remark, that perceptions are technically identified with specific evaluative expressions of natural language, i.e. expressions that characterize values from some scale. The FLn-mathematical basis for all these theories is *fuzzy type theory* because we argue (together with logicians and linguists, cf. [11, 12, 26]) that the first order (fuzzy) logic is not sufficient for capturing semantics of natural language.

We think that there are good reasons to continue this development of FLb and move it further towards human (i.e., commonsense) reasoning, following the methodology of PNL and results obtained in the AI *theory of commonsense reasoning*. Another promising direction is to include also uncertainty in FLb in the sense that has been nicely established in [6]. In this paper, logic is joined with probability theory.

So, we see some of the possible future directions of FLb in the following:

(a) Extend the repertoire of natural language expressions, for which a reasonable working mathematical model can be elaborated.
(b) Find a reasonable class of "intended models" for the theory of intermediate quantifiers.
(c) Extend the theory of generalized quantifiers (cf. [10, 25]) using the formalism of FLn.
(d) Study various forms of commonsense human reasoning and search for a reasonable formalization of them. As a subgoal, extend the list of generalized syllogisms with intermediate quantifiers.
(e) Develop a reasonable formalization of FLb on the basis of which the above mentioned constituents of PNL methodology, namely the World Knowledge Database and Multiagent, Modular Deduction Database could be formed.
(f) Extend the technique started in [6] to be able to include also uncertainty inside FLb.
(g) Study the ways, how other formal systems of FLn (besides fuzzy type theory) could be used for solution of problems raising in FLb and develop them accordingly.

References

1. L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154:34–55, 2005.
2. L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.
3. E. Davis and L. Morgenstern. Introduction:progress in formal commonsense reasoning. *Artifical Intelligence*, 153:1–12, 2004.
4. A. Dvořák and V. Novák. Fuzzy logic deduction with crisp observations. *Soft Computing*, 8:256–263, 2004.
5. A. Dvořák and V. Novák. Formal theories and linguistic descriptions. *Fuzzy Sets and Systems*, 143:169–188, 2004.
6. T. Flaminio and F. Montagna. Mv-algebras with internal states and probabilistic fuzzy logics. *International Journal of Approximate Reasoning*, 2009.
7. J. A. Goguen. The logic of inexact concepts. *Synthese*, 19:325–373, 1968-69.
8. P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.
9. Petr Hájek and Petr Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.
10. E.L. Keenan and D. Westerstå hl. Quantifiers in formal and natural languages. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 837–893. Elsevier, Amsterdam, 1997.
11. P. Materna. *Concepts and Objects*. Acta Philosophica Fennica 63, Helsinki, 1998.
12. P. Materna. *Conceptual Systems*. Logos Verlag, Berlin, 2004.
13. V. Novák. *Fuzzy Sets and Their Applications*. Adam Hilger, Bristol, 1989.
14. V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic I, II. *Kybernetika*, 26:47–66, 134–154, 1990.
15. V. Novák. Towards formalized integrated theory of fuzzy logic. In Z. Bien and K.C. Min, editors, *Fuzzy Logic and Its Applications to Engineering, Information Sciences, and Intelligent Systems*, pages 353–363. Kluwer, Dordrecht, 1995.
16. V. Novák. Perception-based logical deduction. In B. Reusch, editor, *Computational Intelligence, Theory and Applications*, pages 237–250. Springer, Berlin, 2005.
17. V. Novák. Perception-based logical deduction as alternative approximate reasoning method. In *Proc. Int. Conf. FUZZ-IEEE 2005*, pages 1032–1037, Reno, USA, May 2005, 2005.
18. V. Novák. Fuzzy logic theory of evaluating expressions and comparative quantifiers. In *Proc.* 11^{th} *Int. Conf. IPMU, Paris, July 2006*, volume 2, pages 1572–1579, Les Cordeliers, Paris, 2006. Éditions EDK.
19. V. Novák. Which logic is the real fuzzy logic? *Fuzzy Sets and Systems*, 157:635–641, 2006.
20. V. Novák. A comprehensive theory of trichotomous evaluative linguistic expressions. *Fuzzy Sets and Systems*, 159(22):2939–2969, 2008.
21. V. Novák. A formal theory of intermediate quantifiers. *Fuzzy Sets and Systems*, 159(10):1229–1246, 2008. , DOI: 10.1016/j.fss.2007.12.008.
22. V. Novák and S. Lehmke. Logical structure of fuzzy IF-THEN rules. *Fuzzy Sets and Systems*, 157:2003–2029, 2006.
23. V. Novák and I. Perfilieva. On the semantics of perception-based fuzzy logic deduction. *International Journal of Intelligent Systems*, 19:1007–1031, 2004.
24. J. Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.
25. S. Peters and D. Westerståhl. *Quantifiers in Language and Logic*. Claredon Press, Oxford, 2006.
26. P. Sgall, E. Hajičová, and J. Panevová. *The Meaning of the Sentence in Its Syntactic and Pragmatic Aspects*. D. Reidel, Dordrecht, 1986.

Infinitary aggregation

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Abstract. In this paper, based on [12, 18], we present infinitary aggregation functions on sequences possessing some a priori given properties. General infinitary aggregation is also discussed, and the connection with integrals, e.g., Lebesgue, Choquet and Sugeno integrals, is given.

Aggregation of finitely many inputs, directly related to many applications, were investigated in many fields [1–3, 5, 7, 12, 15, 24, 26]. Aggregation of infinitely but still countably many inputs is important in several mathematical areas, such as discrete probability theory, but also in non-mathematical areas, such as decision problems with an infinite jury, game theory with infinitely many players, etc. Though these theoretical tasks seem to be far from reality, they enable a better understanding of decision problems with extremely huge juries, game theoretical problems with extremely many players, etc., see [20, 22, 25].

In our contribution, based on [12, 18], we discuss infinitary aggregation functions on sequences possessing some a priori given properties, such as additivity, comonotone additivity, symmetry, etc. Based on these properties, infinitary OWA operators are discussed, among others, see [23]. On the other side we discuss infinitary aggregation functions $\mathsf{A}^{(\infty)}\colon [0,1]^{\mathbb{N}} \to [0,1]$ related to a given extended aggregation function $\mathsf{A}\colon \cup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$, where special attention is paid to t-norms, t-conorms, and weighted arithmetic means, where a connection with Toeplitz matrix (see [4, 11]) was obtained. Note that the discussion of the infinitary arithmetic mean $\mathsf{AM}^{(\infty)}\colon [0,1]^{\mathbb{N}} \to [0,1]$ can be found in [13, 14].

General infinitary aggregation is also discussed (see [12, 19]), thus extending the results concerning aggregation of infinite sequences. Note that in such case, some restrictions on the domain of aggregation functions is usually necessary. For example, to apply Lebesgue, Choquet or Sugeno integrals, see [21], one should require the measurability of the input function to be aggregated.

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References

1. Aczél, J., Dhombres, J.: Functional equations in several variables, Encyclopedia of Mathematics and its Applications 31, With applications to mathematics, information theory and to the natural and social sciences, Cambridge University Press, Cambridge, 1989.
2. Alsina, C., Frank, M. J., Schweizer, B.: Associative functions, Triangular norms and copulas, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
3. Beliakov, G.,Pradera, A., Calvo, T.: Aggregation Functions: A Guide for Practitioners, Studies in Fuziness and Soft Computing, Springer, Berlin, 2007.
4. Böttcher, A., Silbermann, B.: Analysis of Toeplitz operators, Springer, 1990.
5. Bouchon-Meunier, B. (ed.): Aggregation and fusion of imperfect information, Studies in Fuzziness and Soft Computing. 12. Heidelberg: Physica-Verlag, 1998.
6. Bullen, P. S.: Handbook of means and their inequalities, Mathematics and its Applications 560, Kluwer Academic Publishers Group, Dordrecht, 2003.
7. Calvo, T. and Mayor, G. and Mesiar, R. (eds.), Aggregation operators, Studies Fuzziness Soft Computing 97, Physica, Heidelberg, 2002.
8. Calvo, T., Mesiar, R., Yager, R.R.: Quantitative Weights and Aggregation, IEEE Trans. Fuzzy Syst. 12 (2004), 62-69.
9. Calvo, T., Pradera, A.: Double aggregation operators, Fuzzy Sets and Systems 142(1) (2004), 15-33.
10. Cutello, V., Montero, J.: Recursive connective rules, Int. J. Intelligent Systems 14 (1999), 3-20.
11. Dunford, N., Schwartz, J. T.: Linear Operators I. Interscience, New York-London, 1958.
12. Grabisch, M., Marichal, J. -L., Mesiar, R., Pap, E.: Aggregation Functions, Cambridge University Press (in press).
13. González, L., Muel, E., Mesiar, R.: What is the arithmetic mean of an infinite sequence? Proc. ESTYLF'2002, Leon, 2002, 183–187.
14. González, L., Muel, E., Mesiar, R.: A remark on the arithmetic mean of an infinite sequence. Internat. J. Uncertainty, Fuzziness, Knowledge-Based Systems 10, Suppl. (2002)51–58.
15. Klement, E. P., Mesiar, R., Pap, E.: Triangular norms, Trends in Logic—Studia Logica Library 8, Kluwer Academic Publishers, Dordrecht, 2000.
16. Luo, X., Jennings, N. R.: A spectrum of compromise aggregation operators for multi-attribute decision making, Artificial Intelligence 171 (2007), 161-184.
17. Marques, P., Ricardo, A., Ribeiro, R. A.: Aggregation with generalized mixture operators using weighting functions, Fuzzy Sets and Systems 137 (2003), 43-58.
18. Mesiar, R., Pap, E.: Aggregation of infinite sequences, Information Sciences 178(18) (2008), 3557-3564.
19. Mesiar, R., Thiele, H.: On T-quantifiers and S-quantifiers. In: Novák V., Perfilieva I., eds., Discovering the World with Fuzzy Logic. Physica-Verlag, Heidelberg, 2000, 310–326.
20. Neyman, A.: Values of games with infinitely many players. In: R.J. Aumann, S. Hart (ed.), 2002. "Handbook of Game Theory with Economic Applications," Handbook of Game Theory with Economic Applications, Elsevier, edition 1, volume 3, number 3, chapter 56, 2121-2167.
21. Pap, E.: Null-Additive Set Functions. Kluwer Academic Publishers, Dordrecht- Boston-London, 1995.
22. Rovatti, R., Fantuzzi, C.: s-norm aggregation of infinite collections. Fuzzy Sets and Systems 84 (1996), 255-269.
23. Stupňanova, A.: Infinitary OWA operators, International Conference 70 Years of FCE STU, December 4-5, 2008, Bratislava, Slovakia.
24. Torra, V., Narukawa, Y.: Modeling decisions: Information Fusion and Aggregation Operators, Cognitive Technologies, Springer, 2007.
25. Vallentyne, P., Kagan, Sh.: Infinite Value and Finitely Additive Value Theory. Journal of Philosophy 94 (1997): 5-26.
26. Yager, R. R.: On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans. Syst., Man, Cybern. 18 (1988) 183–190

Towards a theory of a fuzzy rule base interpolation

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1 Introduction

It is well known that a fuzzy rule base is a characterization of a partially given mapping (fuzzy function) between fuzzy universes. For practical applications, it is desirable to interpolate that function in order to compute its values at points (fuzzy or crisp) other than fuzzy sets (nodes) in antecedents of the rule base. Moreover, interpolation requires that in the case of coincidence between a point and a node, a computation method should produce a fuzzy set that coincides with the corresponding one in the consequence part of the rule base.

Moreover, by Lotfi Zadeh, interpolation is a fuzzy logic inference engine. In his early papers [14, 15], he proposed the Generalized Modus Ponens (GMP) in the form of a formal scheme of inference and its possible realization in the form of the Compositional Rule of Inference (CRI). Both schemes were intensively investigated by many authors which resulted in a collection of methods related to a fuzzy rule base interpolation. Let us briefly overview main contributions to the field. In our opinion, all contributions can be classified into the following four groups.

- Technical solutions which are focused on detailed construction of an interpolating value [1–3, 7, 9, 8, 13]. The following steps are common to all proposed solutions in this group: algorithmic construction of an interpolating value, choice of closeness measures in spaces of arguments and dependent values, proving that in the case of closeness between known and arbitrary arguments, the known and computed dependent values are close as well. In the milestone paper [8] and then in [1, 7, 13], it is required that arguments and dependent values have some predefined characteristics (shapes, norms, etc.). In all papers above, the case of sparse data is used for testing.
- Axiomatic approach. A set of axioms has been proposed by Jenei [6], aiming at capturing almost all properties that have been of interest in the previous (to him) publications. The property of linearity has been considered as well. Similar approach has been proposed in [7].
- Theoretical approach. The interpolation problem consists in extending a fuzzy IF-THEN rule base, considered as a partial function between fuzzy set universes, to a total function between these universes. Moreover, a certain criterion should be minimized. According to the chosen criterion a solution and its representation are determined. In [12], an interpolating function is chosen from a Sobolev space of functions. A criterion is analogous to the case of spline interpolation between crisp values: it minimizes a real value measuring smoothness. Although the approach is very interesting, it does not provide any hint how to calculate the interpolating function.

