Technical Report
FLLL–TR–0216

FUZZY.LOGIC.LAB.LINZ

Triangular norms. Position paper III: Continuous t-norms

Erich Peter Klement
Fuzzy Logic Laboratorium Linz-Hagenberg
ep.klement@jku.at

Radko Mesiar
Slovak Technical University, Bratislava & Czech Academy of Sciences, Prague
mesiar@cvt.stuba.sk

Endre Pap
University of Novi Sad
pap@im.ns.ac.yu, pape@eunet.yu

Abstract — This third and last part of a series of position papers on triangular norms (for Parts I and II see [25, 26]) presents the representation of continuous Archimedean t-norms by means of additive generators, and the representation of continuous t-norms by means of ordinal sums with Archimedean summands, both with full proofs. Finally some consequences of these representation theorems in the context of comparison and convergence of continuous t-norms, and of the determination of continuous t-norms by their diagonal sections are mentioned.

Key words: Triangular norm

Submitted for publication

1 Introduction

This is the third and final part of a series of position papers on the state of the art of some particularly important aspects of triangular norms in a condensed form. The monograph [24] provides a rather complete and self-contained overview about triangular norms and their applications.

Part I [25] considered some basic analytical properties of t-norms, such as continuity, and important classes such as Archimedean, strict and nilpotent t-norms. Also the dual operations, the triangular conorms, and De Morgan triples were mentioned. Finally, a short historical overview on the development of t-norms and their way into fuzzy sets and fuzzy logics was given.

Part II [26] is devoted to general construction methods based mainly on pseudo-inverses, additive and multiplicative generators, and ordinal sums, including also some constructions leading to non-continuous t-norms, and to a presentation of some distinguished families of t-norms.

In this third part we first present the representation of continuous Archimedean t-norms by means of additive generators, and then the representation of continuous t-norms by means of ordinal sums with Archimedean summands. These theorems were first proved in the framework of triangular norms in [29]. However, they can be also derived from results in [31] in the framework of semigroups. We include full proofs of the representation theorems mentioned above, since the original sources are not so easily accessible and/or they heavily use the special language of semigroup theory. Finally we include some results and examples which follow from these representation theorems.

Several notions and results from Parts I and II will be needed in this paper, and they can be found there in full detail [25, 26]. For the convenience of the reader, we briefly recall some of them.

Recall that a triangular norm (briefly t-norm) is a binary operation T on the unit interval $[0,1]$ which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $T\colon [0,1]^2 \longrightarrow [0,1]$ such that for all $x,y,z \in [0,1]$:

(T1) $T(x,y) = T(y,x)$,

(T2) $T(x,T(y,z)) = T(T(x,y),z)$,

(T3) $T(x,y) \leq T(x,z)$ whenever $y \leq z$,

(T4) $T(x,1) = x$.

Observe that for a continuous t-norm T the Archimedean property is equivalent to $T(x,x) < x$ for all $x \in\,]0,1[$, and that each continuous Archimedean t-norm is either strict or nilpotent [25, Theorem 6.15]. Given a t-norm T, an element $x \in [0,1]$ is said to be idempotent if $T(x,x) = x$ (clearly, 0 and 1 are idempotent elements of each t-norm, the so-called trivial idempotent elements).

Observe that the pseudo-inverse is defined for arbitrary monotone functions [26, Definition 2.1]. In our special setting with mostly deal with continuous, decreasing function $t\colon [0,1] \longrightarrow [0,\infty]$ with $t(1) = 0$, in which case the pseudo-inverse $f^{(-1)}$ reduces to

$$f^{(-1)}(x) = f^{-1}(\min(x, f(0))).$$

2 Representation of continuous Archimedean t-norms

For the class of all t-norms (which includes non-continuous t-norms and even t-norms which are not Borel measurable) the only existing characterization is by the axioms (T1)–(T4). The important subclass of continuous t-norms, however, has nice representations in terms of one-place functions and ordinal sums.

Theorem 2.1 *For a function* $T\colon [0,1]^2 \longrightarrow [0,1]$ *the following are equivalent:*

(i) T *is a continuous Archimedean t-norm.*

(ii) T *has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function* $t\colon [0,1] \longrightarrow [0,\infty]$ *with* $t(1) = 0$*, which is uniquely determined up to a positive multiplicative constant, such that for all* $(x, y) \in [0,1]^2$

$$T(x,y) = t^{(-1)}(t(x) + t(y)). \tag{2.1}$$

Proof: Assume first that $t\colon [0,1] \longrightarrow [0,+\infty]$ is a continuous, strictly decreasing function with $t(1) = 0$ and that T is constructed by (2.1), i.e., t is an additive generator of T. The commutativity (T1) and the monotonicity (T3) of T are obvious. Also, the boundary condition (T4) holds since for all $x \in [0,1]$

$$T(x,1) = t^{(-1)}(t(x) + t(1)) = t^{(-1)}(t(x)) = x.$$

Concerning the associativity (T2), for all $x, y, z \in [0,1]$ we obtain

$$\begin{aligned} T(T(x,y),z) &= t^{(-1)}(t(T(x,y)) + t(z)) \\ &= t^{(-1)}(t(t^{(-1)}(t(x) + t(y))) + t(z)) \\ &= t^{(-1)}(t(x) + t(y) + t(z)) \\ &= t^{(-1)}(t(x) + t(t^{(-1)}(t(y) + t(z)))) \\ &= t^{(-1)}(t(x) + t(T(y,z))) \\ &= T(x, T(y,z)), \end{aligned}$$

where the third equality is a consequence of

$$t(t^{(-1)}(t(x) + t(y))) = \min(t(x) + t(y), t(0)).$$

To prove the converse, let T be a continuous Archimedean t-norm. Concerning the notion $x_T^{(n)}$ we will use, recall that, for each $x \in [0,1]$, we have $x_T^{(0)} = 1$ and, for $n \in \mathbb{N}$, by recursion

$$x_T^{(n)} = T\big(x, x_T^{(n-1)}\big).$$

Define now for $x \in [0,1]$ and $m, n \in \mathbb{N}$

$$\begin{aligned} x_T^{(\frac{1}{n})} &= \sup\{y \in [0,1] \mid y_T^{(n)} < x\}, \\ x_T^{(\frac{m}{n})} &= \left(x_T^{(\frac{1}{n})}\right)_T^{(m)}. \end{aligned}$$

Since T is Archimedean, we have for all $x \in \,]0,1]$

$$\lim_{n\to\infty} x_T^{(\frac{1}{n})} = 1. \tag{2.2}$$

Note that the expression $x_T^{(\frac{m}{n})}$ is well-defined because of $x_T^{(\frac{m}{n})} = x_T^{(\frac{km}{kn})}$ for all $k \in \mathbb{N}$. If, for some $x \in [0,1]$ and some $n \in \mathbb{N} \cup \{0\}$, we have $x_T^{(n)} = x_T^{(n+1)}$ then, in the standard way by induction, we obtain

$$x_T^{(n)} = x_T^{(2n)} = \big(x_T^{(n)}\big)_T^{(2)}$$

and, since T is continuous Archimedean, $x_T^{(n)} \in \{0,1\}$. This means that we have $x_T^{(n)} > x_T^{(n+1)}$ whenever $x_T^{(n)} \in \,]0,1[$.

