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Unit 3

Fuzzy Relations

Motivation

Fuzzy relations and operations on fuzzy relations are the key to the thorough understanding of fuzzy inference. Their study, therefore, is indispensable.

We start with a short overview of classical relations in order to understand the background.

Classical Relations

Let X and Y be non-empty sets. A subset R of the Cartesian product $X \times Y$ is called a *relation from X to Y*. If $(x, y) \in R$, for some pair (x, y), we say that x is R-related to y.

If $X = Y$, i.e. R is a subset of $X \times X = X^2$, we say that R is a *binary relation on X*.

If, for any $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in R$, we call R a *function from X to Y*.

Example 1: "Owning Cars"

$X = \{$Alice, Bob, Christine, Daniel, Eva, Martin, Sylvia, Thomas$\}$
$Y = \{$BMW 3, Chrysler Voyager, Ford Focus, Mazda 6, Mercedes C, Opel Vectra, Toyota Corolla, Volvo V40, VW Passat$\}$

	BMW 3	Chrysler Voyager	Ford Focus	Mazda 6	Mercedes C	Opel Vectra	Toyota Corolla	Volvo V40	VW Passat
Alice	1	0	0	0	0	0	0	0	0
Bob	0	0	0	1	0	0	0	0	0
Christine	0	0	0	0	0	0	0	1	0
Daniel	0	1	1	0	0	0	0	0	0
Eva	0	0	0	1	0	0	0	0	0
Martin	1	0	0	0	0	0	0	0	0
Sylvia	0	0	1	0	0	0	0	0	1
Thomas	0	0	1	0	1	0	0	0	0

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Example 2: "Having a Relationship"

$X = Y = \{$Alice, Bob, Christine, Daniel, Eva, Martin, Sylvia, Thomas$\}$

	Alice	Bob	Christine	Daniel	Eva	Martin	Sylvia	Thomas
Alice	0	0	0	0	0	0	0	1
Bob	0	0	0	0	0	0	0	0
Christine	0	0	0	1	0	1	0	0
Daniel	0	0	1	0	0	0	0	0
Eva	0	0	0	0	0	0	0	0
Martin	0	0	1	0	0	0	1	0
Sylvia	0	0	0	0	0	1	0	0
Thomas	1	0	0	0	0	0	0	0

Example 3: “Close To”

$X = Y = \{1, \ldots, 8\}$

	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0
3	0	1	1	1	0	0	0	0
4	0	0	1	1	1	0	0	0
5	0	0	0	1	1	1	0	0
6	0	0	0	0	1	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	0	0	1	1

Example 4: "Equality"

$X = Y = \{1, \ldots, 8\}$

	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0
4	0	0	0	1	0	0	0	0
5	0	0	0	0	1	0	0	0
6	0	0	0	0	0	1	0	0
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

Set Operations on Relations

Let R and Q be relations from X to Y.

Intersection $R \cap Q$**:** x is $R \cap Q$-related to y if x is R-related to y *and* x is Q-related to y

Union $R \cup Q$**:** x is $R \cup Q$-related to y if x is R-related to y *or* x is Q-related to y

Complement $\complement R$**:** x is $\complement R$-related to y if x is *not* R-related to y

Inclusion: we say $R \subseteq Q$ if $(x, y) \in R$ always implies $(x, y) \in Q$

Inversion: if R is a binary relation on X, the inverse is defined as

Composition of Relations

Let R be a relation from X to Y and Q be a relation from Y to Z. The composition $R \circ Q$ is defined as a relation from X to Z in the following way:

$$R \circ Q = \{(x, z) \mid \text{there is a } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in Q\}$$

Example

$$X = \{a, b, c\},\ Y = \{r, s, t, u\},\ Z = \{x, y\}$$

R	r	s	t	u
a	1	0	1	0
b	1	1	0	0
c	0	0	0	1

Q	x	y
r	0	0
s	1	0
t	1	1
u	0	1

$$R \circ Q = ?$$

Properties of Binary Relations

Reflexivity: $(x, x) \in R$ for all $x \in X$

Irreflexivity: $(x, x) \notin R$ for all $x \in X$

Symmetry: if $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in X$

Asymmetry: if $(x, y) \in R$ then $(y, x) \notin R$ for all $x, y \in X$

Antisymmetry: if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$ for all $x, y \in X$

Transitivity: if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$ for all $x, y, z \in X$

Completeness: for all $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$

Example 3: “Close To”

$X = Y = \{1, \ldots, 8\}$

	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0
3	0	1	1	1	0	0	0	0
4	0	0	1	1	1	0	0	0
5	0	0	0	1	1	1	0	0
6	0	0	0	0	1	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	0	0	1	1

Which properties does this relation fulfill?

