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Program

9:00–10:00 Session 1 ***(Chair: Roland Richter)***

9:00 Bettina Heise:
Some aspects about imaging and image processing for interferometry and DIC microscopy

9:30 Thomas Natschläger:
Fault Detection with Neuronal Networks

10:00 Coffee Break

10:15–11:15 Session 2 ***(Chair: Bernhard Moser)***

10:15 Fabrizio Durante:
Construction of multivariate statistical models with given partial information

10:45 Peter Sarkoci:
Pairs of Dominating Triangular Norms: A Constructive Viewpoint

Comparison of phase shifting vs. Hilbert transformation in optical phase microscopy plus a two dimensional unwrap algorithm using FFT

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1. Introduction

Interferometric techniques belong to the standard methods in the field of optical metrology to give quantitative results for investigation of transparent objects or measuring surface displacements. Meanwhile holographic interferometric methods have also found its application in the field of microbiology [1] [2] [3], where their qualitative nature gives an advantage to phase contrast or DIC microscopy [4], which have high resolution but can give only qualitative results.
There exist a variety of temporal and spatial phase based interferogram analysis methods. In this paper we concentrate us on phase shifting and phase modulated techniques. Our interferometer enables us to perform both versions in one setup. In both applications we have the problem of unwrapping for the analysis of the recorded fringe patterns. We present a FFT based 2D unwrapping algorithm which allows a simple and fast reconstruction of the phase distribution of the objects. We apply our methods for technical and biological objects.

2. Measuring Principle

Figure 1 depicts the measuring principle for phase shift and Hilbert transform microscopy schematically.

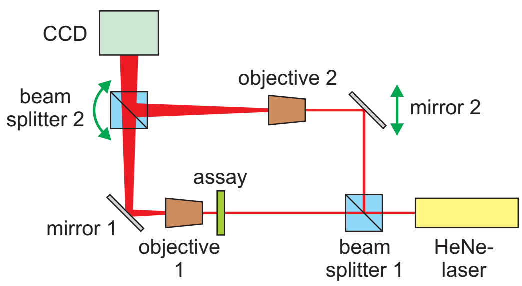

Figure 1: Measuring principle for phase shift and Hilbert transformation microscopy

Phase shift microscope: The mirror 2 is slightly displaced to positions between $\pm\lambda/2$ (λ_{HeNe} = 632 nm). Due to this displacement the optical length in the reference path is changing and so the phases difference $\Delta\varphi$ of the two electromagnetic waves (probe- and reference beam) is changing $\pm\pi$. This causes a modulation of the intensity *I* for each pixel of the CCD camera according to Equ. 1. This kind of modulation could also be done by a Pockels cell.

$$I = \frac{I_1 + I_2}{2} + \sqrt{I_1 I_2}\cos(\varphi + \Delta\varphi) \tag{1}$$

I_1 … intensity of the probe beam
I_2 … intensity of the reference beam
I … intensity on the CCD
$\Delta\varphi$ … phase difference causes by mirror shifting
φ … phase caused by the assay

Figure 2a shows the intensity of one pixel caused by $\Delta\varphi$ (blue line). The green line is the calculated intensity via Görtzel algorithm. The value of interest is the phase φ caused by the assay. The wrapped phases φ for each pixel are represented by the matrix *P* depicted in Fig. 2b.

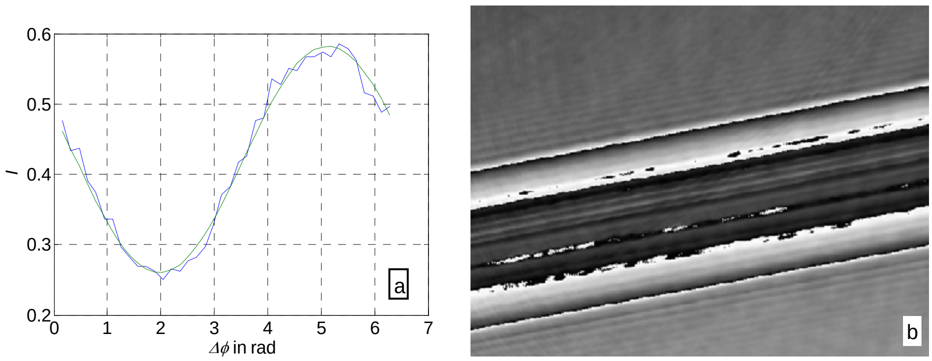

Figure 2: Phase image of a glass fiber taken by the phase shift microscope;
(a) changing of the intensity *I* of one pixel due to $\Delta\varphi$;
(b) whole phase image *P* (wrapped);

In phase sifting techniques, with at least three equidistant spaced shifts by summing up a large number of interferograms [5], with arbitrary phase shifts in ±π range the influence of the background can be reduced [6].

Hilbert transformation microscopy: Slightly rotation of the beam splitter 2 (Fig. 1) causes a one dimensional periodical sinusoidal intensity pattern on the CCD (Fig. 3a). One dimensional Hilbert transformation is used to calculate the wrapped phase angle (Fig. 3b).

