Formal approaches to rule-based systems in medicine: the case of CADIAG-2

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Abstract

There is no established formal framework for expert systems based on weighted IF-THEN rules. We discuss three mathematical models that have been recently proposed by the authors for CADIAG-2—a well-known system of this kind. The three frameworks are based on fuzzy logics, probability theory and possibilistic logic, respectively. CADIAG-2 is used here as a case study to evaluate these frameworks. We point out their use, advantages and disadvantages. In addition, the described models provide insight into various aspects of CADIAG-2.

Keywords: Medical expert systems, t-norm based logics, probability theory, possibilistic logic, CADIAG-2

1. Introduction

Medical expert systems are intended to facilitate the clinical personnel's daily work by deriving useful information from patient data in an automated way. The range of their applicability is wide. Their original and still typical task is to suggest a patient's possible diagnoses on the basis of signs, findings, test results and symptoms.

The expert systems of the CADIAG family (where "CADIAG" abbreviates "Computer-Assisted DIAGnosis") are well-known examples along this line. These systems provide decision support in several fields of internal medicine. They have been developed since the early 80's under the supervision of K.-P. Adlassnig (Medical University of Vienna). In a first version, called CADIAG-1, relationships were formulated as IF-THEN rules and inferences were performed according to classical (two-valued)

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propositional logic [AKLGG]. Each rule had the form of an implication; for instance, a rule could express the fact that a particular pattern of signs and symptoms implies the presence of a disease.

However, the information available to physicians about a patient and about medical relationships is in general not fully determinate. To process information of a more general type, the successor system CADIAG-2 was designed to handle graded inputs and weighted rules [AdKo, AKS, AKSEG, AKSG, LAK]. The rules of CADIAG-2 are syntactically similar to those of the preceding system. However, a weight is associated with each IF-THEN rule, telling to which degree the rule is applicable. Consequently, CADIAG-2 can deal not only with sharp correlations, but also with those that are not so clear. Moreover, medical entities are assigned truth values that can vary continuously between the real numbers 0 and 1 rather than being limited to a simple "false" (i.e. 0) or "true" (i.e. 1). In particular, the system can deal not only with the presence or absence of a symptom, but also with borderline cases. Similarly, not only the clear certainty or exclusion of a diagnose is expressible, but also restricted certainty.

CADIAG-2 consists of a knowledge base and an inference engine. On the basis of initial information about a patient in the form of symptoms, signs, present diseases or any other kind of facts, CADIAG-2's inference engine applies successively the rules in the knowledge base and outputs a weighted list of possible diseases.

Although CADIAG-2 is designed in a transparent way and its overall accuracy, compared with the actual physicians' diagnoses, is high according to [AKSEG], there is no unique way to explain the meaning of the numerical weights associated to the generated diseases. Furthermore the question why these degrees arise in the way they do is open. A critical discussion can be found, e.g., in [DHN]. In fact, the method according to which weights associated to signs, symptoms, or diseases are to be treated depends on the tricky question which formal framework is appropriate for CADIAG-2 or, more generally, for expert systems based on weighted IF-THEN rules.

By a formal framework we mean an inference system based on clear principles in line with the ideas upon which the system is based. Only then, a precise understanding of its inferential mechanism becomes possible. Furthermore, a well-defined framework is a prerequisite for the possibility of checking whether the rules are free of contradiction (that is, whether they are *consistent*) and, provided they are not so, for possible repair strategies. Still another aspect concerns possible extensions or modifications of the knowledge base; the representation of the medical knowledge might be improved. Furthermore we might gain insights into the portability of the inference engine of the system; we recall in this connection the unexpected failure of the attempt to transfer the inferential mechanism of MYCIN—the forefather of all expert systems—to other areas [KrNg].

Many papers deal with foundational aspects of expert systems which, like CADIAG-2, process graded inputs and are based on weighted IF-THEN rules applied from symptoms to diseases. However, to establish a formal framework appropriate for CADIAG-2 turns out to be tricky. Let us mention, for instance, the notable work of J. Yen [Yen] using the Dempster-Shafer theory of evidence. Yen even directly refers to CADIAG-2; he uses a modified version of its knowledge base. Yen's expert system GERTIS is based on the idea to consider the weight of symptoms as belief and to use the diseases for the hypothesis space. The inference is not in accordance with the one of CADIAG-

2, important aspects not being taken into account (this actually was not Yen's aim). In fact, it is highly questionable to interpret the inference of CADIAG-2 in the framework of Dempster and Shafer's theory.

We conclude that there is no established formal framework for CADIAG-2 or related systems. This is a disadvantage indicated, e.g., in [DHN] as an open problem together with the question "what are the weights for diseases calculated by the system?". Based on the progress of recent years in the field of knowledge representation, we might, however, now have better chances to overcome the conceptual difficulties.

In this paper we take the system CADIAG-2 as a case study and discuss three formal frameworks recently introduced in the literature. We point out their use, advantages and disadvantages, and we identify which aspects have been worked on with success and which aspects remain a challenge.

CADIAG-2 and similar systems handle various types of degrees. Degrees are associated to symptoms to account for borderline cases, degrees are associated to diseases to account for graded certainty and degrees are associated to rules to account for non-strict relationships between medical facts. The three formal approaches we deal with can be distinguished by the way in which these degrees are understood and formally treated. In fact, each approach interprets them in a specific way and provides some insight into CADIAG-2.