  In [11, 10] we propose to consider an interpolating function as a member of a space of functions which are represented by fuzzy relations. The criterion is maximization of a similarity expressed by the biresiduation. Moreover, the interpolating function is assumed to be represented by a formula over a set of primitives given by fuzzy sets in the rule base. In contrast to [12], the interpolating function is obtained constructively via solution of the related system of fuzzy relation equations.

It is worth to be remarked that the theoretical approach is close to the technical one in what concerns a computational aspect. However, its significance is in explicit characterization of a set of interpolating functions together with a criterion which picks up just one member of that set.

– Logical approach. Being inspired by Lotfi Zadeh's ideas, the approach has been realized in [5]. It is focused on how a logic of similarity dedicated to interpolation can be defined, by considering different natural consequence relations induced by the presence of a similarity relation on the set of interpretations.

If interpolation of a fuzzy function is focused on finding a fuzzy function which interpolates the given data and has no other restrictions, then it can be solved in a class of fuzzy functions represented by fuzzy relations. In this case, interpolation leads to a problem of solvability of a system of fuzzy relation equations. On the other hand, if an interpolating fuzzy function is used for computation at arbitrary fuzzy points then the problem of interpolation is focused on estimation of closeness between an original and interpolating fuzzy functions. In this case, interpolation becomes a part of a problem of approximation and thus requires a rigorous formulation that includes a choice of a quality of approximation, etc.

In our contribution, we overview the principal literature devoted to problems mentioned above and propose a general framework for the interpolation problem in accordance with the presented theoretical approach. We will explain, why CRI does not guarantee interpolation at nodes, and show how this deficiency can be overcome. We restrict ourselves to those solutions of the interpolation problem which can be expressed by fuzzy relations [10, 11]. In that particular case, the interpolation problem is equivalent to the problem of solvability of a system of fuzzy relation equations. However, we will not consider the problem of solvability in its full depth. We will concentrate on requirements which allow us to pick up a unique solution which is, moreover, representable by a formula. We also propose a solution of the interpolation problem in the case of a sparse rule base and at a point which is disjoint with any of fuzzy sets in antecedents of the fuzzy rule base. Last, but not least, we will show, how the proposed approach relates to semi-linear spaces and their homomorphisms [4].

References

1. P. Baranyi, L.T. Koczy, T.D. Gedeon, A generalized concept for fuzzy rule interpolation, IEEE Trans. Fuzzy Systems 12 (2004) 820837.
2. D. Boixader and Jacas J. Extensionality based approximate reasoning, Int. J. Approximate Reasoning 19 (1998) 221-230.
3. B. Bouchon-Meunier, F. Esteva, L. Godo, M. Rifqi, S.Sandri, A principled approach to fuzzy rule base interpolation using similarity relations. EUSFLAT Conf. 2005: 757-763
4. Di Nola, A. and Lettieri, A. and Perfilieva, I. and V. Novák, Algebraic Analysis of Fuzzy Systems, Algebraic Analysis of Fuzzy Systems 158 (2007) 1–22.
5. D. Dubois, H. Prade, F. Esteva, P. Garcia and L. Godo, A logical approach to interpolation based on similarity relations, Int. J. Approximate Reasoning, 17 (1997) 1-36.
6. S. Jenei, Interpolation and extrapolation of fuzzy quantities revisitedan axiomatic approach, Soft Comput. 5 (2001) 179193.
7. Wen-Hoar Hsiao, Shyi-Ming Chen, Chia-Hoang Lee, A new interpolative reasoning method in sparse rule-based systems, Fuzzy Sets and Systems 93 (1998) 17-22
8. L.T. Koczy and K. Hirota, Approximate reasoning by linear rule interpolation and general approximation, Int. J. Approximate Reasoning 9 (1993), pp. 197225.
9. L.T. Koczy and K. Hirota, Interpolative reasoning with insufficient evidence in sparse fuzzy rule bases, Inform. Sci. 71 (1993), 169201.
10. I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets and Systems 147 (2004) 363–383.
11. I. Perfilieva, S. Gottwald, Solvability and approximate solvability of fuzzy relation equations, Int. J. of General Systems 32 (2003) 361–372.

12. Thomas Vetterlein, Spline interpolation between hyperspaces of convex or fuzzy sets, Fuzzy Sets and Systems 157 (2006) 2472 2481
13. Shi Yan, Masaharu Mizumoto, Wu Zhi Qiao, Reasoning conditions on Koczy's interpolative reasoning method in sparse fuzzy rule bases, Fuzzy Sets and Systems 75 (1995) 63-71
14. L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.
15. L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, II, III, Information Sciences 8-9 (1975) 199–257, 301–357, 43–80.
16. L.A. Zadeh, A theory of approximate reasoning. In: J. Hayes, D. Michie and L.I. Mikulich (Eds), Machine Intelligence 9 (1979) 149-194.

Additive and non-additive discrete aggregation of preferences

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Consider a finite set of alternatives $X = \{x_1,\ldots,x_m\}$ and a set N of agents, here chosen to be finite as well, that is, $N = \{1,\ldots,n\}$. *Preference aggregation* is a key topic in social choice and game theories, as well as, more recently, in artificial intelligence. Broadly speaking, the issue is to unify, in some way, the n individual preferences of agents $i \in N$. Traditionally, these latter take the form of binary relations $\gtrsim_i, i \in N$ over X, where $x_k \gtrsim_i x_l$ reads 'x_k *is at least as good as* x_l *to agent* $i \in N$'. Formally, $\gtrsim_i \subseteq X \times X$ is a subset of ordered pairs, and $x_k \gtrsim_i x_l$ is the short-hand notation for $\{\{x_k\},\{x_k,x_l\}\} \in \gtrsim_i$, where $\{\{x_k\},\{x_k,x_l\}\}$ denotes the ordered pair consisting of any two elements x_k, x_l with x_k being the first (in set theory).

Aggregation of preferences may be conceived in terms of providing some (hopefully non-empty) subset $X^* \subseteq X$ of socially optimal alternatives. To this end, *Pareto-dominance* is a main criterion [3], according to which if an alternative $x_k \in X$ is not Pareto-dominated, that is, if there is no $x_l \in X$ such that $x_l \gtrsim_i x_k$ for all $i \in N$, with strict preference $x_l >_i x_k$ for at least one $i \in N$, then such an alternative is socially optimal, that is, $x_k \in X^*$. Another main criterion is *majority voting*, and definitely several more criteria could be listed [5]. In general, non-emptiness of the set X^* of socially optimal alternatives, however obtained, depends on what assumptions on individual preferences one is prepared to make. Two main such assumptions are *completeness*, that is, for all $i \in N$ and all $x_k, x_l \in X$ either $x_k \gtrsim_i x_l$ or $x_l \gtrsim_i x_k$ or both, and *transitivity*, that is, for all $i \in N$ and all $x_h, x_k, x_l \in X$ if $x_h \gtrsim_i x_k$ and $x_k \gtrsim_i x_l$ then $x_h \gtrsim_i x_l$. A preference relation satisfying both these conditions is commonly said to be *rational* [9].

In a rational preference relation $\gtrsim$ with no indifference, for any two alternatives there is always one *strictly preferred* to the other. Clearly, such a preference corresponds to a *unique permutation* of alternatives, and indeed several *mechanisms* for implementing socially most desirable alternatives require agents to declare one such a strict preference relation or, equivalently, one permutation of alternatives [5]. This paper extends such an approach by allowing players to declare as many permutations of alternatives as they want, with no consistency requirement, of any kind, among them. In fact, an even preliminary idea comes from the observation that the general (and somehow traditional) approach described above, according to which preference aggregation should yield some subset $X^* \subseteq X$ of socially optimal alternatives, may result somehow binding. Specifically, even when $X^* \neq \emptyset$, still no information is provided about how to socially rank sub-optimal alternatives $x \in X \backslash X^*$, which instead could be useful in many circumstances. Furthermore, just like aggregating n real numbers (i.e. integrating a function $f : N \rightarrow \mathbb{R}$) amounts to provide a unique real number according to some criterion (or measure μ, so to obtain $\int f d\mu$), similarly aggregating preferences could well be (and is, indeed, in this paper) conceived in terms of providing a unique social preference $\gtrsim_N$ (i.e. a preference associated with the whole set N of agents) for given individual preferences of agents $\gtrsim_i, i \in N$. That is to say, aggregating any collection of elements, of whatever kind, should consist in providing a unique such an element. Hence when elements are preferences any aggregation technique should yield a preference as well.

Formally, agents' preferences take the form of a *permutation group* $\mathcal{S}_m^i \subseteq \mathcal{S}_m$ which is only required to be a non-empty subset of the *symmetric group of degree m* [1, p. 20], that is, $\mathcal{S}_m^i \neq \emptyset$ for all $i \in N$, with the convention that preferred alternatives come first. Representing a preference relation as a generic non-empty subset of permutations seems an interesting modeling choice [8]. It does not require nor imply completeness. Hence, this latter becomes a special feature that preferences may or may not display, making the whole approach rather general and flexible [4].

Before detailing the adopted additive and non-additive aggregation techniques, it must be specified how a permutation group identifies a (possibly non-complete) preference. In this view, it is crucial to observe that assessing whether $x_k \gtrsim_i x_l$ requires to inspect what positions $\pi(k)$ and $\pi(l)$ are assigned to such two alternatives by *all* permutations $\pi \in \mathcal{S}_m^i$. Concerning strict preference, $x_k >_i x_l$ if $\pi(k) < \pi(l)$ for all $\pi \in \mathcal{S}_m^i$. On the other hand, agent $i \in N$ is indifferent between t alternatives $x_{k_1}, \ldots, x_{k_t}$, denoted $x_{k_1} \sim_i \cdots \sim_i x_{k_t}$, with $1 < t \leq m$, if there exists $l_t \in \{t, t+1, \ldots, m\}$ such that both the following conditions are satisfied:

(i) $\pi(k_{t'}) \in \{l_t - t + 1, l_t - t + 2, \ldots, l_t\}$ for all $t' \in \{k_1, \ldots k_t\}$ and all $\pi \in \mathcal{S}_m^i$,

(ii) for all $t' \in \{k_1, \ldots k_t\}$ and all $l \in \{l_t - t + 1, l_t - t + 2, \ldots, l_t\}$ there are at least $(t-1)!$ distinct $\pi \in \mathcal{S}_m^i$ such that $\pi(k_{t'}) = l$.

In words, there must be t consecutive positions occupied by the t alternatives in each permutation and each alternative must occupy each position in *at least* $(t-1)!$ distinct permutations. Finally, $x_k \sim_i x_k$ for all alternatives $x_k \in X$ and all agents $i \in N$. Transitivity clearly holds: if $x_h >_i x_k >_i x_l$ then $x_h >_i x_l$ as well as $x_h \sim_i x_k \sim_i x_l$ entails $x_h \sim_i x_l$. But completeness does not, because if any pair of alternatives does not fall in any of the above two cases then such two alternatives remain incomparable. On the other hand, a rational (i.e. complete and transitive) preference relation corresponds to a unique *ordered collection of equivalence classes* of alternatives, and therefore if an agent $i \in N$ has a rational preference relation, then the permutation group $\mathcal{S}_m^i$ representing such a preference is the set of all permutations that are *admissible* w.r.t. (with respect to) the unique associated *ordered partition* [2] of X.

