LINZ
2010

31st Linz Seminar on Fuzzy Set Theory

Lattice-Valued Logic and its Applications

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Abstracts

Petr Cintula
Erich Peter Klement
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Editors

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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2010 will be the 31st seminar carrying on this tradition and is devoted to the theme "Lattice-Valued Logic and its Applications". The goal of the seminar is to present and discuss recent advances of mathematical fuzzy logic (understood in the broader framework of lattice-valued logics) and concentrate on its applications in various areas of computer science, linguistics, and philosophy.

A large number of highly interesting contributions were submitted for possible presentation at LINZ 2010. In order to maintain the traditional spirit of the Linz Seminars — no parallel sessions and enough room for discussions — we selected those twenty-six submissions which, in our opinion, fitted best to the focus of this seminar. This volume contains the abstracts of this impressive selection. These regular contributions are complemented by six invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

Petr Cintula
Erich Peter Klement
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Contents

The impact of adding a constant

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Abstract. We consider an extension of [0,1]-Gödel logic by a unary operator o that adds a constant $r \in [0,1]$ fixed for every interpretation. We show that the set of formulas in propositional logic valid for all r is axiomatizable by a Hilbert-Frege system although entailment in this logic is not compact. This is achieved by evaluating the formula under finitely many linear orders. Moreover, we prove—contrary to the case without o—that validity in the corresponding first-order logic is not r. e.

We consider a language $\mathcal{L}_{\mathrm{o}}^{\mathrm{P}}$ that comprises a countably infinite set of propositional variables, connectives $\bot$, $\supset$, $\wedge$, $\vee$ with their usual arities as well as a unary connective o. The semantics of *propositional Gödel logics with* o in $\mathcal{L}_{\mathrm{o}}^{\mathrm{P}}$ is determined by Gödel r*-interpretations* $\mathfrak{I}$; here $r \in [0,1]$ and $\mathfrak{I}$ maps formulas to $[0,1]$ such that

$$\begin{aligned} \mathfrak{I}(A \wedge B) &= \min\{\mathfrak{I}(A), \mathfrak{I}(B)\}, & \mathfrak{I}(\bot) &= 0, \\ \mathfrak{I}(A \vee B) &= \max\{\mathfrak{I}(A), \mathfrak{I}(B)\}, & \mathfrak{I}(A \supset B) &= \begin{cases} 1 & \text{if } \mathfrak{I}(A) \le \mathfrak{I}(B), \\ \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) > \mathfrak{I}(B). \end{cases} \\ \mathfrak{I}(\mathrm{o}(A)) &= \min\{1, r + \mathfrak{I}(A)\}, \end{aligned}$$

A formula A is *valid* if $\mathfrak{I}(A) = 1$ holds for all Gödel r-interpretations $\mathfrak{I}$, $r \in [0,1]$. We introduce the well-known abbreviations $\top := \bot \supset \bot$, $\neg A := A \supset \bot$, $A \prec B := (B \supset A) \supset B$, $A \leftrightarrow B := (A \supset B) \wedge (B \supset A)$ and define the o-powers $\mathrm{o}^0 A := A$, $\mathrm{o}^{n+1} A := \mathrm{o}^n \mathrm{o} A$. We have then, e. g.,

$$\mathfrak{I}(A \prec B) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) < \mathfrak{I}(B) \text{ or } \mathfrak{I}(A) = \mathfrak{I}(B) = 1, \\ \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) \ge \mathfrak{I}(B). \end{cases}$$

We write $R \Vdash S$ and say R *entails* S if for all $r \in [0,1]$ holds that, whenever $\mathfrak{I}$ is an r-interpretation such that $\mathfrak{I}(A) = 1$ for all $A \in R$, we have $\mathfrak{I}(A) = 1$ for all $A \in S$.

One immediately observes that the entailment relation is not compact for $\mathcal{L}_{\mathrm{o}}^{\mathrm{P}}$ since for $R := \{\mathrm{o}^k\, x \supset y;\, k \in \mathbb{N}\}$ and $S := \{y \vee \neg\, \mathrm{o} \bot\}$, we have $R \Vdash S$ but for any finite $E \subseteq R$ we have $E \nVdash S$.

Nevertheless, the set of validities is clearly decidable since this problem can be translated into, e. g., the deciable theory of real closed fields. In this paper, we focus on the axiomatization in a Hilbert-Frege style.

Dummett [1] proved that the set of valid formulas in propositional Gödel logic (without o) is axiomatized by the Hilbert-Frege style proof system (GPL) that consists of the axiom schema of linearity (LIN) $A \supset B \vee B \supset A$ added e. g. to the following system (IPL), which axiomatizes propositional intuitionistic logic:

$A \supset (B \supset A)$,	modus ponens: $\frac{A \quad A \supset B}{B}$,
$(A \wedge B) \supset A$,	$(A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)$,
$(A \wedge B) \supset B$,	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$,
$A \supset (A \vee B)$,	$A \supset (B \supset (A \wedge B))$,
$B \supset (A \vee B)$,	$\perp \supset A$.

One can prove that the set of valid formulas in propositional Gödel logics with o is given by (GPL) plus the following simple axiom schemata:

$$\neg\, \mathrm{o} \perp \vee (A \prec \mathrm{o}\, A), \ \ (A \supset B) \supset (\mathrm{o}\, A \supset \mathrm{o}\, B), (\mathrm{o}\, A \supset \mathrm{o}\, B) \supset ((A \supset B) \vee \mathrm{o}\, A),$$
$$\neg\, \mathrm{o} \perp \supset (\mathrm{o}\, A \leftrightarrow A), (A \prec B) \supset (\mathrm{o}\, A \prec \mathrm{o}\, B), (\mathrm{o}\, A \prec \mathrm{o}\, B) \supset ((A \prec B) \vee \mathrm{o}\, A)$$

The proof employs Dummett's idea to use chains, i. e. linear orderings of propositional variables and their o-powers w. r. t. $\prec$ and $\leftrightarrow$, to evaluate the given formula; cf. also [2]. The proof also reveals how to construct a finite counter-model to a formula if it is not valid. A consequence is that this logic is the intersection of the finitely valued logics when only those $r \in [0, 1]$ are considered where all operations are defined.

First-order Gödel logics with o uses predicate symbols and quantifiers $\forall$, $\exists$ with the usual semantics

$$\mathfrak{I}(\forall x\, A(x)) = \inf\{\mathfrak{I}(A(u));\ u \in |\mathfrak{I}|\},$$
$$\mathfrak{I}(\exists x\, A(x)) = \sup\{\mathfrak{I}(A(u));\ u \in |\mathfrak{I}|\}.$$

For the sake of simplicity, we consider here only the function-free fragment.

Theorem 1. *Then there is an effective embedding $A \mapsto A^e$ from formulas in classical first-order predicate logic to first-order Gödel logics with* o *such that the following conditions are equivalent for any closed classical formula A:*

1. *There is a classical interpretation $\mathfrak{I}'$ such that $\mathfrak{I}'(A) = 0$ and $|\mathfrak{I}'|$ is finite.*
2. *There is $r \in (0, 1]$ and a Gödel r-interpretation $\mathfrak{I}$ such that $\mathfrak{I}(A^e) < 1$.*

We use a variant of Scarpellini's [3] embedding of classical logic into Łukasiewicz logic to show the non-enumerability of the set of the valid formulas. The immediate consequence is that first-order Gödel logics with o is not r. e.

We will also discuss what happens if the logics discussed above are extended with the projection operator $\triangle$ where

$$\mathfrak{I}(\triangle A) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) = 1, \\ 0 & \text{if } \mathfrak{I}(A) = 0. \end{cases}$$

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Extending Cantor–Łukasiewicz set theory with classes

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As conjectured by Skolem [8] and proved by White [10], naïve set theory with the unrestricted axiom schema of comprehension, which is inconsistent over classical logic due to Russell's paradox, turns out to be consistent over infinite-valued Łukasiewicz logic. Hájek [5, 3] studied the theory under the name *Cantor–Łukasiewicz set theory* (denoted by CŁ further on)[1] and showed several negative results on arithmetic over CŁ. Additionally, some basic constructions (such as kernels of fuzzy sets) are in general undefinable in CŁ on pain of contradiction, as any bivalent or finitely-valued operator makes it possible to reproduce Russell's paradox. These facts cast serious doubts on Skolem's conjecture that a large part of mathematics could be formalized in the theory.

Here I suggest to remedy the drawbacks of CŁ by extending the theory with classes, in a similar manner as von Neumann–Bernays–Gödel's classical set theory NBG extends Zermelo–Fraenkel's ZF. Besides a few observations on the features and expressive power of the resulting theory CŁC, I discuss its motivational aspects and compare it with two set theories with classes over classical logic (NBG and Vopěnka's [9] AST).

1 Cantor–Łukasiewicz Set Theory with Classes

Cantor–Łukasiewicz set theory CŁ is a theory over first-order Łukasiewicz infinite-valued logic Ł∀ (see, e.g., [4]) with the only primitive predicate $\in$ and the set comprehension terms $\{x \mid \varphi(x)\}$ governed by the comprehension axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, for all formulae φ. The extension CŁC of CŁ by classes can be defined as follows:

Definition 1. CŁC *is a theory over two-sorted first-order Łukasiewicz logic with the connective* $\triangle$ *(Ł*$\forall_\triangle$*, see, e.g., [4]). The language of* CŁC *consists of:*

[1] In [5] and several follow-up articles, the theory is denoted by $CŁ_0$, while CŁ denotes a certain inconsistent extension of $CŁ_0$. For notational simplicity, we shall use the name CŁ for Hájek's $CŁ_0$, since the inconsistent theory is of a very limited interest.

- *The sort of variables for* sets *(lowercase letters)*
- *The sort of variables for* classes *(uppercase letters)*
- *The primitive membership predicate* $\in$ *between sets (*set membership predicate, *or* set-in-set membership*)*
- *The primitive predicate of membership of sets in classes (*class membership predicate, *or* set-in-class membership*, denoted also by* $\in$*, as the two are always distinguishable by the type of arguments)*
- Set comprehension terms $\{x \mid \varphi\}$ *(of the set sort) for any set formula (see below)* φ
- Class comprehension terms $[x \mid \varphi]$ *(of the class sort) for any formula* φ *of* CŁC

Set formulae *are those that contain no* $\triangle$ *nor any class term. The axioms of* CŁC *are the following, for any set formula* φ *and any formula* ψ*:*

- Set comprehension axioms: $\varphi(y) \leftrightarrow y \in \{x \mid \varphi(x)\}$
- Class comprehension axioms: $y \in [x \mid \psi(x)] \leftrightarrow \psi(y)$
- Class extensionality axioms: $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to (\psi(A) \leftrightarrow \psi(B))$

Classes of CŁC are intended to represent crisp or fuzzy subsets of models of CŁ: class comprehension axioms ensure the existence of any class delimited by a property expressible in the language of CŁC. Notice that the logical vocabulary of CŁC contains the connective $\triangle$, which allows us, i.a., to speak about crisp collections of objects in models. Unrestricted set comprehension, however, only applies to set formulae, in which $\triangle$ is forbidden. In fact, the set fragment of CŁC coincides with CŁ:

Theorem 1. CŁC *is a conservative extension of* CŁ *(therefore is consistent).*

Proof. Every model M of CŁ can be extended to a model M' of CŁC by interpreting class variables as ranging over fuzzy classes of set-objects (i.e., membership functions from the universe of M to the algebra of truth values) and realizing the set-in-class membership predicate accordingly (namely, defining the values of set-in-class membership as the degrees provided by these membership functions): the validity of the axioms of CŁC in M' is easily seen. The conservativeness then follows (by the strong completeness of Ł$\forall$ and Ł$\forall_\triangle$, see, e.g., [6]) from the fact that the truth values of set formulae only regard the elements of M (as set formulae cannot contain class terms and the semantics is compositional).

It can be seen that the axioms for classes are the same as those of Henkin-style monadic second-order fuzzy logic Ł$\forall$, analogous to that of [1, §3]. CŁC can thus be understood as a fuzzy class theory over the universe of CŁ.

Even though a hierarchy of higher-order classes over the CŁ-universe could be introduced in the same way as in [1, §5], many classes of classes (e.g., the partition of Theorem 2(5) below) can be encoded in a rather standard way (cf. [9, §I.5–6] for AST) by first-order relations,[2] understanding a (class) binary relation R together with a class A as encoding the class $\mathcal{K}$ of classes X with $X \in \mathcal{K} \equiv_{\mathrm{df}} (\exists i \in A)(X = [j \mid Rij])$. Obviously, tuples (or set-indexed systems) of classes and usual higher-order class operations (e.g., class intersection or union) can be encoded in CŁC as well.

2 Extensionality and intensionality

In CŁC, classes are construed as extensional (i.e., determined by their membership functions), as they are intended to represent (crisp or fuzzy) collections of objects in models. The axiom of class extensionality indeed ensures that any two classes with the same membership function (i.e., with the same degrees of membership of all elements) are intersubstitutable salva veritate. Since intersubstitutivity (which in Ł$\forall_\triangle$ is a crisp relation) can be regarded as the logical identity (as factoring a model of CŁC by the intersubstitutivity relation does not change the truth values of formulae), we can define:

Definition 2. *In* CŁC*, we define:* $A = B \equiv_{\mathrm{df}} (\forall x)\triangle(x \in A \leftrightarrow x \in B)$.

On the other hand, CŁ-sets are not extensional. Recall from [5] that two different set equalities are introduced in CŁ: the provably crisp *Leibniz equality* $=$ and the (provably fuzzy) *extensional equality* $\approx$ (denoted by $=_{\mathrm{e}}$ in [5] and its follow-ups), defined as $x = y \equiv_{\mathrm{df}} (\forall u)(x \in u \leftrightarrow y \in u)$ and $x \approx y \equiv_{\mathrm{df}} (\forall u)(u \in x \leftrightarrow u \in y)$. Leibniz equality ensures intersubstitutivity salva veritate (so it can be identified with the logical identity predicate), while extensional equality (which will be also called *co-extensionality* further on) does not (though it is also a fuzzy equivalence relation). Leibniz equality implies extensional equality, $x = y \to x \approx y$, but it is inconsistent to assume $x = y \leftrightarrow x \approx y$ in CŁ. Hájek has actually proved in [3] that there are infinitely many set terms which are all provably co-extensional with (e.g.) $\emptyset =_{\mathrm{df}} \{x \mid 0\}$ while being Leibniz non-identical.

Even though CŁ-sets are not extensional, in CŁC we can define their *extensions,* i.e., the classes of their elements:

Definition 3. *In* CŁC *we define the* extension *of a set x as the class* $\mathrm{Ext}\, x =_{\mathrm{df}} [q \mid q \in x]$.

[2] See [5] for handling ordered pairs in CŁ.

The definitions of extension and co-extensionality can be extended to classes by setting $\mathrm{Ext}\, A =_{\mathrm{df}} A$; $A \approx x \equiv_{\mathrm{df}} (\forall q)(q \in A \leftrightarrow q \in x)$ and analogously for $x \approx A$ and $A \approx B$. The following observations are easily obtained:

Theorem 2. CŁC *proves:*

1. $A = B \leftrightarrow \triangle(A \approx B)$*, by the axiom of class extensionality*[3]
2. $x \approx y \leftrightarrow \mathrm{Ext}\, x \approx \mathrm{Ext}\, y$*, and similarly for* $A \approx x$ *and* $A \approx B$
3. $\mathrm{Ext}\, \{x \mid \varphi\} = [x \mid \varphi]$
4. $\approx$ *is a fuzzy equivalence relation which partitions the set universe into fuzzy blocks* $\{x\}_\approx =_{\mathrm{df}} \{q \mid q \approx x\}$ *that satisfy* $\{x\}_\approx \approx \{y\}_\approx \leftrightarrow x \approx y$ *and* $x \in \{x\}_\approx$
5. *The crisp equivalence relation of full co-extensionality* $\triangle(x \approx y)$ *partitions the set universe into crisp class blocks* $[x]_\approx =_{\mathrm{df}} [q \mid \triangle(q \approx x)]$

In contrast to NBG or AST, it is not the case in CŁC that all sets are classes and only some classes are sets. Nevertheless, every set is in CŁC fully co-extensional with a class (namely, its extension), and only some classes are fully co-extensional with sets. This motivates the following definition of (im)proper classes in CŁC:

Definition 4. *In* CŁC*, we say that a class A is* proper *if* $\neg(\exists x)\triangle(x \approx A)$*, and* improper *(or a* set extension*) if* $(\exists x)\triangle(a \approx A)$.

Examples of improper classes are the empty class $\Lambda =_{\mathrm{df}} [x \mid 0]$, the universal class $\mathrm{V} = [x \mid 1]$, and generally $\mathrm{Ext}\, x$ for any set x. By Yatabe's overspill theorem [11], an example of a proper class is the class FN of standard natural numbers (similarly as in AST;[4] details are omitted here for space restrictions).

Though not yet proved for CŁ, a claim analogous to one valid for naïve set theory over the logic BCK (see [7]) has been conjectured by Terui:[5]

$$\text{CŁ} \vdash \{x \mid \varphi\} = \{x \mid \psi\} \text{ iff } \varphi \text{ and } \psi \text{ are syntactically identical}$$

Even though this feature might be viewed as a defect that trivializes CŁ, it would nevertheless make a good sense in CŁC, as it would make the distinction in CŁC between sets and classes parallel Frege's [2] distinction between *Sinn* (sense, or intension) and *Bedeutung* (meaning, or extension): indeed, the extensional CŁC-class $[x \mid \varphi(x)]$ represents the *collection of instances* of the

[3] Though only $x = y \to \triangle(x \approx y)$ is provable for sets by Hájek's result of [3] cited above.

[4] Although the theories differ in that AST has, so to speak, a 'strange' structure of classes over 'common' finite sets, while CŁC has 'common' fuzzy classes over a 'strange' structure of sets.

[5] Yatabe, pers. comm.

property $\varphi(x)$—or its *extension;* while the intensional CŁC-set $\{x \mid \varphi(x)\}$ represents (by Terui's conjecture, exactly; otherwise partly) the *way* the property φ is defined—i.e., its *sense* (or intension). Thus it is not counter-intuitive if, e.g., CŁ proves $\{x \mid \varphi \vee \psi\} \neq \{x \mid \psi \vee \varphi\}$, as the two sets, though co-extensional, are presented in different ways. (Arguably, this is a *desired* feature in naïve set theories.)

3 On the motivation of CŁC

It may be objected that classes destroy the appealing simplicity of the full comprehension principle in CŁ. Nevertheless, they only represent fuzzy or crisp classes that are anyway present in the models of CŁ, and they make it possible to handle many natural constructions (such as kernels of fuzzy sets, including, e.g., FN and $\mathrm{Ker}(\omega)$ as models of arithmetic) within the theory. The features of CŁC (the existence of a universal set, the distinction between intensional sets and extensional classes, the properness of the class of standard natural numbers, etc.) suggest that CŁ-sets may provide a sufficiently rich ground structure for a mathematically non-trivial class theory over CŁ.

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Matrices over residuated lattices, related structures, and applications

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This talk will provide an overview of past and current research in structures related to matrices with entries from complete residuated lattices (alternatively, to binary fuzzy relations with complete residuated lattices as the structures of truth degrees). In particular, the talk will provide an overview of closure and interior structures associated to such matrices and their applications in data analysis. The topics and results covered in this talk include Galois connections, closure and interior operators, induced by the matrices, their axiomatic descriptions, their representation by ordinary Galois connections, closure and interior operators, the lattices of fixpoints of the associated operators, and some other topics.

A particular attention will be paid to the application of the presented structures to formal concept analysis of data with fuzzy attributes and to further topics of theoretical interest which arise in the context of this application, such as sublattices of the fixpoints and particular data dependencies called attribute implications.

The last part of the talk will provide on overview of a recent application of the presented structures in factor analysis of binary data and data with fuzzy attributes. We will present the problem of factor analysis, the role of fixpoints of the operators discussed, basic complexity results about the problem, an approximation algorithm for factorization, its experimental evaluation, and illustrative examples.

Decidability of some problems in a description logic over infinite-valued product logic

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Abstract. In this contribution we will show that satisfiability and validity of concepts in the Fuzzy Description Logic (FDL) based on infinite-valued Product Logic with universal and existential quantifiers (which are not interdefinable) are decidable problems. We give an algorithm that reduces the problem of satisfiability (and validity) of concepts in our FDL to a semantic consequence problem, with finite number of hypothesis, on infinite-valued propositional Product Logic. The proof makes use of a special kind of interpretations, here called quasi-witnessed, that are particularly adequate for the infinite-valued Product Logic.

1 Introduction

For each one of three basic continuous t-norms (minimum, Łukasiewicz and product) a propositional and a first order logical system have been studied in the literature. In this paper we only deal with logics given by the standard semantics in the fuzzy tradition, and not by the general semantics (cf. [1]). The language of these logics takes as primitive connectives the multiplicative conjunction $\odot$, its residuum implication $\rightarrow$ and the falsum constant $\bot$; while the intended semantics of $\odot$ is the corresponding t-norm $\star$, the semantics of $\rightarrow$ is given by the residuum of the t-norm (i.e., by $x \Rightarrow y := \max\{z \in [0,1] : x \star z \leq y\}$) and $\bot$ is interpreted as 0. It is well known that simply using these three connectives we can define a new constant $\top$ as well as new connectives $\wedge$ and $\vee$ whose intended semantics are 1 and the lattice operations over $[0,1]$ with its natural order; and it is common to introduce a negation $\neg$ defined by $\neg\varphi := \varphi \rightarrow \bot$. Complete (for finite theories) Hilbert style axiomatizations for the propositional logics defined by the three basic continuous t-norms can be found in [1]; it is also proved there that the problem of a formula being valid in these logics is, in the three cases, NP-complete. On the other hand, the behaviour of the first order logics introduced by these three t-norms is not so nice: while in the minimum case a recursively axiomatizable logic is obtained, this is not true for the other two: Łukasiewicz t-norm introduces a Π_2-complete logic and product t-norm is even worst introducing a not arithmetical one.

Since classical Description Logic (DL) $\mathcal{ALC}$ can be seen as a fragment of first order classical logic, Hájek proposed in [2] to introduce the fuzzy version, one for each t-norm, of this DL as a fragment of its first order fuzzy logic. In this paper we will use the notation $\star-\mathcal{ALE}$ to denote the fuzzy DL defined by Hájek using the t-norm $\star$. Concepts in these fuzzy DLs are recursively defined from a fix set of concept names and a fix set of role names.

Definition 1. *The set of* concepts *is the smallest set such that:*

1. *every concept name A is a concept,*
2. *$\bot$ and $\top$ are concepts,*
3. *if C, D are concepts and R is a role name, then $C \boxdot D$, $C \to D$, $\forall R.C$ and $\exists R.C$ are concepts.*

Definition 2. *Let $\star$ be a t-norm and let $\Rightarrow$ be its residuum. Then, an* $\star$-interpretation *$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists on a crisp set $\Delta^{\mathcal{I}}$ (called the* domain *of $\mathcal{I}$) and an* interpretation function *$\cdot^{\mathcal{I}}$, which maps every concept C to a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, every role name R to a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$ and such that, for every concepts C, D, every role name R and every element $a \in \Delta^{\mathcal{I}}$, it holds that:*

$$\begin{aligned}
\bot^{\mathcal{I}}(a) &= 0\\
\top^{\mathcal{I}}(a) &= 1\\
(C \boxdot D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \star D^{\mathcal{I}}(a)\\
(C \to D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a)\\
(\forall R.C)^{\mathcal{I}}(a) &= inf\{R^{\mathcal{I}}(a,b) \Rightarrow C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\}\\
(\exists R.C)^{\mathcal{I}}(a) &= sup\{R^{\mathcal{I}}(a,b) \star C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\}
\end{aligned}$$

In his paper, Hájek defines a concept C to be **1-satisfiable in** $\star-\mathcal{ALE}$ if there is some interpretation $\mathcal{I}$ and object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = 1$. This definition can be generalized in the obvious way to **r-satisfiability** (for every $r \in [0,1]$). Analogously, Hájek also defines a concept C to be **valid in** $\star-\mathcal{ALE}$ if for every interpretation $\mathcal{I}$ and object $a \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}(a) = 1$. We will write $\mathrm{Sat}_r^\star$ and $\mathrm{Val}^\star$ to denote the set of concepts that are, respectively, r-satisfiable and valid in $\star-\mathcal{ALE}$.

The main result in [2] says that if $\star$ is Łukasiewicz t-norm, then the sets $\mathrm{Sat}_1^\star$ and $\mathrm{Val}^\star$ are decidable; an easy consequence of this fact is that for every $r \in \mathbb{Q} \cap [0,1]$, $\mathrm{Sat}_r^\star$ is also decidable. The proof consists on two claims. The first claim is a general one: for every t-norm, the problems of 1-satisfiability and validity in finite interpretations are decidable problems. This is proved using a reduction of the problem to the propositional fuzzy logic given by the corresponding t-norm; the idea behind this reduction is the fact that finite interpretations can be codified using a finite number of propositional formulas. The second claim is a particular one of Łukasiewicz: for every $r \in [0,1]$, a concept is r-satisfiable iff it is r satisfiable in some finite interpretation. The proof of this fact is based on the notion of witnessed interpretation (see [2]): using the definition it is trivial that concepts r-satisfiable in a witnessed interpretation are also r-satisfiable in a finite interpretation; and in the case of Łukasiewicz t-norm it is well known that first order formulas (in particular this applies to concepts) r-satisfiable are r-satisfiable in a witnessed interpretation.

2 Main Result

In the rest of the contribution we will focus on the product t-norm, thus from now on $\star$ will always refer to the product t-norm.

Theorem 1 (Product Case). *For every $r \in [0, 1]$, the set Sat_r is decidable; and the set Val is also decidable.*

Thus, although the first order logic is non arithmetical we prove that $\star$–$\mathcal{ALE}$ is much more tractable. The proof of this result follows the same pattern than Hájek's one reducing the problem to a consequence problem in the propositional fuzzy logic (of product t-norm this time), but in this occasion we cannot use witnessed interpretations. The reason is that there are concepts, like

$$\forall R.A \sqcap \neg\forall R.(A \boxdot A) \quad \text{and} \quad \neg\forall R.A \sqcap \neg\exists R.\neg A$$

which are 1-satisfiable, but never in a witnessed interpretation.

3 Sketch of the Proof

First of all we notice that for every $r, s \in (0, 1)$, $\mathrm{Sat}_r = \mathrm{Sat}_s$. This is an immediate consequence of the fact that for every $l \in \mathbb{R}^+$, the function $x \longmapsto x^l$ is an automorphism of the standard product algebra. We will use the notation Sat_i to indicate the set of concepts that are intermediately satisfiable (i.e., 0.5 satisfiable). Therefore, we only need to show the decidability of the sets Sat_0, Sat_1, Sat_i and Val. Moreover, using that

- $C \in \mathrm{Sat}_0$ iff $\neg C \in \mathrm{Sat}_1$,
- $C \in \mathrm{Sat}_i$ iff $C \sqcup \neg C \notin \mathrm{Val}$,

it follows that it would be enough to prove that Sat_1 and Val are decidable.

The proof is based on two steps. The first step tells us that in the product case it is enough to consider quasi-witnessed interpretations (called closed models in [4]). Since our semantics is the standard one and not the general one, we point out that we cannot directly use the results from [4]. However, we can prove Proposition 1 similarly to [1, Theorem 5.4.30] (in the product case all primitive connectives are continuous except for $\Rightarrow$ in the point $(0, 0)$).

Definition 3. *An $\star-$interpretation $\mathcal{I}$ is* quasi-witnessed *when it satisfies*

(wit$\exists$) *for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is some $b \in \Delta^{\mathcal{I}}$ such that*

$$(\exists R.C)^{\mathcal{I}}(a) \;=\; R^{\mathcal{I}}(a,b) \star C^{\mathcal{I}}(b),$$

(qwit$\forall$) *for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ either $(\forall R.C)^{\mathcal{I}}(a) = 0$ or there is some $b \in \Delta^{\mathcal{I}}$ such that*

$$(\forall R.C)^{\mathcal{I}}(a) \;=\; R^{\mathcal{I}}(a,b) \Rightarrow C^{\mathcal{I}}(b).$$

Proposition 1. *Let φ be a first order formula and let $r \in [0,1]$. The following statements are equivalent:*

1. *φ is satisfiable with truth value r in a first-order interpretation over Π,*
2. *φ is satisfiable with truth value r in a first-order interpretation over a 1-generated subalgebra of Π.*

Since the 1-generated subalgebra (unique up to isomorphism) of Π is order discrete (all non-zero elements have a predecessor), we get the following corollary.

Corollary 1. *Let C be a concept and let $r \in [0,1]$. The following statements are equivalent:*

1. *C is satisfiable with truth value r in a Π-interpretation,*
2. *C is satisfiable with truth value r in a quasi-witnessed Π-interpretation.*

The second step of the proof consists on proving that r-satisfiability in a quasi-witnessed Π-interpretation is decidable, and this is proved by a reduction to the propositional product logic $\models_{\Pi}$ with variables

$$Prop := \{p_{R(a,b)} : R \text{ is a role name and } a,b \in \mathrm{Ind}\} \;\cup$$

$$\{p_{C(a)} : C \text{ is a concept and } a \in \mathrm{Ind}\},$$

where Ind is a fix infinite set $\{a_n : n \in \omega\}$. Next step in the proof is an algorithm (it will be explained in the talk) that converts every concept C into two finite sets T_C of Y_C of propositional formulas (with variables over $Prop$), and that satisfies the following condition.

Proposition 2. *Let C be a concept, and let T_C and Y_C be the two finite sets associated. For every $r \in [0,1]$, the following statements are equivalent:*

1. C *is satisfiable with truth value* r *in a quasi-witnessed* Π*-interpretation,*
2. there is some propositional evaluation e *over the set* $Prop$ *such that* $e(p_{C(a_0)}) = r$*,* $e[T_C] = 1$*, and* $e[\psi] \neq 1$ *for every* $\psi \in Y_C$*.*

We can now finish the proof of Theorem 1 because Proposition 2 tells us that

- $C \in \mathrm{Sat}_1$ iff $\{pr(C(a_0))\} \cup pr(T_C) \not\models_{\Pi} \bigvee pr(Y_C)$,
- $C \in \mathrm{Val}$ iff $pr(T_C) \models_{\Pi} pr(C(a_0)) \vee \bigvee pr(Y_C)$.

Hence, we have a reduction of these problems to the semantic consequence problem, with a finite number of hypothesis, in the propositional product logic; which it is known [3, Theorem 3] to be in PSPACE.

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Logics with a (lattice) disjunction and their completeness properties

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Classical logic and many non-classical logics have a disjunction connective, which sometimes is primitive in the presentation or sometimes might be definable by using a formula in two variables. In this talk we will investigate the rôle of disjunction in the proofs of some important properties of these logics, namely completeness with respect to particular algebraic semantics. We will show that some of these proofs can be carried out as well when disjunction in neither primitive nor definable by a single formula, but definable by a (possibly parameterized and infinite) set of formulae, as it happens in some natural examples. In this short abstract we give the basic definitions of the kinds of disjunctions we study, the classes of logics they define, and point to the results we will present about them.

First we need to introduce some notation to deal with some generalized notions of disjunction connectives. Indeed, given a parameterized set of formulae $\nabla(p, q, \overrightarrow{r})$ we define:

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq \omega}\},$$

where $\mathrm{Fm}_{\mathcal{L}}^{\leq \omega}$ denotes the set of all sequences of formulae in the language $\mathcal{L}$. When there are no parameters in the set $\nabla(p, q)$ and it is unitary, we write $\varphi \vee \psi$ instead of $\varphi \nabla \psi$.

Definition 1. *Let* L *be a logic and* ∇ *a (parameterized) set of formulae in two variables. We say that* ∇ *is a* (p-)protodisjunction *if it satisfies:*

(PD) $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$ *and* $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$

A (p-)protodisjunction ∇ *is called a* (p-)disjunction *whenever it satisfies the* Proof by Cases Property*,* PCP *for short:*

If $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$.

A *weak* (p-)disjunction satisfies just a weak form of the *Proof by Cases Property* (wPCP for short): if $\varphi \vdash_{\mathrm{L}} \chi$ and $\psi \vdash_{\mathrm{L}} \chi$, then $\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$. All weak (p-)disjunctions in a given logic are mutually interderivable:

Lemma 1. *Let* L *be a propositional logic and ∇, ∇' parameterized sets of formulae. Assume that ∇ is a weak p-disjunction in* L*. Then: ∇' is a weak p-disjunction in* L *iff $\varphi \nabla \psi \dashv\vdash_{\mathrm{L}} \varphi \nabla' \psi$.*

This level of generality is actually needed as shown by the following examples:

1. Let G be Gödel-Dummett logic (see [3]) and $\mathrm{G}_{\rightarrow}$ its purely implicational fragment. Then the finite set $\{(p \to q) \to q, (q \to p) \to p\}$ is a disjunction in $\mathrm{G}_{\rightarrow}$. We can prove that no single formula can define a weak disjunction in this logic.
2. Consider the logic FL of all pointed residuated lattices (see [4] for more details). This logic has lattice connectives $\wedge$ and $\vee$, truth-constants $\overline{1}$ and $\overline{0}$, and a non-commutative conjunction $\&$ with left and right residua denoted respectively as $\backslash$ and $/$. Given formulae α, φ, one defines the *left conjugate* and the *right conjugate* of φ with respect to α respectively as $\lambda_\alpha(\varphi) = (\alpha \backslash \varphi \& \alpha) \wedge \overline{1}$ and $\rho_\alpha(\varphi) = (\alpha \& \varphi / \alpha) \wedge \overline{1}$. An *iterated conjugate* of φ with respect to the formulae $\alpha_1, \ldots, \alpha_n$ is a composition $\gamma(\varphi) = \gamma_{\alpha_1}(\gamma_{\alpha_2}(\ldots \gamma_{\alpha_n}(\varphi)))$ where $\gamma_{\alpha_i} \in \{\lambda_{\alpha_i}, \rho_{\alpha_i}\}$ for every i. With this notation, a p-disjunction for this logic can be defined by the following infinite set with parameters:

   $$\varphi \nabla \psi = \{\gamma_1(\varphi \wedge \overline{1}) \vee \gamma_2(\psi \wedge \overline{1}) \mid \text{where } \gamma_1, \gamma_2 \text{ are iterated conjugates}\}$$

   Interestingly enough, the lattice connective $\vee$ is a protodisjunction but not a weak disjunction in FL. Moreover, we can prove that there is no finite set of formulae defining a weak disjunction in this logic. In contrast, when we consider the logic FL_e, obtained as the axiomatic extension of FL adding the commutativity axiom for $\&$, the p-disjunction can be simplified to a disjunction connective given by just one formula: $(\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$. Finally, when we add the weakening law and get to FL_{ew}, making $\overline{1}$ the top element in each algebra, the lattice connective $\vee$ becomes a disjunction.

We study the relation between proof by cases and other properties a disjunction is expected to satisfy: commutativity, idempotency and associativity (which, however, are typically also satisfied by conjunction connectives, whereas (PD) and (w)PCP are typically satisfied only by disjunction connectives).

Lemma 2. *Let* L *be a logic and* ∇ *a p-protodisjunction. If* ∇ *satisfies* wPCP*, then it also satisfies the following conditions:*

(C) $\varphi\nabla\psi \vdash_{\mathrm{L}} \psi\nabla\varphi$
(I) $\varphi\nabla\varphi \vdash_{\mathrm{L}} \varphi$
(A) $\varphi\nabla(\psi\nabla\chi) \dashv\vdash_{\mathrm{L}} (\varphi\nabla\psi)\nabla\chi$

We can show that there are logics with a protodisjunction satisfying the conditions (C), (I) and (A) which is not a weak disjunction.[3]

Definition 2. *We call a logic* (weakly) (p-)disjunctional *if it has a (weak) (p-)disjunction. We add the prefix 'finitely' if the (weak) disjunction is definable by a finite set. Furthermore, we call a logic* (weakly) disjunctive *if it has a (weak) disjunction given by a single parameter-free formula.*

The logics mentioned above and some other natural examples show the separation of most of these classes of logics.

Finally, we present a stronger notion of disjunction: a disjunction whose interpretation in the algebraic counterpart of the logic is the supremum w.r.t. the order relation in the algebras. Of course, this idea makes sense only if ∇ is a parameter-free singleton; let us use $\vee$ instead of ∇ in this case. We need the presence of a good generalized implication in the language: a weak p-implication in the sense of [2].

Definition 3. *Let* L *be a logic,* $\Rightarrow$ *a weak p-implication, and* $\vee$ *a formula in two variables. We say that* $\vee$ *is a* lattice protodisjunction for $\Rightarrow$ *if:*

($\vee$1) $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \vee \psi$
($\vee$2) $\vdash_{\mathrm{L}} \psi \Rightarrow \varphi \vee \psi$
($\vee$3) $\varphi \Rightarrow \chi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \vee \psi \Rightarrow \chi$

We say that $\vee$ *is a* lattice disjunction for $\Rightarrow$ *(resp.* lattice weak disjunction for $\Rightarrow$*) if it also has the* PCP *(resp. the* wPCP*).*

By combining ($\vee$1), ($\vee$2), ($\vee$3), and the reflexivity and transitivity of $\Rightarrow$, we can easily show that any lattice protodisjunction for $\Rightarrow$ satisfies the following stronger versions of the properties (C), (I), and (A):

(iC) $\vdash_{\mathrm{L}} \varphi \vee \psi \Rightarrow \psi \vee \varphi$
(iI) $\vdash_{\mathrm{L}} \varphi \vee \varphi \Rightarrow \varphi$
(iA) $\vdash_{\mathrm{L}} (\varphi \vee \psi) \vee \chi \Rightarrow \varphi \vee (\psi \vee \chi)$ and $\vdash_{\mathrm{L}} \varphi \vee (\psi \vee \chi) \Rightarrow (\varphi \vee \psi) \vee \chi$

[3] We can also show the independence of the conditions (C), (I) and (A) of protodisjunctions by several (artificial) examples. Let us just mention a natural example: any substructural non-contractive involutive logic (e.g. linear logic or Łukasiewicz infinite-valued logic) has the multiplicative disjunction $\oplus$ which satisfies conditions (PD), (C), and (A) but not (I).

The classes of lattice protodisjunctions and disjunctions are mutually incomparable: on one hand, recall that the protodisjunction $\vee$ of the logic FL_e (or even FL) is not a disjunction and observe that it is clearly a *lattice* protodisjunction for $\rightarrow$. On the other hand, consider the expansion of FL_e with a connective $\Box$ defined in each algebra as $\Box x = \overline{1}$ for $x \geq \overline{1}$ and $\bot$ otherwise. Then clearly $\Box(p \vee q)$ is a disjunction (since $\Box(p \vee q) \dashv\vdash p \vee q$) but is not a lattice disjunction for $\rightarrow$.

We use the aforementioned notions of disjunction to obtain a number of results concerning completeness of logics w.r.t. particular algebraic semantics and axiomatization of logics with good completeness properties:

- A characterization of implicational semilinear logics defined in [2], that is, logics with a weak p-implication such that it defines a class of totally ordered algebras which is a complete semantics for the logic. This captures and characterizes, to a large extent, the notion of fuzzy logic.
- Axiomatization of the weakest semilinear logic above a given logic.
- Characterization of logics complete with respect to a class of densely ordered algebras.
- Given a class $\mathbb{K}$ of algebras, we characterize strong completeness and finite strong completeness properties of a logic L w.r.t. $\mathbb{K}$ in terms of embedding properties, generalizing results in [1].
- We solve several open problems (e.g. necessity of using $\triangle$ in axiomatizing $\mathrm{MTL}_{\sim}$) and give alternative proof of theorems (e.g. showing that the logic PŁ$'$ is complete w.r.t. chains) from fuzzy logic literature.

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Lattice-valued predicate transformers and interchange systems

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1 From Predicate Transformers to Interchange Systems

In the 1970s, Edsger Dijkstra was concerned with improving the quality of computer programs; perhaps more exactly, he was interested in improving the methodologies which programmers used to develop programs [4]. An outcome of his efforts was the development of predicate transformer semantics, a formal method for studying program semantics.

In developing his ideas, Dijkstra proposed some radical shifts in how programmers could/should think about programs and about developing programs. Instead of thinking in terms of individual values, he felt a programmer should think of properties of values. Also, instead of focusing on inputs, a programmer should focus on outputs.

Thus, a programmer considers what property she or he wants or needs in the output, and then the programmer transforms this predicate for outputs into a corresponding predicate for inputs. This predicate transformer process associates with the desired output predicate an input predicate so that if an input value satisfies the associated input predicate, then the program will produce an output value that will have the desired output property. The output predicate is called a *postcondition* predicate, and the associated input predicate is called a *precondition* predicate. Once an associated input predicate is chosen, then any more restrictive input predicate would also work to produce an output value with the desired property. Dijkstra called the most general or largest input predicate which would produce the desired output the *weakest* precondition predicate.