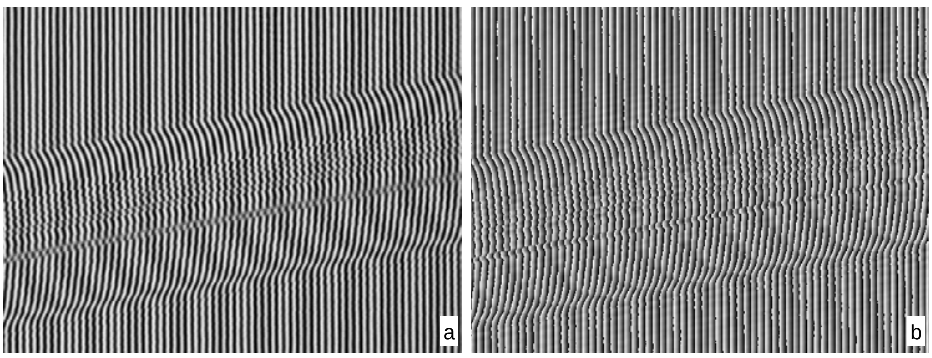

Figure 3: Phase image of a glass fiber taken by the Hilbert transfomation microscope;
(a) intensity image *I*; (b) phase image *P* (wrapped) calculated by Hilbert transformation;

Frequency respectively phase modulated techniques have the advantage that the analysis can be performed with one recorded fringe image. This allows the recording of fast dynamic processes. On the other hand in technical applications they are often disturbed by background structures or diffraction patterns. Hence, a subtraction of the background image or a further calibration is mostly necessary.

The final phase image is evaluated in 3 steps:

1. Scanning of wrapped phase image by phase shift or Hilbert transformation
2. Unwrapping of the phase image by two dimensional unwrapping
3. Subtraction of the regression plan

3. EXPERIMENTAL SETUP

The experimental setup implemented is depicted in Fig. 4. The setup represents both: phase shift and Hilbert transformation microscope. It is possible to take pictures for both techniques by one measurement cycle. The Hilbert transformation needs only one fringe pattern picture to work fine. On the other hand the phase shifting algorithm can handle also fringe pattern.

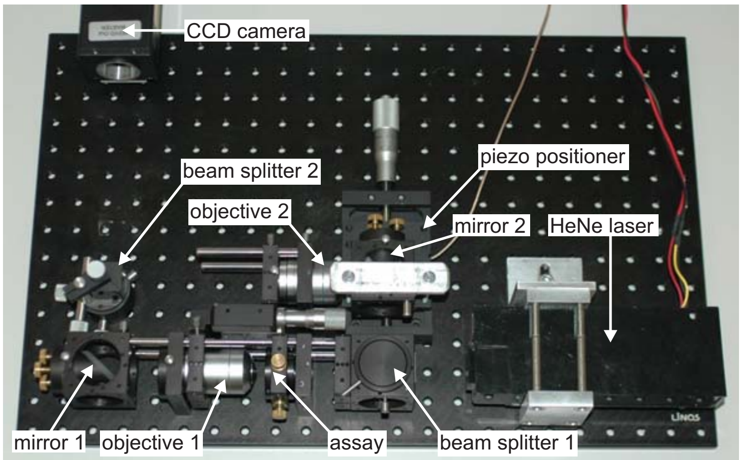

Figure 4: Experimental setup for phase shift and Hilbert transformation microscopy (compare with Fig. 1)

4. TWO DIMENSIONAL UNWRAPPING USING FFT

The phase shifting and Hilbert transformation microscopy yields two dimensional wrapped images where the pixel values represent the phase in the range from $-\pi$ to $+\pi$ (Fig. 1). The real phase of the pixel can exceed this range. The real phase is obtained by adding integer multiply of 2π (also negative and zero).

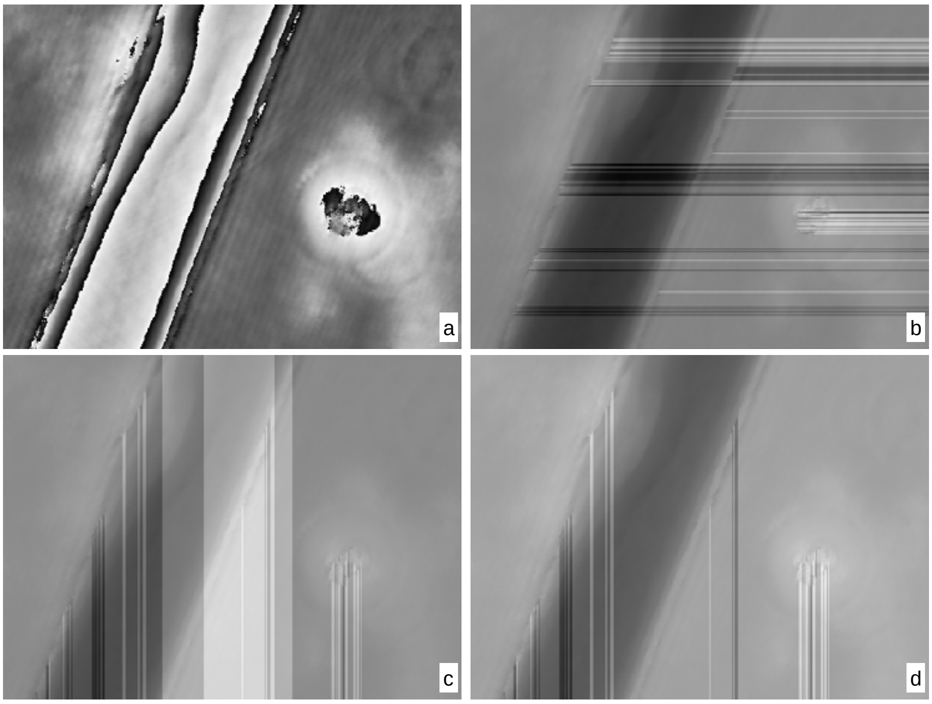

Figure 1: Phase image of a glass fiber; (a) unwraped data; (b) horizontal unwraped;
(c) vertical unwarped, (d) first vertical then horizontal unwraped