The first approach discussed here (Section 3) stresses exclusively the aspect of vagueness. Indeed, CADIAG-2 is often viewed as a "fuzzy expert system", hence it seems natural to identify the inference engine of CADIAG-2 with a particular fuzzy logic in the sense of [Haj].

The second approach (Section 4) stresses the aspect of uncertainty, an aspect that is obviously involved in any prevision process. We interpret here CADIAG-2 probabilistically, motivated by the fact that the weights of the CADIAG-2 rules are often interpreted in accordance with probability theory or generalised versions of it.

The two interpretations above were successfully used for checking the consistency of the system's rules. The checks, described in [CiRu, KPP], allowed the discovery of various errors in the knowledge representation of the rules of CADIAG-2 and also led to concrete proposals on how to measure and repair the inconsistencies [Pic2].

The third approach (Section 5) is the only one which takes both vagueness and uncertainty into account within one single framework. The approach is based on a new logic, recently introduced in [ZeGo], which treats uncertainty along the line of Dubois and Prade's possibilistic logic.

In a concluding section (Section 6) we summarise achievements and open issues. We comment on the question how CADIAG-2 and related systems can profit from these formal frameworks and in which respects further efforts are still desirable.

2. A short introduction to CADIAG-2

In this section we briefly introduce the medical expert system CADIAG-2.

CADIAG-2 consists of a knowledge base and an inference engine. The knowledge base consists of a collection of weighted IF-THEN rules. The rules are based on causal, statistical, definitional or occasionally heuristical expert knowledge about the respective medical entities. They were designed manually; in this respect, the expert systems of the CADIAG family are distinguished from approaches based on an automatic construction of the knowledge base, as proposed, e.g., in [BAB].

The task of the inference engine is to derive information from provided input data by means of the rules contained in the knowledge base. As an input, symptoms, signs, present diseases, or any other kind of facts about a patient, all optionally weighted by means of degrees, are accepted. The output of a run of CADIAG-2 is a weighted list of possible diagnoses for this patient.

A structural operational semantics for CADIAG-2 was introduced in [CiVe]. The resulting calculus, called CadL ("Cadiag Logic"), conveniently describes the mode of operation of CADIAG-2. Accordingly, in this paper we identify the CADIAG-2 inference engine with CadL. However, we will stress the interpretatory aspects and stay at a semiformal level; we refer to [CiVe] for full details concerning CadL.

We start with two finite sets of symbols referring to medical entities: the set S of variables $\sigma_1, \sigma_2, \ldots$ that denote a patient's signs, findings, test results or symptoms (which we will commonly refer to as *symptoms*) and the set D of variables $\delta_1, \delta_2, \ldots$ that refer to diseases and therapies (to which we will often refer as *diagnoses*). We will use the symbols φ, χ, ψ to denote either kind of variable. Furthermore, the *propositions* of CadL, indicated by α, β, γ, are built up from the variables by means of $\wedge$ ("and"), $\vee$ ("or"), and $\sim$("not").

In CadL, a pair (α, t), where α is a proposition and t is an element of the real unit interval $[0, 1]$, is called a *graded proposition*. Let us explain the intended meaning of graded propositions by examples.

Example 2.1. Assume that σ denotes the property "having fever". This is a vague property; that is, not under all circumstances it is appropriate to say that the patient has fever or the patient does not have fever; there are borderline cases. If the patient's body temperature is above, say, $38.5°C$, s/he has clearly fever, and this fact is expressed by $(\sigma, 1)$. If the patient's temperature is below, say $37.5°C$, s/he has no fever, expressible by $(\sigma, 0)$. Any temperature in between $37.5°C$ and $38.5°C$ is a borderline case of fever. We choose in these cases a value between 0 and 1. For instance, a temperature of $37.8°C$ may result in the expression $(\sigma, 0.3)$.

Example 2.2. Let δ denote a disease. Then (δ, c) is the statement that we are certain about the presence of δ to the degree c. In particular, $(\delta, 1)$ means full certainty, and the smaller c is, the less certain we are that δ is present. Furthermore, $c = 0$ plays an extra role. $(\delta, 0)$ is a statement of full certainty just like $(\delta, 1)$, but this time about the absence of δ.

The input of a run of CADIAG-2 is a set of medical entities together with a weight. We represent an input in CadL by a collection of graded propositions:

$$(\sigma_1, t_1), \ldots, (\sigma_k, t_k), \quad (\delta_1, c_1), \ldots, (\delta_l, c_l).$$

which expresses the available information about a patient.

We now turn to the knowledge base of CADIAG-2 (KB for short). Each rule contained in the KB represents a relationship between a pattern of medical entities on the one hand and a single medical entity on the other hand. The former is called *antecedent* and the latter *consequent*. Each rule is furthermore endowed with a weight, which is a number $d \in (0, 1]$. Accordingly, a rule is expressed in CadL by a graded implication between a possibly compound proposition and a single variable, possibly negated. Rules representing a relationship between two (possibly negated) variables are called *binary*, whereas rules representing a relationship between a possibly complex proposition and a variable are called *compound*.

There are three types of rules. A graded implication of the form

$$(\alpha \to \varphi,\ d), \tag{c}$$

where $d > 0$, represents a rule of type "confirming to the degree d", or "(c)" for short. An example, taken from CADIAG-2'KB is the following binary rule:

IF *strongly increased number of urinary proteines* THEN *systemic lupus erythematosus* with the degree $d = 0.2$.