One can associate with each input or output predicate a subset of the set of inputs or the set of outputs, respectively. Predicate transformers can be easily understood if one thinks of a program as a (partial) function and thinks of the function as a set of ordered pairs. If, for example, X is the set of inputs and Y is the set of outputs, then a program is a function $p : X \to Y$. If we let Q be a postcondition predicate, then the corresponding weakest precondition predicate is

$$\{x \in X \,|\, p(x) \in Q\}$$

In this deterministic setting, predicate transformer semantics is straightforward. If $\mathcal{P}$ is the set of predicates for X, $\mathcal{Q}$ is the set of predicates for Y, and $Q \in \mathcal{Q}$, then we can define the predicate transformer to be $p^{\leftarrow} : \mathcal{Q} \to \mathcal{P}$ such that

$$p^{\leftarrow}(Q) = \{x \in X \,|\, p(x) \in Q\}$$

Predicate transformer semantics becomes more interesting when one works in a nondeterministic setting, i.e., when one allows programs to be relations from X to Y. Then for a given postcondition, it is not immediately clear what the corresponding (weakest) precondition should be.

Two standard approaches for handling nondeterminism are *angelic* and *demonic*. The motivation for the two terms comes from the nature of nondeterminism. For a given input there is often a choice for the corresponding output. However, for a given execution of a program with a given input we assume that only one of the possible outputs will be produced. When a postcondition Q is chosen, there may be input values x such that some corresponding outputs will be in Q and some will not be, i.e., sometimes the output will satisfy the postcondition predicate and sometimes the output will not satisfy the postcondition predicate. Expressed mathematically, if $r : X \to Y$ is a nondeterministic program and if $Q \subset Y$ is a postcondition predicate, then there may exist $x \in X$ such that for $xr = \{y \in Y \,|\, (x, y) \in r\}$

$$xr \cap Q \neq \varnothing \text{ but } xr \not\subset Q$$

Hence, depending on the choice of output $y \in xr$, the postcondition Q will or will not be satisfied. Thus, should x be in the postcondition associated with Q or not? Using angelic semantics, x is in the corresponding precondition because one assumes an "angelic" choice will be made by selecting an output in Q. Using demonic semantics, x is not in the corresponding semantics because one assumes a "demonic" choice would be made to select an output not in Q. Since when using predicate transformers, one always wants to get an output in Q, we have the following two definitions.

Let $r_a^{\leftarrow} : \mathcal{Q} \to \mathcal{P}$ be the angelic predicate transformer for $r : X \to Y$. Then for $Q \in \mathcal{Q}$,

$$r_a^{\leftarrow}(Q) = \{x \in X \,|\, xr \cap Q \neq \varnothing\}$$

Let $r_d^{\leftarrow} : \mathcal{Q} \to \mathcal{P}$ be the demonic predicate transformer for $r : X \to Y$. Then for $Q \in \mathcal{Q}$,

$$r_d^{\leftarrow}(Q) = \{x \in X \mid xr \subset Q\}$$

There are naturally occurring situations when one would want to expand the possible outputs to include lattice-valued variations. In some cases, these variations themselves could produce the the relational nature (as opposed to the functional nature) of the program. In other cases, the lattice-valued variations would be in addition to the the relational or multifunctional nature of the original program. An example of the latter would be a security system which itself includes multiple activities. For example, when a security system is activated, it may be due to potential harm to computers, to a company's in-house software, or to company data. Thus, there is a natural relational or multifunctional nature of the problem. Additionally, depending on the nature and severity of the security threat, there may be a practical needs for lattice-valued responses. If, for example, the threat is minimal, it would probably not be fiscally responsible to respond by imposing the highest level of security measures. Thus, we want to consider possible definitions for lattice-valued predicate transformers.

In what follows, we consider lattice-valued angelic and demonic predicate transformers, and in the development, we try to understand predicate transformers in a topological systems setting. To do this, we review how programming semantics may be viewed in a topological setting. In the 1980s, M. Smyth [5] and others began thinking of the sets of predicates as topologies. In order to use the predicates to reason about programs, there needs to be a logical structure associated with the collections of predicates. This logical structure may be developed in terms of affirmative finite observations [8]. Thus, if a statement is true, we want to be able to determine this in finite time.

Of course, we want our logic to support finite conjunctions and disjunctions. Thus, thinking of predicates as subsets, we want the collection of predicate subsets to be closed under finite intersections and finite unions. A natural follow-up question is whether the collection of predicate subsets could or should be closed under arbitrary intersections and arbitrary unions. Working with our logic of affirmative finite observations, we can consider having the collection of predicate subsets closed under infinite unions because if an element is in an infinite union, then it is in, at least, one of the sets, and as soon as one of the sets containing the element is processed, we can stop processing. However, there is a problem with closure under infinite intersections because if an element is in an infinite intersection, this fact can only be verified by processing all the infinite subsets, and this can not happen in finite time.

The empty set and the whole set can also be considered predicates in our affirmative and finitely observable logic because no element is in the empty set and all elements are in the whole set. Thus, it is natural to think of the collection of predicates as a topology.

We modify our notation slightly, and replace $r : X \to Y$ with $R : X \to \wp(Y)$ where for $x \in X$, $R(x) = xr$. Thus, our program is a function R from the set of inputs to the powerset of outputs. We would like to look at the predicate transformers in a topological systems setting; see [8].

Definition 1. *A* topological system *is an ordered triple* $(X, A, \models)$*, where* $(X, A) \in |\mathbf{Set} \times \mathbf{Loc}|$ *and* $\models$ *is a satisfaction relation on* (X, A) *, meaning that* $\models$ *is a relation from* X *to* A *such that both the following join and meet interchange laws respectively hold:*

$$\text{if } S \text{ is a subset of } A\text{, then } x \models \bigvee S \text{ iff } \exists\, a \in S,\ x \models a,$$

$$\text{if } S \text{ is a finite subset of } A\text{, then } x \models \bigwedge S \text{ iff } \forall a \in S,\ x \models a.$$

It should be noted that given a topological space (X, T), the ordered triple $(X, T, \in)$ is a topological system with $\models$ taken as the membership relationship.

Continuous functions between topological systems are ordered pairs

$$(f, \varphi) : (X, A, \models_1) \to (Y, B, \models_2)$$

where $f : X \to Y$ is a set function and $\varphi : A \to B$ is a localic morphism satisfying the condition that for all $x \in X$ and all $b \in B$,

$$f(x) \models_2 b \text{ if and only if } x \models_1 \varphi^{op}(b).$$

When this last condition is satisfied, we say that the interchange property holds.

Vickers [8] developed topological systems, in part, to create a more natural setting for studying and applying topological ideas in computer science applications. Thus, it is appropriate to try to understand predicate transformers in a topological systems setting. We begin with a generalization; we start with $(X, T, \in)$, where X is the set of input values and where T is the lattice of predicates on X.

For the codomain for the angelic predicate transformer, we propose $(\wp(Y), W, \models_a)$, where W is the lattice of predicates on Y and where $\models_a$ is a satisfaction relation on $\wp(Y)$ i.e., $\models_a \subset \wp(Y) \times W$ such that for $N \in \wp(Y)$ and $V \in W$

$$N \models_a V \Leftrightarrow N \cap V \neq \varnothing$$

For the codomain for the demonic predicate transformer, we propose $(\wp(Y), W, \models_d)$, where $\models_d$ is a satisfaction relation on $\wp(Y)$ such that for $N \in \wp(Y)$ and $V \in W$

$$N \models_d V \Leftrightarrow N \subset V$$

The proposed continuous function for the angelic case is

$$(R, \varphi_a) : (X, T, \in) \to (\wp(Y), W, \models_a)$$

where $\varphi_a^{op} : W \to T$ such that for $V \in W$

$$\varphi_a^{op}(V) = \{x \in X \mid xr \cap V \neq \varnothing\}$$

Similarly, the proposed continuous function for the angelic case is

$$(R, \varphi_d) : (X, T, \in) \to (\wp(Y), W, \models_d)$$

where $\varphi_d^{op} : W \to T$ such that for $V \in W$

$$\varphi_d^{op}(V) = \{x \in X | \, xr \subset V\}$$

For both (R, φ_a) and (R, φ_d), the morphism interchange property holds. For example, for $x \in X$ and $V \in W$,

$$x \in \varphi_a^{op}(V) \Leftrightarrow xr \cap V \neq \varnothing$$

$$\Leftrightarrow R(x) \cap V \neq \varnothing$$

$$\Leftrightarrow R(x) \models_a V$$

Though discussing angelic and demonic predicate transformers in the context of topological systems is in the spirit of [8], beginning with $(Y, W, \in)$ a topological system does not imply that either $(\wp(Y), W, \models_a)$ or $(\wp(Y), W, \models_d)$ is a topological system. Further, neither φ_a nor φ_d need be a localic morphism. Thus, in keeping with the motivation behind [8], we generalize topological systems and their continuous functions to *interchange systems* and a *morphism interchange property*.

An interchange system is a triple $(X, A, \models)$ where X is a set, A is a structured set, and $\models$ is a relation from X to A, i.e., $\models \subset X \times A$. Further, the relation $\models$ must satisfy properties so that the structure of A may be transferred to the powerset of A. Morphisms between interchange systems are pairs $(f, g) : (X, A, \models_1) \to (Y, B, \models_2)$ such that $f : X \to Y$ is a set function, $g : A \to B$ is a function such that $g^{op} : B \to A$ is a concrete structure preserving map, and for each $x \in X$ and each $b \in B$,

$$x \models_1 g^{op}(b) \Leftrightarrow f(x) \models_2 b$$

When this last biconditional holds, we say that the pair (f, g) satisfies the morphism interchange property.

What exactly is the predicate transformer interchange system setting? Said differently, what structure is imposed upon or is needed for $(\wp(Y), \mathcal{Q}, \models)$ to be considered a viable predicate transformer interchange system codomain? In addition to the basic interchange and morphism interchange properties, we have the following. For the angelic case, $\models_a$ is closed under arbitrary unions, and φ_a^{op} preserves arbitrary unions. For the demonic case, $\models_d$ is closed under arbitrary intersections, and φ_d^{op} preserves arbitrary intersections.

We have considered the interchange systems $(\wp(Y), W, \models_a)$ and $(\wp(Y), W, \models_d)$. We now want to address a related question. If we begin with a topological system $(X, A, \models)$, can we in a "natural" way define angelic and/or demonic interchange systems on the powerset $(\wp(X), A, \models')$. Note that although we are trying to "lift" from X to $\wp(X)$ and we are modifying our satisfaction relation, we are not changing the A.

For $C \in \wp(X)$ and $a \in A$, define $(\wp(X), A, \models_\exists)$ such that

$$C \models_\exists a \text{ iff } \exists x \in C,\ x \models a$$

and define $(\wp(X), A, \models_\forall)$ such that

$$C \models_\forall a \text{ iff } \forall x \in C,\ x \models a$$

For $C \in \wp(X)$ and $S \subset A$,

$$C \models_\exists \bigvee S \Leftrightarrow \exists a \in S,\ C \models_\exists a$$

Therefore, $(\wp(X), A, \models_\exists)$ has characteristics of $(\wp(Y), W, \models_a)$; $\models_\exists$ interchanges arbitrary joins.

The situation is similar for $(\wp(X), A, \models_\forall)$. For $S \subset A$ and for $C \in \wp(X)$,

$$C \models_\forall \bigwedge S \Leftrightarrow \forall a \in S,\ C \models_\forall A$$

Thus, $(\wp(X), A, \models_\forall)$ has the characteristics of $(\wp(Y), W, \models_d)$; $\models_\forall$ interchanges arbitrary meets.

2 Lattice-Valued Extensions of Transformers and Systems

It was seen in the previous section that predicate transformers give rise to interchange morphisms between interchange systems. This section outlines how each of these ideas can be reformulated in a lattice-valued context, and, further, these lattice-valued extensions relate to each other in a way that extends the relationship of the previous section to a lattice-valued setting.

We first tackle the matter of generalizing multifunctions and the associated angelic and demonic predicate transformers. Throughout this discussion, L is a frame and associated Heyting implication is denoted $\rightarrow$. Let X, Y be sets. In this section we regard a nondeterministic program not as a multifunction $R : X \rightarrow Y$, but equivalently as a function $R : X \rightarrow \wp(Y)$. In this way, the L-valued counterpart is obvious: an *L-valued multirelation* is a mapping $R : X \rightarrow L^Y$ which associates with each input from X an *L-valued (post-condition) predicate* on the set Y of outputs. This sets up our discussion of L-valued nondeterminism even though we are expressing it using functions..

Using powerset-valued mappings to represent multifunctions, the previous section respectively defined for each $R : X \rightarrow \wp(Y)$ the associated angelic and demonic predicate transformers

$$\begin{aligned} \varphi_a^{op} &: \wp(Y) \rightarrow \wp(X) \quad \text{by} \quad \varphi_a^{op}(Q) = \{x \in X : R(x) \cap Q \neq \varnothing\}, \\ \varphi_d^{op} &: \wp(Y) \rightarrow \wp(X) \quad \text{by} \quad \varphi_d^{op}(Q) = \{x \in X : R(x) \subset Q\}. \end{aligned}$$

For the L-valued case, we respectively define for each $R : X \rightarrow L^Y$ the associated *L-valued angelic and demonic predicate transformers*

$$\begin{aligned} \varphi_a^{op} &: L^Y \rightarrow L^X \quad \text{by} \quad \varphi_a^{op}(q)(x) = \bigvee_{y \in Y} (R(x)(y) \wedge q(y)), \\ \varphi_d^{op} &: L^Y \rightarrow L^X \quad \text{by} \quad \varphi_d^{op}(q)(x) = \bigwedge_{y \in Y} (R(x)(y) \rightarrow q(y)), \end{aligned}$$

where we have abused the notation by using the same variations on φ as before, counting on context to keep matters clear. We also use in the sequel the notation φ_a and φ_d for the non-concrete morphisms from L^X to L^Y

It is convenient to have the fibre-map $G_X : \wp(X) \rightarrow L^X$ defined by $G_X(A) = \chi_A$, where this map assumes that L is a consistent frame, in which case G_X is an order-embedding.

Theorem 1. *The following hold for $R : X \rightarrow L^Y$:*

1. *Let L be consistent. Then it is the case that $\varphi_a^{op} \circ G_Y = G_X \circ \varphi_a^{op}$, $\varphi_d^{op} \circ G_Y = G_X \circ \varphi_d^{op}$. Restated, the L-valued predicate transformers extend the traditional predicate transformers.*
2. *The L-valued φ_a^{op} preserves arbitrary joins.*
3. *The L-valued φ_d^{op} preserves arbitrary meets.*

Statements (2) and (3) of the theorem justify the monikers “angelic” and “demonic” for the L-valued transformers: in the traditional setting angelic transformers preserve arbitrary unions and demonic transformers preserve arbitrary intersections; and indeed, in the traditional setting any map between families of predicates which preserves arbitrary unions [intersections] is termed angelic [demonic, respectively].

Now given an L-valued nondeterministic program $R : X \to L^Y$ and a family $\mathcal{Q} \subset L^Y$ of L-valued predicates on the output set Y, we use R and $\mathcal{Q}$ to construct associated *L-valued interchange systems* and *L-interchange morphisms* similarly to the previous section.

We begin by putting

$$\mathcal{P}_a \equiv (\varphi_a^{op})^{\to} (\mathcal{Q}) = \{\varphi_a^{op} (q) : q \in \mathcal{Q}\} ,$$
$$\mathcal{P}_d \equiv \left(\varphi_d^{op}\right)^{\to} (\mathcal{Q}) = \left\{\varphi_d^{op} (q) : q \in \mathcal{Q}\right\} .$$

Then $\mathcal{P}_a, \mathcal{P}_d \subset L^X$ are families of predicates on X; in fact, $\mathcal{P}_a$ [$\mathcal{P}_d$] is a complete join [meet] semi-lattice of L^X—see the Lemma below.

Now to complete the construction of the L-valued "angelic" and "demonic" interchange systems which respectively use $(X, \mathcal{P}_a)$, $(X, \mathcal{P}_d)$ as the ground objects, we define respectively the following *angelic* and *demonic L-valued satisfaction relations*:

$$\models^a : X \times \mathcal{P}_a \to L \quad \text{by} \quad \models^a (x, p) = p (x) ,$$
$$\models^d : X \times \mathcal{P}_d \to L \quad \text{by} \quad \models^d (x, p) = p (x) ,$$

With the the L-valued angelic interchange system $(X, \mathcal{P}_a, \models^a)$ and demonic interchange system $\left(X, \mathcal{P}_d, \models^d\right)$ in hand, we next consider the corresponding L-valued angelic and demonic interchange systems which use $\left(L^Y, \mathcal{Q}\right)$ as the ground object by respectively defining the following angelic and demonic L-valued satisfaction relations:

$$\models_a : L^Y \times \mathcal{Q} \to L \quad \text{by} \quad \models_a (b, q) = \textstyle\bigvee_{y \in Y} (b (y) \wedge q (y)) ,$$
$$\models_d : L^Y \times \mathcal{Q} \to L \quad \text{by} \quad \models_d (b, q) = \textstyle\bigwedge_{y \in Y} (b (y) \to q (y)) .$$

These constructions yield L-valued interchange systems $\left(L^Y, \mathcal{Q}, \models_a\right)$ and $\left(L^Y, \mathcal{Q}, \models_d\right)$.

More generally, we have the following notion:

Definition 2. *Let L be a frame. An* L-valued interchange system $(X, \mathcal{A}, \models)$ $isatriple$, *where* $(X, \mathcal{A})$ *is a* ground object *from* $\mathbf{Set} \times \mathbf{PoSet}^{op}$ *and the (*L-valued*)* satisfaction relation $\models$ *is a relation from* X *to*$\mathcal{A}$, *i.e.,* $\models \; : X \times \mathcal{A} \to L$ *is a mapping.*

Lemma 1. *It follows that* $\models^a$ *and* $\models_a$ *satisfy the join-interchange law, i.e.,:*

$$\models^a \left(x, \bigvee_{\gamma \in \Gamma} p_\gamma \right) = \bigvee_{\gamma \in \Gamma} \models^a (x, p_\gamma) ,$$

$$\models_a \left(x, \bigvee_{\gamma \in \Gamma} p_\gamma\right) = \bigvee_{\gamma \in \Gamma} \models_a (x, p_\gamma)\,;$$

and it follows that $\models^d$ *and* $\models_d$ *both satisfy the meet-interchange law, i.e.,*

$$\models^d \left(x, \bigwedge_{\gamma \in \Gamma} p_\gamma\right) = \bigwedge_{\gamma \in \Gamma} \models^d (x, p_\gamma)\,,$$

$$\models_d \left(x, \bigwedge_{\gamma \in \Gamma} p_\gamma\right) = \bigwedge_{\gamma \in \Gamma} \models_d (x, p_\gamma)\,.$$

We need the following definition.

Definition 3. *Let* $(f, \varphi) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ *be a morphism in* $\mathbf{Set} \times \mathbf{PoSet}^{op}$. *Then* $(f, \varphi) : (X, \mathcal{A}, \models) \to (Y, \mathcal{B}, \models)$ *is an* interchange morphism *from* $(X, \mathcal{A}, \models)$ *to* $(Y, \mathcal{B}, \models)$ *if the following* interchange condition *holds:*

$$\forall x \in X,\, \forall b \in \mathcal{B},\; \models (x, \varphi^{op}(b)) = \;\models (f(x), b)\,.$$

Theorem 2. *The following hold:*

1. *The pair* (R, φ_a) *is an interchange morphism from* $(X, \mathcal{P}_a, \models^a)$ *to* $\left(L^Y, \mathcal{Q}, \models_a\right)$.
2. *The pair* (R, φ_d) *is an interchange morphism from* $\left(X, \mathcal{P}_d, \models^d\right)$ *to* $\left(L^Y, \mathcal{Q}, \models_d\right)$.

Finally, we note that in [2, 3, 6, 7] topological systems were extended to L-topological systems, and that approach to fuzzification can be applied directly to the interchange systems and morphisms constructed in the last section *without* first fuzzifying the multifunction and predicate transformers as above. And as we will see, in a "soft way", we achieve the constructions essentially done in the paragraphs above, which more or less says that the "diagram of constructions" modeling this abstract commutes.

To the explore that the claim of the previous paragraph, let $R : X \to \wp(Y)$ be a multifunction written as a mapping and let $\mathcal{Q} \subset \wp(Y)$ be a family of traditional attributes on Y. Recall that from Section 1 we respectively have the angelic and demonic systems

$$(X, \mathcal{P}_a, \in)\,,\; (X, \mathcal{P}_d, \in)\,,$$

where in this crisp context $\mathcal{P}_a$ and $\mathcal{P}_d$ are defined as the subfamilies of $\wp(X)$ which are respectively the images of the traditional angelic φ_a^{op} and demonic φ_d^{op}. The fuzzification approach of [2, 6] defines the L-valued extension of $\in$ in each case to be precisely $\models^a$ and $\models^d$. Further, the interchange condition for each morphism generated by $R : X \to \wp(Y)$, namely

$$x \in \varphi_a^{op}(V) \Leftrightarrow R(x) \models_a V, \quad x \in \varphi_d^{op}(V) \Leftrightarrow R(x) \models_d V,$$

respectively, extends to the L-valued case *à la* [2, 6] to say

$$\models^a (x, \varphi_a^{op}(q)) = \models_a (R(x), q), \quad \models^d \left(x, \varphi_d^{op}(q)\right) = \models_d (R(x), q),$$

which are precisely the conditions satisfied by the L-valued interchange morphisms of the Theorem just above.

To sum up, the above considerations in combination with Statement (1) of Theorem 1 above mean that the fuzzification scheme of [2, 3, 6, 7] as applied to Section 1 yields the same interchange systems and interchange morphisms as the fuzzification scheme of the earlier part of this Section yields when applied to Section 1.

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Conditional measures: An alternative to Cox functional equation

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In 1946, R.T. Cox [3] tried to justify the notion of probability as a measure of belief from first principles. Relying on the Boolean algebra structure of events, he proposed three basic postulates on a degree of belief $g(A|B) \in [0,1]$, where A, B are events A, B in an Boolean algebra S.

1. $g(A \cap C|B) = F(g(A|C \cap B), g(C|B))$ (if $C \cap B \neq \emptyset$);
2. $g(A^c|B) = n(g(A|B)), B \neq \emptyset$, where A^c is the complement to A.
3. Function F is twice differentiable, with continuous second derivative and function n is twice differentiable.

On this basis, Cox claimed that $g(A|B)$ must be isomorphic to a probability measure. This alleged result has been used ad nauseam in various areas such as statistics, decision theory and artificial intelligence to justify probability measures as the only reasonable way of representing belief by numbers. For instance Jaynes [10] emphatically asserts this result as one building block of his probability theory, just requiring function F to be strictly increasing in both places. Paris[7] proves one version of this theorem. More recently Joe Halpern[8, 9] has shown that Cox result does not hold on finite sets, and even requires the addition of technical postulates in the infinite setting in order to be restored. Postulate 3 is not sufficient, and need to be completed. Hence probability theory does not follow from the Boolean setting of events.

There are commonsense objections to this setting, one for each postulate. Postulate 3 is clearly technical hence the most debatable from an intuitive point of view. Against postulate 2 it can be objected that in the presence of incomplete information the conditional belief in an event need not be computed from the conditional belief of the opposite event. This postulates fails in most uncertainty theories, such as imprecise probability, evidence and possibility theory, and this is what makes them attractive. As to the basic postulate 1, the reason why Cox writes it is because he knows the form of Bayes conditioning as the quotient of two probabilities.

Another way of understanding conditional probability is to see it as the probability of a conditional event. This is the view pioneered by De Finetti[4] and pursued by Coletti and Scozzafava [2]. De Finetti justifies conditional probability based on a betting scheme tolerating the possibility of conditioning on an event of zero probability. It is possible to generalize conditional probability à la De Finetti to other uncertainty measures [1].

There is yet another stream of literature on conditional events $A|B$, initiated by Goodman and Nguyen [6], viewing them as pairs of Boolean events of the form $(A \cap B, A \cup B^c)$ corresponding to conjunction and material implication respectively. The underlying idea is to distinguish between examples $A \cap B$ of a rule "if B then A" and counterexamples $A^c \cap B$. It is clear that conditional probability $P(A|B)$ can then be written in the form $h(P(A \cap B), P(A \cup B^c))$ with $h(x, y) = \frac{x}{x+1-y}$, where $x \leq y$. So an alternative setting to Cox problem, which sounds as natural as his own, is to find monotonic set functions g, and continuous functions $h : U \to [0, 1]$, where $U := \{(x, y) \in [0, 1]^2 | x \leq y\}$, $n : [0, 1] \to [0, 1]$ such that:

1. $g(A|B) = h(g(A \cap B), g(A \cup B^c))$
2. $g(A^c|B) = n(g(A|B)), B \neq \emptyset$ (the only Cox postulate we keep, having probability measures in mind).
3. Function F is monotonic increasing in both places and function n is monotonic decreasing.
4. $g((A|B)|C) = g(A|B \cap C)$, if $B \cap C \neq \emptyset$.

The last axiom is very natural as it says that the conditioning event is the conjunction of all pieces of evidence collected so far. For capacities that do not obey postulate 2, the two sets of axioms clearly lead to two distinct forms of conditioning. For instance, the expression $\frac{g(A \cap B)}{g(A \cap B)+1-g(A \cup B^c)}$ induced by the second framework does not reduce to the form $\frac{g(A \cap B)}{g(B)}$ enforced by Cox framework. The first kind of conditioning (using our set of axioms) is in agreement with imprecise probability theory and robust statistics, while the one induced by the Cox framework is more in line with Dempster rule of conditioning with belief functions [11]. The two conditioning rules also differ in possibility theory [5].

The following properties can be requested for function h:

1. $h(0, x) = 0$ for $x \in [0, 1[$ (since $g(\emptyset|B) = 0$);
2. $h(x, 1) = 1$ for $x \in]0, 1]$(since $g(A|B) = 1$ if $B \subseteq A$);

3. $h(x, x) = x$ for all $x \in [0, 1]$
(since if $B = S, g(A|B) = g(A) = g(A \cap B) = g(A \cup B^c))$;
4. $h(x, y) = 1 - h(1 - y, 1 - x)$ for all $(x, y) \in U \setminus \{(0, 1)\}$, from autoconjugation of g, choosing $n(x) = 1 - x$.

Therefore, there exists an idempotent symmetric sum $h' : [0, 1]^2 \to [0, 1]$ such that the restriction of h' to U is h, and h' possesses the listed properties.

According to Silvert [12], there exists a binary operation $*$ on $[0, 1]$ such that

$$h'(x, y) = \frac{x * y}{x * y + (1 - x) * (1 - y)}.$$

Obviously, we have $h'(0, x) = 0$ if and only if $0 * x = 0$ for all $x \in [0, 1[$. Notice that, using such an operation $*$, $h'(x, 1) = 1$ is automatically satisfied for $x \in]0, 1]$. Moreover choosing $*$ to be the minimum yields the h function found from usual probabilistic conditioning.

The additional reduction of iterated conditioning axiom 4 is instrumental for further restricting the choice of function h. In particular it enforces the usual conditioning definition for probability measures.

This presentation will show preliminary results obtained on this problem, which for the most part remains open.

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Modalities and many-valued: modelling uncertainty measures and similarity-based reasoning and application to fuzzy description logics

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The talk will begin remembering results on papers [22, 15, 16] where the authors model uncertainty measures as fuzzy modalities on a many-valued residuated logics. the basic idea is that uncertainty measures is not a truth degree but could be interpreted as a truth degree of the fuzzy sentence "the probability of φ". The main properties being that the language does not allow nested modalities and their semantic is not defined as generalized Kripke models in strict sense. We will give the example of probabilities (possibilities, belief functions) over classical propositions and its generalizations to uncertainty measures on many-valued (fuzzy) propositions.

But the main goal of the talk is modal many-valued fuzzy logics defined by Kripke models and some applications. Modal many-valued logics is a topic that has deserved few attention until the nineties. The first known papers are the papers [12, 13] of Fitting where the author defines a modal system over a logic of a finite Heyting algebra and give a complete axiomatization of them. In its late paper Fitting justify it in a very nice an elegant semantics based on a multi-agent system each one using classical logic and having a preference relation. He defined the modal operators based on semantics and his definitions is the ones used in later papers on the topic as generalizations modal operators on many-valued systems based on Kripke semantics (with many-valued worlds and many-valued accessibility relations). In these approaches Modal many-valued language is built taking the language of the many-valued logic $(\wedge, \vee, \&, \rightarrow_{\&}, \neg_{\&}, 0, 1)$ plus at least one the usual modal operators (necessity $\Box$ and possibility $\Diamond$) and its semantic is defined by generalized Kripke models taking many-valued evaluation in each world $w \in W$ and many-valued accessibility relations $S : W \times W \rightarrow [0, 1]$. The evaluation of modal operators are given (following Fitting [12, 13]) by

- $V(\Box\varphi, w) = \bigwedge\{R(w, w') \rightarrow V(\varphi, w') : w' \in W\}$.

– $V(\Diamond\varphi, w) = \bigvee\{R(w, w')\&V(\varphi, w') : w' \in W\}$.

There are different attempts and approaches to motivate and study modal logic formalisms based on Kripke semantics to the many-valued setting. Roughly speaking we can classify the approaches in three groups depending how the corresponding Kripke frames look like, in the sense of how many-valuedness affects the worlds and the accesibility relations. Next we describe these three groups and comment about our work in each of them.

A first group (see e.g. [5, 10, 28]) is formed by those logical systems whose class of Kripke frames are such that their *worlds are classical* (i.e. they follow the rules of classical logic) but their *accessibility relations are many-valued*, with values in some suitable lattice A. In such a case, the usual approach to capture the many-valuedness of an accessibility relation $R : W \times W \to A$ is by considering the induced set of classical accessibility relations $\{R_a \mid a \in A\}$ defined by the different level-cuts of R, i.e. $\langle w, w' \rangle \in R_a$ iff $R(w, w') \geq a$. At the syntactical level, one then introduces as many (classical) necessity operators $\Box_a$ (or possibility operators $\Diamond_a$) as elements a of the lattice A, interpreted by (classical) relations R_a. Therefore, in this kind of approach, one is led to a multi-modal language but where (both modal and non modal) formulas are Boolean in each world.

In this setting we will present as example the similarity-based reasoning developed in [8–10, 6, 11]. The starting point is the paper by Ruspini [27] about a possible semantics for fuzzy set theory. He develops the idea that we could represent a fuzzy concept by its set of prototypical elements (which will have fully membership to the corresponding fuzzy set) together with a similarity relation giving the degree of similarity of each element of the universe to the closest prototype. This degree is taken then as the membership to the fuzzy set. From this basic idea, we will see three graded entailments (approximate, proximity and strong) that can be represented in a multi-modal systems with frames where the (graded) accessibility is given by a fuzzy similarity relation on pairs of worlds, and for which we have proved completeness in several cases [7].

A second group of approaches are the ones whose corresponding Kripke frames have *many-valued worlds*, evaluating propositional variables in a suitable lattice of truth-values A, but with *classical accessibility relations* (see e.g. [23, 20, 24]). In this case, we have languages with only one necessity and/or possibility operator ($\Box, \Diamond$), but whose truth-evaluation rules in the worlds is many-valued, so modal (and non-modal formulas) are many-valued.

Finally, a third group of approaches are *fully many-valued*, in the sense that in their Kripke frames, both worlds and accessibility relations are many-valued, again over a suitable lattice A. In that case, some approaches (like [12, 13, 25, 4]) have a language with a single necessity/possibility operator ($\Box, \Diamond$), and some (like [26, 2]) consider a multi-modal language with a family of indexed operators $\Box_a$ and $\Diamond_a$ for each $a \in A$, interpreted in the Kripke models via the level-cuts R_a of a many-valued accessibility relations R. Actually, these two kinds of approaches are not always equivalent, in the sense that the operator $\Box$ and the set of operators $\{\Box_a \mid a \in A\}$ are not always interdefinable (or analogously with the possibility operators).

In this setting we will present recent results a summary [2, 4] on minimum modal many-valued logic over the logic of a finite residuated lattice and modal Gödel logic respectively.

Next we will sketch what Fuzzy description logic could be following the proposal of Hájek in [18, 19] and developed in [14]. Finally we propose a research proposal which main goal to be the study of n-graded Description Logics (depending of the underlying logic and the expressiveness of the description language we want), a topic for which we have at hand many results: canonical completeness of first order finite-valued residuated logic, modal many-valued results, decidability results and many possible reasoning algorithms.

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Vagueness and degrees of truth: Scenes from a troubled marriage

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Since antiquity the seeming inadequacy of simple logical principles in face of vague predicates has puzzled philosophers and logicians, as witnessed by the well known *sorites paradox* (see, e.g., [20]). My favorite version of the paradox seems to prove that I will stay forever young: Nobody will deny (1) that I was young immediately after birth. Moreover, it seems reasonable to accept (2) that when I am young at time t then I am also young at time $t + \varepsilon$, where ε, say, is a millisecond. But then iterated applications of instantiation and modus ponens (starting with an instance of t immediately after my birth) allow me to conclude that (3) I am young now and will stay young in future.

The standard reaction of proponents of fuzzy logic as a logic for reasoning with vague predicates is to point out that the paradox can be resolved by assuming that the inductive premise (2) of a sorites argument is not absolutely true, but only true to some (high) degree. Modifying also modus ponens accordingly then blocks the derivation of 3 from 1 and 2. However, most philosophers and linguists, but also many logicians that have dealt with the *sorites paradox* and with reasoning under vagueness in general have pointed out that the proposed solution via degrees of truth comes at a very high price. Not only do we have to forsake the clarity and simplicity of classical logic, but, by solving one problem in a particular way, we only seem to have created a host of new problems. To name just a few: How should one understand degrees of truth? How to argue that a certain choice of many-valued truth functions for connectives is adequate? Why should one insist on truth functionality at all? What are the criteria for the adequateness of such a 'fuzzy model', in particular with respect to observed linguistic behavior?

The prolific and lively debate on adequate models of reasoning under vagueness in analytic philosophy has resulted in a number of competing theories that explicitly reject degrees of truth as a basic notion: epistemicism maintains that vagueness is a particular form of ignorance (e.g., [20, 17]); supervaluationism, which comes in various different versions (e.g., [7, 18, 11, 19]), refers to a space of precisifications that has to be taken into account in inferences involving vague predicates; contextualism emphasizes the role of rapidly changing conversational records in determining the acceptability of vague statements (e.g., [15, 14]); pragmatistic theories seek to model reasoning under vagueness by way of language choices (e.g., [1]). Indeed, until recently, no systematic defense of a fuzzy logic based theory of vagueness, that could compete with the rich and thoroughgoing analysis of various problems and concepts offered by the cited philosophers (or by linguistic accounts of vagueness for that matter) has been attempted. However, with the appearance of *Vagueness and Degrees of Truth* by Nicholas J.J. Smith [16] a well presented, densely argued, multi-layered theory of vagueness involving deductive fuzzy logic is now on record.

In our presentation we briefly want to assess the state of the art in the ongoing debate on theories of vagueness in analytic philosophy. In particular, we will review some central aspects of Smith's degree based theory, called 'fuzzy plurivaluationism.' This account of vagueness does indeed refer to t-norm based fuzzy logics, as propagated and developed by Petr Hájek [10] and many colleagues since almost two decades. However, it has to be emphasized that fuzzy plurivaluationism does not amount to a straightforward vindication of mathematical fuzzy logic as a formalism for reasoning under vagueness. Rather Smith convincingly mitigates a number of difficulties with degree based models of vagueness by augmenting fuzzy models with a kind of nondeterministic evaluation referring to whole sets of permissible models, instead of focusing on single intended models. In this manner essential features of supervaluationism — for which Smith prefers to use the term plurivaluationism — are combined with truth functional fuzzy logics.

In the second part of the presentation we want to advertise our preferred approach to interpet linearly ordered 'degrees of truth' and to justify corresponding truth functions. This approach has been initiated by Robin Giles already in the 1970s [9, 8], who in turn refers to Paul Lorenzen's dialogue game based characterization of constructive reasoning [12]. A central feature of Giles's model of reasoning is the separation of (1) the analysis of logical connectives and (2) the interpretation of 'fuzzy' atomic assertions. To this aim the stepwise reduction of logically complex assertions to their atomic components (1) is guided by Lorenzen-style dialogue rules that regulate idealized debates between a proponent and an opponent of an assertion. As for 2, the two players agree to pay a fixed amount of money to the opposing player for each incorrect statement they make. The (in)correctness of stating an atomic sentence p is decided by an elementary (yes/no) experiment E_p associated with p. 'Fuzziness' arises from the stipulation that the experiments may be dispersive, i.e., yield different results upon repetition; only a fixed success probability is known for E_p. Giles demonstrated that an initial statement F can be asserted by the proponent without having to expect a loss of money, independently of the probabilities assigned to the elementary experiments, if and only if F is valid in Łukasiewicz logic $\mathbf{Ł}_\infty$.

We will indicate how Giles's betting *cum* dialogue game scenario can be generalized to characterize also the two other fundamental t-norm based fuzzy logics, namely Product logic **P** and Gödel logic **G** [3, 5]. Moreover, we will point out that winning strategies in dialogue games are systematically related to cut-free proofs in so-called hypersequent systems [6, 3]. Finally, this presents an opportunity to allude also to very recent results that aim at a characterization of the family of logics that can be extracted from particularly simple and transparant variants of Giles's game. This family turns out to include, besides $\mathbf{Ł}_\infty$, Abelian logic **A** [13, 2], cancellative hoop logic **CHL** [4], and all finite valued Łukasiewicz logics $\mathbf{Ł}_n$, but not neither Product logic **P** nor Gödel logic **G**.

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Residuated lattices with applications to logic

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Abstract. Residuated lattices arise naturally in many contexts, including algebra, logic and linguistics [10]. I this talk I will survey some of the basic results about residuated lattices and their applications and outline some recent research directions.

A residuated lattice is a lattice-ordered monoid for which multiplication is residuated on both sides, and is often expanded by an additional constant.

The algebraic example of residuated lattices that initiated their independent study is that of the lattice of (two sided) ideals of a ring with unit, under usual multiplication of ideals. This lattice is actually bounded, with the ring itself serving as both the top and as the multiplicative identity (the residuated lattice is integral), and modular, but it is neither distributive in general, nor is the multiplication commutative.It was in this context that Ward and Dilworth [18] defined what under the more general definition of Blount and Tsinakis [4] we now refer to as integral and commutative residuated lattices. A related example is formed by the submodules over an integral domain of the associated field of fractions.

A much simpler algebraic example is that of the powerset of a monoid, under the element-wise operation on the subsets. Also, a natural example is that of binary relations on a set, or more generally (the reduct of) a relation algebra. Actually, the powerset construction works even if the original structure is only a partial monoid, and if we consider a Brandt groupoid we obtain an example of a relation algebra. In that sense these two examples are related. If we stipulate that the monoid of a residuated lattice is a group then we obtain as examples all lattice-ordered groups [1].

Residuated lattices arise also very naturally in logic, where implication is modeled by the residual operation(s) [11]. As a consequence, Boolean algebras and Heyting algebras, algebraic models of propositional classical and intuitionistic logic, respectively, are examples of residuated lattices, the latter being exactly the ones for which multiplication is the meet operation in the lattice. Algebraic models of other non-classical, including fuzzy, logics are also residuated lattices. Thus, as examples we have MV-algebras [5], related to Łukasiewicz infinitely valued logic, BL-algebras [15], related to Hájek basic logic, models of relevance logic, as well models of certain fragments of linear logic.

Complete residuated lattices are definitionally equivalent to quantales [17] (or to frames/locales if multiplication is the same as meet). The corresponding categories are not, however, the same, as we consider different fundamental operations and thus different (homo)morphisms. Partly in this form, residuated lattices can also be applied for solving the isomorphism problem in abstract algebraic logic, by considering modules over quantales (or quantaloids, to include π-institutions); the residuation operations are however used explicitly in this application. Finally, residuated lattices and certain of their reducts/subvarieties are considered in linguistics in relation to Lambek's calculus.

Even though residuated lattices include a wide and diverse range of examples, they possess interesting properties and are amenable to fruitful mathematical study, which yields results to various applications [4, 10]. These include the investigation of global properties of substructural non-classical logics, including a study of maximally consistent such [7] and the axiomatization of the intersection of two such logics [6]. Also, they provide to tools for analyzing the structure of (generalized) MV and BL-algebras [2, 13, 9].

A relatively recent research direction is the development of relational semantics for substructural logics. These provide representation theorems for residuated lattices and connect directly algebra with proof theory [3]. Residuated frames [12, 8] can be used to recast proof-theoretic ideas in a general setting from which algebraic results can be obtained. Time permitting, I will also mention some recent results about generalizations of relation algebras, viewed as expansions of residuated lattices.

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On Klaua's first approach toward graded identity and graded membership in set theory

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1 Introduction

Without any influence by or relationship to L.A. Zadeh and his seminal paper [6] the German mathematician D. Klaua presented in his papers [2, 3] from 1965 onwards two versions for a cumulative hierarchy of *many-valued* sets.