Figure 1bcd show that one-dimensional unwrapping fails to build the real phase. Therefore an alternative algorithm for two-dimensional unwrapping is presented.

The algorithm relays on a mechanical model. The pixels of the phase image are assumed to be solid elements. Each element is linked with its neighbors via springs (Figure 1 a and c).

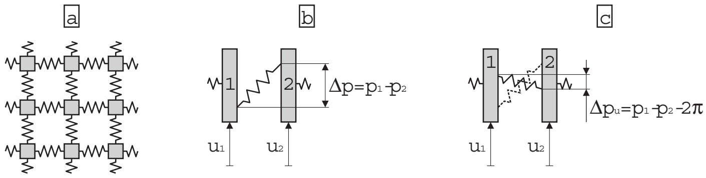

Figure 1: Mechanical model; (a) top view; (b) side view of two elements; (c) unwrapped side view

The difference of the phase between two adjacent elements Δp is in the range from -2π to $+2\pi$. Now Δp is limited to the range $[-\pi\ +\pi]$ by equation 2 (Figure 1 c).

$$\Delta p_u = \mathrm{mod}(\Delta p + \pi, 2\pi) - \pi \qquad (2)$$

The static position of the elements u can be calculated by solving a set of linear equations (one for every pixel!). For normal image sizes this isn't suitable. But set of equations can be written in the form of Eqn. 3-5.

$$L = \left(\mathrm{mod}\left(P * \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \pi, 2\pi\right) - \pi \right) * \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} + \left(\mathrm{mod}\left(P * \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \pi, 2\pi\right) - \pi \right) * \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad (3)$$

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4}$$

$$L = U * H \tag{5}$$

P … wrapped phase
H … Laplace filter operand
L … unwrapped Laplace filtered phase P
U … unwrapped phase

The set of Eqn. 3-5 assume that the boundary of the phase image P is equal zero. Equation 5 can be solved in the Fourier domain by Equ. 6.

$$U = \Im^{-1}\left\{\frac{\Im\{L\}}{\Im\{H\}}\right\} \tag{6}$$

The mean intensity of the Laplace filter L is zero. To avoid a division by zero in the Fourier domain $\Im\{H\}(0,0)$ is set to one. The Fourier transformation works for periodical signals. This can be used to correct the boundary condition by mirroring the phase image in both directions to fourfold size before calculation the unwrapped phase U. This is according to boundary condition of second kind [9]. Figure 3a shows the unwrapped phase U (Fig. 3a) of the phase image P.

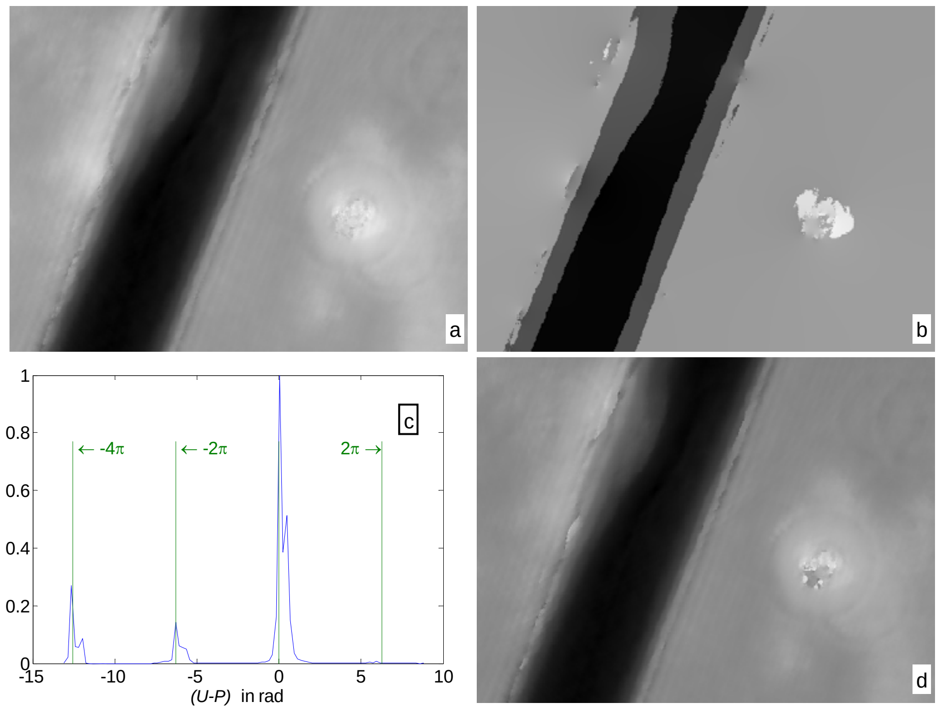

Figure 3: Phase image of a glass fiber; (a) unwraped data *U*; (b) *U-P*;
(c) histogram of *U-P*, (d) *U* with rounded *U-P*

