Advances in Knowledge-Based Technologies

Proceedings of the
Master and PhD Seminar
Winter term 2010/11, part 1

Softwarepark Hagenberg
SCCH, Room 0/2
8 November 2010

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Program

Session 1. Chair: Bernhard Moser

14:00	Verena Schlager: Coherence Probe Microscopy 3D Image Processing - Review and Recent Results
14:30	Bettina Heise: Monogenic Signal and Phase Filter: Applications in Image Processing and Optics
14:45	Swanhild Bernstein: Texture Analysis based on Monogenic Signals and Emperical Mode Decomposition

Session 2. Chair: Roland Richter

15:30	Monika Pekárová: Special Classes of Distorted Generated Copulas
16:00	Tomáš Bacigál, Vladimír Jágr: Convex combination in dependence structure of Archimax copulas and comparison study on real data
16:30	Bernhard Moser: Characterization of the Unit-Ball of the Discrepancy Norm

Coherence Probe Microscopy 3D Image Processing

Review and Recent Results

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November 8, 2010

Abstract

Optical coherence tomography (OCT), a technique originally proposed for applications in the field of biomedical diagnostics, is shown to be an efficient measurement technique for a multitude of problems posed in technical engineering and material research [1]. Coherence Probe Microscopy (CPM) should bring us to a still more efficient technique for analyzing surfaces of organic or polymer material.
The mathematical side of the CPM research is to increase the image quality and the analyze of the images. For the first part image enhancement techniques like denoising filters, binarization techniques or background correction play a quite important role. The second part splits into retrieving orientation informations of the inner structure and clustering the images into different groups depending on their inner structure, e.g. for detecting failure images.
In this presentation I will give a review about this different considerations of my master thesis. Additionally I will present the recent results in orientation analysis based on the Riesz-Laplace wavelet transform [2, 3, 4].

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Monogenic Signal and Phase Filter: Applications in Image Processing and Optics

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Abstract: Mathematical methods, based on monogenic signal theory, offer an alternative in field of image processing. Whereas analytic signals are well-known as 1D-approach in electrical engineering and signal processing, for 2D-signals (as e.g. images) the Riesz transform possesses a functionality comparable to the 1D-Hilbert transform [1]. We encounter examples of these methods e.g. for corner/edge detection [2], for fringe analysis [3], and in an generalized way for texture analysis [4]. The computed features, as local energy, local phase and frequency, or local orientation of structures within the image, can be used for the description and classification of the textures under investigation in the following.

The question may be posed how monogenic signal processing can be combined with multi-scale/ multi-resolution techniques. Different approaches are described in literature: Monogenic signal and wavelet-based methods [5], monogenic signal and scale-space methods [6], and monogenic signal and empirical mode decomposition [7].

In optics, spiral phase filtering or vortex filtering [8], applying a phase filter with a helical phase function in Fourier domain, represents the physical analogue to the mathematical Riesz transform in image processing. However, it should be noted that in optical imaging the complex-valued electric field is the quantity of interest to be influenced, in contrast to the scalar-valued intensity modified in image processing. Therefore, the obtained effects will differ: in optics these filters find use e.g. for laser beam shaping to provide so-called donut modes [9] or for contrast modifications during the imaging process; in image processing they are applied e.g. for extraction of salient image points [10] or for local phase/frequency-based analysis of the recorded patterns.

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Texture Analysis based on Monogenic Signals and Emperical Mode Decomposition

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Analytic signal representation as introduced by Gabor plays an important role in optical signal processing and in coherence theory of optical fields. Several definitions for extending the notion of representation to two and more dimensions have appeared. We will present the model of monogenic signals which is one generalization. Combining the signal and its Riesz transformed result yields the monogenic signal

$$f_M(x) = f(x) + (h_R * f)(x).$$

Applying the Poisson kernel to the original signal results in a smoothed signal which, for all scale parameters s, results in a Poisson scale-space

$$p(x;s) = (f * h_P)(x) \quad \text{and} \quad q(x;s) = (f * h_Q)(x),$$

where s denotes the scale parameter, $p(x;s)$ is the Poisson scale-space and h_P indicates the scalar-valued Poisson kernel and h_Q denotes the conjugated Poisson kernel. The expressions for the local amplitude and local phase in this case are

$$A_M(x;s) = \sqrt{p(x;s)^2 + |q(x;s)|^2} \quad \text{and} \quad \Phi_M(x;s) = \frac{q(x;s)}{|q(x;s)|} \arctan\left(\frac{|q(x;s)|}{p(x;s)}\right).$$

The monogenic scale-space is an alternative to the Gaussian scale-space. Coupling methods of differential geometry tensor algebra, monogenic signal and quadrature filter, a general model for 2D image structures can be obtained as the monogenic extension of a curvature tensor.

The analysis of emperical data in order to detect and parameterize multiscale patterns and shapes is an important problem. The so-called Hilbert-Huang transform consits in the following: first, one decomposes iteratively the times series into empirical adaptive nonlinear modes (instrinsic mode functions (IMF)) which exhibits nonlinear shapes and patterns . Second, the Hilbert spectrum analysis of the IMFs provides the localized time-frequency spectrum and the extraction of instantaneous frequencies.

Special classes of distorted generated copulas

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November 8, 2010

Abstract

Based on a recent representation of copulas invariant under univariate conditioning, a new class of copulas linked to a distortion of the identity function is introduced and studied.

1 INTRODUCTION

Copulas [19] link univariate marginal distribution functions into a joint distribution function of the corresponding random vector. In this paper we will deal with bivariate copulas only. Recall that a function $C\colon [0,1]^2 \to [0,1]$ is a (bivariate) copula whenever it is grounded, $C(x,y) = 0$ whenever $0 \in \{x,y\}$, it has neutral element 1, $C(x,y) = x \wedge y$, whenever $1 \in \{x,y\}$ and it is 2-increasing, $C(x+\epsilon, y+\delta) - C(x, y+\delta) \geq C(x+\epsilon, y) - C(x,y)$ for all $x, y, \epsilon, \delta \in [0,1]$ such that $x+\epsilon, y+\delta \in [0,1]$. Three basic copulas Π, M, W given by $\Pi(x,y) = xy$, $M(x,y) = x \wedge y$, $W(x,y) = (x+y-1) \vee 0$, express the independence, total comonotone dependence ($Y = \varphi(X)$ for an increasing function φ) and total countermonotone dependence ($Y = \eta(X)$ for a decreasing function η) of the univariate random variables X and Y, respectively. For modelling purposes, the knowledge of a large class of copulas is required. Thus several parametric classes of copulas have been introduced. For an overview we recommend monographs [9, 17]. It seems so that the most prominent class of copulas is the class of Archimedean copulas together with their M-ordinal sums.