This paper offers a closer look at the first one of these approaches by Klaua which since its presentation in [2, 4] never has been discussed any more. Our main emphasis shall be (i) on an interesting simultaneous inductive definition of a graded membership and a graded identity relation, and (ii) on some disadvantages which may be the main reason to abandon this approach.

2 Boolean Valued Models for Set Theory

For later reference we have to mention a type of Boolean interpretations for the language of ZF set theory which has been introduced by D. Scott and R. Solovay in 1967 for the purpose of independence proofs in set theory, cf. e.g. [5, 1]. The first-order language of ZF has the predicate symbols $=, \in$.

An inductive definition through the ordinals is used to define for a fixed complete Boolean algebra $\mathbf{B}$ (with carrier B) a sequence of sets $V_\alpha^{\mathbf{B}}$ by

$$V_\alpha^{\mathbf{B}} =_{\text{def}} \left\{ u \in {}^{\mathsf{dom}(u)}B \mid (\exists\, \xi < \alpha)(\mathsf{dom}(u) \subseteq V_\xi^{\mathbf{B}}) \right\} \tag{1}$$

for each ordinal α. And the "union" of all these sets $V_\alpha^{\mathbf{B}}$ becomes the universe of discourse $V^{\mathbf{B}}$ of an B-interpretation $\mathfrak{V}^{\mathbf{B}}$:

$$V^{\mathbf{B}} =_{\text{def}} \{u \mid \exists\, \xi(u \in V_\xi^{\mathbf{B}})\}\,. \tag{2}$$

What remains to be determined to get a B-interpretation $\mathfrak{V}^{\mathbf{B}}$ for ZF set theory completed are the B-valued relations which correspond to the predicate symbols

$=, \in$ of the language of ZF:

$$[\![a \,\hat{\in}\, b]\!] =_{\text{def}} \sup_{u \in \mathsf{dom}(b)} \Big(b(u) * [\![a \,\hat{=}\, u]\!] \Big)\,, \tag{3}$$

$$[\![a \,\hat{=}\, b]\!] =_{\text{def}} \inf_{u \in \mathsf{dom}(a)} \Big(a(u) \rightarrowtail [\![u \,\hat{\in}\, b]\!] \Big) * \inf_{v \in \mathsf{dom}(b)} \Big(b(v) \rightarrowtail [\![v \,\hat{\in}\, a]\!] \Big) \tag{4}$$

for all $a, b \in V^{\mathbf{B}}$.

Here $*$ is the Boolean conjunction, i.e. the meet operation, and $\rightarrowtail$ the Boolean implication function, i.e. relative pseudocomplement.

3 Klaua's first universe of fuzzy sets

For the first one of his hierarchies, D. Klaua [2, 4] made a, not really important, restriction: his hierarchy did have only ω levels.

He started from some infinite (crisp) set U of objects with a *graded* identity relation $\equiv$, i.e. a relation which is reflexive, symmetric and $\&_{\mathsf{L}}$-transitive for the Łukasiewicz arithmetic conjunction $\&_{\mathsf{L}}$ and its associated implication $\to_{\mathsf{L}}$, defined by the Łukasiewicz t-norm $*_{\mathsf{L}}$ and their residuation operation $\rightarrowtail_{\mathsf{L}}$. This means he assumes

$$\models \quad x \equiv x\,, \qquad \models \quad x \equiv y \to_{\mathsf{L}} y \equiv x$$

together with

$$\models \quad x \equiv y \;\&_{\mathsf{L}}\; y \equiv z \to_{\mathsf{L}} x \equiv z\,.$$

Then he forms, with reference to the standard (crisp) power set operation $\mathbb{P}$ the hierarchy

$$V_U^*(0) = U \times \{0\}\,, \quad V_U^*(n+1) = \mathbb{P}(V_U^*(n)) \times \{1\}\,, \quad V_U^* = \bigcup_{n<\omega} V_U^*(n)\,.$$

The members of $E_U = V_U^* \setminus V_U^*(0)$ are Klaua's *many-valued sets*, the members of $V_U^*(0)$ his *urelements*.

It is completely inessential for the construction of this hierarchy V_U^* that the construction proceeds only through the members of ω. This construction could proceed through all the ordinals to yield a more extended hierarchy $\widehat{V_U^*}$ with the same set $V_U^*(0)$ of urelements.

Additionally let us call the many-valued sets from $V_\emptyset$ the *pure* sets.

We prove that there is a canonical embedding $\widehat{V_\emptyset^*} \mapsto \widehat{V_\emptyset^{**}} \subseteq V^{\mathbf{B}}$, of the pure sets of the universes V_U^* into each one of the Boolean valued universes $V^{\mathbf{B}}$.

4 Graded membership and graded identity

For this kind of set theoretic universe V_U^* Klaua introduces a graded, i.e. many-valued *identity* predicate $=_w$ together with a graded *membership* predicate $\in_w$ and a graded inclusion $\subseteqq_w$. By simultaneous induction on the rank ordering of this hierarchy these predicates may be defined as follows:

$$[\![x =_w y]\!] = \begin{cases} [\![\mathsf{pr}_1(x) \equiv \mathsf{pr}_1(y)]\!], & \text{if } x, y \in V_U^*(0) \\ 0\,, & \text{if } x, y \text{ are of different rank} \\ [\![x \subseteqq_w y \wedge y \subseteqq_w x]\!], & \text{if } x, y \in E_U \text{ are of equal rank} \end{cases}$$

with

$$[\![x \in_w y]\!] = \sup_{v \in \mathsf{pr}_1(y)} [\![x =_w v]\!] \qquad \text{for all } y \in E_U \tag{5}$$

and

$$[\![x \subseteqq_w y]\!] = \inf_{u \in \mathsf{pr}_1(x)} [\![u \in_w y]\!] \qquad \text{for all } x, y \in E_U. \tag{6}$$

All these constructions can be done with reference to any MTL-algebra and its operations $\sqcap, \sqcup, *, \rightarrowtail$ in place of $\min, \max, *_{\mathsf{L}}, \rightarrowtail_{\mathsf{L}}$.

Proposition 1. *Over the common part $\widehat{V_\emptyset^{**}} \subseteq V^{\mathbf{B}}$ of all the universes of Boolean valued sets, Klaua's version of graded interpretations $\in_w, =_w$ of membership and equality coincides with the Scott/Solovay version $\hat{\in}, \hat{=}$.*

Historically it is interesting to recognize that the famous simultaneous definition of Boolean valued versions $\hat{\in}, \hat{=}$ of membership and equality, given by D. Scott and R. Solovay in 1967, has been foreshadowed in essential structural details by the Klaua paper [2] in 1965.

5 Properties of this graded identity

Proposition 2. *For arbitrary $x, y, z \in \widehat{V_U^*}$ one has over any MTL-algebra $\mathbf{M}$:*

$$\models \quad z =_w x \,\&\, x \in_w y \rightarrow z \in_w y\,, \tag{7}$$

and over any BL-algebra $\mathbf{M}$:

$$\models \quad x \in_w y \,\&\, y =_w z \rightarrow x \in_w z\,. \tag{8}$$

Theorem 1. *For fuzzy sets $a, b \in E_U$ one has over any MTL-algebra $\mathbf{M}$:*

$$\models \quad a =_w b \;\leftrightarrow\; \forall z (z \in_w a \leftrightarrow z \in_w b)\,, \tag{9}$$

and over any BL-algebra $\mathbf{M}$ also:

$$\models \quad a =_w b \;\leftrightarrow\; \forall z (a \in_w z \leftrightarrow b \in_w z)\,. \tag{10}$$

Over BL-algebras this gives the interesting coincidence

$$\models \quad \forall z(z \in_w a \leftrightarrow z \in_w b) \;\leftrightarrow\; a =_w b \;\leftrightarrow\; \forall z(a \in_w z \leftrightarrow b \in_w z)$$

and thus indicates that this simultaneous definition of $=_w, \in_w$ offers a suitable set theoretic setting.

6 Disadvantages of comprehension

Unfortunately the approch of [2, 4] does not allow to prove suitable generalized versions at least of all the comprehension axioms of ZF.

As already mentioned in [2, 4], it is e.g. impossible to prove the existence of a suitable intersection. Particularly does one have

$$\not\models \quad \exists x \forall z(z \in_w x \leftrightarrow z \in_w a \wedge z \in_w b)\,.$$

The crucial point here is that the universe $\widehat{V_U^*}$ does not have sufficiently many objects: this will be explained in more detail.

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Associativity, commutativity and symmetry in residuated structures

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Residuated structures are among the most frequently considered algebras in many valued mathematics and logics. The main character of these structures consists in a partially ordered set with a pair of operations $\otimes$ and $\rightarrow$ that form an adjunction, according to the fundamental models of conjunction and implication in (many-valued) logics. Sometimes these algebras are approached from the point of view of one of the two terms of the adjunction above mentioned.
The present contribution deals with the approach based on the point of view of implicative algebras (see [2]) renamed weak extended-order algebras (shortly w-eo algebras) and denoted $(L, \rightarrow, \top)$ in [1]; in fact the binary operation $\rightarrow$ is nothing but an extension of a partial order in L with greatest element $\top$, in the sense that for all $a, b \in L$ the following equivalence

$$a \rightarrow b = \top \iff a \leq b$$

holds. These are $(2, 0)$-algebras characterized by the conditions

- $a \rightarrow a = a \rightarrow \top = \top$
- if $a \rightarrow b = b \rightarrow a = \top$ then $a = b$
- if $a \rightarrow b = b \rightarrow c = \top$ then $a \rightarrow c = \top$.

Every integral residuated structure (i.e. having the greatest element $\top$ as an, at least one sided, unit) can be handled by such an approach that has shown to allow important comprehension, discussion and development of the fundamental properties that most frequently are assumed in residuated structures.
Usually the only motivation of the conditions required in residuated structures depends on what is needed in their application fields; on the contrary, extended order algebras introduced and studied in [1] aim to motivate and justify in their own structure the commonly asked requirements such as order-completeness, associativity, commutativity and symmetry.
It is noteworthy that just like every lattice-structure on a set L is completely determined by the underlying order relation $\leq$ in L, it can be seen, since [1], that

the properties and even the existence of an integral residuated structure on the upper bounded poset $(L, \leq, \top)$ are determined by the way the order relation is extended to get an implication in L with true value $\top$.

The fundamental result that every extended-order algebra (w-eo algebra whose implication is antitonic in the first and isotonic in the second argument) can be embedded in its MacNeille completion allows to consider the completeness condition a not too strong assumption, which is very important in most applications. Moreover it is possible to recognize since the first step of extending the order relation of $(L, \leq, \top)$ whether the obtained implication $\rightarrow$ originates a complete residuated structure, in particular a product $\otimes$, and which properties they have. Particular attention is devoted to associativity, commutativity and symmetry (as a good substitute of commutativity) of the product with a critical view of their motivation.
The relevance of these properties toward the algebraic operations corresponding to the fundamental connectives of implication, negation, conjunction and disjunction is discussed. It becomes clear that commutativity is not a fundamental condition, which is a quite established acquirement, while symmetry (which leads to a product distributive over joins in both arguments, having $\top$ as a unit) is much better motivated.
In fact, if one assumes, as it is quite natural, that the implication $\rightarrow$ satisfy left and right distributivity conditions in the first and in the second argument, respectively, then these conditions are preserved by the MacNeille completion and provide an isotonic and an antitonic Galois connection involving $\rightarrow$.
The first one generates the adjoint product-conjunction that is distibutive over joins on the right side and has $\top$ as a right unit. The second one determines a further implication $\rightsquigarrow$ related to $\rightarrow$ by the equivalence

$$a \leq b \rightarrow c \;\Leftrightarrow\; b \leq a \rightsquigarrow c.$$

$(L, \rightarrow, \top)$ is symmetrical if and only if $(L, \rightsquigarrow, \top)$ is a w-eo algebra that induces the same order on L. Equivalently, $(L, \rightarrow, \top)$ is symmetrical if and only if the product is distributive over joins in both arguments and has $\top$ as a unit, which motivates the chosen terminology.

But the main efforts of this work tend to exploit the possibility of dropping the associativity assumption, which is instead a well established requirement in all the approaches to structures related to logical connectives, including residuated lattices, quantales, t-norms and (monoidal) closed categories.
So, from the starting point of the implication of a suitable extended-order algebra we deduce operations corresponding to conjunction, negation and disjunction and discuss their properties, without assuming up to some point commutativity, associativity and even symmetry. Of course, under the assumption of

symmetry, without commutativity, there are different, though related, negations and disjunctions; relevant properties of the latter ones depend more heavily on the involutive character of the negations than on the assumption of associativity.

The effective power of associativity turns out rather to allow the ”strong” version of several properties, provided that those are satisfied in their ”weak” version: examples are given in [1].
As a relevant example, if $(L, \rightarrow, \top)$ is complete with a distributive implication, the adjoint product $\otimes$ and the dual implication $\rightsquigarrow$ can be obtained which are related to $\rightarrow$ by ”weak” conditions expressed by the equivalences

(i) $a \rightarrow (b \rightarrow c) = \top \;\Leftrightarrow\; (b \otimes a) \rightarrow c = \top$
(ii) $a \rightsquigarrow (b \rightarrow c) = \top \;\Leftrightarrow\; b \rightarrow (a \rightsquigarrow c) = \top.$

Then it can be seen that the associativity assumption is equivalent to ask that the ”strong” version of (i) and (ii) are satisfied; more precisely, the following are equivalent

1. $(L, \rightarrow, \top)$ is associative
2. for all $a, b, c \in L$: $a \rightarrow (b \rightarrow c) = (b \otimes a) \rightarrow c$
3. for all $a, b, c \in L$: $a \rightsquigarrow (b \rightarrow c) = b \rightarrow (a \rightsquigarrow c)$.

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Mathematical fuzzy logic and axiomatic arithmetic

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This will be a survey of some attempts to generalize axiomatic systems of the arithmetic of natural number (like Peano and Robinson arithmetic) by investigating them or their analogs in the frame of fuzzy predicate logic. First there are results on adding to classical Peano arithmetic a truth predicate Tr and the schema $\varphi \equiv Tr(\bar{\varphi})$ where $\bar{\varphi}$ is a code (Gödel number) of the formula φ. Classically it is contradictory but over Łukasiewicz logic it is consistent. Second some fuzzy axiom systems for addition and multiplication of natural numbers will be presented that are extremely weak but still essentially undecidable: each consistent axiomatizable extension of such a theory is undecidable (and hence incomplete, thus the famous Gödel's second incompleteness theorem holds for them). No deep knowledge of metamathematics of arithmetic will be assumed to be known.

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Intransitivity and Vagueness

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Abstract. There are many examples in the literature that suggest that indistinguishability is intransitive, despite the fact that the indistinguishability relation is typically taken to be an equivalence relation (and thus transitive). It is shown that if the uncertainty perception and the question of when an agent *reports* that two things are indistinguishable are both carefully modeled, the problems disappear, and indistinguishability can indeed be taken to be an equivalence relation. Moreover, this model also suggests a logic of *vagueness* that seems to solve many of the problems related to vagueness discussed in the philosophical literature. In particular, it is shown here how the logic can handle the *sorites paradox*.

1 Introduction

While it seems that indistinguishability should be an equivalence relation and thus, in particular, transitive, there are many examples in the literature that suggest otherwise. For example, tasters cannot distinguish a cup of coffee with one grain of sugar from one without sugar, nor, more generally, a cup with $n + 1$ grains of sugar from one with n grains of sugar. But they can certainly distinguish a cup with 1,000 grains of sugar from one with no sugar at all.

These intransitivities in indistinguishability lead to intransitivities in preference. For example, consider someone who prefers coffee with a teaspoon of sugar to one with no sugar. Since she cannot distinguish a cup with n grains from a cup with $n + 1$ grains, she is clearly indifferent between them. Yet, if a teaspoon of sugar is 1,000 grains, then she clearly prefers a cup with 1,000 grains to a cup with no sugar.

There is a strong intuition that the indistinguishability relation should be transitive, as should the relation of equivalence on preferences. Indeed, transitivity is implicit in our use of the word "equivalence" to describe the relation on

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preferences. Moreover, it is this intuition that forms the basis of the partitional model for knowledge used in game theory (see, e.g., [1]) and in the distributed systems community [5]. On the other hand, besides the obvious experimental observations, there have been arguments going back to at least Poincaré [18] that the physical world is not transitive in this sense. In this paper, I try to reconcile our intuitions about indistinguishability with the experimental observations, in a way that seems (at least to me) both intuitively appealing and psychologically plausible. I then go on to apply the ideas developed to the problem of *vagueness*.

To understand the vagueness problem, consider the well-known *sorites paradox*: If $n + 1$ grains of sand make a heap, then so do n. But 1,000,000 grains of sand are clearly a heap, and 1 grain of sand does not constitute a heap. Let **Heap** to be a predicate such that **Heap**(n) holds if n grains of sand arranged in a pyramidal shape make a heap. What is the extension of **Heap**? That is, for what subset of natural numbers does **Heap** hold? Is this even well defined? Clearly the set of numbers for which **Heap** holds is upward closed: if n grains of sand is a heap, then surely $n + 1$ grains of sand is a heap. Similarly, the set of grains of sand which are not a heap is downward closed: if n grains of sand is not a heap, then $n-1$ grains of sand is not a heap. However, there is a fuzzy middle ground, which is in part the reason for the paradox. The relationship of the vagueness of **Heap** to indistinguishability should be clear: n grains of sand are indistinguishable from $n + 1$ grains. Indeed, just as **Heap** is a vague predicate, so is the predicate **Sweet**, where **Sweet**(n) holds if a cup of coffee with n grains of sugar is sweet. So it is not surprising that an approach to dealing with intransitivity has something to say about vagueness.

The rest of this paper is organized as follows. In Section 2 I discuss my solution to the intransitivity problem. In Section 3, I show how this solution can be applied to the problem of vagueness. There is a huge literature on the vagueness problem. Perhaps the best-known approach in the AI literature involves fuzzy logic, but fuzzy logic represents only a small part of the picture; the number of recent book-length treatments, including [15, 16, 21, 23], give a sense of the activity in the area. I formalize the intuitions discussed in Section 2 using a logic for reasoning about vague propositions, provide a sound a complete axiomatization for the logic, and show how it can deal with problems like the sorites paradox. I compare my approach to vagueness to some of the leading alternatives in Section 4. Finally, I conclude with some discussion in Section 5.

2 Intransitivity

Clearly part of the explanation for the apparent intransitivity in the sugar example involves differences that are too small to be detected. But this can't be the whole story. To understand the issues, imagine a robot with a simple sensor for sweetness. The robot "drinks" a cup of coffee and measures how sweet it is. Further imagine that the robot's sensor is sensitive only at the 10-grain level. Formally, this means that a cup with 0–9 grains results in a sensor reading of 0, 10–19 grains results in a sensor reading of 1, and so on. If the situation were indeed that simple, then indistinguishability would in fact be an equivalence relation. All cups of coffee with 0–9 grains of sugar would be indistinguishable, as would cups of coffee with 10–19 grains, and so on. However, in this simple setting, a cup of coffee with 9 grains of sugar would be distinguishable from cups with 10 grains.

To recover intransitivity requires two more steps. The first involves dropping the assumption that the number of grains of sugar uniquely determines the reading of the sensor. There are many reasons to drop this assumption. For one thing, the robot's sensor may not be completely reliable; for example, 12 grains of sugar may occasionally lead to a reading of 0; 8 grains may lead to a reading of 1. A second reason is that the reading may depend in part on the robot's state. After drinking three cups of sweet coffee, the robot's perception of sweetness may be dulled somewhat, and a cup with 112 grains of sugar may result in a reading of 10. A third reason may be due to problems in the robot's vision system, so that the robot may "read" 1 when the sensor actually says 2. It is easy to imagine other reasons; the details do not matter here. All that matters is what is done about this indeterminacy. This leads to the second step of my "solution".

To simplify the rest of the discussion, assume that the "indeterminacy" is less than 4 grains of sugar, so that if there are actually n grains of sugar, the sensor reading is between $\lfloor (n-4)/10 \rfloor$ and $\lfloor (n+4)/10 \rfloor$.[1] It follows that two cups of coffee with the same number of grains may result in readings that are not the same, but they will be at most one apart. Moreover, two cups of coffee which differ by one grain of sugar will also result in readings that differ by at most one.

The robot is asked to compare the sweetness of cups, not sensor readings. Thus, we must ask when the robot *reports* two cups of coffee as being of equivalent sweetness. Given the indeterminacy of the reading, it seems reasonable that two cups of sugar that result in a sensor reading that differ by no more than one are reported as indistinguishable, since they could have come from cups of coffee with the same number of grains of sugar. It is immediate that reports

[1] $\lfloor x \rfloor$, the floor of x, is the largest integer less than or equal to x. Thus, for example, $\lfloor 3.2 \rfloor = 3$.

of indistinguishability will be intransitive, even if the sweetness readings themselves clearly determine an equivalence relation. Indeed, if the number of grains in two cups of coffee differs by one, then the two cups will be reported as equivalent. But if the number of grains differs by at least eighteen, then they will be reported as inequivalent.

Of course, I would like to argue that what applies to robots applies to people as well. The "indistinguishability problem" comes from confounding reports of perceptions with the perceptions themselves. Reports of relative sweetness (and, more generally, reports about perceptions) exhibit intransitivity; there are cases when, given three cups of sugar, say a, b, and c, an agent will report that a and b are equivalent in sweetness, as are b and c, but will report that c is sweeter than a. Nevertheless, the underlying "perceived sweetness" relation can be taken to be transitive. But what exactly is "perceived sweetness"? To make sense of this, we must assume that an agent has some internal analogue of a sensor; the perceived sweetness is then the sensor reading. (Of course, the "sensor reading" might well correspond to the firing of certain neurons.) Note that, in general, the perceived sweetness of a cup of coffee will depend on more than just the number of grains of sugar in the cup; it will also depend on the agent's subjective state just before drinking the coffee and perhaps some other factors. Thus, rather than considering a **Sweeter-Than** relation where **Sweeter-Than**(n, n') holds if a cup of coffee with n grains is reported as sweeter than one with n' grains of sugar, we should consider a **Sweeter-Than**$'$ relation, where **Sweeter-Than**$'((c, w), (c', w'))$ holds if cup of coffee c tried by the agent in world w (where the world includes the time, features of the agent's state such as how many cups of coffee she has had recently, and whatever other features are relevant to the agent's perception) is perceived as sweet as cup of coffee c' tried by the agent in world w'. The latter relation is transitive almost by definition; the former relation may not even be well defined. For some pairs (n, n'), an agent may sometimes report a cup of n grains of sugar to be sweeter than one with n', and at other times report a cup with n' grains of sugar to be sweeter than (or indistinguishable from) one with n grains. It is perfectly consistent to have intransitivities in reports of sweetness although there is no intransitivity in actual perceptions.

3 Vagueness

The term "vagueness" has been used somewhat vaguely in the literature. A common interpretation has been to take a term is said to be vague if its use varies both between and within speakers. (According to Williamson [23], this interpretation of vagueness goes back at least to Peirce [17], and was also used by Black

[3] and Hempel [13].) In the language of the previous section, this would make P vague if, for some a, some agents may report $P(a)$ while others may report $\neg P(a)$ and, indeed, the same agent may sometimes report $P(a)$ and sometimes $\neg P(a)$. While this is a consequence of vagueness, it does not seem to quite capture the notion. For example, agents may disagree as a result of one of them making a silly mistake; for similar reasons, an agent may give different answers at different times as a result of having made what he later feels is a silly mistake the first time. We would not want to call a predicate vague in this case.[2] I return to this issue in Section 3. For now, rather than trying to give a precise definition of vagueness, I present a formal logic of vagueness, that allows us to reason about vague and context-sensitive notions, without trying to distinguish them.

3.1 A Modal Logic of Vagueness: Syntax and Semantics

To reason about vagueness, I consider a modal logic $\mathcal{L}_n^{DR}$ with two families of modal operators: $R_1, \ldots, R_n$, where $R_i\varphi$ is interpreted as "agent i reports φ", and $D_1, \ldots, D_n$, where $D_i\varphi$ is interpreted as "according to agent i, φ is definitely the case". For simplicity, I consider only a propositional logic; there are no difficulties extending the syntax and semantics to the first-order case. As the notation makes clear, I allow multiple agents, since some issues regarding vagueness (in particular, the fact that different agents may interpret a vague predicate differently) are best considered in a multi-agent setting.

Start with a (possibly infinite) set of primitive propositions. More complicated formulas are formed by closing off under conjunction, negation, and the modal operators $R_1, \ldots, R_n$ and $D_1, \ldots, D_n$.

A *vagueness structure* M has the form $(W, P_1, \ldots, P_n, \pi_1, \ldots, \pi_n)$, where P_i is a nonempty subset of W for $i = 1, \ldots, n$, and π_i is an interpretation, which associates with each primitive proposition a subset of W. Intuitively, P_i consists of the worlds that agent i initially considers plausible. For those used to thinking probabilistically, the worlds in P_i can be thought of as those that have prior probability greater than ϵ according to agent i, for some fixed $\epsilon \geq 0$.[3] A simple class of models is obtained by taking $P_i = W$ for $i = 1, \ldots, n$; however, as we shall see, in the case of multiple agents, there are advantages to allowing $P_i \neq W$. Turning to the truth assignments π_i, note that it is somewhat

[2] I thank Zoltan Szabo for pointing out this example.

[3] In general, the worlds that an agent considers plausible depends on the agent's subjective state. That is why I have been careful here to say that P_i consists of the worlds that agent i *initially* considers plausible. P_i should be thought of as modeling the agent i's prior beliefs, before learning whatever information led to the agent i to its actual subjective state. It should shortly become clear how the model takes into account the fact that the agent's set of plausible worlds changes according to the agent's subjective state.

nonstandard in modal logic to have a different truth assignment for each agent; this different truth assignment is intended to capture the intuition that the truth of formulas like **Sweet** is, to some extent, dependent on the agent, and not just on objective features of the world.

I assume that $W \subseteq O \times S_1 \times \ldots S_n$, where O is a set of objective states, and S_i is a set of subjective states for agent i. Thus, worlds have the form $(o, s_1, \ldots, s_n)$. Agent i's subjective state s_i represents i's perception of the world and everything else about the agent's makeup that determines the agent's report. For example, in the case of the robot with a sensor, o could be the actual number of grains of sugar in a cup of coffee and s_i could be the reading on the robot's sensor. Similarly, if the formula in question was **Thin(TW)** ("Tim Williamson is thin", a formula often considered in [23]), then o could represent the actual dimensions of TW, and s_i could represent the agent's perceptions. Note that s_i could also include information about other features of the situation, such as the relevant reference group. (Notions of thinness are clearly somewhat culture dependent and change over time; what counts as thin might be very different if TW is a sumo wrestler.) In addition, s_i could include the agent's cutoff points for deciding what counts as thin, or what counts as red. In the case of the robot discussed in Section 2, the subjective state could include its rule for deciding when to report something as sweet.[4]

If p is a primitive proposition then, intuitively, $(o, s_1, \ldots, s_n) \in \pi_i(p)$ if i would consider p true if i knew exactly what the objective situation was (i.e., if i knew o), given i's possibly subjective judgment of what counts as "p-ness". Given this intuition, it should be clear that all that should matter in this evaluation is the objective part of the world, o, and (possibly) agent i's subjective state, s_i. In the case of the robot, whether $(o, s_1, \ldots, s_n) \in \pi_i(\textbf{Sweet})$ clearly depends on how many grains of sugar are in the cup of coffee, and may also depend on the robot's perception of sweetness and its cutoff points for sweetness, but does not depend on other robots' perceptions of sweetness. Note that the robot may give different answers in two different subjective states, even if the objective state is the same and the robot knows the objective state, since both its perceptions of sweetness and its cutoff point for sweetness may be different in the two subjective states.

I write $w \sim_i w'$ if w and w' agree on agent i's subjective state, and I write $w \sim_o w'$ if w and w' agree on the objective part of the state. Intuitively, the $\sim_i$ relation can be viewed as describing the worlds that agent i considers possible.

[4] This partition of the world into objective state and subjective states is based on the "runs and systems" framework introduced in [11] (see [5] for motivation and discussion). The framework has been used to analyze problems ranging from distributed computing [5] to game theory [9] to belief revision [7]. More recently, it has been applied to the Sleeping Beauty problem [10].

Put another way, if $w \sim_i w'$, then i cannot distinguish w from w', given his current information. Note that the indistinguishability relation is transitive (indeed, it is an equivalence relation), in keeping with the discussion in Section 2. I assume that π_i depends only on the objective part of the state and i's subjective state, so that if $w \in \pi_i(p)$ for a primitive proposition p, and $w \sim_i w'$ and $w \sim_o w'$, then $w' \in \pi_i(p)$. Note that j's state (for $j \neq i$) has no effect on i's determination of the truth of p. There may be some primitive propositions whose truth depends only on the objective part of the state (for example, **Crowd**(n), which holds if there are at least n people in a stadium at a given time, is such a proposition). If p is such an objective proposition, then $\pi_i(p) = \pi_j(p)$ for all agents i and j, and, if $w \sim_o w'$, then $w \in \pi_i(p)$ iff $w' \in \pi_i(p)$.

I next define what it means for a formula to be true. The truth of formulas is relative to both the agent and the world. I write $(M, w, i) \models \varphi$ if φ is true according to agent i in world w. In the case of a primitive proposition p,

$$(M, w, i) \models p \quad \text{iff} \quad w \in \pi_i(p).$$

I define $\models$ for other formulas by induction. For conjunction and negation, the definitions are standard:

$$\begin{array}{l} (M, w, i) \models \neg\varphi \quad \text{iff} \quad (M, w, i) \not\models \varphi; \\ (M, w, i) \models \varphi \wedge \psi \quad \text{iff} \quad (M, w, i) \models \varphi \text{ and } (M, w, i) \models \psi. \end{array}$$

In the semantics for negation, I have implicitly assumed that, given the objective situation and agent i's subjective state, agent i is prepared to say, for every primitive proposition p, whether or not p holds. Thus, if $w \notin \pi_i(p)$, so that agent i would not consider p true given i's subjective state in w if i knew the objective situation at w, then I am assuming that i would consider $\neg p$ true in this world. This assumption is being made mainly for ease of exposition. It would be easy to modify the approach to allow agent i to say (given the objective state and i's subjective state), either "p holds", "p does not hold", or "I am not prepared to say whether p holds or p does not hold".[5] However, what I am explicitly avoiding here is taking a fuzzy-logic like approach of saying something like "p is true to degree .3". While the notion of degree of truth is certainly intuitively appealing, it has other problems. The most obvious in this context is where the .3 is coming from. Even if p is vague, the notion "p is true to degree .3" is precise. It is not clear that introducing a continuum of precise propositions

[5] The resulting logic would still be two-valued; the primitive proposition p would be replaced by a family of three primitive propositions, p_y, p_n, and $p_?$, corresponding to "p holds", "p does not hold", and "I am not prepared to say whether p holds or does not hold", with a semantic requirement (which becomes an axiom in the complete axiomatization) stipulating that exactly one proposition in each such family holds at each world.

to replace the vague proposition p really solves the problem of vagueness. Having said that, there is a natural connection between the approach I am about to present and fuzzy logic; see Section 4.2.

Next, I consider the semantics for the modal operators R_j, $j = 1, \ldots, n$. Recall that $R_j\varphi$ is interpreted as "agent j reports φ". Formally, I take $R_j\varphi$ to be true if φ is true at all plausible states j considers possible. Thus, taking $\mathcal{R}_j(w) = \{w' : w \sim_j w'\}$,

$$(M, w, i) \models R_j\varphi \quad \text{iff} \quad (M, w', j) \models \varphi \text{ for all } w' \in \mathcal{R}_j(w) \cap P_j.$$

The use of P_j allows reports to be mistaken. That is, we may have $(M, w, i) \models \neg\varphi \wedge R_j\varphi$ if $w \notin P_j$.

Note that, in evaluating $R_j\varphi$ from i's point of view at world w, we evaluate the truth of φ *according to* j at all worlds w' that j considers possible at w (i.e., those worlds $w' \in \mathcal{R}_j(w) \cap P_j$). Thus, the truth of $R_j\varphi$ at world w is independent of i; all agents agree on the truth value of $R_j\varphi$ at w. This may seem a little strange at first, since it implicitly assumes that all agents "know" the worlds w' that j considers possible at w and j's interpretation of φ_j at w'. But this is a standard concern in all multi-agent logics of knowledge and belief, and is dealt with the same way in all of them: i's uncertainty about j's interpretation or about the worlds that j considers possible is modeled by having other worlds w' that i considers possible at w where the worlds that j considers possible and/or j's interpretation is different from w.

Of course, for a particular formula φ, an agent may neither report φ nor $\neg\varphi$. An agent may not be willing to say either that TW is thin or that TW is not thin. Note that, effectively, the set of plausible states according to agent j given the agent's subjective state in world w can be viewed as the worlds in P_j that are indistinguishable to agent j from w. Essentially, the agent j is updating the worlds that she initially considers plausible by intersecting them with the worlds she considers possible, given her subjective state at world w.

Note that, in general, agents can give conflicting reports; that is, a formula such as $R_i p \wedge R_j \neg p$ is consistent. This can happen, for example, if P_i and P_j are disjoint, or if $\pi_i(p)$ is disjoint from $\pi_j(p)$. However, if agents i and j both consider all worlds possible and agree on their interpretation of all primitive propositions, then they cannot give conflicting reports.

Finally, φ is definitely true at state w if the truth of φ is determined by the objective state at w:

$$(M, w, i) \models D_j\varphi \quad \text{iff} \quad (M, w', j) \models \varphi \text{ for all } w' \text{ such that } w \sim_o w'.$$

A formula is said to be *agent-independent* if its truth is independent of the agent. That is, φ is agent-independent if, for all worlds w,

$$(M, w, i) \models \varphi \quad \text{iff} \quad (M, w, j) \models \varphi.$$

As we observed earlier, objective primitive propositions (whose truth depends only on the objective part of a world) are agent-independent; it is easy to see that formulas of the form $D_j\varphi$ and $R_j\varphi$ are as well. If φ is agent-independent, then I often write $(M, w) \models \varphi$ rather than $(M, w, i) \models \varphi$.

3.2 A Modal Logic of Vagueness: Axiomatization and Complexity

It is easy to see that R_j satisfies the axioms and rules of the modal logic KD45.[6] It is also easy to see that D_j satisfies the axioms of KD45. It would seem that, in fact, D_j should satisfy the axioms of S5, since its semantics is determined by $\sim_j$, which is an equivalence relation. This is not quite true. The problem is with the so-called *truth axiom* of S5, which, in this context, would say that anything that is definitely true according to agent j is true. This would be true if there were only one agent, but is not true with many agents, because of the different π_i operators.

To see the problem, suppose that p is a primitive proposition. It is easy to see that $(M, w, i) \models D_i p \Rightarrow p$ for all worlds w. However, it is not necessarily the case that $(M, w, i) \models D_j p \Rightarrow p$ if $i \neq j$. Just because, according to agent i, p is definitely true according to agent j, it does not follow that p is true *according to agent* i. What is true in general is that $D_j\varphi \Rightarrow \varphi$ is valid for *agent-independent* formulas. Unfortunately, agent independence is a semantic property. To capture this observation as an axiom, we need a syntactic condition sufficient to ensure that a formula is necessarily agent independent. I observed earlier that formulas of the form $R_j\varphi$ and $D_j\varphi$ are agent-independent. It is immediate that Boolean combination of such formulas are also agent-independent. Say that a formula is *necessarily agent-independent* if it is a Boolean combination of formulas of the form $R_j\varphi$ and $D_{j'}\varphi'$ (where the agents in the subscripts may be the same or different). Thus, for example, $(\neg R_1 D_2 p \wedge D_1 p) \vee R_2 p$ is necessarily agent-independent. Clearly, whether a formula is necessarily agent-independent depends only on the syntactic form of the formula. Moreover, $D_j\varphi \Rightarrow \varphi$ is valid for formulas that are necessarily agent-independent. However, this axiom does not capture the fact that $(M, w, i) \models D_i\varphi \Rightarrow \varphi$ for all worlds w. Indeed, this

[6] For modal logicians, perhaps the easiest way to see this is to observe a relation $\mathcal{R}_j$ on worlds can be defined consisting of all pairs (w, w') such that $w \sim_j w'$ and $w' \in P_j$. This relation, which characterizes the modal operator R_j, is easily seen to be Euclidean and transitive, and thus determines a modal operator satisfying the axioms of KD45.

fact is not directly expressible in the logic, but something somewhat similar is. For arbitrary formulas $\varphi_1, \ldots, \varphi_n$, note that at least one of $D_i\varphi_1 \Rightarrow \varphi_1, \ldots,$ $D_n\varphi_n \Rightarrow \varphi_n$ must be true respect to each triple (M, w, i), $i = 1, \ldots, n$. Thus, the formula $(D_1\varphi_1 \Rightarrow \varphi_1) \vee \ldots \vee (D_n\varphi_n \Rightarrow \varphi_n)$ is valid. This additional property turns out to be exactly what is needed to provide a complete axiomatization.

Let AX be the axiom system that consists of the following axioms Taut, R1–R4, and D1–D6, and rules of inference $\mathrm{Nec}_{\mathrm{R}}$, $\mathrm{Nec}_{\mathrm{D}}$, and MP:

Taut. All instances of propositional tautologies.
R1. $R_j(\varphi \Rightarrow \psi) \Rightarrow (R_j\varphi \Rightarrow R_j\psi)$.
R2. $R_j\varphi \Rightarrow R_jR_j\varphi$.
R3. $\neg R_j\varphi \Rightarrow R_j\neg R_j\varphi$.
R4. $\neg R_j(\mathit{false})$.
D1. $D_j(\varphi \Rightarrow \psi) \Rightarrow (D_j\varphi \Rightarrow D_j\psi)$.
D2. $D_j\varphi \Rightarrow D_jD_j\varphi$.
D3. $\neg D_j\varphi \Rightarrow D_j\neg D_j\varphi$.
D4. $\neg D_j(\mathit{false})$.
D5. $D_j\varphi \Rightarrow \varphi$ if φ is necessarily agent-independent.
D6. $(D_1\varphi_1 \Rightarrow \varphi_1) \vee \ldots \vee (D_n\varphi_n \Rightarrow \varphi_n)$.
Nec$_{\mathrm{R}}$**.** From φ infer $R_j\varphi$.
Nec$_{\mathrm{D}}$**.** From φ infer $D_j\varphi$.
MP. From φ and $\varphi \Rightarrow \psi$ infer ψ.

Using standard techniques of modal logic, it is can be shown that AX characterizes $\mathcal{L}_n^{DR}$.

Theorem 1. *AX is a sound and complete axiomatization with respect to vagueness structures for the language* $\mathcal{L}_n^{DR}$.

This shows that the semantics that I have given implicitly assumes that agents have perfect introspection and are logically omniscient. Introspection and logical omniscience are both strong requirements. There are standard techniques in modal logic that make it possible to give semantics to R_j that is appropriate for non-introspective agents. With more effort, it is also possible to avoid logical omniscience. (See, for example, the discussion of logical omniscience in [5].) In any case, very little of my treatment of vagueness depends on these properties of R_j.

The complexity of the validity and satisfiability problem for the $\mathcal{L}_n^{DR}$ can also be determined using standard techniques.

Theorem 2. *For all* $n \geq 1$*, determining the problem of determining the validity (or satisfiability) of formulas in* $\mathcal{L}_n^{DR}$ *is PSPACE-complete.*

Proof: The validity and satisfiability problems for KD45 and S5 in the case of two or more agents is known to be PSPACE-complete [12]. The modal operators R_j and D_j act essentially like KD45 and S5 operators, respectively. Thus, even if there is only one agent, there are two modal operators, and a straightforward modification of the lower bound argument in [12] gives the PSPACE lower bound. The techniques of [12] also give the upper bound, for any number of agents. ∎

3.3 Capturing Vagueness and the Sorites Paradox

Although I have described this logic as one for capturing features of vagueness, the question still remains as to what it means to say that a proposition φ is vague. I suggested earlier that a common view has been to take φ to be vague if, in some situations, some agents report φ while others report $\neg\varphi$, or if the same agent may sometimes report φ and sometimes report $\neg\varphi$ in the same situation. Both intuitions can be captured in the logic. As we have seen, it is perfectly consistent that $(M, w) \models R_i\varphi \wedge R_j\neg\varphi$ if $i \neq j$; that is, the logic makes it easy to express that two agents may report different things regarding φ. Expressing the second intuition requires a little more care; it is certainly not consistent to have $(M, w) \models R_j\varphi \wedge R_j\neg\varphi$. However, a more reasonable interpretation of the second intuition is to say that in the same *objective* situation, an agent i may both report φ and report $\neg\varphi$. It is consistent that there are two worlds w and w' such that $w \sim_o w'$, $(M, w) \models R_j\varphi$, and $(M, w') \models R_j\neg\varphi$. In the case of one agent, under this interpretation, φ is taken to be vague if $(M, w) \models \neg D_j\neg R_j\varphi \wedge \neg D_j\neg R_j\neg\varphi$. It is easy to show that, as a consequence, $(M, w) \models \neg D_j R_j\varphi$. This statement just says that the objective world does not determine an agent's report. In particular, a formula such as $\varphi \wedge \neg D_j R_j\varphi$ is consistent; if φ is true then an agent will not necessarily report it as true. This can be viewed as one of the hallmarks of vagueness. I return to this point in Section 4.5.

