Unit 1

Motivation and Basics of Classical Logic

Motivation

- In our everyday life, we use vague, qualitative, im-precise linguistic terms like "small", "hot", "around two o'clock"

- Even very complex and crucial human actions are de-cisions are based on such concepts, e.g. in

  - Process control

  - Driving

  - Financial/business decisions

  - Law and justice

Motivation (cont'd)

- These imprecise terms and the way they are processed, therefore, play a crucial role in everyday life

- To have a mathematical model which is able to express the complex semantics of such terms, hence, would lead to more intelligent systems and open completely new opportunities

- Concepts in classical mathematics and technology are inadequate to provide such models

L. A. Zadeh (1965)

"More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. [. . .] Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction."

Fuzzy Logic and Fuzzy Sets

- Fuzzy logic is a generalized kind of logic

- Fuzzy sets are the key to the semantics of vague lin-guistic terms

- Fuzzy sets are based on fuzzy logic

To have a solid basis for studying fuzzy logic and fuzzy sets, a short overview of classical logic and set theory is necessary.

Classical Two-Valued Logic

- The set of truth values consists of two elements: $\{0, 1\}$

- There are two basic binary operations $\wedge, \vee$ and the unary complement $\neg$

- Other logical operations, e.g. the implication $\rightarrow$, the logical equivalence $\leftrightarrow$, and the exclusive or, can be constructed from the three basic operations $\wedge, \vee, \neg$

Propositional Logic

- Propositions are either true or false

- A proposition may either be an atomic propositional variable $p_1, p_2, \ldots$ or a compound expression $(p \wedge q)$, $(p \vee q)$, or $\neg p$, where p and q are propositions

- The truth value of a proposition is evaluated by assigning a truth value to each propositional variable and evaluating the formula "from the inside to the outside" applying the logical operations

Truth Tables of the Three Basic Logical Operations

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

p	$\neg p$
0	1
1	0

Basic Properties

The following holds for all assignments of p, q, and r:

1. $p \wedge q = q \wedge p$, $p \vee q = q \vee p$ (commutativity)

2. $p \wedge (q \wedge r) = (p \wedge q) \wedge r$, $p \vee (q \vee r) = (p \vee q) \vee r$ (associativity)

3. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (distributivity)

4. $p \wedge 1 = p$, $p \vee 0 = p$ (neutral elements)

5. $p \wedge 0 = 0$, $p \vee 1 = 1$ (absorption)

6. $p \wedge p = p$, $p \vee p = p$ (idempotence)

7. $\neg(\neg p) = p$ (involution)

8. $\neg(p \wedge q) = \neg p \vee \neg q$, $\neg(p \vee q) = \neg p \wedge \neg q$ (De Morgan laws)

9. $p \wedge \neg p = 0$, $p \vee \neg p = 1$ (excluded middle)

Basics of Set Theory

- A set A is a collection of objects belonging to a given universe X, where, for each possible object x from the universe X, it is decidable whether it belongs to the set A or not; if so, we say $x \in A$; if not, we say $x \notin A$

- A set A is a subset of B if all objects in A are in B as well; if so, we say $A \subseteq B$; we say $A \subset B$ if there is at least one element in B which is not in A

- The set of all subsets of X is denoted with $\mathcal{P}(X)$

- The empty set, which does not contain any object, is denoted with $\emptyset$

Operations on Sets

- The intersection of two sets A and B is the set of objects from X which belong both to A and B; we denote this set with $A \cap B$

- The union of two sets A and B is the set of objects from X which belong at least to one of the sets A and B; we denote this set with $A \cup B$

- The complement of a set A is the set of objects from X which do not belong to A; we denote this set with C_A

Connection to Logic

- $A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$

- $A \cup B = \{x \in X \mid x \in A \vee x \in B\}$

- $\complement A = \{x \in X \mid x \notin A)\} = \{x \in X \mid \neg(x \in A)\}$

- $A \subseteq B$ if and only if $(x \in A) \to (x \in B)$ for all $x \in X$

Basic Properties

The following holds for all sets A, B, C on the same universe X:

1. $A \cap B = B \cap A$, $A \cup B = B \cup A$ (commutativity)

2. $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity)

3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributivity)

4. $A \cap X = A$, $A \cup \emptyset = A$ (neutral elements)

5. $A \cap \emptyset = \emptyset$, $A \cup X = X$ (absorption)

6. $A \cap A = A$, $A \cup A = A$ (idempotence)

7. $\complement(\complement A) = A$ (involution)

8. $\complement(A \cap B) = \complement A \cup \complement B$, $\complement(A \cup B) = \complement A \cap \complement B$ (De Morgan laws)

Characteristic Function

The characteristic function of a set A is defined as follows (for all $x \in X$):

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Connection to logic:

- $\chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x)$

- $\chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x)$

- $\chi_{\complement A}(x) = \neg \chi_A(x)$

- $A \subseteq B$ if and only if $(\chi_A(x) \to \chi_B(x)) = 1$ for all $x \in X$

What are Fuzzy Sets?

The idea behind fuzzy logic is to replace the set of truth values $\{0, 1\}$ by the entire unit interval $[0, 1]$.

A fuzzy set on a universe X is represented by a function which maps each element $x \in X$ to a degree of membership from the unit interval $[0, 1]$. These so-called membership functions are direct generalizations of characteristic functions.