A graded implication of the form

$$(\sim\varphi \to \sim\psi,\ 1), \tag{ao}$$

where φ and ψ are variables, is of type "always occurring", or "(ao)" for short. It formalises the fact that if φ is not present in the patient then ψ is not present either.

The following binary rule is of type (ao):

IF NOT *negative Waaler-Rose test* THEN NOT *Juvenile idiopathic arthritis* *(seronegative), polyarticular form*

Finally, a graded implication of the form

$$(\varphi \to \sim\psi,\ 1), \tag{me}$$

where φ and ψ are again variables, is of type "mutually exclusive", or "(me)" for short. It expresses the fact that if φ is present then ψ is fully excluded.

For example

IF *lupus erythematodes cells* THEN NOT *Morbus Behçet*

As our last step in this section, we have to explain how CADIAG-2 infers information from a given input by means of the rules contained in the knowledge base. The inference rules of CadL can be divided into two groups: the evaluation and manipulation rules. The former serve to determine the value associated to a compound proposition from the values of the variables being given. The manipulation rules mirror the three types of rules in the KB of CADIAG-2.

Below, the operations $\bar{\wedge}$, $\bar{\vee}$, $\bar{\sim}$ are applied to values from $[0,1]$ and denote the minimum, maximum and standard negation (i.e. $x \mapsto 1-x$) respectively.

The *evaluation rules* are

$$(\wedge_1)\ \frac{(\alpha, s) \quad (\beta, t)}{(\alpha \wedge \beta, s \,\bar{\wedge}\, t)} \qquad (\wedge_2)\ \frac{(\alpha, 0)}{(\alpha \wedge \beta, 0)} \qquad (\wedge_3)\ \frac{(\beta, 0)}{(\alpha \wedge \beta, 0)}$$

$$(\vee_1)\ \frac{(\alpha, s) \quad (\beta, t)}{(\alpha \vee \beta, s \,\bar{\vee}\, t)} \qquad (\vee_2)\ \frac{(\alpha, r)}{(\alpha \vee \beta, r)} \qquad (\vee_3)\ \frac{(\beta, r)}{(\alpha \vee \beta, r)}$$

$$(\sim)\ \frac{(\alpha, t)}{(\sim\alpha,\ \bar{\sim} t)}$$

for any proposition α, β of CadL and $s, t \in [0,1]$, $r \in (0,1]$.

The *manipulation rules* are

$$(\mathrm{c})\ \frac{(\alpha \to \varphi,\ d) \quad (\alpha, t)}{(\varphi, d \,\bar{\wedge}\, t)} \text{ where } d, t > 0$$

$$(\mathrm{me})\ \frac{(\psi \to \sim\varphi,\ 1) \quad (\psi, 1)}{(\varphi, 0)} \qquad (\mathrm{ao})\ \frac{(\sim\psi \to \sim\varphi,\ 1) \quad (\psi, 0)}{(\varphi, 0)}$$

for any $\alpha, \varphi, \psi \in \mathcal{F}$ such that φ and ψ are atomic.

A theory $\mathcal{T}$ of CadL is a set of graded propositions and graded implications. Clearly, the graded propositions contained in a theory describe the information known about a patient and the graded implications represent the rules in the KB of CADIAG-2. A run of CADIAG-2 corresponds to an inference made in CadL from $\mathcal{T}$.

For each disease δ, the relevant question is whether from the given theory one can infer (δ, c) by means of the rules in CadL, for some c (i.e., whether CadL proves (δ, c)). In fact, it might be possible that CadL proved both (δ, c) and (δ, c') for some c' distinct from c. In this case only the best result is of interest, where "best" refers to the following definition.

Definition 2.3. Let $c, c' \in [0,1]$. We say that c is *better* than c' (in symbols $c \succcurlyeq c'$) if either $c \geq c' > 0$ or $c = 0$ and $c' \neq 1$. CadL proves (δ, c) *optimally* from a theory $\mathcal{T}$ if CadL proves (δ, c) and $c \succcurlyeq c'$ for all c' such that CadL proves (δ, c').

The result of a run of CADIAG-2 is a set of graded diseases. A graded disease (δ, c) appears in this list exactly if CadL optimally proves (δ, c) from $\mathcal{T}$.

We note that it may be the case that both $(\delta, 0)$ and $(\delta, 1)$ are provable. The CADIAG-2 inference engine considers this situation as an error causing the program to quit. Here, we assume that the theory $\mathcal{T}$ does not prove both $(\alpha, 0)$ and $(\alpha, 1)$ for any proposition α.

We conclude by providing an example of a short inference done by CADIAG-2.

Example 2.4. Assume that the following is known about a patient:

σ_1: *"run-in" pain*; σ_2: *prior osteotomy of a joint* ("Run-in" pain means pain when taking the first steps in the morning or after resting.)
Consider furthermore the disease δ: *Primary osteoarthritis*.

The KB of CADIAG-2 contains the following two rules:

- IF σ_1 THEN δ with the degree 0.4.
- IF σ_2 THEN δ with the degree 0.2.