For more details we recommend [17, 20]. Note only that by C_f we denote an Archimedean copula $C_f\colon [0,1]^2 \to [0,1]$ given by

$$C_f(x,y) = f^{(-1)}(f(x) + f(y)), \tag{1}$$

where $f\colon [0,1] \to [0,\infty]$ is a continuous strictly decreasing convex function satisfying $f(1) = 0$ and $f^{(-1)}\colon [0,\infty] \to [0,1]$ is the pseudo-inverse of f, $f^{(-1)}(u) = f^{-1}(\min(u, f(0)))$. The function f is called a generator of copula C_f and we denote by $\mathcal{F}$ the set of all generators.

We introduce some well-known examples of parametric families of Archimedean copulas, compare [9, 17]:

i) For real $\lambda \neq 0$, define $f_\lambda\colon [0,1] \to [0,\infty]$ by $f_\lambda(x) = \frac{x^{-\lambda}-1}{\lambda}$. Then $f_\lambda \in \mathcal{F}$ whenever $\lambda \geq -1$. Adding $f_0 = f_\Pi$, $f_\Pi(x) = -\log x$, the Clayton family $(C_{f_\lambda})_{\lambda \geq -1}$ is recognized, with $C_{f_{-1}} = W, C_{f_0} = \Pi$ and $C_{f_1} = H$ (Ali-Mikhail-Haq copula) given by $H(x,y) = \frac{xy}{x+y-xy}$ whenever $xy \neq 0$.

ii) The Gumbel family can be seen as $(C_{f_\Pi^\lambda})_{\lambda \geq 1}$.

iii) The Yager family can be seen as $(C_{f_W^\lambda})_{\lambda \geq 1}$.

The idea of a modification of formula (1) by means of a dependence function known from the extreme value copulas (EV-copulas, in short) was proposed in [2] as Archimax copulas. We give some more details on Archimax copulas in Section 3. On the other side, a recent study of copulas invariant under univariate conditioning initiated in [14] and [8] was completed by Durante and Jaworski in [5], providing a complete description of copulas with the above mentioned property. In the next section, we give more details on this result. The main aim of this paper is a generalization of construction provided by the results of Durante and Jaworski. Motivated by Archimax copulas, we introduce a new class of DUCS (Distorted Univariate Conditioning Stable) copulas in Section 3. In Section 4, several examples and properties of DUCS copulas are discussed.

2 Copulas invariant under univariate conditioning

We will consider copulas invariant under left univariate truncation only, and we will call them briefly copulas invariant under univariate conditioning. Copulas invariant under univariate conditioning are linked to g-ordinal sums based on the product copula and introduced in [14].

Proposition 2.1 *Let $f \in \mathcal{F}$ and let $\bar{f}\colon [0,1] \to [0,\infty]$ be given by $\bar{f}(x) = f(1-x)$. Then the functions $C_{(f)}$, $C_{(\bar{f})}\colon [0,1]^2 \to [0,1]$ given by*

$$C_{(f)}(x,y) = x f^{(-1)}\left(\frac{f(y)}{x}\right) \tag{2}$$

whenever $x \in]0,1]$, and

$$C_{(\bar{f})}(x,y) = x\left(1 - f^{(-1)}\left(\frac{\bar{f}(y)}{x}\right)\right) \tag{3}$$

whenever $x \in]0,1]$, are copulas, which are invariant under univariate conditioning.

Note that for any $f \in \mathcal{F}$ and $c > 0$, $C_{(cf)} = C_{(f)}$. Moreover, $C_{(f)} \le \Pi$ and for each $x \in]0,1[$ there is $y \in]0,1[$ so that $C_{(f)}(x,y) < xy$. Similarly, $C_{(\bar{f})} \ge \Pi$ and for each $x \in]0,1[$ there is $y \in]0,1[$ such that $C_{(\bar{f})}(x,y) > xy$. Observe also that for each $(x,y) \in [0,1]^2$, $C_{(\bar{f})}(x,y) = x - C_{(f)}(x,1-y)$, i.e., $C_{(\bar{f})}$ is a flipping of the copula $C_{(f)}$, compare [4, 17].

Theorem 2.2 *A copula $C\colon [0,1]^2 \to [0,1]$ is invariant under univariate conditioning if and only if C is a g-ordinal sum where each summand C_k, $k \in \mathcal{K}$, satisfies $C_k \in \{C_{(f_k)}, C_{(\bar{f}_k)}\}$ for some $f_k \in \mathcal{F}$.*

The main aim of our paper is a generalization of copulas introduced in Proposition 2.1, and thus we give now some examples.

Example 2.3

i) *Let $f = f_W$. Then $C_{(f_W)} = C_{f_W} = W$, and $C_{(\bar{f}_W)} = M$.*

ii) *For $p \in]0,1]$, define $h_p\colon [0,1] \to [0,\infty]$ by $h_p(x) = (1-x^p)^{\frac{1}{p}}$. Then $h_p \in \mathcal{F}$ and $C_{(h_p)} = C_{f_{-p}}$ is a Clayton copula with parameter $-p \in [-1,0[$ (see Introduction item (i)). Observe that $C_{f_{-p}} \le \Pi$.*

iii) *For $\lambda \in]0,\infty[$, define $g_\lambda\colon [0,1] \to [0,\infty]$ by $g_\lambda(x) = ((1-x)^{-\lambda} - 1)^{-\frac{1}{\lambda}}$. Then $g_\lambda \in \mathcal{F}$ and $C_{(\bar{g}_\lambda)} = C_{f_\lambda}$ is a Clayton copula with parameter $\lambda \in]0,\infty[$. Note that then $C_{f_\lambda} \ge \Pi$.*

iv) *Recall that $f_1\colon [0,1] \to [0,\infty]$ in Introduction item (i) is given by $f_1(x) = \frac{1}{x} - 1$. Then $C_{(f_1)}(x,y) = \frac{x^2 y}{1-y+xy}$, and $C_{(\bar{f}_1)}(x,y) = \frac{xy}{x+y-xy} = C_{f_1}(x,y)$.*