Example 4: “Equality”

$X = Y = \{1, \ldots, 8\}$

	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0
4	0	0	0	1	0	0	0	0
5	0	0	0	0	1	0	0	0
6	0	0	0	0	0	1	0	0
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

Which properties does this relation fulfill?

Example 5: “At most as large as”

$X = Y = \{1, \ldots, 8\}$

	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1
3	0	0	1	1	1	1	1	1
4	0	0	0	1	1	1	1	1
5	0	0	0	0	1	1	1	1
6	0	0	0	0	0	1	1	1
7	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	1

Which properties does this relation fulfill?

Example 5: "Much smaller than"

$X = Y = \{1, \ldots, 8\}$

	1	2	3	4	5	6	7	8
1	0	0	0	0	1	1	1	1
2	0	0	0	0	0	1	1	1
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	0	0	1
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0

Which properties does this relation fulfill?

Special Types of Binary Relations

Preordering:

reflexive and transitive

Weak Ordering:

reflexive, transitive, and complete

Equivalence Relation:

reflexive, symmetric, and transitive

Ordering:

reflexive, antisymmetric, and transitive

Linear Ordering:

reflexive, antisymmetric, transitive, and complete

Strict Ordering:

Some Basic Results

- R is transitive if and only if $R \circ R \subseteq R$
- $R \cup \{(x,x) \mid x \in X\}$ is reflexive
- $R \cap \complement\{(x,x) \mid x \in X\}$ is irreflexive
- $R \cap R^{-1}$ and $R \cup R^{-1}$ are symmetric
- $R \cap \complement R^{-1}$ is asymmetric
- $R \cup \complement R^{-1}$ is complete

Fuzzy Relations

Let X and Y be non-empty sets. A fuzzy subset R of the Cartesian product $X \times Y$ is called a *fuzzy relation from X to Y*. For $(x, y) \in R$, for some pair (x, y), $\mu_R(x, y)$ is the degree to which x is R-related to y.

If $X = Y$, i.e. R is a fuzzy subset of $X \times X = X^2$, we say that R is a *binary fuzzy relation on X*.

Set Operations on Fuzzy Relations

Let R and Q be fuzzy relations from X to Y and let (T, S, N) be a De Morgan triple.

T**-Intersection** $R \cap_T Q$**:** $\mu_{R \cap_T Q}(x, y) = T\big(\mu_R(x, y), \mu_Q(x, y)\big)$

S**-Union** $R \cup_S Q$**:** $\mu_{R \cup_S S}(x, y) = S\big(\mu_R(x, y), \mu_Q(x, y)\big)$

N**-Complement** $\complement_N R$**:** $\mu_{\complement_N R}(x, y) = N\big(\mu_R(x, y)\big)$

Inclusion: we say $R \subseteq Q$ if and only if $\mu_R(x, y) \leq \mu_Q(x, y)$ for all $(x, y) \in X \times Y$

Inversion: $\mu_{R^{-1}}(x, y) = \mu_R(y, x)$

Example 6: "Likes"

$X = Y = \{$Alice, Bob, Christine, Daniel, Eva, Martin, Sylvia, Thomas$\}$

	Alice	Bob	Christine	Daniel	Eva	Martin	Sylvia	Thomas
Alice	1.0	0.0	0.0	0.3	0.5	0.1	0.0	1.0
Bob	0.0	1.0	0.0	0.0	0.0	1.0	0.0	0.0
Christine	0.5	0.3	1.0	1.0	0.0	1.0	0.7	0.3
Daniel	0.3	0.2	1.0	1.0	0.5	0.0	0.2	0.5
Eva	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Martin	0.8	0.8	1.0	0.9	0.4	0.9	1.0	0.6
Sylvia	0.2	0.3	0.5	0.5	0.4	1.0	0.9	0.0
Thomas	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Composition of Relations

Let R be a fuzzy relation from X to Y and Q be a fuzzy relation from Y to Z. The T-composition $R \circ_T Q$ is defined as a fuzzy relation from X to Z in the following way:

$$\mu_{R \circ_T Q}(x, y) = \mathsf{sup}\{T\big(\mu_R(x, y), \mu_Q(y, z)\big) \mid y \in Y\}$$

Example

$$X = \{a, b, c\},\ Y = \{r, s, t, u\},\ Z = \{x, y\}$$

R	r	s	t	u
a	1.0	0.2	0.9	0.2
b	0.8	1.0	0.0	0.2
c	0.0	0.3	0.2	0.9

Q	x	y
r	0.3	0.4
s	1.0	0.6
t	0.9	0.8
u	0.3	1.0

$$R \circ_{T_{\mathbf{M}}} Q = ? \qquad R \circ_{T_{\mathbf{L}}} Q = ?$$

Properties of Binary Fuzzy Relations

Reflexivity: $\mu_R(x, x) = 1$ for all $x \in X$

Irreflexivity: $\mu_R(x, x) = 0$ for all $x \in X$

Symmetry: $\mu_R(x, y) = \mu_R(y, x)$ for all $x, y \in X$

T**-Asymmetry:** $T(\mu_R(x, y), \mu_R(y, x)) = 0$ for all $x, y \in X$

T**-Transitivity:** $T(\mu_R(x, y), \mu_R(y, z)) \leq \mu_R(x, z)$ for all $x, y, z \in X$

