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Triangular norms. Position paper III: continuous t-norms

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Abstract

This third and last part of a series of position papers on triangular norms (for Parts I and II see (E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Position paper I: basic analytical and algebraic properties, Fuzzy Sets and Systems, in press; E.P. Klement, R. Mesiar, E. Pap, Triangular norms. Position paper II: general constructions and parameterized families, submitted for publication) presents the representation of continuous Archimedean t-norms by means of additive generators, and the representation of continuous t-norms by means of ordinal sums with Archimedean summands, both with full proofs. Finally some consequences of these representation theorems in the context of comparison and convergence of continuous t-norms, and of the determination of continuous t-norms by their diagonal sections are mentioned.

Keywords: Continuous triangular norm; Additive generator; Ordinal sum

1. Introduction

This is the third and final part of a series of position papers on the state of the art of some particularly important aspects of triangular norms in a condensed form. The monograph [23] provides a rather complete and self-contained overview about triangular norms and their applications.

Part I [24] considered some basic analytical properties of t-norms, such as continuity, and important classes such as Archimedean, strict and nilpotent t-norms. Also the dual operations, the

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triangular conorms, and De Morgan triples were mentioned. Finally, a short historical overview on the development of t-norms and their way into fuzzy sets and fuzzy logics was given.

Part II [25] is devoted to general construction methods based mainly on pseudo-inverses, additive and multiplicative generators, and ordinal sums, including also some constructions leading to non-continuous t-norms, and to a presentation of some distinguished families of t-norms.

In this third part we first present the representation of continuous Archimedean t-norms by means of additive generators, and then the representation of continuous t-norms by means of ordinal sums with Archimedean summands. These theorems were first proved in the framework of triangular norms in [28]. However, they can also be derived from results in [30] in the framework of semigroups. We include full proofs of the representation theorems mentioned above, since the original sources are not so easily accessible and/or they heavily use the special language of semigroup theory. Finally we include some results and examples which follow from these representation theorems.

Several notions and results from Parts I and II will be needed in this paper, and they can be found there in full detail [24,25]. For the convenience of the reader, we briefly recall some of them.

Recall that a triangular norm (briefly t-norm) is a binary operation T on the unit interval $[0,1]$ which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $T:[0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$:

(T1) $T(x,y) = T(y,x)$,
(T2) $T(x,T(y,z)) = T(T(x,y),z)$,
(T3) $T(x,y) \leqslant T(x,z)$ whenever $y \leqslant z$,
(T4) $T(x,1) = x$.

Observe that for a continuous t-norm T the Archimedean property is equivalent to $T(x,x) < x$ for all $x \in]0,1[$, and that each continuous Archimedean t-norm is either strict or nilpotent [24, Theorem 6.15]. Given a t-norm T, an element $x \in [0,1]$ is said to be idempotent if $T(x,x) = x$ (clearly, 0 and 1 are idempotent elements of each t-norm, the so-called trivial idempotent elements).

Observe that the pseudo-inverse is defined for arbitrary monotone functions [25, Definition 2.1]. In our special setting we mostly deal with continuous, decreasing function $t:[0,1] \to [0,\infty]$ with $t(1) = 0$, in which case the pseudo-inverse $t^{(-1)}$ reduces to

$$t^{(-1)}(x) = t^{-1}(\min(x,t(0))).$$

2. Representation of continuous Archimedean t-norms

For the class of all t-norms (which includes non-continuous t-norms and even t-norms which are not Borel measurable) the only existing characterization is by the axioms (T1)–(T4). The important subclass of continuous t-norms, however, has nice representations in terms of one-place functions and ordinal sums.

Theorem 2.1. *For a function $T:[0,1]^2 \to [0,1]$ the following are equivalent*:

(i) *T is a continuous Archimedean t-norm.*

(ii) *T has a continuous additive generator*, *i.e.*, *there exists a continuous*, *strictly decreasing function* $t:[0,1]\to[0,\infty]$ *with* $t(1)=0$, *which is uniquely determined up to a positive multiplicative constant*, *such that for all* $(x,y)\in[0,1]^2$

$$T(x,y)=t^{(-1)}(t(x)+t(y)). \tag{1}$$

Proof. Assume first that $t:[0,1]\to[0,+\infty]$ is a continuous, strictly decreasing function with $t(1)=0$ and that T is constructed by (1), i.e., t is an additive generator of T. The commutativity (T1) and the monotonicity (T3) of T are obvious. Also, the boundary condition (T4) holds since for all $x\in[0,1]$

$$T(x,1)=t^{(-1)}(t(x)+t(1))=t^{(-1)}(t(x))=x.$$

Concerning the associativity (T2), for all $x,y,z\in[0,1]$ we obtain

$$\begin{aligned}T(T(x,y),z)&=t^{(-1)}(t(T(x,y))+t(z))\\&=t^{(-1)}(t(t^{(-1)}(t(x)+t(y)))+t(z))\\&=t^{(-1)}(t(x)+t(y)+t(z))\\&=t^{(-1)}(t(x)+t(t^{(-1)}(t(y)+t(z))))\\&=t^{(-1)}(t(x)+t(T(y,z)))\\&=T(x,T(y,z)),\end{aligned}$$

where the third equality is a consequence of

$$t(t^{(-1)}(t(x)+t(y)))=\min(t(x)+t(y),t(0)).$$

To prove the converse, let T be a continuous Archimedean t-norm. Concerning the notion $x_T^{(n)}$ we will use, recall that, for each $x\in[0,1]$, we have $x_T^{(0)}=1$ and, for $n\in\mathbb{N}$, by recursion

$$x_T^{(n)}=T(x,x_T^{(n-1)}).$$

Define now for $x\in[0,1]$ and $m,n\in\mathbb{N}$

$$\begin{aligned}x_T^{(1/n)}&=\sup\{y\in[0,1]\mid y_T^{(n)}<x\},\\x_T^{(m/n)}&=(x_T^{(1/n)})_T^{(m)}.\end{aligned}$$