With these modeling choices, a preference aggregation technique (for n-cardinal agent sets) is intended to be a mapping $\Phi : \left(2^{\mathcal{S}_m} \backslash \{\emptyset\}\right)^n \rightarrow 2^{\mathcal{S}_m} \backslash \{\emptyset\}$ providing, given any n-profile $\mathcal{S}_m^1, \ldots, \mathcal{S}_m^n$ of permutation groups, some social permutation group $\mathcal{S}_m^N$. This means selecting some non-empty subset of *socially optimal* permutations. Hence, the general method may consist in evaluating the *social worth* $\mathcal{W}(\pi)$ (determined according to some criterion, see below) of each permutation $\pi \in \mathcal{S}_m$, and next choosing as optimal those whose worth is maximal, that is,

$$\mathcal{S}_m^N = \left\{\hat{\pi} \in \mathcal{S}_m : \mathcal{W}(\hat{\pi}) \geq \mathcal{W}(\pi) \text{ for all } \pi \in \mathcal{S}_m\right\}.$$

The next step is to suitably define some real-valued function $\mathcal{W} : \mathcal{S}_m \rightarrow \mathbb{R}_+$ quantifying the social worth of permutations. To this end, the basic tool is the *distance between permutations*, which in combinatorial theory is traditionally chosen to be the analog of the ℓ_∞ norm [1, p. 25, ex. 14]. Yet, in the present setting the analog of the ℓ_2 norm seems more appropriate. In any case, it is clear that any permutation of m elements is, in fact, a m-vector whose coordinates are the first m natural numbers (in some order), which therefore sum up to $\binom{m+1}{2}$ [7, p. 6], and thus several other metrics on $\mathcal{S}_m$ may well be conceived. Formally,

$$d(\pi, \sigma) = \left(\sum_{1 \leq k \leq m} \Big(\pi(k) - \sigma(k)\Big)^2\right)^{\frac{1}{2}} \text{ for all } \pi, \sigma \in \mathcal{S}_m.$$

For each $\pi \in \mathcal{S}_m$ define $f_\pi : N \rightarrow [\varepsilon_m, 1]$ by

$$f_\pi(i) = \frac{1}{1 + \min_{\sigma \in \mathcal{S}_m^i} d(\pi, \sigma)},$$

$$\varepsilon_m = \left[1 + \left(\sum_{0 \le k \le \lfloor \frac{n}{2} \rfloor} (n-2k-1)^2\right)^{\frac{1}{2}}\right]^{-1} = \left[1 + \max_{\pi,\sigma \in \mathcal{S}_m} d(\pi,\sigma)\right]^{-1}.$$

Inspection reveals that $f_\pi(i)$ is a strictly decreasing function of the distance separating permutation π from (a closest element in) the permutation group $\mathcal{S}_n^i$ of each agent $i \in N$. Accordingly, any discrete integral of f_π w.r.t. some suitable additive measure provides a social worth of π. For example, if one wants to assign more weight to agents with more flexible preferences, then one conceivable additive measure $p : N \to \Delta^{n-1}$ (where Δ^{n-1} denotes the $n-1$-dimensional simplex whose extreme points or vertexes correspond to players) is

$$p(i) = \frac{|\mathcal{S}_m^i|}{\sum_{j \in N} |\mathcal{S}_m^j|}.$$

Accordingly, the social worth $\mathcal{W}(\pi)$ of any permutation $\pi \in \mathcal{S}_m$ is given by

$$\mathcal{W}(\pi) = \int f_\pi dp = \sum_{1 \le i \le n} f_\pi(i)p(i).$$

In order to allow for non-additive aggregation, a *fuzzy measure* $\gamma : 2^N \to [0,1]$, with $\gamma(\emptyset) = 1 - \gamma(N) = 0$ and $\gamma(A) \le \gamma(A')$ for all $A \subseteq A' \in 2^N = \{B : B \subseteq N\}$, must be specified. Again, if one wants to assign more weight to *coalitions* with more flexible preferences, then one conceivable fuzzy or non-additive measure is

$$\gamma(A) = \frac{|\bigcup_{i \in A} \mathcal{S}_m^i|}{|\bigcup_{j \in N} \mathcal{S}_m^j|}.$$

Accordingly, the social worth may be given by the *discrete Choquet integral* [6]

$$\mathcal{W}(\pi) = \int^C f_\pi d\gamma = \sum_{1 \le i \le n} [f_\pi((i)) - f_\pi((i-1))]\, \gamma(\{(i)(i+1),\ldots,(n)\}),$$

where $(\cdot) : N \to N$ is any relabeling of agents satisfying $f_\pi((i)) \ge f_\pi((i-1))$ for $1 \le i \le n$, with $f_\pi((0)) := 0$.

Finally, each permutation π identifies a *coalitional game* [12] $v_\pi : 2^N \to [\varepsilon_m, 1]$ defined by , $v(\emptyset) := 0$ and

$$v_\pi(A) = \frac{1}{1 + \min_{\sigma \in \mathcal{S}_m^A} d(\pi,\sigma)} \text{ for all } A \in 2^N, \emptyset \ne A,$$

where $\mathcal{S}_m^A = \bigcup_{i \in A} \mathcal{S}_m^i$. This game is clearly monotone, that is, $A \subseteq B \Rightarrow v_\pi(A) \le v_\pi(B)$ for all coalitions $A, B \in 2^N$. Accordingly, it may be the integrated w.r.t. γ by means of the *extended Choquet integral* [10]

$$\int^{EC} v_\pi d\gamma.$$

In particular, two distinct such extended integrals may be used. One obtains by integrating along any *maximal chain* of coalitions through which v_π displays minimal increases and, given this, γ^* displays maximal increases, where γ^* is the *dual fuzzy measure* of γ, defined by $\gamma^*(A) = 1 - \gamma(N \backslash A)$ for all $A \in 2^N$. The other obtains by means of the Möbius inversion μ^{v_π} of v_π given by [11]

$$\mu^{v_\pi}(A) = \sum_{B \subseteq A} (-1)^{|A \backslash B|} v_\pi(B) \text{ for all } A \in 2^N.$$

In both cases the resulting integral $\int^{EC}$ is an extension of $\int^{C}$, because if v_π is additive, then $\int^{EC} v_\pi d\gamma = \int^{C} f_{v_\pi} d\gamma$ for all measures γ, where f_{v_π} denotes the restriction of v_π to singletons.

References

1. M. Aigner. *Combinatorial Theory*. Springer-Verlag, 1979.
2. G. Andrews. *The Theory of Partitions (Encyclopedia of Mathematics and its Applications, G.-C. Rota editor)*. Cambridge University Press, 1984.
3. R. J. Aumann. The core of a cooperative game without side payments. *Transactions of the American Mathematical Society*, 98:539552, 1961.
4. F. Brandt and P. Harrenstein. Dominance in social choice and coalitional game theory. Technical report, PAMAS research group, 2008.
5. V. Danilov and A. Sotskov. *Social Choice Mechanisms*. Springer, 2002.
6. M. Grabisch, M. Murofushi, and M. Sugeno. *Fuzzy Measures and Integrals - Theory and Applications*. Physica-Verlag (Springer), 2000. (Studies in fuzziness and soft computing vol. 40).
7. R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics - A Foundation for Computer Science*. Addison-Wesley, 1994.
8. C. Lamboray. A comparison between the prudent order and the ranking obtained with Borda's, Copeland's, Slater's and Kemeny's rules. *Mathematical Social Sciences*, (54):116, 2007.
9. A. Mas-Colell, M. D. Whinston, and J. R. Green. *Microeconomic Theory*. Oxford University Press, 1995.
10. G. Rossi. Extending the choquet integral. In *New Dimensions in Fuzzy Logic and Technology, Proceedings of the 5th EUSFLAT Conference*, volume I, pages 273–280, 2007. (http://www.eusflat.org/publications/proceedings/EUSFLAT_2007/ papers/Rossi_Giovanni_(23).pdf).
11. G.-C. Rota. On the foundations of combinatorial theory I: theory of Möbius functions. *Z. Wahrscheinlichkeitsrechnung u. verw. Geb.*, 2:340–368, 1964.
12. L. S. Shapley. A value for *n*-person games. In H. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games*, volume 2, pages 307–317. Princeton University Press, 1953.

A discussion about representation models for imprecise quantities and probabilities

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1 Introduction

This paper is based on topics discussed by D. Dubois and H. Prade in his contribution to the 2005 Linz Seminar, entitled *Fuzzy intervals versus fuzzy numbers: is there a missing concept in fuzzy set theory?* [3]. In Fuzzy Set Theory, the usual representation of imprecise quantities and proportions is by means of fuzzy numbers. Fuzzy numbers are defined as fuzzy subsets of numbers (real numbers unless other type is specified) verifying they are normal, convex, and have finite support. Usually they are defined by means of triangular membership functions, though other shapes are also employed. Fuzzy numbers represent ill-defined quantities like *around 3* or *approximately between 1.5 and 2.3* by means of soft, fuzzy restrictions on the set of numbers. They are important since we humans are used to express quantities this way, and hence we find it intuitive to be given an imprecise result expressed by a fuzzy number. Among the most important applications of fuzzy numbers we can mention fuzzy control, in which restrictions on the numerical domain of variables are usually expressed by fuzzy numbers that appear in the rules. They have also been employed in order to extend the mathematical notion of measure to imprecise (fuzzy) sets, such like cardinalities and probabilities.

Recently, some alternatives to the usual representation of imprecise quantities and proportions as fuzzy numbers have been proposed, namely Gradual Numbers [4] and RL-numbers [7]. As discussed in [3], both approaches consider that the so-called fuzzy numbers are in fact *fuzzy intervals* and provide a definition of imprecise quantities in which each possible precise approximation or representative is a crisp number instead of a crisp interval. As noted in [3], this was a topic of unresolved debates in early Linz Seminars, in particular in the first Linz Seminar in 1979.

Gradual and RL-numbers are essentially the same, with small differences regarding the starting point in their development and the way they are summarized and interpreted. In this work we argue about differences and analogies, advantages and disadvantages, between fuzzy numbers and the new proposals, and we conclude that they are different but complementary approaches. Finally, we make a proposal about when and how to employ each one in practice, and we recall an approximation to probability and statistics with fuzzy events on the basis of RL-numbers proposed in [5].

2 Gradual and RL-numbers

2.1 Gradual numbers

Gradual numbers were suggested in [2] and introduced under that name in [4]. Gradual numbers are seen as particular cases of *gradual/fuzzy elements*, the latter characterized by an assignment function from a complete lattice L of membership grades with top 1 and bottom 0 to a set X (were 0 has no image). Hence, a gradual (real) number is an assignment from $L\backslash\{0\}$ to $\mathbb{R}$. Typically, L is some subset of the unit interval.

The arithmetic of Gradual numbers is defined as follows [4]: given two Gradual numbers represented by their assignment function $\mathcal{A}_x$, $\mathcal{A}_y$, and a crisp arithmetic operation $*$, we obtain a Gradual number as:

$$\mathcal{A}_{x*y}(\alpha) = \mathcal{A}_x(\alpha) * \mathcal{A}_y(\alpha) \tag{1}$$

The algebraic structure and properties of Gradual numbers with operation $*$ is the same as those of the corresponding crisp numbers with the same operation.

2.2 RL-numbers

The concept of RL-number comes from the notion of *RL-representation* [6]. RL stands for *restriction level*, represented by values in $(0,1]$ that correspond to the degree to which we are strict in the definition of the property. This notion is similar to that of Gradual set as explained above by interpreting values in the lattice L as restriction levels, with two main differences:

- First, all the crisp operations between sets are extended to the case of RL representations by operating in each level independently. Though this idea is also employed with Gradual sets and Gradual numbers, the former is always interpreted as the representation of a certain fuzzy set, requiring a mapping of degrees in order to preserve consistency with the usual fuzzy set operations. On the contrary, RL-representations are not intended to be a different way of representing fuzzy sets, but an alternative way of representation of which fuzzy sets are a particular case (every $\alpha \in L\backslash\{0\}$ is assigned the corresponding α-cut of the fuzzy set, verifying the usual nested inclusion). No mapping of degrees is considered and any operation and definition is extended from the crisp case by operating in each level individually. This way, crisp properties are maintained, remarkably all those related to the complement operation (negation of properties) like the Excluded Middle axioms.
- In RL-representations, fuzzy sets are seen both as particular cases and also as a way to *summarize* the information given by the RL-representation by accumulating the evidence associated to each element. A fuzzy set can be obtained from a RL-representation by assigning a each level $\alpha \in L\backslash\{0\}$ an evidence mass given by the difference with the next level, and then adding up for each element the mass of those levels in which the element appears. This way of obtaining a fuzzy set from a RL-representation differs also from the one proposed in [4]. In both cases, different RL-representations/Gradual sets can yield the same fuzzy set.

RL-numbers arise in a natural way as measures on RL-representations and are identical to Gradual numbers, except in the way they are summarized as fuzzy subsets of numbers, as just explained.

3 Comparison

From the point of view of representation, as explained in [3, 2, 4], fuzzy numbers are in fact fuzzy intervals, so their semantics is that of the extension of intervals to the case of fuzzy information. A fuzzy interval represented by a Gradual/RL-representation yields an interval in each level of L. This notion is interpreted in [2, 4] as a *crisp interval of Gradual numbers*. The notion of RL-interval is that of a RL-representation in which we have an interval in each level L without the nested requirement of fuzzy sets.

From the point of view of arithmetic, arithmetic of fuzzy numbers has several well-known problems coming from the fact that they are extending a crisp arithmetic of intervals, that does not verify all the ordinary properties of arithmetic of numbers [4]. As it is well known, imprecision grows in every calculation. An important problem arises in the case of fuzzy integers when employing the usual Extension Principle for arithmetics, since counterintuitive results may be obtained. For example, as indicated in [4], the difference between a fuzzy integer and itself yields in general a fuzzy set centered around 0 and not the crisp value 0. But, even worst, some intuitive results of operations between fuzzy integers are not fuzzy integers, because convexity is lost. For example, if we multiply a fuzzy integer with support between 2 and 4 by the crisp number 2, we obtain a fuzzy integer with support between 4 and 8 in which odd numbers are in the support. This is clearly counterintuitive [1]. On the contrary, the arithmetic of Gradual/RL-numbers allows the fuzziness to diminish along operations (even to the extent that the operation between non-crisp Gradual/RL-numbers can be a crisp number [7]), and all the arithmetic properties are preserved. In [6, 7] we propose a measure of fuzziness for Gradual/RL-numbers. Remarkably, this imprecision can be bounded simply by fixing L.