Now choose and fix an arbitrary element $a \in \,]0,1[$, and define the function $h\colon \mathbb{Q} \cap [0,\infty[\longrightarrow [0,1]$ by $h(r) = a_T^{(r)}$. Since T is continuous and since (2.2) holds, h is a continuous function. Moreover, we have for all $x \in [0,1]$ and $m,n,p,q \in \mathbb{N}$

$$\begin{aligned} x_T^{(\frac{m}{n}+\frac{p}{q})} &= x_T^{(\frac{mq+np}{nq})} \\ &= \big(x_T^{(\frac{1}{nq})}\big)_T^{(mq+np)} \\ &= T\Big(\big(x_T^{(\frac{1}{nq})}\big)_T^{(mq)}, \big(x_T^{(\frac{1}{nq})}\big)_T^{(np)}\Big) \\ &= T\big(x_T^{(\frac{m}{n})}, x_T^{(\frac{p}{q})}\big) \end{aligned}$$

and, as a consequence, for all $r,s \in \mathbb{Q} \cap [0,\infty[$

$$h(r+s) = a_T^{(r+s)} = T(a_T^{(r)}, a_T^{(s)}) \le a_T^{(r)} = h(r),$$

i.e., h is also non-increasing. The function h is even strictly decreasing on the preimage of $]0,1]$ since for all $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q} \cap [0,\infty[$ with $h(\frac{m}{n}) > 0$ we get

$$h(\tfrac{m}{n} + \tfrac{p}{q}) \le h\big(\tfrac{mq+1}{nq}\big) = \big(a_T^{(\frac{1}{nq})}\big)_T^{(mq+1)} < \big(a_T^{(\frac{1}{nq})}\big)_T^{(mq)} = h(\tfrac{m}{n}).$$

The monotonicity and continuity of h on $\mathbb{Q} \cap [0,\infty[$ allows us to extend it uniquely to a function $\overline{h}\colon [0,\infty] \longrightarrow [0,1]$ via

$$\overline{h}(x) = \inf\{h(r) \mid r \in \mathbb{Q} \cap [0,x]\}.$$

Then $\overline{h}$ is continuous and non-increasing, and we have for all $x,y \in [0,\infty]$

$$\overline{h}(x+y) = T(\overline{h}(x), \overline{h}(y)).$$

Moreover, $\overline{h}$ is strictly decreasing on the preimage of $]0,1]$. Define the function $t\colon [0,1] \longrightarrow [0,\infty]$ by

$$t(x) = \sup\{y \in [0,\infty] \mid \overline{h}(y) > x\}$$

with the usual convention $\sup \emptyset = 0$ (observe that t is just the pseudo-inverse of $\overline{h}$ and vice versa). Then t is continuous, strictly decreasing, and satisfies $t(1) = 0$ [24, Remark 3.4]. A combination of all the arguments so far yields that t is indeed a continuous additive generator of T since for each $(x,y) \in [0,1]^2$

$$T(x,y) = T(\overline{h}(t(x)), \overline{h}(t(y))) = \overline{h}(t(x)+t(y)) = t^{(-1)}(t(x)+t(y)).$$

To show that the continuous additive generator t of T constructed above is unique up to a positive multiplicative constant, assume that the two functions $t_1, t_2 \colon [0,1] \longrightarrow [0,\infty]$ are both continuous additive generators of T, i.e., we have for each $(x,y) \in [0,1]^2$ the equality

$$t_1^{(-1)}(t_1(x) + t_1(y)) = t_2^{(-1)}(t_2(x) + t_2(y)).$$

Substituting $u = t_2(x)$ and $v = t_2(y)$, we obtain that, for all $u, v \in [0, t_2(0)]$ satisfying $u + v \in [0, t_2(0)[$,

$$t_1 \circ t_2^{(-1)}(u) + t_1 \circ t_2^{(-1)}(v) = t_1 \circ t_2^{(-1)}(u+v). \tag{2.3}$$

Then from the continuity of t_1 and $t_2^{(-1)}$ it follows that (2.3) holds for all $u, v \in [0, t_2(0)]$ with $u + v \in [0, t_2(0)]$.

Equation (2.3) is a Cauchy functional equation (see [2]), whose continuous, strictly increasing solutions $t_1 \circ t_2^{(-1)} \colon [0, t_2(0)] \longrightarrow [0,\infty]$ must satisfy $t_1 \circ t_2^{(-1)} = b \cdot \mathrm{id}_{[0,t_2(0)]}$ for some $b \in \,]0,\infty[$. As a consequence, we get $t_1 = b \cdot t_2$ for some $b \in \,]0,\infty[$, thus completing the proof. □

Because of the special form of the pseudo-inverse $t^{(-1)}$, the representation (2.1) in Theorem 2.1 can also be written as

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))).$$

We already have seen in [25, Proposition 6.13 and Theorem 6.17] that a continuous Archimedean t-norm is either strict or nilpotent, a distinction which can be made also with the help of their additive generators. Indeed, generators t of strict t-norms satisfy $t(0) = \infty$ while generators of nilpotent t-norms satisfy $t(0) < \infty$ [26, Corollary 2.8].

Recall that for the product $T_{\mathbf{P}}$ and for the Łukasiewicz t-norm $T_{\mathbf{L}}$ additive generators $t : [0,1] \longrightarrow [0,\infty]$ are given by, respectively,

$$t(x) = -\log x,$$
$$t(x) = 1 - x.$$

Based on the proof of Theorem 2.1, it is possible to give some constructive way to obtain additive generators of continuous Archimedean t-norms. As an illustrating example, we include the following result of [11] (compare also [1, 34, 4]) for the case of strict t-norms which can be derived in a straightforward manner from the proof of Theorem 2.1.