3 DUCS copulas

Based on the description of EV-copulas (extreme value copulas), see [21] or overview chapter [7], Capéraà et al. [2] have introduced Archimax copulas as a common generalization of Archimedean copulas and EV-copulas. Recall that for $f \in \mathcal{F}$, an Archimax copula $C_{f,D}\colon [0,1]^2 \to [0,1]$ is given by

$$C_{f,D}(x,y) = f^{(-1)}\left((f(x)+f(y))D\left(\frac{f(x)}{f(x)+f(y)}\right)\right), \tag{4}$$

with convention $\frac{0}{0} = \frac{\infty}{\infty} = 1$. Here $D\colon [0,1] \to [0,1]$ is a dependence function which is convex and satisfies $x \vee (1-x) \le D(x) \le 1$ for all $x \in [0,1]$. Evidently, for the strongest dependence funtction $D^*\colon [0,1] \to [0,1], D^*(x) = 1$, the Archimax copula C_{f,D^*} is just the Archimedean copula $C_f, C_{f,D^*} = C_f$. On the other side, for the weakest dependence function $D_*\colon [0,1] \to [0,1], D_*(x) = x \vee (1-x)$, for any $f \in \mathcal{F}$ it holds $C_{f,D_*} = M$. For $D \ne D^*$, Archimax copulas $C_{f,D}$ can be seen as distorted Archimedean copulas. Inspired by this observation, we propose to consider distorted univariate conditioning stable copulas, briefly DUCS copula.

Proposition 3.1 *Let $f \in \mathcal{F}$ and let $d\colon [0,1] \to [0,1]$ be a function. Define $C_{(f,d)}\colon [0,1]^2 \to [0,1]$ by*

$$C_{(f,d)}(x,y) = x f^{(-1)}\left(\frac{f(y)}{d(x)}\right), \tag{5}$$

with convention $\frac{0}{0} = 0$. Then:

i) *$C_{(f,d)}$ is grounded*

ii) *1 is neutral element of $C_{(f,d)}$ if and only if $d(1) = 1$*

iii) *$C_{(f,d)}$ is a copula for any $f \in \mathcal{F}$ if and only if there is a function $\tilde{d}\colon [0,1] \to [0,1]$ so that $d(x)\tilde{d}(x) = x$ for all $x \in [0,1]$, and both d and $\tilde{d}$ are non-decreasing on $]0,1]$.*

Note that given $d\colon [0,1] \to [0,1]$ such that there is a function $\tilde{d}\colon [0,1] \to [0,1]$ satisfying $d(x)\tilde{d}(x) = x$ for all $x \in [0,1]$, necessarily $d(x) \geq x$, $\tilde{d}(x) \geq x$ and both d and $\tilde{d}$ are continuous and positive on $]0,1]$. If $d(0) = 0$, the value $\tilde{d}(0)$ can be chosen arbitrarily. However, in order to have the uniqueness of the relation of d and $\tilde{d}$, we will consider continuous d and $\tilde{d}$ only. Evidently, then $\tilde{}$ can be seen as duality, $\tilde{\tilde{d}} = d$.

Denote by $\mathcal{D}$ the set of all continuous non-decreasing functions $d\colon [0,1] \to [0,1]$ such that there is a continuous non-decreasing function $\tilde{d}\colon [0,1] \to [0,1]$ for which $d(x)\tilde{d}(x) = x$ for all $x \in [0,1]$. Elements of $\mathcal{D}$ will be called distortions. Clearly $d \in \mathcal{D}$ if and only if $\tilde{d} \in \mathcal{D}$. Now we are ready to define DUCS copulas.

Definition 3.2 *A copula $C\colon [0,1]^2 \to [0,1]$ is called a DUCS copula whenever there is a generator $f \in \mathcal{F}$ and a distortion $d \in \mathcal{D}$ so that $C = C_{(f,d)}$.*

Example 3.3

i) *The strongest distortion $d^* \in \mathcal{D}$ is given by $d^*(x) = 1$. For any $f \in \mathcal{F}$, $C_{(f,d^*)} = \Pi$, i.e., the product copula is the strongest DUCS copula. Moreover, $d_* = (\tilde{d^*})$ the weakest distortion is given by $d_*(x) = x$, and $C_{(f,d_*)} = C_{(f)}$ for any generator $f \in \mathcal{F}$ (observe the striking similarity with the bounds of Archimax copulas).*

ii) *For any $d \in \mathcal{D}$, the copula $C_{(f_W,d)}\colon [0,1]^2 \to [0,1]$ is given by $C_{(f_W,d)}(x,y) = \max(0, x + (y-1)\tilde{d}(x))$. Take a parametric family $(d_{(\alpha)})_{\alpha \in [0,1]}$ of distortions, $d_{(\alpha)}(x) = \frac{x}{\alpha + (1-\alpha)x}$. Then $\tilde{d}_{(\alpha)}(x) = \alpha + (1-\alpha)x$, and $C_{(f_W,d_{(\alpha)})}(x,y) = \max(0, \alpha(x+y-1) + (1-\alpha)xy)$. Observe that the family $(C_{(f_W,d_{(\alpha)})})_{\alpha \in [0,1]}$ is a parametric family of Archimedean copulas continuous and decreasing in parameter α, with extremal elements $W = C_{(f_W,d_{(1)})}$ and $\Pi = C_{(f_W,d_{(0)})}$. Note that the generator $f_{(\alpha)}$ of $C_{(f_W,d_{(\alpha)})}$ for $\alpha < 1$ is given by $f_{(\alpha)}(x) = -\log(\alpha + (1-\alpha)x)$.*

iii) *For $\alpha \in]0,1[$, define $d_{\{\alpha\}}\colon [0,1] \to [0,1]$ by $d_{\{\alpha\}}(x) = \max(\alpha, x)$. Then $d_{\{\alpha\}} \in \mathcal{D}$, and the DUCS copula $C_{(f_W,d_{\{\alpha\}})}\colon [0,1]^2 \to [0,1]$ is given by*

$$C_{(f_W,d_{\{\alpha\}})}(x,y) = \begin{cases} \max(0, \frac{x}{\alpha}(y-1+\alpha)) & \text{if} \quad x \in [0,\alpha], \\ W(x,y) & \textit{elsewhere}. \end{cases} \tag{6}$$

Observe that this copula is a W-ordinal sum copula as introduced in [16], see also [3, 6], $C_{(f_W,d_{\{\alpha\}})} = W - (\langle 0, \alpha, \Pi \rangle)$.

4 Properties and examples of DUCS copulas

DUCS copulas are based on generators from $\mathcal{F}$ and distortions from $\mathcal{D}$. The structure of $\mathcal{F}$, especially construction methods for generators, were deeply studied in [1]. Concerning the distortions set $\mathcal{D}$, we have the next important result.