All these facts are expressed by a theory of CadL as follows:

$$(\sigma_1, 1), \quad (\sigma_2, 1), \quad (\sigma_1 \to \delta,\ 0.4) \quad (\sigma_2 \to \delta,\ 0.2). \tag{1}$$

Using the (c) manipulation rule, CadL proves $(\delta, 0.4)$ as well as $(\delta, 0.2)$. Consequently, CadL proves $(\delta, 0.4)$ optimally.

The result $(\delta, 0.4)$ in turn means that there is a good reason to consider the presence of δ, which is however far from being certain.

Challenges for formal frameworks

A formal framework for CADIAG-2 or similar systems should ideally be able to provide inferences in the same way the system does, overcome conceptual difficulties and preserve the advantages of the system. In particular the results should be kept transparent: together with each output, CADIAG-2 provides a chain of arguments, beginning with the input information and leading step by step to the result.

In the following we will discuss and compare three different approaches, recently introduced by the authors to formalise CADIAG-2. For each of them we will systematically check whether

- the formalism can draw the conclusions that CADIAG-2 does;
- the formalism interprets the various types of degrees in an appropriate way and justifies the inferential mechanism of CADIAG-2;
- the inferences in the formalism are comprehensible, that is, the intended meaning of each inference rule in terms of medical relationships is clearly understood;
- the formalism gives the possibility of checking the consistency of the rules of CADIAG-2.

We have specified CadL as an operational semantics for CADIAG-2. Thus CadL will conveniently serve as the reference to which the formalisms presented in the sequel will be compared.

3. CADIAG-2 and fuzzy logic

Our first approach to providing CADIAG-2 with a theoretical basis leads to the field which is most often mentioned in connection with this expert system: fuzzy logic.

CADIAG-2 is indeed usually presented as an example of a *fuzzy expert system*; see, e.g., the monographs [KlFo] and [Zim]. The truth values for a patient's symptoms in CADIAG-2 are indeed mainly determined by means of fuzzy sets; 600 fuzzy sets are predefined in the system. Moreover, the inferential mechanism of CADIAG-2 is close to Zadeh's max-min rule [Zad]; cf. [AKSG, DHN].

We note at this point that calling CADIAG-2 and related systems "fuzzy" expert systems gives rise to a common misunderstanding. Calling an expert system "fuzzy" seems to suggest that the system processes genuinely vague information. It is, on the one hand, the very aim of CADIAG-2 to take the vagueness of medical information appropriately into account. On the other hand, however, CADIAG-2 accepts as its input only information that is crisp. For instance, CADIAG-2 allows to deal with notions like "having high fever". But the input for a run of CADIAG-2 is necessarily a *pairing* of this information with a truth degree; and the input "having high fever" together with a real $t \in [0, 1]$ refers to a precise body temperature and consequently a crisp information. To deal with (genuinely) vague information and with uncertainty of vague notions, several approaches exist that are conceptually quite different from those discussed here; see, e.g., [FGM] and the references given there.

The approach discussed in this section is described in [CiVe]. Fuzzy logics are understood here in the sense of Hájek [Haj], that is, as t-norm-based many-valued logics. These logics are based on truth degrees taken from the real unit interval $[0, 1]$; furthermore, the conjunction is interpreted by a t-norm and the implication by the corresponding residuum.[1] Generally speaking, fuzzy logics fit conceptually well to CADIAG-2. The system uses indeed the same set $[0, 1]$ of truth degrees. Furthermore, fuzzy logics are truth functional; the value of a compound formula is calculated from its constituents. CADIAG-2 does the same, the conjunction being interpreted by the Gödel t-norm (i.e. the minimum).

To justify the decision to use fuzzy logic as a framework for CADIAG-2, some care is nevertheless needed. Degrees appear in CADIAG-2 at three different levels: attached to symptoms, attached to diseases and attached to rules.

A symptom σ is endowed with the value $t \in [0, 1]$ in case that the actual situation fits to σ to the possibly limited degree t. In this case we have a perfect conceptual correspondence between the expert system and fuzzy logic.

The cases of diagnoses and rules however are different. The weights express an uncertainty, not a degree of presence, in these cases. In the present approach, however, all weights are interpreted as degrees of compatibility, not only when attached to symptoms. In order to justify such an interpretation we argue as follows: let δ denote a diagnose and $c \in (0, 1]$. The statement (δ, c) means that we are confident to the degree c that a patient suffers from δ. (δ, c) is the conclusion of an anamnesis and a physical examination of the patient; based on the known facts, c is the degree to which an expert is inclined to consider the presence of δ. Thus, the value c is the result of a comparison; the expert compares his knowledge about the patient to a pattern of symptoms that would clearly imply that δ is present. In this sense, c is the degree to which the actual facts are *compatible* with the conjecture that δ is present.

A similar, and even more direct, argument applies for the weight of a rule. We understand $(\alpha \to \delta,\ d)$ as the statement that α fits to the conjecture that δ is present to the degree d.

[1] A *t*-norm is a commutative, associative, in both arguments monotonically increasing function $*\colon [0, 1]^2 \to [0, 1]$ such that $1 * x = x$ for all $x \in [0, 1]$. The residuum of $*$ is a function $\Rightarrow_*\colon [0, 1]^2 \to [0, 1]$ where $x \Rightarrow_* y = \max\{z \mid x * z \leq y\}$.