Corollary 2.2 *Let T be a strict t-norm. Fix an arbitrary element $x_0 \in \,]0,1[$, and define the function $t \colon [0,1] \longrightarrow [0,\infty]$ by*

$$t(x) = \inf \left\{ \tfrac{m-n}{k} \;\middle|\; m,n,k \in \mathbb{N} \quad \text{and} \quad (x_0)_T^{(m)} < T\big((x_0)_T^{(n)}, x_T^{(k)}\big) \right\}.$$

Then t is an additive generator of T.

Example 2.3 If we consider the Hamacher product T [15] defined by

$$T(x,y) = \frac{xy}{x + y - xy}$$

whenever $(x, y) \neq (0, 0)$, observe that we get (taking into account $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$) for all $(x, y) \in [0, 1]^2$

$$T(x, y) = \frac{1}{\frac{1}{x} + \frac{1}{y} - 1}$$

and, for each $x \in [0, 1]$ and each $n \in \mathbb{N}$

$$x_T^{(n)} = \frac{1}{\frac{n}{x} - n + 1}.$$

For $x_0 = 0.5$ the inequality

$$(x_0)_T^{(m)} < T\big((x_0)_T^{(n)}, x_T^{(k)}\big)$$

is easily seen to be equivalent to $m - n > k\big(\frac{1}{x} - 1\big)$, yielding the additive generator $t\colon [0, 1] \longrightarrow [0, \infty]$ of T specified by

$$\begin{aligned} t(x) &= \inf\left\{ \tfrac{m-n}{k} \;\middle|\; m, n, k \in \mathbb{N} \text{ and } \tfrac{m-n}{k} > \tfrac{1}{x} - 1 \right\} \\ &= \frac{1-x}{x}. \end{aligned}$$

The representation of continuous Archimedean t-norms given in Theorem 2.1 is based on the addition on the interval $[0, +\infty]$. There is a completely analogous representation thereof based on the multiplication on $[0, 1]$, thus leading to a representation of continuous Archimedean t-norms by means of multiplicative generators [26, Section 2]. By duality, there are also representations of continuous Archimedean t-conorms by means of additive generators and multiplicative generators, respectively.

Remark 2.4 (i) If T is a continuous Archimedean t-norm with additive generator $t\colon [0, 1] \longrightarrow [0, \infty]$, then the function $\theta\colon [0, 1] \longrightarrow [0, 1]$ defined by $\theta(x) = e^{-t(x)}$ is a multiplicative generator of T.

(ii) If S is a continuous Archimedean t-conorm then the dual t-norm T is continuous Archimedean and, therefore, has an additive generator $t\colon [0, 1] \longrightarrow [0, \infty]$. Then $s\colon [0, 1] \longrightarrow [0, \infty]$ defined by $s(x) = t(1 - x)$ is an additive generator of S, and $\xi\colon [0, 1] \longrightarrow [0, 1]$ defined by $e^{-t(1-x)}$ is a multiplicative generator of S.

(iii) Given a continuous Archimedean t-norm T and a strictly increasing bijection $\varphi\colon [0, 1] \longrightarrow [0, 1]$, it is clear that the function $T_\varphi\colon [0, 1]^2 \longrightarrow [0, 1]$ given by

$$T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$$

is a continuous Archimedean t-norm too. By Theorem 2.1, there are additive generators $t, t_\varphi\colon [0, 1] \longrightarrow [0, \infty]$ of T and T_φ, respectively. Taking into account [26, Proposition 2.9], t_φ equals $t \circ \varphi$ up to a multiplicative constant.

It is straightforward that each isomorphism $\varphi\colon [0, 1] \longrightarrow [0, 1]$ preserves (among many other properties) the continuity, the strictness and the existence of zero divisors. Therefore, each t-norm which is isomorphic to a strict or to a nilpotent t-norm, itself is strict or nilpotent, respectively.

Conversely, if T_1 and T_2 are two strict t-norms with additive generators t_1 and t_2 (which are bijective functions from $[0, 1]$ into $[0, \infty]$ in this case), respectively, then $\varphi\colon [0, 1] \longrightarrow [0, 1]$ given by $\varphi = t_1^{-1} \circ t_2$ is a strictly increasing bijection and $T_2 = (T_1)_\varphi$. If T_1 and T_2 are two nilpotent t-norms with additive generators t_1 and t_2, respectively, then we have $T_2 = (T_1)_\varphi$, where the strictly increasing bijection $\varphi\colon [0,1] \longrightarrow [0,1]$ is given by $\varphi = t_1^{-1} \circ \left(\frac{t_1(0)}{t_2(0)} \cdot t_2\right)$ (observe that in this case the two functions t_1 and $\frac{t_1(0)}{t_2(0)} \cdot t_2$ can be viewed as bijections from $[0,1]$ into $[0, t_1(0)]$).

We therefore have shown the following result:

Lemma 2.5 *Two continuous Archimedean t-norms are isomorphic if and only if they are either both strict or both nilpotent.*

An immediate consequence of Remark 2.4(iii) and Lemma 2.5 is that the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ are not only prototypical examples of strict and nilpotent t-norms, respectively, but that each continuous Archimedean t-norm is isomorphic either to $T_{\mathbf{P}}$ or to $T_{\mathbf{L}}$:

Theorem 2.6 *(i) A function $T\colon [0,1]^2 \longrightarrow [0,1]$ is a strict t-norm if and only if it is isomorphic to the product $T_{\mathbf{P}}$.*

(ii) A function $T\colon [0,1]^2 \longrightarrow [0,1]$ is a nilpotent t-norm if and only if it is isomorphic to the Łukasiewicz t-norm $T_{\mathbf{L}}$.

Each multiplicative generator $\theta\colon [0,1] \longrightarrow [0,1]$ of a strict t-norm T can be viewed as an isomorphism between $T_{\mathbf{P}}$ and T, i.e., $T = (T_{\mathbf{P}})_\theta$. In particular, this means that there are infinitely many isomorphisms between $T_{\mathbf{P}}$ and T. On the other hand, if T is a nilpotent t-norm with additive generator $t\colon [0,1] \longrightarrow [0,\infty]$, then there is a unique isomorphism $\varphi\colon [0,1] \longrightarrow [0,1]$ between $T_{\mathbf{L}}$ and T, namely, $\varphi = 1 - \frac{1}{t(0)} \cdot t$.

Recall that each continuous t-norm T satisfying $T(x,x) < x$ for all $x \in \,]0,1[$ is Archimedean [24, Proposition 2.15].

Corollary 2.7 *If T is a continuous t-norm with trivial idempotent elements only, i.e., $T(x,x) = x$ only if $x \in \{0,1\}$, then T is Archimedean and, therefore, has a continuous additive generator.*

Remark 2.8 Note that the representation in Theorem 2.1 holds for continuous Archimedean t-norms only. However, there are several possibilities to show the existence of continuous additive generators for a function $T\colon [0,1]^2 \longrightarrow [0,1]$ under weaker hypotheses than in Theorem 2.1.