While I take the consistency of formulas such as $R_i\varphi \wedge R_j\neg\varphi$ and $\varphi \wedge \neg D_j R_j\varphi$ to be a characteristic feature of a vague predicate φ, I do not view this as the definition of vagueness. For example, if φ is the statement "there are 25 children playing in the room", then an agent j may not notice all 25, and hence not report there are 25 children in the room. Moreover, if agent i observes all 25 children, and thus reports that there are 25 children, agent i and agent j's reports differ. Hence neither $R_j\varphi$ nor $D_j R_j\varphi$ may hold, although "there are 25 children in the room" would not typically be taken to be vague. Similarly, if ψ is a context-sensitive statement such as "TW is the leftmost person in the lineup", then an agent i might report φ to be true in some states although not in others, although ψ is not at all vague.

Having borderline cases has often been taken to be a defining characteristic of vague predicates. Since I am considering a two-valued logic, propositions do not have borderline cases: at every world, either φ is true or it is false. However, it is not the case that φ is either *definitely* true or false. That is, there are borderline cases between $D\varphi$ and $D\neg\varphi$. But the fact that neither $D\varphi$ and $D\neg\varphi$ holds cannot be taken to be a definition of vagueness either; an agent may be uncertain about the number of children in a room (and thus not be prepared to say that it is definitely 25 or definitely not 25), even though the statement "there are 25 children in a room" is not vague.

I believe that perhaps the best characterization of vagueness is that vague predicates satisfy sorites-like paradoxes. Very roughly speaking, a unary predicate P is vague if there exist N domain elements $d_1, \ldots, d_N$, all of which differ slightly in some dimension relevant to P, such that

1. there is common agreement that $P(d_1)$;
2. there is common agreement that $\neg P(d_N)$;
3. there is common agreement that if $P(d_j)$, then $P(d_{j'})$ for $j' < j$; and
4. there is common agreement that if $\neg P(d_j)$, then $\neg P(d_{j'})$ for $j' > j$.

We may also want to add a fifth condition, which is meant to capture the intuition of "borderline cases":

5. For some intermediate domain elements d in the sequence (that is, for some domain elements d_j with $1 < j < N$), an agent finds it difficult to categorize d_j as satisfying P or $\neg P$.

These conditions are indeed very rough. For example, to make them precise, one would have to make clear what it means for a dimension to be "relevant". But even ignoring that, there are some subtleties involved in these statements, subtleties that the logic I have introduced can help clarify. What does it mean that there is "common agreement that $P(d_1)$"? It seems reasonable to say that this means, in a state that includes domain element d_1, all agents would report $P(d_1)$. With only one agent in the picture, we can get an analogue to this statement: in all states that include d_1, the agent would report $P(d_1)$. That is, we would expect $D_i R_i P(d_1)$ to hold for all agents i in all models that include d_1 where P is given the intended interpretation. (Note that these statements all make sense even if there is no objective truth to the statement $P(d)$ for any domain element d.) The second statement can be expressed in the logic in a similar way. Perhaps the most reasonable interpretation of the third statement is that if an agent i would report $P(d_j)$ in a particular situation, then he would also report $P(d_{j'})$ for $j' > j$ in the same situation; similarly for the fourth statement. If we take the difficulty of categorizing o as meaning that in some circumstances the agent i reports $P(o)$ and in some circumstances he reports $\neg P(o)$, then the fifth statement becomes $\neg D_i R_i P(o) \wedge \neg D_i R_i \neg P(o)$.

Although this rough definition applies only to unary predicates, it should be clear that it can be modified to deal with predicates of arbitrary arity. The definition presumes a reasonably large number of domain elements. I do not believe that vagueness is an issue if there are only three domain elements. On the other hand, I interpret "domain element" somewhat liberally here. For example, suppose that I have a car in my driveway, and I keep chipping pieces away from it until eventually (after a large but finite number of chips) it becomes a pile of metal shards. Initially it is a car; at the end, it is not. I would be comfortable taking the domain here to include a different element denoting the car after n chips, for the various values of n. We can then consider whether the "Car" predicate applies to each one.[7]

With this background, let us now see how the framework can deal with the sorites paradox. The sorites paradox is typically formalized as follows:

1. **Heap**(1,000,000).
2. $\forall n > 1($**Heap**$(n) \Rightarrow$ **Heap**$(n-1))$.
3. $\neg$**Heap**(1).

It is hard to argue with statements 1 and 3, so the obvious place to look for a problem is in statement 2, the inductive step. And, indeed, most authors have, for various reasons, rejected this step (see, for example, [4, 21, 23] for typical discussions). As I suggested in the introduction, it appears that rejecting the inductive step requires committing to the existence of an n such that n grains of sand is a heap and $n-1$ is not. While I too reject the inductive step, it does *not* follow that there is such an n in the framework I have introduced here, because I do not assume an objective notion of heap (whose extension is the set of natural numbers n such that n grains of sands form a heap). What constitutes a heap in my framework depends not only on the objective aspects of the world (i.e., the number of grains of sand), but also on the agent and her subjective state.

To be somewhat more formal, assume for simplicity that there is only one agent. Consider models where the objective part of the world includes the number of grains of sand in a particular pile of sand being observed by the agent, and the agent's subjective state includes how many times the agent has been asked whether a particular pile of sand constitutes a heap. What I have in mind here is that sand is repeatedly added to or removed from the pile, and each time this is done, the agent is asked "Is this a heap?". Of course, the objective part of the world may also include the shape of the pile and the lighting conditions, while the agent's subjective state may include things like the agent's sense perception

[7] I thank Zoltan Szabo for pointing out this example.

of the pile under some suitable representation. Exactly what is included in the objective and subjective parts of the world do not matter for this analysis.

In this setup, rather than being interested in whether a pile of n grains of sand constitutes a heap, we are interested in the question of whether, when viewing a pile of n grains of sand, the agent would report that it is a heap. That is, we are interested in the formula **Pile**$(n) \Rightarrow R($**Heap**$)$, which I hereafter abbreviate as $S(n)$. The formula **Pile**(n) is true at a world w if, according to the objective component of w, there are in fact n grains of sand in the pile. Note that **Pile** is not a vague predicate at all, but an objective statement about the number of grains of sand present.[8] By way of contrast, the truth of **Heap** at world w depends on both the objective situation in w (how many grains of sand there actually are) and the agent's subjective state in w.

There is no harm in restricting to models where $S(1,000,000)$ holds in all worlds and $S(1)$ is false in all worlds where the pile actually does consist of one grain of sand. If there are actually 1,000,000 grains of sand in the pile, then the agent's subjective state is surely such that she would report that there is a heap; and if there is actually only one grain of sand, then the agent would surely report that there is not a heap. We would get the paradox if the inductive step, $\forall n > 1(S(n) \Rightarrow S(n-1))$, holds in all worlds. However, it does not, for reasons that have nothing to do with vagueness. Note that in each world, $\mathbf{Pile}(n)$ holds for exactly one value of n. Consider a world w where there is 1 grain of sand in the pile and take $n = 2$. Then $S(2)$ holds vacuously (because its antecedent **Pile**(2) is false), while $S(1)$ is false, since in a world with 1 grain of sand, by assumption, the agent reports that there is not a heap.

The problem here is that the inductive statement $\forall n > 1(S(n) \Rightarrow S(n-1))$ does not correctly capture the intended inductive argument. Really what we mean is more like "if there are n grains of sand and the agent reports a heap, then when one grain of sand is removed, the agent will still report a heap".

Note that removing a grain of sand changes both the objective and subjective components of the world. It changes the objective component because there is one less grain of sand; it changes the subjective component even if the agent's sense impression of the pile remains the same, because the agent has been asked one more question regarding piles of sand. The change in the agent's subjective state may not be uniquely determined, since the agent's perception of a pile of $n-1$ grains of sand is not necessarily always the same. But even if it is uniquely determined, the rest of my analysis holds. In any case, given that the world changes, a reasonable reinterpretation of the inductive statement might be "For all worlds w, if there are n grains of sand in the pile in w, and the

[8] While I am not assuming that the agent knows the number of grains of sand present, it would actually not affect my analysis at all if the agent was told the exact number.

agent reports that there is a heap in w, then the agent would report that there is a heap in all the worlds that may result after removing one grain of sand." This reinterpretation of the inductive hypothesis cannot be expressed in the logic, but the logic could easily be extended with dynamic-logic like operators so as to be able to express it, using a formula such as

$$\textbf{Pile}(n) \wedge R(\textbf{Heap}) \Rightarrow [\text{remove 1 grain}](\textbf{Pile}(n-1) \wedge R(\textbf{Heap}).$$

Indeed, with this way of expressing the inductive step, there is no need to include **Pile**(n) or **Pile**$(n-1)$ in the formula; it suffices to write $R(\textbf{Heap}) \Rightarrow [\text{remove 1 grain}]R(\textbf{Heap})$.

Is this revised inductive step valid? Again, it is not hard to see that it is not. Consider a world where there is a pile of 1,000,000 grains of sand, and the agent is asked for the first time whether this is a heap. By assumption, the agent reports that it is. As more and more grains of sand are removed, at some point the agent (assuming that she has the patience to stick around for all the questions) is bound to say that it is no longer a heap.[9]

Graff [8] points out that a solution to the sorites paradox that denies the truth of the inductive step must deal with three problems:

- The semantic question: If the inductive step is not true, is its negation true? If so, then is there a sharp boundary where the inductive step fails? If not, then what revision of classical logic must be made to accommodate this fact?
- The epistemological question: If the inductive step is not true, why are we unable to say which one of its instances is not true?
- The psychological question: If the inductive step is not true, then why are we so inclined to accept it?

I claim that the solution I have presented here handles the first two problems easily, and suggests a plausible solution for the third. For the semantic question, as I have observed, although the inductive argument fails, there is no fixed n at which it fails. The n at which it fails may depend on the person and (even in the case that there is only one person in the picture), may depend on the state of that person. The answer that someone gives to the question the first time it is asked may be different from the answer given the kth time it is asked, even if all objective features of the world remain the same. The logic has this feature

[9] There may well be an in-between period where the agent is uncomfortable about having to decide whether the pile is a heap. As I observed earlier, the semantics implicitly assumes that the agent is willing to answer all questions with a "Yes" or "No", but it is easy to modify things so as to allow "I'm not prepared to say". The problem of vagueness still remains: At what point does the agent first start to say "I'm not prepared to say"?

despite being two-valued (although it extends classical logic both by allowing modal operators and allowing the truth of a formula to depend on the agent).

The answer to the epistemological question is essentially the same as that for the semantic question. We cannot say at which n the induction fails because there is no fixed n at which it fails. The n depends on features on the subjective state of the person being asked (for example, how many she has been asked before). Note that this claim that can be confirmed easily experimentally. We can ask different people a series of questions and see when their answer change from "heap" to "not heap". We can also ask the same person such a series of questions, with different starting points (so that different numbers of questions have been asked at the point when, say, a pile of 10,000 grains is reached). Clearly, the change will not always come at the same value of n in all these cases.

A convincing answer to the psychological question requires a deeper understanding of how people answer questions involving universal quantification. One possible answer may be that if a statement of the form $\forall x \varphi(x)$ is true for "almost all" instances of x, then people are inclined to accept $\forall x \varphi(x)$. To test this would require making precise what "almost all" means. But even if this could be made precise, it seems to me that this is not quite how people deal with universals. For example, suppose we are interested not in whether there is a heap, but whether there is at least one grain of sand. Consider the statement "For all worlds w, if there is more than one grain of sand in the pile in w, then there is more than one grain of sand after removing one grain of sand." I do not think that people would be inclined to accept this statement. If we are interested in worlds where there can be up to 1,000,000 grains of sand, the statement is certainly true for almost all of them. Nevertheless, it would be rejected because it is so easy to think of a counterexample.

Thus, it seems that for someone to accept a statement of the form $\forall x \varphi(x)$, it does not suffice that there exist very few counterexamples. It must be difficult to think of counterexamples. To the extent that this is true, the question is then why people find it hard to think of counterexamples to the statement "For all worlds w, if there are n grains of sand in the pile in w, and the agent reports that there is a heap in w, then the agent would report that there is a heap in all the worlds that may result after removing one grain of sand." Note that the quantification here is over worlds, not over n. Part of the problem is that it is hard to enumerate the worlds systematically, since a world includes both the objective state and the agent's subjective state. (Note that, although I focused on the case where the agent's subjective state consisted only of the number of times the question has been asked, it is far from clear that the agent would make this restriction when asked the question.) I conjecture that, when looking for counterexamples, people implicitly consider only worlds where they are asked the question the first time. I admit that this is only a conjecture, but it does not seem so implausible. After all, in practice, people are not asked a series of sorites questions. They are typically asked only once. Moreover, it does not immediately leap to mind that the response might depend on how many times the question has been asked. It would be interesting to actually test what situations people consider focus on when trying to answer the universal. In any case, if this conjecture is true, my solution to the psychological question rests on another assumption that should be easy to test, and is one I alluded to earlier: whatever people answer the first time they are asked the question, they will continue to give the same answer after one grain of sand is removed. People rarely change their mind between the first and second question in a sorites series.

Unlike the answers to the semantic and epistemological questions, which are essentially matters of logic, the answer to the psychological question is one that requires psychological experiments to verify. But I claim that this is as it should be.

4 Relations to Other Approaches

In this section I consider how the approach to vagueness sketched in the previous section is related to other approaches to vagueness that have been discussed in the literature. As I said earlier, there is a huge literature on the vagueness problem, so I focus here on approaches that are somewhat in the same spirit as mine.

4.1 Context-Dependent Approaches

My approach for dealing with the sorites paradox is perhaps closest to what Graff [8] has called *context-dependent* approaches, where the truth of a vague predicate depends on context. The "context" in my approach can be viewed as a combination of the objective state and the agent's subjective state. Although a number of papers have been written on this approach (see, for example, [8, 14, 20]), perhaps the closest in spirit to mine is that of Raffman [19].

In discussing sorites-like paradoxes, Raffmman considers a sequence of colors going gradually from red to orange, and assumes that to deal with questions like "if patch n is red, then so is patch $n-1$", the agent makes pairwise judgments. She observes that it seems reasonable that an agent will always place patches n and $n+1$, judged at the same time, in same category (both red, say, or both orange). However, it is plausible that patch n will be assigned different colors when paired with $n-1$ than when paired with $n+1$. This observation (which I agree is likely to be true) is easily accommodated in the framework that I have presented here: If the agent's subjective state includes the perception of two adjacent color patches, and she is asked to assign both a color, then she will almost surely assign both the same color. Raffman also observes that the color judgment may depend on the colors that have already been seen as well as other random features (for example, how tired/bored the agent is), although she does not consider the specific approach to the sorites paradox that I do (i.e., the interpretation of the inductive step of the paradox as "if, *the first time I am asked*, I report that $P(n)$ holds, then I will also report that $P(n-1)$ holds if asked immediately afterwards").

However, none of the context-dependent approaches use a model that explicitly distinguishes the objective features of the world from the subjective features of a world. Thus, they cannot deal with the interplay of the "definitely" and "reports that" operators, which plays a significant role in my approach. By and large, they also seem to ignore issues of higher-order vagueness, which are well dealt with by this interplay (see Section 4.4).

4.2 Fuzzy Logic

Fuzzy logic [24] seems like a natural approach to dealing with vagueness, since it does not require a predicate be necessarily true or false; rather, it can be true to a certain degree. As I suggested earlier, this does not immediately resolve the problem of vagueness, since a statement like "this cup of coffee is sweet to degree .8" is itself a crisp statement, when the intuition suggests it should also be vague.

Although I have based my approach on a two-valued logic, there is a rather natural connection between my approach and fuzzy logic. We can take the degree of truth of a formula φ in world w to be the fraction of agents i such that $(M, w, i) \models \varphi$. We expect that, in most worlds, the degree of truth of a formula will be close to either 0 or 1. We can have meaningful communication precisely because there is a large degree of agreement in how agents interpret subjective notions thinness, tallness, sweetness.

Note that the degree of truth of φ in $(o, s_1, \ldots, s_n)$ does not depend just on o, since $s_1, \ldots, s_n$ are not deterministic functions of o. But if we assume that each objective situation o determines a probability distribution on tuples $(s_1, \ldots, s_n)$ then, if n is large, for many predicates of interest (e.g., **Thin**, **Sweet**, **Tall**), I expect that, as an empirical matter, the distribution will be normally distributed with a very small variance. In this case, the degree of truth of such a predicate P in an objective situation o can be taken to be the expected degree of truth of P, taken over all worlds $(o, s_1, \ldots, s_n)$ whose first component is o.

This discussion shows that my approach to vagueness is compatible with assigning a degree of truth in the interval $[0, 1]$ to vague propositions, as is done in fuzzy logic. Moreover non-vague propositions (called *crisp* in the fuzzy logic literature) get degree of truth either 0 or 1. However, while this is a way of giving a natural interpretation to degrees of truth, and it supports the degree of truth of $\neg\varphi$ being 1 minus the degree of truth of φ, as is done in fuzzy logic, it does not support the semantics for $\wedge$ typically taken in fuzzy logic, where the degree of truth of $\varphi \wedge \psi$ is taken to be the minimum of the degree of truth of φ and the degree of truth of ψ. Indeed, under my interpretation of degree of truth, there is no functional connection between the degree of truth of φ, ψ, and $\varphi \wedge \psi$

4.3 Supervaluations

The D operator also has close relations to the notion of *supervaluations* [6, 22]. Roughly speaking, the intuition behind supervaluations is that language is not completely precise. There are various ways of "extending" a world to make it precise. A formula is then taken to be true at a world w under this approach if it is true under all ways of extending the world. Both the R_j and D_i operators have some of the flavor of supervaluations. If we consider just the objective component of a world o, there are various ways of extending it with subjective components $(s_1, \dots, s_n)$. $D_i\varphi$ is true at an objective world o if $(M, w, i) \models \varphi$ for all worlds w that extend o. (Note that the truth of $D_j\varphi$ depends only on the objective component of a world.) Similarly, given just a subjective component s_j of a world, $R_j\varphi$ is true of s_j if $(M, w, i) \models \varphi$ for all worlds that extend s_i. Not surprisingly, properties of supervaluations can be expressed using R_j or D_j. Bennett [2] has defined a modal logic that formalizes the supervaluation approach.

4.4 Higher-Order Vagueness

In many approaches towards vagueness, there has been discussion of *higher-order vagueness* (see, for example, [6, 23]). In the context of the supervaluation approach, we can say that $D\varphi$ ("definitely φ") holds at a world w if φ is true in all extensions of w. Then $D\varphi$ is not vague; at each world, either $D\varphi$ or $\neg D\varphi$ (and $D\neg D\varphi$) is true (in the supervaluation sense). But using this semantics for definitely, it seems that there is a problem. For under this semantics, "definitely φ" implies "definitely definitely φ" (for essentially the same reasons that $D_i\varphi \Rightarrow D_iD_i\varphi$ in the semantics that I have given). But, goes the argument, this does not allow the statement "This is definitely red" to be vague. A rather awkward approach is taken to dealing with this by Fine [6] (see also [23]), which allows different levels of interpretation.

I claim that the real problem is that higher-order vagueness should not be represented using the modal operator D in isolation. Rather, a combination of D and R should be used. It is not interesting particularly to ask when it is definitely the case that it is definitely the case that something is red. This is indeed true exactly if it is definitely red. What is more interesting is when it is definitely the case that agent i would report that an object is definitely red. This is represented by the formula $D_iR_iD_i$**Red**. We can iterate and ask when i would report that it is definitely the case that he would report that it is definitely the case that he would report it is definitely red, i.e., when $D_iR_iD_iR_iD_i$**Red** holds, and so on. It is easy to see that D_iR_ip does not imply $D_iR_iD_iR_ip$; lower-order vagueness does not imply higher-order vagueness. Since I have assumed that agents are introspective, it can be shown that higher-order vagueness implies lower-order vagueness. In particular, $D_iR_iD_iR_i\varphi$ does imply $D_iR_i\varphi$. (This follows using the fact that $D_i\varphi \Rightarrow \varphi$ and $R_iR_i\varphi \Rightarrow R_i\varphi$ are both valid.) The bottom line here is that by separating the R and D operators in this way, issues of higher-order vagueness become far less vague.

4.5 Williamson's Approach

One of the leading approaches to vagueness in the recent literature is that of Williamson; see [23, Chapters 7 and 8] for an introduction. Williamson considers an epistemic approach, viewing vagueness as ignorance. Very roughly speaking, he uses "know" where I use "report". However, he insists that it cannot be the case that if you know something, then you know that you know it, whereas my notion of reporting has the property that R_i implies R_iR_i. It is instructive to examine the example that Williamson uses to argue that you cannot know what you know, to see where his argument breaks down in the framework I have presented.

Williamson considers a situation where you look at a crowd and do not know the number of people in it. He makes what seem to be a number of reasonable assumptions. Among them is the following:

> I know that if there are exactly n people, then I do not know that there are not exactly $n-1$ people.

This may not hold in my framework. This is perhaps easier to see if we think of a robot with sensors. If there are n grains of sugar in the cup, it is possible that a sensor reading compatible with n grains will preclude there being $n-1$ grains. For example, suppose that, as in Section 2, there are n grains of sugar, and the robot's sensor reading is between $\lfloor (n-4)/10 \rfloor$ and $\lfloor (n+4)/10 \rfloor$. If there are in fact 16 grains of sugar, then the sensor reading could be 2 ($= \lfloor (16+4)/10 \rfloor$).

But if the robot knows how its sensor works, then if its sensor reading is 2, then it knows that if there are exactly 16 grains of sand, then (it knows that) there are not exactly 15 grains of sugar. Of course, it is possible to change the semantics of R_i so as to validate Williamson's assumptions. But this point seems to be orthogonal to dealing with vagueness.

Quite apart from his treatment of epistemic matters, Williamson seems to implicitly assume that there is an objective notion of what I have been calling subjectively vague notions, such as red, sweet, and thin. This is captured by what he calls the *supervenience thesis*, which roughly says that if two worlds agree on their objective part, then they must agree on how they interpret what I have called subjective propositions. Williamson focuses on the example of thinness, in which case his notion of supervenience implies that "If x has exactly the same physical measurements in a possible situation s as y has in a possible situation t, then x is thin in s if and only if y is thin in t" [23, p. 203]. I have rejected this viewpoint here, since, for me, whether x is thin depends also on the agent's subjective state. Indeed, rejecting this viewpoint is a central component of my approach to intransitivity and vagueness.

Despite these differences, there is one significant point of contact between Williamson's approach and that presented here. Williamson suggests modeling vagueness using a modal operator C for *clarity*. Formally, he takes a model M to be a quadruple (W, μ, α, π), where W is a set of worlds and π is an interpretation as above (Williamson seems to implicitly assume that there is a single agent), where μ is a metric on W (so that μ is a symmetric function mapping $W \times W$ to $[0, \infty)$ such that $\mu(w, w') = 0$ iff $w = w'$ and $\mu(w_1, w_2) + \mu(w_2, w_3) \leq \mu(w_1, w_3)$), and α is a non-negative real number. The semantics of formulas is defined in the usual way; the one interesting clause is that for C:

$$(M, w) \models C\varphi \text{ iff } \quad (M, w') \models \varphi \text{ for all } w' \text{ such that } \mu(w, w') \leq \alpha.$$

Thus, $C\varphi$ is true at a world w if φ is true at all worlds within α of w.

The intuition for this model is perhaps best illustrated by considering it in the framework discussed in the previous section, assuming that there is only one proposition, say **Tall(TW)**, and one agent. Suppose that **Tall(TW)** is taken to hold if TW is above some threshold height t^*. Since **Tall(TW)** is the only primitive proposition, we can take the objective part of a world to be determined by the actual height of TW. For simplicity, assume that the agent's subjective state is determined by the agent's subjective estimate of TW's height (perhaps as a result of a measurement). Thus, a world can be taken to be a tuple (t, t'), where t is TW's height and t' is the agent's subjective estimate of the height. Suppose that the agent's estimate is within $\alpha/2$ of TW's actual height, so that the set W of possible worlds consists of all pairs (t, t') such that $|t - t'| \leq \alpha/2$.

Assume that all worlds are plausible (so that $P = W$). It is then easy to check that $(M,(t,t')) \models DR(\mathbf{Tall}(\mathbf{TW}))$ iff $t \geq t^* + \alpha$. That is, the agent will definitely say that TW is Tall iff TW's true height is at least α more than the threshold t^* for tallness, since in such worlds, the agent's subjective estimate of TW's height is guaranteed to be at least $t^* + \alpha/2$.

To connect this to Williamson's model, suppose that the metric μ is such that $\mu((t,t'),(u,u')) = |t-u|$; that is, the distance between worlds is taken to be the difference between TW's actual height in these worlds. Then it is immediate that $(M,(t,t')) \models C(\mathbf{Tall(TW)})$ iff $t \geq t^* + \alpha$. In fact, a more general statement is true. By definition, $(M,(t,t')) \models C\varphi$ iff $(M,(u,u')) \models \varphi$ for all $(u,u') \in W$ such that $|t-u| \leq \alpha$. It is easy to check that $(M,(t,t')) \models DR\varphi$ iff $(M,(u,u')) \models \varphi$ for all $(u,u') \in W$ such that $|t-u'| \leq \alpha/2$. Finally, a straightforward calculation shows that, for a fixed t,

$$\{u : \exists u'((u,u') \in W, |t-u| \leq \alpha)\} = \{u : \exists u'((u,u') \in W, |t-u'| \leq \alpha/2)\}.$$

Thus, if φ is a formula whose truth depends just on the objective part of the world (as is the case for **Tall(TW)** as I have defined it) then $(M,(t,t')) \models C\varphi$ iff $(M,(t,t') \models DR\varphi$.

Williamson suggests that a proposition φ should be taken to be vague if $\varphi \wedge \neg C\varphi$ is satisfiable. In Section 3.3, I suggested that $\varphi \wedge \neg DR\varphi$ could be taken as one of the hallmarks of vagueness. Thus, I can capture much the same intuition for vagueness as Williamson by using DR instead of C, without having to make what seem to me unwarranted epistemic assumptions.

5 Discussion

I have introduced what seems to me a natural approach to dealing with intransitivity of preference and vagueness. Although various pieces of the approach have certainly appeared elsewhere, it seems that this particular packaging of the pieces is novel. The approach leads to a straightforward logic of vagueness, while avoiding many of the problems that have plagued other approaches. In particular, it gives what I would argue is a clean solution to the semantic, epistemic, and psychological problems associated with vagueness, while being able to deal with higher-order vagueness.

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On some generalized L-valued uniform structures

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Uniform spaces and quasi-uniform spaces in L-valued setting can be found in [3–8]. Garcia et. al. [10] introduced uniform spaces in a unifying framework of GL-monoid to include both Lowen Uniformity and Hutton type uniformities. Our interest lies on some generalisations not considered in the earlier setups.

The setting of the paper is a fuzzy lattice L $(L, \leq, \bigwedge, \bigvee, ')$ with order reversing involution. We refer to [1, 4, 9] for basic definitions.

1 L-semi uniformity

We introduce an L-semi-pseudo-metric on L^X as a generalization of L- fuzzy p. metric [9] as follows:

Definition 1. *An L-semi-pseudo-metric on L^X is a mapping $P : L^X \times L^X \to [0, +\infty]$ satisfying the axioms*

(SEM 1) $B \subseteq A \Rightarrow P(A, B) = \underline{0}$,
$B \neq \underline{0} \Rightarrow P(\underline{0}, B) = +\infty$.

(SEM 2) $P(A, B) \leq P(A, C) + P(C, B)$.

(SEM 3) $A, B \neq \underline{0} \Rightarrow P(A, B) = \bigcup_{d\in\beta^*(B)} \bigcap_{e\in\beta^*(A)} P(e, d)$.

(SEM 4) "$P(A, C) < r \Rightarrow C \subseteq B$" $\Leftrightarrow$ "$P(B', D) < r \Rightarrow D \subseteq A'$".

(SEM 5) $A \subseteq B \Rightarrow P(B, C) \leq P(A, C)$.

Definition 2. *A mapping $int : L^X \to L^X$ is an interior operator on L^X, if it fulfills the following conditions:*

(IO1) $int(\underline{1}) = \underline{1}$.

(IO2) $int(A) \subseteq A, \ \forall A \in L^X$.

(IO3) $int(A \bigcap B) = int(A) \bigcap int(B), \ \forall A, B \in L^X$.

L^X together with an interior operator 'int' shall be called an interior space.

An interior operator 'int' is an L-topological interior operator if it further satisfies:

(IO4) $int(int(A)) = int(A), \; \forall\, A \in L^X$.

Definition 3. *Semi diagonal of* $L^X \otimes L^X$ *is a mapping* $\Delta : L^X \to L^X$ *such that* $\Delta(A) = A$.

Definition 4. *Let* $\mathscr{U}^*$ *be the collection of all maps* $U : L^X \to L^X$ *which satisfy:*

s1) $\Delta \subseteq U$.

s2) $U(\bigcup_\lambda V_\lambda) = \bigcup_\lambda U(V_\lambda), \quad V_\lambda \in L^X$.

For any $U \in \mathscr{U}^*$*, we say* $(x_\alpha, y_\beta) \in U \Leftrightarrow y_\beta \in U(x_\alpha)$*, where* $x_\alpha,\; y_\beta \in L^X$.

The following definition is from [4]:

Definition 5. *For any* $U \in \mathscr{U}^*$, $U^r(x_\alpha) = \bigcap\{y_\beta \mid U(y'_\beta) \subseteq x'_\alpha\}$.

Then $U^r \in \mathscr{U}^*$ and $(U^r)^r = U$. If $U = U^r$, then U is symmetric.
Further, the following notions are introduced:

Definition 6. *An L-semi-quasi-uniformity* $\mathscr{U}$ *on* L^X *is a non empty subfamily of* $\mathscr{U}^*$ *satisfy the following:*

(SQ1) $U \bigcap U \in \mathscr{U}, \;\; \forall U,\, V \in \mathscr{U}$.

(SQ2) If $V \in \mathscr{U}^*$ *such that* $U \subseteq V$*, for some* $U \in \mathscr{U}$*, then* $V \in \mathscr{U}$.

Definition 7. *We shall call a non-empty subfamily* $\mathscr{B}$ *of* $\mathscr{U}^*$ *to be a base for some L-semi-quasi-uniformity* $\mathscr{U}$*, if for any* $U \in \mathscr{U}$*, there is* $B \in \mathscr{B}$ *such that* $B \subseteq U$.

A non-empty subfamily $\mathscr{B}$ of $\mathscr{U}^*$ is a base for some L-semi-quasi-uniformity $\mathscr{U}$, if it satisfies the following:

(SQ1′) For any $U,\, V \in \mathscr{B}$*, there is* $W \in \mathscr{B}$ *such that* $W \subseteq U \bigcap V$.

Definition 8. *We shall say that a base* $\mathscr{B}$ *for an L- semi-quasi-uniformity* $\mathscr{U}$ *on* L^X *is an L- semi-uniformity if it satisfies the following:*

(SQ3) For any $B \in \mathscr{B}$ *implies* $B^r \in \mathscr{B}$.

The collection of symmetric members of $\mathscr{U}$ is a base for $\mathscr{U}$.
The following definition was proposed in [2]:

Definition 9. *We say that a base* $\mathscr{B}$ *for an L-semi-quasi-uniformity on* L^X *is an L-local quasi-uniformity iff* $\forall\, U \in \mathscr{B}$ *and* $\forall\, x_\alpha \in L^X$ *there exists* $V \in \mathscr{B}$ *such that* $V \circ V(x_\alpha) \subseteq U(x_\alpha)$*. If* $\mathscr{U}$ *is an L-semi-quasi-uniformity which is L-local quasi-uniformity, we shall call* $\mathscr{U}$ *to be an L-local quasi-uniformity.*

Proposition 1. [4] *Let* $i : L^X \to L^X$ *be an L-topological interior operator. Then* $\mathbb{F} = \{V \in L^X \mid i(V) = V\}$ *is an L-topology and* $i(V)$ *is the interior of* V *in* $(L^X, \mathbb{F})$.

Theorem 1. *Let* $(L^X, \mathscr{U})$ *be an L-semi-quasi-uniform space and* $\mathscr{B}$ *be any base for* $\mathscr{U}$. *The mapping* $int : L^X \to L^X$ *defined by,* $int(A) = \bigcup\{x_\alpha \mid \exists\, V \in \mathscr{B}$ s.t. $V(x_\alpha) \subseteq A\}$, *is an interior operator on* L^X.

Every L-semi-quasi-uniformity therefore generates an interior space.

Lemma 1. *Let* $(L^X, \mathscr{U})$ *be an L-semi-quasi-uniform space. Then '* int*' is an L-topological interior operator under the following condition:*

For any $U \in \mathscr{U}$ *and* $x_\alpha \in L^X$, *there exists* $V \in \mathscr{U}$ *such that to each* $y_\beta \in V(x_\alpha)$ *there corresponds* $W \in \mathscr{U}$ *with* $W(y_\beta) \subseteq U(x_\alpha)$.

Now by proposition 1 we have the following:

Theorem 2. *Every L-semi-quasi-uniformity with condition in lemma 1 generates an L-topological space.*

Definition 10. *The interior operator generated by an L- semi-quasi-uniformity* $\mathscr{U}$ *is the interior operator generated by* int.

In particular, for any $x_\alpha \in L^X$, $\mathscr{N}_{x_\alpha} = \{U(x_\alpha) \mid U \in \mathscr{U}\}$ is the neighborhood system at x_α in the generating interior space. If the family $\{\mathscr{N}_{x_\alpha} \mid x_\alpha \in L^X\}$ is a neighborhood system for some L-topology $\mathbb{F}$, we say that $\mathbb{F}$ is the L-topology generated by $\mathscr{U}$.

It now follows that "*Every L-local quasi-uniformity is an L-semi-quasi-uniformity with condition given in lemma 1*." We then have the following result:

Theorem 3. *Every L-local quasi-uniform space generates an L-topological space.*

Remark 1. The converse of the theorem 3 is also true.

Definition 11. *Let* $(L^X, \mathscr{U})$ *and* $(L^Y, \mathscr{V})$ *be L-semi-uniform spaces, a function* $f : (L^X, \mathscr{U}) \to (L^Y, \mathscr{V})$ *is L-semi-uniformly continuous iff for every* $V \in \mathscr{V}$, *there exists* $U \in \mathscr{U}$ *such that* $\widehat{f}(U) \subseteq V$, *where* $\widehat{f}(x_\alpha, y_\beta) = (f(x_\alpha), f(y_\beta))$.

The following result shall follow:

Theorem 4. *Fuzzy L-semi-uniformly continuous functions on L-semi-uniform spaces are continuous with respect to the relative interior spaces.*

2 L-semi-pseudo metrization

For any L-semi-pseudo-metric space (L^X, P) and $0 < s$, the mapping $U_s : L^X \to L^X$ defined by,

$$U_s(A) = \bigcup\{C \in L^X \mid P(A, C) < s\}, \ \forall A \in L^X$$

satisfies the condition (s1) and (s2) and that for any s, t satisfying $0 < s < t$ implies $U_s \subseteq U_t$.
The following result now follows:

Theorem 5. *Every L-semi-pseudo-metric induces an L-semi-uniformity.*

Now calling an L-semi-uniform space $(L^X, \mathscr{U})$ to be L-semi-pseudo-metrizable if there is an L-semi-pseudo-metric inducing the same interior operator generated by $\mathscr{U}$, we have the following:

Theorem 6. *An L-semi-uniform space $(I^X, \mathscr{U})$ is an L-semi-pseudo metric iff $\mathscr{U}$ has a countable base.*

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Quantaloids as categorical basis for many valued mathematics

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The purpose of this talk is to explain that enriched categories over quantaloids form a categorical basis for many valued mathematics. The point of departure are two arguments:

Argument 1. Mathematical structures beyond the *Special Adjoint Functor Theorem* have usually a poor structure — e.g. Birkhoff-Neumann logic for quantum mechanics.

Argument 2. The mathematical meaning of transitivity is composition.

If we put these arguments together, then quantaloids are the natural categorical basis for algebraic as well as topological structures in many valued mathematics. In fact quantaloids are enriched categories over the monoidal closed category $s\ell$ of complete lattices with arbitrary join preserving maps. It is interesting to note that quantales induce quantaloids in at least four different ways — this means that quantales themselves do not uniquely determine the categorical basis for many valued mathematics. Hence various seemly unrelated mathematical structures appear as $\mathcal{Q}$-categories (cf. [2, 3, 12]) where $\mathcal{Q}$ is a quantaloid induced by a quantale in one or another way:

1. Preorder sets are **2**-categories (where the quantaloid $\mathcal{Q}$ is induced by the Boolean algebra **2**).
2. Partially ordered sets are separated **2**-categories.
3. Non-symmetric metric spaces are separated $\mathcal{Q}$-categories where $\mathcal{Q}$ is induced by $[0, 1]$ provided with the usual multiplication (see [8]).
4. Ω-sets are symmetric $\mathcal{Q}$-categories where $\mathcal{Q}$ is induced by a frame Ω (cf. [12]).
5. M-valued sets are symmetric $\mathcal{Q}$-categories where $\mathcal{Q}$ is induced by a commutative strictly two-sided quantale M (cf. [5]).

Moreover, in the non-commutative setting (e.g. the quantale is given by the spectrum of non-commutative C^*-algebras) pre-Q-sets are $\mathcal{Q}$-categories where the objects of $\mathcal{Q}$ are given by all stable elements of Q (cf. [6]). If the underlying

quantaloid is involutive, then the symmetrization process of pre-Q-sets holds. Hence pre-Q-sets generate quite naturally Q-sets which form a basis for sheaf theory on quantales (cf. [7]).
Finally, the filter monad exists on the category **Cat**($\mathcal{Q}$) of $\mathcal{Q}$-categories. Hence **Cat**($\mathcal{Q}$) has topological space objects.

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Cancellative residuated lattices arising on 2-generated submonoids of natural numbers

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It was proved in [3] that there are only two atoms in the subvariety lattice of cancellative residuated lattices. The first one is the variety of Abelian ℓ-groups $\mathcal{CLG}$ which is known to be generated by the additive ℓ-group of integers $\mathbf{Z}$ (due to Weinberg [6]). The second atom $\mathcal{CLG}^-$ is the variety generated by the negative cone $\mathbf{Z}^-$ of $\mathbf{Z}$ (see [3]). The negative cone $\mathbf{Z}^-$ is a residuated chain whose monoidal reduct is 1-generated. In this talk we will study residuated chains with a slightly more complex structure than $\mathbf{Z}^-$, namely the residuated chains arising on 2-generated submonoids of $\mathbf{Z}^-$. However, it turns out that the investigation of such algebras looks unfamiliar because it needs a lot of notions which are dual to well-known notions for natural numbers. Due to this fact we formulate the results in the dual term-wise equivalent setting. More precisely, we will consider all dually residuated chains arising on 2-generated submonoids of natural numbers. Then we will describe a mutual position of varieties generated by these chains. As a consequence we find out which of them generate a cover of $\mathcal{CLG}^-$.

A dual integral commutavive residuated lattice (shortly dual ICRL) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, +, \dot{-}, 0\rangle$, where $\langle L, \wedge, \vee, 0\rangle$ is a lattice with a bottom element 0, $\langle L, +, 0\rangle$ is a commutative monoid and for all $x, y, z \in L$ we have

$$x + y \geq z \text{ iff } x \geq z \dot{-} y\,.$$

It is easy to see that the dual ICRLs are term-wise equivalent to integral commutative residuated lattices. We say that a dual ICRL is *cancellative* if its monoidal

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reduct is cancellative, i.e., if it satisfies the identity $(x + y) \dot{-} y \approx x$. A dual ICRL is called *divisible* if the join is definable as $x \vee y \approx (y \dot{-} x) + x$. By $\mathbf{N}$ we denote the dual of $\mathbf{Z}^-$, i.e., $\mathbf{N} = \langle \mathbb{N}, \wedge, \vee, +, \dot{-}, 0 \rangle$, where $x \dot{-} y = (x - y) \vee 0$. Note that $\mathbf{N}$ is cancellative and divisible.