The use of fuzzy logic may independently be motivated from the close formal relationship between the logic CadL and those fuzzy logics in whose formulas truth degrees are used explicitly. Syntactically closely related to CadL, we may mention fuzzy logics with evaluated syntax; see [Pav, NPM, Haj]. Somewhat more expressive than fuzzy logics with evaluated syntax are those whose language contains truth constants. Fuzzy logics of this kind have been studied extensively; see, for instance, [EGHN, EGN].

Here, we will choose a logic of the latter kind: a fuzzy logic with rational truth constants. With this choice, however, a problem arises. In CADIAG-2, a pair (α, t) means that the degree of presence of α equals t. In fuzzy logic, however, (α, t) is reasonably defined as being satisfied if the degree of presence of α is *at least* t. Under the latter interpretation all rules are sound but the rule $(\sim)$ for the negation. Indeed, if $t \in (0, 1]$ is a lower bound for the truth value of α then, taking into account the interpretation of $\sim$ by $t \mapsto 1 - t$, we conclude that $1 - t$ is an upper bound of the truth value of $\sim \alpha$.

This problem can be overcome. We restrict the notion of proof in CadL in a way which is not restrictive in practice. We allow the application of the rule $(\sim)$ only in case that the truth value of all occurring variables is actually fixed and not only lower-bounded.

Definition 3.1. We call a proof in CadL from a theory $\mathcal{T}$ *regular* if the following condition holds. Let (α, t) be the assumption of a rule $(\sim)$, and let φ be an atom appearing in α. Then either (φ, s) is contained in $\mathcal{T}$ for some $s \in [0, 1]$ or (φ, s) is contained in the proof for some $s \in \{0, 1\}$.

The logic GZL

Our reasoning framework is a logic that we call Gödel-Zadeh logic, GZL for short. GZL is a fragment of Gödel logic enriched with standard negation, the operator Δ of [Baa] and rational truth constants.

The *atomic propositions* of GZL are countably many variables $\varphi_1, \varphi_2, \ldots$ as well as constants $\bar{t}$ for each rational number $t \in [0, 1]$. The *lattice propositions* of GZL are built up from the atomic propositions by means of the binary connectives $\wedge$ and $\vee$, the unary connective $\sim$ and the modality Δ. A *comparing proposition* of GZL is a pair of two lattice propositions α and β, denoted by $\alpha \to \beta$.

Definition 3.2. A *valuation* v of GZL maps the lattice propositions to $[0, 1]$ as follows: (i) $v(\bar{t}) = t$ for each rational $t \in [0, 1]$, (ii) $v(\alpha \wedge \beta) = v(\alpha) \bar{\wedge} v(\beta)$, (iii) $v(\alpha \vee \beta) = v(\alpha) \bar{\vee} v(\beta)$, (iv) $v(\sim \alpha) = \bar{\sim} v(\alpha)$, (v) $v(\Delta\alpha) = \bar{\Delta} v(\alpha)$, where $\bar{\vee}$, $\bar{\sim}$, $\bar{\wedge}$ are as in Section 2 and

$$\bar{\Delta} t = \begin{cases} 1 & \text{if } t = 1, \\ 0 & \text{else} \end{cases} \tag{2}$$

for $t \in [0, 1]$. We say that a valuation v *satisfies* a comparing proposition $\alpha \to \beta$ if $v(\alpha) \leq v(\beta)$.

A theory of GZL is a set of comparing propositions. We say that a theory $\mathcal{T}$ *semantically implies* a proposition $\alpha \to \beta$ if any valuation satisfying every element of $\mathcal{T}$ satisfies $\alpha \to \beta$ as well.

How do we use GZL in order to emulate the inference engine of CADIAG-2? We now describe how an inference in CadL translates to GZL.

Assume a theory $\mathcal{T}$ be given, consisting of graded propositions and graded implications, coding the input and the rules, respectively. Each pair (φ, t) is translated into $\bar{t} \to \varphi$ and $\varphi \to \bar{t}$. A rule of type (c) $(\alpha \to \beta,\ d)$ is translated into $\alpha \wedge \bar{d} \to \beta$. A rule of type (me) $(\psi \to {\sim}\varphi,\ 1)$ is translated into $\Delta\psi \to {\sim}\varphi$. A rule of type (ao) $({\sim}\psi \to {\sim}\varphi,\ 1)$ is translated into $\Delta({\sim}\psi) \to {\sim}\varphi$.

The relationship between CadL and GZL can be described as follows.

Theorem 3.3 ([CiVe])**.** *Let $\mathcal{T}$ be the theory of* CadL *associated to an input of CADIAG-2. Let $\mathcal{T}'$ be the corresponding theory of* GZL. *If there is a regular proof of (α, t) from $\mathcal{T}$ in* CadL*, then $\mathcal{T}'$ semantically implies $\bar{t} \to \alpha$ if $t > 0$ and of $\alpha \to \bar{0}$ if $t = 0$.*

To translate inferences in CadL into syntactic proofs of GZL an adequate proof system for GZL was provided in [CiVe]. This uses sequents-of-relations, a generalisation of Gentzen's sequents introduced in [BaFe]. In our context, a *sequent-of-relations* $\mathcal{G}$ is a multiset of ordered triples

$$\alpha_1 \triangleleft_1 \beta_1 \ \mid\ \ldots\ \mid\ \alpha_n \triangleleft_n \beta_n,$$

where α_i and β_i are formulas of GZL and $\triangleleft_i \in \{<, \leq\}$ for $i = 1, \ldots, n$.