For example, it is possible to drop the commutativity [T1] [31] (see also [24, Theorem 2.43]) or to weaken the associativity [T2] [6, 28]. In the case of left-continuous t-norms, either the Archimedean property [27] or the existence of a (not necessarily continuous) additive generator [39] implies the existence of a continuous additive generator. In the case of a strictly monotone Archimedean t-norm T, the continuity of T in the point $(1,1)$ is sufficient [14].

3 Representation of continuous t-norms

The construction of a new semigroup from a family of given semigroups using ordinal sums goes back to A. H. Clifford [8] (see also [9, 17, 35]), and it is based on ideas presented in [10, 22]. It has been successfully applied to t-norms in [13, 25, 29, 37].

Definition 3.1 Let $(T_\alpha)_{\alpha\in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm T defined by

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\frac{x-a_\alpha}{e_\alpha-a_\alpha}, \frac{y-a_\alpha}{e_\alpha-a_\alpha}\right) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

is called the *ordinal sum* of the *summands* $\langle a_\alpha, e_\alpha, T_\alpha\rangle$, $\alpha \in A$, and we shall write

$$T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}.$$

Observe that the index set A is necessarily finite or countably infinite. It also may be empty, in which case the ordinal sum equals the idempotent t-norm $T_{\mathbf{M}}$.

Note that the representation of continuous Archimedean t-norms by means of multiplicative generators can be derived directly from more general results for I-semigroups (see [24, 29, 31, 38]). Similarly, the following representation of continuous t-norms by means of ordinal sums follows also from results of [31] in the context of I-semigroups.

Theorem 3.2 *A function* $T\colon [0,1]^2 \longrightarrow [0,1]$ *is a continuous t-norm if and only if* T *is an ordinal sum of continuous Archimedean t-norms.*

Proof: Obviously, each ordinal sum of continuous t-norms is a continuous t-norm.

Conversely, if T is a continuous t-norm, we first show that the set I_T of all idempotent elements of T is a closed subset of $[0,1]$. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of idempotent elements of T which converges to some $x \in [0,1]$, then the continuity of T implies

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_n, x_n) = T(x,x),$$

so x is also an idempotent element of T, and I_T is closed.

In the case $I_T = [0,1]$ we have $T = T_{\mathbf{M}}$, i.e., an empty ordinal sum. If $I_T \neq [0,1]$ it can be written as the (non-trivial) union of a finite or countably infinite family of pairwise disjoint open subintervals $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ where, of course, each a_α and each e_α (but no element in $]a_\alpha, e_\alpha[$) is an idempotent element of T.

For the time being, assume that $A \neq \emptyset$ and fix an arbitrary $\alpha \in A$. Then the monotonicity of T implies that for all $(x,y) \in [a_\alpha, e_\alpha]^2$

$$a_\alpha = T(a_\alpha, a_\alpha) \leq T(x,y) \leq T(e_\alpha, e_\alpha) = e_\alpha$$

and for all $x \in [a_\alpha, 1]$

$$a_\alpha = T(a_\alpha, a_\alpha) \leq T(x, a_\alpha) \leq T(1, a_\alpha) = a_\alpha,$$

showing that $([a_\alpha, e_\alpha], T|_{[a_\alpha,e_\alpha]^2})$ is a semigroup with annihilator a_α and with trivial idempotent elements only (actually, a_α acts as annihilator on $[a_\alpha, 1]$). Because of the monotonicity and continuity of T we also have for each $\alpha \in A$

$$\{T(z, e_\alpha) \mid z \in [0,1]\} = [0, e_\alpha],$$

which means that each $x \in [0, e_\alpha]$ can be written as $x = T(z, e_\alpha)$ for some $z \in [0,1]$. This, together with the associativity of T, implies that

$$T(x, e_\alpha) = T(T(z, e_\alpha), e_\alpha) = T(z, T(e_\alpha, e_\alpha)) = T(z, e_\alpha) = x,$$

showing that e_α acts as neutral element on $[0, e_\alpha]$ and, subsequently, also in the I-semigroup $([a_\alpha, e_\alpha], T|_{[a_\alpha, e_\alpha]^2})$.

Let $\varphi_\alpha \colon [0,1] \longrightarrow [a_\alpha, e_\alpha]$ be the strictly increasing bijection given by

$$\varphi_\alpha(x) = a_\alpha + (e_\alpha - a_\alpha) \cdot x,$$

then for each $\alpha \in A$ the function $T_\alpha \colon [0,1]^2 \longrightarrow [0,1]$ defined by

$$T_\alpha(x, y) = \varphi_\alpha^{-1}(T(\varphi_\alpha(x), \varphi_\alpha(y))$$

is a continuous t-norm which has only trivial idempotent elements, and which is also Archimedean because of Corollary 2.7. A simple computation verifies that for all $\alpha \in A$ and for all $(x, y) \in [a_\alpha, e_\alpha]^2$ we have

$$T(x, y) = a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\big(\tfrac{x - a_\alpha}{e_\alpha - a_\alpha}, \tfrac{y - a_\alpha}{e_\alpha - a_\alpha}\big).$$

If $(x, y) \in [0,1]^2$ (without loss of generality we may assume $x \leq y$) is contained in none of the squares $[a_\alpha, e_\alpha]^2$ then there exists some idempotent element $b \in [x, y]$ which acts as neutral element on $[0, b]$ and as annihilator on $[b, 1]$, and we have

$$T(x, y) = T(T(x, b), y) = T(x, T(b, y)) = T(x, b) = x = \min(x, y),$$

completing the proof that $T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}$.

The uniqueness of the representation of T is an immediate consequence of the one-to-one correspondence between the set of idempotent elements of T and the family of intervals $(]a_\alpha, e_\alpha[)_{\alpha \in A}$.

□

The combination of Theorem 3.2 and of the results of Section 2 yields the following representations of continuous t-norms:

Corollary 3.3 *For a function $T \colon [0,1]^2 \longrightarrow [0,1]$ the following are equivalent:*

(i) T is a continuous t-norm.

(ii) T is isomorphic to an ordinal sum whose summands contain only the t-norms $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$.