As noted in [2, 4, 7], fuzzy intervals combine both fuzziness (in that the representation varies from one level to another) and imprecision (in that in each level we have an interval in general, and not a single value). Hence, Gradual/RL-numbers are better suited for representing imprecise real quantities and, in particular, to extend the notion of measure to the case of fuzzy sets (or, in general, RL-representations). Fuzzy numbers (intervals) are not, as already noted for cardinality of fuzzy sets in [1] and can be found in early proposals for fuzzy cardinality like that in [8]. On his turn, fuzzy intervals are the natural extension to the case of fuzzy information of the notion of interval and what it represents, in particular an useful tool for representing restrictions.

Finally, let us note that fuzzy numbers (intervals) are easier to interpret and understand for a human user. This is very important since in many applications, input and/or output of the systems is provided by/to an human expert.

We conclude that fuzzy numbers and Gradual/RL-numbers are different but complementary. All are useful, but for different purposes; they have the same usefulness as intervals and numbers, respectively, in the crisp case. More specifically we propose the following:

- Gradual/RL-numbers are the correct choice for extending measures (cardinality, probability of fuzzy events, etc.) to the case of fuzzy information represented by either fuzzy sets, in particular, or RL-representations in general. Any such measure on a fuzzy set should assign a Gradual/RL-number when the (possibly fuzzy) set/event we want to measure is well known (possibly by a representation as a fuzzy set).
- Fuzzy numbers (intervals) are an useful, correct, intuitive way to define fuzzy restrictions with semantics of fuzzy interval like “around x” or “approximately between x and y”. This is useful in two different situations:
  - when the imprecise quantity we want to represent is ill-known, for example if it is a measure provided by a human as the answer to a question “what is the probability ...”, or if the (possibly fuzzy) set/event we want to measure is not well known, like in the case of imprecise probabilities. This case has to deal with the well-known problems of arithmetic of fuzzy intervals, and
  - when providing meaningful information to a human. Fuzzy intervals are better suited than Gradual/RL-numbers for this purpose. In this sense, a lot of work about approximation of a Gradual/RL-number by a fuzzy interval remain to be done. A suitable approach for that can be to first summarize the Gradual/RL-number as a fuzzy subset of numbers (not necessarily a fuzzy number), and then to approximate that fuzzy subset by a fuzzy number. Different proposals for the first step can be found in [4, 7]. An approach for the second step is suggested in [7] on the basis of a previous work and will be studied in the future.

Finally, following the first of our conclusions, probability and statistics based on RL-numbers have been introduced in [5]. On the basis of the ideas of RL-representations, we perform statistical inference and tests in each level independently. A future work for us is to study in deep this research avenue.

References

1. M. Delgado, M.J. Martín-Bautista, D. Sánchez, and M.A. Vila. A probabilistic definition of a nonconvex fuzzy cardinality. *Fuzzy Sets and Systems*, 126(2):41–54, 2002.
2. D. Dubois and H. Prade. Fuzzy elements in a fuzzy set. In *Proc. IFSA'05*, volume 12, pages 55–60. 2005.
3. D. Dubois and H. Prade. Fuzzy intervals versus fuzzy numbers: Is there a missing concept in fuzzy set theory? In *Linz Seminar 2005 Abstracts*, pages 45–46. 2005.
4. D. Dubois and H. Prade. Gradual elements in a fuzzy set. *Soft Computing*, 12:165–175, 2008.
5. D. Sánchez, M. Delgado, and M.A. Vila. A restriction level approach to probability and statistics. In *ESTYLF 2008*, 2008.
6. D. Sánchez, M. Delgado, and M.A. Vila. A restriction level approach to the representation of imprecise properties. In *IPMU 2008*, pages 153–159, 2008.
7. D. Sánchez, M. Delgado, and M.A. Vila. RL-numbers: An alternative to fuzzy numbers for the representation of imprecise quantities. In *FUZZ-IEEE 2008*, pages 2058–2065, 2008.
8. L. A. Zadeh. A theory of approximate reasoning. *Machine Intelligence*, 9:149–194, 1979.

Axioms for fuzzy implications: dependence and independence

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A fuzzy implication is an extension of the classical binary implication which plays important roles in fuzzy set theory [3], [8], [10], [11] as well as in many applications such as fuzzy morphology in image processing [6], [7], [9] and association rules in data mining [13].
A fuzzy implication is defined as a $[0,1]^2 \to [0,1]$ mapping I which satisfies the four axioms:

FA: the first place antitonicity:
$(\forall (x_1, x_2, y) \in [0,1]^3)(x_1 < x_2 \Rightarrow I(x_1, y) \geq I(x_2, y))$;
SI: the second place isotonicity:
$(\forall (x, y_1, y_2) \in [0,1]^3)(y_1 < y_2 \Rightarrow I(x, y_1) \leq I(x, y_2))$;
DF: dominance of falsity of antecedent: $(\forall x \in [0,1])(I(0,x) = 1)$;
DT: dominance of truth of consequent: $(\forall x \in [0,1])(I(x,1) = 1)$.

Besides these four defining axioms, there are many other potential properties of a fuzzy implication to fulfill certain requirements among which the following eight axioms are widely proposed in the literature [1], [2], [4], [5], [8], [12]:

NT: neutrality of truth: $(\forall x \in [0,1])(I(1,x) = x)$;
EP: exchange principle: $(\forall (x,y,z) \in [0,1]^3)(I(x, I(y,z)) = I(y, I(x,z)))$;
OP: ordering principle: $(\forall (x,y) \in [0,1]^2)(I(x,y) = 1 \Leftrightarrow x \leq y)$;
SN: $(\forall x \in [0,1])(I(x,0)$ defines a strong negation);
CB: consequent boundary: $(\forall (x,y) \in [0,1]^2)(I(x,y) \geq y)$;
ID: identity: $(\forall x \in [0,1])(I(x,x) = 1)$;
CP: contrapositive principle: $(\forall (x,y) \in [0,1]^2)(I(x,y) = I(N(y), N(x)))$, where N is a strong fuzzy negation;
CO: continuity: I is a continuous mapping.

Many authors have worked on different fuzzy implications with different axioms and have found out some interrelationship between these eight axioms [1], [2], [5], [8]. But the results are far from complete. Our aim of this work is to determine a full view of the interrelationship between these eight axioms. The result helps us to determine a classification on the class of fuzzy implications.

References

1. M. Baczyński, Residual implications revisited. Notes on the Smets-Magrez Theorem, Fuzzy Sets and Systems 145 (2004) 267-277.

2. H. Bustince, P. Burillo, F. Soria, Automorphisms, negations and implication operators, Fuzzy Sets and Systems 134 (2003) 209-229.
3. H. Bustince, M. Pagola, E. Barrenechea, Construction of fuzzy indices from fuzzy DI-subsethood measures: Application to the global comparison of images, Information Sciences 177 (2007) 906-929.
4. J.C. Fodor, Contrapositive symmetry of fuzzy implications, Fuzzy Sets and Systems 69 (1995) 141-156.
5. J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Netherlands, 1994.
6. B. Jayaram, Rule reduction for efficient inferencing in similarity based reasoning, International Journal of Approximate Reasoning 48 (2008) 156-173.
7. E.E. Kerre, M. Nachtegael, Fuzzy Techniques in Image Processing, Physica-Verlag, New York, 2000.
8. G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic, Theory and Applications, Prentice Hall, USA, 1995.
9. M. Nachtegael, H. Heijmans, D. Van der Weken, E. Kerre, Fuzzy Adjunctions in Mathematical Morphology, Proc. of JCIS 2003, North Carolina, USA, sept, (2003) 202-205.
10. D. Ruan, E.E. Kerre, Fuzzy implication operators and generalized fuzzy method of cases, Fuzzy Sets and Systems 54 (1993) 23-37.
11. D. Ruan, E.E. Kerre, Fuzzy IF-THEN Rules in Computational Intelligence: Theory and Applications, Kluwer Academic Publishers, Boston 2000.
12. A survey on Fuzzy Implication Functions, M. Mas, M. Monserrat, J. Torrens, E. Trillas, IEEE Transactions on Fuzzy Systems 15(6) (2007) 1107-1121.
13. P. Yan, G. Chen, Discovering a cover set of ARsi with hierarchy from quantitative databases, Information Sciences 173 (2005) 319-336.

Embedding topology into algebra

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1 Introduction

This paper continues our study of a (possible) single framework in which to treat both variable-basis fuzzy topological spaces in the sense of [18] and the respective algebraic structures underlying their topologies. Originally suggested by both localic theory and fuzzy set theory, the problem has a long history.

In 1959 D. Papert and S. Papert constructed an adjunction between the categories **Top** of topological spaces and $\mathbf{Frm}^{op}$ the dual of the category **Frm** of frames [13]. The adjunction was described more succinctly by J. Isbell in [8], where he introduced the name *locale* for the objects of $\mathbf{Frm}^{op}$ and considered the category **Loc** of locales as a substitute for **Top**. In 1982 P. Johnstone gave a coherent statement to localic theory in his famous book "Stone Spaces" [9]. Using the logic of finite observations S. Vickers introduced in [27] the notion of *topological system* to get a single framework in which to treat both spaces and locales.

On the other hand, the pioneering papers of C. L. Chang [2] and R. Lowen [11] started the theory of *fixed-basis* fuzzy topological spaces. In 1983 S. E. Rodabaugh introduced the category **FUZZ** of *variable-basis* fuzzy topological spaces [15]. Since then it is known as the category **C-Top** of variable-basis lattice-valued topological spaces [18, 19]. Both fixed- and variable-basis topologies induced many researchers to study their properties [5–7, 16–21]. In particular, in [4] J. T. Denniston and S. E. Rodabaugh considered functorial relationships between lattice-valued topology and topological systems. Using fuzzy topological spaces and crisp topological systems they encountered some problems. This manuscript aims at providing another approach to the question.

In a series of papers [22, 24, 26] we studied the categories of fixed- as well as variable-basis topological spaces over an arbitrary variety of algebras generalizing the approaches of C. L. Chang (resp. R. Lowen) and S. E. Rodabaugh. The basic point of our investigations was an attempt to do fuzzy mathematics with non-ordered structures. The motivation for the problem was provided by [23], where we introduced the notion of *fuzzy object* in a concrete category $(\mathbf{A}, U)$ over $\mathbf{X}$ as an U-structured arrow $X \xrightarrow{f} UA$. The new notion not only generalizes the standard ones used in the literature, but also provides a wide range of new (sometimes even unexpected) examples and applications in the field of algebra and topology [23]. In particular, our definition does not require A to be a lattice or even to be ordered.

In [25] we introduced the notion of *variable-basis topological system* over an arbitrary variety of algebras. By analogy with J. T. Denniston and S. E. Rodabaugh we considered functorial relationships between the categories of variable-basis topological spaces and variable-basis topological systems. In particular, we constructed a full embedding of the former category into the latter one. Our slogan was: while considering fuzzy topological spaces, one should consider fuzzy topological systems.

We also showed that unlike the category of variable-basis topological spaces which is topological over its ground category, the category of variable-basis topological systems has that property iff the respective underlying functor is an isomorphism and posed the question on the nature of the latter category (i.e., whether it is algebraic). It is the purpose of the current paper to answer the question (partly) in the affirmative, thereby providing a full embedding of topology into algebra.

The necessary categorical background can be found in [1, 12]. For algebraic notions we recommend [3, 12]. It is expected from the reader to be acquainted with basic concepts of category theory, e.g., with that of an adjoint situation.

2 Variable-basis topological spaces

In this section we recall from [24] the definition of the category of variable-basis topological spaces over an arbitrary variety of algebras. Notice that our definition generalizes (in fact goes in line with) the respective one of S. E. Rodabaugh [18, 19]. Start by recalling the concept of *variety* [3, 12].

Definition 1. *Let* $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ *be a (possibly proper) class of cardinal numbers. An* Ω-algebra *is a pair* $(A,(\omega^A_\lambda)_{\lambda\in\Lambda})$ *(denoted by* A*) consisting of a set* A *and a family of maps* $A^{n_\lambda} \xrightarrow{\omega^A_\lambda} A$*. An* Ω-homomorphism $(A,(\omega^A_\lambda)_{\lambda\in\Lambda}) \xrightarrow{f} (B,(\omega^B_\lambda)_{\lambda\in\Lambda})$ *is a map* $A \xrightarrow{f} B$ *such that* $f\circ\omega^A_\lambda = \omega^B_\lambda \circ f^{n_\lambda}$*.* $\mathbf{Alg}(\Omega)$ *is the category of* Ω*-algebras and* Ω*-homomorphisms. Let* $\mathcal{M}$ *(resp.* $\mathcal{E}$*) be the class of* Ω*-homomorphisms with injective (resp. surjective) underlying maps. A* variety of Ω-algebras *(also called a* variety*) is a full subcategory of* $\mathbf{Alg}(\Omega)$ *closed under the formation of products,* $\mathcal{M}$*-subobjects (subalgebras) and* $\mathcal{E}$*-quotients (homomorphic images). The objects (resp. morphisms) of a variety will be referred to as* algebras *(resp.* homomorphisms*).*

The categories **Frm**, **SFrm**, **Quant** and **SQuant** of frames, semiframes, quantales and semi-quantales (popular in lattice-valued topology) are varieties [19].