S**-Completeness:** $S(\mu_R(x, y), \mu_R(y, x)) = 1$ for all $x, y \in X$

Strong Completeness: $\max(\mu_R(x, y), \mu_R(y, x)) = 1$ for all $x, y \in X$

Example 4: "Similarity"

$X = Y = \{1, \ldots, 8\}$

	1	2	3	4	5	6	7	8
1	1.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0
2	0.5	1.0	0.5	0.0	0.0	0.0	0.0	0.0
3	0.0	0.5	1.0	0.5	0.0	0.0	0.0	0.0
4	0.0	0.0	0.5	1.0	0.5	0.0	0.0	0.0
5	0.0	0.0	0.0	0.5	1.0	0.5	0.0	0.0
6	0.0	0.0	0.0	0.0	0.5	1.0	0.5	0.0
7	0.0	0.0	0.0	0.0	0.0	0.5	1.0	0.5
8	0.0	0.0	0.0	0.0	0.0	0.0	0.5	1.0

Which properties does this relation fulfill?

Some Basic Results

Let (T, S, N) be a De Morgan triple and let $\bar{E}$ denote the classical equality.

- R is T-transitive if and only if $R \circ_T R \subseteq R$
- $R \cup_S \bar{E}$ is reflexive
- $R \cap_T \complement_N \bar{E}$ is irreflexive
- $R \cap_T R^{-1}$ and $R \cup_S R^{-1}$ are symmetric
- $R \cap_T \complement_N R^{-1}$ is T-asymmetric if $T(x, N(x)) = 0$ holds (first law of excluded middle)
- $R \cup_S \complement_N R^{-1}$ is strongly complete if $S(x, N(x)) = 1$ holds (second law of excluded middle)

Special Types of Binary Fuzzy Relations

T-Preordering:

reflexive and T-transitive

Weak T-Ordering:

reflexive, T-transitive, and strongly complete

T-Equivalence Relation:

reflexive, symmetric, and T-transitive

Strict T-Ordering:

irreflexive, T-asymmetric, and T-transitive

Other Classes: Fuzzy Orderings

Given a t-norm T and a T-equivalence E on a non-empty domain X, a fuzzy relation L on X is called *T-E-ordering* if and only if it is T-transitive and has the following two properties for all $x, y, z \in X$:

E-Reflexivity: $\mu_E(x, y) \leq \mu_L(x, y)$

T-E-Antisymmetry: $T\big(\mu_L(x, y), \mu_L(y, x)\big) \leq \mu_E(x, y)$

Other Classes: Resemblance Relations

Given a distance measure $d : X^2 \to \mathbb{R}_0^+$ on a non-empty domain X, a fuzzy relation R on X is called *d-resemblance relation* if and only if it is reflexive and symmetric and has the following property for all $x, y, r, s \in X$:

Compatibility: $d(x, y) \leq d(r, s)$ implies $\mu_R(x, y) \geq \mu_R(r, s)$

Images and Preimages of Functions

Let $f : X \rightarrow Y$ be a function and A be a subset of X. Then the *image of* A *w.r.t.* f is defined as follows:

$$f(A) = \{y \in Y \mid \text{ there is an } x \in A \text{ such that } y = f(x)\}$$

Let B be a subset of Y. Then the *preimage of* Y *w.r.t.* f is defined as

$$f^{-1}(B) = \{x \in X \mid \text{ there is a } y \in B \text{ such that } y = f(x)\}.$$

Images and Preimages of Relations

Let R be a relation from X to Y and A be a subset of X. Then the *image of* A *w.r.t.* R is defined as follows:

$$R(A) = \{y \in Y \mid \text{ there is an } x \in A \text{ such that } (x, y) \in R\}$$

Let B be a subset of Y. Then the *preimage of* Y *w.r.t.* R is defined as

$$R^{-1}(B) = \{x \in X \mid \text{ there is a } y \in B \text{ such that } (x, y) \in R\}.$$

Example

$$X = \{a, b, c\},\ Y = \{r, s, t, u\},\ A = \{a, b\},\ B = \{r\}$$

R	r	s	t	u
a	1	0	1	0
b	1	1	0	0
c	0	0	0	1

$$R(A) = ? \qquad R^{-1}(B) = ?$$

General Method to Compute $R(A)$

Suppose that R is a relation from X to Y and that A is a subset of X.