Proposition 4.1 *Let $A\colon [0,1]^n \to [0,\infty]$ be a continuous idempotent homogeneous aggregation function. Then for any $d_1, \cdots, d_n \in \mathcal{D}$, also the function $d\colon [0,1] \to [0,1]$ given by $d(x) = A(d_1(x), \cdots, d_n(x))$ is a distortion.*

As a corollary of Proposition 4.1, $\mathcal{D}$ is a convex class which is also a lattice with top element d^* and bottom element d_*. Note that a similar conclusion holds for the class of DUCS copulas with a fixed generator f.

Corollary 4.2 *Let $f \in \mathcal{F}$ and $d_1, d_2 \in \mathcal{D}$. For DUCS copulas $C_{(f,d_1)}$ and $C_{(f,d_2)}$, denote $C = C_{(f,d_1)} \vee C_{(f,d_2)}$ and $D = C_{(f,d_1)} \wedge C_{(f,d_2)}$. Then both C and D are DUCS copulas, $C = C_{(f,d_1 \vee d_2)}$ and $D = C_{(f,d_1 \wedge d_2)}$.*

For special distortions we can get interesting DUCS copulas.

Proposition 4.3 *Let $d^{(\lambda)} \in \mathcal{D}$ be given by $d^{(\lambda)}(x) = x^{\frac{1}{\lambda}}, \lambda \in [1,\infty[$. Then, for any $f \in \mathcal{F}$, $C_{(f,d^{(\lambda)})} = C_{(f^\lambda)}$.*

Proposition 4.4 *Let $d_{[\alpha]} \in \mathcal{D}$ be given by $d_{[\alpha]}(x) = \frac{x}{\alpha} \wedge 1, \alpha \in]0,1]$. Note that $d_{[\alpha]} = \tilde{d}_{\{\alpha\}}$, (see Example 3.3 iii). Then, for any $f \in \mathcal{F}$, $C_{(f,d_{[\alpha]})} = g - (\langle 0, \alpha, C_f \rangle)$, i.e., DUCS copula $C_{(f,d_{[\alpha]})}$ is a g-ordinal sum.*

Remark 4.5 *For any distortion* $d \in \mathcal{D}$ *and constant* $\alpha \in]0,1]$, *the function* $d_{\langle\alpha\rangle} \colon [0,1] \to [0,1]$ *given by* $d_{\langle\alpha\rangle}(x) = \frac{d(\alpha x)}{d(\alpha)}$ *is also a distortion (formally, conditional distortion), and* $(\tilde{d}_{\langle\alpha\rangle})(x) = \frac{\tilde{d}(\alpha x)}{\tilde{d}(\alpha)}$, *i.e.,* $(\tilde{d}_{\langle\alpha\rangle}) = (\tilde{d})_{\langle\alpha\rangle}$. *After some processing concerning the univariate conditioning, see [14], it can be shown that the left conditioning with threshold* α *of DUCS copula* $C_{(f,d)}$ *is just the DUCS copula* $C_{(f,d_{\langle\alpha\rangle})}$, *i.e.,*$(C_{(f,d)})_{(\alpha)} = C_{(f,d_{\langle\alpha\rangle})}$.

Moreover, $d_{\langle\alpha\rangle} = d$ *for all* $\alpha \in]0,1]$ *yields the Cauchy equation* $d(\alpha x) = d(\alpha)d(x)$, *with solution* $d(x) = x^{\frac{1}{\lambda}}$, $\lambda \in [1,\infty[$, *i.e.* $d = d^{(\lambda)}$, *see Proposition 4.3. Thus the only univariate conditioning invariant DUCS copulas are copulas* $C_{(f,d^{(\lambda)})} = C_{(f^{\lambda})}$.

For more details and proofs we recommend [15].

ACKNOWLEDGEMENT

This work was partially supported by APVV LPP - 0004-07, VEGA 1/0496/08, APVV-0443-07.

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Convex combination in dependence structure of Archimax copulas and comparison study on real data

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November 4, 2010

Abstract

In this paper we focus on asymmetric copulas that are able to model relationship of non-exchangeable random variables. Investigation is aimed at relation between powers of aditive generators and dependence functions of Archimax copulas. We also propose some new construction methods for dependence functions and give application to real hydrological data.

1 Introduction

Copula is a function which allows modelling dependence structure between stochastic variables. In recent years copulas turned out to be a promising tool in multivariate modelling, mostly with applications in actuarial sciences and hydrology. The main advantage is that the copula approach can split the problem of constructing multivariate distributions into a part containing the marginal distribution functions and a part containing the dependence structure. These two parts can be studied and estimated separately and then rejoined to form a multivariate distribution function.

In the simplest, bivariate case, copula is a function $C\colon [0,1]^2 \to [0,1]$ which satisfies the boundary conditions, $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for $t \in [0,1]$ (uniform margins), and the 2-increasing property, $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ for all $u_1 \leq u_2, v_1 \leq v_2 \in [0,1]$. Copula is symmetric if $C(u,v) = C(v,u)$ for all $(u,v) \in [0,1]^2$ - then it can model exchangeable random variables - otherwise is asymmetric.

In the contribution we summarize our recent research, for more proofs and examples please refer to [3].

2 Archimax copulas

The most used symmetric models are Archimedean copulas [14], i.e., copulas $C_\varphi\colon [0,1]^2 \to [0,1]$ expressable in the form

$$C_\varphi(u,v) = \varphi^{(-1)}\left(\varphi(u) + \varphi(v)\right), \tag{1}$$

where $\varphi\colon [0,1] \to [0,\infty]$ is a continuous strictly decreasing convex function satisfying $\varphi(1) = 0$ (such φ is called a generator), and its pseudo-inverse $\varphi^{(-1)}\colon [0,\infty] \to [0,1]$ is given by

$$\varphi^{(-1)}(t) = \varphi^{-1}\left(\min(\varphi(0), t)\right). \tag{2}$$

For nice overview of fitting Archimedean copulas to real data we recommend [5] and references therein.