(iii) There is a family $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ and a family $h_\alpha \colon [a_\alpha, e_\alpha] \longrightarrow [0, \infty]$ of continuous, strictly decreasing functions with $h_\alpha(e_\alpha) = 0$ for each $\alpha \in A$ such that for all $(x, y) \in [0,1]^2$

$$T(x, y) = \begin{cases} h_\alpha^{(-1)}(h_\alpha(x) + h_\alpha(y)) & \text{if } (x, y) \in [a_\alpha, e_\alpha]^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{3.1}$$

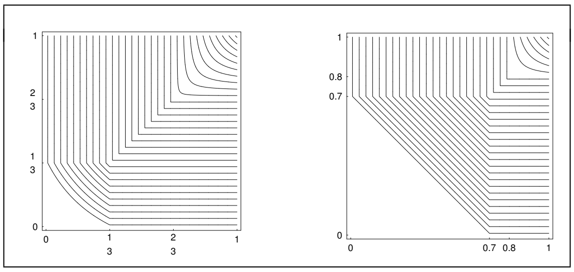

Figure 1: Contour plots of the isomorphic t-norms T (left) and $T_{0.7,0.8}$ from Example 3.4

Example 3.4 Consider the continuous t-norm T (see Figure 1) given by

$$T(x,y) = \begin{cases} \max\left(\frac{3x+3y+9xy-1}{6}, 0\right) & \text{if } (x,y) \in \left[0, \frac{1}{3}\right]^2, \\ \frac{4x+4y-3xy-4}{9x+9y-9xy-8} & \text{if } (x,y) \in \left[\frac{2}{3}, 1\right]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

This t-norm T can be written as the ordinal sum $(\langle 0, \frac{1}{3}, T_1\rangle, \langle \frac{2}{3}, 1, T_2\rangle)$ with T_1 and T_2 being given by

$$T_1(x,y) = \max\left(\tfrac{x+y+xy-1}{2}, 0\right),$$
$$T_2(x,y) = \frac{xy}{x+y-xy}.$$

Observe that the nilpotent t-norm T_1 was introduced in [40], and that the strict t-norm T_2 is the Hamacher product $T_0^{\mathbf{H}}$ [26], and that the functions t_1, t_2 given by

$$t_1(x) = -\log \tfrac{1+x}{2},$$
$$t_2(x) = \frac{1-x}{x}.$$

are continuous additive generators of T_1 and T_2, respectively. Defining the functions $h_1\colon \left[0, \frac{1}{3}\right] \longrightarrow [0,\infty]$ and $h_2\colon \left[\frac{2}{3}, 1\right] \longrightarrow [0,\infty]$ by

$$h_1(x) = -\log \tfrac{1+3x}{2},$$
$$h_2(x) = \frac{3-3x}{3x-2},$$

we can represent our t-norm T in the form (3.1). For any numbers $a, b \in \,]0,1[$ with $a < b$ consider the t-norm $T_{ab} = (\langle 0, a, T_{\mathbf{L}}\rangle, \langle b, 1, T_{\mathbf{P}}\rangle)$ (see Figure 1). Then T is isomorphic to T_{ab}, i.e., we have $T = (T_{ab})_\varphi$ where the strictly increasing bijection $\varphi\colon [0,1] \longrightarrow [0,1]$ is given by

$$\varphi(x) = \begin{cases} a\,\frac{\log(1+3x)}{\log 2} & \text{if } x \in \left[0, \frac{1}{3}\right], \\ a + (b-a)(3x-1) & \text{if } x \in \left]\frac{1}{3}, \frac{2}{3}\right], \\ b + (1-b)e^{\frac{3x-3}{3x-2}} & \text{otherwise.} \end{cases}$$

Analogous representations for continuous t-conorms can be obtained by duality (making the necessary changes, e.g., replacing $\min$ by $\max$).

4 Consequences of the representation theorems

Theorems 2.1 and 3.2 simplify the work with continuous t-norms in the sense that it suffices to consider (a family of) continuous Archimedean t-norms and, subsequently, their additive generators. In particular, the additive generator (which is a one-place function) of a continuous Archimedean t-norm T carries all the information of the whole t-norm T.

Knowing the structure of continuous t-norms allows us also to deduce general properties from partial information. For instance, if for a continuous t-norm T and for some $x_0 \in \,]0,1[$ the vertical section $f\colon [0,1] \longrightarrow [0,1]$ given by $f(y) = T(x_0, y)$ is strictly monotone and satisfies $f(y) < y$ for all $y \in \,]0,1]$, then T is a strict t-norm.

In this section, we demonstrate the impact of Theorems 2.1 and 3.2 on the problems of (pointwise) comparison and convergence of continuous t-norms, and on the determination of continuous t-norms by their diagonal sections.

The following necessary and sufficient condition for the comparison of continuous Archimedean t-norms can be found in [38, Lemma 5.5.8] (see also [24, Theorem 6.2], for the special case of strict t-norms it was proved first in [36] (see also [5]).

Theorem 4.1 *Let T_1 and T_2 be two continuous Archimedean t-norms with additive generators $t_1, t_2\colon [0,1] \longrightarrow [0,\infty]$, respectively. The following are equivalent:*

(i) $T_1 \leq T_2$.

(ii) The function $t_1 \circ t_2^{-1}\colon [0, t_2(0)] \longrightarrow [0,\infty]$ is subadditive, i.e., for all $u, v \in [0, t_2(0)]$ with $u + v \in [0, t_2(0)]$ we have

$$t_1 \circ t_2^{-1}(u + v) \leq t_1 \circ t_2^{-1}(u) + t_1 \circ t_2^{-1}(v).$$

There exist criteria (some of which are only sufficient) for the comparability of continuous Archimedean t-norms which sometimes are easier to check than the subadditivity in Theorem 4.1. The following sufficient conditions can be derived easily from Theorem 4.1 (recall that a function $f\colon\ [a,b] \longrightarrow [0,\infty]$ is called concave if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in [a,b]$ and for all $\lambda \in [0,1]$) and from [16, (103)].

Corollary 4.2 *Let T_1 and T_2 be two continuous Archimedean t-norms with additive generators $t_1, t_2\colon [0,1] \longrightarrow [0,\infty]$, respectively. Then we have $T_1 \leq T_2$ if one of the following conditions is satisfied:*

(i) The function $t_1 \circ t_2^{-1}\colon [0, t_2(0)] \longrightarrow [0,\infty]$ is concave.

(ii) The function $f\colon\ ]0, t_2(0)] \longrightarrow [0,\infty]$ defined by

$$f(x) = \frac{(t_1 \circ t_2^{-1})(x)}{x}$$

is non-increasing.

(iii) The function

$$\frac{t_1'}{t_2'}\colon\ ]0,1[\ \longrightarrow [0,\infty[$$

is non-decreasing.