Let $a, b \in \mathbb{N}$. Then $\mathbf{M}(a, b) = \langle M(a, b), \wedge, \vee, +, \dot{-}, 0 \rangle$ is the dual ICRL living on $M(a, b)$, i.e., $\wedge, \vee$ induce the usual order on $\mathbb{N}$ and

$$x \dot{-} y = \min\{z \in M(a, b) \mid z \geq x - y\} .$$

It is clear that if a, b are not coprime then $\mathbf{M}(a, b) \cong \mathbf{M}(a/d, b/d)$ for $d = \gcd(a, b)$ since $\dot{-}$ is fully determined by the monoidal operation and the order. Thus we consider only coprime generators a, b. Let $a, b \in \mathbb{N}$ such that $a < b$. It is obvious that $a \in \{0, 1\}$ implies $\mathbf{M}(a, b) \cong \mathbf{N}$. Thus we exclude also these possibilities. The following is the set of interesting generators:

$$Gen = \{\langle a, b \rangle \in \mathbb{N}^2 \mid 1 < a < b,\ a, b \text{ coprime}\} .$$

Proposition 1. *For each $\langle a, b \rangle \in Gen$ the algebra $\mathbf{M}(a, b)$ is a simple, non-divisible, cancellative, dual ICRL.*

Let $\mathbf{A}$ be an algebra. Then $\mathcal{V}(\mathbf{A})$ denotes the variety generated by $\mathbf{A}$. Given $\langle a, b \rangle, \langle c, d \rangle \in Gen$, it is possible to find identities which hold in $\mathbf{M}(a, b)$ and do not hold in $\mathbf{M}(c, d)$ provided that $\langle a, b \rangle \neq \langle c, d \rangle$. Thus we obtain the following theorem.

Theorem 1. *Let $\langle a, b \rangle, \langle c, d \rangle$ be two different pairs from the set Gen. Then $\mathcal{V}(\mathbf{M}(a, b)) \neq \mathcal{V}(\mathbf{M}(c, d))$.*

In order to describe the mutual position of varieties $\mathcal{V}(\mathbf{M}(a, b))$ in the subvariety lattice, we need to characterize subalgebras of $\mathbf{M}(a, b)$. Let $\langle a, b \rangle \in Gen$. Then the remainder on integer division of b by a is denoted $\rho_a(b)$. We will show that the proper nontrivial subalgebras of $\mathbf{M}(a, b)$ are isomorphic to $\mathbf{N}$ if a is prime or $\rho_a(b) \neq 1$. In the remaining cases (i.e., a is not prime and $\rho_a(b) = 1$) the subalgebras of $\mathbf{M}(a, b)$ are completely determined by the divisors of a.

Theorem 2. *Let $\langle a, b \rangle \in Gen$. If a is prime or $\rho_a(b) \neq 1$ then each nontrivial proper subalgebra of $\mathbf{M}(a, b)$ is isomorphic to $\mathbf{N}$.*

Theorem 3. *Let $\langle a, b \rangle \in Gen$ such that $\rho_a(b) = 1$. For each divisor d of a there is a nontrivial subalgebra of $\mathbf{M}(a, b)$ isomorphic to $\mathbf{M}(a/d, b)$ and each nontrivial subalgebra of $\mathbf{M}(a, b)$ is isomorphic to $\mathbf{M}(a/d, b)$ for a divisor d of a.*

Using the latter theorems and Jónsson's lemma, it is possible to prove that some of the algebras $\mathbf{M}(a, b)$ generate an almost minimal variety.

Theorem 4. *Let $\langle a, b\rangle \in Gen$. If a is prime or $\rho_a(b) \neq 1$, then $\mathcal{V}(\mathbf{M}(a, b))$ is a cover of $\mathcal{V}(\mathbf{N})$.*

Thus there are infinitely many covers of $\mathcal{V}(\mathbf{N})$ among varieties $\mathcal{V}(\mathbf{M}(a, b))$ for $\langle a, b\rangle \in Gen$. In fact, the remaining varieties of this type do not generate covers of $\mathcal{V}(\mathbf{N})$. We will prove it by describing their mutual position in the subvariety lattice.

Theorem 5. *Let $\langle a, b\rangle, \langle c, d\rangle \in Gen$ such that $\rho_a(b) = \rho_c(d) = 1$. Then we have $\mathcal{V}(\mathbf{M}(c, d)) \subseteq \mathcal{V}(\mathbf{M}(a, b))$ iff c divides a and $d = b$.*

Using the latter theorem, we obtain also the converse of Theorem 4.

Theorem 6. *Let $\langle a, b\rangle \in Gen$. Then $\mathcal{V}(\mathbf{M}(a, b))$ is a cover of $\mathcal{V}(\mathbf{N})$ iff a is prime or $\rho_a(b) \neq 1$.*

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On the structural description of involutive $\mathbf{FL}_e$-algebras

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1 Introduction and preliminaries

Residuated lattices and substructural logics are subjects of intense investigation. Algebraic semantics of substructural logics are classes of residuated lattices. We shall investigate involutive FL_e-algebras in this paper, and will establish structural description for some subclasses.

For any binary operation $\circledast$ (on a poset) which is commutative and non-decreasing one can define its *residuum* $\rightarrow_\circledast$ by $x \circledast y \leq z \iff x \rightarrow_\circledast z \geq y$. The displayed equivalence is often referred to as *adjointness conditions*. If $\rightarrow_\circledast$ exists, it has an equivalent description, namely, $\rightarrow_\circledast$ is the unique binary operation on the poset such that we have $x \rightarrow_\circledast y = \max\{z \mid x \circledast z \leq y\}$. Let $\mathcal{C} = \langle X, \leq, \bot, \top \rangle$ be a bounded poset. A *involution* over $\mathcal{C}$ is an order reversing bijection on X such that its composition by itself is the identity map of X. Involutions are continuous in the order topology of $\mathcal{C}$. *T-conorms* (resp. *t-norms*) over $\mathcal{C}$ are commutative monoids on X with unit element $\bot$ (resp. $\top$). T-conorms and t-norms are *duals* of one another. That is, for any involution $'$ and t-conorm $\oplus$ over $\mathcal{C}$, the operation $\odot$ on X defined by $x \odot y = (x' \oplus y')'$ is a t-norm over $\mathcal{C}$. Vice versa, for any involution $'$ and t-norm $\odot$ over $\mathcal{C}$, the operation $\oplus$ on X defined by $x \oplus y = (x' \odot y')'$ is a t-conorm over $\mathcal{C}$. *Uninorms* over $\mathcal{C}$ [6, 1] are commutative monoids on X with unit element e (which may be different from $\bot$ and $\top$). Every uninorm over $\mathcal{C}$ has an *underlying t-norm* $\odot$ and *t-conorm* $\oplus$ acting on the subdomains $[\bot, e]$ and $[e, \top]$, respectively. That is, for any uninorm $\circledast$ over $\mathcal{C}$, its restriction to $[\bot, e]$ is a t-norm over $[\bot, e]$, and its restriction to $[e, \top]$ is a t-conorm over $[e, \top]$.

Definition 1. *$\mathcal{U} = \langle X, \circledast, \leq, \bot, \top, e, f, \rangle$ is called an* involutive FL_e-algebra *if*

1. *$\mathcal{C} = \langle X, \leq, \bot, \top \rangle$ is a bounded poset,*
2. *$\circledast$ is a uninorm over $\mathcal{C}$ with neutral element e,*

3. *for every* $x \in X$, $x \to_{\circledast} f = \max\{z \in X \mid x \circledast z \leq f\}$ *exists, and*
4. *for every* $x \in X$, *we have* $(x \to_{\circledast} f) \to_{\circledast} f = x$.

We will call $\circledast$ *an involutive uninorm. It is not difficult to see that every involutive uninorm is residuated (see [5]) and hence* $\circledast$ *is isotone (see [4]). Therefore,* $' : X \to X$ *given by*

$$x' = x \to_{\circledast} f$$

is an order-reversing involution. Denote

$$X^{+} = \{x \in X \mid x \geq e\} \qquad \text{and} \qquad X^{-} = \{x \in X \mid x \leq e\}.$$

If $\mathcal{C}$ *is linearly ordered, we call* $\mathcal{U}$ *an involutive* FL_e-chain. $\mathcal{U}$ *is called* finite *if* X *is a finite set.*

2 General Observations

Lemma 1. *In any involutive* FL_e*-algebra* $\mathcal{U} = \langle X, \circledast, \leq, \bot, \top, e, f, \rangle$*, for* $x, y \in X$ *the following statements hold true:*

1.

$$f' = e, \tag{1}$$

$$x \circledast y = \big(x \to_{\circledast} y'\big)', \tag{2}$$

$$\bot \circledast X = \bot,^{1} \tag{3}$$

$$\top \circledast [e, \top] = \top, \tag{4}$$

$$(x \to_{\circledast} \bot) \circledast (x \to_{\circledast} \bot)' = \bot. \tag{5}$$

If $\bot < x \to_{\circledast} \bot$ *then*

$$x \to_{\circledast} \bot \not\leq f. \tag{6}$$

If $x \leq x \to_{\circledast} \bot$ *then*

$$(x \to_{\circledast} \bot)' \leq x \to_{\circledast} \bot. \tag{7}$$

2. *Assume* $e \geq f$. *Then we have*

$$x' \circledast y' \leq (x \circledast y)'. \tag{8}$$

If in addition $c \leq f$, c *is idempotent then*
(a) $\mathcal{U}$ *has a subalgebra on* $[c, c']$,

[1] A shorter notation for $\bot \circledast x = \bot$ for $x \in X$. Complex multiplication will be used extensively in the sequel.

(b) for $x \geq c$ we have

$$(x \to_{\circledast} c) \circledast (x \to_{\circledast} c)' = c, \tag{9}$$

(c) if $c \leq x \leq x \to_{\circledast} c$ then

$$(x \to_{\circledast} c)' \leq x \to_{\circledast} c. \tag{10}$$

Theorem 1. *Let $\langle X, \circledast, \leq, \bot, \top, e, f, \rangle$ be an involutive FL$_e$-algebra, $\otimes$ its underlying t-norm and $\oplus$ its underlying t-conorm acting on X^+ and X^-, respectively. Then $\otimes$ and $\oplus$ uniquely determine $\circledast$ on $X^+ \times X^-$ via*

$$x \circledast y = \begin{cases} (x \to_{\oplus} y')', \text{ if } x \leq y' \\ (y \to_{\otimes} x')', \text{ if } x > y' \end{cases} \tag{11}$$

Corollary 1. *If there are no elements in X which are incomparable with e in an involutive FL$_e$-algebra $\langle X, \circledast, \leq, \bot, \top, e, f, \rangle$ then the underlying t-norm and t-conorm of $\circledast$ uniquely determine $\circledast$.*

Theorem 1 motivates the following construction:

Definition 2. *Let X_1 and X_2 be two partially ordered sets such that the ordinal sum $os\langle X_1, X_2\rangle$ of X_1 and X_2 (that is putting X_1 under X_2 so to say) is bounded and has an order reversing involution $'$. Let $\otimes$ be a t-norm on X_1, $\oplus$ be a t-conorm on X_2. Denote*

$$\mathcal{U}_{\otimes}^{\oplus} = \langle os\langle X_1, X_2\rangle, \circledast, \leq, \bot, \top, e, f\rangle$$

where

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \to_{\oplus} y')' & \text{if } (x \in X_2,\, y \in X_1, \text{ and } x \leq y') \text{ or } (y \in X_2,\, x \in X_1, \text{ and } x \leq y') \\ (y \to_{\otimes} x')' & \text{if } (x \in X_2,\, y \in X_1, \text{ and } x > y') \text{ or } (y \in X_2,\, x \in X_1, \text{ and } x > y') \end{cases} . \tag{12}$$

In general $\circledast$ given in (12) will not be an involutive uninorm. But if $\circledast$ is any involutive uninorm with underlying t-norm $\otimes$ and underlying t-conorm $\oplus$ then, by Corollary 1, $\circledast$ must have the form (12).

After the above general structural description we will focus on which pair of a t-norm and a conorm is suitable for defining an involutive uninorm via (12).

3 Finite involutive $\mathbf{FL}_e$-chains

Definition 3. *Consider a finite involutive FL$_e$-chain $\mathcal{U}$ and denote the cardinality of its universe by n. Clearly, $\mathcal{U}$ is order-isomorphic to a finite involutive FL$_e$-chain with universe $\{1, 2, \ldots, n\} \subset \mathbf{N}$, denote it by $\langle \{1, 2, \ldots, n\}, \circledast, \leq, 1, n, e, f\rangle$. Call $e - f$ the* rank *of $\mathcal{U}$. It is easy to see that the rank is well-defined.*

Standing assumption: Because of the order-isomorphism which was mentioned in Definition 3, without loss of generality, in the sequel we will consider finite involutive FL$_e$-chains *solely* on the universe $\{1, 2, \ldots, n\}$,[2] and will employ the shorter notation

$$\mathcal{U}_n = \langle \{1, 2, \ldots, n\}, \circledast, \leq, e, f\rangle.^3$$

Lemma 2. *Let $\mathcal{U}_n$ be any finite involutive FL$_e$-chain.*

1. *We have*
$$f = n + 1 - e.^4 \tag{13}$$
2. *We have that $rank(\mathcal{U}_n)$ is necessarily even if n is odd, and vice versa.*
3. *If $e \neq 1$ then $2 \circledast 2 \in \{1, 2\}$.*
4. *If $n \geq 3$, $rank(\mathcal{U}_n) \geq 0$, and $2 \circledast 2 = 2$ then $\langle \{2, \ldots, n-1\}, \circledast, \leq, e, f\rangle$ is a subalgebra of $\mathcal{U}_n$.*

Definition 4. *Let $\otimes$ be a t-norm on $\{1, 2, \ldots, e\}$, $\oplus$ be a t-conorm on $\{e, e+1, \ldots, n\}$, and let $x' = n + 1 - x$ for $x \in \{1, 2, \ldots, n\}$. Denote*

$$\mathcal{U}_\otimes^\oplus = \langle \{1, 2, \ldots, n\}, \circledast, \leq, e, f\rangle$$

where

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \leq e \\ x \oplus y & \text{if } x, y \geq e \\ (x \to_\oplus y')' & \text{if } (x \geq e,\ y \leq e, \text{ and } x \leq y') \text{ or } (y \geq e,\ x \leq e, \text{ and } x \leq y') \\ (y \to_\otimes x')' & \text{if } (x \geq e,\ y \leq e, \text{ and } x > y') \text{ or } (y \geq e,\ x \leq e, \text{ and } x > y') \end{cases}. \tag{14}$$

[2] Here n being any natural number $n \geq 1$.
[3] Without writing the top and the bottom elements.
[4] Here and in the sequel $+$ and $-$ refer to addition and subtraction of natural numbers, respectively.

Consider a finite involutive FL_e-chain $\mathcal{U}_n = \langle\{1, 2, \ldots, n\}, \circledast, \leq, e, f\rangle$ and denote its underlying t-norm (which acts on $\{1, 2, \ldots, e\}$) and its underlying t-conorm (which acts on $\{e, e+1, \ldots, n\}$) by $\otimes$ and $\oplus$, respectively. By Corollary 1 we have $\mathcal{U}_n = \mathcal{U}_{\otimes}^{\oplus}$ (given in Definition 2).

Below we give a characterization for the pairs of a t-norm and a conorm which are suitable for defining an involutive uninorm via (12) provided that the rank of the finite algebra is 0, 1, or 2.

Theorem 2. *We have that $\circledast$ is the monoidal operation of a finite involutive FL_e-chain with rank 0 (resp. rank 1) iff n is odd (resp. n is even) and*

$$x \circledast y = \begin{cases} \min(x, y) & \text{if } x \leq y' \\ \max(x, y), & \text{if } x > y' \end{cases} \tag{15}$$

Definition 5. *Call a finite involutive FL_e-algebra $\langle\{1, 2, \ldots, n\}, \circledast, \leq, e, f, \rangle$ $\top\bot$-indecomposable if it has no subalgebra on $[2, \ldots, n-1]$. Call an involutive FL_e-algebra simple if it has no proper subalgebra.*

Theorem 3. *There is a one-to-one correspondence between $\top\bot$-indecomposable involutive uninorms with rank 2 on n-element chains and conorm operations on $\frac{n-1}{2}$-element chains given as follows:*

Let $\odot$ be the t-norm operation on $\{1, 2, \ldots, \frac{n+3}{2}\}$ given by

$$x \odot y = \begin{cases} 1 & \text{if } x, y < \frac{n+3}{2} \\ \min(x, y) & \text{otherwise} \end{cases}. \tag{16}$$

1. *For any involutive uninorm on $\{1, \ldots, n\}$ with $rank = 2$, its underlying t-norm is equal to $\odot$.*
2. *For any conorm operation $\oplus$ on $\{\frac{n+3}{2}, \frac{n+3}{2}+1, \ldots, n\}$, the monoidal operation of $\mathcal{U}_{\odot}^{\oplus}$ given in Definition 4 is an involutive uninorm on $\{1, \ldots, n\}$ with $rank = 2$.*

Corollary 2. *Denote $\mathcal{C}_n$ the number of conorm operations on an n-element chain. The number of $\top\bot$-indecomposable involutive uninorms on an n-element chain with rank 2 equals to $\mathcal{C}_{\frac{n-1}{2}}$. The number of involutive uninorms on an n-element chain with rank 2 equals to $\sum_{i=1}^{\frac{n-1}{2}} \mathcal{C}_i$.*

4 Involutive uninorms on $[0, 1]$ with $e = f$

Definition 6. *For any commutative residuated lattice on a complete and dense chain* $\langle X, \leq, \oplus, \to_{\oplus}, 1\rangle$*, define* $\circledbullet : X \times X \to X$ *by* $x \circledbullet y = \inf\{u \oplus v \mid u > x, v > y\}$*, and call it the* skewed pair *of* $\oplus$*. For any commutative co-residuated chain* $\langle X, \leq, \circledbullet, \to_{\circledbullet}, 1\rangle$*, define* $\oplus : X \times X \to X$ *by* $x \oplus y = \sup\{u \circledbullet v \mid u < x, v < y\}$*, and call it the* skewed pair *of* $\oplus$*. Call* $(\oplus, \circledbullet)$ *a* skew pair.

Definition 7. *Let* $(L_2, \leq)$ *be a complete, dense chain and* $L_1 \subseteq L_2$*. Let* $(L_1, \oplus, \to_{\oplus}, \leq, \top)$ *be a commutative residuated lattice on a complete and dense chain, and let* $'$ *be an order reversing involution on* L_2*. The operation* $\odot$ *is said to be* dual *to* $\oplus$ *with respect to* $'$ *if* $\odot$ *is a binary operation on* $(L_1)' = \{x' \mid x \in L_1\}$ *given by* $x \odot y = (x' \oplus y')'$*. We say that the operation* $\circledbullet$ *is* skew dual *to the residuated operation* $\oplus$ *with respect to* $'$ *if* $\circledbullet$ *is the dual of the skewed pair of* $\oplus$*.*

Theorem 4. *[5] Any involutive uninorm on* $[0, 1]$ *with* $e = f$ *can be represented by (11) where its undelying t-norm and t-conorm are skew-duals.*

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T-actions on bounded lattices

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T-norms were introduced by Schweizer and Sklar [14] in the framework of probabilistic metric spaces. Several authors have studied with t-norms on bounded lattices. For more detail, we refer [2], [7], [8], [9], [10], [12], [13].

In this paper, we introduce an action of a bounded lattice on a partially ordered set. Later on, we define a new order on a partially ordered set by using this action. In particular, when we suppose that the bounded lattice and the partially ordered set are the same sets and the function is a t-norm T, we show that this function is an action function again. So, this action is called a t-action and the order on the bounded lattice is denoted by $\preceq_T$.

In our contribution, we focus to determine that the subset of the bounded lattice is a lattice with respect to $\preceq_T$ under some special circumstances. Using this idea, we obtain some new conclusions. Especially, we show that the set of all idempotent elements of t-norm T is a complete lattice with respect to $\preceq_T$.

Definition 1. *Let L be a bounded lattice and $(S, \leq)$ be a partially ordered set. Let T be a t-norm on L. An action of L on S is a function $\lambda : L \times S \longrightarrow S$ such that for all $x \in S$, $\ell_1, \ell_2 \in L$*

a1. $\lambda(1, x) = x$
a2. $\lambda(T(\ell_1, \ell_2), x) = \lambda(\ell_1, \lambda(\ell_2, x))$
a3. *If $\ell_1 \leq \ell_2$, then $\lambda(\ell_1, x) \leq \lambda(\ell_2, x)$*

Typically, we will not mention λ and we will write ℓx instead of $\lambda(\ell, x)$. Now, let us define the following binary relation on S:

$$x \preceq y :\Leftrightarrow \exists \ell \in L \quad such\ that \quad \ell y = x$$

Proposition 1. *The binary relation $\preceq$ is a partially ordered relation on S.*

Example 1. Let L be a bounded lattice, T be a t-norm on L and $S = L$. Then

$$\lambda : L \times L \longrightarrow L$$
$$(\ell, x) \longmapsto \lambda(\ell, x) := T(\ell, x)$$

is an action on L. The action obtained this way is called t-action on L. Denote by $\preceq_T$ the order constructed from t-action on L and have as follows:

$$x \preceq_T y :\Leftrightarrow \exists \ell \in L \ : \ T(\ell, y) = x.$$

Generally, in this study, we will take $S = L$ and investigate t-actions.

Proposition 2. *Let L be a bounded lattice, T be a t-norm on L and λ be an action on $S = L$. Then $T = \lambda$. Hence T is unique t-action on L.*

Example 2. Let $L = [0, 1]$ and T^{nM} be nilpotent minimum t-norm on $[0, 1]$. Then, $(L, \preceq_{T^{nM}})$ is a meet-semilattice, but not a join-semilattice.

Proposition 3. *Let L be a lattice and T be any t-norm on L. If $a \preceq_T b$ for $a, b \in L$, then $T(a, c) \preceq_T T(b, c)$ for every $c \in L$.*

Proposition 4. *Let $(L, \leq)$ be a bounded lattice and T be a t-norm on L. If $(L, \preceq_T)$ is a chain, then T is a divisible t-norm; i.e, $\leq = \preceq_T$.*

Theorem 1. *Let L be a complete lattice and T be any t-norm on L. Then $a \wedge_T b = T(a, b)$ for every $a, b \in H_T$ and $\bigvee_T\{a_\tau | \tau \in Q\} = \bigvee\{a_\tau | \tau \in Q\}$ for every $\{a_\tau | \tau \in Q\} \subseteq H_T$. $(H_T, \preceq_T)$ is a complete lattice.*

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Conjunctors with special properties

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Many-valued propositional logics work with binary operations acting on the sets of truth values. Most often, a lattice $(L, \leq, 0, 1)$ of truth values and monotone extensions of the classical Boolean operations (conjunction, disjunction, implication, ...) are considered [16]. Restricting ourselves to fuzzy logics and extensions of the Boolean conjunction, the standard lattice $([0,1], \leq, 0, 1)$ and a monotone extension $C\colon [0,1]^2 \to [0,1]$, called *conjunctor*, of the Boolean conjunction are used. Depending on the type of conjunctor to be studied, additional properties of C may be required, leading to special classes of fuzzy logics. Conjunctors are not only used in logics, but also (maybe under different names and with additional properties) in other mathematical fields.

Unfortunately, in general no universally accepted terminology is available, and different names for the same object often indicate the area of origin for these names. In the field of *aggregation functions* [17], conjunctors are binary aggregation functions with annihilator 0. We briefly summarize some classes of well-known conjunctors characterized by special properties, including some important results (for more information see [8]).

(i) Conjunctors with neutral element 1, i.e., satisfying $C(1,x) = C(x,1) = x$ for all $x \in [0,1]$, are called *semicopulas* [3] (but also *weak t-norms* [32] or, simply, *conjunctors* [34]).
(ii) An associative semicopula is called a *pseudo-t-norm* [12]. Note that the class of $\sup$- (or $\inf$-)closures of pseudo-t-norms equals the class of semicopulas [9].
(iii) A most important class of conjunctors is the class of *triangular norms* (*t-norms*) introduced in [29]. Triangular norms are associative commutative semicopulas (see the monographs [1, 20, 30] or [21–23]). From [23, Corollary 3.3] (which is based on [26]) we know that a function $T\colon [0,1] \to$

$[0,1]$ is a continuous t-norm if and only if there is a family $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ and a family $h_\alpha\colon [a_\alpha, e_\alpha] \to [0,\infty]$ of continuous, strictly decreasing functions with $h_\alpha(e_\alpha) = 0$ for each $\alpha \in A$ such that for all $(x,y) \in [0,1]^2$

$$T(x,y) = \begin{cases} h_\alpha^{-1}(\min(h_\alpha(x) + h_\alpha(y), h_\alpha(a_\alpha))) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Consequently, a continuous t-norm $T\colon [0,1]^2 \to [0,1]$ satisfies $T(x,x) < x$ for all $x \in]0,1[$ if and only if there is a strictly decreasing continuous function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ (a so-called *additive generator* of T) such that

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))). \tag{1}$$

Such t-norms are called *continuous Archimedean t-norms*.

(iv) *Copulas* [31] are supermodular semicopulas, i.e., they satisfy

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y}) \tag{2}$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$, and they play an important role in probability theory (for details see [28]). Recall that associative copulas are exactly 1-Lipschitz t-norms. If they are also Archimedean, this property is equivalent to the convexity of the corresponding additive generator [27].

(v) 1-Lipschitz conjunctors, i.e., conjunctors satisfying

$$|C(x,y) - C(x^*,y^*)| \leq |x - x^*| + |y - y^*| \tag{3}$$

for all $(x,y), (x^*,y^*) \in [0,1]^2$, are called *quasi-copulas* [2, 15]. The class of sup- (or $\inf$-)closures of quasi-copulas equals the class of copulas.

(vi) Associative commutative conjunctors with neutral element $e \in]0,1[$ are called *conjunctive uninorms* [13]. A conjunctive uninorm $U\colon [0,1]^2 \to [0,1]$ which is cancellative on $]0,1[^2$ and continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$ is generated by a strictly increasing bijection $u\colon [0,1] \to [-\infty,\infty]$ via $U(x,y) = u^{-1}(u(x) + u(y))$, using the convention $\infty + (-\infty) = -\infty$. Such uninorms are called *representable*.

Remark 1. (i) Continuous t-norms are the conjunctors in BL-logics [18], left-continuous t-norms model conjunction in MTL-logics [11], and pseudo-t-norms play a key role in pseudo-BL-algebras [5, 6]. Representable uninorms were used in [14] as models for a conjunction.

(ii) Copulas provide the link between the marginal distributions of a random vector and its joint distribution [31] ("conjunction" of marginal distributions), and quasi-copulas are important for the construction of preference structures [4].

(iii) In [25] the one-to-one correspondence between fuzzy equivalence relations and fuzzy partitions based on a conjunctor C was investigated. Although usually C was required to be a t-norm (see, e.g., [19]), and the use of semicopulas was proposed in [33], only in [25] it was shown that for a sound axiomatics of C-based fuzzy equivalence relations and of C-based fuzzy partitions we need
   - a conjunctor C satisfying $\max(C(x,1), C(1,x)) = x$ for all $x \in [0,1]$ in the case of fuzzy equivalence relations;
   - a conjunctor C satisfying, for all $x \in [0,1]$, $C(1,x) = x$ and the existence of $y \in [0,1]$ such that $C(x,1) \leq y$ and $C(y,1) \leq x$ in the case of fuzzy partitions;
   - a commutative semicopula C if a one-to-one correspondence between C-based fuzzy equivalence relations and C-based fuzzy partitions is required.

An interesting generalization of convex functions in one variable to higher dimensions called ultramodular functions was proposed in [24]. We will study this concept in the framework of conjunctors.

Definition 1. A conjunctor $C\colon [0,1]^2 \to [0,1]$ is called *ultramodular* if for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0,1]^2$ with $\mathbf{x} + \mathbf{y} + \mathbf{z} \in [0,1]^2$ we have

$$C(\mathbf{x} + \mathbf{y} + \mathbf{z}) + C(\mathbf{x}) \geq C(\mathbf{x} + \mathbf{y}) + C(\mathbf{x} + \mathbf{z}). \tag{4}$$

Proposition 1. (i) *An aggregation function $C\colon [0,1]^2 \to [0,1]$ is a supermodular conjunctor if and only if there is a copula $K\colon [0,1]^2 \to [0,1]$ and non-decreasing functions $f, g\colon [0,1] \to [0,1]$ satisfying $f(1) = g(1) = 1$ and $K(f(0), g(0)) = 0$ such that, for all $(x,y) \in [0,1]^2$, $C(x,y) = K(f(x), g(y))$ (see* [10]).

(ii) *An aggregation function $C\colon [0,1]^2 \to [0,1]$ is an ultramodular conjunctor if and only if all of its one-dimensional sections are convex and if there is a copula $K\colon [0,1]^2 \to [0,1]$ such that, for all $(x,y) \in [0,1]^2$, $C(x,y) = K(C(x,1), C(1,y))$.*

Note that, for each ultramodular copula $K\colon [0,1]^2 \to [0,1]$ and for all non-decreasing convex functions $\varphi, \psi\colon [0,1] \to [0,1]$ satisfying $\varphi(0) = \psi(0) = 0$ and $\varphi(1) = \psi(1) = 1$, the function $C\colon [0,1]^2 \to [0,1]$ given by $C(x,y) = K(\varphi(x), \psi(y))$ is an ultramodular conjunctor. The ultramodular copulas form a convex compact set with smallest element W (lower Fréchet-Hoeffding bound) and greatest element Π (product).

Proposition 2. (i) *A semicopula $C\colon [0,1]^2 \to [0,1]$ is ultramodular if and only if it is a copula with convex one-dimensional sections.*

(ii) *If* $C\colon [0,1]^2 \to [0,1]$ *is an ultramodular associative semicopula then it is an Archimedean copula and, provided that its additive generator* $t\colon [0,1] \to [0,\infty]$ *is two times differentiable,* $\frac{1}{t'}$ *is non-increasing and convex.*

If we denote by $\mathcal{T}$ the set of all additive generators of continuous Archimedean t-norms, then for each $t\colon [0,1] \to [0,\infty]$ in $\mathcal{T}$ the function $C_t\colon [0,1]^2 \to [0,1]$ given by

$$C_t(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x \cdot t^{-1}\left(\min\left(t(0), \frac{t(y)}{x}\right)\right) & \text{otherwise,} \end{cases} \tag{5}$$

is a semicopula. Because of [7], C_t is a copula if and only if t is convex.

Proposition 3. *Suppose that* $t \in \mathcal{T}$ *is two times differentiable and that* $\frac{1}{t'}$ *is non-increasing and convex. Then* C_t *given by* (5) *is an ultramodular copula.*

Example 1. Define $t_1, t_2\colon [0,1] \to [0,\infty]$ by $t_1(x) = -\log x$ and $t_2(x) = \frac{1}{x} - 1$. Then, for $(x,y) \in]0,1] \times [0,1]$ we have

$$C_{t_1}(x,y) = x \cdot y^{\frac{1}{x}} \qquad \text{and} \qquad C_{t_2}(x,y) = \frac{x^2 y}{1 - y + xy}.$$

Both C_{t_1} and C_{t_2} are ultramodular copulas, but only t_1 satisfies the hypotheses of Proposition 3 (note that $\frac{1}{t_2'}$ is concave).

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The Value of the Two Values

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Abstract. Bilattices have proven again and again to be extremely rich structures from a logical point of view. As a matter of fact, even if we fix the canonical notion of many-valued entailment and consider the smallest non-trivial bilattice, distinct logics may be defined according to the chosen ontological or epistemological reading of the underlying truth-values. This note will explore the consequence relations of two very natural variants of Belnap's well-known 4-valued logic, and delve into their interrelationship. The strategy will be that of reformulating those logics using only two 'logical values', by way of uniform classic-like semantical and proof-theoretical frameworks, with the help of which such logics can be more easily compared to each other.

1 Introduction

Consider the order-bilattice $(\mathcal{V}, \leq_1, \leq_2)$ where $\mathcal{V} = \{t, \top, \bot, f\}$, the 'truth order' $\leq_1$ has t as its greatest element and f as its least element, as well as intermediate mutually incomparable elements $\top$ and $\bot$, and the 'information order' $\leq_2$ has $\top$ as its greatest element and $\bot$ as its least element, as well as intermediate mutually incomparable elements t and f. Construing $\mathcal{V}$ as a set of 'truth-values', we may consider the algebraic structures $\mathcal{L}_i = (\mathcal{V}, \wedge_i, \vee_i, \neg_i)$, for $i = 1, 2$, where $\wedge_i$ (resp. $\vee_i$) denotes the meet (resp. join) under $\leq_i$, and $\neg_i$ is an order-reversing involution for $\leq_i$ having the intermediate elements, in each case, as fixed-points. It is easy to see that these algebraic structures are 'interlaced', i.e., the operators of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) are all monotone with respect to $\leq_2$ (resp. $\leq_1$). Even stronger than that, all distributive laws hold between the two meets and the two joins. Morever, $\neg_1$ (called 'negation') and $\neg_2$ (called 'conflation') obviously commute, that is, $\neg_1\neg_2 x = \neg_2\neg_1 x$. To make the above structures more expressive we will also add their least and upper bounds as zero-ary operators. The result will be a strongly symmetric bidimensional structure:

$$\mathcal{B} = (\mathcal{V}, \wedge_1, \wedge_2, \vee_1, \vee_2, \neg_1, \neg_2, t, \top, \bot, f)$$

Let $\Gamma \cup \Delta$ be a collection of formulas from the term algebra $\mathcal{T}_{\mathcal{B}}$ corresponding to the structure $\mathcal{B}$, and let Hom denote the set of all homomorphisms from $\mathcal{T}_{\mathcal{B}}$

into $\mathcal{B}$. This is known as a 'truth-functional interpretation', and the meaning of each operator is said to be fixed by a 'truth-table'. The canonical notion of entailment $\models$ over the above mentioned truth order assumes some partition of the truth-values $\mathcal{V}$ into sets $\mathcal{D}$ (called 'designated values') and $\mathcal{U}$ (called 'undesignated values'), and $\Gamma \models \Delta$ is said to hold iff, for every $v \in \mathsf{Hom}$, either $v(\Gamma) \cap \mathcal{U} \neq \varnothing$ or $v(\Delta) \cap \mathcal{D} \neq \varnothing$. One should notice that such notion is remarkably sensitive to the choice of designated / undesignated values. There are at least three such choices, though, that would seem to make immediate sense from the viewpoint of the truth order:

[$\mathcal{V}_b$] $\mathcal{D}_b = \{t, \top, \bot\}$ and $\mathcal{U}_b = \{f\}$
[$\mathcal{V}_{e\ell}$] $\mathcal{D}_{e\ell} = \{t, \top\}$ and $\mathcal{U}_{e\ell} = \{\bot, f\}$
[$\mathcal{V}_n$] $\mathcal{D}_n = \{t\}$ and $\mathcal{U}_n = \{\top, \bot, f\}$

Choice [$\mathcal{V}_{e\ell}$] has in fact been intensely investigated in the literature, and the corresponding entailment relation, $\models_{e\ell}$, is known to be adequate for the so-called 'first-degree entailment'. It is both paraconsistent and paracomplete. On the other hand, the entailment relation $\models_b$, that corresponds to [$\mathcal{V}_b$] is paraconsistent but not paracomplete, and the exact opposite is the case for the entailment relation $\models_n$, that corresponds to [$\mathcal{V}_n$]. A reasonable rationale for the choice [$\mathcal{V}_b$], according to the ordinary 'truth-degree interpretation', is that one might be dealing with vague states-of-affairs in which some values should not be ascertained to be 'false', yet they are 'not quite true'. Analogously, for [$\mathcal{V}_n$], there may be other kinds of inexact states-of-affairs in which some values should not be ascertained to be 'true', yet they are 'not quite false'.

The present study will show in more detail what do such entailment relations have in common, and how do they differ from each other. The comparison will be made simpler when we recast the logics involved in terms of semantics and proof-systems that mention only *two* logical values or *two* syntactic labels, as it happens in classical logic.

2 A Closer Look at the Logical Operators

From the semantical point of view, a logical operator $\&$ called *conjunction* is often used to internalize at the object-language level a collection of properties that we attribute to the meta-linguistic 'and', such as:

[and_1] $v(\alpha\&\beta) \in \mathcal{D} \Rightarrow v(\alpha) \in \mathcal{D}$ and $v(\beta) \in \mathcal{D}$
[and_2] $v(\alpha\&\beta) \in \mathcal{D} \Leftarrow v(\alpha) \in \mathcal{D}$ and $v(\beta) \in \mathcal{D}$

Similarly, a logical operator $||$ called *disjunction* is often used to internalize properties that we attribute to the meta-linguistic 'or', such as:

$[\mathsf{or}_1]\quad v(\alpha \,||\, \beta) \in \mathcal{D} \;\Rightarrow\; v(\alpha) \in \mathcal{D}$ or $v(\beta) \in \mathcal{D}$
$[\mathsf{or}_2]\quad v(\alpha \,||\, \beta) \in \mathcal{D} \;\Leftarrow\; v(\alpha) \in \mathcal{D}$ or $v(\beta) \in \mathcal{D}$

Obviously, the use of a classical meta-language together with the above assumed partition of the truth-values in two classes allows us to immediately rewrite $[\mathsf{and}_2]$ and $[\mathsf{or}_2]$ as:

$[\mathsf{and}_2]\quad v(\alpha \& \beta) \in \mathcal{U} \;\Rightarrow\; v(\alpha) \in \mathcal{U}$ or $v(\beta) \in \mathcal{U}$
$[\mathsf{or}_2]\quad v(\alpha \,||\, \beta) \in \mathcal{U} \;\Rightarrow\; v(\alpha) \in \mathcal{U}$ and $v(\beta) \in \mathcal{U}$

As it turns out, according to $\models_{e\ell}$, each operator $\wedge_i$ enjoys properties $[\mathsf{and}_1]$ and $[\mathsf{and}_2]$, and each operator $\vee_i$ enjoys properties $[\mathsf{or}_1]$ and $[\mathsf{or}_2]$, for $i = 1, 2$. However, according to either $\models_b$ or $\models_n$, this only holds good for $i = 1$, that is, for the logical operators defined according to the truth-order $\leq_1$. Indeed, for both the latter entailment relations, on what concerns the operators defined according to the information order $\leq_2$, it can easily be checked that $\wedge_2$ enjoys property $[\mathsf{and}_2]$ but fails property $[\mathsf{and}_1]$, while $\vee_2$ enjoys property $[\mathsf{or}_1]$ but fails property $[\mathsf{or}_2]$. One can also argue (having more space!) that $\neg_2$ only behaves like a real *negation* according to $\models_{e\ell}$, but not according to $\models_b$ or to $\models_n$. This much for the similarities between $\models_b$ and $\models_n$. To properly understand how they differ, we will concentrate in what follows in the case of $\models_b$, as $\models_n$ may be seen to produce entirely dual results.

It's not overemphasizing to insist here that a quick look at the truth-tables of $\wedge_2$, with an eye at its behavior according to the entailment relations $\models_b$ and $\models_n$ will not immediately reveal what inferences are to be validated by one of these relations, and not by the other... It is equally far from obvious that the proof systems to be extracted from those truth-tables will be somehow comparable, as they might make some syntactic use of the specific partitions of $\mathcal{V}_b$ and $\mathcal{V}_n$... How should one proceed, then, for the comparison?

3 An Alternative Bivalent Semantics and an Adequate Proof-System for It

Let's now leave behind the idea that semantics should be presented truth-functionally in terms of a set of homomorphisms between two similar algebras, and start instead just from a set of two 'logical values' $\mathcal{V}_2 = \{1, 0\}$, partitioned into $\mathcal{D}_2 = \{1\}$ and $\mathcal{U}_2 = \{0\}$. A *bivalent* semantics Sem, now, will be just a collection of 'bivaluation' mappings of the form $b : \mathcal{T}_{\mathcal{B}} \longrightarrow \mathcal{V}_2$. Given any such collection of mappings, again, a canonical notion of entailment can be defined as usual: $\Gamma \models \Delta$ iff, for every $b \in \mathsf{Sem}$, either $b(\Gamma) \cap \mathcal{U}_2 \neq \varnothing$ or $b(\Delta) \cap \mathcal{D}_2 \neq \varnothing$.

To constructively provide a bivalent characterization of the values in $\mathcal{V}_b$, one should be able to distinguish between the many truth-values in $\mathcal{D}_b$, and in particular between the two intermediate values. That can be done, though, if one finds a one-variable formula $\theta(p)$ such that, given some $v_1, v_2 \in \mathsf{Hom}$ such that $v_1(p) = \top$ and $v_2(p) = \bot$, we have that $v_1(\theta(p)) \in \mathcal{U}_b$ yet $v_2(\theta(p)) \in \mathcal{D}_b$. This way we can map each 'algebraic value' in $\mathcal{D}_b$ into the 'logical value' 1, the only truth-value that belongs to $\mathcal{D}_2$, while still being able to 'tell the difference' when we are dealing with each of them. As it happens, the language of $\mathcal{B}$ is indeed sufficiently expressive to 'separate' these two values: just consider the formula $\theta_1(p)$ as $f \wedge_2 p$, and notice that $v_1(\theta_1(p)) \in \mathcal{U}_b$ when $v_1(p) = \top$, but $v_2(\theta_1(p)) \in \mathcal{D}_b$ when $v_2(p) = \bot$. Clearly, the joint combination of p with $\neg_1 p$ and with $\theta_1(p)$ can help in distinguishing each pair of truth-values from $\mathcal{D}_b$, and we shall make use of that in what follows.