1. **Cylindric Extension:** Define a dummy relation R' from X to Y as
$$R' = \{(x, y) \mid x \in A\};$$
2. **Intersect:** Compute an intermediate relation $R'' = R \cap R'$;
3. **Project R'' onto Y:** Define a set B as
$$B = \{y \in Y \mid \text{ there is an } x \in X \text{ such that } (x, y) \in R''\};$$

The set B is then exactly $R(A)$.

General Method to Compute $R^{-1}(B)$

Suppose that R is a relation from X to Y and that B is a subset of Y.

1. **Cylindric Extension:** Define a dummy relation R' from X to Y as
$$R' = \{(x, y) \mid y \in B\};$$
2. **Intersect:** Compute an intermediate relation $R'' = R \cap R'$;
3. **Project R'' onto X:** Define a set A as
$$A = \{x \in X \mid \text{ there is a } y \in Y \text{ such that } (x, y) \in R''\};$$

The set A is then exactly $R^{-1}(B)$.

Example

$X = Y = \{1, \ldots, 8\}, A = \{4, 5, 7\}$

R	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1
3	0	0	1	1	1	1	1	1
4	0	0	0	1	1	1	1	1
5	0	0	0	0	1	1	1	1
6	0	0	0	0	0	1	1	1
7	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	1

$$R(A) = ?$$

Step 1: Cylindric Extension R'

$A = \{4, 5, 7\}$

R'	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
6	0	0	0	0	0	0	0	0
7	1	1	1	1	1	1	1	1
8	0	0	0	0	0	0	0	0

Step 2: Intersection $R'' = R \cap R'$

R	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1
3	0	0	1	1	1	1	1	1
4	0	0	0	1	1	1	1	1
5	0	0	0	0	1	1	1	1
6	0	0	0	0	0	1	1	1
7	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	1

R'	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
6	0	0	0	0	0	0	0	0
7	1	1	1	1	1	1	1	1
8	0	0	0	0	0	0	0	0

Step 3: Projection of R'' onto Y

R''	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	1	1	1	1	1
5	0	0	0	0	1	1	1	1
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	0

$$R(A) = \{4, 5, 6, 7, 8\}$$

Images and Preimages of Fuzzy Relations

Let R be a fuzzy relation from X to Y and A be a fuzzy subset of X. Then the T*-image of* A *w.r.t.* R is defined as follows:

$$\mu_{R_T(A)}(y) = \sup\{T(\mu_A(x), \mu_R(x,y)) \mid x \in X\}$$

Let B be a fuzzy subset of Y. Then the T*-preimage of* Y *w.r.t.* R is defined as

$$\mu_{R_T^{-1}(B)}(x) = \sup\{T(\mu_R(x,y), \mu_B(y)) \mid y \in Y\}$$

General Method to Compute $R_T(A)$

Suppose that R is a fuzzy relation from X to Y and that A is a fuzzy set on X.

1. **Cylindric Extension:** Define a dummy relation R' from X to Y as

$$\mu_{R'}(x, y) = \mu_A(x);$$

2. **Intersect:** Compute intermediate relation $R'' = R \cap_T R'$;
3. **Project R'' onto Y:** Define a fuzzy set B as

$$\mu_B(y) = \sup\{\mu_{R''}(x, y) \mid x \in X\}$$

The set B is then exactly $R_T(A)$.

General Method to Compute $R_T^{-1}(B)$

Suppose that R is a fuzzy relation from X to Y and that B is a fuzzy set on Y.

1. **Cylindric Extension:** Define a dummy relation R' from X to Y as
$$\mu_{R'}(x, y) = \mu_B(y);$$
2. **Intersect:** Compute intermediate relation $R'' = R \cap_T R'$;
3. **Project R'' onto X:** Define a fuzzy set A as
$$\mu_A(x) = \sup\{\mu_{R''}(x, y) \mid y \in Y\}$$

The set A is then exactly $R_T^{-1}(B)$.

Example

$$X = \{a, b, c\},\ Y = \{r, s, t, u\}$$

R	r	s	t	u
a	1.0	0.2	0.9	0.2
b	0.8	1.0	0.0	0.2
c	0.0	0.3	0.2	0.9

x	$\mu_A(x)$
a	1.0
b	0.4
c	0.0

y	$\mu_B(y)$
r	0.0
s	0.3
t	0.7
u	0.1

$$R_{T_{\mathbf{L}}}(A) = ? \qquad R^{-1}_{T_{\mathbf{L}}}(B) = ?$$

Example

$X = Y = \{1, \ldots, 8\}$

R	1	2	3	4	5	6	7	8
1	0.0	0.0	0.0	0.5	1.0	1.0	1.0	1.0
2	0.0	0.0	0.0	0.0	0.5	1.0	1.0	1.0
3	0.0	0.0	0.0	0.0	0.0	0.5	1.0	1.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.5	1.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