Going further, among few classes of asymmetric copulas, convenient enough to model non-exchangeable random variables, we focus on the class of Archimax copulas [4] built up from a generator φ and a convex function $A\colon [0,1] \to [0,1]$, $\max(t, 1-t) \le A(t) \le 1$ for all $t \in [0,1]$, called dependence function. Then the corresponding Archimax copula is given by

$$C_{\varphi,A}(u,v) = \varphi^{(-1)}\left[(\varphi(u)+\varphi(v))\, A\left(\frac{\varphi(u)}{\varphi(u)+\varphi(v)}\right)\right] \quad \text{for all } u,v \in [0,1] \tag{3}$$

(with conventions $0/0 = \infty/\infty = 0$, where $\varphi^{(-1)}$ is given by (2). Observe that Archimax copulas contains as special subclasses all Archimedean copulas (then $A \equiv 1$) and all extreme value copulas [15], in short EV copulas (then $\varphi(t) = -\log(t)$). For the weakest dependence function $A = A_*$, $A_*(t) = \max(t, 1-t)$, we have $C_{\varphi,A_*} = M$, the strongest copula of comonotone dependence, independently of the generator φ. Moreover, it is easy to check that an Archimax copula $C_{\varphi,A}$ is symmetric if and only if $A(t) = A(1-t)$ for all $t \in [0,1]$ (i.e., A is symmetric wrt. axis $x = 1/2$).

Now, to introduce new facts, suppose that φ is a generator of a copula C_φ. Then also φ^λ, $\lambda > 1$, is a generator of a copula C_{φ^λ}. As an example recall the Gumbel family of copulas $\left(C^G_{(\lambda)}\right)_{\lambda\in[1,\infty[}$, generated by generators $\varphi^G_{(\lambda)}\colon\ [0,1] \to [0,\infty]$, $\varphi^G_{(\lambda)}(t) = (-\log t)^\lambda$, which bears from the product copula Π generated by $\varphi^G_{(1)}$.

Proposition 2.1. *Let $\varphi\colon\ [0,1] \to [0,\infty]$ be a generator of a copula C_φ. For any dependence function A, and any $\lambda \ge 1$, the Archimax copula $C_{\varphi^\lambda,A}$ is also an Archimax copula based on generator φ, i.e., $C_{\varphi^\lambda,A} = C_{\varphi,B_{(A,\lambda)}}$, where $B_{(A,\lambda)}\colon\ [0,1] \to [0,1]$ is a dependence function given by*

$$B_{(A,\lambda)}(t) = A_{(\lambda)}(t)\left[A\left(\left(\frac{t}{A_{(\lambda)}(t)}\right)^\lambda\right)\right]^{1/\lambda}, \tag{4}$$

with $A_{(\lambda)}\colon\ [0,1] \to [0,1]$, $A_{(\lambda)}(t) = \left(t^\lambda + (1-t)^\lambda\right)^{1/\lambda}$.

Dependence function $A_{(\lambda)}$, $\lambda \in [0,1]$, are called Gumbel dependence functions due to the fact that $C^G_{(\lambda)} = C_{\varphi^G_{(1)},A_{(\lambda)}}$. Observe that the Archimedean copula C_{φ^λ} is just an Archimax copula based on φ and $A_{(\lambda)}$, $C_{\varphi^\lambda} = C_{\varphi,A_{(\lambda)}}$, independently of the generator φ. Proposition 2.1 has an important impact for the structure of Archimax copulas. For any generator $\varphi\colon\ [0,1] \to [0,\infty]$, classes $\mathcal{A}_{\varphi^\lambda}$ of Archimax copulas based on generator φ^λ, $\lambda \in [1,\infty[$, are nested, and $\mathcal{A}_{\varphi^\lambda} \subsetneqq \mathcal{A}_{\varphi^\mu}$ whenever $1 \le \mu < \lambda \le \infty$, where $\mathcal{A}_{\varphi^\infty} = \bigcap_{\lambda=1}^\infty \mathcal{A}_{\varphi^\lambda} = \{M\}$. Therefore it is important to know the basic form η of each generator φ, $\varphi = \eta^\lambda$ with $\lambda \ge 1$, where $\eta\colon\ [0,1] \to [0,\infty]$ is a generator such that for any $\lambda \in]0,1[$, η^λ is no more convex. Such generators η will be called basic generators and they correspond to Archimedean copulas C_η such that for any $p > 1$, the corresponding L_p-norm $\|C_\eta\|_p > 1$ (for more details we recommend [1, 13]).

Proposition 2.2. *Let $\varphi\colon\ [0,1] \to [0,\infty]$ be a generator. Let $\alpha = \inf\left\{\frac{\varphi(x)\varphi''(x)}{(\varphi'(x))^2} \,\middle|\, x \in]0,1[\text{ and } \varphi'(x), \varphi''(x) \text{ exist}\right\}$. Then $\eta = \varphi^{1/p}$, where $p = \frac{1}{1-\alpha}$, is a basic generator.*

Based on Propositions 2.1 and 2.2, we propose to fit Archimax copulas based on basic generators η only. Thus before choosing the appropriate candidates for fitting of a generator, one should check their basic forms. The next lemma gives a sufficient condition for a generator η to be basic.

Lemma 2.3. *Let $\eta\colon\ [0,1] \to [0,\infty]$ be a generator and let $\eta'(1^-) \neq 0$. Then η is a basic generator.*

Proof. Due to continuity of η and $\eta(1) = 0$, if $\eta'(1^-) \neq 0$ then $\alpha = \inf \Big\{ \frac{\eta(x)\eta''(x)}{(\eta'(x))^2} \Big| \ x \in]0,1[$ and $\eta'(x), \eta''(x)$ exist$\Big\} = 0$ and thus $p = 1$. □

Example 2.4.

(i) *For each Gumbel generator $\varphi^G_{(\lambda)}$, the corresponding basic generator is $\eta = \varphi^G_{(1)}$ (the generator of the product copula).*

(ii) *The weekest copula $C^{(p)}$ which has minimal L_p-norm, $\|C^{(p)}\|_p = 1$, $p \in [1, \infty[$, is an Archimedean copula generated by a generator $\varphi^Y_{(p)} : \ [0,1] \to [0, \infty]$, $\varphi^Y_{(p)}(x) = (1-x)^p$ (Y stands for Yager family, see [16], more details on L_p-norms and copulas can be found in [1]). Again, for any $p \in [1, \infty[$, the corresponding basic generator $\eta = \varphi^Y_{(1)}$ is unique (generator of the lower Frechet-Hoeffding bound W).*

(iii) *Based on Lemma 2.3 one can quickly check that the families of Clayton, Frank, Ali-Mikhail-Haq (see [9, 14]), are generated by basic generators only.*

(iv) *From two-parameter families given in [9], for example BB1 generator $\varphi(t) = (t^{-a} - 1)^b$ with $a > 0, b \geq 1$ gains its basic form only for $b = 1$, and thus would result in strict Clayton copula.*