$\mathcal{G}$ is *satisfied* by some valuation of GZL v if $v(\alpha_i) \triangleleft_i v(\beta_i)$ for some i; $\mathcal{G}$ is *valid* in GZL if satisfied by all valuations of GZL.

Definition 3.4. Axioms and rules of SeqGZL are the following, where $\mathcal{G}$ is an arbitrary side sequent-of-relations and $\triangleleft, \triangleleft_1, \triangleleft_2 \in \{<, \leq\}$:

Axioms

$$\frac{}{\alpha \leq \alpha}\ (A1) \qquad \frac{}{\bar{s} \triangleleft \bar{t}}\ (A2), \text{ where } s \triangleleft t$$

Logical Rules

$$\frac{\mathcal{G} \mid \alpha \triangleleft \gamma \mid \beta \triangleleft \gamma}{\mathcal{G} \mid \alpha \wedge \beta \triangleleft \gamma}\ (\wedge\triangleleft) \qquad \frac{\mathcal{G} \mid \gamma \triangleleft \alpha \qquad \mathcal{G} \mid \gamma \triangleleft \beta}{\mathcal{G} \mid \gamma \triangleleft \alpha \wedge \beta}\ (\triangleleft\wedge)$$

$$\frac{\mathcal{G} \mid \alpha \triangleleft \gamma \qquad \mathcal{G} \mid \beta \triangleleft \gamma}{\mathcal{G} \mid \alpha \vee \beta \triangleleft \gamma}\ (\vee\triangleleft) \qquad \frac{\mathcal{G} \mid \gamma \triangleleft \alpha \mid \gamma \triangleleft \beta}{\mathcal{G} \mid \gamma \triangleleft \alpha \vee \beta}\ (\triangleleft\vee)$$

$$\frac{\mathcal{G} \mid {\sim}\beta \triangleleft \alpha}{\mathcal{G} \mid {\sim}\alpha \triangleleft \beta}\ ({\sim}\triangleleft) \qquad \frac{\mathcal{G} \mid \beta \triangleleft {\sim}\alpha}{\mathcal{G} \mid \alpha \triangleleft {\sim}\beta}\ (\triangleleft{\sim}) \qquad \frac{\mathcal{G} \mid \beta \triangleleft \alpha}{\mathcal{G} \mid {\sim}\alpha \triangleleft {\sim}\beta}\ ({\sim}\triangleleft{\sim})$$

$$\frac{\mathcal{G} \mid \alpha < 1 \mid 1 \leq \beta}{\mathcal{G} \mid \Delta\alpha \leq \beta}\ (\Delta \leq) \qquad \frac{\mathcal{G} \mid \alpha \leq 0 \mid 1 \leq \beta}{\mathcal{G} \mid \alpha \leq \Delta\beta}\ (\leq \Delta)$$

$$\frac{\mathcal{G} \mid \alpha < 1 \qquad \mathcal{G} \mid 0 < \beta}{\mathcal{G} \mid \Delta\alpha < \beta}\ (\Delta <) \qquad \frac{\mathcal{G} \mid \alpha < 1 \qquad \mathcal{G} \mid 1 \leq \beta}{\mathcal{G} \mid \alpha < \Delta\beta}\ (< \Delta)$$

Rules for Constants

$$\frac{\mathcal{G} \mid \alpha \triangleleft \overline{0.5}}{\mathcal{G} \mid \alpha \triangleleft {\sim}\alpha}\ (\triangleleft\tfrac{1}{2}) \qquad \frac{\mathcal{G} \mid \overline{0.5} \triangleleft \alpha}{\mathcal{G} \mid {\sim}\alpha \triangleleft \alpha}\ (\tfrac{1}{2}\triangleleft)$$

$$\frac{\mathcal{G} \mid \alpha \triangleleft \bar{0}}{\mathcal{G} \mid \alpha \triangleleft \beta}\ (\triangleleft 0) \qquad \frac{\mathcal{G} \mid \bar{1} \triangleleft \beta}{\mathcal{G} \mid \alpha \triangleleft \beta}\ (1\triangleleft)$$

$$\frac{\mathcal{G} \mid \overline{\bar{\sim} t} \triangleleft \alpha}{\mathcal{G} \mid {\sim}\bar{t} \triangleleft \alpha}\ ({\sim}\mathrm{c}\triangleleft) \qquad \frac{\mathcal{G} \mid \alpha \triangleleft \overline{\bar{\sim} t}}{\mathcal{G} \mid \alpha \triangleleft {\sim}\bar{t}}\ (\triangleleft{\sim}\mathrm{c})$$

Structural Rules

$$\frac{\mathcal{G} \mid \alpha \triangleleft \beta \mid \alpha \triangleleft \beta}{\mathcal{G} \mid \alpha \triangleleft \beta}\ (\mathrm{EC}) \qquad \frac{\mathcal{G} \mid \alpha \leq \beta \qquad \mathcal{G} \mid \gamma \triangleleft_1 \delta}{\mathcal{G} \mid \gamma \triangleleft_1 \beta \mid \alpha \triangleleft_2 \delta}\ (\mathrm{com})$$

$$\frac{\mathcal{G}}{\mathcal{G} \mid \alpha \triangleleft \beta}\ (\mathrm{EW}) \qquad \frac{\mathcal{G} \mid \alpha < \beta \qquad \mathcal{G} \mid \beta \leq \alpha}{\mathcal{G}}\ (\mathrm{cut})$$

The definition of *proof* in SeqGZL is the usual one.