Since T is Archimedean, we have for all $x\in\,]0,1]$

$$\lim_{n\to\infty}x_T^{(1/n)}=1. \tag{2}$$

Note that the expression $x_T^{(m/n)}$ is well-defined because of $x_T^{(m/n)}=x_T^{(km/kn)}$ for all $k\in\mathbb{N}$. If, for some $x\in[0,1]$ and some $n\in\mathbb{N}\cup\{0\}$, we have $x_T^{(n)}=x_T^{(n+1)}$ then, in the standard way by induction, we obtain

$$x_T^{(n)}=x_T^{(2n)}=(x_T^{(n)})_T^{(2)}$$

and, since T is continuous Archimedean, $x_T^{(n)}\in\{0,1\}$. This means that we have $x_T^{(n)}>x_T^{(n+1)}$ whenever $x_T^{(n)}\in\,]0,1[$.

Now choose and fix an arbitrary element $a \in]0,1[$, and define the function $h : \mathbb{Q} \cap [0,\infty[\to [0,1]$ by $h(r) = a_T^{(r)}$. Since T is continuous and since (2) holds, h is a continuous function. Moreover, we have for all $x \in [0,1]$ and $m,n,p,q \in \mathbb{N}$

$$\begin{aligned} x_T^{(m/n)+(p/q)} &= x_T^{((mq+np)/nq)} \\ &= (x_T^{(1/nq)})_T^{(mq+np)} \\ &= T((x_T^{(1/nq)})_T^{(mq)}, (x_T^{(1/nq)})_T^{(np)}) \\ &= T(x_T^{(m/n)}, x_T^{(p/q)}) \end{aligned}$$

and, as a consequence, for all $r,s \in \mathbb{Q} \cap [0,\infty[$

$$h(r+s) = a_T^{(r+s)} = T(a_T^{(r)}, a_T^{(s)}) \leqslant a_T^{(r)} = h(r),$$

i.e., h is also non-increasing. The function h is even strictly decreasing on the preimage of $]0,1]$ since for all $m/n,\ p/q \in \mathbb{Q} \cap [0,\infty[$ with $h(m/n) > 0$ we get

$$h\left(\frac{m}{n} + \frac{p}{q}\right) \leqslant h\left(\frac{mq+1}{nq}\right) = (a_T^{(1/nq)})_T^{(mq+1)} < (a_T^{(1/nq)})_T^{(mq)} = h\left(\frac{m}{n}\right).$$

The monotonicity and continuity of h on $\mathbb{Q} \cap [0,\infty[$ allows us to extend it uniquely to a function $\bar{h} : [0,\infty] \to [0,1]$ via

$$\bar{h}(x) = \inf\{h(r) \mid r \in \mathbb{Q} \cap [0,x]\}.$$

Then $\bar{h}$ is continuous and non-increasing, and we have for all $x,y \in [0,\infty]$

$$\bar{h}(x+y) = T(\bar{h}(x), \bar{h}(y)).$$

Moreover, $\bar{h}$ is strictly decreasing on the preimage of $]0,1]$. Define the function $t : [0,1] \to [0,\infty]$ by

$$t(x) = \sup\{y \in [0,\infty] \mid \bar{h}(y) > x\}$$

with the usual convention $\sup \emptyset = 0$ (observe that t is just the pseudo-inverse of $\bar{h}$ and vice versa). Then t is continuous, strictly decreasing, and satisfies $t(1) = 0$ [23, Remark 3.4]. A combination of all the arguments so far yields that t is indeed a continuous additive generator of T since for each $(x,y) \in [0,1]^2$

$$T(x,y) = T(\bar{h}(t(x)), \bar{h}(t(y))) = \bar{h}(t(x)+t(y)) = t^{(-1)}(t(x)+t(y)).$$

To show that the continuous additive generator t of T constructed above is unique up to a positive multiplicative constant, assume that the two functions $t_1, t_2 : [0,1] \to [0,\infty]$ are both continuous additive generators of T, i.e., we have for each $(x,y) \in [0,1]^2$ the equality

$$t_1^{(-1)}(t_1(x)+t_1(y)) = t_2^{(-1)}(t_2(x)+t_2(y)).$$

Substituting $u = t_2(x)$ and $v = t_2(y)$, we obtain that, for all $u,v \in [0,t_2(0)]$ satisfying $u+v \in [0,t_2(0)[$,

$$t_1 \circ t_2^{(-1)}(u) + t_1 \circ t_2^{(-1)}(v) = t_1 \circ t_2^{(-1)}(u+v). \tag{3}$$

Then from the continuity of t_1 and $t_2^{(-1)}$ it follows that (3) holds for all $u,v \in [0,t_2(0)]$ with $u+v \in [0,t_2(0)]$.

Eq. (3) is a Cauchy functional equation (see [2]), whose continuous, strictly increasing solutions $t_1 \circ t_2^{(-1)}: [0, t_2(0)] \to [0, \infty]$ must satisfy $t_1 \circ t_2^{(-1)} = b \cdot \mathrm{id}_{[0, t_2(0)]}$ for some $b \in]0, \infty[$. As a consequence, we get $t_1 = bt_2$ for some $b \in]0, \infty[$, thus completing the proof. □

Because of the special form of the pseudo-inverse $t^{(-1)}$, representation (1) in Theorem 2.1 can also be written as

$$T(x, y) = t^{-1}(\min(t(x) + t(y), t(0))).$$

We already have seen in [24, Proposition 6.13 and Theorem 6.17] that a continuous Archimedean t-norm is either strict or nilpotent, a distinction which can be made also with the help of their additive generators. Indeed, generators t of strict t-norms satisfy $t(0) = \infty$ while generators of nilpotent t-norms satisfy $t(0) < \infty$ [25, Corollary 2.8].