3 Some construction methods for dependence functions

Based on some known dependence functions, it is desirable to be able to construct new dependence functions to increase the fitting potential of our Archimax copulas buffer. Recall that for dependence functions $A_1, \ldots, A_n$ also their convex sum $A = \sum_{i=1}^n \lambda_i A_i$ is a dependence function. Inspired by the bivariate construction [10] and based on the recent results [12], consider dependence functions $A_1, \ldots, A_n$. Then the corresponding EV copulas $C_{A_1}, \ldots, C_{A_n} \colon \ [0,1]^2 \to [0,1]$ are given by

$$C_{A_i}(u,v) = \exp\left((\log u + \log v) A_i \left(\frac{\log u}{\log u + \log v} \right) \right). \tag{5}$$

Take arbitrary two probability vectors $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0,1]^n$, $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$. Then due to [12] the function $C \colon \ [0,1]^2 \to [0,1]$ given by

$$C(u,v) = \prod_{i=1}^n C_i \left(u^{a_i}, v^{b_i} \right) \tag{6}$$

is also a copula. Note that EV copulas are characterized by the power stability $C\left(u^\lambda, v^\lambda\right) = (C(u,v))^\lambda$ for any $\lambda \in]0, \infty[$, $u, v \in [0,1]$. It is then easy to see that C given by (6) is also an EV copula, and thus there is a dependence function A so that $C = C_A$. For processing purposes, denote $t = \frac{\log u}{\log u + \log v}$. Then $\log v = \frac{1-t}{t} \log u$ and $\log u + \log v = \frac{\log u}{t}$. Moreover,

$$C(u,v) = \exp\left((\log u + \log v) A \left(\frac{\log u}{\log u + \log v} \right) \right) = \exp\left(\frac{\log u}{t} A(t) \right). \tag{7}$$

On the other hand, due to (6),

$$\begin{aligned} C(u,v) &= \prod_{i=1}^n \exp\left(\left(a_i \log u + b_i \frac{1-t}{t} \log u \right) A_i \left(\frac{a_i \log u}{a_i \log u + b_i \frac{1-t}{t} \log u} \right) \right) \\ &= \exp\left(\frac{\log u}{t} \sum_{i=1}^n (t a_i + (1-t) b_i) A_i \left(\frac{t a_i}{t a_i + (1-t) b_i} \right) \right). \end{aligned} \tag{8}$$

Comparing (7) and (8), we see that

$$A(t) = \sum_{i=1}^{n} (ta_i + (1-t)b_i)\, A_i \left(\frac{ta_i}{ta_i + (1-t)b_i} \right). \tag{9}$$

What was just shown is the following construction method.

Proposition 3.1. *Let $A_1, \ldots, A_n$ be dependence functions. Then for any probability vectors $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0,1]^n$, also the function $A\colon [0,1] \to [0,1]$ given by (9) is a dependence function.*

Observe that the formula (9) can be deduced by induction from the original formula given in [10], see also [8] dealing with A_1, A_2 and $\alpha, \beta \in [0,1]$. Then the function $A\colon [0,1]^2 \to [0,1]$ given by

$$\begin{aligned} A(t) &= (\alpha t + \beta(1-t))\, A_1 \left(\frac{\alpha t}{\alpha t + \beta(1-t)} \right) + \\ &+ ((1-\alpha)t + (1-\beta)(1-t))\, A_2 \left(\frac{(1-\alpha)t}{(1-\alpha)t + (1-\beta)(1-t)} \right) \end{aligned}$$

is a dependence function. Moreover, if $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then the formula (9) turns into the standard convex sum $A(t) = \sum_{i=1}^{n} a_i A_i(t)$. Evidently, this method allows to introduce asymmetric Archimax copulas even if starting from symmetric Archimax copulas.

Inspired by [2] where construction methods for generators of Archimedean copulas were discussed, we propose one more construction method for dependence function. For a dependence function A, denote by B a $[0,1] \to [0,1]$ function given by $B(t) = A(t) - 1 + t$. Each such B is characterized by its convexity, non-decreasingness and boundary conditions

$$\max(0, 2t-1) \le B(t) \le t.$$

The pseudo-inverse $B^{(-1)}\colon [0,1] \to [0,1]$ of B is given by

$$B^{(-1)}(u) = \sup\{t \in [0,1] \,|\, B(t) \le u\},$$

and it is characterized by concavity, non-decreasingness and boundary conditions

$$u \le B^{(-1)}(u) \le \frac{u+1}{2}. \tag{10}$$

Consider dependence functions $A_1, \ldots, A_n$ and related functions $B_1^{(-1)}, \ldots, B_n^{(-1)}$. Then the convex combination $\sum_{i=1}^{n} \lambda_i B_i^{(-1)}$ is concave, non-decreasing and satisfy the boundary conditions (10), and thus there is a dependence function A such that its related function $B^{(-1)}$ is just equal to $\sum_{i=1}^{n} \lambda_i B_i^{(-1)}$. This fact proves our next construction method.

Proposition 3.2. *Let $A_1, \ldots, A_n$ be dependence functions and let $(\lambda_1, \ldots, \lambda_n) \in [0,1]^n$ be a probability vector. Then the function $A\colon [0,1] \to [0,1]$ given by*

$$A(t) = \left(\sum_{i=1}^{n} \lambda_i B_i^{(-1)} \right)^{(-1)} (t) + 1 - t \tag{11}$$

is a dependence function.

4 Application

To examine performance of new models we consider two kinds of bivariate hydrological data. One is constitued by monthly average flow rate of two rivers (one is tributary to another) comprising 660 realisations, another sequence of 113 entries comes from annual summer term maxima of a river flow with corresponding flood volume (Figure 1).

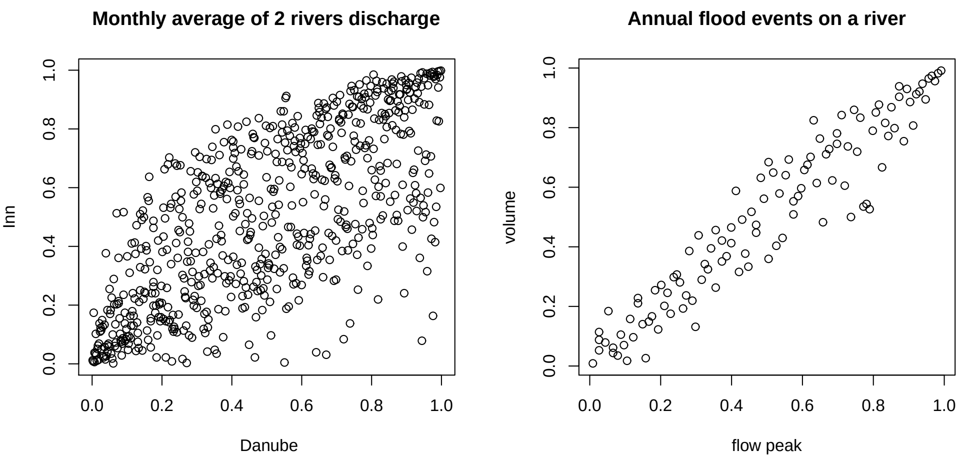

Figure 1: Scatter plot of data after transformation by corresponding empirical marginals.