Figure 3b shows the image *U-P*. These are the values added to *P* to get the unwrapped phase. The histogram of *U-P* (Fig. 3c) demonstrates that the unwrapping is mainly an addition of multiples of 2π to the wrapped phase. Fig. 3d is the rounded values *U-P* of Fig 3 b to multiples of 2π plus the unwrapped phase *P* (Fig. 1a). Keep in mind that there is no loss of information between Fig. 1a and Fig. 3d.

5. RESULTS

6. CONCLUSIONS

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- The two dimensional unwrapping algorithm works fine also for mean phase images.
- The algorithm is simple and fast (the most time is needed to calculate the two dimensional FFT)

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3D ANALYSIS OF LIPID DROPLETS IN DIC IMAGES AND FLUORESCENCE IMAGE STACKS

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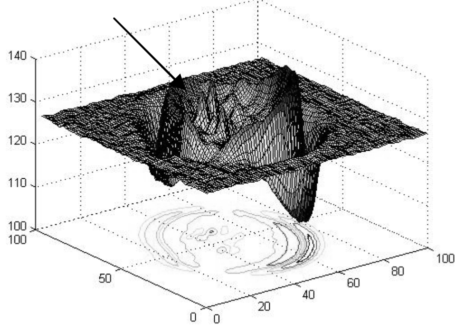
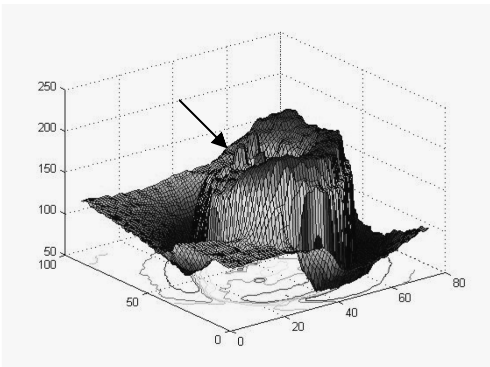
KEY WORDS: Lipid droplets, DIC, deconvolution, wavelets

The characterization of the spatial lipid droplet distribution and aggregation in yeast cells plays an important rule for investigation of fat metabolism. To avoid multiple staining only lipid droplets are fluorescently marked, whereas the whole yeast cells are imaged by DIC microscopy simultaneously. By conventional DIC microscopy there is no linear relation between the measured intensity and the original phase gradient because of combined amplitude and phase response of. this type of microscopy. Although several proposals for technical improvements of DIC microscopy based on phase shifting [1;2] or different shear directions [3] exist, commercial microscopes specialized for fluorescence imaging are often equipped only with a combined conventional DIC imaging modality. After linearization of the problem we can perform a deconvolution to get approximately quantitative values for the optical path length (OPL) map from the measured phase gradient. We use two different approaches for deconvolution: the first is based on a Maximum Likelihood deconvolution algorithm, the second approach is an iterative projection based method to reconstruct the phase values of the cells. Lipid droplets can be clearly recognized as peaks in OPL maps due to their slightly different refractive index.



3D OPL reconstruction of a yeast cell by ML deconvolution



3D OPL reconstruction of a yeast cell by iterative projection based deconvolution

Additionally to the DIC images, fluorescence image stacks of the stained lipid droplets are analyzed. The goal is to reconstruct the 3D configuration of the droplets inside the cell. À trous wavelets based techniques are successfully used for spot detection in 2D fluorescence microscopy images[4]. The technique is particularly well suited for the detection of isotropic features. Due to the spherical appearance of the lipid droplets, we apply the 3D version of the à trous wavelets to the image stack, combined with hard threshold shrinkage. A brief statistical analysis of the detected droplets features is performed.
The potential for a more accurate analysis of the distribution of lipid droplets by combining the two techniques presented above is discussed in the conclusion of this work.

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Fault detection for rotating machines with neural networks

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Abstract

In this contribution we describe an approach for fault detection and isolation for rotating machines. The method is based on using application specific characteristic frequencies of the sliding FFT as inputs for a neural network whose outputs will be used to classify the state (ok, failure, kind of failure) of the monitored machine.

This approach was successfully applied in a case study whose goal it was do monitor a piston pump. In this study we were able to achieve a classification accuracy of more than 99% when using labeled data.

However in a real situation one is faced with the situation that only data from the normal state of the machine is available (e.g. recorded during the setting up of the machine). We show that also under this circumstances it is possible to train a neural network which models the normal state and to derive an error signal from the network outputs which allows to determine the detection of a failure. However no classification is possible in this situation. We will present the highly satisfactory results achieved with this approach in the above mentioned case study.