Recall that for the product $T_{\mathbf{P}}$ and for the Łukasiewicz t-norm $T_{\mathbf{L}}$ additive generators $t : [0, 1] \to [0, \infty]$ are given by, respectively,

$$t(x) = -\log x,$$
$$t(x) = 1 - x.$$

Based on the proof of Theorem 2.1, it is possible to give some constructive way to obtain additive generators of continuous Archimedean t-norms. As an illustrating example, we include the following result of [11] (compare also [1,4,33]) for the case of strict t-norms which can be derived in a straightforward manner from the proof of Theorem 2.1.

Corollary 2.2. *Let T be a strict t-norm. Fix an arbitrary element $x_0 \in]0, 1[$, and define the function $t : [0, 1] \to [0, \infty]$ by*

$$t(x) = \inf \left\{ \frac{m-n}{k} \,\middle|\, m, n, k \in \mathbb{N} \text{ and } (x_0)_T^{(m)} < T((x_0)_T^{(n)}, x_T^{(k)}) \right\}.$$

Then t is an additive generator of T.

Example 2.3. If we consider the Hamacher product T [15] defined by

$$T(x, y) = \frac{xy}{x + y - xy},$$

whenever $(x, y) \neq (0, 0)$, observe that we get (taking into account $1/\infty = 0$ and $1/0 = \infty$) for all $(x, y) \in [0, 1]^2$

$$T(x, y) = \frac{1}{(1/x) + (1/y) - 1}$$

and, for each $x \in [0, 1]$ and each $n \in \mathbb{N}$

$$x_T^{(n)} = \frac{1}{(n/x) - n + 1}.$$

For $x_0 = 0.5$ the inequality

$$(x_0)_T^{(m)} < T((x_0)_T^{(n)}, x_T^{(k)})$$

is easily seen to be equivalent to $m-n > k((1/x)-1)$, yielding the additive generator $t:[0,1]\to[0,\infty]$ of T specified by

$$t(x) = \inf\left\{ \frac{m-n}{k} \,\middle|\, m,n,k\in\mathbb{N} \text{ and } \frac{m-n}{k} > \frac{1}{x}-1 \right\} = \frac{1-x}{x}.$$

The representation of continuous Archimedean t-norms given in Theorem 2.1 is based on the addition on the interval $[0,+\infty]$. There is a completely analogous representation thereof based on the multiplication on $[0,1]$, thus leading to a representation of continuous Archimedean t-norms by means of multiplicative generators [25, Section 2]. By duality, there are also representations of continuous Archimedean t-conorms by means of additive generators and multiplicative generators, respectively.

Remark 2.4. (i) If T is a continuous Archimedean t-norm with additive generator $t:[0,1]\to[0,\infty]$, then the function $\theta:[0,1]\to[0,1]$ defined by $\theta(x)=\mathrm{e}^{-t(x)}$ is a multiplicative generator of T.

(ii) If S is a continuous Archimedean t-conorm then the dual t-norm T is continuous Archimedean and, therefore, has an additive generator $t:[0,1]\to[0,\infty]$. Then $s:[0,1]\to[0,\infty]$ defined by $s(x)=t(1-x)$ is an additive generator of S, and $\xi:[0,1]\to[0,1]$ defined by $\xi(x)=\mathrm{e}^{-t(1-x)}$ is a multiplicative generator of S.

(iii) Given a continuous Archimedean t-norm T and a strictly increasing bijection $\varphi:[0,1]\to[0,1]$, it is clear that the function $T_\varphi:[0,1]^2\to[0,1]$ given by

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x),\varphi(y)))$$

is a continuous Archimedean t-norm too. By Theorem 2.1, there are additive generators $t, t_\varphi:[0,1]\to[0,\infty]$ of T and T_φ, respectively. Taking into account [25, Proposition 2.9], t_φ equals $t\circ\varphi$ up to a multiplicative constant.

It is straightforward that each isomorphism $\varphi:[0,1]\to[0,1]$ preserves (among many other properties) the continuity, the strictness and the existence of zero divisors. Therefore, each t-norm which is isomorphic to a strict or to a nilpotent t-norm, itself is strict or nilpotent, respectively.

Conversely, if T_1 and T_2 are two strict t-norms with additive generators t_1 and t_2 (which are bijective functions from $[0,1]$ into $[0,\infty]$ in this case), respectively, then $\varphi:[0,1]\to[0,1]$ given by $\varphi=t_1^{-1}\circ t_2$ is a strictly increasing bijection and $T_2=(T_1)_\varphi$. If T_1 and T_2 are two nilpotent t-norms with additive generators t_1 and t_2, respectively, then we have $T_2=(T_1)_\varphi$, where the strictly increasing bijection $\varphi:[0,1]\to[0,1]$ is given by $\varphi=t_1^{-1}\circ((t_1(0)/t_2(0))t_2)$ (observe that in this case the two functions t_1 and $(t_1(0)/t_2(0))t_2$ can be viewed as bijections from $[0,1]$ into $[0,t_1(0)]$).