Fix a variety **A** and an algebra Q. Given a set X, Q^X the *Q-powerset of X* is an algebra with operations lifted point-wise from Q, i.e., $(\omega^{Q^X}_\lambda(\langle p_i\rangle_{n_\lambda}))(x) = \omega^Q_\lambda(\langle p_i(x)\rangle_{n_\lambda})$. Every map $X \xrightarrow{f} Y$ provides the standard image and preimage operators on the respective powersets $\mathcal{P}(X) \xrightarrow{f^\rightarrow} \mathcal{P}(Y)$ and $\mathcal{P}(Y) \xrightarrow{f^\leftarrow} \mathcal{P}(X)$. Moreover, there exists the *Zadeh preimage operator* $Q^Y \xrightarrow{f^\leftarrow_Q} Q^X$ defined by $f^\leftarrow_Q(p) = p\circ f$ [17, 19, 28]. On the other hand, every homomorphism $A \xrightarrow{g} B$ can be lifted to a map $A^X \xrightarrow{g^X_\rightarrow} B^X$ defined by $g^X_\rightarrow(p) = g\circ p$. Both $Q^Y \xrightarrow{f^\leftarrow_Q} Q^X$ and $A^X \xrightarrow{g^X_\rightarrow} B^X$ are homomorphisms.

For convenience sake from now on we use the following notations [4, 18, 19]. The dual of the category **A** is denoted by **LoA** (the "**Lo**" comes from "localic"). Its objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Given a localic homomorphism φ, the respective homomorphism is denoted by φ^{op} and vice versa.

Let $(X,A) \xrightarrow{(f,\varphi)} (Y,B)$ be a $\mathbf{Set}\times\mathbf{LoA}$-morphism. There exists the *Rodabaugh preimage operator* $B^Y \xrightarrow{(f,\varphi)^\leftarrow} A^X$ defined by $(f,\varphi)^\leftarrow(p) = \varphi^{op}\circ p\circ f$ [4, 16–21]. Since $f^\leftarrow_A \circ (\varphi^{op})^Y_\rightarrow = (f,\varphi)^\leftarrow = (\varphi^{op})^X_\rightarrow \circ f^\leftarrow_B$, $(f,\varphi)^\leftarrow$ is a homomorphism. By analogy with [18, 19] we introduce the category of variable-basis topological spaces.

Definition 2. *Given a subcategory* **C** *of* **LoA***,* **C-Top** *is the category, the objects of which (called* **C**-topological spaces *or* **C**-spaces*) are triples* (X,A,τ)*, where* (X,A) *is a* $\mathbf{Set}\times\mathbf{C}$*-object and* τ *is a subalgebra of* A^X*. Morphisms* $(X,A,\tau) \xrightarrow{(f,\varphi)} (Y,B,\sigma)$ *are* $\mathbf{Set}\times\mathbf{C}$*-morphisms* $(X,A) \xrightarrow{(f,\varphi)} (Y,B)$ *such that* $((f,\varphi)^\leftarrow)^\rightarrow(\sigma) \subseteq \tau$ *(continuity). The forgetful functor* $\mathbf{C}\text{-}\mathbf{Top} \xrightarrow{|-|} \mathbf{Set}\times\mathbf{C}$ *is* $|(X,A,\tau) \xrightarrow{(f,\varphi)} (Y,B,\sigma)| = (X,A) \xrightarrow{(f,\varphi)} (Y,B)$*.*

In this paper we restrict ourselves to the case $\mathbf{C} = \mathbf{LoA}$, calling $\mathbf{LoA}$-spaces by spaces and $\mathbf{LoA}$-continuity by continuity.

3 Variable-basis topological systems

This section introduces our main category of study, i.e., the category of variable-basis topological systems [25]. Its definition generalizes the respective one of S. Vickers [27].

Definition 3. *Given a subcategory* $\mathbf{C}$ *of* $\mathbf{LoA}$*,* $\mathbf{C}$-$\mathbf{TopSys}$ *is the category, the objects of which (called* $\mathbf{C}$-topological systems *or* $\mathbf{C}$-systems*) are tuples* $D = (\mathrm{pt}D, \Sigma D, \Omega D, \models)$*, where* $(\mathrm{pt}D, \Sigma D, \Omega D)$ *is a* $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$*-object and* $\mathrm{pt}D \times \Omega D \xrightarrow{\models} \Sigma D$ *is a map (called* ΣD-satisfaction relation on $(\mathrm{pt}D, \Omega D)$*) such that for every* $x \in \mathrm{pt}D$*,* $\Omega D \xrightarrow{\models(x,-)} \Sigma D$ *is a homomorphism. Morphisms* $D_1 \xrightarrow{f=(\mathrm{pt}f,(\Sigma f)^{op},(\Omega f)^{op})} D_2$ *are* $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$*-morphisms* $(\mathrm{pt}D_1, \Sigma D_1, \Omega D_1) \xrightarrow{f} (\mathrm{pt}D_2, \Sigma D_2, \Omega D_2)$ *such that for every* $x \in \mathrm{pt}D_1$ *and every* $d \in \Omega D_2$*,* $\Sigma f(\models_2(\mathrm{pt}f(x), d)) = \models_1(x, \Omega f(d))$ *(continuity). The forgetful functor* $\mathbf{C}$-$\mathbf{TopSys} \xrightarrow{|-|} \mathbf{Set} \times \mathbf{C} \times \mathbf{C}$ *is* $|D_1 \xrightarrow{f} D_2| = (\mathrm{pt}D_1, \Sigma D_1, \Omega D_1) \xrightarrow{f} (\mathrm{pt}D_2, \Sigma D_2, \Omega D_2)$.

In this paper we restrict ourselves to the case $\mathbf{C} = \mathbf{LoA}$, calling $\mathbf{LoA}$-systems by systems and $\mathbf{LoA}$-continuity by continuity.

Definition 4. Q-$\mathbf{TopSys}$ *is the subcategory of* $\mathbf{LoA}$-$\mathbf{TopSys}$ *with objects all systems* D *with* $\Sigma D = Q$ *and morphisms all system morphisms* f *such that* $\Sigma f = 1_Q$.

Lemma 1. *The subcategory* Q-$\mathbf{TopSys}$ *is full iff* $\mathbf{A}(Q,Q) = \{1_Q\}$*. If* $\mathtt{I}$ *is an initial object in* $\mathbf{A}$*, then* $\mathtt{I}$-$\mathbf{TopSys}$ *is full.*

Since $2 = \{\bot, \top\}$ is an initial object in $\mathbf{Frm}$, the full subcategory 2-$\mathbf{TopSys}$ of $\mathbf{Loc}$-$\mathbf{TopSys}$ is isomorphic to the category $\mathbf{TopSys}$ of S. Vickers. Given a set K, the subcategory K-$\mathbf{TopSys}$ of $\mathbf{LoSet}$-$\mathbf{TopSys}$ is isomorphic to the category $\mathtt{Chu}(\mathbf{Set}, K)$ of *Chu spaces* over K [14]. K-$\mathbf{TopSys}$ is full iff K is the empty set or a singleton.

4 Topological spaces versus topological systems

In this section we show that the category $\mathbf{LoA}$-$\mathbf{Top}$ is isomorphic to a full regular mono-coreflective subcategory of the category $\mathbf{LoA}$-$\mathbf{TopSys}$.

Lemma 2. *There exists a full embedding* $\mathbf{LoA}$-$\mathbf{Top} \xhookrightarrow{E_T} \mathbf{LoA}$-$\mathbf{TopSys}$ *given by the formula*

$$E_T((X,A,\tau) \xrightarrow{(f,\varphi)} (Y,B,\sigma)) = (X,A,\tau,\models_1) \xrightarrow{(f,\varphi,((f,\varphi)^{\leftarrow})^{op})} (Y,B,\sigma,\models_2)$$

where $\models_i(z,p) = p(z)$.

Proof. For continuity of $E_T(f,\varphi)$ notice that $\models_1(x,(f,\varphi)^{\leftarrow}(p)) = \models_1(x, \varphi^{op} \circ p \circ f) = \varphi^{op} \circ p \circ f(x) = \varphi^{op}(\models_2(f(x),p))$. As for fullness, given any continuous $E_T(X,A,\tau) \xrightarrow{f} E_T(Y,B,\sigma)$, $(\Omega f(p))(x) = \models_1(x, \Omega f(p)) = \Sigma f(\models_2(\mathrm{pt}f(x),p)) = (\Sigma f \circ p \circ \mathrm{pt}f)(x) = ((\mathrm{pt}f,(\Sigma f)^{op})^{\leftarrow}(p))(x)$, i.e., $\Omega f(p) = (\mathrm{pt}f,(\Sigma f)^{op})^{\leftarrow}$. Therefore $(X,A,\tau) \xrightarrow{(\mathrm{pt}f,(\Sigma f)^{op})} (Y,B,\sigma)$ is continuous and $E_T(\mathrm{pt}f,(\Sigma f)^{op}) = f$. $\square$

Lemma 3. *There exists a functor* **LoA-TopSys** $\xrightarrow{\text{Spat}}$ **LoA-Top** *with*

$$\text{Spat}(D_1 \xrightarrow{f} D_2) = (\text{pt}D_1, \Sigma D_1, \tau) \xrightarrow{(\text{pt}f,(\Sigma f)^{op})} (\text{pt}D_2, \Sigma D_2, \sigma)$$

where $\tau = \{\models_1(-,b) \,|\, b \in \Omega D_1\}$.

Proof. By analogy with S. Vickers we call $\models_1(-,b)$ the *extent* of b. To show that $\text{Spat}D$ is a space we notice that

$$(\omega_\lambda^{(\Sigma D)^{\text{pt}D}}(\langle\models_1(-,b_i)\rangle_{n_\lambda}))(x) = \omega_\lambda^{\Sigma D}(\langle\models_1(x,b_i)\rangle_{n_\lambda}) = \models_1(x,\omega_\lambda^{\Omega D}(\langle b_i\rangle_{n_\lambda})) = (\models_1(-,\omega_\lambda^{\Omega D}(\langle b_i\rangle_{n_\lambda})))(x).$$

As for continuity of $\text{Spat}f$, given the extent of some $d \in \Omega D_2$, $((\text{pt}f,(\Sigma f)^{op})^{\leftarrow}(\models_2(-,d)))(x) = (\Sigma f \circ \models_2(-,d) \circ \text{pt}f)(x) = \Sigma f(\models_2(\text{pt}f(x),d)) = \models_1(x,\Omega f(d)) = (\models_1(-,\Omega f(d)))(x)$. □

Lemmas 2 and 3 provide the following theorem.

Theorem 1. *The functor* Spat *is a right-adjoint-left-inverse of the functor* E_T.

Proof. Every system D has a map $\Omega D \xrightarrow{\Phi} \{\models(-,b) \,|\, b \in \Omega D\}$ defined by $\Phi(b) = \models(-,b)$. Since Φ is a coequalizer of the homomorphisms $\text{Ker}\,\Phi \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B$ with $\pi_i(b_1,b_2) = b_i$, it is a regular epimorphism in **A**. Moreover, straightforward computations show that $E_T\,\text{Spat}D \xrightarrow{e=(1_{\text{pt}D},1_{\Sigma D},\Phi^{op})} D$ provides an E_T-(co-universal) map for D. □

Corollary 1. *The category* **LoA-Top** *is isomorphic to a full regular mono-coreflective subcategory of the category* **LoA-TopSys***.*

5 Embedding topology into algebra

The previous section provided the (full) embedding **LoA-Top** $\overset{E_T}{\hookrightarrow}$ **LoA-TopSys**. With a little effort one can see that the category **LoA-Top** is topological over its ground category [24]. On the other hand, the category **LoA-TopSys** is topological over its ground category iff the forgetful functor $|-|$ is an isomorphism [25]. The following theorem explains the reason.

Theorem 2. *The forgetful functor* $|-|$ *of the category* **LoA-TopSys** *creates isomorphisms and is adjoint but the category itself is (Epi, Mono-Source)-factorizable, therefore,* (**LoA-TopSys**$,|-|$) *is essentially algebraic. If* **A***-epimorphisms are onto, then* $|-|$ *preserves extremal epimorphisms and thus the category in question is algebraic.*

Notice that epimorphisms are onto neither in the category **Frm** of frames nor in the category **Quant** of quantales [10] (unfortunately, we can not say anything on epimorphisms in the category **SQuant** of semi-quantales). It is still an open question whether the functor $|-|$ preserves extremal epimorphisms. The only result obtained so far is as follows.