Construction of multivariate statistical models with given partial information

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1 Introduction

For our purpose, a *multivariate statistical model* $\mathcal{S}$ is formed by the pair $(\mathcal{P}, \mathbf{X})$, where:

- $\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P})$ is a probability space (in the classical Kolmogorov's sense);
- $\mathbf{X} = (X_1, \dots, X_n)$ is a vector of $n \geq 2$ continuous random variables (=r.v.'s) taking values on $\mathbb{R}$.

To each multivariate statistical model, we can associate an n–dimensional distribution function (=d.f.) $F : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x_1, x_2, \dots, x_n) = \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n).$$

From F we can derive, for each $i \in \{1, 2, \dots, n\}$, the d.f. F_i of each component of $\mathbf{X}$ defined by

$$F_i(x) = \mathbb{P}(X_i \leq x) = F(+\infty, \dots, +\infty, x_i, +\infty, \dots, +\infty).$$

Such F_i are called *(univariate) marginals* of F.

In practice, F expresses the behaviour of the random phenomena that we would model, being the probability space just a mathematical fiction (see [9] for a complete discussion about this point of view).

For many years, a problem of interest to statisticians has been the construction of special families of multivariate d.f.'s that can be, conveniently, fitted to real data in order to describe our random phenomena. Specifically, a multivariate d.f. contains two kinds of information: the behaviour of each component of the random vector, captured by the marginals, and the dependence among these components. In view of the following Theorem due to A. Sklar [12], these two aspects can be treated separately.

Theorem 1. *Let $X_1, X_2, \dots, X_n$ be r.v.'s with joint d.f. F and marginal d.f.'s $F_1, F_2, \dots, F_n$. Then there exists a d.f. $C_n : \mathbb{I}^n \to \mathbb{I}$ whose univariate marginals are uniformly distributed on $\mathbb{I} := [0, 1]$, called copula, such that, for all $\mathbf{x} \in \mathbb{R}^n$,*

$$F(\mathbf{x}) = C_n(F_1(x_1), F_2(x_2), \dots, F_n(x_n)). \tag{1}$$

Conversely, if C_n is an n–copula and $F_1, F_2, \dots, F_n$ are univariate d.f.'s, then the function F defined by (1) is an n–d.f. with marginals $F_1, F_2, \dots, F_n$.

In particular, the second part of the Sklar's Theorem takes a great importance. In fact, if *the marginals are known*, then the choice of a suitable multivariate statistical model can be restricted to the construction of a suitable family of copulas. This fact has been recently discovered in many statistical applications. For many years, in fact, multivariate models had been often constructed either under the assumption of the independence of their components or by assuming that the components are connected by a multivariate normal distribution. Copulas, instead, allow to study models with a more flexible and wide range of dependence. For an overview of the applications of copulas, see [1, 6, 7, 10].

In this paper, we introduce some definitions and basic properties about copulas and, then, we construct a new family of multivariate copulas, which can be used in the construction of a family of multivariate d.f.'s with prescribed marginals.

2 Multivariate copulas

Let n be in $\mathbb{N}$, $n \geq 2$, and denote by $\mathbf{x} = (x_1, \dots, x_n)$ any point in $\mathbb{R}^n$. An *n–dimensional copula* (shortly, *n–copula*) is a mapping $C_n : \mathbb{I}^n \to \mathbb{I}$ satisfying the following conditions:

(C1) $C_n(\mathbf{u}) = 0$ whenever $\mathbf{u} \in \mathbb{I}^n$ has at least one component equal to 0;

(C2) $C_n(\mathbf{u}) = u_i$ whenever $\mathbf{u} \in \mathbb{I}^n$ has all the components equal to 1 except the i–th one, which is equal to u_i;

(C3) C_n is *n–increasing*, viz., for each n–box $B = \times_{i=1}^n [u_i, v_i]$ in $\mathbb{I}^n$ with $u_i \leq v_i$ for each $i \in \{1, \dots, n\}$,

$$V_{C_n}(B) := \sum_{\mathbf{z} \in B} \operatorname{sgn}(\mathbf{z}) C_n(\mathbf{z}) \geq 0, \tag{2}$$

where the sum is taken over all vertices $\mathbf{z}$ in B, $z_i \in \{u_i, v_i\}$ for each i in $\{1, 2, \dots, n\}$, and $\operatorname{sgn}(\mathbf{z})$ equals -1, if the number of u_i's among the coordinates of $\mathbf{z}$ is odd, and equals 1, otherwise.

Notice that, if $C_n : \mathbb{I}^n \to \mathbb{I}$ admits derivatives up to order n, then property (C3) is equivalent to

$$\frac{\partial C_n(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} \geq 0$$

for every $\mathbf{u} \in \mathbb{I}^n$.

For the case $n = 2$, property (C3) is equivalent to

$$C_2(u_1, u_2) + C_2(v_1, v_2) \geq C_2(u_1, v_2) + C_2(u_2, v_1)$$

for all $u_1 \leq v_1$ and $u_2 \leq v_2$.