family	generator $\varphi_{\boldsymbol{\theta}}(t)$	parameter range	limiting case (Archimed.)
Gumbel	$(-\log(t))^{\theta_1}$	$[1,\infty]$	$\{1\}$ Π, $\{\infty\}$ M
Clayton	$t^{-\theta_1}-1$	$]0,\infty]$	$\{0\}$ Π, $\{\infty\}$ M
Frank	$-\log\left(\frac{e^{-\theta_1 t}-1}{e^{-\theta_1}-1}\right)$	$\Re$	$\{-\infty\}$ W, $\{0\}$ Π, $\{\infty\}$ M
Joe	$-\log\left(1-(1-t)^{\theta_1}\right)$	$[1,\infty]$	$\{1\}$ Π, $\{\infty\}$ M
BB1	$(t^{-\theta_1}-1)^{\theta_2}$	$]0,\infty]\times[1,\infty]$	$\{0,1\}$ Π, $\{\infty,\infty\}$ M
	dependence function $A_{\boldsymbol{\theta}}(t)$		limiting case (EV)
Mixed	$\theta_1 t^2-\theta_1 t+1$	$[0,1]$	$\{0\}$ Π
Gumbel (logistic)	$\left(t^{\theta_1}+(1-t)^{\theta_1}\right)^{1/\theta_1}$	$[1,\infty]$	$\{1\}$ Π, $\{\infty\}$ M
Hüsler Reiss	$t*\Phi\left(\frac{1}{\theta_1}+\frac{\theta_1}{2\log(t/(1-t))}\right)+$ $+(1-t)\Phi\left(\frac{1}{\theta_1}-\frac{\theta_1}{2\log(t/(1-t))}\right)$ Φ is CDF of standard normal	$[0,\infty]$	$\{1\}$ Π, $\{\infty\}$ M
Tawn (asymmetric logistic)	$1-\theta_2+(\theta_2-\theta_1)t+$ $+\left((\theta_1 t)^{\theta_3}+(\theta_2(1-t))^{\theta_3}\right)^{\frac{1}{\theta_3}}$	$[0,1]\times[0,1]\times$ $\times[0,\infty]$	$\{0,0,1\}$ Π, $\{1,1,\infty\}$ M
LPL	$\begin{cases}1-\frac{1-b}{a}t & t\leq a-c\\ \frac{b-a}{1-a}+\frac{1-b}{1-a}t & t\geq a+c\\ At^2+Bt+C & \text{otherwise}\end{cases}$ $A=\frac{(1-b)}{4(1-a)ac}$ $B=\frac{2(1-b)(2ac-a-c)}{4(1-a)ac}$ $C=\frac{2(1+b)ac+(1-b)c^2-(b+4c-1)a^2}{4(1-a)ac}$ $a=\theta_1,\ c=\theta_3\min(a,1-a)$ $b=\max(a,1-a)(1-\theta_2)+\theta_2$	$[0,1]\times[0,1]\times$ $\times[0,1]$	$\{0,.,.\}$ $\{1,.,.\}$ $\{.,1,.\}$ Π $\{0.5,0,0\}$ M

Table 1: Overview of parametric families used to construct Archimax copula.

Tables 2 and 3 summarize competition of new construction methods alongside well-established models (for overview see Table 1, [14][9]) and related construction methods [2]. Besides parameters and maximized value of log-likelihood function [7] we provide the corresponding estimation time and test statistic S_n of GOF test [6] (not p-values which are of lesser efficiency for comparison purposes). Parameters were chosen from a grid of 21 values in each dimension of $\boldsymbol{\theta}$, specifically the sequence $seq = \{0, 0.05, \ldots, 1\}$ was stretched by formula $min + (max - min)seq^{pow}$ to fit a reasonable range of parameters for each model. Those parameters bounded by finite values min and max were left in regular grid, i.e. $pow = 1$, otherwise we replaced infinity by 10 and set $pow = 2$ so that grid thin out from point representing independence. Parameters other than bounded by unit interval were rounded to one decimal place. Only the range giving positive dependence is considered. Values in parentheses are fixed during estimation, square brackets indicate construction method of dependence function, in particular [bi] denotes biconvex combination given by Proposition 3.1 for $n = 2$, [li] represents special case when $a_i = b_i$ ($i = 1, 2$), and [inv] refers to Proposition 3.2. So far we implemented construction procedures for two dependence functions only and their individual parameters are estimated separately (in advance) from weighting parameters of their combination.

Software is implemented in R and published at www.math.sk/wiki/bacigal.

5 Conclusion

As seen from our results, given the two different data sets, the newly proposed construction methods give better fit in case of dependence functions with roughly equal fitting performance. Note that the best results for fixed number of parameters are given by Archimax construction with both generator and dependence function, from which we may judge that majority of well-established models in Archimedean and EV branch capture mutually different dependence structure, in other words they complement one another well. The few exceptions that follow from Proposition 2.1 are equivalences of Archimedean copula with Gumbel generator and EV copula with Gumbel dependence function, or equivalence of BB1 and Archimax copula with Clayton generator and Gumbel dependence function. In our software actually the estimation of Archimedean part is faster which may evoke a demand for some alternative to Proposition 2.1 in reverse order.

ACKNOWLEDGEMENT

This work was partially supported by APVV-LPP-0004-07, VEGA 1/0080/10, SK-AT-0020-08 and ”Program na podporu mladych vyskumnikov, STU v Bratislave”.