The following result consists exactly in providing a way of writing down a bivalent description of the initial 4-valued truth-tables of $\mathcal{B}$ from the point of view of $\models_b$. Because there are many operators and the full description is therefore quite long, we will leave it to an Appendix.

Theorem 1. A sound and complete bivalent semantics for the paraconsistent logic behind choice $[\mathcal{V}_b]$ is given by the collection Sem_2^b of all bivaluations that respect all the clauses that can be found in the Appendix. □

To wit, the above theorem guarantees that $\Gamma \models_{\mathsf{Sem}_2^b} \Delta$ iff $\Gamma \models_b \Delta$.

If the mentioned clauses on bivaluations do not look as if they're defining an actual decision procedure for inferences related to $\models_b$, this might be just the time to directly use them to formulate instead an analytic proof system that will do the job. This can indeed be done in the following way.

Theorem 2. A sound and complete collection of tableau rules for the logic behind choice $[\mathcal{V}_b]$ is given by reading the bivalent clauses used in Theorem 1 (see Appendix) as two-signed tableau rules, in an appropriate way, namely:

- each expression of the form $b(\varphi) = w$ is rewritten as a signed formula $w{:}\varphi$;
- a '$\Rightarrow$' separates the head of a tableau rule (to the left) from its conclusions (to the right);
- each ',' is understood as separating nodes (signed formulas) from the same branch;
- an '$|$' at the right of a '$\Rightarrow$' demarcates bifurcations in the output of a given rule;
- an expression of the form '$h_1, \ldots, h_n \Rightarrow *$' denotes a closure rule. □

The class of rules that results from the above is clearly 'classic-like'. However, differently from what happens in the case of usual tableau systems for classical propositional logic, the blind application of such rules to test the validity of a given inference might not be terminating, if rules are chosen in a particularly unfortunate order. To fix that, and guarantee that the tableau system will indeed be analytic, one had better attach to it a convenient 'proof strategy'. Basically, the problem resides in the case where more than one rule is applicable to the same head. This might happen here if we recall that a formula of the form $\theta_1(\alpha)$ is actually just an abbreviation for $f \wedge_2 \alpha$. So, if a node of the form $w{:}\theta_1(\diamond(\overrightarrow{\alpha}))$

needs to be analyzed, one should decide whether it will be better to apply to it the rule $\mathsf{biv}[\wedge_2]\langle w\rangle$ or the rule $\mathsf{biv}[\theta_1\diamond]\langle w\rangle$. Our proof strategy in this case will choose to always apply $\mathsf{biv}[\theta_1\diamond]\langle w\rangle$ first, as it will guarantee that a certain non-canonical measure of complexity keeps decreasing. Moreover, according to this same proof strategy, for nodes of the form $w{:}\theta_1(p)$ or $w{:}\neg p$, where p is an atom, no further rule should be applied.

Now, the reformulation of $\models_n$ in terms of a bivalent semantics and the association of an analytic proof system for the corresponding logic can be done in exactly the same way as we did for $\models_b$, and the same can be done also for $\models_{e\ell}$. It will be a completely automatic task, then, to use the classic-like proof system associated, say, to $\models_n$ in order to test the validity of theorems and rules of the logic given, say, by $\models_b$. Moreover, the introduction of a suitable implication into $\mathcal{B}$ can simplify the above tasks, in making the underlying language even more expressive. The full paper will show details of all the above, and also show how much of the deeper interest behind the 4-valued approach can be thoroughly retained in the present classic-like bivalent / two-signed approach.

Appendix

Here one can find the exhaustive bivalent description of the 4-valued logic that corresponds to choice $[\mathcal{V}_b]$ and its associated entailment relation $\models_b$.

In the meta-linguistic notation below, a ',' replaces an 'and', a '|' replaces an 'or', a '⇒' replaces an 'if-then' assertion, and a '⋇' represents the absurd, so that a clause such as $\mathsf{biv}[C2]$ should be read as saying that states $b(\alpha) = 0$ and $b(\theta_1(\alpha)) = 1$ cannot simultaneously obtain, for any bivaluation b and formula α.

The expressions α and β, below, denote arbitrary formulas from $\mathcal{T}_\mathcal{B}$.

. . . .

$\mathsf{biv}[\wedge_1]\langle 1\rangle \quad b(\alpha \wedge_1 \beta) = 1 \;\Rightarrow$

$\quad (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 0\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1)$

$\quad \mid (b(\alpha) = 1\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 0\,,\; b(\theta_1(\beta)) = 1)$

$\quad \mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$

$\quad \mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$

$\mathsf{biv}[\wedge_1]\langle 0\rangle \quad b(\alpha \wedge_1 \beta) = 0 \;\Rightarrow$

$\quad (b(\alpha) = 0\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0)$

$\quad \mid (b(\beta) = 0\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$

$\quad \mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$

$\quad \mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$

$\mathsf{biv}[\neg_1\wedge_1]\langle 1\rangle \quad b(\neg_1(\alpha \wedge_1 \beta)) = 1 \;\Rightarrow$
$\quad (b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0)$
$\quad \mid (b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 1)$
$\quad \mid (b(\beta) = 1\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 1)$

$\mathsf{biv}[\neg_1\wedge_1]\langle 0\rangle \quad b(\neg_1(\alpha \wedge_1 \beta)) = 0 \;\Rightarrow$
$\quad b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 0\ ,\ b(\theta_1(\alpha)) = 1\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 0\ ,\ b(\theta_1(\beta)) = 1$

$\mathsf{biv}[\theta_1\wedge_1]\langle 1\rangle \quad b(\theta_1(\alpha \wedge_1 \beta)) = 1 \;\Rightarrow\; b(\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 1\ ,\ b(\beta) = 1\ ,\ b(\theta_1(\beta)) = 1$

$\mathsf{biv}[\theta_1\wedge_1]\langle 0\rangle \quad b(\theta_1(\alpha \wedge_1 \beta)) = 0 \;\Rightarrow$
$\quad (b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0)$
$\quad \mid (b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$

. . . .

$\mathsf{biv}[\wedge_2]\langle 1\rangle \quad b(\alpha \wedge_2 \beta) = 1 \;\Rightarrow$
$\quad (b(\alpha) = 1\ ,\ b(\beta) = 1)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 1)$
$\quad \mid (b(\beta) = 1\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 1)$

$\mathsf{biv}[\wedge_2]\langle 0\rangle \quad b(\alpha \wedge_2 \beta) = 0 \;\Rightarrow$
$\quad (b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 0\ ,\ b(\theta_1(\alpha)) = 1\ ,\ b(\beta) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\alpha) = 0\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 0\ ,\ b(\theta_1(\beta)) = 1)$
$\quad \mid (b(\alpha) = 0\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$

$\mathsf{biv}[\neg_1\wedge_2]\langle 1\rangle \quad b(\neg_1(\alpha \wedge_2 \beta)) = 1 \;\Rightarrow$
$\quad (b(\alpha) = 0\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0)$
$\quad \mid (b(\beta) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 1)$
$\quad \mid (b(\beta) = 1\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 1)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$

$\mathsf{biv}[\neg_1\wedge_2]\langle 0\rangle \quad b(\neg_1(\alpha \wedge_2 \beta)) = 0 \;\Rightarrow$
$\quad (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 0\ ,\ b(\theta_1(\alpha)) = 1\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 0\ ,\ b(\theta_1(\beta)) = 1)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 0\ ,\ b(\theta_1(\alpha)) = 1\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 0\ ,\ b(\theta_1(\beta)) = 1)$

$\mathsf{biv}[\theta_1\wedge_2]\langle 1\rangle \quad b(\theta_1(\alpha \wedge_2 \beta)) = 1 \;\Rightarrow$
$\quad (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 0\ ,\ b(\theta_1(\alpha)) = 1\ ,\ b(\beta) = 1)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 0\ ,\ b(\theta_1(\beta)) = 1)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 1)$
$\quad \mid (b(\beta) = 1\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 1)$

$\mathsf{biv}[\theta_1\wedge_2]\langle 0\rangle \quad b(\theta_1(\alpha \wedge_2 \beta)) = 0 \;\Rightarrow$
$\quad (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\alpha) = 0\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 1\ ,\ b(\neg_1\beta) = 0\ ,\ b(\theta_1(\beta)) = 1)$
$\quad \mid (b(\alpha) = 0\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\alpha) = 1\ ,\ b(\neg_1\alpha) = 0\ ,\ b(\theta_1(\alpha)) = 1\ ,\ b(\beta) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$
$\quad \mid (b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$

. . . .

$\mathsf{biv}[\vee_1]\langle 1\rangle \quad b(\alpha \vee_1 \beta) = 1 \;\Rightarrow\; (b(\alpha) = 1 \ \mid\ b(\beta) = 1)$

$\mathsf{biv}[\vee_1]\langle 0\rangle \quad b(\alpha \vee_1 \beta) = 0 \;\Rightarrow$
$\quad (b(\alpha) = 0\ ,\ b(\neg_1\alpha) = 1\ ,\ b(\theta_1(\alpha)) = 0\ ,\ b(\beta) = 0\ ,\ b(\neg_1\beta) = 1\ ,\ b(\theta_1(\beta)) = 0)$

biv$[\neg_1\vee_1]\langle 1\rangle$ $\quad b(\neg_1(\alpha \vee_1 \beta)) = 1 \;\Rightarrow$
$(b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$
$\mid (b(\alpha) = 0\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 0\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$

biv$[\neg_1\vee_1]\langle 0\rangle$ $\quad b(\neg_1(\alpha \vee_1 \beta)) = 0 \;\Rightarrow$
$(b(\alpha) = 1\,,\; b(\neg_1\alpha) = 0\,,\; b(\theta_1(\alpha)) = 1)$
$\mid (b(\beta) = 1\,,\; b(\neg_1\beta) = 0\,,\; b(\theta_1(\beta)) = 1)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$

biv$[\theta_1\vee_1]\langle 1\rangle$ $\quad b(\theta_1(\alpha \vee_1 \beta)) = 1 \;\Rightarrow$
$(b(\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1)$
$\mid (b(\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$

biv$[\theta_1\vee_1]\langle 0\rangle$ $\quad b(\theta_1(\alpha \vee_1 \beta)) = 0 \;\Rightarrow$
$(b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$

. . . .

biv$[\vee_2]\langle 1\rangle$ $\quad b(\alpha \vee_2 \beta) = 1 \;\Rightarrow$
$(b(\alpha) = 1\,,\; b(\beta) = 1)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 0\,,\; b(\theta_1(\alpha)) = 1)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0)$
$\mid (b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$
$\mid (b(\beta) = 1\,,\; b(\neg_1\beta) = 0\,,\; b(\theta_1(\beta)) = 1)$

biv$[\vee_2]\langle 0\rangle$ $\quad b(\alpha \vee_2 \beta) = 0 \;\Rightarrow$
$(b(\alpha) = 0\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\beta) = 0\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$
$\mid (b(\alpha) = 0\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 0\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$

biv$[\neg_1\vee_2]\langle 1\rangle$ $\quad b(\neg_1(\alpha \vee_2 \beta)) = 1 \;\Rightarrow$
$(b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0)$
$\mid (b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$
$\mid (b(\alpha) = 1\,,\; b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$

biv$[\neg_1\vee_2]\langle 0\rangle$ $\quad b(\neg_1(\alpha \vee_2 \beta)) = 0 \;\Rightarrow$
$(b(\alpha) = 1\,,\; b(\neg_1\alpha) = 0\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$
$\mid (b(\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\neg_1\beta) = 0\,,\; b(\theta_1(\beta)) = 1)$

biv$[\theta_1\vee_2]\langle 1\rangle$ $\quad b(\theta_1(\alpha \vee_2 \beta)) = 1 \;\Rightarrow$
$(b(\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1\,,\; b(\beta) = 1\,,\; b(\theta_1(\beta)) = 1)$

biv$[\theta_1\vee_2]\langle 0\rangle$ $\quad b(\theta_1(\alpha \vee_2 \beta)) = 0 \;\Rightarrow$
$(b(\neg_1\alpha) = 1\,,\; b(\theta_1(\alpha)) = 0)$
$\mid (b(\neg_1\beta) = 1\,,\; b(\theta_1(\beta)) = 0)$

. . . .

biv$[\neg_1]\langle 1\rangle$ $\quad b(\neg_1\alpha) = 1 \;\Rightarrow\; b(\alpha) = 1 \mid (b(\alpha) = 0\,,\; b(\theta_1(\alpha)) = 0)$

biv$[\neg_1]\langle 0\rangle$ $\quad b(\neg_1\alpha) = 0 \;\Rightarrow\; (b(\alpha) = 1\,,\; b(\theta_1(\alpha)) = 1)$

biv$[\neg_1\neg_1]\langle 1\rangle$ $\quad b(\neg_1\neg_1\alpha) = 1 \;\Rightarrow\; b(\alpha) = 1$

biv$[\neg_1\neg_1]\langle 0\rangle$ $\quad b(\neg_1\neg_1\alpha) = 0 \;\Rightarrow\; b(\alpha) = 0$

$\mathsf{biv}[\theta_1\neg_1]\langle 1\rangle \quad b(\theta_1(\neg_1\alpha)) = 1 \;\Rightarrow$
$(b(\alpha) = 0\;,\; b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 0)$
$\mid\; (b(\alpha) = 1\;,\; b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 1)$

$\mathsf{biv}[\theta_1\neg_1]\langle 0\rangle \quad b(\theta_1(\neg_1\alpha)) = 1 \;\Rightarrow$
$(b(\alpha) = 1\;,\; b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 0)$
$\mid\; (b(\alpha) = 1\;,\; b(\neg_1\alpha) = 0\;,\; b(\theta_1(\alpha)) = 1)$

....

$\mathsf{biv}[\neg_2]\langle 1\rangle \quad b(\neg_2\alpha) = 1 \;\Rightarrow\; b(\alpha) = 1$

$\mathsf{biv}[\neg_2]\langle 0\rangle \quad b(\neg_2\alpha) = 0 \;\Rightarrow\; b(\alpha) = 0$

$\mathsf{biv}[\neg_1\neg_2]\langle 1\rangle \quad b(\neg_2\alpha) = 1 \;\Rightarrow$
$(b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 0)$
$\mid\; (b(\alpha) = 1\;,\; b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 1)$

$\mathsf{biv}[\neg_1\neg_2]\langle 0\rangle \quad b(\neg_2\alpha) = 0 \;\Rightarrow\; (b(\alpha) = 1\;,\; b(\neg_1\alpha) = 0\;,\; b(\theta_1(\alpha)) = 1)$

$\mathsf{biv}[\theta_1\neg_2]\langle 1\rangle \quad b(\theta_1(\neg_2\alpha)) = 1 \;\Rightarrow$
$(b(\alpha) = 0\;,\; b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 0)$
$\mid\; (b(\alpha) = 1\;,\; b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 1)$

$\mathsf{biv}[\theta_1\neg_2]\langle 0\rangle \quad b(\theta_1(\neg_2\alpha)) = 1 \;\Rightarrow$
$(b(\alpha) = 1\;,\; b(\neg_1\alpha) = 1\;,\; b(\theta_1(\alpha)) = 0)$
$\mid\; (b(\alpha) = 1\;,\; b(\neg_1\alpha) = 0\;,\; b(\theta_1(\alpha)) = 1)$

....

$\mathsf{biv}[t] \quad b(t) = 0 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\neg_1 t] \quad b(\neg_1 t) = 1 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\theta_1 t] \quad b(\theta_1(t)) = 0 \;\Rightarrow\; ⁂$

$\mathsf{biv}[\top] \quad b(\top) = 0 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\neg_1\top] \quad b(\neg_1\top) = 0 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\theta_1\top] \quad b(\theta_1(\top)) = 1 \;\Rightarrow\; ⁂$

$\mathsf{biv}[\bot] \quad b(\bot) = 0 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\neg_1\bot] \quad b(\neg_1\bot) = 0 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\theta_1\bot] \quad b(\theta_1(\bot)) = 0 \;\Rightarrow\; ⁂$

$\mathsf{biv}[f] \quad b(f) = 1 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\neg_1 f] \quad b(\neg_1 f) = 0 \;\Rightarrow\; ⁂$
$\mathsf{biv}[\theta_1 f] \quad b(\theta_1(f)) = 1 \;\Rightarrow\; ⁂$

....

$\mathsf{biv}[C0] \quad (b(\alpha) = 0\;,\; b(\alpha) = 1) \;\Rightarrow\; ⁂$
$\mathsf{biv}[C1] \quad (b(\alpha) = 0\;,\; b(\neg_1\alpha) = 0) \;\Rightarrow\; ⁂$
$\mathsf{biv}[C2] \quad (b(\alpha) = 0\;,\; b(\theta_1(\alpha)) = 1) \;\Rightarrow\; ⁂$
$\mathsf{biv}[C3] \quad (b(\neg_1\alpha) = 0\;,\; b(\theta_1(\alpha)) = 0) \;\Rightarrow\; ⁂$

Fuzzy topology and Łukasiewicz logics from the viewpoint of duality theory

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1 Introduction

Our aim is to explore relationships between many-valued logic (see, e.g., [8, 9, 2]) and fuzzy topology (see, e.g., [13, 3, 7]) from the viewpoint of duality theory (see, e.g., [10]). In particular we consdier fuzzy topological dualities for the algebras of Łukasiewicz n-valued logic $Ł_n^c$ with truth constants and for the algebras of modal Łukasiewicz n-valued logic $MŁ_n^c$ with truth constants, where $MŁ_n^c$ is based on modal Łukasiewicz n-valued logic (without truth constants except $0, 1$) in [14] (modal many-valued logic is considered also in [5]). We emphasize that fuzzy topology naturally arises in the context of many-valued logic.

Stone duality for Boolean algebras (see, e.g., [10]) is one of the most important results in algebraic logic and states that there is a categorical duality between Boolean algebras (i.e., the algebras of classical propositional logic) and Boolean spaces (i.e., zero-dimensional compact Hausdorff spaces). Since both many-valued logic and fuzzy topology can be considered as based on the same idea that there are more than two truth values, it is natural to expect that there is a duality between the algebras of many-valued logic and "fuzzy Boolean spaces." Stone duality for Boolean algebras was extended to Jónsson-Tarski duality (see, e.g., [1]) between modal algebras and relational spaces (or descriptive general frames), which is another classical theorem in duality theory. Thus, it is also natural to expect that there is a duality between the algebras of modal many-valued logic and "fuzzy relational spaces."

We realize the above expectations in the cases of $Ł_n^c$ and $MŁ_n^c$. We first develop a categorical duality between the algebras of $Ł_n^c$ and $\mathbf{n}$-fuzzy Boolean spaces (see Definition 9), which is a generalization of Stone duality for Boolean algebras to the $\mathbf{n}$-valued case via fuzzy topology. This duality is developed based on the following insights:

1. The spectrum of an algebra of $Ł_n^c$ (i.e., the set of prime $\mathbf{n}$-filters of it) can be naturally equipped with a certain $\mathbf{n}$-fuzzy topology, where $\mathbf{n}$-filter is deduc-

tive filter in the sense of [6] and its primeness is defined in the usual way (i.e., by the condition that $x \vee y \in P$ implies either $x \in P$ or $y \in P$).

2. The notion of clopen subset of Boolean space in Stone duality for Boolean algebras corresponds to that of continuous function from $\mathbf{n}$-fuzzy Boolean space to $\mathbf{n}$ ($= \{0, 1/(n-1), 2/(n-1), ..., 1\}$) equipped with the $\mathbf{n}$-fuzzy discrete topology in the duality for the algebras of $Ł_n^c$. This means that the zero-dimensionality of $\mathbf{n}$-fuzzy topological spaces is defined in terms of continuous function into $\mathbf{n}$ (see Definition 8).

Moreover, based on the duality for the algebras of $Ł_n^c$, we develop a categorical duality between the algebras of $MŁ_n^c$ and $\mathbf{n}$-fuzzy relational spaces (see Definition 12), which is a generalization of Jónsson-Tarski duality for modal algebras to the $\mathbf{n}$-valued case via fuzzy topology. Note that an $\mathbf{n}$-fuzzy relational space is also defined in terms of continuous functions into $\mathbf{n}$ (see Definition 12).

There have been some studies on dualities for algebras of many-valued logics (see, e.g., [4, 11, 12, 14]). However, they are based on the ordinary topology and therefore do not reveal relationships between many-valued logic and fuzzy topology. By the results in this paper, we can notice that fuzzy topological spaces naturally arise as spectrums of algebras of some many-valued logics and that there are categorical dualities connecting fuzzy topology and those many-valued logics which generalize Stone and Jónsson-Tarski dualities via fuzzy topology.

2 Fuzzy topological duality for algebras of $Ł_n^c$

Definition 1. *Let* $\mathbf{n}$ *denote* $\{0, 1/(n-1), 2/(n-1), ..., 1\}$. *We equip* $\mathbf{n}$ *with all constants* $r \in \mathbf{n}$ *and the operations* $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp})$ *defined as follows:*

$$
\begin{aligned}
x \wedge y &= \min(x, y); \\
x \vee y &= \max(x, y); \\
x * y &= \max(0, x + y - 1); \\
x \wp y &= \min(1, x + y); \\
x \rightarrow y &= \min(1, 1 - (x - y)); \\
x^{\perp} &= 1 - x.
\end{aligned}
$$

Let $\mathbf{n}^S$ denote the set of all functions from S to $\mathbf{n}$. Operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, r \in \mathbf{n})$ on $\mathbf{n}^S$ are defined pointwise ($r \in \mathbf{n}$ is considered as a constant function on S whose value is always r).

Definition 2 **([3, 7]).** *For a set* S *and a subset* $\mathcal{O}$ *of* $\mathbf{n}^S$, $(S, \mathcal{O})$ *is an (stratified)* $\mathbf{n}$*-fuzzy space iff the following hold:*

1. *$r \in \mathcal{O}$ for any $r \in \mathbf{n}$;*
2. *if $\mu_1, \mu_2 \in \mathcal{O}$ then $\mu_1 \wedge \mu_2 \in \mathcal{O}$;*
3. *if $\mu_i \in \mathcal{O}$ for $i \in I$ then $\bigvee_{i \in I} \mu_i \in \mathcal{O}$,*

Then, we call $\mathcal{O}$ the $\mathbf{n}$-fuzzy topology of $(S, \mathcal{O})$, and an element of $\mathcal{O}$ an open $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$. An $\mathbf{n}$-fuzzy set λ on S is a closed $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$ iff $\lambda = \mu^{\perp}$ for some open $\mathbf{n}$-fuzzy set μ on $(S, \mathcal{O})$.

Definition 3. *Let S_1 and S_2 be $\mathbf{n}$-fuzzy spaces. Then, a function $f : S_1 \rightarrow S_2$ is continuous iff, for any open $\mathbf{n}$-fuzzy set μ on S_2, $f^{-1}(\mu)$, which is defined as $\mu \circ f$, is an open $\mathbf{n}$-fuzzy set on S_1.*

Definition 4. *Let $(S, \mathcal{O})$ be an $\mathbf{n}$-fuzzy space. Then, an open basis $\mathcal{B}$ of $(S, \mathcal{O})$ is a subset of $\mathcal{O}$ such that the following holds: (i) $\mathcal{B}$ is closed under $\wedge$; (ii) for any $\mu \in \mathcal{O}$, there are $\mu_i \in \mathcal{B}$ for $i \in I$ with $\mu = \bigvee_{i \in I} \mu_i$.*

Definition 5. *An $\mathbf{n}$-fuzzy space S is Kolmogorov iff, for any $x, y \in S$ with $x \neq y$, there is an open $\mathbf{n}$-fuzzy set μ on S with $\mu(x) \neq \mu(y)$.*

Definition 6. *Let 1 denote the constant function on S whose value is always 1. Then, S is compact iff, if $1 = \bigvee_{i \in I} \mu_i$ for open $\mathbf{n}$-fuzzy sets μ_i on S, then there is a finite subset J of I such that $1 = \bigvee_{i \in J} \mu_i$.*

We equip $\mathbf{n}$ with the discrete $\mathbf{n}$-fuzzy topology (i.e., $\mathbf{n}^S$).

Definition 7. *Let S be an $\mathbf{n}$-fuzzy space. Then, $\mathrm{Cont}(S)$ is defined as the set of all continuous functions from S to $\mathbf{n}$. We endow $\mathrm{Cont}(S)$ with the pointwise operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, r \in \mathbf{n})$.*

Definition 8. *For an $\mathbf{n}$-fuzzy space S, S is zero-dimensional iff $\mathrm{Cont}(S)$ forms an open basis of S.*

Definition 9. *For an $\mathbf{n}$-fuzzy space S, S is an $\mathbf{n}$-fuzzy Boolean space iff S is zero-dimensional, compact and Kolmogorov.*

The following theorem is a main result. A homomorphism of algebras of $Ł_n^c$ is defined as a function which preserves all the operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, r \in \mathbf{n})$.

Theorem 1. *The category of algebras of $Ł_n^c$ and their homomorphisms is dually equivalent to the category of $\mathbf{n}$-fuzzy Boolean spaces and continuous functions.*

3 Fuzzy topological duality for algebras of $\mathrm{MŁ}_n^c$

As in [14, 5], we define modal Łukasiewicz n-valued logic $\mathrm{MŁ}_n^c$ with truth constants by using a many-valued version of Kripke semantics. The connectives of $\mathrm{MŁ}_n^c$ are a unary connective $\Box$ plus those of $\mathrm{Ł}_n^c$ (i.e., $\wedge, \vee, *, \wp, \rightarrow, (\text{-})^\perp, r \in \mathbf{n}$). $\mathbf{Form}_\Box$ denotes the set of formulas of $\mathrm{MŁ}_n^c$.

Definition 10. *Let (W, R) be a Kripke frame (i.e., R is a relation on a set W). Then, e is a Kripke $\mathbf{n}$-valuation on (W, R) iff e is a function from $W \times \mathbf{Form}_\Box$ to $\mathbf{n}$ which satisfies: For each $w \in W$ and $\varphi, \psi \in \mathbf{Form}_\Box$,*

1. $e(w, \Box\varphi) = \bigwedge\{e(w', \varphi)\ ;\ wRw'\}$;
2. $e(w, \varphi @ \psi) = e(w, \varphi) @ e(w, \psi)$ *for* $@ = \wedge, \vee, *, \wp, \rightarrow$;
3. $e(w, \varphi^\perp) = (e(w, \varphi))^\perp$;
4. $e(w, r) = r$ *for* $r \in \mathbf{n}$.

Then, (W, R, e) is called an $\mathbf{n}$-valued Kripke model. Define $\mathrm{MŁ}_n^c$ as the set of all those formulas $\varphi \in \mathbf{Form}_\Box$ such that $e(w, \varphi) = 1$ for any $\mathbf{n}$-valued Kripke model (W, R, e) and any $w \in W$.

Definition 11. *Let (S, R) be a Kripke frame and f a function from S to $\mathbf{n}$. Define $\Box_R f : S \rightarrow \mathbf{n}$ by $(\Box_R f)(x) = \bigwedge\{f(y)\ ;\ xRy\}$.*

For a Kripke frame (S, R) and an $\mathbf{n}$-fuzzy set μ on S, an $\mathbf{n}$-fuzzy set $R^{-1}[\mu]$ on S is defined by $R^{-1}[\mu](x) = \bigvee\{\mu(y)\ ;\ xRy\}$ for $x \in S$.

Definition 12. *An $\mathbf{n}$-fuzzy relational space is defined as a tuple (S, R) such that S is an $\mathbf{n}$-fuzzy Boolean space and that a relation R on S satisfies the following conditions:*

1. *if* $\forall f \in \mathrm{Cont}(S)((\Box_R f)(x) = 1 \Rightarrow f(y) = 1)$ *then* xRy;
2. *if* $\mu \in \mathrm{Cont}(S)$, *then* $R^{-1}[\mu] \in \mathrm{Cont}(S)$.

Definition 13. *A continuous bounded morphism $f : (S_1, R_1) \rightarrow (S_2, R_2)$ between $\mathbf{n}$-fuzzy relational spaces is defined as a continuous function $f : S_1 \rightarrow S_2$ which satisfies the following conditions:*

1. *if* xR_1y *then* $f(x)R_2f(y)$;
2. *if* $f(x_1)R_2x_2$ *then there is* $y_1 \in S_1$ *such that* $x_1R_1y_1$ *and* $f(y_1) = x_2$.

The following theorem is the other main result. A homomorphism of algebras of $\mathrm{MŁ}_n^c$ is defined as a function which preserves all the operations: $\wedge, \vee, *, \wp, \rightarrow, (\text{-})^\perp, r \in \mathbf{n}$ and $\Box$.

Theorem 2. *The category of algebras of $\mathrm{MŁ}_n^c$ and their homomorphisms is dually equivalent to the category of $\mathbf{n}$-fuzzy relational spaces and continuous bounded morphisms.*

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Decidability for lattice-valued logics

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Decidability problems – ascertaining whether or not an effective method exists for determining membership of a given class – have long played a prominent role in both logic and algebra, bridging the gap between abstract presentations and computational methods. Perhaps most significant are the decidability of validity for a given logic and (which amounts to the same thing for algebraizable logics) the equational theory of some class of algebras. Intriguingly, to address such problems for lattice-valued logics, in particular, substructural and fuzzy logics based on classes of residuated lattices, methods from both fields – logic and algebra – appear to be essential.

On the one hand, syntactic approaches – typically involving cut-elimination for Gentzen systems – have been used to prove decidability for the full Lambek calculus, fragments of linear logic, and relevance logics, as well as interesting classes of algebras such as distributive residuated lattices and lattice ordered abelian groups (see, e.g., [5, 9]). On the other hand, algebraic methods such as establishing the (strong) finite model property or finite embeddability property, have been used to prove decidability for a wide range of logics and classes of algebras obeying some kind of integrality (weakening) or idempotency (contraction) conditions (see, e.g., [1, 5]) . The intention of this talk is to describe both the syntactic and algebraic approaches to establishing decidability, illustrated with examples of lattice-valued logics taken from a wide range of sources.

Recall that a *pointed commutative residuated lattice* (PCRL for short) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 1, 0\rangle$ such that: $\langle A, \wedge, \vee\rangle$ is a lattice; $\langle A, \cdot, 1\rangle$ is a commutative monoid; and $x \cdot y \leq z$ iff $x \leq y \to z$ for all $x, y, z \in A$. Also, $\neg x = x \to 0$; $x^0 = 1$; and $x^{n+1} = x \cdot x^n$ for $n \in \mathbb{N}$. $\mathbf{A}$ is *integral* if $x \leq 1$ for all $x \in A$; *idempotent* if $x = x^2$ for all $x \in A$; and *involutive* if $\neg\neg x = x$ for all $x \in A$. The following lattice-valued logics, all based on classes of (expanded) PCRLs, will be considered in this talk:

– *Nelson's constructive logic with strong negation.* It has been shown by Spinks and Veroff in [10] that the variety of Nelson algebras, algebraic semantics for constructive logic with strong negation N, is term-equivalent to the variety of *Nelson residuated lattices*: integral involutive PCRLs satisfying

$$((x^2 \to y) \wedge ((\neg y)^2 \to \neg x)) \to (x \to y) \approx 1.$$

A sequent calculus is obtained for this variety in [7] by extending a system (essentially $\mathsf{InFL_{ew}}$ or AMALL, see, e.g., [5]) for integral involutive PCRLs with a single structural rule. Using the translation of [10], this is then also a calculus for the logic N that provides, among other things, a new and easy proof of decidability, and can be extended to obtain calculi and decidability proofs for logics such as nilpotent minimum logic NM and Łukasiewicz three-valued logic $\mathsf{Ł_3}$.

- *Casari's comparative logics.* Some logics, not fuzzy in the sense of being characterized by chains, nevertheless have certain "fuzzy features". In particular, logics for comparative reasoning introduced by Casari in the 1980s provide an alternative truth degree semantics for modeling vagueness [4]. Algebras for these logics, called lattice-ordered pregroups, have degrees of both truth and falsity related by an involutive negation, and (possibly) intermediate degrees between. In the language of residuated lattices, they are involutive PCRLs satisfying

$$0 \approx 0 \cdot 0 \quad \text{and} \quad x \to x \approx 1.$$

Decidability is established for Casari's basic comparative logic (and some variants) in [6] by combining a sequent calculus for linear logic without exponentials ($\mathsf{InFL_e}$ or MALL) with a hypersequent calculus for lattice-ordered abelian groups.

- *Gödel modal logics.* In the general framework of Bou et al. [2], fuzzy modal logics can be based on Kripke models where the accessibility relation between worlds are either Boolean-valued (crisp) or many-valued (fuzzy) and propositional connectives operate as usual for a fixed logic at an individual world, while the values of formulas $\Box\varphi$ and $\Diamond\varphi$ are based on the values of φ at accessible (to some degree) worlds. Axiomatizations for Gödel "K" modal logics with the $\Box$ or the $\Diamond$ modality (so far not both) have been provided by Caicedo and Rodríguez in [3]. In particular, the algebras of the fragments with $\Box$ (which coincide) are term-equivalent to integral idempotent PCRLs expanded with a unary operator $\Box$ satisfying:

$$\Box(x \wedge y) = \Box x \wedge \Box y; \qquad 1 \approx \Box 1; \qquad \neg\neg\Box x \leq \Box\neg\neg x.$$

A proof-theoretic proof of decidability for this logic, indeed PSPACE-completeness, is given in [8], subsequently extended to the (different) diamond fragments.

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EQ-logics: fuzzy logics based on fuzzy equality

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Mathematical logic has been for many years developed on the basis of implication as the main connective. In eighties, another direction has been initiated which is called equational logic (see [5, 6]). The emphasis in it lays in the style of proofs in which substitution of equals for equals, instead of modus ponens is used. Thus, equality, or equivalence, assumes an important role instead. Recently, a new book developing classical boolean logic on the basis of equivalence as the main (not sole) connective (see [15]) has been published. Besides others, the logic in this book is developed in "equational style", which means that proofs proceed as sequences of equations (in fact, equivalences) using equational style axioms and special inference rules (without modus ponens). The motivation there stems from the algorithmic approach. The idea of equational style of logic, however, is much older and goes to G. W. Leibnitz (cf. [2]). Let us also recall the the type theory (higher order logic) of L. Henkin in [9] who developed it using identity as a sole connective.

This brought an idea whether also fuzzy logic could be developed on the basis of fuzzy equality as the principal connective, where by fuzzy equality we mean, in fact, a fuzzy equivalence, thus generalizing classical equivalence.

In this case, there are two possibilities for the choice of the necessary algebraic structure of truth values. First, we can take it to be a residuated lattice (cf., e.g., [7]). The residuum $\rightarrow$ operation is a natural interpretation of implication. This is a primary connective while the equivalence is interpreted by a biresiduation $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ is a derived operation.

It would be, however, unnatural to interpret the basic connective (in our case the fuzzy equality) by a derived operation. Therefore a special kind of algebra called EQ-algebra (see [10, 13]) has been introduced, in which fuzzy equality is the basic operation and implication is derived from it. Let us recall this algebra.

Definition 1. *EQ-algebra $\mathcal{E}$ is an algebra of type (2, 2, 2, 0), i.e.*

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle,$$

where for all $a, b, c, d \in E$:

(E1) $\langle E, \wedge, \mathbf{1}\rangle$ *is a commutative idempotent monoid (i.e.* $\wedge$*-semilattice with top element* $\mathbf{1}$*). We put* $a \leq b$ *iff* $a \wedge b = a$*, as usual.*
(E2) $\langle E, \otimes, \mathbf{1}\rangle$ *is a commutative monoid and* $\otimes$ *is isotone w.r.t.* $\leq$*.*
(E3) $a \sim a = \mathbf{1}$*,*
(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$*,*
(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$*,*
(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$*,*
(E7) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$*,*
(E8) $a \otimes b \leq a \sim b$*.*

The operation $\wedge$ is called meet (infimum), $\otimes$ is called product and $\sim$ is a fuzzy equality. Notice, that EQ-algebra has been generalized later ($\otimes$ needs not be either commutative or associative) and it is called by semicopula-based EQ-algebra (see [4]) in more detail).

Formal many-valued logic built on EQ-algebras called EQ-logic[1] has been introduced in [14]. The basic connectives of it are equivalence $\equiv$, conjunction $\boldsymbol{\wedge}$ and fusion $\boldsymbol{\&}$. Implication is defined as

$$A \Rightarrow B := (A \boldsymbol{\wedge} B) \equiv A. \tag{1}$$

A more detailed investigation showed that the structure of truth values for EQ-logic must be "good" to be strong enough. It means that the property $a \sim \mathbf{1} = a$ must hold in it.

In this paper we continue the work on EQ-logics, both propositional as well as predicate first order ones. The goal is to show a possible direction in the development of mathematical fuzzy logics, in which axioms are formed as identities. We focus on three types of propositional EQ-logics — basic, involutive and prelinear one and prove completeness of each of them. We also provide proofs of basic properties in equational style.

The basic EQ-logic seems to be the simplest logic definable on the basis of EQ-algebras. It has the following axioms:

(EQ1) $(A \equiv \top) \equiv A$
(EQ2) $A \boldsymbol{\wedge} B \equiv B \boldsymbol{\wedge} A$
(EQ3) $(A \boldsymbol{\wedge} B) \boldsymbol{\wedge} C \equiv A \boldsymbol{\wedge} (B \boldsymbol{\wedge} C)$
(EQ4) $A \boldsymbol{\wedge} A \equiv A$
(EQ5) $A \boldsymbol{\wedge} \top \equiv A$
(EQ6) $A \boldsymbol{\&} \top \equiv A$
(EQ7) $\top \boldsymbol{\&} A \equiv A$
(EQ8a) $((A \boldsymbol{\wedge} B) \boldsymbol{\&} C) \Rightarrow (B \boldsymbol{\&} C)$

[1] The term "fuzzy equational logic" has been introduced in [1] in a much narrower meaning.

(EQ8b) $(C \,\&(A \wedge B)) \Rightarrow (C \,\& B)$
(EQ8) $((A \wedge B) \equiv C) \,\&(D \equiv A) \Rightarrow (C \equiv (D \wedge B))$
(EQ9) $(A \equiv B) \,\&(C \equiv D) \Rightarrow (A \equiv C) \equiv (D \equiv B)$
(EQ10) $(A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$
(EQ11) $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B)$

($\top$ is logical truth). The following are inference rules:

$$(EA) \quad \frac{A, A \equiv B}{B} \qquad (L) \quad \frac{B \equiv C}{A[p := B] \equiv A[p := C]}$$

By $A[p := B]$ we denote a formula resulting from A by replacing all occurrences of a propositional variable p in A by the formula B. The rule (EA) is the *equanimity rule* and (L) is the *Leibnitz rule* (cf. [15]). Semantics of this logic is formed by good semicopula-based EQ-algebras.

Involutive EQ-logic (IEQ-logic), unlike basic EQ-logic, contains falsity $\bot$ and is characterized by the property of double negation. The truth, negation and disjunction are defined as follows:

$$\top := \bot \equiv \bot,$$
$$\neg A := A \equiv \bot,$$
$$A \vee B := \neg(\neg A \wedge \neg B).$$

Axioms of IEQ-logic are (EQ2)–(EQ11) and the following:

(EQ13) $(A \,\& B) \,\& C \equiv A \,\&(B \,\& C)$,
(EQ14) $(A \wedge \bot) \equiv \bot$,
(EQ15) $\neg\neg A \equiv A$.

Semantics is formed by non-commutative involutive EQ-algebras (IEQ-algebras are EQ-algebras with $\neg\neg a = a$ where $\neg a = a \sim \mathbf{0}$).

The third class of the considered EQ-logics is prelinear EQ-logic. Its axioms are (EQ1)–(EQ14) plus

(EQ16) $(A \Rightarrow B) \vee (D \Rightarrow (D \,\&(C \Rightarrow (B \Rightarrow A) \,\& C)))$,

where the disjunction $\vee$ is defined as

$$A \vee B := ((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A).$$

Semantics is formed by good non-commutative prelinear EQ-algebras (prelinear EQ-algebras are EQ-algebras with prelinear property, see [3]).

EQ-logics lie even deeper than the MTL-algebra-based (core) fuzzy logics (cf. [8]). Our results raise some interesting philosophical questions. For example, (fuzzy) equality seem to be a more fundamental (and simpler) concept than (fuzzy) implication. Our results may shed light on the long-existing question regarding the essence of implication. Another question: why are the mentioned "goodness axiom" and, at least, "separateness" ($a \sim b = \mathbf{1}$ iff $a = b$) necessary in logic, even though the general character of equality does not enforce them?

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Galois connections in semilinear spaces

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1 Introduction

The aim of this contribution is twofold: to help in formalization and unification of tools and methods used in the theory of fuzzy relation equations, and to propose a theory which generalizes the theory of linear spaces. As it is known from the literature, there are at least two types of systems of fuzzy relation equations which differ in types of composition [2, 1, 6, 7]. However, there are similar, but in some sense dual results about their solvability, and about structures of their solution sets. On the other hand, there is a profound theory of linear spaces where the problem of solvability of systems of linear equations is entirely solved. Thus, our motivation was to find a proper generalization of the theory of linear spaces which can be a theoretical platform for analysis of systems of fuzzy relation equations.