Theorem 3.5 ([CiVe])**.** *Let $\mathcal{T} = \{\alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n\}$ be a finite theory of* GZL *and let $\alpha \to \beta$ be a comparing proposition of* GZL*. $\mathcal{T}$ semantically implies $\alpha \to \beta$ if and only if there is a proof in* SeqGZL *of $\alpha \leq \beta$ from $\{\alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n\}$.*

Notice that SeqGZL is an *analytic calculus* for GZL, that is, proofs in SeqGZL can be easily built bottom up as they proceed by stepwise decomposition of the formulas (sequent-of-relations) to be proved. Recall that analytic calculi are key for developing automated reasoning methods (theorem provers).

Corollary 3.6. *Let $\mathcal{T}$ be the theory of* CadL *associated to an input of CADIAG-2. Let $\mathcal{T}'$ be the corresponding set of sequents-of-relations. If there is a regular proof of (α, t) from $\mathcal{T}$ in* CadL*(cf. Def. 3.1), then there is a proof in* SeqGZL *of $\bar{t} \leq \alpha$ if $t > 0$ and of $\alpha \leq \bar{0}$ if $t = 0$.*

The example below shows how to use SeqGZL to emulate the behaviour of CADIAG-2 and recall the interpretation of the numerical values of CADIAG-2 in the fuzzy logic approach.

Example 3.7. Let us see how the derivation in CadL shown in Example 2.4 is realised in SeqGZL. The CadL theory (1) translates into the following sequents-of-relations of SeqGZL:

$$\bar{1} \leq \sigma_1, \quad \bar{1} \leq \sigma_2, \quad \sigma_1 \wedge \overline{0.4} \leq \delta \quad \sigma_2 \wedge \overline{0.2} \leq \delta. \tag{3}$$

From $\bar{1} \leq \sigma_1$, we can derive $\overline{0.4} \leq \sigma_1 \wedge \overline{0.4}$ and then

$$\overline{0.4} \leq \delta. \tag{4}$$

Similarly, we can use the information about σ_2 to derive $\overline{0.2} \leq \delta$; but this formula is already implied by (4).

The result (4) means that a truth degree of at least 0.4 is assigned to the presence of δ, i.e. that (at least) 0.4 is the degree to which, based on the actual facts (3), it is reasonable to assume that δ is present.

3.1. Discussion

In order to decide if our formal framework is appropriate for the expert system CADIAG-2, we first compare the logic GZL with the original system with regard to their strength. By Theorem 3.3, we know that GZL is able to draw all conclusion that CadL does and therefore to emulate CADIAG-2.

This most important requirement being fulfilled, the primary achievement of our framework is its conceptual clarity. Our logic is built on solid grounds: GZL is semantically based and rules are such that inferences are correct and complete with respect to the chosen semantics (Theorem 3.5).

A side effect of this achievement is the following drawback. Endowing CADIAG-2 with a semantically based logic, we inevitably add more interrelations between facts than used by the original system, which did not care about a semantic reference. In fact, GZL is strictly stronger than CadL. As an example, consider a knowledge base containing the rules $(\alpha \to \beta,\ 1)$ and $({\sim}\alpha \to \beta,\ 1)$. Then GZL can conclude that β has at least the truth value 0.5, while CadL does not. The example is certainly artificial; still, there is no criterion known to us to rule out this kind of example. We also do not know if in practice the additional strength of GZL would have any impact.

Furthermore, our framework allows a consistent interpretation of the numerical values processed by the system. All values are interpreted in a uniform way: as degrees of compatibility. As regards the degrees of presence, that is, the values associated to symptoms, this choice is satisfying without doubt. As regards the degrees of uncertainty, that is, the values associated to rules or diagnoses, our choice is admittedly arguable, but justifiable. Recall our argument: a rule $(\alpha \to \delta,\ d)$ expresses that on the basis of α, the presence of δ is suggested to the degree d. The degree d is a degree of compatibility; d is the degree to which the information α fits to the conjecture that δ holds. Further general considerations of this issue can be found in our concluding Section 6.

Relying on fuzzy logic, we cannot provide a satisfactory justification of the way truth degrees are calculated. The problematic rule in this respect is the rule (c) and all we can say is the following. When interpreting all the degrees in the system as compatibilities, their precise values are less important than their relative order. The rule (c) is in fuzzy logic known as the generalised (or fuzzy) modus ponens [Ger] and in its general version, the resulting value is calculated by means of a t-norm. The Gödel (i.e. minimum) t-norm, which we use here, is the only one that does not involve any calculation but for which only the relative order of the conjuncts matters. To say anything more specific in favour of the rule (c) seems hard and might not be possible at all. For its arbitrariness, fuzzy logic itself has been subjected to criticism since the time it was invented.

CADIAG-2 provides, together with each output, a chain of arguments, beginning with the information provided by the user and leading step by step to the result. Proofs in CadL are also largely comprehensible. In contrast, within SeqGZL, results cannot be presented in the same smooth way; this advantage of CADIAG-2 is not preserved. Like in the case of any proof system based on sequent calculi or their variants, proof rules have a rather technical character, a reasonable translation of which to natural language is largely impossible. For instance, the relational signs $\leq, <$ in SeqGZL just denote relations between real numbers; the original nature of the implication in a rule contained in a knowledge base gets inevitably lost.