We therefore have shown the following result:

Lemma 2.5. *Two continuous Archimedean t-norms are isomorphic if and only if they are either both strict or both nilpotent.*

An immediate consequence of Remark 2.4(iii) and Lemma 2.5 is that the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ are not only prototypical examples of strict and nilpotent t-norms, respectively, but that each continuous Archimedean t-norm is isomorphic either to $T_{\mathbf{P}}$ or to $T_{\mathbf{L}}$:

Theorem 2.6. (i) *A function $T:[0,1]^2 \to [0,1]$ is a strict t-norm if and only if it is isomorphic to the product $T_{\mathbf{P}}$.*

(ii) *A function $T:[0,1]^2 \to [0,1]$ is a nilpotent t-norm if and only if it is isomorphic to the Łukasiewicz t-norm $T_{\mathbf{L}}$.*

Each multiplicative generator $\theta:[0,1] \to [0,1]$ of a strict t-norm T can be viewed as an isomorphism between $T_{\mathbf{P}}$ and T, i.e., $T = (T_{\mathbf{P}})_\theta$. In particular, this means that there are infinitely many isomorphisms between $T_{\mathbf{P}}$ and T. On the other hand, if T is a nilpotent t-norm with additive generator $t:[0,1] \to [0,\infty]$, then there is a unique isomorphism $\varphi:[0,1] \to [0,1]$ between $T_{\mathbf{L}}$ and T, namely, $\varphi = 1 - (1/t(0))t$.

Recall that each continuous t-norm T satisfying $T(x,x) < x$ for all $x \in]0,1[$ is Archimedean [23, Proposition 2.15].

Corollary 2.7. *If T is a continuous t-norm with trivial idempotent elements only, i.e., $T(x,x) = x$ only if $x \in \{0,1\}$, then T is Archimedean and, therefore, has a continuous additive generator.*

Remark 2.8. Note that the representation in Theorem 2.1 holds for continuous Archimedean t-norms only. However, there are several possibilities to show the existence of continuous additive generators for a function $T:[0,1]^2 \to [0,1]$ under weaker hypotheses than in Theorem 2.1.

For example, it is possible to drop the commutativity (T1) [30] (see also [23, Theorem 2.43]) or to weaken the associativity (T2) [6,27]. In the case of left-continuous t-norms, either the Archimedean property [26] or the existence of a (not necessarily continuous) additive generator [38] implies the existence of a continuous additive generator. In the case of a strictly monotone Archimedean t-norm T, the continuity of T at the point $(1,1)$ is sufficient for the existence of a continuous additive generator [14].

3. Representation of continuous t-norms

The construction of a new semigroup from a family of given semigroups using ordinal sums goes back to A. H. Clifford [8] (see also [9,17,34]), and it is based on ideas presented in [10,21]. It has been successfully applied to t-norms in [13,24,28,36].

Definition 3.1. Let $(T_\alpha)_{\alpha\in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm T defined by

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha)T_\alpha\left(\dfrac{x-a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y-a_\alpha}{e_\alpha - a_\alpha}\right) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is called the *ordinal sum of the summands* $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, $\alpha \in A$, and we shall write

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}.$$

Observe that the index set A is necessarily finite or countably infinite. It also may be empty, in which case the ordinal sum equals the idempotent t-norm $T_{\mathbf{M}}$.

Note that the representation of continuous Archimedean t-norms by means of multiplicative generators can be derived directly from more general results for I-semigroups (see [23,28,30,37]). Similarly, the following representation of continuous t-norms by means of ordinal sums follows also from results of [30] in the context of I-semigroups.

Theorem 3.2. *A function* $T : [0,1]^2 \to [0,1]$ *is a continuous t-norm if and only if* T *is an ordinal sum of continuous Archimedean t-norms.*

Proof. Obviously, each ordinal sum of continuous t-norms is a continuous t-norm.

Conversely, if T is a continuous t-norm, we first show that the set I_T of all idempotent elements of T is a closed subset of $[0,1]$. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of idempotent elements of T which converges to some $x \in [0,1]$, then the continuity of T implies

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_n, x_n) = T(x,x),$$

so x is also an idempotent element of T, and I_T is closed.

In the case $I_T = [0,1]$ we have $T = T_{\mathbf{M}}$, i.e., an empty ordinal sum. If $I_T \neq [0,1]$ it can be written as the (non-trivial) union of a finite or countably infinite family of pairwise disjoint open subintervals $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ where, of course, each a_α and each e_α (but no element in $]a_\alpha, e_\alpha[$) is an idempotent element of T.

For the time being, assume that $A \neq \emptyset$ and fix an arbitrary $\alpha \in A$. Then the monotonicity of T implies that for all $(x,y) \in [a_\alpha, e_\alpha]^2$

$$a_\alpha = T(a_\alpha, a_\alpha) \leqslant T(x,y) \leqslant T(e_\alpha, e_\alpha) = e_\alpha$$

and, for all $x \in [a_\alpha, 1]$

$$a_\alpha = T(a_\alpha, a_\alpha) \leqslant T(x, a_\alpha) \leqslant T(1, a_\alpha) = a_\alpha,$$

showing that $([a_\alpha, e_\alpha], T|_{[a_\alpha,e_\alpha]^2})$ is a semigroup with annihilator a_α and with trivial idempotent elements only (actually, a_α acts as an annihilator on $[a_\alpha, 1]$). Because of the monotonicity and continuity of T we also have for each $\alpha \in A$

$$\{T(z, e_\alpha) \mid z \in [0,1]\} = [0, e_\alpha],$$

which means that each $x \in [0, e_\alpha]$ can be written as $x = T(z, e_\alpha)$ for some $z \in [0,1]$. This, together with the associativity of T, implies that

$$T(x, e_\alpha) = T(T(z, e_\alpha), e_\alpha) = T(z, T(e_\alpha, e_\alpha)) = T(z, e_\alpha) = x,$$

showing that e_α acts as a neutral element on $[0, e_\alpha]$ and, subsequently, in the I-semigroup $([a_\alpha, e_\alpha], T|_{[a_\alpha,e_\alpha]^2})$.