We denote by $\mathcal{C}_n$ the set of all n–dimensional copulas ($n \geq 2$). For every $C_n \in \mathcal{C}_n$ and for every $\mathbf{u} \in \mathbb{I}^n$, we have that

$$W_n(\mathbf{u}) \leq C_n(\mathbf{u}) \leq M_n(\mathbf{u}), \tag{3}$$

where

$$W_n(\mathbf{u}) := \max\left\{\sum_{i=1}^{n} u_i - n + 1, 0\right\}, \qquad M_n(\mathbf{u}) := \min\{u_1, u_2, \dots, u_n\}.$$

Notice that M_n is in $\mathcal{C}_n$, but W_n is in $\mathcal{C}_n$ only for $n = 2$. Another important n–copula is the product $\Pi_n(\mathbf{u}) := \prod_{i=1}^{n} u_i$. The following result characterizes some properties of random vectors in terms of copulas.

Theorem 2. *Let $X_1, X_2, \dots, X_n$ be continuous r.v.'s with copula C_n.*

- *$X_1, X_2, \dots, X_n$ are independent if, and only if, $C_n = \Pi_n$.*
- *Each of the r.v.'s $X_1, X_2, \dots, X_n$ is a strictly increasing function of any of the others if, and only if, $C_n = M_n$.*
- *If $\alpha_1, \alpha_2, \dots, \alpha_n$ are strictly increasing mappings, respectively, on $RanX_1$, $RanX_2$, $\dots, RanX_n$, then C_n is the copula of $(\alpha_1(X_1), \dots, \alpha_n(X_n))$.*

For more details about copulas, see [8, 11].

3 A new family of copulas

Given a continuous function $f\colon \mathbb{I} \to \mathbb{I}$, we define the mapping $C_f^n : \mathbb{I}^n \to \mathbb{I}$ given by

$$C_f^n(u_1, u_2, \dots, u_n) = u_{[1]} \prod_{i=2}^{n} f(u_{[i]}), \tag{4}$$

where $u_{[1]}, \dots, u_{[n]}$ denote the components of $(u_1, u_2, \dots, u_n) \in \mathbb{I}^n$ rearranged in increasing order, i.e. for instance

$$u_{[1]} = \min(u_1, u_2, \dots, u_n) \quad \text{and} \quad u_{[n]} = \max(u_1, u_2, \dots, u_n).$$

It is easy to note that C_f^n is symmetric, viz. it is invariant under any permutation of his arguments. Moreover, Π_n and M_n can be constructed by means of (4): it suffices to take $f(t) = t$ and $f(1) = 1$, respectively. The following result characterizes the copulas of type (4).

Theorem 3. *Let $f\colon \mathbb{I} \longrightarrow \mathbb{I}$ be a continuous function and let C_f^n be the function defined by* (4)*. Then C_f^n is an n-copula if, and only if,*

(i) *$f(1) = 1$;*

(ii) *f is increasing;*

(iii) *the function $t \to f(t)/t$ is decreasing on* $(0, 1]$.

Example 1. Let α be in $\mathbb{I}$ and consider the function $f(t) = \alpha t + \overline{\alpha}$, with $\overline{\alpha} := 1 - \alpha$. Then, the n-copula C_f^n, denoted by C_α, is given by

$$C_\alpha(u_1, u_2, \dots, u_n) = u_{[1]} \prod_{i=2}^{n} (\alpha u_{[i]} + \overline{\alpha}).$$

In particular, for $n = 2$, we obtain a convex combination of Π_2 and M_2.

Example 2. Let α be in $\mathbb{I}$ and consider the function $f(t) = t^\alpha$. Then, the n-copula C_f^n, denoted by C_α, is given by

$$C_\alpha(\mathbf{u}) = (\min(u_1, u_2, \ldots, u_n))^{1-\alpha} \prod_{i=1}^{n} u_i^\alpha,$$

which is a (weighted) geometric mean of the copulas Π_n and M_n. This family generalizes the Cuadras-Augé family of bivariate copulas (see [2]). Notice that every copula C_α is a *multivariate extreme copula*, viz. for every $t > 0$ $C_\alpha(u_1^t, u_2^t, \ldots, u_n^t) = (C_\alpha(u_1, u_2, \ldots, u_n))^t$ [8].

In Table 1, we collect several examples of generators of copulas of type (4).

Table 1: Some generators for the new class of n-copulas.

Generator	Parameters
$\min(\alpha t, 1)$	$\alpha \geq 1$
$1 - (1-t)^\alpha$	$\alpha \geq 1$
$\dfrac{(1+\alpha)t}{\alpha t + 1}$	$\alpha \geq 0$
$\dfrac{1 - \exp(-\alpha t)}{1 - \exp(-\alpha)}$	$\alpha > 0$
$\dfrac{\beta t}{\beta t + \alpha(1-t)}$	$0 < \alpha \leq \beta \leq 1$
$\dfrac{\sin(\alpha t)}{\sin \alpha}$	$0 \leq \alpha \leq \pi/2$

Finally, we give a statistical interpretation for copulas of type (4).

Let $W_1, W_2, \ldots, W_n$, Z be $n+1$ independent random variables such that, for all $i \in \{1, 2, \ldots, n\}$, W_i has d.f. f satisfying parts (i), (ii) and (iii) in Theorem 3, and Z has d.f. $g(t) = t/f(t)$ (note that $g(1) = 1$ and g is increasing since $f(t)/t$ is decreasing). Consider the random variables $U_i = \max(W_i, Z)$, for all $i = 1, 2, \ldots, n$. Then, for every $(u_1, u_2, \ldots, u_n)$, the d.f. of the random vector $(U_1, U_2, \ldots, U_n)$ is given by

$$P(U_1 \leq u_1, \ldots, U_n \leq u_n) = u_{[1]} \prod_{i=2}^{n} f(u_{[i]}),$$

and, hence, it is a copula of type (4).