We discovered that the theory of Galois connections can be successfully used in characterization of solvability and solutions sets of systems of linear-like equations in semilinear spaces. If solvability is connected with characterization of vectors of right-hand sides then there exists a Galois connection between a set of admissible right-hand sides and a solutions set. Moreover, on the basis of this theory, two types of systems of linear-like equations in semilinear spaces are dual, so that only one of them should be investigated.

2 Idempotent Semilinear Spaces

We recall that a linear (vector) space is a special case of a module over a ring, i.e. a linear space is a unitary module over a field [8]. In this contribution, we will be dealing with a unitary semimodule over a commutative semiring [5, 3] which will be called a *semilinear space*. Moreover, our semilinear space will be an idempotent structure with respect to its main operation.

Definition 1. *Let* $\mathcal{R} = (R, +, \cdot, 0, 1)$ *be a commutative semiring and* $\mathcal{V} = (V, +, \bar{0})$ *a commutative monoid. We say that* $\mathcal{V}$ *is a (left) semilinear space over* $\mathcal{R}$ *if an external (left) multiplication* $\lambda : \bar{x} \mapsto \lambda\bar{x}$ *where* $\lambda \in R$ *and* $\bar{x} \in V$ *is* *defined. Moreover, the following mutual properties are fulfilled for all $\bar{x}, \bar{y} \in V$ and $\lambda, \mu \in R$:*

(SLS1) $\lambda(\bar{x} + \bar{y}) = \lambda\bar{x} + \lambda\bar{y}$*,*
(SLS2) $(\lambda + \mu)\bar{x} = \lambda\bar{x} + \mu\bar{x}$*,*
(SLS3) $(\lambda \cdot \mu)\bar{x} = \lambda(\mu\bar{x})$*,*
(SLS4) $1\bar{x} = \bar{x}$*,*
(SLS5) $\lambda\bar{0} = \bar{0}$*.*

Since only left semilinear spaces will be considered, we will omit the word "left" in the name of this structure. Moreover we will simply write a semilinear space instead of a semilinear space over $\mathcal{R}$ if $\mathcal{R}$ is clear from the context. Elements of a semilinear space will be distinguished by overline.

Example 1. Let $\mathcal{R} = (R, +, \cdot, 0, 1)$ be a commutative semiring. Denote R^n $(n \geq 1)$ the set of n-dimensional vectors whose components are elements of R, i.e. $R^n = \{\bar{x} = (x_1, \ldots, x_n) \mid x_1 \in R, \ldots, x_n \in R\}$. Let $\bar{0} = (0, \ldots, 0)$ and

$$\bar{x} + \bar{y} = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

Then $\mathcal{R}^n = (R^n, +, \bar{0})$ is a commutative monoid. For any $\lambda \in R$, let us define external multiplication $\lambda\bar{x}$ by

$$\lambda\bar{x} = \lambda(x_1, \ldots, x_n) = (\lambda \cdot x_1, \ldots, \lambda \cdot x_n).$$

Then $\mathcal{R}^n$ is a semilinear space over $\mathcal{R}$.

Semilinear space $\mathcal{R}^n$, $n \geq 1$, (see Example 1) will be called *vectorial semilinear space* over $\mathcal{R}$.

Definition 2. *Semilinear space $\mathcal{V}$ over $\mathcal{R}$ is called idempotent if both operations $+$ in $\mathcal{V}$ and $\mathcal{R}$ are idempotent.*

Let $\mathcal{V} = (V, +, \bar{0})$ be an idempotent semilinear space. Then

$$\bar{x} \leq \bar{y} \iff \bar{x} + \bar{y} = \bar{y}, \tag{1}$$

is the *natural order* on $\mathcal{V}$. Thence, $(V, \leq)$ is a bounded $\vee$-semilattice where $\bar{x} \vee \bar{y} = \bar{x} + \bar{y} = \sup\{\bar{x}, \bar{y}\}$ and $\bar{0}$ is a bottom element.

It may happen (see Example 2 below) that two idempotent semilinear spaces $\mathcal{V}_1 = (V, +_1, \bar{0}_1)$ and $\mathcal{V}_2 = (V, +_2, \bar{0}_2)$ with the same support V determine *dual* (or reverse) natural orders $\leq_1$ and $\leq_2$ on V, i.e.

$$\bar{x} \leq_1 \bar{y} \iff \bar{y} \leq_2 \bar{x}.$$

In this case, $\leq_2$ is used to be denoted as $\geq_1$. With respect to $\leq_1$, $(\mathcal{V}_2, \geq_1)$ is a $\wedge$-semilattice with the top element $\bar{0}_2$ where $\bar{x} \wedge \bar{y} = \bar{x} +_2 \bar{y} = \inf\{\bar{x}, \bar{y}\}$. We will call $\mathcal{V}_1$ a $\vee$-*semilinear space*, and $\mathcal{V}_2$ a $\wedge$-*semilinear space*. Moreover, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are idempotent semilinear spaces over the same semiring then we will call them *dual*. It is easy to see that for dual semilinear spaces, the *Principle of Duality* for ordered sets holds true.

Example 2. Let $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ be an integral, residuated, commutative l-monoid and $\mathcal{L}_\vee = (L, \vee, *, 0, 1)$ a commutative $\vee$-semiring. L^n $(n \geq 1)$ is a set of n-dimensional vectors as in Example 1.

1. Algebra $\mathcal{L}^n_\vee = (L^n, \vee, \bar{0})$ is an idempotent commutative monoid where $\bar{0} = (0, \dots, 0) \in L^n$, and for any $\bar{x}, \bar{y} \in L^n$,

$$\bar{x} \vee \bar{y} = (x_1, \dots, x_n) \vee (y_1, \dots, y_n) = (x_1 \vee y_1, \dots, x_n \vee y_n).$$

   The order on $\mathcal{L}^n_\vee$ is determined by $\vee$ so that $\bar{x} \leq \bar{y}$ if and only if $x_1 \leq y_1, \dots, x_n \leq y_n$. For any $\lambda \in L$, let us define external multiplication $\lambda\bar{x}$ by

$$\lambda\bar{x} = \lambda(x_1, \dots, x_n) = (\lambda * x_1, \dots, \lambda * x_n).$$

   Then $\mathcal{L}^n_\vee$ is an (idempotent) $\vee$-semilinear space over $\mathcal{L}_\vee$.
2. Algebra $\mathcal{L}^n_\wedge = (L^n, \wedge, \bar{1})$ is an idempotent commutative monoid where $\bar{1} = (1, \dots, 1) \in L^n$, and for any $\bar{x}, \bar{y} \in L^n$,

$$(x_1, \dots, x_n) \wedge (y_1, \dots, y_n) = (x_1 \wedge y_1, \dots, x_n \wedge y_n).$$

   The natural order on $\mathcal{L}^n_\wedge$ is determined by $\wedge$, and it is dual to $\leq$ which was introduced on L^n in the case 1. above. We will denote the natural order on $\mathcal{L}^n_\wedge$ by $\leq^d$ so that $\bar{x} \leq^d \bar{y}$ if and only if $\bar{x} \geq \bar{y}$ or if and only if $x_1 \geq y_1, \dots, x_n \geq y_n$. For any $\lambda \in L$, let us define external multiplication $\lambda \diagdown \bar{x}$ by

$$\lambda \diagdown (x_1, \dots, x_n) = (\lambda \diagdown x_1, \dots, \lambda \diagdown x_n).$$

   Then $\mathcal{L}^n_\wedge$ is an (idempotent) $\wedge$-semilinear space over $\mathcal{L}_\vee$.

$\vee$-semilinear space $\mathcal{L}^n_\vee$ and $\wedge$-semilinear space $\mathcal{L}^n_\wedge$ are dual.

2.1 Galois Connections in Semilinear Spaces

Two dual idempotent semilinear spaces can be connected by Galois connections.

Theorem 1. *(i) Let $\mathcal{L}^n_\vee$ be a $\vee$-semilinear space, and $\mathcal{L}^n_\wedge$ be a $\wedge$-semilinear space, both over $\mathcal{L}_\vee$. Then for each $\lambda \in L$, mappings $\bar{x} \mapsto \lambda\bar{x}$ and $\bar{y} \mapsto \lambda \diagdown \bar{y}$ establish a Galois connection between $(\mathcal{L}^n_\vee, \leq)$ and $(\mathcal{L}^n_\wedge, \leq^d)$.*

(ii) Let $\mathcal{L}_{\vee}^m$, $m \geq 1$, be a $\vee$-semilinear space, and $\mathcal{L}_{\wedge}^n$, $n \geq 1$, a $\wedge$-semilinear space, both over $\mathcal{L}_{\vee}$. Then for each $n \times m$ matrix $A = (a_{ij})$ with components from L, mappings $h_A : L^m \to L^n$ and $g_A : L^n \to L^m$ given by

$$h_A(\bar{x})_i = a_{i1} * x_1 \vee \cdots \vee a_{im} * x_m, \quad i = 1, \ldots, n, \tag{2}$$

and

$$g_A(\bar{y})_j = (a_{1j} \to y_1) \wedge \cdots \wedge (a_{nj} \to y_n), \quad j = 1, \ldots, m, \tag{3}$$

establish a Galois connection between $(\mathcal{L}_{\vee}^n, \leq)$ and $(\mathcal{L}_{\wedge}^n, \leq^d)$.

3 Systems of Equations in Semilinear Spaces $\mathcal{L}_{\vee}^m$ and $\mathcal{L}_{\wedge}^n$

3.1 System of Equations in Semilinear Space $\mathcal{L}_{\vee}^m$

Let $\mathcal{L}_{\vee}^m$ where $m \geq 1$ be a $\vee$-semilinear space over $\mathcal{L}_{\vee}$, and $\mathcal{L}_{\wedge}^n$ where $n \geq 1$ be a $\wedge$-semilinear space over $\mathcal{L}_{\vee}$. Let $n \times m$ matrix $A = (a_{ij})$, vector $\bar{b} = (b_1, \ldots, b_n) \in L^n$, and vector $\bar{d} = (b_1, \ldots, d_m) \in L^m$ have components from L. The following systems of equations

$$\begin{aligned} a_{11} * x_1 \vee \cdots \vee a_{1m} * x_m &= b_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots& \\ a_{n1} * x_1 \vee \cdots \vee a_{nm} * x_m &= b_n, \end{aligned} \tag{4}$$

and

$$\begin{aligned} (a_{11} \to y_1) \wedge \cdots \wedge (a_{n1} \to y_n) &= d_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots& \\ (a_{1m} \to y_1) \wedge \cdots \wedge (a_{nm} \to y_n) &= d_m, \end{aligned} \tag{5}$$

are considered with respect to unknown vectors $\bar{x} = (x_1 \ldots, x_m) \in L^m$ and $\bar{y} = (y_1 \ldots, y_n) \in L^n$. By (2) and (3), systems (4) and (5) can be represented respectively as follows:

$$h_A(\bar{x}) = \bar{b}.$$

and

$$g_A(\bar{y}) = \bar{d}.$$

3.2 Solvability in terms of Galois Connection

Below, we will give some results about system (4), its solvability and solutions. By the Principle of Duality for dual semilinear spaces, similar but dual results can be proved for system (5).

Let system (4) be specified by $n \times m$ matrix A and vector $\bar{b} \in L^n$. Then solvability of (4) depends on a relationship between A and $\bar{b}$. We can prove the following

Theorem 2. *Let A be a given matrix, and h_A and g_A establish a Galois connection between semilinear spaces $\mathcal{L}_\vee^m$ and $\mathcal{L}_\wedge^n$. Then system (4) is solvable if and only if $\bar{b}$ is a closed element of $\mathcal{L}_\wedge^n$ with respect to the closure operator $g_A \circ h_A$, or if and only if*

$$\bar{b} = h_A(g_A(\bar{b})).$$

Corollary 1. *Let the conditions of Theorem 2 be fulfilled. Then $\bar{b}$ is a closed element of $\mathcal{L}_\wedge^n$ with respect to $g_A \circ h_A$ if and only if there exists $\bar{x} \in L^m$ such that $h_A(\bar{x}) = \bar{b}$.*

Theorem 3. *Let A be a given matrix, $g_A \circ h_A$ a closure operator on $\mathcal{L}_\wedge^n$. The set of closed elements of $\mathcal{L}_\wedge^n$ with respect to $g_A \circ h_A$ is a semilinear subspace of $\mathcal{L}_\vee^n$.*

Theorem 4. *Let system (4) be specified by $n \times m$ matrix A and vector $\bar{b} \in L^n$. Moreover, let $\bar{b}$ be a closed element of $\mathcal{L}_\wedge^n$ with respect to $g_A \circ h_A$. Then $g_A(\bar{b})$ is a solution of system (4).*

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Web-geometric approach to totally ordered monoids

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The semantical part of the monoidal t-norm based logic (MTL) [3] is represented by MTL-algebras. MTL-algebras are lattice-based, however, it is known that each one is a subdirect product of MTL-chains [3]. Therefore, in order to investigate their monoidal operations, it is usually sufficient to focus on integral tomonoids. In this paper, we intend to investigate tomonoids, more specifically, we intend to give a visual interpretation of their associativity.

We recall that a *magma* [2] is an algebraic structure $(M, *)$ on a set M where $*\colon M \times M \to M$ is a binary operation. A *totally ordered magma*, or a *togma* for short, is a structure $(M, *, \leq)$ where $(M, \leq)$ is a chain and $(M, *)$ is a magma with the operation $*$ isotone with respect to $\leq$. A *monoid* is a structure $(M, *, 1)$ where $(M, *)$ is a magma, $*$ is an associative operation, and 1 is a neutral element of $*$. A *totally ordered monoid*, or a *tomonoid* for short, is a structure $(M, *, 1, \leq)$ where $(M, *, 1)$ is a monoid and $(M, *, \leq)$ is a togma. A tomonoid is *integral* if the neutral element is also the top element. A *quasigroup* is a magma $(M, *)$ in which the equations $a * x = b$ and $y * a = b$ have unique solutions x, y for every a and b in M. Finally, a *loop* is a structure $(M, *, 1)$ where $(M, *)$ is a quasigroup and 1 is a neutral element of $*$.

In our approach, we are inspired by web geometry [1]. This branch of differential geometry offers several concepts and tools which are known to characterize algebraic properties of *loops* in a surprisingly transparent geometric way. Web geometry introduces the notion of a *3-web* (Figure 1) which is (in our simpler case) a system of three families of foliations of the plane such that each pair of them defines local coordinates. Important 3-webs are the *regular* ones; they are homeomorphic images of webs where all foliations are systems of parallel

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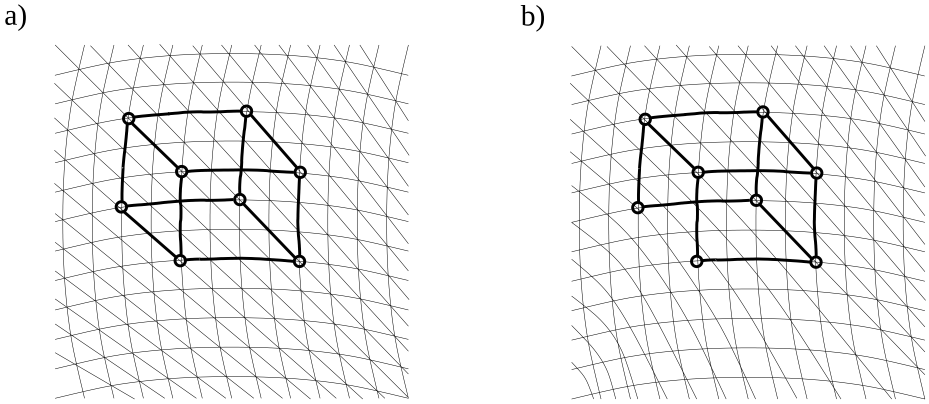

Fig. 1. Illustration of a 3-web in a two-dimesional plane: a) a 3-web which satisfies the Reidemeister closure condition and b) a 3-web which violates it.

lines. Web geometry defines various *closure conditions* which characterize regularity of 3-webs. Namely, it is the *Reidemeister closure condition* [6] which is illustrated in Figure 1-a. The algebraic structures, which are closely related to 3-webs, are quasigroups and loops. Taking a quasigroup $(M, *)$ we can easily define a 3-web on $M \times M$ by three families of curves given by the equations $x = a$, $y = b$, and $x * y = c$, respectively, for fixed $a, b, c \in M$. In the other way round, having a 3-web defined on a plane Ω, defining a point $p \in \Omega$, and denoting two families of foliations as coordinates and one as "level sets", an operation can be defined on the set of "level sets". It can be shown that this operation is a loop; it is called a *local loop*. Regularity of 3-webs are closely related to the algebraic properties of the corresponding local loops. Namely, associative loops (i.e., groups) are characterized by the Reidemeister closure condition (and their 3-webs are known to be regular).

Loops are, however, very specific structures and the tools described above cannot be used directly in the case of tomonoids. Nevertheless, still there are some similarities (the neutral element, for example) and this approach can be adopted also for tomonoids as we will show in the sequel.

First, we define the tools which we will be using in order to investigate the geometry of tomonoids. Let $(M, *, \leq)$ be a togma. A *rectangle* (Figure 2-a) is a set of four points $P = \{a, b\} \times \{c, d\} \subset M \times M$. Note that the order of the coordinates does not play a role here. Let $(u, v) \in M \times M$, and let

$$P = \{a, b\} \times \{c, d\} \subset M \times M \,,$$
$$R = \{e, f\} \times \{g, h\} \subset M \times M$$

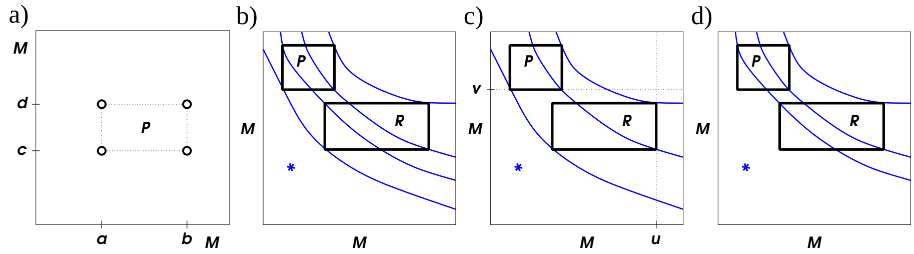

Fig. 2. Illustration of a togma $(M, *, \leq)$ expressed by the system of its level sets. a) a rectangle $P\{a, b\} \times \{c, d\}$, b) a pair of equivalent restangles $P \approxeq R$, c) a pair of (u, v)-local rectangles P, R; they are strongly aligned according to (u, v), i.e., $P \simeq^{uv} R$, d) a pair of aligned rectangles $P \approx R$.

be two rectangles. A pair of rectangles P, R is said to be (u, v)-*local* (Figure 2-c) if $d = v$ and $f = u$. (Easily, for a given pair of rectangles, there are up to four distinct points according to which they can be considered as local.) We say that the rectangles P, R are *equivalent* (Figure 2-b) (and we denote it by $P \approxeq R$) if the functional values at the corresponding pairs of vertices are equal, i.e., if:

$$\begin{aligned} a * c &= e * g\,, \\ a * d &= e * h\,, \\ b * c &= f * g\,, \\ \text{and } b * d &= f * h\,. \end{aligned}$$

Beside $\approxeq$, we define also three weaker relations on the set of rectangles. We say that the rectangles P, R are *strongly aligned according to* $(u, v) \in M \times M$ (Figure 2-c) (denoted by $P \simeq^{uv} R$) if they are (u, v)-local and if

$$\begin{aligned} a * v &= e * h\,, \\ b * c &= u * g\,, \\ \text{and } b * v &= u * h\,. \end{aligned}$$

By other words, it is required that the functional values at the corresponding pairs of vertices are equal except of the pair where u or v does not appear. Suppose that $a * c \leq a * d \leq b * c \leq b * d$. We say that the rectangles P, R are *aligned* (Figure 2-d) (denoted by $P \approx R$) if

$$\begin{aligned} a * d &= e * h\,, \\ b * c &= f * g\,, \\ \text{and } b * d &= f * h\,, \end{aligned}$$

i.e., if the functional values at the corresponding pairs of vertices are equal except of the pair with the lowest functional value. Finally, we say that the rectangles P, R are *weakly aligned* (denoted by $P \sim R$) if the relation $P \approxeq R$ is violated for at most one pair of vertices (it does not matter which one). Obviously, all the introduced relations are equivalences. Moreover, $\approxeq \subseteq \simeq^{uv} \subseteq \approx \subseteq \sim$ on the set of (u, v)-local rectangles.

The following result characterizes the associativity of a general tomonoid:

Proposition 1. *Let $(M, *, 1, \leq)$ be a togma with a neutral element 1. Then the operation $*$ is associative (and M is a tomonoid) if and only if*

$$P \simeq^{11} R \Rightarrow P \approxeq R$$

for all $(1, 1)$-local rectangles $P, R \subset M \times M$.

The togmas and tomonoids, where "continuity" plays a role, allow to formulate stronger results:

Proposition 2. *Let $(M, *, \leq)$ be a togma with $*$ associative (i.e., a totally ordered semigroup). Let, for some fixed $u, v \in M$, the mappings*

$$M \to M\colon\ x \mapsto x * v\,,$$
$$M \to M\colon\ y \mapsto u * y$$

be bijective. Then, necessarily,

$$P \simeq^{uv} R \Rightarrow P \approxeq R$$

for all (u, v)-local rectangles $P, R \subset M \times M$.

Corollary 1. *Let $(M, *, 1, \leq)$ be a togma with a neutral element 1. Let, moreover, regardless of $u, v \in M$, the mappings*

$$M \to M\colon\ x \mapsto x * v\,,$$
$$M \to M\colon\ y \mapsto u * y\,,$$

are bijective. (Note that, in such a case, M is a totally ordered loop.) Then the operation $*$ is associative (and M is a totally ordered group) if and only if

$$P \sim R \Rightarrow P \approxeq R$$

for all the rectangles $P, R \subset M \times M$.

This result corresponds directly with the Reidemeister closure condition and the result given by web geometry.

Proposition 3. *Let* $(M, *, 1, \leq)$ *be an integral tomonoid. Let the functions*

$$M \to \downarrow\{v\}\colon\ x \mapsto x * v\,,$$
$$M \to \downarrow\{u\}\colon\ y \mapsto u * y$$

be surjections for some fixed $u, v \in M$*. Then, necessarily,*

$$P \simeq^{uv} R \Rightarrow P \approxeq R$$

for all (u, v)*-local rectangles* $P, R \subset \downarrow\{u\} \times \downarrow\{v\}$*.*

The presented proposition requires that certain sections of the tomonoid are surjections. Note that this requirement is a counterpart of the continuity (according to the order topology) of the sections. The following result reveals the geometric properties of "continuous" integral tomonoids which are the monoidal operations of BL-algebras [4, 5].

Corollary 2. *Let* $(M, *, 1, \leq)$ *be a togma with a neutral element* 1 *and let* 1 *be also the top element. Let, moreover,* $*$ *be "continuous", i.e., let the functions*

$$M \to \downarrow\{v\}\colon\ x \mapsto x * v\,,$$
$$M \to \downarrow\{u\}\colon\ y \mapsto u * y\,,$$

be surjections for all fixed $u, v \in M$*. Then the operation* $*$ *is associative (and* M *is an integral tomonoid) if and only if*

$$P \approx R \Rightarrow P \approxeq R$$

for all the rectangles $P, R \subset M \times M$*.*

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On convergences of lattice valued random elements with applications

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Relatively uniform convergence of weighted sums of random elements taking values in a σ-complete Banach lattice with the σ-property has been studied in [2]. It has been shown that the usual assumptions of independent and identically distributed random elements can be replaced by weaker conditions to obtain a fruitful theory. The results obtained are new even for real valued random elements. We will provide a broader discussion on convergence of lattice valued random elements.

We will provide the estimation methods based on method of moments and lattice diffusion related annealing algorithms. The lattice moments of random vectors has been introduced by [3]. The latter is based on a well known properties of atomic diffusion (see [4]) and could a competing estimation technique on a lattice. We will relate such a method to the known least squares lattice method.

The applications of the given structures in reliability, artificial intelligence and modeling of systems with observations suffering from spatial deformations and stochastic loads will be also presented.

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Powerset operator foundations for categorically-algebraic fuzzy set theories

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1 Introduction

The paper sets forth in detail categorically-algebraic (catalg) foundations for the operations of taking the image and preimage of (fuzzy) sets, coined as *forward* and *backward powerset operators*. Motivated by an open question of S. E. Rodabaugh, we construct a monad on the category of sets, the algebras of which generate the fixed-basis forward powerset operator of L. A. Zadeh. On the next step, we provide a direct lifting of the backward powerset operator using the notion of categorical *biproduct*. The obtained framework is readily extended to the variable-basis case, justifying the powerset theories currently popular in the fuzzy community. At the end of the paper, our general variety-based setting postulates the requirements, under which a convenient variety-based powerset theory can be developed, suitable for employment in all areas of fuzzy mathematics dealing with fuzzy powersets, including fuzzy algebra, logic and topology.

2 Quantale modules

This section constructs a monad on the category **Set** of sets and maps, the algebras of which generate the fixed-basis forward powerset operator of L. A. Zadeh [24], answering the question of S. E. Rodabaugh [10, Open Question 6.17] on its existence.

Definition 1. *Given a quantale Q, a* (left) Q-module *is a $\bigvee$-semilattice A with an action $Q \times A \xrightarrow{*} A$ such that*

1. $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ *for every* $q \in Q$, $S \subseteq A$;
2. $(\bigvee S) * a = \bigvee_{s \in S}(s * a)$ *for every* $a \in A$, $S \subseteq Q$;
3. $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ *for every* $q_1, q_2 \in Q$, $a \in A$.

A Q-module homomorphism $A \xrightarrow{\varphi} B$ *is a $\bigvee$-preserving map such that $\varphi(q * a) = q * \varphi(a)$ for every $a \in A$, $q \in Q$. $(Q\text{-}\mathbf{Mod}, |-|)$ is the construct of*

Q-modules and their homomorphisms. A Q-module A over a unital quantale (Q, e) is called unital *provided that $e * a = a$ for every $a \in A$. Given a unital quantale Q,* **U**Q-**Mod** *is the full subcategory of* Q-**Mod** *of all unital Q-modules.*

Motivated by the category (**U**)R-**Mod** of (unital) left modules over a ring R with identity [2], the topic provides a rich source for investigation [8, 9, 14, 20, 21].

Theorem 1. *For every quantale Q, there exists a unital quantale $Q[e]$ such that the categories* Q-**Mod** *and* **U**$Q[e]$-**Mod** *are isomorphic.*

Meta-mathematically restated, the category Q-**Mod** as an entity is redundant in mathematics and therefore we restrict our attention to the case of unital modules only.

Theorem 2. *The underlying functor of* **U**Q-**Mod** *has a left adjoint.*

Proof. Given a set X, the set Q^X of all maps $X \xrightarrow{\alpha} Q$ with the point-wise structure is a unital Q-module. There exists a map $X \xrightarrow{\eta_X} |Q^X|$, $(\eta_X(x))(y) = e$, if $x = y$; otherwise, $(\eta_X(x))(y) = \bot$, which is the universal arrow since $\alpha = \bigvee_{x \in X}(\alpha(x) * \eta_X(x))$ for every $\alpha \in Q^X$ and thus, every map $X \xrightarrow{f} |A|$ has a lift $Q^X \xrightarrow{\overline{f}} A$, $\overline{f}(\alpha) = \bigvee_{x \in X}(\alpha(x) * f(x))$. □

Corollary 1. *There exists an adjoint situation $(\eta, \epsilon) : F \dashv |-| :$* **U**$Q$-**Mod** $\to$ **Set**.

The functor F of Corollary 1 lifts the fixed-basis forward powerset operator of L. A. Zadeh, justifying its correctness without involving the technique of [10, 12, 13]. It provides the traditional forward powerset operator in case of $Q = \{\bot, \top\}$.

Definition 2. *The adjoint situation of Corollary 1 induces a monad $\mathsf{T} = (T, \eta, \mu)$ on* **Set** *defined by $T = |-|F$, $\mu = |-|\epsilon F$ and called the Q-*powerset monad*, providing the standard powerset monad for $Q = \{\bot, \top\}$.*

Theorem 3. *The comparison functor* **U**Q-**Mod** $\xrightarrow{K}$ **Set**$^{\mathsf{T}}$ *is a concrete isomorphism and therefore* **U**Q-**Mod** *is a monadic construct.*

Theorem 3 gives rise to a meta-mathematical result answering the above-mentioned open question of S. E. Rodabaugh.

Meta-Theorem 1 *Given a unital quantale Q, there exists a monad on* **Set***, the algebras of which generate the fixed-basis forward powerset operator in the sense of L. A. Zadeh.*

The backward powerset operator can be lifted as well, the approach based on the notion of categorical *biproduct*, the motivating push given by **U**Q-**Mod**.

Lemma 1. *The category* **U**Q-**Mod** *has biproducts.*

Proof. Let $(A_i)_{i\in I}$ be a set-indexed family of unital Q-modules, with the product of their underlying sets $\mathcal{P} = (\prod_{i\in I} |A_i|, (\pi_i)_{i\in I})$. The point-wise structure on $\prod_{i\in I} |A_i|$ gives a **U**Q-**Mod**-product. Given $j \in I$, define $A_j \xrightarrow{\mu_j} \prod_{i\in I} A_i$, $\mu_j(a) = (a_i)_{i\in I}$ with $a_i = a$, if $i = j$; otherwise, $a_i = \bot$, and get a **U**Q-**Mod**-sink $\mathcal{C} = ((\mu_i)_{i\in I}, \prod_{i\in I} A_i)$. $\mathcal{C}$ is a coproduct of $(A_i)_{i\in I}$ since for every **U**Q-**Mod**-sink $\mathcal{T} = (A_i \xrightarrow{\varphi_i} B)_{i\in I}$, the map $\prod_{i\in I} A_i \xrightarrow{\varphi} B$, $\varphi((a_i)_{i\in I}) = \bigvee_{i\in I} \varphi_i(a_i)$ is the unique **U**Q-**Mod**-morphism such that $\varphi \circ \mathcal{C} = \mathcal{T}$. $\square$

3 Quantaloids

The hom-sets of the category R-**Mod** can be supplied with the structure of an abelian group, morphism composition acting distributively from the left and from the right. In addition, this category has finite biproducts. It can be shown that these two seemingly unrelated properties are linked. By Lemma 1, the category **U**Q-**Mod** has set-indexed biproducts. This section shows that this gives an extra property to the hom-set structure in question. We begin by introducing enriched categories, suitable for the new setting. The concept has taken its proper place in mathematics quite a long time ago [14].

Definition 3. *A* quantaloid *is a category* **Q** *with* hom*-sets being* $\bigvee$*-semilattices, composition of morphisms preserving* $\bigvee$ *in both variables.*

Quantaloids are precisely the categories enriched in the category **CSLat**$(\bigvee)$ [7].

Lemma 2. *Every quantaloid is a pointed category, where the zero morphisms are the bottom elements of the respective* hom*-sets.*

The next result provides a generalization of [6, Proposition 40.12], replacing finite biproducts ($\bigoplus$) with the set-indexed ones.

Theorem 4. *Every pointed category* **C** *with biproducts has a unique quantaloid structure, given for every subset* $S \subseteq \mathbf{C}(A, B)$ *by each of the following formulas* $A \xrightarrow{\bigvee S} B = A \xrightarrow{\Delta} \bigoplus_{s\in S} A \xrightarrow{[s]_{s\in S}} B = A \xrightarrow{\langle s\rangle_{s\in S}} \bigoplus_{s\in S} B \xrightarrow{\nabla} B = A \xrightarrow{\Delta} \bigoplus_{s\in S} A \xrightarrow{\bigoplus_{s\in S} s} \bigoplus_{s\in S} B \xrightarrow{\nabla} B.$

Corollary 2. *If* $\mathbf{C}$ *is a pointed category with products or coproducts, then* $\mathbf{C}$ *has biproducts iff there exists a unique quantaloid structure on* $\mathbf{C}$.

Thus, every pointed category with biproducts is a quantaloid, encoding the non-categorical $\bigvee$-semilattice structure on $\hom$-sets with the categorical notion of biproduct.

4 Functors induced by biproducts

This section shows two functors produced by biproducts, lifting the powerset operators. To be in line with the results of [10–13], the functors have a specific pattern.

Lemma 3. *A subcategory* $\mathbf{C}$ *of a pointed category* $\mathbf{D}$ *with biproducts gives the functors:*

1. $\mathbf{Set} \times \mathbf{C} \xrightarrow{(-)^{\rightarrow}} \mathbf{D}$, $((X,A) \xrightarrow{(f,\varphi)} (Y,B))^{\rightarrow} = \bigoplus_{x\in X} A \xrightarrow{(f,\varphi)^{\rightarrow}} \bigoplus_{y\in Y} B$, $(f,\varphi)^{\rightarrow} \circ \mu^X_x = \mu^Y_{f(x)} \circ \varphi$ *for every* $x \in X$. *If* $y \in Y$, $\pi^Y_y \circ (f,\varphi)^{\rightarrow} = \varphi \circ (\bigvee\{\pi^X_x \mid f(x) = y\})$.
2. $\mathbf{Set} \times \mathbf{C}^{op} \xrightarrow{(-)^{\leftarrow}} \mathbf{D}^{op}$, $((X,A) \xrightarrow{(f,\varphi)} (Y,B))^{\leftarrow} = \bigoplus_{y\in Y} B \xrightarrow{(f,\varphi)^{\leftarrow}} \bigoplus_{x\in X} A$ *with* $\pi^X_x \circ (f,\varphi)^{\leftarrow} = \varphi^{op} \circ \pi^X_{f(x)}$ *for every* $x \in X$. *If* $y \in Y$, $(f,\varphi)^{\leftarrow} \circ \mu^Y_y = (\bigvee\{\mu^X_x \mid f(x) = y\}) \circ \varphi^{op}$.

Moreover, for every pair $A \xrightarrow{\varphi} B \xrightarrow{\psi^{op}} A$ *of* $\mathbf{D}$*-morphisms:* $\varphi \circ \psi^{op} \leqslant 1_B$ *and* $1_A \leqslant \psi^{op} \circ \varphi$ *iff* $(f,\varphi)^{\rightarrow} \circ (f,\psi)^{\leftarrow} \leqslant 1_{\bigoplus_{y\in Y} B}$ *and* $1_{\bigoplus_{x\in X} A} \leqslant (f,\psi)^{\leftarrow} \circ (f,\varphi)^{\rightarrow}$ *for every map* $X \xrightarrow{f} Y$. *If* $(X,A) \xrightarrow{(f,\varphi)} (Y,B)$ *is a morphism in* $\mathbf{Set} \times \mathbf{C}^{op}$ *(resp.* $\mathbf{Set} \times \mathbf{C}$*) with* $X \xrightarrow{f} Y$ *bijective, then* $(f,\varphi)^{\leftarrow} = (f^{-1},\varphi^{op})^{\rightarrow}$ *(resp.* $(f,\varphi)^{\rightarrow} = (f^{-1},\varphi^{op})^{\leftarrow}$*).*

Corollary 3. *An object of a pointed category* $\mathbf{D}$ *with biproducts gives the functors:*

1. $\mathbf{Set} \xrightarrow{(-)^{\rightarrow}_A} \mathbf{D}$, $(X \xrightarrow{f} Y)^{\rightarrow}_A = \bigoplus_{x\in X} A \xrightarrow{f^{\rightarrow}_A} \bigoplus_{y\in Y} A$, $f^{\rightarrow}_A \circ \mu^X_x = \mu^Y_{f(x)}$ *for every* $x \in X$. *If* $y \in Y$, $\pi^Y_y \circ f^{\rightarrow}_A = \bigvee\{\pi^X_x \mid f(x) = y\}$.
2. $\mathbf{Set} \xrightarrow{(-)^{\leftarrow}_A} \mathbf{D}^{op}$, $(X \xrightarrow{f} Y)^{\leftarrow}_A = \bigoplus_{y\in Y} A \xrightarrow{f^{\leftarrow}_A} \bigoplus_{x\in X} A$, $\pi^X_x \circ f^{\leftarrow}_A = \pi^Y_{f(x)}$ *for every* $x \in X$. *If* $y \in Y$, $f^{\leftarrow}_A \circ \mu^Y_y = \bigvee\{\mu^X_x \mid f(x) = y\}$.

Moreover, $f^{\rightarrow}_A \circ f^{\leftarrow}_A \leqslant 1_{\bigoplus_{y\in Y} A}$ *and* $1_{\bigoplus_{x\in X} A} \leqslant f^{\leftarrow}_A \circ f^{\rightarrow}_A$ *for every map* $X \xrightarrow{f} Y$. *Given a bijective map* $X \xrightarrow{f} Y$, $(f^{-1})^{\leftarrow}_A = f^{\rightarrow}_A$ *and* $(f^{-1})^{\rightarrow}_A = f^{\leftarrow}_A$.

The category $\mathbf{U}Q$-$\mathbf{Mod}$ provides the crisp powerset operators as well as the fuzzy approaches of L. A. Zadeh [24] and S. E. Rodabaugh [12].

5 Copowers versus free objects in constructs

The last section, clarifying completely the nature of point-set lattice-theoretic powerset theories, has left one important point untouched. We already know two ways of obtaining the forward powerset operator: either through a monad on **Set** (Meta-Theorem 1), or employing the technique of biproducts (Lemma 3). Corollary 2 disguised the partial order on hom-sets of a given category through biproducts. This section does the job for the Q-powerset monad. The result is based on [1, Exercise 10R] running as follows.

Lemma 4. *Let $(\mathbf{C}, |-|)$ be a construct with $|-|$ representable by an object A. For every $\mathbf{C}$-object B and every set X, B is a free over X iff B is an Xth copower of A.*

Corollary 4. *For a pointed construct $(\mathbf{C}, |-|)$ with biproducts, equivalent are:*

1. *$\mathbf{C}$ has an object A free over a singleton;*
2. *the underlying functor $|-|$ is representable by A;*
3. *$(-)^{\rightarrow}_{A}$ is a left adjoint to $|-|$.*

If $\mathbf{C}$ is an equationally presentable category of structured sets and structure-preserving maps, then each of the above items implies equivalence of $\mathbf{C}$ to a monadic construct.

The monad in question is generated by the fixed-basis forward powerset operator of Corollary 3. How does it relate to the respective operator of L. A. Zadeh?

Lemma 5. *Let $(\mathbf{C}, |-|)$ be a construct of structured sets and structure-preserving maps which is a quantaloid. If $\mathbf{C}$ has an object A free over a singleton, then every $\mathbf{C}$-object B can be equipped with a $\bigvee$-semilattice structure preserved by $\mathbf{C}$-morphisms. Moreover, given $\varphi, \psi \in \mathbf{C}(B, C)$, $\varphi \leqslant \psi$ iff $\varphi(b) \leqslant \psi(b)$ for every $b \in B$.*

Corollary 5. *Let $(\mathbf{C}, |-|)$ be a pointed construct of structured sets and structure-preserving maps, which has biproducts and an object A free over a singleton. Given a map $X \xrightarrow{f} Y$, $(f^{\rightarrow}_{A}(\alpha))(y) = \bigvee\{\alpha(x) \mid f(x) = y\}$ for every $\alpha \in A^{X}$, $y \in Y$.*

The results provide an important consequence which improves Meta-Theorem 1.

Meta-Theorem 2 *Given an equationally presentable pointed construct of structured sets and structure-preserving maps, which has biproducts and an object A free over a singleton, there exists a monad on* **Set**, *the algebras of which generate the fixed-basis forward powerset operator $(-)^{\rightarrow}_{A}$ in the sense of L. A. Zadeh.*

6 Variety-based powerset operators and their induced theories

Following our trend on developing a purely catalg outlook on fuzzy mathematics [15, 16, 18, 19, 22], this section presents a new setting for powerset theories. The cornerstone of the approach is the notion of *algebra*. The structure is to be thought of as a set with a family of operations defined on it, satisfying certain identities, e.g., semigroup, monoid, group and also complete lattice, frame, quantale.

Definition 4. *Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a class of cardinal numbers. An Ω-*algebra *is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ consisting of a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ called n_λ-*ary operations *on A. An Ω-*homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{f} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ *is a map $A \xrightarrow{f} B$ such that $f \circ \omega_\lambda^A = \omega_\lambda^B \circ f^{n_\lambda}$. $(\mathbf{Alg}(\Omega), |-|)$ is the construct of Ω-algebras and Ω-homomorphisms. For $\mathcal{M}$ (resp. $\mathcal{E}$) being the class of Ω-homomorphisms with injective (resp. surjective) underlying maps, a* variety of Ω-algebras *is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients. The objects (resp. morphisms) of a variety are called* algebras *(resp.* homomorphisms*). Given a variety $\mathbf{A}$, its dual category is denoted by $\mathbf{LoA}$. Given an algebra A, $\mathbf{S}_A$ is the subcategory of $\mathbf{A}$ with the only morphism 1_A. Given a subclass $\Omega' \subseteq \Omega$, an Ω'-*reduct *of $\mathbf{A}$ is a pair $(||-||, \mathbf{B})$, where $\mathbf{B}$ is a variety of Ω'-algebras and $\mathbf{A} \xrightarrow{||-||} \mathbf{B}$ is a concrete functor.*

We introduce requirements on a variety $\mathbf{A}$ allowing to develop a fruitful theory.