Lemma 4. *Given an extremal epimorphism* $D_1 \xrightarrow{e} D_2$ *in* **LoA-TopSys***, both* pt$e$ *and* Σe *are extremal epimorphisms in the respective categories.*

Lemma 2 and Theorem 2 provide the (full) embedding of the (topological) category **LoA-Top** into the essentially algebraic category **LoA-TopSys**, i.e., an embedding of “topology” into “algebra”.

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References

1. J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories: the Joy of Cats*, Repr. Theory Appl. Categ. **17** (2006), 1–507.
2. C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.
3. P. M. Cohn, *Universal Algebra*, D. Reidel Publ. Comp., 1981.
4. J. T. Denniston and S. E. Rodabaugh, *Functorial relationships between lattice-valued topology and topological systems*, submitted to Quaest. Math.
5. U. Höhle, *Fuzzy sets and sheaves. Part I: Basic concepts*, Fuzzy Sets Syst. **158** (2007), no. 11, 1143–1174.
6. U. Höhle, *Fuzzy sets and sheaves. Part II: Sheaf-theoretic foundations of fuzzy set theory with applications to algebra and topology*, Fuzzy Sets Syst. **158** (2007), no. 11, 1175–1212.
7. U. Höhle and A. P. Šostak, *Axiomatic Foundations of Fixed-Basis Fuzzy Topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Dordrecht: Kluwer Academic Publishers, 1999, pp. 123–272.
8. J. R. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5–32.
9. P. T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.
10. D. Kruml and J. Paseka, *Algebraic and Categorical Aspects of Quantales*, Handbook of Algebra (M. Hazewinkel, eds.), vol. 5, Elsevier B.V, 2008, pp. 323–362.
11. R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), 621–633.
12. E. G. Manes, *Algebraic theories*, Springer-Verlag, 1976.
13. D. Papert and S. Papert, *Sur les treillis des ouverts et les paratopologies.*, Semin. de Topologie et de Geometrie differentielle Ch. Ehresmann 1 (1957/58), No.1, p. 1-9, 1959.
14. V. Pratt, *Chu spaces*, School on category theory and applications. Lecture notes of courses, Coimbra, Portugal, July 13-17, 1999. Coimbra: Universidade de Coimbra, Departamento de Matemática. Textos Mat., Sér. B. 21, 39-100 (1999).
15. S. E. Rodabaugh, *A categorical accommodation of various notions of fuzzy topology*, Fuzzy Sets Syst. **9** (1983), 241–265.
16. S. E. Rodabaugh, *Point-set lattice-theoretic topology*, Fuzzy Sets Syst. **40** (1991), no. 2, 297–345.
17. S. E. Rodabaugh, *Powerset operator based foundation for point-set lattice-theoretic (poslat) fuzzy set theories and topologies*, Quaest. Math. **20** (1997), no. 3, 463–530.
18. S. E. Rodabaugh, *Categorical Foundations of Variable-Basis Fuzzy Topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Dordrecht: Kluwer Academic Publishers, 1999, pp. 273–388.
19. S. E. Rodabaugh, *Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics*, Int. J. Math. Math. Sci. **2007** (2007), Article ID 43645, 71 pages, doi:10.1155/2007/43645.
20. S. E. Rodabaugh and A. Pultr, *Category theoretic aspects of chain-valued frames: part I: categorical foundations*, Fuzzy Sets Syst. **159** (2008), 501–528.
21. S. E. Rodabaugh and A. Pultr, *Category theoretic aspects of chain-valued frames: part II: applications to lattice-valued topology*, Fuzzy Sets Syst. **159** (2008), 529–558.
22. S. Solovjovs, *From quantale algebroids to topological spaces*, Abstracts of the 29th Linz Seminar on Fuzzy Set Theory (E. P. Klement, S. E. Rodabaugh, and L. N. Stout, eds.), Johannes Kepler Universität, Linz, 2008, pp. 98–101.
23. S. Solovjovs, *On a Categorical Generalization of the Concept of Fuzzy Set: Basic Definitions, Properties, Examples*, VDM Verlag Dr. Müller, 2008.
24. S. Solovyov, *Categorical frameworks for variable-basis sobriety and spatiality*, to appear in Proc. of Int. Conf. on Topological Algebras and their Applications (ICTAA) 2008.
25. S. Solovyov, *Variable-basis topological systems versus variable-basis topological spaces*, submitted to Soft Comput.
26. S. Solovyov, *Sobriety and spatiality in varieties of algebras*, Fuzzy Sets Syst. **159** (2008), no. 19, 2567–2585.
27. S. Vickers, *Topology via Logic*, Cambridge University Press, 1989.
28. L. A. Zadeh, *Fuzzy sets*, Inf. Control **8** (1965), 338–365.

On the link between chance and truth

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The starting point for fuzzy set theory and many formalizations of fuzzy logic is the idea of substituting the usual two truth values with many truth values.

In any case the concept of truth remains at the root of the mathematical description of fuzzy systems.

To address the argument of the seminar I would like to analyze the following question: is truth a primitive concept?

In my presentation I will try to show that chance is a more primitive concept, witness the fact that we can recover truth starting from chance but not viceversa. This fact has some interesting consequences. Let me start from an interesting story.

When Goedel arrives at Princeton (1940) he has in his suitcase almost all his logical works that will be published in the future. Therefore after his arrival in the USA his main interest is philosophical.

What remains of this work two articles ready to be published and thousands of pages of philosophical notes.

Almost nothing of this material has been published by the author. As reported by Goedel one of the motivations is that he was unable to give a logical formalization to this material.

Without entering in the complex and often obscure metaphysics left by Goedel, for the porpoise of these notes it is sufficient to say that his system was based on one fundamental assumption: chance does not exist.

At Princeton Goedel encounters very often Einstein, that was involved in problems very close the the fundamental assumption made by Goedel. In fact Einstein was searching for a deterministic completion of quantum theory.

Both failed in their attempts, because Goedel was not able to obtain a clear and formal representation of his ideas, while Bell and Kochen-Specker theorems show that Einstein's hypothesis is wrong. Nonetheless I believe that the open problem they left us is very interesting, and its investigation can give us some insights on the meaning of the concept of truth value.

To start the analysis let me mention some aspects of the work on a stochastic semantics of logic, more precisely a stochastic interpretation of proof semantics of linear logic. The interested reader can find all the technical details at the following address: http://www.isib.cnr.it/infor/papers/stat.pdf.

In the stochastic semantics for logic every formula is interpreted into a coherent sets of observables (trajectories of a suitable stochastic process). Let X be the set of possible observables of a formula ϕ. It is assumed that to every observable x it is associated a statistical measure σ_x^X over the possible outcomes.

For instance for an atomic formula P a coherent set of observables has the form $\{x : \sigma_x^X = \sigma_{\bar{x}}^X\}$, where $\bar{x}$ is the available observable. For this reason the interpretation of logic depends on the observed data i.e. the trajectories $\overline{x_1}, \ldots, \overline{x_n}, \ldots$ relative to the atomic formulae $P_1, \ldots, P_n, \ldots$ Changing the available observables $\overline{x_1},...,\overline{x_n}$ the interpretation changes. Therefore we can say that the interpretation of logic is generated by random choices.

In logic, there are two different semantics: the interpretation of formulae (truth-valued semantics) and the interpretation of proofs (coherent spaces, λ-calculus). In stochastic semantics the two aspects are unified, we can start from the interpretation of formulae, interpreted into sets of available observables, and obtain a semantics where also proofs can be interpreted, again into sets of observables of the proven formula.

The important property, i.e. validity of stochastic semantics, is that, while the interpretation of formulae always depend on random choices, i.e. the available observable, for proofs random choices are irrelevant.

More precisely it is proved that if ϕ is a provable formula then the interpretation of a proof π of ϕ does not depend on the observables used to interpret the proven formula and π is represented by a $true$ formula in a suitable topos of presheaves. It is also shown that a $true$-formula in this topos is a time-uncertainty invariant property, i.e. a property that remains $true$ if time passes and information changes due to randomness.

Let us assume that mathematics is reliable, i.e. that mathematics is able to forecast events that have not yet been observed. There are a lot of examples of this fact.

Under this hypothesis we can assume that the laws of mathematics respects some important symmetries of the universe. Due to the fact that the laws of mathematics are chance invariants, as the validity theorem shows, we can assume that also the symmetries of universe are chance invariants, therefore we can conclude in accordance with the ideas of Einstein (and Goedel) that ”God does not play dice with the universe”.

What remains to understand is why and where Einstein and Goedel were wrong, and here truth-values enter into play.

Both of them negate the importance and existence of chance.

I believe that this is the root of their error.

To explain this point let me use the following metaphor.

If you have a sphere you can easily recover its center of symmetry: a point in the space. But if you have a point in the space it is impossible to recover the sphere that has that point as its center of symmetry. Therefore if we want to reconstruct the sphere from its center the only way is to reduce the sphere to its center, i.e. to negate its existence. Using the above metaphor let me explain how truth can be seen as the center of the sphere and chance the sphere itself.

We have seen that proofs and computations (due to the Curry-Howard isomorphism) are chance invariant.

Note that in the statistical interpretation of logic there is only one atomic formula that has a chance independent interpretation: 1 that is interpreted by the trivial stochastic process (the one described by the trivial filtration whose algebras are always equal to $\{\emptyset,\Omega\}$). Moreover, in the truth-valued semantics of linear logic a formula ϕ is true iff 1 belongs to its truth value.

From this, by completeness theorem, also truth is a chance invariant, it does not depend on chance like observables, hence it does not vary due to time and uncertainty changes.

Therefore we can characterize truth starting from chance: truth is a chance-invariant, but it is impossible to describe chance starting from truth, simply because truth, being a chance-invariant, has no explicit link with random choices.

For every atomic proposition different from 1, different observables define different interpretations. Therefore, in stochastic semantics, an atomic proposition can never be true, and its interpretation depends always on random choices that generate the available observable $\bar{x}$.

The possibility of describing the link between proofs and chance invariants is based on the fact that the interpretation of formulae depends on random choices, in fact we must make logic vary depending on random choices to describe the invariants of this variability (the laws of chance).

If we use the concept of truth in the interpretation of logic it is impossible to describe the symmetries of chance because using truth we cannot reconstruct the variability of the processes governed by chance.

Therefore, I believe that both Einstein and Goedel missed the starting point: we must start from the analysis of the variability of chance-dependent systems to describe the symmetries that allow us to understand why ”God does not play dice with the universe”.

Aggregation in topological spaces

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Using a notion of a Lipschiz continous function (see [4]) we can define aggregation operator on the support which is a topological space. If the underlying space is metrizable we can obtain a wide range of aggregation operators. For instance we can get operators as studied by [3] where classical Euclidean geometry and topology play a crucial role. Or we can simply put support to be a metric fractal with Haussdorf dimension between 1 and 2, getting the copula with fractal support (as given by [2]). However, when support is not metrizable, the problem is more complicated. And varies from T2, regular and fully normal topological space (such aggregation is discussed by [5]) to a topological fractal. For the construction of copulas on the latter one, only partial answers exist until now. And these complexities will be particularly discussed. For instance, for construction of Haussdorf dimension in topological space one can employ the Neuman canonical bornological structure (see [1]). Furhermore, we will need a topological contractivity and adequate iterating functions system. The applications of the given structures in reliability and modeling of systems with observations suffering from spatial deformations and stochastic loads will be also presented.

References

1. Almeida J. and Barreira L. (2002). Hausdorff Dimension in Convex Bornological Spaces, Journal of Mathematical Analysis and Applications, Volume 268, Number 2, 590-601
2. Fredricks G.A., Nelsen R.B. and Rodriguez-Lallena J.A. (2005). Copulas with fractal supports, Insurance: Mathematics and Economics 37 (2005) 4248.
3. E. P. Klement and A. Kolesárová, Intervals of 1-Lipschitz aggregation operators, quasi-copulas, and copulas with given affine section, Monatsh. Math. 152, 151-167 (2007)
4. I. Kupka, A topological analogue of the Lipschitz condition, Topology proceedings, Vol.32, No. 1, 2008, 1–7.
5. Stehlík M. (2008). Lipschitz aggregation operators in topological spaces, Research Report.

The partial-algebra method for the representation of algebras related to fuzzy logics

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1 Introduction

The reconstruction of a partially ordered group from a partial subalgebra of it, has been a popular topic within the field of quantum structures. This is amazing; in order to develop a reasonable theory, properties have to be assumed which exclude the most interesting examples from quantum theory.

The situation is different in fuzzy logics. The mentioned additional properties fit perfectly; assuming them, leads directly into the realm of algebras corresponding to fuzzy logics. A broader interest in the topic did not develop, though. The machinery of universal algebra is at most partially applicable to partial algebras.

With this contribution, we are going to give an overview of results concerning the representation of partial algebas by po-groups, and to demonstrate the benefit for the development of a structure theory for algebras occurring in fuzzy logics. In particular, we exhibit how easy it can be – in principle – to construct a group from a partial algebra, and how close the scope of this representation theory is to the context of fuzzy logics.

