An observation on (un)decidable theories in fuzzy logic (abstract)

Petr Hájek

Institute of Computer Science, Acad. Sci. 182 07 Prague
hajek@cs.cas.cz

Mathematical fuzzy logic (fuzzy logic as a kind of mathematical logic) has been presented at most Linz meetings (the present author lectured about it in years 1996, 1997, 2000, 2003, 2004, 2005, 2007). The present contribution continues this by presenting a (surprising) fact on decidability and undecidability of fuzzy theories. In classical logic the extension of a decidable theory T by a single axiom φ is again a decidable theory thanks to the deduction theorem: $T, \varphi \vdash \psi$ iff $T \vdash \varphi \to \psi$. (It follows that a decidable theory has a decidable complete extension, which is useful e.g. for the proof of essential undecidability some theories.) Also for some fuzzy logics (Gödel logic, logics with Baaz's Delta) provability in T, φ recursively reduces to provability in T and decidability of T implies decidability of (T, φ) due to specific deduction theorem of these logics. But in general the answer is negative. We are going to present a decidable theory T over Łukasiewicz logic and its extension (T, φ) which is undecidable (but of course recursively axiomatizable). We shall construct T and φ in Łukasiewicz *propositional* logic but it gives trivially a example in predicate logic (propositional variables understood as nullary predicates, then each formula is logically equivalent to a quantifier free formula).

Let $R(n, m)$ be a recursive relation on natural numbers such that its existential projection $(\exists n)R(n, m)$ is not recursive. Let T be a theory over Łukasiewicz propositional logic whose language consists of propositional variables q, p_n (n positive natural) and whose axioms are $q^n \to p_m$ for all n, m with $R(n, m)$. (q^n is $q\& \ldots \&q$, n conjuncts, as usual.)

The theory has a trivial crisp model evaluating q by 1 and evaluating p_m by 1 iff $(\exists n)R(n, m)$, otherwise evaluating p_m by 0.

Theorem. $(T, q) \vdash p_m$ iff $(\exists n)R(n, m)$, hence (T, q) is undecidable.
(Easy.)

Theorem. The theory T is decidable.

Surprisingly difficult. Hint: The set of all formulas φ provable in T is of course Σ_1 (recursively enumerable). One can show that also the set of formulas unprovable in T is Σ_1: one can recursively reduce the problem of T-unprovability of a formula to the satisfiability problem of open formulas in the ordered field of reals, the latter being decidable (even PSPACE, [1]).

Remark. (1) The theorem trivially holds for BL; simply add the schema $\neg\neg\alpha \to \alpha$ for each α to the axioms of T.

(2) In my paper [3] a weak arithmetic is defined over the fuzzy logic BL∀ and Gödel's incompleteness theorem for it is proved (each axiomatizable extension of this arithmetic consistent over BL∀ is incomplete in the sense of fuzzy logic) and it is claimed that essential undecidability follows. But the *problem* of the existence of a decidable complete extension of a decidable theory over the logic BL∀ seems to remain open as well as the problem whether the weak arithmetic over BL∀ is essentially undecidable.

References

1. Canny J: Some algebraic and geometric computations in PSPACE. In Proc. 20th Ann. ACM Symp. Theory of Computing, pages 460–467, 1988.
2. Hájek P.: Metamathematics of fuzzy logic. Kluwer 1998.
3. Hájek P.: Mathematical Fuzzy Logic and Natural Numbers. Fundamenta Informaticae 81 (2007) p. 155-163.