x	$\mu_A(x)$
1	0.0
2	0.4
3	1.0
4	0.5
5	0.0
6	0.0
7	0.0
8	0.0

$$R_{T_{\mathbf{M}}}(A) = ?$$

Step 1: Cylindric Extension R'

x	$\mu_A(x)$
1	0.0
2	0.4
3	1.0
4	0.5
5	0.0
6	0.0
7	0.0
8	0.0

R'	1	2	3	4	5	6	7	8
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
4	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Step 2: Intersection $R'' = R \cap_{T_\mathbf{M}} R'$

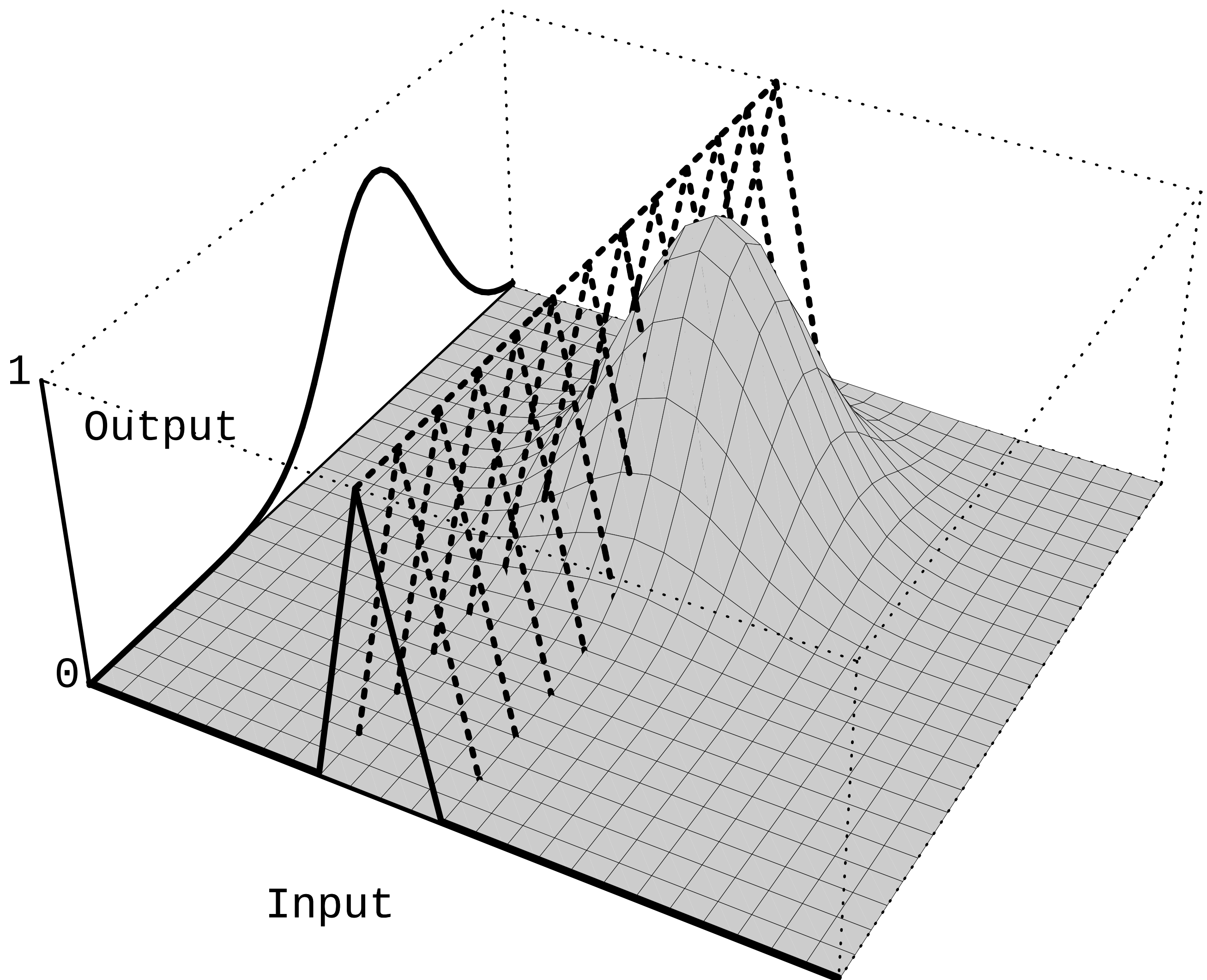
R''	1	2	3	4	5	6	7	8
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.4	0.4	0.4	0.4
3	0.0	0.0	0.0	0.0	0.0	0.5	1.0	1.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.5
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Step 3: Projection of R'' onto Y