Definition 5. *A variety $\mathbf{A}$ is called* convenient *provided that there exists an Ω'-reduct $\mathbf{B}$ of $\mathbf{A}$ satisfying the following properties:*

1. *$\mathbf{B}$ is equationally presentable;*
2. *$\mathbf{B}$ is pointed and has biproducts;*
3. *$\mathbf{B}$ has an algebra free over a singleton.*

Unlike the authors of [4, 10, 12, 13], we fix two ground categories instead of one.

Definition 6. *Given a convenient variety $\mathbf{A}$, the* ground categories *for the variety-based powerset theories are fixed to $\mathbf{Set} \times \mathbf{A}$ and $\mathbf{Set} \times \mathbf{LoA}$.*

The turning point in our theory is the definition of the powerset operators.

Definition 7. *Given a subcategory $\mathbf{C}$ of a convenient variety $\mathbf{A}$,*

1. *the* forward variety-based powerset operator w.r.t. **C** *is the functor* **Set** $\times$ **C** $\xrightarrow{(-)^{\rightarrow}_{\mathbf{C}}}$ **B**, $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))^{\rightarrow}_{\mathbf{C}} = ((X_1, ||A_1||) \xrightarrow{(f,||\varphi||)} (X_2, ||A_2||))^{\rightarrow}_{\mathbf{B}}$ *where* $(-)^{\rightarrow}_{\mathbf{B}}$ *is the functor of Lemma 3(1);*
2. *the* backward variety-based powerset operator w.r.t. **C** *is the functor* **Set** $\times$ $\mathbf{C}^{op} \xrightarrow{(-)^{\leftarrow}_{\mathbf{C}}}$ **LoB**, $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))^{\leftarrow}_{\mathbf{C}} = ((X_1, ||A_1||) \xrightarrow{(f,||\varphi||)} (X_2, ||A_2||))^{\leftarrow}_{\mathbf{B}}$*, where* $(-)^{\leftarrow}_{\mathbf{B}}$ *is the functor of Lemma 3(2).*

Some properties of the new functors make their use considerably easier.

Lemma 6. *Given a subcategory* **C** *of a convenient variety* **A** *and a* **Set** $\times$ $\mathbf{C}^{op}$*-morphism* $(X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2)$, $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))^{\leftarrow}_{\mathbf{C}} = C_2^{X_2} \xrightarrow{(f,\varphi)^{\leftarrow}_{\mathbf{C}}} C_1^{X_1}$ *with* $(f, \varphi)^{\leftarrow}_{\mathbf{C}}(\beta) = \varphi^{op} \circ \beta \circ f$. $(f, \varphi)^{\leftarrow}_{\mathbf{C}}$ *is an* **A***-homomorphism and thus, the codomain of* $(-)^{\leftarrow}_{\mathbf{C}}$ *is* **LoA**.

Definition 8. *Given a subcategory* **C** *of a convenient variety* **A**, *an operation* ω_λ *with* $n_\lambda \in \Omega \setminus \Omega'$ *is called* $\bigvee_{\mathbf{C}}$-compatible *provided that every* **C***-object* C *satisfies the identity* $\bigvee_{j\in J} \omega^A_\lambda(\langle c_{ij}\rangle_{n_\lambda}) = \omega^A_\lambda(\langle\bigvee_{j\in J} c_{ij}\rangle_{n_\lambda})$ *for every* $\{c_{ij} \mid i \in n_\lambda, j \in J\} \subseteq C$. $\mathbf{A}_{\bigvee_{\mathbf{C}}}$ *is the subcategory of* **A** *with objects those of* **A**, *and morphisms those* $\bigvee$*-preserving maps* $|A_1| \xrightarrow{f} |A_2|$ *which are* **B***-homomorphisms and preserve* $\bigvee_{\mathbf{C}}$*-compatible operations.*

The condition of $\bigvee$-compatibility is a rather restrictive one, e.g., the meet-operation $\wedge$ in **Frm** (*frames*) is not $\bigvee_{\mathbf{Frm}}$-compatible.

Lemma 7. *Given a subcategory* **C** *of a convenient variety* **A** *and a* **Set** $\times$ **C***-morphism* $(X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2)$, $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))^{\rightarrow}_{\mathbf{C}} = C_1^{X_1} \xrightarrow{(f,\varphi)^{\rightarrow}_{\mathbf{C}}} C_2^{X_2}$ *with the property* $((f, \varphi)^{\rightarrow}_{\mathbf{C}}(\alpha))(x_2) = \bigvee\{\varphi \circ \alpha(x_1) \mid f(x_1) = x_2\}$. $(f, \varphi)^{\rightarrow}_{\mathbf{C}}$ *is an* $\mathbf{A}_{\bigvee_{\mathbf{C}}}$*-morphism, and every operation* ω_λ, $n_\lambda \in \Omega \setminus \Omega'$ *preserved by all maps of the form* $(f, \varphi)^{\rightarrow}_{\mathbf{C}}$ *is* $\bigvee_{\mathbf{C}}$*-compatible, that fixes the codomain of* $(-)^{\rightarrow}_{\mathbf{C}}$ *at* $\mathbf{A}_{\bigvee_{\mathbf{C}}}$.

All preliminaries done, we introduce variety-based powerset theories, taking the approach of [13, Definition 3.5] as a good motivating example.

Definition 9. *Given a subcategory* **C** *of a convenient variety* **A**, *a* **C**-powerset theory *is the tuple* $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)^{\rightarrow}_{\mathbf{C}}, (-)^{\leftarrow}_{\mathbf{C}})$. *The triple* $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)^{\rightarrow}_{\mathbf{C}})$ *(resp.* $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)^{\leftarrow}_{\mathbf{C}})$*) is called a* forward *(resp.* backward*)* **C**-powerset theory. *The* underlying theory *of a* **C***-powerset theory* P *is the tuple* $|\mathsf{P}| = (\mathbf{A}, \mathbf{C}, |-| \circ (-)^{\rightarrow}_{\mathbf{C}}, |-| \circ (-)^{\leftarrow}_{\mathbf{C}})$.

Example 1.

1. $\mathcal{P} = (\mathbf{CSLat}(\bigvee), \mathbf{S_2}, (-)^{\rightarrow}_{\mathbf{S_2}}, (-)^{\leftarrow}_{\mathbf{S_2}})$ is the standard crisp powerset theory.
2. $\wp = (\mathbf{CSLat}(\bigwedge), \mathbf{S_2}, (-)^{\rightarrow}_{\mathbf{S_2}}, (-)^{\leftarrow}_{\mathbf{S_2}})$ is a non-standard crisp powerset theory.
3. $\mathsf{P} = (\mathbf{Frm}, \mathbf{S}_I, (-)^{\rightarrow}_{\mathbf{S}_I}, (-)^{\leftarrow}_{\mathbf{S}_I})$ is the fixed-basis fuzzy approach of L. A. Zadeh.
4. $\mathsf{P} = (\mathbf{Quant}, \mathbf{S}_L, (-)^{\rightarrow}_{\mathbf{S}_L}, (-)^{\leftarrow}_{\mathbf{S}_L})$ is the fixed-basis L-fuzzy approach of J. A. Goguen. The machinery can be generalized to an arbitrary convenient variety $\mathbf{A}$.
5. $\mathsf{P} = (\mathbf{SQuant}, \mathbf{C}, (-)^{\rightarrow}_{\mathbf{C}}, (-)^{\leftarrow}_{\mathbf{C}})$ is the variable-basis approach of S. E. Rodabaugh.
6. $\mathsf{P} = (\mathbf{FuzLat}, \mathbf{FuzLat}, (-)^{\leftarrow}_{\mathbf{FuzLat}})$ is the variable-basis approach of P. Eklund.
7. $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)^{\leftarrow}_{\mathbf{C}})$ provides our former variety-based approach.

The backward part of the underlying theories produced by $\mathcal{P}$ and $\wp$ is the same, whereas the forward one is essentially different justifying the next definition.

Definition 10. *The* dual P^{op} *of a given powerset theory* P *is the theory obtained from the dual partial order on* hom-*sets of the reduct* $\mathbf{B}$ *of the convenient variety* $\mathbf{A}$.

Example 2. The powerset theories $\mathcal{P}$ and $\wp$ are dual.

Two important examples of the application of the new concept are provided by the theories of *variety-based topological spaces* and *systems* [3, 5, 17, 23].

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Interval-valued logics

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In [5], the authors introduced Interval-Valued Monoidal Logic[1] (IVML). Its language is the language of Höhle's Monoidal Logic (ML,[3]) enriched with two unary connectives $\Box$ and $\Diamond$, and a constant $\overline{u}$. Its axioms are those of ML plus 15 new ones describing the behaviour of $\Box$, $\Diamond$ and $\overline{u}$. The deduction rules are modus ponens (MP, from ϕ and $\phi \to \psi$ infer ψ), generalization (G, from ϕ infer $\Box\phi$) and monotonicity of $\Diamond$ (M$\Diamond$, from $\phi \to \psi$ infer $\Diamond\phi \to \Diamond\psi$).
In some way, ML can be seen as a special case of IVML. Indeed, it can be proven [7] that for all sets $T \cup \{\phi\}$ of ML-formulae, $T \vdash_{ML} \phi$ iff $\{\chi' | \chi \in T\} \vdash_{IVML} \phi'$ (where ψ' is the IVML-formula obtained by substituting $\Box p$ in ψ for every proposition variable p in ψ).
IVML is sound and complete with respect to the variety of triangle algebras. These are algebraic structures that describe interval-valued residuated lattices (IVRLs): (closed) interval-valued bounded lattices endowed with a product and implication that satisfy the residuation principle, such that the sublattice of exact intervals (i.e., intervals consisting of one element) is closed under product and implication. Table 1 shows to which mappings and interval in an IVRL the connectives and constant in IVML correspond. Also the notations in triangle algebras are included, in the second column. The soundness and completeness of

Table 1. Semantic meaning of $\Box$, $\Diamond$ and $\overline{u}$.

IVML	triangle algebra	IVRL
$\Box$	ν	$p_v : [x,y] \mapsto [x,x]$
$\Diamond$	μ	$p_h : [x,y] \mapsto [y,y]$
$\overline{u}$	u	$[0,1]$

[1] IVML was called Triangle Logic in [5], but was recently renamed [7].

IVML w.r.t. triangle algebras and the connection between triangle algebras and IVRLs explains why this logic was called interval-valued.
IVML (and its extensions) enjoys the following deduction theorem:
$T \cup \{\phi\} \vdash_{IVML} \psi$ iff there is an integer n such that $T \vdash_{IVML} (\Box\phi)^n \to \psi$.

Numerous extensions of IVML can be defined. One of them is Interval-Valued Monoidal T-norm based Logic (IVMTL), which compares to IVML in more or less the same way as MTL [2] compares to ML. IVMTL (introduced in [6] under the name Pseudo-linear Triangle Logic) is IVML extended with the axiom scheme $(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$. The semantics of this logic are pseudo-prelinear triangle algebras, i.e., triangle algebras in which the subalgebra of exact elements is an MTL-algebra. IVMTL and its extensions are even pseudo-chain complete [6], which means that we can restrict its semantics to pseudo-linear triangle algebras, i.e., triangle algebras in which the subalgebra of exact elements is an MTL-chain. Recently it was proven [7] that IVMTL (and all other interval-valued counterparts of fuzzy logics that satisfy the real-chain embedding property [1, 4]) is even standard complete: we can further restrict the semantics to triangle algebras on $\mathcal{L}^I$, which is a lattice on the set of subintervals of the unit interval.

Remark that it is of course also possible to extend IVML with the axiom scheme $(\phi \to \psi) \vee (\psi \to \phi)$. The resulting logic is sound and complete with respect to prelinear triangle algebras. It was proven in [6] that prelinear triangle algebras are exactly triangle algebras in which the subalgebra of exact elements is a Boolean algebra. That is why we called this logic Interval-Valued Classical Propositional Calculus (IVCPC). Using the pseudo-chain completeness in this case shows that IVCPC is actually three-valued (because a (non-trivial) linear Boolean algebra has two elements).

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Possibilistic logic for graded properties

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Abstract. We introduce a logic for reasoning under uncertainty about properties whose presence may vary between "false" and "true" continuously. Uncertainty is understood in the sense of Dubois and Prade's Possibilistic Logic. Graded properties are modelled in a Boolean algebra of regular open sets of a topological space, in a way that properties with distinct but close degrees are not necessarily interpreted by disjoint sets.

1 Introduction

In this note we present a logic for reasoning under uncertainty. In general, uncertainty can be understood in many different ways [3]. Here, we treat uncertainty according to the well-known approach of Dubois and Prade: up to inessential notational differences, our framework is propositional Possibilistic Logic [1]. This choice provides us a calculus which is reasonably strong and in suitable contexts well applicable.

We deal with graded implications, written $\alpha_1, \ldots, \alpha_k \overset{d}{\Rightarrow} \beta$. Here, $\alpha_1, \ldots, \alpha_k$, β denote crisp properties and are consequently modelled by elements of a Boolean algebra. The value $d \in [0, 1]$ expresses uncertainty in a quantitative way, the background being a possibly insufficient amount of information. In our setting, d is the degree to which the agent is able to tell from his knowledge that $\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \neg\beta$ can be excluded: $d = 1$ means sure exclusion, in which case the implication clearly holds; $d = 0$ means not to assert anything; and any value in between expresses a tendency.

Our concern is to formalise reasoning about properties which are not in all situations clearly true or false. Such properties, commonly called "vague", are naturally modelled by fuzzy sets. We note that we do not intend to formalise reasoning about vague properties themselves. We will rather deal with the case that vague properties appear together with a degree of presence. Work closely related to ours can be found in [2] and there is further ongoing work in this direction.

We will shortly outline our approach, whose detailed description is in progress [5]. We start with a finite collection of symbols $\varphi, \ldots$, interpreted by fuzzy sets.

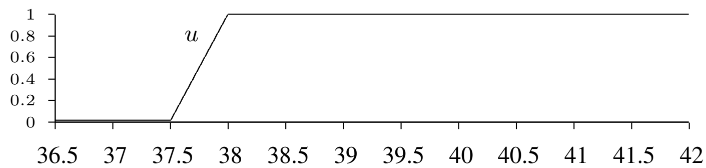
They can be combined by the connectives $\wedge, \vee, \sim$ interpreted by the pointwise minimum, maximum, and standard negation, respectively. Next, such expressions are endowed with an explicit degree of presence. For instance, the pair (φ, t) means that the property modelled by φ holds to the degree $t \in [0, 1]$. (φ, t) is crisp, and as such it is formally treated in the framework of Possibilistic Logic.

What makes our calculus specific is the interpretation of a pair (φ, t). Let φ be interpreted by the fuzzy set $u \colon S \to [0, 1]$. We might be tempted to associate with (φ, t) the set $[u]_t = \{s \in S : u(s) = t\}$. To axiomatise the resulting logic is however difficult; in addition it is conceptually problematic to assume an infinite, even uncountable, set of mutually exclusive situations.

We have to find an alternative solution. Intuitively speaking, we are guided by the idea to model (φ, t), where t runs over the real unit interval, in a way that the "smooth" transition between "φ" and "non-φ" is somehow reflected in the model, and still there should no infinite set of mutually exclusive properties involved. Our proposal is simple; we interpret (φ, s) and (φ, t) as overlapping if s and t differs by less than some fixed value ζ. Our next, still provisorily solution is associate with (φ, t) the larger set $[u]_{(t-\zeta, t+\zeta)} = \{s \in S : t - \zeta < u(s) < t + \zeta\}$. To include the statement that φ is clearly false or true, we moreover extend the set of syntactically usable truth values from $[0, 1]$ to $[-\zeta, 1 + \zeta]$.

We are already close to our actual definition, but modifications are still necessary. Boolean operations between sets as shown would lead again to sets of the form $[u]_t$. However, once degrees of presence are modelled in a "tolerant" way it would not make much sense still to have "point-like" interpretations available.

We opt for certain topological restrictions. Consider the case of a fuzzy set like the one which models "having fever":



W.r.t. the standard topologies, u is continuous. Furthermore for $0 < t < 1$ the set $[u]_t$ has an empty interior, whereas $[u]_{(t-\zeta, t+\zeta)}$ is open. Indeed, $[u]_{(t-\zeta, t+\zeta)}$ is even regular open, that is, the open interior of a closed set. Finally, the associated crisp properties are modelled by the closed sets $[u]_0$, and $[u]_1$, whose open interiors are again regular open.

In what follows, we will not work with a Boolean algebra of subsets of S. We rather assume that the universe S is a topological space. Let $\mathcal{R}(S)$ be the set of all regular open sets of S; then $(\mathcal{R}(S); \cap, \vee, ^{\perp}, \emptyset, S)$ is a Boolean algebra, where, for $A, B \in \mathcal{R}(S)$, $A \vee B = (A \cup B)^{-\circ}$ and $A^{\perp} = S \backslash A^{-}$.

The actual interpretation of (φ, t) will be an element of $\mathcal{R}(S)$, namely

$$[u]_t^\zeta \;=\; [u]_{(t-\zeta,t+\zeta)'}{}^{-\circ},$$

where $R' = \{(r \vee 0) \wedge 1 \colon\, r \in R\}$ for $R \subseteq [-\zeta, 1+\zeta]$.

2 The logic $\mathtt{IG}^\zeta$

The logic $\mathtt{IG}^\zeta$ is an extension of Possibilistic Logic to include graded properties. Let us fix a $\zeta \in (0, \frac{1}{2})$.

Definition 1. *Let M be a collection of continuous fuzzy sets over a T_1-space S, containing the constant 0 and constant 1 fuzzy sets and closed under pointwise minimum, maximum, and standard negation. Assume furthermore that for $u \in M$ and $t \in (0,1)$, the interior of $[u]_t$ is empty.*

Let $\mathcal{R}_M$ be the Boolean subalgebra of $\mathcal{R}(S)$ generated by $[u]_t^\zeta$, where $u \in M$ and $t \in [-\zeta, 1+\zeta]$. Let $\varrho \colon\; \mathcal{R}_M \to [0,1]$ be such that (i) $\varrho(1) = 0$, (ii) $\varrho(a) = 1$ *iff* $a = 0$, *and* (iii) $\varrho(a \vee b) = \varrho(a) \wedge \varrho(b)$.

The pair (M, ρ) is called a regular Kleene uncertainty algebra; ϱ is called a rejection function on $\mathcal{R}_M$.

The language of $\mathtt{IG}^\zeta$ follows the two-layer concept. Let $N \geq 1$. The *gradable propositions* are built up from symbols $\varphi_1, \ldots, \varphi_N, \bar{0}, \bar{1}$ by means of $\wedge, \vee, \sim$. *Graded propositions* are of the form (φ, t), where φ is a gradable proposition and $t \in [-\zeta, 1+\zeta]$. Finally, the *propositions* are Boolean combinations of graded propositions, and implications are of the form $\alpha_1, \ldots, \alpha_k \overset{d}{\Rightarrow} \beta$ for propositions $\alpha_1, \ldots, \alpha_k, \beta$ and a value $d \in [0,1]$.

Let (M, ϱ) be a regular Kleene uncertainty algebra. An evaluation v for $\mathtt{IG}^\zeta$ maps gradable propositions to M in the expected way; a graded proposition (φ, t) is mapped to $[v(\varphi)]_t^\zeta$; and the remaining propositions are mapped to $\mathcal{R}_M$ in the expected way. Finally, an implication $\alpha_1, \ldots, \alpha_k \overset{d}{\Rightarrow} \beta$ is satisfied by v if $\varrho(v(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \neg\beta)) \geq d$.

We axiomatise $\mathtt{IG}^\zeta$ as follows. The rules called basic in the sequel are those of Possibilistic Logic [4].

Definition 2. *The following are the basic rules of $\mathtt{IG}^\zeta$, where α, β, γ are propositions, Γ is a finite set of propositions, and $c, d \in [0,1]$:*

$$\bot \overset{d}{\Rightarrow} \alpha \qquad \alpha \overset{d}{\Rightarrow} \alpha \qquad \alpha \overset{d}{\Rightarrow} \top \qquad \alpha, \neg\alpha \overset{d}{\Rightarrow} \bot$$

$$\alpha \overset{0}{\Rightarrow} \beta \qquad \frac{\Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma \overset{c}{\Rightarrow} \alpha} \text{ where } c < d \qquad \frac{\Gamma \overset{c}{\Rightarrow} \alpha \quad \alpha \overset{d}{\Rightarrow} \beta}{\Gamma \overset{c \wedge d}{\Rightarrow} \beta}$$

$$\frac{\Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma, \beta \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \alpha, \beta \overset{d}{\Rightarrow} \gamma}{\Gamma, \alpha \wedge \beta \overset{d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \overset{c}{\Rightarrow} \alpha \quad \Gamma \overset{d}{\Rightarrow} \beta}{\Gamma \overset{c \wedge d}{\Rightarrow} \alpha \wedge \beta}$$

$$\frac{\Gamma, \alpha \overset{c}{\Rightarrow} \gamma \quad \Gamma, \beta \overset{d}{\Rightarrow} \gamma}{\Gamma, \alpha \vee \beta \overset{c \wedge d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma \overset{d}{\Rightarrow} \alpha \vee \beta} \qquad \frac{\Gamma \overset{d}{\Rightarrow} \beta}{\Gamma \overset{d}{\Rightarrow} \alpha \vee \beta}$$

$$\frac{\alpha \overset{d}{\Rightarrow} \beta}{\neg\beta \overset{d}{\Rightarrow} \neg\alpha} \qquad \frac{\neg\alpha \overset{d}{\Rightarrow} \beta}{\neg\beta \overset{d}{\Rightarrow} \alpha} \qquad \frac{\alpha \overset{d}{\Rightarrow} \neg\beta}{\beta \overset{d}{\Rightarrow} \neg\alpha}$$

The following are the fuzzy-set rules of $\mathtt{IG}^\zeta$*, where* ϕ, ψ *are gradable propositions,* α *is a proposition,* Γ *is a finite set of propositions, and* $s, t \in [-\zeta, 1+\zeta]$*:*

$$(\phi, s) \overset{1}{\Rightarrow} \neg(\phi, t) \text{ where } |s-t| \geq 2\zeta$$

$$(\phi, s) \overset{1}{\Rightarrow} (\phi, t) \text{ where } -\zeta \leq s \leq t < \zeta \text{ or } 1-\zeta < t \leq s \leq 1+\zeta$$

$$(\phi, r) \overset{1}{\Rightarrow} (\phi, s) \vee (\phi, t) \text{ where } s \leq r \leq t \leq s + 2\zeta$$

$$(\phi, r), (\phi, s) \overset{1}{\Rightarrow} (\phi, t) \text{ where } r \leq t \leq s$$

$$\neg(\phi, s_1), \ldots, \neg(\phi, s_k) \overset{1}{\Rightarrow} \bot \text{ where } s_1 < \zeta;\ s_2 - s_1, \ldots, s_k - s_{k-1} \leq 2\zeta;\ s_k > 1-\zeta$$

$$\frac{\Gamma, (\phi \wedge \psi, s \wedge t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\phi, s), (\psi, t) \overset{1}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\phi \wedge \psi, t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\phi, r), (\psi, s) \overset{1}{\Rightarrow} \alpha} \text{ where } r, s \geq t + 2\zeta$$

$$\frac{\Gamma, \neg(\phi \wedge \psi, t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\phi, s) \overset{1}{\Rightarrow} \alpha} \text{ where } s + 2\zeta \leq t \qquad \frac{\Gamma, \neg(\phi \wedge \psi, t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{1}{\Rightarrow} \alpha} \text{ where } s + 2\zeta \leq t$$

$$\frac{\Gamma, (\phi \vee \psi, s \vee t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\phi, s), (\psi, t) \overset{1}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\phi \vee \psi, t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\phi, r), (\psi, s) \overset{1}{\Rightarrow} \alpha} \text{ where } r + 2\zeta, s + 2\zeta \leq t$$

$$\frac{\Gamma, \neg(\phi \vee \psi, t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\phi, s) \overset{1}{\Rightarrow} \alpha} \text{ where } s \geq t + 2\zeta \qquad \frac{\Gamma, \neg(\phi \vee \psi, t) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{1}{\Rightarrow} \alpha} \text{ where } s \geq t + 2\zeta$$

$$\frac{\Gamma, (\phi, c) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\sim\phi, \sim c) \overset{1}{\Rightarrow} \alpha} \qquad \frac{\Gamma, (\sim\phi, c) \overset{1}{\Rightarrow} \alpha}{\Gamma, (\phi, \sim c) \overset{1}{\Rightarrow} \alpha}$$

A theory $\mathcal{T}$ *is called consistent if* $\mathcal{T} \vdash \top \overset{d}{\Rightarrow} \bot$ *implies* $d = 0$*.*

Theorem 1. *Let* $\mathcal{T}$ *be a consistent finite theory of* $\mathtt{IG}^\zeta$ *and* $\Gamma \overset{e}{\Rightarrow} \delta$ *an implication of* $\mathtt{IG}^\zeta$*.* $\mathcal{T}$ *semantically entails* $\Gamma \overset{e}{\Rightarrow} \delta$ *if and only if* $\mathcal{T}$ *proves* $\Gamma \overset{e}{\Rightarrow} \delta$*.*

We finally note that by means of an additional rule called the *smoothness rule* we can achieve that the rejection function is, in a natural sense, continuous. In this case, the rejection function is induced by a continuous function on the universe. The additional extension is particularly useful for the application of $\mathtt{IG}^{\varsigma}$ in the field of medical expert systems.

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Measure-free conditioning in MV-chains and additive measures

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Measure-free conditioning of *events* as elements of some suitable stuctured set $\mathbb{L}$ works in two steps. In a first step, *conditional events* "a given b" are defined in terms of the events $a, b \in \mathbb{L}$ as well-defined elements of some structured set. In a second step, the uncertainty of such conditional events is expressed by elements of the real unit interval as values of a suitable measure. Therefore, in measure-free conditioning we look for the "measure of a conditional event (a given b)" rather than for the "(conditional measure of an event a) given a fixed event b". The latter interpretation would be in the spirit of the classical approach in probability theory without defining conditional events and, therefore, is not considered here.

For events from a *Boolean algebra* $\mathbb{L}$, Goodman, Nguyen and Walker defined the conditional events as the lattice-intervals $[a \wedge b, b \to a]$ with $b \to a = b' \vee a$ and showed that the set of such conditionals forms a (semi-simple) MV-algebra, see [1], Section 4.3, Theorem 1. The author generalized this interval based definition of conditional events

$$(a \parallel b) = [a \wedge b, b \to a]$$

for events a, b from an *MV-algebra* $\mathbb{L}$ where now the residuation $b \to a = b' \sqcup a$ is expressed by the dual semigroup operation $\sqcup$ interpreted as "union", and observed that, vice versa, any interval can be written as conditional event, i.e. $[a, b] = (a \parallel b \to a)$. For details and related topics see [6, 7]. In our joint papers [2, 3] with Höhle, we introduced the *canonical extension* $\tilde{\mathbb{L}}$ of the MV-algebra $\mathbb{L}$ of events as the set of all pairs (a, b) of events a, b with $a \leq b$ equipped with a "canonical" *Girard algebra* structure, so called because of some analogy to *Girard quantales* from [5, 11]. All results for the canonical extension can be rewritten for conditional events because, additionally to the above mentioned one-to-one correspondence, the lattice-intervals can be identified with the pairs of its endpoints. In this sense the MV-algebra extension of any Boolean algebra of events from [1] is obtained as corollary. Furthermore, on any Girard algebra we introduced a *conditioning operator* $|$ as a binary operation fulfilling some "natural" axioms and we proved that the canonical extension of any MV-algebra of events always admits such conditioning operators which lead to the conditional events $(a \parallel b) \in \tilde{\mathbb{L}}$.

Moreover, there are several classes of MV-algebras $\mathbb{L}$ which admit conditioning operators $|$. In these cases, its values $(a \mid b) \in \mathbb{L}$ can also be considered as conditional events, but they are special events.

In the second step of measure-free conditioning we start with an additive measure (so-called *state* in [4]) m on an MV-algebra $\mathbb{L}$ of events. If $\mathbb{L}$ admits a conditioning operator $|$ then m can be applied directly to $(a \mid b)$. Boolean algebras $\mathbb{L}$ do not admit conditioning operators, but in [2] we proved that m has a unique additive measure extension $\tilde{m}$ on the MV-algebra extension $\tilde{\mathbb{L}}$ of conditional events $(a \parallel b)$. For non-Boolean MV-algebras $\mathbb{L}$ there do not exist such additive measure extensions but only so-called *weakly additive* measure extensions $\tilde{m}$ on the Girard algebra $\tilde{\mathbb{L}}$ where the classical additivity property is required only for all MV-subalgebras M of $\tilde{\mathbb{L}}$, see [8]. In [9] we characterized all weakly additive measures $\tilde{m}$ on the canonical extension $\tilde{\mathbb{L}}$ of any *finite MV-chain* $\mathbb{L}$. In [10] we generalized these results to any *finite MV-algebra*.

The aim of this talk is to generalize the main results from [9] to any *MV-chain* $\mathbb{L}$ and to apply them to conditional events $(a \parallel b) \in \tilde{\mathbb{L}}$. Finally, in the prominent case of the real unit interval $\mathbb{L} = [0, 1]$ with Łukasiewics' MV-algebra structure, these results are compared with the following three types of conditional events $(a \mid b) \in \mathbb{L}$ where the conditioning operator $|$ is induced by the *mean value functions* C_i given by resp.

$$C_1(\alpha,\beta) = \frac{\beta}{1+\beta-\alpha}\ , \quad C_2(\alpha,\beta) = \frac{\alpha+\beta}{2}\ , \quad C_3(\alpha,\beta) = \left\{\begin{array}{l} \beta \textit{ if} \quad \beta \le \frac{1}{2} \\ \frac{1}{2} \textit{ if } \alpha < \frac{1}{2} < \beta \\ \alpha \textit{ if} \quad \frac{1}{2} \le \alpha \end{array}\right\}$$

for $\alpha \le \beta$.

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Lattice-valued algebra and logic with potential applications in linguistic valued decision making

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Abstract: This note outlines the main academic ideas, objectives and research work we have been working for the last 15 years on lattice-valued algebra, lattice-valued logic and reasoning systems. It also gives some potential application of this research topic into linguistic valued based information processing and decision making from the logical point of view. Moreover, it presents an overall framework of the research and development of lattice-valued algebra and logic with some illustrated open problems in this topic.

Keywords: lattice-valued logic; lattice implication algebra; linguistic truth-valued algebra; linguistic truth-valued logic; decision making

1 Introduction

The logical foundation is vital for uncertainty reasoning from the symbolism point of view. It is analogous to the way in which classical logic provides a foundation for certain reasoning. As pointed out by Zagare ([1], p.103): "Without a logically consistent theoretical structure to explain them, empirical observations are impossible to evaluate; without a logically consistent theoretical structure to constrain them, original and creative theories are of limited utility; and without a logically consistent argument to support them, even entirely laudable conclusions… lose much of their intellectual force," we can, only through an exploration of the underlying logic, ascertain the consistency and completeness of our analyses.

Lattice-valued logic, e.g., [2, 3] and among others, as one of the most important many-valued logics, extends the chain-type truth-valued field to general lattice in which the truth-values are incompletely comparable with each other. Some researchers claim that chains can be applied in most cases, but very often the assumption is an oversimplification of reality due to the ignoring the incomparable elements. It is rather hard to directly provide the logical foundation to deal with incomparable information. In fact, relations in the real world are rarely linear. Incomparability is an important type of uncertainty often associated with human's intelligent activities in practice. This raises an overall uncertainty of objects due to missing information, ambiguity or conflicting evaluations, but it is not easily handled through conventional methods because of its complexity. The lattice structure is a useful and well-developed branch of abstract algebra for modelling the ordering relations in the real world, which is almost indispensable in explaining complex phenomena in an easy way [4, 5]. Hence, lattice-valued logic plays an important and promising research role that provides an alternative logical ground and approach to deal with both imprecision and incomparability.

Based on the above-mentioned academic ideals, we have been investigating how to deal with the incomparability in the intelligent information processing from the symbolism point of view since 1993. We have closely followed our academic routine from lattice-valued logical algebra — lattice implication algebra (LIA), as well as the corresponding lattice valued logic systems, lattice-valued approximate reasoning theory, and to lattice-valued automated reasoning theory and methods (for more details we refer to [6] and references therein). One of the fundamental goals is to provide practical and efficient inference methods and algorithms based on scientific and reasonable logic systems for dealing with both imprecision and incomparability in the intelligent information processing. We briefly outline some of our recent research framework on how to use LIA and its logic and reasoning schemes into linguistic valued information processing and decision making in the following section.

2 Application of LIA as a Linguistic Truth-Valued Algebra

Human beings cannot be seen as a precision mechanism. They usually express world knowledge using natural language with full of vague and imprecise concepts. Words, in different natural languages, sometimes seem difficult to distinguish their boundary, but their meaning of common usage can be understood. Moreover, there are some “vague overlap districts” among some words, which cannot be strictly linearly ordered, e.g., *highly true* and *slightly false* are incomparable, and *approximately true*, *possibly true*, *more or less true* are also incomparable. One cannot collapse that structure into a linearly ordered structure, because then one would impose an ordering on them, which was originally not present. This means the set of linguistic values may not be strictly linearly ordered.

Although there have been some investigations on the algebraic structure of linguistic truth values together with some applications in decision making and social science [7-10] and references therein, it still lacks a formalism for development of logic systems based on linguistic truth values, approximate reasoning and automated reasoning based on linguistic truth-valued logic systems. Among others, one of the key, substantial and also essential problems has not been paid sufficient attention, let alone be solved. That is, how to choose a comparatively appropriate linguistic truth-valued algebraic structure, which can provide a comparatively appropriate interpretation for the logical formulae in linguistic truth value logic systems, and accordingly provide a strict theoretical foundation, as well as a convenient, practical, and effective underlying semantic structure to automated uncertain reasoning based on linguistic truth-valued logic, and various kinds of corresponding intelligent information processing systems.

To attain this goal we propose to characterize the set of linguistic truth-values by a lattice-valued algebra structure, specially taking the LIA structure as one of alternatives, i.e., use the LIA to construct the structure of linguistic value sets in natural language. We summarize the basic framework as follows.

Definition 1 Let L_m, L_n be two LIAs and $L_m=\{a_1,\ldots, a_m\}$: $a_1\leq,\ldots\leq a_m$, $L_n=\{b_1,\ldots, b_n\}$: $b_1\leq,\ldots\leq b_n$, $a_i \rightarrow a_j = a_{m\wedge(m-i+j)}$, $a_i'= a_i \rightarrow a_1$, $b_k \rightarrow b_l = b_{n\wedge(n-k+l)}$, $b_k'= b_k \rightarrow b_1$. Define the product of L_m and L_n as follows: $L_m\times L_n =\{(a, b)|\ a\in L_m,\ b \in L_n\}$. The operations on $L_m\times L_n$ are defined respectively as follows: $(a_i, b_k)\vee(a_j, b_l)=(a_i\vee a_j, b_k\vee b_l)$, $(a_i, b_k)\wedge(a_j, b_l)=(a_i\wedge a_j, b_k\wedge b_l)$, $(a_i, b_k)\rightarrow(a_j, b_l)=(a_i\rightarrow a_j, b_k\rightarrow b_l)$, $(a_i, b_j)'=(a_i', b_j')$. Then $(L_m\times L_n, \vee, \wedge, \rightarrow, ', (a_1, b_1), (a_m, b_n))$ is a LIA, denoted by $L_{m\times n}$. If n=2, its Hasse Diagram is depicted in Fig. 1.

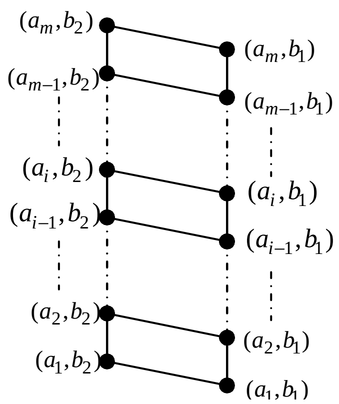

Fig. 1 Hasse Diagram of $L_{m\times 2}$

Example 1 (Lattice-valued algebra of linguistic terms with 18 elements modelled by LIA) Let a set of linguistic modifiers AD={Slightly (Sl for short), Somewhat (So), Rather (Ra), Almost (Al), Exactly (Ex), Quite (Qu), Very (Ve), Highly (Hi), Absolutely (Ab)} with the ordering relationship Sl<So<Ra<Al<Ex<Qu<Ve<Hi<Ab. We also define a set of meta truth values MT={True (Tr), False (Fa)}, where Fa<Tr. The set of linguistic values by combining AD and MT, denoted as L-LIA, forms a lattice with the boundary. And we define $\wedge$, $\vee$, implication $\rightarrow$ and complement operation $'$ on this lattice according to the LIA structure, which forms a linguistic truth-valued lattice implication algebra:

Let a mapping f: L-LIA $\rightarrow L_{18}=L_9\times L_2$ be defined as follows (L_{18} is defined as in Definition 1): f(Ab, Tr)= (a_9, b_2), f(Hi, Tr)= (a_8, b_2), f(Ve, Tr)= (a_7, b_2), f(Qu, Tr)= (a_6, b_2), f(Ex, Tr)= (a_5, b_2), f(Al, Tr)= (a_4, b_2), f(Ra, Tr)= (a_3, b_2), f(So, Tr)= (a_3, b_2), f(Sl, Tr)= (a_1, b_2), f(Sl, Fa)= (a_9, b_1), f(So, Fa)= (a_8, b_1), f(Ra, Fa)= (a_7, b_1), f(Al, Fa)= (a_6, b_1), f(Ex, Fa)= (a_5, b_1), f(Qu, Fa)= (a_4, b_1), f(Ve, Fa)= (a_3, b_1), f(Hi, Fa)= (a_2, b_1), f(Ab, Fa)= (a_1, b_1). Then f is a bijection. Denote its inverse mapping as f^{-1}. In addition, if for any $x, y\in$ L-LIA, $x\vee y= f^{1}(f(x) \vee f(y))$, $x\wedge y= f^{1}(f(x) \wedge f(y))$, $x\rightarrow y= f^{1}(f(\mathrm{x})\rightarrow f(y))$, $x'=f^{1}((f(x))')$, then it is easy to prove that (*L-LIA*, $\vee, \wedge, \rightarrow, '$)(still denoted as L) is a LIA, and L is isomorphic to L_{18}, i.e., f is an isomorphic mapping from L onto L_{18}. Notice that, for example, f(Hi, Tr)= (a_8, b_2) and f(Sl, Fa)= (a_9, b_1), and according to the Hasse diagram in Fig. 1 that (a_8, b_2) // (a_9, b_1) (// means incomparable), so does (Hi, Tr) and (Sl, Fa), i.e., Highly True // Slightly False, which intuitively is true knowing that both are incomparable in terms of their meanings in natural language. For the detailed work, we refer to [10].

In general, we conjecture that the domain of a linguistic-valued algebra (LA) can be represented as a lattice. Thus, a linguistic-valued logic is a logic in which the truth degree of an assertion is a linguistic value in LIA. A key insight behind the linguistic-valued logic scheme is that we can use natural language to express a logic in which the truth values of propositions are expressed as linguistic values in natural language terms such as *true*, *very true*, *very false*, and *false*, instead of a numerical scale. The lattice-valued logic system with truth-values in L-LIA is called linguistic truth-valued logic system, which leads to the linguistic truth approximate reasoning. It will be based on the direct reasoning in natural language which offers the advantage of not requiring the linguistic approximation step and the definition of the membership functions of the linguistic terms as in traditional fuzzy logic; also will treat vague information in its true format. In addition this proposed procedure has another advantage, i.e., the handling of incomparable linguistic terms in logical systems.

Information aggregation is important in decision making systems. In [11], Yager discussed the effect of the importance degrees in the types of aggregation *Max* and *Min* and proposed a general specification of the requirements that any importance transformation function must satisfy in both types of aggregations. It has been proved that the implication operator in a LIA satisfied those conditions of the importance transformation function, they can be used to capture the transformation between the weights and the individual ratings in *Min*-type aggregation, this, actually, provides one direct application of L-LIA into the linguistic information aggregation and decision making.

3 Conclusions

The questions we proposed in Section 2 are still open, we believe that it is feasible and reasonable to use lattice-valued algebra and lattice-valued logic to establish strict linguistic truth-valued logic and various kinds of corresponding linguistic information processing systems, based on what have been done so far about lattice-valued algebra, lattice-valued logic by different researches, certainly also relying on a continuous work on this direction.

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