2 How the partial algebras arise

We are interested in an analysis of the algebraic semantics of fuzzy logics. Our general motivation has been to understand more clearly what fuzzy logics actually “talk” about. Such an analysis is not of a purely academic interest, but might help us to understand why fuzzy logics are successful in applications like medical decision support.

We consider residuated lattices, understood in accordance with [11], except that we add integrality to the definition. Taking the order reversed to what is common in logics, where stronger statements correspond to smaller elements of the algebra in use, a residuated lattice is an algebra $(L; \wedge, \vee, \oplus, \otimes, \oslash, 0)$ such that (i) $(L; \wedge, \vee, 0)$ is a lattice with 0, (ii) $(L; \oplus, 0)$ is a monoid, and (iii) for any $a, b, c \in L$, $a \leq b \oplus c$ if and only if $a \otimes b \leq c$ if and only if $a \oslash c \leq b$.

We define the partial operation $+$ on a residuated lattice L as follows: For $a, b \in L$, put

$$a + b = a \oplus b \quad \text{if } a \text{ is the smallest element } x \text{ such that } x \oplus b = a \oplus b$$
$$\text{and } b \text{ is the smallest element } y \text{ such that } a \oplus y = a \oplus b,$$

and if this condition is not fulfilled, we let $a + b$ undefined.

Note that for any $a \in L$, $a + 0$ is always defined, and in fact, it may happen no other sums exist. However, it is the very nature of residuated lattices that enough such sums exist that the total operation $\oplus$ is uniquely determined by $+$. Namely, note that for any pair $a, b \in L$, we may find elements $a' \leq a$ and $b' \leq b$ such that $a' + b'$ is defined and equals $a \oplus b$; take $a' = (a \oplus b) \oslash b$ and $b' = (a \oplus b) \otimes a'$. It follows that

$$a \oplus b \;=\; \max\{a' + b' \colon\; a' \leq a,\, b' \leq b, \text{ and } a' + b' \text{ is defined}\}.$$

In other words, if we know the structure of the partial algebra $(L;\wedge,\vee,+,0)$, we know the structure of the residuated lattice from which we started.

3 What basic properties the partial algebras have

The transition from a residuated lattice $(L;\wedge,\vee,\oplus,\otimes,\oslash,0)$ to the partial algebra $(L;\wedge,\vee,+,0)$, is certainly reasonable only if the new structure has some interesting properties not shared with the original one. The key property obtained is cancellativity: $a+c=b+c$ or $c+a=c+b$ implies $a=b$. We note that if cancellativity does already hold in the original algebra, the transition is trivial, a case which has been dealt with in [8].

Generalizing a remark in [15], we may state the following.

Theorem 1. *Let $(L;\wedge,\vee,+,0)$ be the partial algebra associated to a residuated lattice. Then, for any $a,b,c\in L$, the following properties hold:*

(P1) $(L;\wedge,\vee,0)$ *is a lower-bounded lattice.*
(P2) $+$ *is a partial binary operation such that:*
 (i) *If $(a+b)+c$ and $a+(b+c)$ are both defined, then $(a+b)+c=a+(b+c)$;*
 (ii) $a+0=0+a=a$.
(P3) *If $a+c$ and $b+c$ are both defined, then $a\le b$ if and only if $a+c\le b+c$.*
 If $c+a$ and $c+b$ are both defined, then $a\le b$ if and only if $c+a\le c+b$.

Condition (P3) expresses the cancellativity, even in a sharpened form.

One should be aware of what in general does not hold. Namely, only a weak version of associativity holds; the existence of $(a+b)+c$ and $a+(b+c)$ is not assumed to be equivalent. Furthermore, the partial order need not be the natural one; $a\le b$ does not necessarily mean that $a+x=b$, or $x+a=b$, for some x.

4 How the representation works

Dealing with a cancellative operation, we naturally ask if it is possible to isomorphically embed the partial algebra $(L;\wedge,\vee,+,0)$ into the positive cone of some lattice-ordered group $(G;\wedge,\vee,+,0)$. To determine the exactly required algebraic properties of the algebra is impossible, but already known sufficient conditions cover a number of cases. Let us describe the situation where the construction works the smoothest.

Namely, let us assume that L does have the properties mentioned at the end of the last section, that is, fulfils the stronger form of associativity, and is naturally ordered w.r.t. $+$. Then L is what has been called a generalized pseudoeffect algebra, or GPE-algebra for short [5].

A further, crucial property of L is automatic: the Riesz decomposition property (RDP_0), saying that if $a\le b+c$, then $a=b'+c'$ for some $b'\le b$ and $c'\le c$. We may proceed as follows:

- Let $\mathcal{G}(L)$ be the group freely generated by L, subject to the conditions $a+b=c$ if this equality holds in L.
- Let $\mathcal{C}(L)$ be the subsemigroup generated by the range of the natural embedding of L into $\mathcal{G}(L)$.

That's all. Under the conditions mentioned, L does not collaps; we may consider L as a subset of $\mathcal{G}(L)$. Furthermore, $\mathcal{G}(L)$ becomes a lattice-ordered group by taking $\mathcal{C}(L)$ as its positive cone; L is located within the latter. The lattice operations performed in L and in $\mathcal{G}(L)$, coincide.

5 Where the method is applicable

In the indicated, or slightly modified, way, we may derive

(i) Mundici's representation theorem of MV-algebras [12];
(ii) Dvurečenskij's representation theorem of pseudo-MV algebras [2]; see [4];
(iii) Bosbach's representation theorem of cone algebras [1]; see [16].

It is furthermore not difficult to include an additional product, to represent

(iv) ŁΠ-algebras [7]; see [13];
(v) PŁ-, PŁ'-, and PŁ'$_\Delta$-algebras [9]; see [13].

By similar methods, we get

(vi) a particularly easy proof of the representation theorem for BL-algebras; see [14];
(vii) Dvurečenskij's representation theorem for linearly ordered pseudo-BL algebras [3] as well as a representation of their implicational subreducts; see [16].

6 How to extend the method

The po-group embedding method is not limited to the case in which the po-group is generated by the partial algebra as indicated. As pointed out in [15], there are numerous further cases, found by an inspection of left-continous t-norm algebras. As an example, let $\odot$ be the rotation-annihilation of two Łukasiewicz t-norms [10], and let $\oplus$ be the corresponding t-conorm. Consider the lexicographical product $\mathbb{R} \times_{\text{lex}} \mathbb{R}$ of two copies of the naturally ordered reals; let

$$\begin{aligned} L = \{(a,b) \in (\mathbb{R} \times_{\text{lex}} \mathbb{R})^+ \colon\ & a = 0,\ 0 \leq b \leq 1 \\ & \text{or } 0 < a < 1,\ b = 0 \\ & \text{or } a = 1,\ -1 \leq b \leq 0\}, \end{aligned}$$

and let $+$ be the group addition restricted to those pairs in L whose sum is again in L. Then $(L; \wedge, \vee, +, (0,0))$ is isomorphic to the partial algebra associated to the residuated lattice $([0,1]; \wedge, \vee, \oplus, \ominus, \ominus, 0)$, where $\ominus$ is the dual of the residual implication of $\odot$.

The question if there are reasonable conditions characterizing residuated lattices embeddable into a po-group, offers a worthwile field of investigation.

References

1. B. Bosbach, Concerning cone algebras, *Algebra Univers.* 15 (1982), 58 - 66.
2. A. Dvurečenskij, Pseudo-MV algebras are intervals in ℓ-groups, *J. Austr. Math. Soc.* 72 (2002), 427 - 445.
3. A. Dvurečenskij, Aglianò–Montagna type decomposition of linear pseudohoops and its applications, *J. Pure Appl. Algebra* 211 (2007), 851 - 861.
4. A. Dvurečenskij, T. Vetterlein, Pseudoeffect algebras I, II, *Int. J. Theor. Phys.* 40 (2001), 685 - 701, 703 - 726.
5. A. Dvurečenskij, T. Vetterlein, Generalised pseudoeffect algebras; in: A. Di Nola, G. Gerla (eds.), "Lectures on Soft Computing and Fuzzy Logic", Springer-Verlag, Berlin 2001; 89 - 111.
6. A. Dvurečenskij, T. Vetterlein, Algebras in the positive cone of po-groups, *Order* 19 (2002), 127 - 146.
7. F. Esteva, L. Godo, Putting together Łukasiewicz and product logics, *Mathware Soft Comput.* 6 (1999), 219 - 234.
8. R. Horčík, Algebraic Properties of Fuzzy Logics, PhD Thesis, Czech Technical University, Prague 2004.
9. R. Horčík, P. Cintula, Product Łukasiewicz logic, *Arch. Math. Logic* 43 (2004), 477 - 503.

10. S. Jenei, Structure of left-continuous triangular norms with strong induced negations. II. Rotation-annihilation construction. *J. Appl. Non-Classical Logics* 11 (2001), 351 - 366.
11. P. Jipsen, C. Tsinakis, A survey of residuated lattices, in: J. Martínez (ed.), "Ordered algebraic structures. Proceedings of the conference on lattice-ordered groups and f-rings (Gainesville 2001)", Kluwer Acad. Publ., Dordrecht 2002; 19 - 56.
12. D. Mundici, Interpretation of AF $C^{\star}$-algebras in Łukasiewicz sentential calculus, *J. Funct. Anal.* 65 (1986), 15 - 63.
13. T. Vetterlein, Partial algebras for Łukasiewicz logics and its extensions, *Arch. Math. Logics* 44 (2005), 913 - 933.
14. T. Vetterlein, Weak effect algebras, *Algebra Univ.* 58 (2008), 129 - 143.
15. T. Vetterlein, MTL-algebras constructed from partially ordered groups, book chapter (eds.: E. P. Klement, S. Gottwald, U. Höhle), to appear.
16. T. Vetterlein, Pseudo-BCK algebras as partial algebras, in preparation.

Measure-free conditioning and extensions of additive measures on finite MV-algebras

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The problem of conditioning can be interpreted in two different ways. One possibility is to take up the classical idea, i. e. defining the “conditional probability” of an event a, given another event b and a probability P on a Boolean algebra $\mathbb{L}$ of events, by

$$P(a \mid b) = \frac{P(a \cap b)}{P(b)} \quad \text{if} \quad P(b) \neq 0. \tag{1}$$

Many researchers followed this line, see e. g. Kroupa [8]. He defined a “conditional state” of a, given b and a state (additive measure) m on an MV-algebra $\mathbb{L}$ with product $\cdot$, by

$$m(a \mid b) = \frac{m(a \cdot b)}{m(b)} \quad \text{if} \quad m(b) \neq 0$$

and showed that $m(. \mid b)$ is a state on $\mathbb{L}$, fixing b. See also the paper [3], where Renyi's axiomatic approach is extended. But let us emphasize that in all these papers $(a \mid b)$ is not a well-defined object but only a symbol used to define the conditional probability resp. state. This line will not be considered in this talk.

In contrast to this approach, we follow the other very different approach of **measure-free conditioning** in the sense of Goodman, Nguyen and Walker [5]. In a first step, they defined the “conditional event a given b” for events a, b in a Boolean algebra $\mathbb{L}$ as the lattice-interval

$$(a \mid b) = [a \cap b,\ b' \cup a]$$

and showed that the set of such conditionals forms a (semi-simple) MV-algebra, see [5], Section 4.3, Theorem 1. In a second step, they extended a given uncertainty measure ν on $\mathbb{L}$ to an uncertainty measure μ on the set of the conditionals $(a \mid b)$ by

$$\mu(a \mid b) = M(\nu(a \cap b),\ \nu(b' \cup a))$$

with respect to some given (mean value) function M. In a third step, they observed that for a probability $\nu = P$ the choice

$$M(x, y) = \frac{x}{1 + x - y} \quad \text{if} \quad (x, y) \neq (0, 1) \tag{2}$$

leads to $\mu(a \mid b) = P(a \mid b)$ from (1), i. e. that the uncertainty measure μ assigned to the conditional events is compatible with the classical conditional probability. See also the paper [4] in the context of non-monotonic logic.

Now, the purpose of this talk is twofold. In the first part, following this year's first motto "Where do we stand?", we present briefly the generalizations of the three steps of measure-free conditioning sketched above. All this was presented by the author in several talks since 1987 and in papers, partially jointly with U. Höhle, see e. g. [7]. Replacing the Boolean algebra $\mathbb{L}$ by an MV-algebra, we found that the conditionals form a Girard algebra. We used this name for a bounded, integral, residuated, commutative l-monoid with involutory residual complement, see [6], because of some analogy to Girard quantales, see [10], Chapter 6. Denoting by $\vee, \wedge$ the lattice operations, by $\sqcap$ the semigroup operation, by $'$ the residual complement and by $\sqcup$ the De Morgan dual, a Girard algebra results to be an MV-algebra if and only if the divisibility property $a \sqcap (a' \sqcup b) = a \wedge b$ holds, see [6], where $\sqcup, \sqcap$ resp. $'$ are Chang's operations $+, \cdot$ resp. $^-$ from [1]. Furthermore, we found that the structure of a Girard algebra is closed with respect to measure-free conditioning. But it remained open the problem of how to extend the additivity of an uncertainty measure m on an MV-algebra to a measure extension $\tilde{m}$ on the resulting Girard algebra extension of conditionals. The additivity on MV-algebras has a clear meaning, see e. g. [11] in the context of admissibility or [9] in the context of quantum theory, where the name state is used. Problems concerning additivity on a Girard algebra $\mathbb{L}$ were discussed in [12], where we introduced the **weak additivity of an uncertainty measure** m on $\mathbb{L}$ if and only if m is additive on all MV-subalgebras of $\mathbb{L}$.