R''	1	2	3	4	5	6	7	8
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.4	0.4	0.4	0.4
3	0.0	0.0	0.0	0.0	0.0	0.5	1.0	1.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.5
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

x	$\mu_{R_{T_{\mathbf{M}}}(A)}(x)$
1	0.0
2	0.0
3	0.0
4	0.0
5	0.4
6	0.5
7	1.0
8	1.0

Graphical Representation of $R_T(A)$



Example with Infinite X

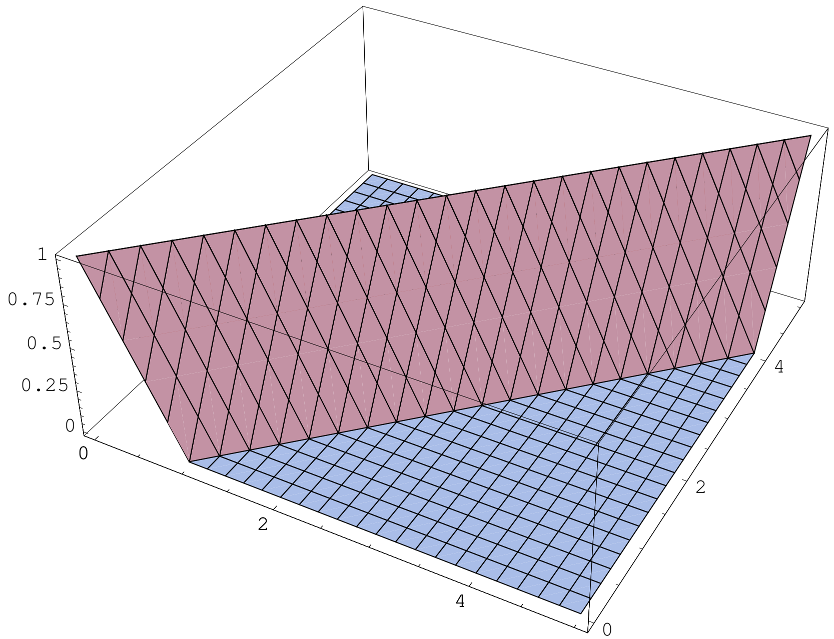
$$X = \mathbb{R}$$

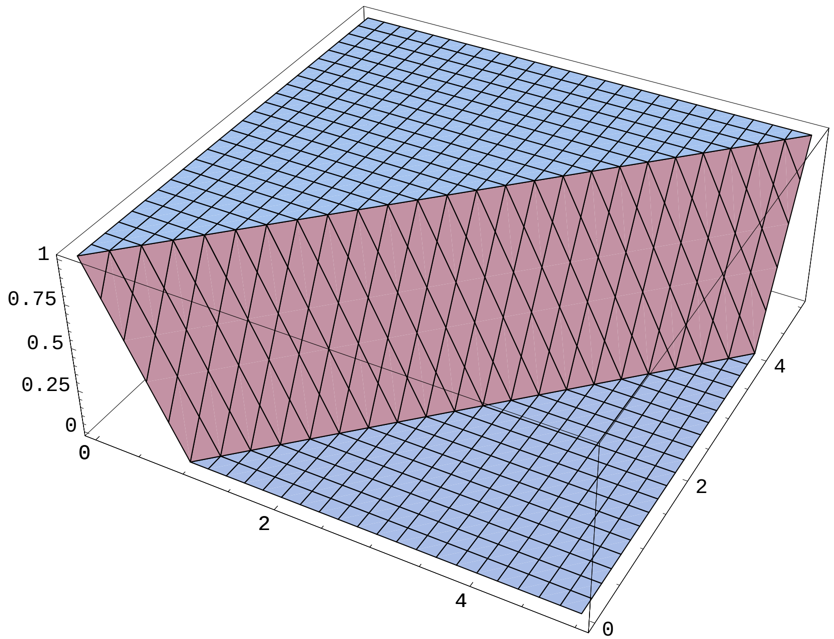
$$\mu_E(x,y) = \max(1 - |x-y|, 0)$$

$$\mu_L(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1 - x + y, 0) & \text{otherwise} \end{cases}$$