Let $\varphi_\alpha : [0,1] \to [a_\alpha, e_\alpha]$ be the strictly increasing bijection given by

$$\varphi_\alpha(x) = a_\alpha + (e_\alpha - a_\alpha)x,$$

then for each $\alpha \in A$ the function $T_\alpha : [0,1]^2 \to [0,1]$ defined by

$$T_\alpha(x,y) = \varphi_\alpha^{-1}(T(\varphi_\alpha(x), \varphi_\alpha(y))$$

is a continuous t-norm which has only trivial idempotent elements, and which is also Archimedean because of Corollary 2.7. A simple computation verifies that for all $\alpha \in A$ and for all $(x,y) \in [a_\alpha, e_\alpha]^2$ we have

$$T(x,y) = a_\alpha + (e_\alpha - a_\alpha) T_\alpha\left(\frac{x - a_\alpha}{e_\alpha - a_\alpha}, \frac{y - a_\alpha}{e_\alpha - a_\alpha}\right).$$

If $(x,y) \in [0,1]^2$ (without loss of generality we may assume $x \leqslant y$) is contained in none of the squares $[a_\alpha, e_\alpha]^2$ then there exists some idempotent element $b \in [x,y]$ which acts as a neutral element on $[0,b]$ and as an annihilator on $[b,1]$, and we have

$$T(x,y) = T(T(x,b),y) = T(x,T(b,y)) = T(x,b) = x = \min(x,y),$$

completing the proof that $T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}$.

The uniqueness of the representation of T is an immediate consequence of the one-to-one correspondence between the set of idempotent elements of T and the family of intervals $(]a_\alpha, e_\alpha[)_{\alpha \in A}$. □

The combination of Theorem 3.2 and of the results of Section 2 yields the following representations of continuous t-norms:

Corollary 3.3. *For a function $T : [0,1]^2 \to [0,1]$ the following are equivalent*:

(i) *T is a continuous t-norm.*
(ii) *T is isomorphic to an ordinal sum whose summands contain only the t-norms $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$.*
(iii) *There is a family $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ and a family $h_\alpha : [a_\alpha, e_\alpha] \to [0,\infty]$ of continuous, strictly decreasing functions with $h_\alpha(e_\alpha) = 0$ for each $\alpha \in A$ such that for all $(x,y) \in [0,1]^2$*

$$T(x,y) = \begin{cases} h_\alpha^{(-1)}(h_\alpha(x) + h_\alpha(y)) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \quad (4)$$

Example 3.4. Consider the continuous t-norm T (see Fig. 1) given by

$$T(x,y) = \begin{cases} \max\left(\dfrac{3x + 3y + 9xy - 1}{6}, 0\right) & \text{if } (x,y) \in [0, \frac{1}{3}]^2, \\ \dfrac{4x + 4y - 3xy - 4}{9x + 9y - 9xy - 8} & \text{if } (x,y) \in [\frac{2}{3}, 1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

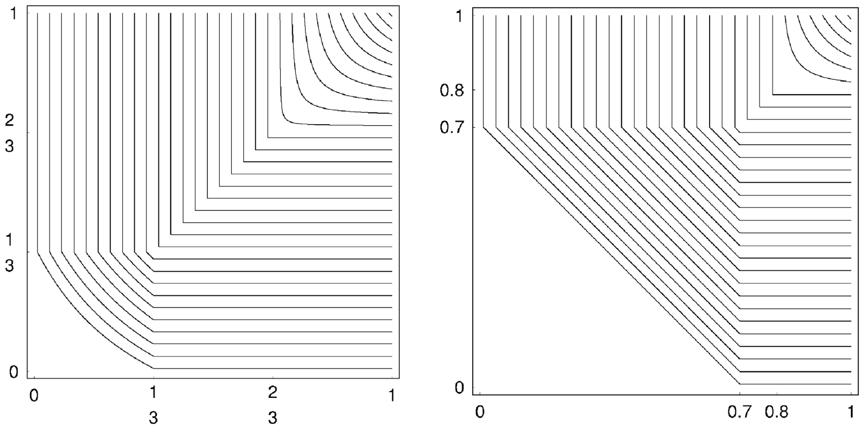

Fig. 1. Contour plots of the isomorphic t-norms T (left) and $T_{0.7,0.8}$ from Example 3.4.

This t-norm T can be written as the ordinal sum $(\langle 0,1/3,T_1\rangle,\langle 2/3,1,T_2\rangle)$ with T_1 and T_2 being given by

$$T_1(x,y)=\max\left(\frac{x+y+xy-1}{2},0\right),$$
$$T_2(x,y)=\frac{xy}{x+y-xy}.$$