For more details about this class, see [3, 4, 5].

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Pairs of Dominating Triangular Norms: The Constructive Viewpoint

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1 Introduction

The concept of dominance has been introduced within the framework of probabilistic metric spaces for triangle functions and for building cartesian products of probabilistic metric spaces [16]. Afterwards the dominance of t-norms was studied in connection with construction of fuzzy equivalence relations [2, 3, 17] and construction of fuzzy orderings [1]. Later on, the concept of dominance was extended to the more general class of aggregation operators [8,10]. The dominance of aggregation operators emerges when investigating which aggregation procedures applied to the system of T-transitive fuzzy relations yield a T-transitive fuzzy relation again [8] or when seeking aggregation operators which preserve the extensionality of fuzzy sets with respect to given T-equivalence relations [9].

Definition 1 *Let $(P, \geq)$ be a poset and let $A\colon P^m \to P$, $B\colon P^n \to P$ be two operations defined on P with arity m and n, respectively. Then we say that A dominates B ($A \gg B$ in symbols) if each matrix $(x_{i,j})$ of type $m \times n$ over P satisfies*

$$\begin{aligned} &A(B(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), \ldots, B(x_{m,1}, x_{m,2}, \ldots, x_{m,n})) \quad \geq \\ &B(A(x_{1,1}, x_{2,1}, \ldots, x_{m,1}), \ldots, A(x_{1,n}, x_{2,n}, \ldots, x_{m,n})). \end{aligned}$$

Let us recall that a t-norm [7, 16] is a monotone, associative and commutative binary operation $T\colon [0,1]^2 \to [0,1]$ with neutral element 1. In our contribution we pay attention mainly to these prototypical triangular norms:

$$\begin{aligned} T_{\mathbf{M}}(x,y) &= \min(x,y), \\ T_{\mathbf{P}}(x,y) &= xy, \\ T_{\mathbf{L}}(x,y) &= \max(0, x+y-1), \end{aligned}$$

We say that a t-norm T_1 is stronger than a t-norm T_2($T_1 \geq T_2$ in symbols) if any $x, y \in [0,1]$ satisfy $T_1(x,y) \geq T_2(x,y)$. We use the notation $T_1 > T_2$ whenever simultaneously $T_1 \geq T_2$ and $T_1 \neq T_2$ hold. One can easily show that each t-norm is weaker than $T_{\mathbf{M}}$ and stronger than $T_{\mathbf{D}}$. Particularly, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$

satisfy $T_{\mathbf{M}} > T_{\mathbf{P}} > T_{\mathbf{L}} > T_{\mathbf{D}}$. It is obvious that $\geq$ is a partial order on the set of all t-norms, i.e., the reflexive, antisymmetric and transitive relation.

By Definition 1 we have that two t-norms T_1 and T_2 satisfy $T_1 \gg T_2$ iff for each $x, y, u, v \in [0,1]$

$$T_1(T_2(x,y), T_2(u,v)) \geq T_2(T_1(x,u), T_1(y,v)). \tag{1}$$

It is easy to show that each t-norm T satisfies $T_{\mathbf{M}} \gg T$, $T \gg T_{\mathbf{D}}$ and $T \gg T$. If $T_1 \gg T_2$ then by inequality (1), the neutrality of 1 and the commutativity of t-norms we have that any $y, u \in [0,1]$ satisfy

$$\begin{aligned} T_1(y,u) = \; & T_1(T_2(1,y), T_2(u,1)) \; \geq \\ \geq \; & T_2(T_1(1,u), T_1(y,1)) \; = T_2(u,y) = T_2(y,u) \end{aligned}$$

so that $T_1 \geq T_2$, see [7]. This means that satisfaction of $T_1 \geq T_2$ is a necessary condition for $T_1 \gg T_2$ or, in other words, that dominance is a subrelation of $\geq$. The converse implication, however, does not hold. For example in the family of Hamacher [5, 6] or Frank t-norms [4], any two nonextremal members are comparable while no one of them dominates the other [13]. Dominance of t-norms is moreover an antisymmetric relation which is a consequence of the fact that it is a subrelation of the antisymmetric relation $\geq$. It was a question if the dominance of t-norms is also transitive [16, Problem 12.11.3]. It revealed oneself only recently, that dominance is not transitive even on the relatively restricted class of continuous t-norms [12, 14].

2 Ordinal Sum T-Norms

Let $[a_1, a_2]$ and $[b_1, b_2]$ be intervals of real numbers. By an *order isomorphism* from $[a_1, a_2]$ onto $[b_1, b_2]$ we mean any *increasing bijection* from the first interval onto the second one. Let I be a closed interval, we denote by ψ_I the unique *affine* order isomorphism from I onto $[0,1]$. For a binary operation $O\colon [c,d]^2 \to [c,d]$, not necessarily a t-norm, and for the order isomorphism $\varphi\colon [a,b] \to [c,d]$ we define a new operation

$$(O)_\varphi\colon [a,b]^2 \to [a,b]\colon (x,y) \mapsto \varphi^{-1}\big(O(\varphi(x), \varphi(y))\big)$$

which we call the *φ-transform* of the operation O.