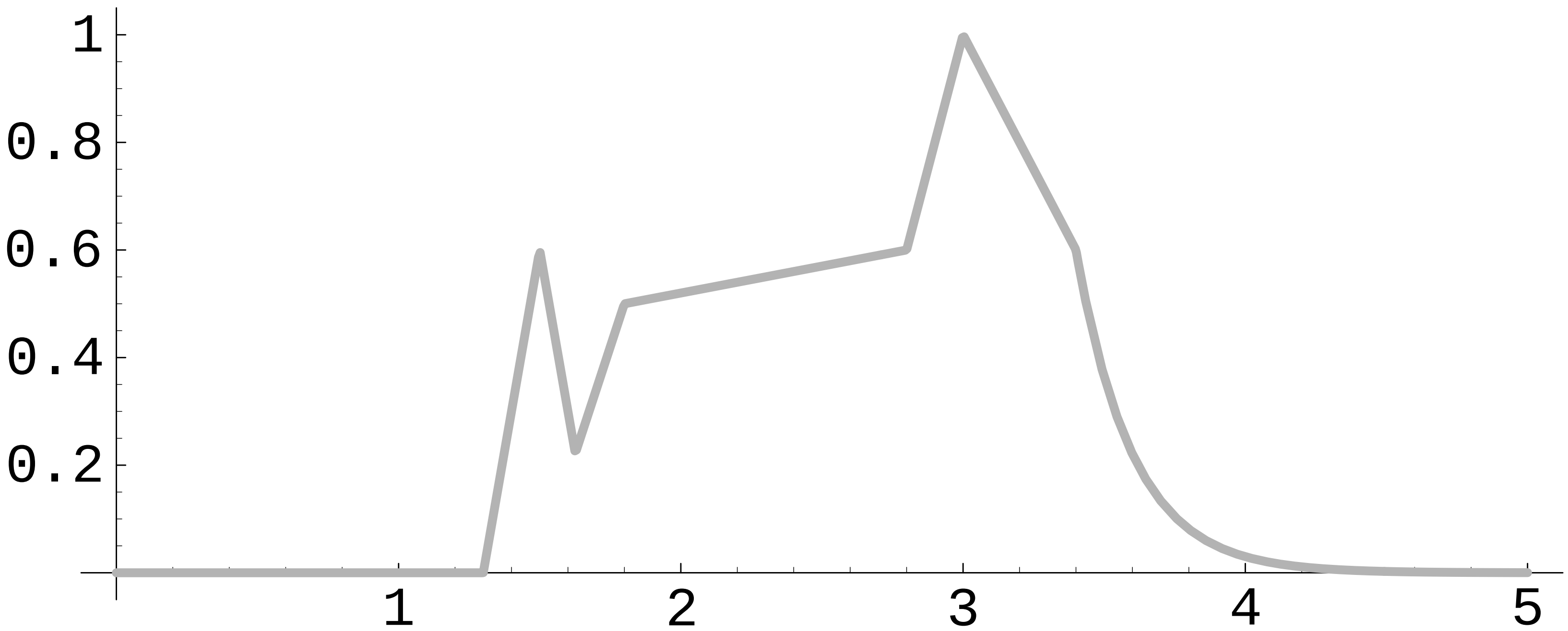
E is a $T_{\mathbf{L}}$-equivalence; L is a $T_{\mathbf{L}}$-E-ordering;