Example 4.3 In [26, Example 2.10(i)] we have seen that for each continuous Archimedean t-norm T with additive generator $t\colon [0,1] \longrightarrow [0,\infty]$, and for each $\lambda \in\ ]0,\infty[$, the function $t^\lambda\colon [0,1] \longrightarrow [0,\infty]$ is an additive generator of a continuous Archimedean t-norm which was denoted there $T^{(\lambda)}$. Now we are able to show that the family $(T^{(\lambda)})_{\lambda\in]0,\infty[}$ is strictly increasing with respect to the parameter λ. Indeed, for $\lambda,\mu \in\ ]0,\infty[$ the composite function $t^\lambda \circ (t^\mu)^{-1}\colon [0,t(0)^\mu] \longrightarrow [0,\infty]$ is given by

$$t^\lambda \circ (t^\mu)^{-1}(x) = x^{\frac{\lambda}{\mu}},$$

and it is concave whenever $\lambda \le \mu$, showing that $(T^{(\lambda)})_{\lambda\in]0,\infty[}$ is a strictly increasing family of t-norms. Consequently, the families of Yager t-norms [41], of Aczél-Alsina t-norms [3], and of Dombi t-norms [12] are strictly increasing families of t-norms.

A nontrivial problem was the monotonicity of the family of Frank t-norms $(T^{\mathbf{F}}_\lambda)_{\lambda\in[0,\infty]}$ [13]. A first proof thereof appeared in [7, Proposition 1.12]. In the following we give a simpler proof [23] based on Corollary 4.2(iii) (see also [24, Proposition 6.8]).

Proposition 4.4 *The family $(T^{\mathbf{F}}_\lambda)_{\lambda\in[0,\infty]}$ of Frank t-norms is strictly decreasing.*

Proof: Recall that $T^{\mathbf{F}}_0 = T_{\mathbf{M}}$, $T^{\mathbf{F}}_1 = T_{\mathbf{P}}$, whose additive generator $t^{\mathbf{F}}_1$ is given by $t^{\mathbf{F}}_1(x) = -\log x$, and $T^{\mathbf{F}}_\infty = T_{\mathbf{L}}$ whose additive generator $t^{\mathbf{F}}_\infty$ is given by $t^{\mathbf{F}}_\infty(x) = 1-x$. For each $\lambda \in\ ]0,1[\cup]1,\infty[$, $T^{\mathbf{F}}_\lambda$ is a strict t-norm, and its additive generator $t^{\mathbf{F}}_\lambda$ is given by $t^{\mathbf{F}}_\lambda(x) = \log\frac{\lambda-1}{\lambda^x-1}$.

Trivially we have $T^{\mathbf{F}}_0 = T_{\mathbf{M}} > T^{\mathbf{F}}_\lambda$ for all $\lambda \in\ ]0,\infty]$. From

$$\frac{(t^{\mathbf{F}}_\infty)'}{(t^{\mathbf{F}}_\lambda)'}(x) = \begin{cases} x & \text{if } \lambda = 1, \\ \frac{\lambda^x-1}{\lambda^x \log\lambda} & \text{if } \lambda \in\ ]0,1[\,\cup\,]1,\infty[\,, \end{cases}$$

it follows that for each $\lambda \in\ ]0,\infty[$ the function $\frac{(t^{\mathbf{F}}_\infty)'}{(t^{\mathbf{F}}_\lambda)'}$ is non-decreasing, implying $T^{\mathbf{F}}_\infty \le T^{\mathbf{F}}_\lambda$ and, since $T^{\mathbf{F}}_\infty$ is nilpotent and $T^{\mathbf{F}}_\lambda$ is strict, even $T^{\mathbf{F}}_\infty < T^{\mathbf{F}}_\lambda$.

Now let us show that $T^{\mathbf{F}}_\mu \le T^{\mathbf{F}}_\lambda$ whenever $1 < \lambda < \mu < \infty$. Observe that for all $x \in\ ]0,1[$ we get

$$\frac{(t^{\mathbf{F}}_\mu)'}{(t^{\mathbf{F}}_\lambda)'}(x) = \frac{\mu^x(\lambda^x-1)\log\mu}{\lambda^x(\mu^x-1)\log\lambda} = \frac{\log\mu}{\log\lambda}\cdot\frac{1-\left(\frac{1}{\lambda}\right)^x}{1-\left(\frac{1}{\mu}\right)^x}.$$

Then $\frac{(t^{\mathbf{F}}_\mu)'}{(t^{\mathbf{F}}_\lambda)'}$ is non-decreasing on $]0,1[$ if and only if

$$\left(1-\left(\tfrac{1}{\mu}\right)^x\right)\left(\tfrac{1}{\lambda}\right)^x\log\tfrac{1}{\lambda} \le \left(1-\left(\tfrac{1}{\lambda}\right)^x\right)\left(\tfrac{1}{\mu}\right)^x\log\tfrac{1}{\mu},$$

i.e., if and only if we have the inequality

$$\frac{\left(\frac{1}{\lambda}\right)^x\log\frac{1}{\lambda}}{\left(\frac{1}{\mu}\right)^x\log\frac{1}{\mu}} \ge \frac{1-\left(\frac{1}{\lambda}\right)^x}{1-\left(\frac{1}{\mu}\right)^x}. \tag{4.1}$$

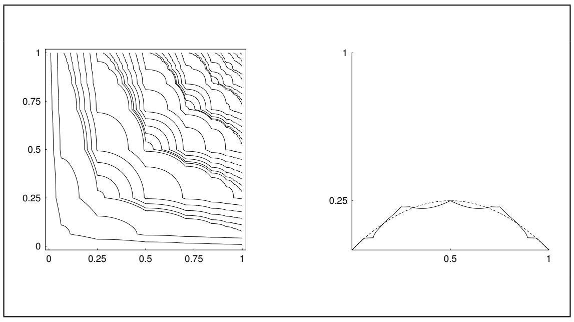

Figure 2: Contour plot of the strict t-norm T (left) considered in Example 4.5 together with the incomparable opposite diagonal sections of T and $T_{\mathbf{P}}$ (right)

Consider now the functions $f, g\colon\ ]0,1[\ \longrightarrow [0,\infty[$ which are defined by $f(x) = 1 - \left(\frac{1}{\lambda}\right)^x$ and $g(x) = 1 - \left(\frac{1}{\mu}\right)^x$. Then, by the Cauchy Mean Value Theorem, for each $x \in\ ]0,1[$ there exists a $y \in\ ]0,x[$ such that

$$\frac{1-\left(\frac{1}{\lambda}\right)^x}{1-\left(\frac{1}{\mu}\right)^x} = \frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(y)}{g'(y)} = \frac{\left(\frac{1}{\lambda}\right)^y \log\frac{1}{\lambda}}{\left(\frac{1}{\mu}\right)^y \log\frac{1}{\mu}} < \frac{\left(\frac{1}{\lambda}\right)^x \log\frac{1}{\lambda}}{\left(\frac{1}{\mu}\right)^x \log\frac{1}{\mu}}.$$

This proves inequality (4.1) and, consequently, the function $\frac{(t_\mu^{\mathbf{F}})'}{(t_\lambda^{\mathbf{F}})'}$ is non-decreasing, i.e., $T_\mu^{\mathbf{F}} \le T_\lambda^{\mathbf{F}}$ and, because of $T_\mu^{\mathbf{F}} \ne T_\lambda^{\mathbf{F}}$, even $T_\mu^{\mathbf{F}} < T_\lambda^{\mathbf{F}}$ in this case. Similarly we can show $T_1^{\mathbf{F}} < T_\lambda^{\mathbf{F}}$ for all $\lambda \in\ ]1,\infty[$.