Let $\{T_i\}$ be a (possibly countably infinite) system of t-norms indexed by $i \in \mathcal{I}$. Let $\{I_i\}$ be a system of intervals $I_i = [a_i, b_i] \subseteq [0,1]$ with pairwise disjoint interiors, indexed by the same set. We define an *ordinal sum t-norm* given by $\{T_i\}$ and $\{I_i\}$ to be a function

$$T(x,y)\colon [0,1]^2 \to [0,1]\colon (x,y) \mapsto \begin{cases} (T_i)_{\psi_{I_i}}(x,y) & \text{if } x, y \in I_i \\ T_{\mathbf{M}}(x,y) & \text{otherwise} \end{cases}. \tag{2}$$

Symbolically, we denote this t-norm by $(\langle a_i, b_i, T_i \rangle)_{i \in \mathcal{I}}$. Triangular norms T_i are the so called *summand operations* and intervals $I_i = [a_i, b_i]$ are *summand carriers*. Note that this definition is sound, although in one-point overlaps of summand carriers there are two independent ways how to define value of the ordinal sum. Moreover, ordinal sum of (continuous) t-norm is a (continuous) t-norm again [7].

Many results related to the dominance of ordinal sum t-norms appeared only recently. Some of them are necessary and sufficient conditions for dominance of ordinal t-norms in general [11, 12]. The others characterize the dominance relation in the special classes of ordinal sum t-norms [12, 15]. The main result of the first kind allows to treat the dominance relation between ordinal sum t-norms summand-wisely [11]:

Theorem 2 Let $\{T_1\}$ be an ordinal sum t-norm , the first one with summand operations $\{T_{1,i}\}$ and summand carriers $I_{1,i}$ indexed by $i \in \mathcal{I}_1$. Analogically, let T_2 be an ordinal sum t-norm given by summands $\{T_{2,i}\}$ and summand carriers $\{I_{2,i}\}$ indexed by $i \in \mathcal{I}_2$. Then $T_1 \gg T_2$ if and only if

- $T_1 \geq T_2$, and
- $\left(T_1 \restriction_{I^2_{2,i}}\right)_{\psi^{-1}_{I_{2,i}}} \gg T_{2,i}$ for each $i \in I_2$.

■

Let us recall that an *idempotent element* of the t-norm T is any $x \in [0,1]$ such that $T(x,x) = x$. We denote $\mathrm{Idp}T$ the set of all idempotent elements of T. If T is an ordinal sum t-norm with summand carriers I_i with $i \in \mathcal{I}$ then

$$\mathrm{Idp}T \supseteq [0,1] \setminus \bigcup_{i \in \mathcal{I}} I_i^\circ \tag{3}$$

where I_i° is the interior of I_i. Another important result is a necessary condition relating the structure of idempotent elements [11]:

Theorem 3 If a t-norm T_1 dominates the t-norm T_2, then $\mathrm{Idp}T_1$ is closed with respect to T_2. ■

If we restrict ourselves to ordinal sums of special type, Theorem 3 can be strengthened [15]:

Theorem 4 Let T_1, T_2 be ordinal sum t-norms which involve $T_{\mathbf{L}}$ as their only summand operation. Then $T_1 \gg T_2$ if and only if $T_1 \geq T_2$ and $\mathrm{Idp}T_1$ is closed with respect to T_2. ■

Theorem 5 Let T_1, T_2 be ordinal sum t-norms which involve $T_{\mathbf{P}}$ as their only summand operation. Then $T_1 \gg T_2$ if and only if $T_1 \geq T_2$ and $\mathrm{Idp}T_1$ is closed with respect to T_2. ■

Observe that, formally, both these statements have the same structure. Both of them reduce the difficult question of dominance to much easier question whether some special set is closed with respect some operation. That allows us to use both these results as a construction method for special ordinal sum t-norms which are in the relationship of dominance.

3 Construction Methods

If the ordinal sum t-norm involve either $T_{\mathbf{L}}$ or $T_{\mathbf{P}}$ as the only summand operation, the whole structure of summand carriers is determined completely by the set of idempotent elements. More precisely, the inclusion (3) changes to the indentity

$$\mathrm{Idp}\, T = [0,1] \setminus \bigcup_{i \in \mathfrak{I}} I_i^{\circ}.$$

Therefore in order to construct pairs of dominating ordinal sums by means of Theorem 4 and Theorem 5 it is sufficient to construct subsets of the unit interval closed either with respect to $T_{\mathbf{L}}$ or with respect to $T_{\mathbf{P}}$. For that purpose the following lemma is useful.

Theorem 6 Let $M \subseteq [0, \infty]$ be a set closed with respect to the standard addition. Let $k > 0$ be an arbitrary positive constant. Then

- the set $M_k = \{1 - \frac{x}{k} \mid x \in M \cap [0,k]\} \cup \{0,1\}$ is closed with respect to $T_{\mathbf{L}}$,
- the set $M_* = \{e^{-x} \mid x \in M\} \cup \{0,1\}$ is closed with respect to $T_{\mathbf{P}}$.

■

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