Graphical Representation of E and L

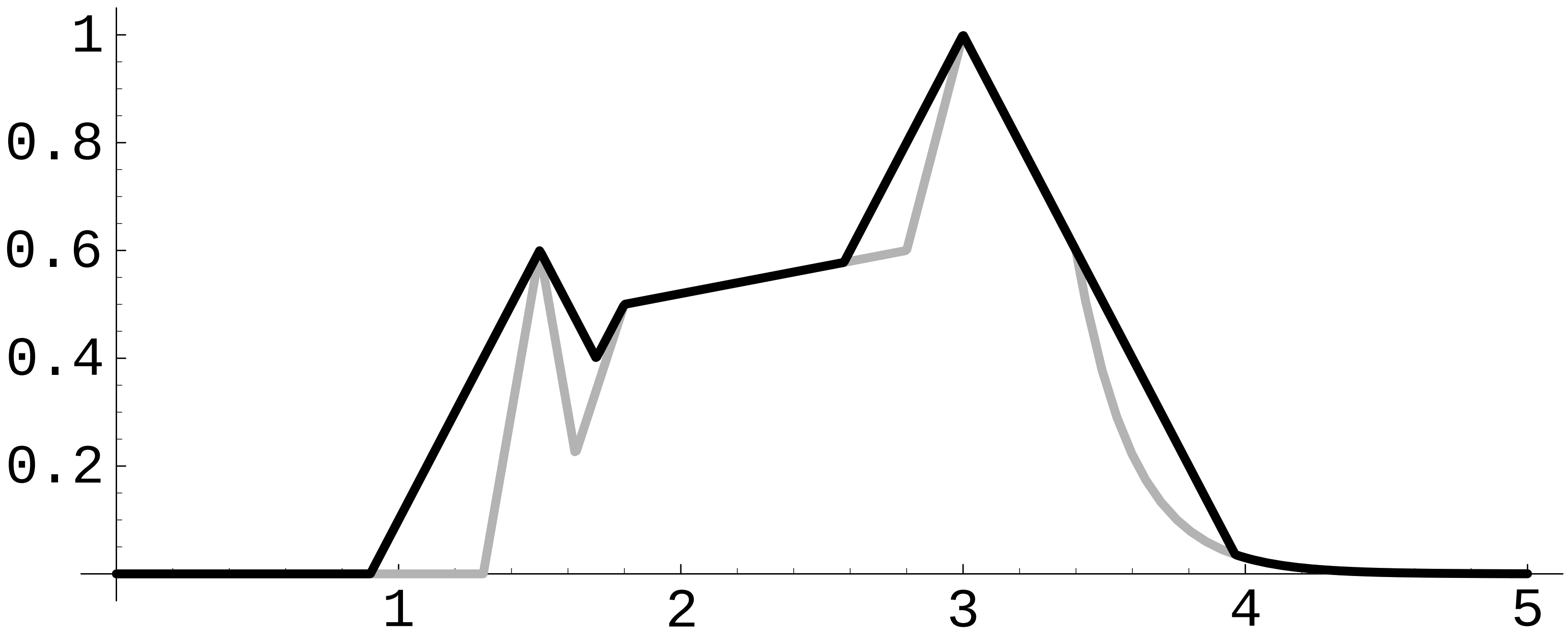




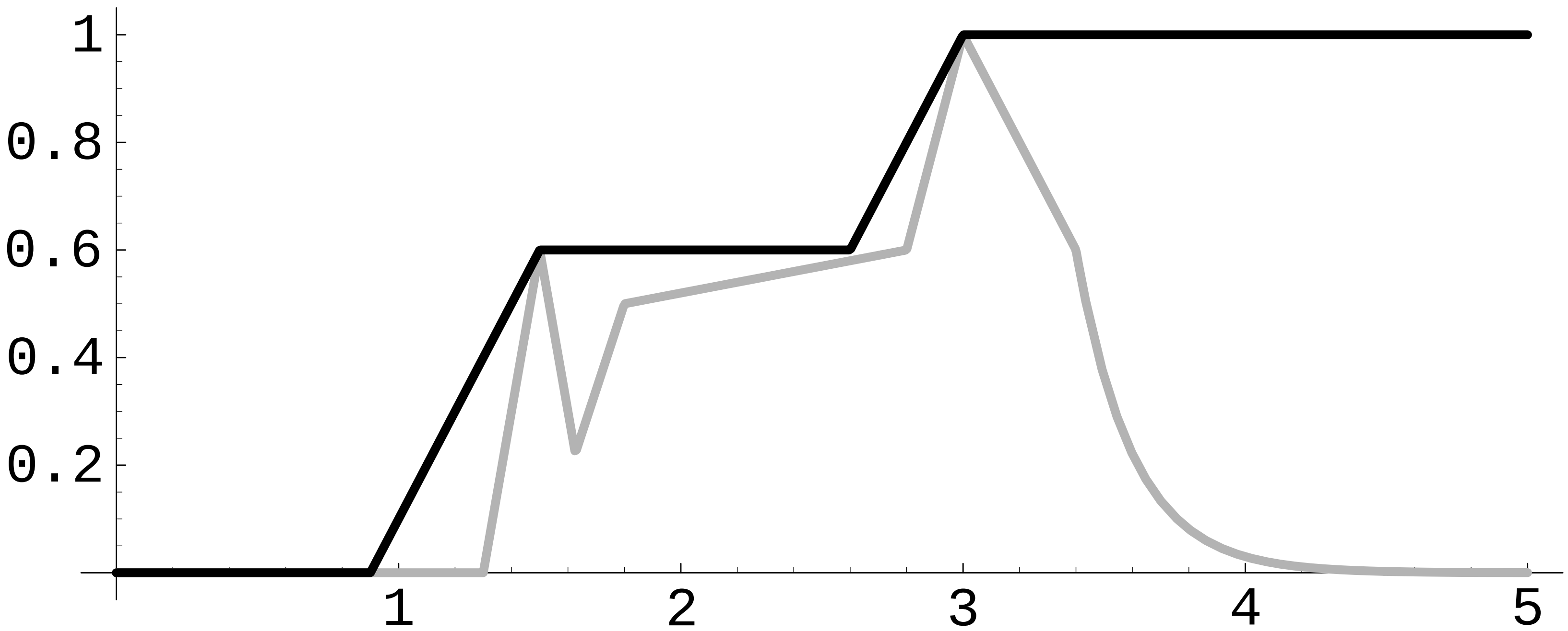
Example: Fuzzy Set A



Example: $E_{T_{\mathbf{L}}}(A)$



Example: $L_{T_{\mathbf{L}}}(A)$



Example: $L_{T_{\mathbf{L}}}^{-1}(A)$