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generator		dependence function		log-lik	time	GOF
family	par.	family	par.	$L(\boldsymbol{\theta})$	[sec]	S_n
Gumbel	1.8			221.6	3	0.2112
Clayton	1.6			199.5	3	0.2559
Frank	5.6			223.4	4	0.0943
Joe	2.1			171.6	4	0.6144
BB1	0.6 1.6			245.5	55	0.0460
		Mixed	1.00	216.0	3	0.2987
		Gumbel	1.8	221.6	4	0.2112
		HüslerReiss	1.6	215.6	14	0.2026
		LL	0.75 0.70 (0.05)	35.6	400	3.1640
		LPL sym.	(0.50) 0.10 0.80	221.1	400	0.1886
		LPL	0.55 0.05 0.75	215.6	8100	0.1359
		Tawn	0.70 1.00 2.4	244.4	6700	0.1874
Gumbel	1.2	Mixed	0.95	225.5	92	0.1256
Gumbel	1.8	Gumbel	1.1	224.0	110	0.1276
Gumbel	1.8	HüslerReiss	0.6	222.5	340	0.1266
Clayton	0.6	Mixed	0.85	*246.9*	90	0.0588
Clayton	0.6	Gumbel	1.6	245.6	107	0.0471
Clayton	0.6	HüslerReiss	1.2	242.1	330	0.0584
Frank	2.5	Mixed	0.85	239.4	107	0.0717
Frank	3.6	Gumbel	1.4	238.6	125	0.0673
Frank	4.2	HüslerReiss	0.9	236.3	330	0.0686
Joe	1.2	Mixed	0.95	215.8	65	0.2065
Joe	1.2	Gumbel	1.8	212.5	76	0.1341
Joe	1.2	HüslerReiss	1.2	203.8	230	0.3392
BB1	0.6 1.0	Mixed	0.85	246.9	2300	0.0588
BB1	0.6 1.0	Gumbel	1.6	245.5	2600	0.0460
BB1	0.6 1.6	HüslerReiss	0.0	245.5	7400	0.0460
Gum–Cla	0.95			229.6	9	0.1316
Gum–Fra	0.10			230.8	11	0.1080
Gum–Joe	1.00			221.6	10	0.2112
Cla–Fra	0.00			223.4	11	0.0943
Cla–Joe	0.10			220.4	10	0.1683
Fra–Joe	0.90			226.6	12	0.1133
BB1–Gum	1.00			245.5	10	0.0460
BB1–Cla	1.00			245.5	10	0.0460
BB1–Fra	1.00			245.5	12	0.0460
BB1–Joe	1.00			245.5	11	0.0460
		[li] Mix–Gum	0.00	221.6	22	0.2112
		[inv]	0.00	221.6	330	0.2112
		[bi]	0.10 0.00	221.6	450	0.2873
		[li] Mix–Hüs	0.35	218.2	36	0.2328
		[inv]	0.55	217.0	1800	0.2522
		[bi]	0.10 0.00	*225.0*	770	0.2767
		[li] Gum–Hüs	0.85	221.7	37	0.2098
		[inv]	0.90	221.7	1900	0.2103
		[bi]	0.85 0.95	222.0	790	0.2472
Clayton	0.6	[li] Mix–Gum	1.00	246.9	470	0.0588
Clayton	0.6	[li] Mix–Hüs	0.90	246.9	770	0.0649
Clayton	0.6	[li] Gum–Hüs	1.00	245.6	780	0.0471

Table 2: Estimation summary for 2 rivers flow rate.

generator		dependence function		log-lik	GOF
family	par.	family	par.	$L(\boldsymbol{\theta})$	S_n
Gumbel	4.8			128.4	0.0203
Clayton	4.2			93.8	0.1094
Frank	10.0			106.0	0.0885
Joe	6.1			112.1	0.0796
BB1	0.2 4.2			129.4	0.0201
		Mixed	1.00	66.5	0.5975
		Gumbel	4.8	128.4	0.0203
		HüslerReiss	4.9	128.7	0.0225
		LL	0.55 0.50 (0.05)	38.6	0.8873
		LPL sym.	(0.50) 0.00 0.50	112.4	0.0928
		LPL	0.50 0.00 0.50	112.4	0.0928
		Tawn	1.00 1.00 4.8	128.4	0.0203
Gumbel	2.8	Mixed	1.00	128.6	0.0198
Gumbel	4.8	Gumbel	1.0	128.4	0.0203
Gumbel	3.2	HüslerReiss	1.2	128.8	0.0188
Clayton	2.0	Mixed	1.00	108.5	0.0659
Clayton	0.2	Gumbel	4.2	129.4	0.0205
Clayton	0.4	HüslerReiss	4.2	129.6	0.0173
Frank	10.0	Mixed	1.00	128.5	0.0116
Frank	3.6	Gumbel	3.2	131.2	0.0132
Frank	4.2	HüslerReiss	3.0	*131.8*	0.0124
Joe	3.6	Mixed	1.00	117.4	0.0491
Joe	1.2	Gumbel	4.2	127.4	0.0216
Joe	1.2	HüslerReiss	4.2	127.5	0.0248
BB1	0.2 2.4	Mixed	1.00	129.6	0.0211
BB1	0.2 1.0	Gumbel	4.2	129.5	0.0185
BB1	0.2 2.4	HüslerReiss	1.6	130.0	0.0171
Gumbel	2.4	LPL sym.	(0.50) 0.00 0.85	130.2	0.0201
Clayton	1.6	LPL sym.	(0.50) 0.15 0.45	121.4	0.0299
Frank	7.2	LPL sym.	(0.50) 0.00 0.80	130.5	0.0137
Gum–Cla	1.00			128.4	0.0203
Gum–Fra	1.00			128.4	0.0203
Gum–Joe	1.00			128.4	0.0203
Cla–Fra	0.00			106.0	0.0885
Cla–Joe	0.00			112.1	0.0796
Fra–Joe	0.00			112.1	0.0796
BB1–Gum	1.00			129.4	0.0201
		[li] Mix–Gum	0.00	128.4	0.0203
		[inv]	0.00	128.4	0.0203
		[bi]	0.00 0.00	128.4	0.0203
		[li] Mix–Hüs	0.00	128.7	0.0225
		[inv]	0.00	128.7	0.0225
		[bi]	0.00 0.00	128.7	0.0225
		[li] Gum–Hüs	0.30	128.7	0.0219
		[inv]	0.25	128.7	0.0220
		[bi]	0.85 0.90	*129.6*	0.0254

Table 3: Estimation summary for summer flood data.

Characterization of the Unit-Ball of the Discrepancy Norm

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Abstract

Hermann Weyl's discrepancy norm turns out to show interesting mathematical properties as dissimilarity measure of signals.
Of special interest is to study the dissimilarity of time or space shifted signals. For the discrepancy norm monotonicity and a Lipschitz property can be proven at least for non-negative signals as encountered in image processing.
The motivating question is whether there are also other norms having such properties. In this talk a deeper insight into this norm is outlined by studying its unit ball as geometric object and by showing how it can be understood as the result of a projection of the unit cube in an extended space.