Observe that the nilpotent t-norm T_1 was introduced in [39], and that the strict t-norm T_2 is the Hamacher product $T_0^{\mathbf{H}}$ [25], and that the functions t_1,t_2 given by

$$t_1(x)=-\log\frac{1+x}{2},$$
$$t_2(x)=\frac{1-x}{x}.$$

are continuous additive generators of T_1 and T_2, respectively. Defining the functions $h_1:[0,1/3]\to[0,\infty]$ and $h_2:[2/3,1]\to[0,\infty]$ by

$$h_1(x)=-\log\frac{1+3x}{2},$$
$$h_2(x)=\frac{3-3x}{3x-2},$$

we can represent our t-norm T in form (4). For any numbers $a,b\in\,]0,1[$ with $a<b$ consider the t-norm $T_{ab}=(\langle 0,a,T_{\mathbf{L}}\rangle,\langle b,1,T_{\mathbf{P}}\rangle)$ (see Fig. 1). Then T is isomorphic to T_{ab}, i.e., we have $T=(T_{ab})_\varphi$ where the strictly increasing bijection $\varphi:[0,1]\to[0,1]$ is given by

$$\varphi(x)=\begin{cases} a\,\dfrac{\log(1+3x)}{\log 2} & \text{if } x\in[0,\frac{1}{3}],\\ a+(b-a)(3x-1) & \text{if } x\in\,]\frac{1}{3},\frac{2}{3}],\\ b+(1-b)e^{(3x-3)/(3x-2)} & \text{otherwise.}\end{cases}$$

Analogous representations for continuous t-conorms can be obtained by duality (making the necessary changes, e.g., replacing min by max).

4. Consequences of the representation theorems

Theorems 2.1 and 3.2 simplify the work with continuous t-norms in the sense that it suffices to consider (a family of) continuous Archimedean t-norms and, subsequently, their additive generators. In particular, the additive generator (which is a one-place function) of a continuous Archimedean t-norm T carries all the information of the whole t-norm T.

Knowing the structure of continuous t-norms allows us also to deduce general properties from partial information. For instance, if for a continuous t-norm T and for some $x_0 \in]0,1[$ the vertical section $f:[0,1]\to[0,1]$ given by $f(y)=T(x_0,y)$ is strictly monotone and satisfies $f(y)<y$ for all $y\in]0,1]$, then T is a strict t-norm.

In this section, we demonstrate the impact of Theorems 2.1 and 3.2 on the problems of (pointwise) comparison and convergence of continuous t-norms, and on the determination of continuous t-norms by their diagonal sections.

The following necessary and sufficient condition for the comparison of continuous Archimedean t-norms can be found in [37, Lemma 5.5.8] (see also [23, Theorem 6.2], for the special case of strict t-norms it was first proved in [35] (see also [5]).

Theorem 4.1. *Let T_1 and T_2 be two continuous Archimedean t-norms with additive generators $t_1,t_2:[0,1]\to[0,\infty]$, respectively. The following are equivalent*:

(i) $T_1 \leqslant T_2$.
(ii) *The function $t_1\circ t_2^{-1}:[0,t_2(0)]\to[0,\infty]$ is subadditive, i.e., for all $u,v\in[0,t_2(0)]$ with $u+v\in[0,t_2(0)]$ we have*

$$t_1\circ t_2^{-1}(u+v) \leqslant t_1\circ t_2^{-1}(u)+t_1\circ t_2^{-1}(v).$$

There exist criteria (some of which are only sufficient) for the comparability of continuous Archimedean t-norms which sometimes are easier to check than the subadditivity in Theorem 4.1. The following sufficient conditions can be derived easily from Theorem 4.1 and from [16, (103)] (recall that a function $f:[a,b]\to[0,\infty]$ is called concave if

$$f(\lambda x+(1-\lambda)y) \geqslant \lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and for all $\lambda\in[0,1]$).

Corollary 4.2. *Let T_1 and T_2 be two continuous Archimedean t-norms with additive generators $t_1,t_2:[0,1]\to[0,\infty]$, respectively. Then we have $T_1\leqslant T_2$ if one of the following conditions is satisfied*:

(i) *The function $t_1\circ t_2^{-1}:[0,t_2(0)]\to[0,\infty]$ is concave.*

(ii) *The function* $f :]0, t_2(0)] \to [0, \infty]$ *defined by*

$$f(x) = \frac{(t_1 \circ t_2^{-1})(x)}{x}$$

is non-increasing.

(iii) *The function*

$$\frac{t_1'}{t_2'} :]0, 1[\to [0, \infty[$$

is non-decreasing.

Example 4.3. In [25, Example 2.10(i)] we have seen that for each continuous Archimedean t-norm T with additive generator $t : [0,1] \to [0,\infty]$, and for each $\lambda \in]0,\infty[$, the function $t^\lambda : [0,1] \to [0,\infty]$ is an additive generator of a continuous Archimedean t-norm which was denoted there $T^{(\lambda)}$. Now we are able to show that the family $(T^{(\lambda)})_{\lambda \in]0,\infty[}$ is strictly increasing with respect to the parameter λ. Indeed, for $\lambda, \mu \in]0,\infty[$ the composite function $t^\lambda \circ (t^\mu)^{-1} : [0, t(0)^\mu] \to [0,\infty]$ is given by

$$t^\lambda \circ (t^\mu)^{-1}(x) = x^{\lambda/\mu},$$

and it is concave whenever $\lambda \leqslant \mu$, showing that $(T^{(\lambda)})_{\lambda \in]0,\infty[}$ is a strictly increasing family of t-norms. Consequently, the families of Yager t-norms [40], of Aczél–Alsina t-norms [3], and of Dombi t-norms [12] are strictly increasing families of t-norms.

A nontrivial problem was the monotonicity of the family of Frank t-norms $(T_\lambda^{\mathbf{F}})_{\lambda \in [0,\infty]}$ [13]. A first proof thereof appeared in [7, Proposition 1.12]. In the following we give a simpler proof [22] based on Corollary 4.2(iii) (see also [23, Proposition 6.8]).