The case $0 < \lambda < \mu \le 1$ can be transformed into $1 \le \frac{1}{\mu} < \frac{1}{\lambda} < \infty$, and the case $0 < \lambda < 1 < \mu < \infty$ is proved combining the two latter cases. □

The comparison of arbitrary continuous t-norms is much more complicated, and it is fully described in [23] (see also [24, Theorem 6.12]).

When comparing t-norms it is evident that the incomparability of their diagonal sections implies the incomparability of the t-norms themselves. The converse, however, is not true in general, not even in the case of continuous Archimedean t-norms.

Example 4.5 Consider the function $t\colon [0,1] \longrightarrow [0,\infty]$ defined by (the index n may be any number in $\mathbb{Z}$)

$$t(x) = \begin{cases} \infty & \text{if } x = 0, \\ 2^n(2-(4x^{\frac{1}{2^n}}-1)^2) & \text{if } x \in \left[\frac{1}{2^{2^{n+1}}}, \frac{1}{2^{2^n}}\right[, \\ 0 & \text{if } x = 1, \end{cases}$$

then t is an additive generator of some strict t-norm T. A simple computation shows that the diagonal sections of T and $T_{\mathbf{P}}$ coincide, but the opposite diagonal sections $d_T, d_{T_{\mathbf{P}}}\colon [0,1] \longrightarrow [0,1]$ given by $d_T(x) = T(x, 1-x)$ and $d_{T_{\mathbf{P}}}(x) = T_{\mathbf{P}}(x, 1-x)$ are incomparable (see Figure 2).

This shows that different continuous t-norms may have identical diagonal sections. Note that there are methods to describe all continuous t-norms having a given diagonal section [21, 24, 30]. Here we only mention one of these methods applied to strict t-norms [21, 30] (see also [24, Proposition 7.11]:

Proposition 4.6 *Let $\delta\colon [0,1] \longrightarrow [0,1]$ be a strictly increasing bijection such that $\delta(x,x) < x$ for all $x \in \,]0,1[$. Then a continuous t-norm T has diagonal section δ if and only if T is strict and the function $t\colon [0,1] \longrightarrow [0,\infty]$ given by*

$$t(x) = \begin{cases} \infty & \text{if } x = 0, \\ 2^n \cdot f\big(\delta^{(-n)}(x)\big) & \text{if } x \in \big]\delta^{(n+1)}(0.5), \delta^{(n)}(0.5)\big], \\ 0 & \text{if } x = 1, \end{cases}$$

is an additive generator of T, where $f\colon [\delta(0.5), 0.5] \longrightarrow [1,2]$ is a strictly decreasing bijection, $\delta^{(0)} = \mathrm{id}_{[0,1]}$, $\delta^{(n)} = \delta \circ \delta^{(n-1)}$ whenever $n \in \mathbb{N}$, and $\delta^{(n)} = \big(\delta^{(-n)}\big)^{-1}$ whenever $-n \in \mathbb{N}$.

As a consequence of Proposition 4.6, two different strict t-norms with the same diagonal section are necessarily incomparable, compare also Example 4.5 (the same result holds for arbitrary continuous t-norms).

Additive generators characterize also analytical properties of the continuous Archimedean t-norms. For instance, a continuous Archimedean t-norm is 1-Lipschitz if and only if it has a convex additive generator [32, 38].

Also, convergence properties can be expressed by means of additive generators [19] (see also [24, Corollary 8.21]).

Proposition 4.7 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of continuous Archimedean t-norms and let T be a continuous Archimedean t-norm. Then the following are equivalent:*

(i) $\lim\limits_{n\to\infty} T_n = T$.

(ii) There exists a sequence of additive generators $(t_n\colon [0,1] \longrightarrow [0,\infty])_{n\in\mathbb{N}}$ of $(T_n)_{n\in\mathbb{N}}$ such that the restriction

$$\big(\lim_{n\to\infty} t_n\big)|_{]0,1]}$$

coincides with the restriction of some additive generator of T to $]0,1]$.

Note that, whenever in Proposition 4.7 the limit t-norm T is strict, then $\lim\limits_{n\to\infty} t_n$ is an additive generator of T.

For example, for each $n > 1$ the function $t_n\colon [0,1] \longrightarrow [0,\infty]$ given by

$$t_n(x) = \tfrac{2}{\log(1+\sqrt{n})} \log \tfrac{n-1}{n^x-1}$$

is an additive generator of the (strict) Frank t-norm $T_n^{\mathbf{F}}$ [13]. Then for all $x \in \,]0,1]$ we have $\lim\limits_{n\to\infty} t_n(x) = 1 - x$, i.e., the sequence $(t_n|_{]0,1]})_{n>1}$ converges to the restriction of an additive generator of the Łukasiewicz t-norm $T_{\mathbf{L}}$ to $]0,1]$. Therefore, the sequence $(T_n^{\mathbf{F}})_{n>1}$ converges to $T_{\mathbf{L}} (= T_\infty^{\mathbf{F}})$.

Finally we mention that the continuous Archimedean t-norms form a dense subclass of the class of all continuous t-norms (with respect to the uniform topology), i.e., each continuous t-norm can be approximated by some continuous Archimedean t-norm with arbitrary precision [20, 24, 33]. More precisely we have:

Theorem 4.8 *Let T be a continuous t-norm. Then for each $\varepsilon > 0$ there is a strict t-norm T_1 and a nilpotent t-norm T_2 such that for all $(x, y) \in [0, 1]^2$*

$$|T(x, y) - T_1(x, y)| < \varepsilon,$$
$$|T(x, y) - T_2(x, y)| < \varepsilon.$$

Acknowledgment

This work was supported by two European actions (CEEPUS network SK-42 and COST action 274) as well as by the grants VEGA 1/8331/01, APVT-20-023402 and MNTRS-1866.