Therefore, in the second part of this talk and following this year's second motto "Where do we go?", we present the details of such extensions for **finite MV-algebras** as sketched in the following. Let $\mathbb{L}$ be a finite MV-algebra and $h : \mathbb{L} \longrightarrow \mathbb{L}_1 \times \ldots \times \mathbb{L}_v$ an MV-algebra isomorphism to a finite product of finite MV-chains $\mathbb{L}_i$, see e. g. [2], Proposition 3.6.5, where for the following it does not matter whether the $\mathbb{L}_i$ are considered as Łukasiewicz' MV-chains or not. Denote by $\tilde{\mathbb{L}}$ the canonical Girard algebra extension $\{(a,b) \in \mathbb{L} \times \mathbb{L} : a \leq b\}$ of $\mathbb{L}$, see [7], and similar $\tilde{\mathbb{L}}_i$ resp. $(\mathbb{L}_1 \widetilde{\times \ldots \times} \mathbb{L}_v)$. Then $h(a) = (a_1, \ldots, a_v)$ induces a Girard algebra isomorphism $\tilde{h} : \tilde{\mathbb{L}} \longrightarrow (\mathbb{L}_1 \widetilde{\times \ldots \times} \mathbb{L}_v)$ given by $\tilde{h}(a,b) = (h(a), h(b))$. Furthermore, $\tilde{g} : (\mathbb{L}_1 \widetilde{\times \ldots \times} \mathbb{L}_v) \longrightarrow \tilde{\mathbb{L}}_1 \times \ldots \times \tilde{\mathbb{L}}_v$, given by $\tilde{g}((a_1, \ldots, a_v), (b_1, \ldots, b_v)) = ((a_1, b_1), \ldots, (a_v, b_v))$, is also a Girard algebra isomorphism. Then any MV-subalgebra $\mathbb{M}$ of $\tilde{\mathbb{L}}$ is isomorphic to the product $\tilde{g}(\tilde{h}(\mathbb{M})) = \mathbb{M}_1 \times \ldots \times \mathbb{M}_v$ of MV-chains $\mathbb{M}_i$ which are the only MV-subalgebras of $\tilde{\mathbb{L}}_i$. Now, any uncertainty measure m on $\mathbb{L}$ induces an uncertainty measure m_h on $\mathbb{L}_1 \times \ldots \times \mathbb{L}_v$, given by

$$m_h(a_1, \ldots, a_v) = m(h^{-1}(a_1, \ldots, a_v)) = m(a).$$

Abusing the notation, we will write also m instead of m_h.
By analogy abusing the notation, we obtain that an extension $\tilde{m}$ on $\tilde{\mathbb{L}}$ induces measures $\tilde{m}$ on $(\mathbb{L}_1 \widetilde{\times \ldots \times} \mathbb{L}_v)$ resp. $\tilde{\mathbb{L}}_1 \times \ldots \times \tilde{\mathbb{L}}_v$, given by

$$\tilde{m}(a,b) = \tilde{m}((a_1, \ldots, a_v), (b_1, \ldots, b_v)) = \tilde{m}((a_1, b_1), \ldots, (a_v, b_v)).$$

Additionally, any additive measure m on $\mathbb{L}$ results to be a convex combination of the unique measures on the $\mathbb{L}_i$, i. e.

$$m(a) = \sum_{i=1}^{v} \mu_i \cdot m(a_i), \quad \mu_i \geq 0, \ \sum_{i=1}^{v} \mu_i = 1.$$

Then the weakly additive measure extensions $\tilde{m}$ on $\tilde{\mathbb{L}}$ are determined by the values

$$\tilde{m}(a,b) = \sum_{i=1}^{v} \mu_i \cdot \tilde{m}(a_i, b_i)$$

for all $(a_i, b_i) \in \mathbb{M}_i$ for each MV-subalgebra (MV-chain) $\mathbb{M}_i$ of $\tilde{\mathbb{L}}_i$. In [13] it is shown that

$$\tilde{m}(a_i, b_i) = M(m(a_i),\, m(b_i))$$

with respect to the mean value function

$$M(x,y) = \frac{y}{1+y-x}\,,$$

which is completely different from that in (2).
Furthermore, the positive and negative results concerning strongly additive measure extensions from [12, 13] can be generalized.
Finally, all notions and results can be applied to conditional events, given by

$$(a \mid b) = (a \wedge b,\, b' \sqcup a),$$

with the identification of a lattice interval $[a,b]$ in $\mathbb{L}$ with the element (a,b) of $\tilde{\mathbb{L}}$.
Several examples will illustrate the results.
It remains open the problem:
How to generalize the results from a finite to an arbitrary MV-algebra $\mathbb{L}$?

References

1. C.C. Chang. Algebraic analysis of many valued logics. *Trans. Amer. Math. Soc.*, 88:467-490, 1958.
2. R.L.O. Cignoli, I.M.L. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-valued Reasoning*. Trends in Logic, Kluwer, 2000.
3. A. Di Nola, G. Georgescu, and A. Lettieri. Conditional states in finite-valued logics. In D. Dubois et al., editors, *Fuzzy Sets, Logics and Reasoning about Knowledge*, pages 161–174. Kluwer, 1999.
4. D. Dubois and H. Prade. Conditioning, non-monotonic logic and non-standard uncertainty models. In I.R. Goodman, M.M. Gupta, H.T. Nguyen, and G.S. Rogers, editors, *Conditional Logic in Expert Systems*, pages 115–158. North-Holland, 1991.
5. I.R. Goodman, H.T. Nguyen, and E.A. Walker. *Conditional Inference and Logic for Intelligent Systems – A Theory of Measure-free Conditioning*. North-Holland, 1991.
6. U. Höhle. Commutative, residuated l-monoids. In U. Höhle and E.P. Klement, editors, *Nonclassical logics and Their Applications to Fuzzy Subsets*, pages 53–106. Kluwer, 1995.
7. U. Höhle and S. Weber. On Conditioning Operators. In U. Höhle and S. Rodabaugh, editors, *Mathematics of Fuzzy Sets – Logic, Topology and Measure Theory*, pages 653–673- Kluwer, 1999.
8. T. Kroupa. Conditional probability on MV-algebras. *Fuzzy Sets and Systems*, 149:369–381, 2005.
9. D. Mundici. Logic of Infinite Quantum Systems. *International Journal of Theoretical Physics*, 32:1941–1955, 1993.
10. K.I. Rosenthal. *Quantales and Their Applications*. Pitman Research Notes in Mathematics 234, Longman, 1990.
11. S. Weber. Uncertainty measures, decomposability and admissibility. *Fuzzy Sets and Systems*, 40:, 395–405, 1991.
12. S. Weber. Uncertainty Measures – Problems Concerning Additivity. *Fuzzy Sets and Systems*, 160:371-383, 2009. Available online 21 October 2008.
13. S. Weber. A complete characterization of all weakly additive measures and of all valuations on the canonical extension of any finite MV-chain. Submitted to the special issue 29th Linz Seminar 2008 of Fuzzy Sets and Systems.

Generalized quantifiers in logic and language

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The aim of this talk is to present the current state of generalized quantifier theory, in particular as it applies to issues in natural language semantics. After a brief historical overview, I present the standard model-theoretic notion of a quantifier, and some questions and results concerning the properties, expressive power, etc. of first-order logics with added generalized quantifiers. I then look at the subclass of quantifiers that typically turn up in language, and their special properties. As illustrative examples I choose (i) quantifiers and negation, (ii) monotonicity properties, and (iii) possessive quantifiers (as in "At least one of most logicians' papers deals with quantifiers". Finally, some open questions and ideas for research are sketched.

Constructing t-norms from a given behaviour on join-irreducible elements

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Triangular norms (t-norms for short) were originally studied in the theory of probabilistic metric spaces in order to generalize the classical triangle inequality to this field [7, 12]. Later on, they played an important role as interpretation of the conjunction in many-valued logics [4], in particular in fuzzy logics [8]. An important class of t-norms is the class of sup-preserving t-norms which play a major role, particularly in residuated lattices [4]. Triangular conorms are introduced as dual notion of triangular norms [10]. There have been some construction methods of t-norms on various classes of lattices [11, 5, 9, 6]. Some lattices can be generated by a class of elements: join- or meet-irreducible elements and some others [1, 2]. In this contribution, we focus on constructing t-norms on complete lattices from a given behavior on join-irreducible elements. We present the sup-extension method to describe the behavior of a t-norm on a finite distributive lattice by means of join-irreducible elements by the following theorem:

Theorem. Let L a finite distributive lattice and $J(L)^* = J(L) \cup \{0,1\}$. If T is a t-norm on $J(L)^*$, then the function $\widetilde{T}$ defined as follows:

$$\widetilde{T}(x,y) = \bigvee_{j \in \eta(x)} \bigvee_{k \in \eta(y)} T(j,k)$$

where $\eta(x)=\{i \in J(L)^* | i \leq x\}=J(L)^* \cap \downarrow x$ for all $x \in L$, is a t-norm on L.

We provide a method to construct t-conorms by carrying out the dual method on meet-irreducible elements. We also obtain some inf-preserving t-conorms on principle ideals of a lattice from given inf-preserving t-conorms. We show that if T is a t-norm on a complete lattice L and every join-irreducible element of L is idempotent, then $T = \wedge$. We give a method to construct t-norms on a product of distributive lattices $L = L_1 \times L_2 \times \ldots \times L_n$. Giving a partititon of the set of join-irreducible elements of $L = L_1 \times L_2 \times \ldots \times L_n$, we show that for a given t-norm on join-irreducible elements, the restriction to each set that forms the partition is not necessarily a t-norm, but a t-subnorm. Moreover, we partition the set of join-irreducible elements $J(L^{[n]})$ of a power of chains $L^{[n]}$, given in [13], into n sets such that

$$\begin{aligned}
J_1 &= \{(0,0,\ldots,0,a) \mid a \in L\} \\
J_2 &= \{(0,0,\ldots,a,a) \mid a \in L\} \\
&\vdots \\
J_{n-1} &= \{(0,a,\ldots,a,a) \mid a \in L\} \\
J_n &= \{(a,a,\ldots,a,a) \mid a \in L\}.
\end{aligned}$$

and provide a method to construct a t-norm from given t-norms on the parts of this partition. We show that if L is finite, then the constructed t-norm is sup-preserving. In an interval valued lattice L, the set $D_L = \{[x,x] \mid x \in L\}$ is called the diagonal of L [3]. Our last result on the characterization of sup-preserving t-norms on $L^{[n]}$ extends result of [3]: Any sup-preserving t-norm T on $L^{[n]}$ can be characterized by its behaviour on J_n and $T((0,\ldots,0,1),(0,\ldots,0,1))$ if it is closed on the diagonal.

References

1. G. Birkhof. *Lattice Theory*. American Mathematical Society, Providence, RI, 1967.
2. B.A. Davey and H.A. Priestley. *Introduction to Lattices and Orders*. Cambridge University Press, Cambridge, United States, 1990.
3. B. Van Gasse, C. Cornelis, G. Deschrijver, and E. Kerre. A characterization of interval-valued residuated lattices. *International Journal of Approximate Reasoning*, 49:478–487, 2008.
4. P. Hajek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht, Holland, 1998.
5. S. Jenei and B. De Baets. On the direct decomposability of t-norms on product lattices. 139:699–707, 2003.
6. F. Karacal and Y. Sagiroglu. Infinitely $\vee$-distributive t-norms on complete lattices and pseudo-complements. *Fuzzy Sets and Systems*, 160:32–43, 2009.
7. K. Menger. Statistical metrics. *Proceeding of National Academy of Sciences*, 28:535–537, 1942.
8. H.T. Nguyen and E. Walker. *A First Course in Fuzzy Logic*. CRC Press, Boca Raton, Florida, 1997.
9. S. Saminger. On ordinal sums of triangular norms on bounded lattices. *Fuzzy Sets and Systems*, 157:1403–1416, 2006.
10. B. Schweizer and A. Sklar. Associative functions and statistical triangle inequalities. *Publ. Math. Debrecen*, 8:169–186, 1961.
11. B. Schweizer and A. Sklar. *Associative Functions and Abstract Semigroups*. Publ. Math. Debrecen, Amsterdam, Holland, 1963.
12. B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. North Holland, Amsterdam, Holland, 1983.
13. C. L. Walker and E. A. Walker. Automorphisms of powers of linearly ordered sets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 14:77–85, 2006.

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