Proposition 4.4. *The family* $(T_\lambda^{\mathbf{F}})_{\lambda \in [0,\infty]}$ *of Frank t-norms is strictly decreasing.*

Proof. Recall that $T_0^{\mathbf{F}} = T_{\mathbf{M}}$, $T_1^{\mathbf{F}} = T_{\mathbf{P}}$, whose additive generator $t_1^{\mathbf{F}}$ is given by $t_1^{\mathbf{F}}(x) = -\log x$, and $T_\infty^{\mathbf{F}} = T_{\mathbf{L}}$ whose additive generator $t_\infty^{\mathbf{F}}$ is given by $t_\infty^{\mathbf{F}}(x) = 1 - x$. For each $\lambda \in]0,1[\cup [1,\infty]$, $T_\lambda^{\mathbf{F}}$ is a strict t-norm, and its additive generator $t_\lambda^{\mathbf{F}}$ is given by $t_\lambda^{\mathbf{F}}(x) = \log(\lambda - 1)/(\lambda^x - 1)$.

Trivially we have $T_0^{\mathbf{F}} = T_{\mathbf{M}} > T_\lambda^{\mathbf{F}}$ for all $\lambda \in]0,\infty]$. From

$$\frac{(t_\infty^{\mathbf{F}})'}{(t_\lambda^{\mathbf{F}})'}(x) = \begin{cases} x & \text{if } \lambda = 1, \\ \dfrac{\lambda^x - 1}{\lambda^x \log \lambda} & \text{if } \lambda \in]0,1[\cup]1,\infty[, \end{cases}$$

it follows that for each $\lambda \in]0,\infty[$ the function $(t_\infty^{\mathbf{F}})'/(t_\lambda^{\mathbf{F}})'$ is non-decreasing, implying $T_\infty^{\mathbf{F}} \leqslant T_\lambda^{\mathbf{F}}$ and, since $T_\infty^{\mathbf{F}}$ is nilpotent and $T_\lambda^{\mathbf{F}}$ is strict, even $T_\infty^{\mathbf{F}} < T_\lambda^{\mathbf{F}}$.

Now let us show that $T_\mu^{\mathbf{F}} \leqslant T_\lambda^{\mathbf{F}}$ whenever $1 < \lambda < \mu < \infty$. Observe that for all $x \in]0,1[$ we get

$$\frac{(t_\mu^{\mathbf{F}})'}{(t_\lambda^{\mathbf{F}})'}(x) = \frac{\mu^x(\lambda^x - 1)\log \mu}{\lambda^x(\mu^x - 1)\log \lambda} = \frac{\log \mu}{\log \lambda}\,\frac{1 - (1/\lambda)^x}{1 - (1/\mu)^x}.$$

Then $(t_\mu^{\mathbf{F}})'/(t_\lambda^{\mathbf{F}})'$ is non-decreasing on $]0,1[$ if and only if

$$\left(1-\left(\frac{1}{\mu}\right)^x\right)\left(\frac{1}{\lambda}\right)^x \log\frac{1}{\lambda} \leqslant \left(1-\left(\frac{1}{\lambda}\right)^x\right)\left(\frac{1}{\mu}\right)^x \log\frac{1}{\mu},$$

i.e., if and only if we have the inequality

$$\frac{(1/\lambda)^x \log\frac{1}{\lambda}}{(1/\mu)^x \log\frac{1}{\mu}} \geqslant \frac{1-(1/\lambda)^x}{1-(1/\mu)^x}. \tag{5}$$

Consider now the functions $f,g:]0,1[\to[0,\infty[$ defined by $f(x)=1-(1/\lambda)^x$ and $g(x)=1-(1/\mu)^x$. Then, by the Cauchy mean value theorem, for each $x\in]0,1[$ there exists a $y\in]0,x[$ such that

$$\frac{1-(1/\lambda)^x}{1-(1/\mu)^x} = \frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(y)}{g'(y)} = \frac{(1/\lambda)^y\log(1/\lambda)}{(1/\mu)^y\log(1/\mu)} < \frac{(1/\lambda)^x\log(1/\lambda)}{(1/\mu)^x\log(1/\mu)}.$$

This proves inequality (5) and, consequently, the function $(t_\mu^{\mathbf{F}})'/(t_\lambda^{\mathbf{F}})'$ is non-decreasing, i.e., $T_\mu^{\mathbf{F}}\leqslant T_\lambda^{\mathbf{F}}$ and, because of $T_\mu^{\mathbf{F}}\neq T_\lambda^{\mathbf{F}}$, even $T_\mu^{\mathbf{F}}<T_\lambda^{\mathbf{F}}$ in this case. Similarly we can show $T_1^{\mathbf{F}}<T_\lambda^{\mathbf{F}}$ for all $\lambda\in]1,\infty[$.

The case $0<\lambda<\mu\leqslant 1$ can be transformed into $1\leqslant 1/\mu<1/\lambda<\infty$, and the case $0<\lambda<1<\mu<\infty$ is proved combining the two latter cases. □

The comparison of arbitrary continuous t-norms is much more complicated, and it is fully described in [22] (see also [23, Theorem 6.12]).

When comparing t-norms it is evident that the incomparability of their diagonal sections implies the incomparability of the t-norms themselves. The converse, however, is not true in general, not even in the case of continuous Archimedean t-norms.

Example 4.5. Consider the function $t:[0,1]\to[0,\infty]$ defined by (the index n may be any number in $\mathbb{Z}$)

$$t(x)=\begin{cases}\infty & \text{if } x=0,\\ 2^n(2-(4x^{1/2^n}-1)^2) & \text{if } x\in\left[\dfrac{1}{2^{2^{n+1}}},\dfrac{1}{2^{2^n}}\right[,\\ 0 & \text{if } x=1,\end{cases}$$

then t is an additive generator of some strict t-norm T. A simple computation shows that the diagonal sections of T and $T_{\mathbf{P}}$ coincide, but the opposite diagonal sections $d_T,d_{T_{\mathbf{P}}}:[0,1]\to[0,1]$ given by $d_T(x)=T(x,1-x)$ and $d_{T_{\mathbf{P}}}(x)=T_{\mathbf{P}}(x,1-x)$ are incomparable (see Fig. 2), implying the incomparability of T and $T_{\mathbf{P}}$.