References

[1] J. Aczél. Sur les opérations definies pour des nombres réels. *Bull. Soc. Math. France*, 76:59–64, 1949.

[2] J. Aczél. *Lectures on Functional Equations and their Applications*. Academic Press, New York, 1966.

[3] J. Aczél and C. Alsina. Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements. *Methods Oper. Res.*, 48:3–22, 1984.

[4] C. Alsina. On a method of Pi-Calleja for describing additive generators of associative functions. *Aequationes Math.*, 43:14–20, 1992.

[5] C. Alsina and J. Gimenez. Sobre L-órdenes entre t-normas estrictas. *Stochastica*, 8:85–89, 1984.

[6] B. Bacchelli. Representation of continuous associative functions. *Stochastica*, 10:13–28, 1986.

[7] D. Butnariu and E. P. Klement. *Triangular Norm-Based Measures and Games with Fuzzy Coalitions*. Kluwer Academic Publishers, Dordrecht, 1993.

[8] A. H. Clifford. Naturally totally ordered commutative semigroups. *Amer. J. Math.*, 76:631–646, 1954.

[9] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups*. American Mathematical Society, Providence, 1961.

[10] A. C. Climescu. Sur l'équation fonctionelle de l'associativité. *Bull. École Polytechn. Iassy*, 1:1–16, 1946.

[11] R. Craigen and Z. Páles. The associativity equation revisited. *Aequationes Math.*, 37:306–312, 1989.

[12] J. Dombi. A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets and Systems*, 8:149–163, 1982.

[13] M. J. Frank. On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. *Aequationes Math.*, 19:194–226, 1979.

[14] P. Hájek. Observations on the monoidal t-norm logic. *Fuzzy Sets and Systems*, 132:107–112, 2002.

[15] H. Hamacher. *Über logische Aggregationen nicht-binär explizierter Entscheidungskriterien*. Rita G. Fischer Verlag, Frankfurt, 1978.

[16] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, Cambridge, 1952.

[17] K. H. Hofmann and J. D. Lawson. Linearly ordered semigroups: Historic origins and A. H. Clifford's influence. In Hofmann and Mislove [18], pages 15–39.

[18] K. H. Hofmann and M. W. Mislove, editors. *Semigroup Theory and its Applications*, volume 231 of *London Math. Soc. Lecture Notes*. Cambridge University Press, Cambridge, 1996.

[19] S. Jenei. On Archimedean triangular norms. *Fuzzy Sets and Systems*, 99:179–186, 1998.

[20] S. Jenei and J. C. Fodor. On continuous triangular norms. *Fuzzy Sets and Systems*, 100:273–282, 1998.

[21] C. Kimberling. On a class of associative functions. *Publ. Math. Debrecen*, 20:21–39, 1973.

[22] F. Klein-Barmen. Über gewisse Halbverbände und kommutative Semigruppen II. *Math. Z.*, 48:715–734, 1942–43.

[23] E. P. Klement, R. Mesiar, and E. Pap. A characterization of the ordering of continuous t-norms. *Fuzzy Sets and Systems*, 86:189–195, 1997.

[24] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Kluwer Academic Publishers, Dordrecht, 2000.

[25] E. P. Klement, R. Mesiar, and E. Pap. Triangular norms. Position paper I: Basic analytical and algebraic properties. *Fuzzy Sets and Systems* (in press), also FLLL-TR-0208.

[26] E. P. Klement, R. Mesiar, and E. Pap. Triangular norms. Position paper II: General constructions and parameterized families. Submitted for publication (FLLL-TR-0215).

[27] A. Kolesárová. A note on Archimedean triangular norms. *BUSEFAL*, 80:57–60, 1999.

[28] G. M. Krause. *A strengthened form of Ling's theorem on associative functions*. PhD thesis, Illinois Institute of Technology, Chicago, 1981.

[29] C. M. Ling. Representation of associative functions. *Publ. Math. Debrecen*, 12:189–212, 1965.

[30] R. Mesiar and M. Navara. Diagonals of continuous triangular norms. *Fuzzy Sets and Systems*, 104:35–41, 1999.

[31] P. S. Mostert and A. L. Shields. On the structure of semi-groups on a compact manifold with boundary. *Ann. of Math., II. Ser.*, 65:117–143, 1957.

[32] R. Moynihan. On τ_T semigroups of probability distribution functions II. *Aequationes Math.*, 17:19–40, 1978.

[33] H. T. Nguyen, V. Kreinovich, and P. Wojciechowski. Strict Archimedean t-norms and t-conorms as universal approximators. *Internat. J. Approx. Reason.*, 18:239–249, 1998.

[34] P. Pi-Calleja. Las ecuaciones funcionales de la teoria de magnitudes. In *Proceedings Segundo Symposium de Matemática, Villavicencio, Mendoza, Coni, Buneos Aires*, pages 199–280, 1954.

[35] G. B. Preston. A. H. Clifford: an appreciation of his work on the occasion of his sixty-fifth birthday. *Semigroup Forum*, 7:32–57, 1974.

[36] B. Schweizer and A. Sklar. Associative functions and statistical triangle inequalities. *Publ. Math. Debrecen*, 8:169–186, 1961.

[37] B. Schweizer and A. Sklar. Associative functions and abstract semigroups. *Publ. Math. Debrecen*, 10:69–81, 1963.

[38] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. North-Holland, New York, 1983.

[39] P. Viceník. Non-continuous generated t-norms. In: S. E. Rodabaugh and E. P. Klement, editors. *Topological and Algebraic Structures in Fuzzy Sets: Recent Developments in the Mathematics of Fuzzy Sets*, Kluwer Academic Publishers, Dordrecht (in press).

[40] S. Weber. A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms. *Fuzzy Sets and Systems*, 11:115–134, 1983.

[41] R. R. Yager. On a general class of fuzzy connectives. *Fuzzy Sets and Systems*, 4:235–242, 1980.

Erich Peter Klement
Fuzzy Logic Laboratorium Linz-Hagenberg
Department of Algebra, Stochastics and
Knowledge-Based Mathematical Systems
Johannes Kepler University
A-4040 Linz, Austria
ep.klement@jku.at

Radko Mesiar
Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering
Slovak Technical University
SK-81 368 Bratislava, Slovakia
mesiar@cvt.stuba.sk
and
Institute of Information Theory and Automation
Czech Academy of Sciences
CZ-182 08 Prague 8, Czech Republic

Endre Pap
Institute of Mathematics
University of Novi Sad
YU-21000 Novi Sad, Yugoslavia
pap@im.ns.ac.yu, pape@eunet.yu