This shows that different continuous t-norms may have identical diagonal sections. Note that there are methods to describe all continuous t-norms having a given diagonal section [20,23,29]. Here we only mention one of these methods applied to strict t-norms [20,29] (see also [23, Proposition 7.11]):

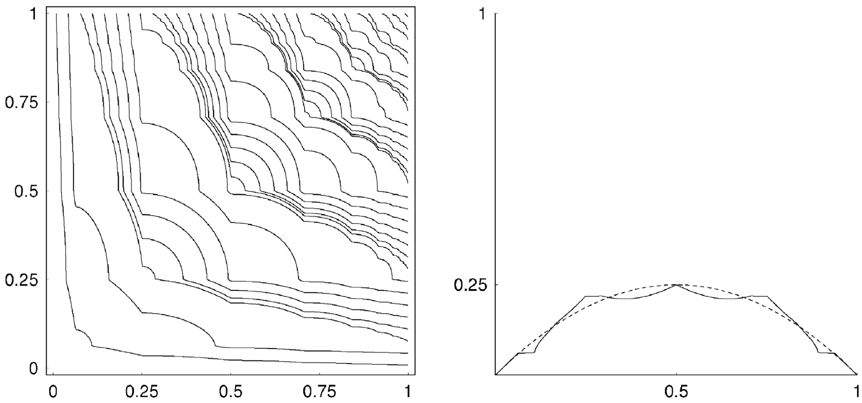

Fig. 2. Contour plot of the strict t-norm T (left) considered in Example 4.5 together with the incomparable opposite diagonal sections of T and T_{P} (right).

Proposition 4.6. *Let* $\delta:[0,1]\to[0,1]$ *be a strictly increasing bijection such that* $\delta(x,x)<x$ *for all* $x\in\,]0,1[$. *Then a continuous t-norm* T *has diagonal section* δ *if and only if* T *is strict and the function* $t:[0,1]\to[0,\infty]$ *given by*

$$t(x)=\begin{cases}\infty & \text{if } x=0,\\ 2^n f(\delta^{(-n)}(x)) & \text{if } x\in\,]\delta^{(n+1)}(0.5),\delta^{(n)}(0.5)],\\ 0 & \text{if } x=1,\end{cases}$$

is an additive generator of T, *where* $f:[\delta(0.5),0.5]\to[1,2]$ *is a strictly decreasing bijection,* $\delta^{(0)}=\mathrm{id}_{[0,1]}$, $\delta^{(n)}=\delta\circ\delta^{(n-1)}$ *whenever* $n\in\mathbb{N}$, *and* $\delta^{(n)}=(\delta^{(-n)})^{-1}$ *whenever* $-n\in\mathbb{N}$.

As a consequence of Proposition 4.6, two different strict t-norms with the same diagonal section are necessarily incomparable, compare also Example 4.5 (the same result holds for arbitrary continuous t-norms).

Additive generators characterize also analytical properties of the continuous Archimedean t-norms. For instance, a continuous Archimedean t-norm is 1-Lipschitz if and only if it has a convex additive generator [31,37].

Also, convergence properties can be expressed by means of additive generators [18] (see also [23, Corollary 8.21]).

Proposition 4.7. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a sequence of continuous Archimedean t-norms and let* T *be a continuous Archimedean t-norm. Then the following are equivalent*:

(i) $\lim_{n\to\infty} T_n = T$.

(ii) *There exists a sequence of additive generators* $(t_n:[0,1]\to[0,\infty])_{n\in\mathbb{N}}$ *of* $(T_n)_{n\in\mathbb{N}}$ *such that the restriction*

$$\left(\lim_{n\to\infty} t_n\right)\Big|_{]0,1]}$$

coincides with the restriction of some additive generator of T *to* $]0,1]$.

Note that, whenever in Proposition 4.7 the limit t-norm T is strict, then $\lim_{n\to\infty} t_n$ is an additive generator of T.

For example, for each $n>1$ the function $t_n : [0,1] \to [0,\infty]$ given by

$$t_n(x) = \frac{1}{2\log(1+\sqrt{n})} \log \frac{n-1}{n^x - 1}$$

is an additive generator of the (strict) Frank t-norm $T_n^{\mathbf{F}}$ [13]. Then for all $x \in]0,1]$ we have $\lim_{n\to\infty} t_n(x) = 1-x$, i.e., the sequence $(t_n|_{]0,1]})_{n>1}$ converges to the restriction of an additive generator of the Łukasiewicz t-norm $T_{\mathbf{L}}$ to $]0,1]$. Therefore, the sequence $(T_n^{\mathbf{F}})_{n>1}$ converges to $T_{\mathbf{L}} (= T_\infty^{\mathbf{F}})$.

Finally we mention that the continuous Archimedean t-norms form a dense subclass of the class of all continuous t-norms (with respect to the uniform topology), i.e., each continuous t-norm can be approximated by some continuous Archimedean t-norm with arbitrary precision [19,23,32]. More precisely we have:

Theorem 4.8. *Let T be a continuous t-norm. Then for each $\varepsilon > 0$ there is a strict t-norm T_1 and a nilpotent t-norm T_2 such that for all $(x, y) \in [0,1]^2$*

$$|T(x,y) - T_1(x,y)| < \varepsilon,$$
$$|T(x,y) - T_2(x,y)| < \varepsilon.$$

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