We summarise that we have endowed CADIAG-2 with a conceptually clear framework, which can well replace the original inference engine. Certainly, progress in one respect can mean a step back in other respects. The interpretability of proofs in fuzzy logic is the topic of a foundational debate in which a progress would have a positive impact for our present application.

Rules Check

As mentioned in the introduction, formal frameworks for rule-based systems may serve to perform various checks on their rules.

Indeed, an important achievement following from the interpretation of CADIAG-2 within fuzzy logic is the reformulation of the consistency check of its KB as a satisfiability problem in Logic; the actual check was performed in [CiRu] for a large portion of the rules. In analogy with the approach in [MoAd], where the rules of CADIAG-1 were translated into formulas of first-order classical logic, CADIAG-2 rules are translated in [CiRu] into suitable formulas of a first-order fuzzy logic. The use of a first-order framework is motivated by the need to associate medical entities with unary (i.e. monadic) predicates. For instance, a statement like "the symptom σ_i is present in a patient a" is identified with the atomic formula $S_i(a)$, which in turn assumes values in $[0, 1]$. The use of a fuzzy logic is due to the interpretation of the values assigned to compound medical entities.

As shown in [CiVe] the fuzzy logic that comes closest to the concepts underlying CADIAG-2 is $\mathsf{RGL}_\sim$, that is, Gödel logic extended by Baaz's Δ, truth constants and an involutive negation (GZL being a fragment of $\mathsf{RGL}_\sim$). The translation of CADIAG-2's rules was based on the meaning of the connectives in $\mathsf{RGL}_\sim$ and on the *sigma-count* interpretation[2] of their weights proposed in [AKS, AKSEG, AKSG]: Given a rule $(\alpha \to \beta, d)$, the interpretation of d amounts to

$$d = \frac{\Sigma_{a \in \mathrm{P}} v(\alpha(a)) \barwedge v(\beta(a))}{\Sigma_{a \in \mathrm{P}} v(\alpha(a))}, \tag{5}$$

where P represents a set of patients, v a valuation as in Definition 3.2 and thus $v(\alpha(a))$ and $v(\beta(a))$ the degrees to which the possibly compound entity α and the atomic entity β apply to patient $a \in \mathrm{P}$.

Recall that in fuzzy logics the weights of the rules are interpreted as lower bounds. Hence two rules assigning different degrees s and t to a same diagnosis (with $0 < s, t <$

[2]The database used for the creation of the knowledge base of CADIAG-2 did not contain as many patients as necessary for the calculation of all weights on the basis of (5). Therefore most of the rule weights were actually estimated by different means, mainly on the basis of physicians' advices.

1) are not contradictory. For this reason the truth constants were not considered in the logic used for the consistency check of [CiRu]: $\mathsf{G}^{\sim}$, that is (first-order) Gödel logic extended by an involutive negation[3]. Compound propositions of CADIAG-2 were therefore interpreted using the connectives of $\mathsf{G}^{\sim}$. The translation is consistency-preserving: the non-existence of a valuation for all the translated formulas (that is, of a valuation assigning the value 1 to all of them) implies errors in the knowledge base.

Though the satisfiability problem (SAT) for the monadic fragment of $\mathsf{G}^{\sim}$ is undecidable, SAT for the set of formulas of $\mathsf{G}^{\sim}$ formalizing most of the rules in the KB of CADIAG-2 turned out to be not only decidable, but even decidable within classical logic. For instance this was the case of formulas corresponding to the *binary rules* in the KB of CADIAG-2. For these formulas existing theorem provers and (counter)model generators for classical first-order logic could therefore be used; the former detected unsatisfiable sets of formulas (recall that in classical logic a set formulas is unsatisfiable if and only if its negation is valid), and the well known (counter)model generators *Mace 4* was used to detect the minimal sets of unsatisfiable formulas. As a result, 11 minimal groups of inconsistent rules were found. For instance, the following group:

(a) IF *Chorea minor*
THEN NOT *Reactive arthritis*

(b) IF NOT *Reactive arthritis*
THEN NOT *Rheumatic fever*

(c) IF *Chorea minor*
THEN *Rheumatic fever*
with the degree 0.99.

Indeed from the assumption "*Chorea minor*", two almost opposite conclusions, are derived; the diagnose "*Rheumatic fever*" is both excluded and confirmed to the degree 0.99, meaning that it is almost sure.

In order to check the consistency of the full knowledge base of CADIAG-2, classical logic is not enough. Nevertheless the translation of the rules as proposed in [CiRu] still applies and the satisfiability of the resulting formulas of $\mathsf{G}^{\sim}$ was shown in [BCP] to be decidable (and even NP-complete, as in the case of classical, propositional logic). A full check could therefore be performed provided that powerful (counter)model generators are designed for $\mathsf{G}^{\sim}$, capable of handling the 20,000 rules of CADIAG-2. Note however that the majority of the rules in the KB of CADIAG-2 is binary and a check of the latter, as done in [CiRu], is therefore very useful. Only 81 rules out of 20,000 are compound. The compound rules contain a complete specification of the rheumatological diseases and due to the use in CADIAG-2 of the (derivable) connectives "at least n out of m" and "at most n out of m", some of these rules are rather complex (e.g., when expressed in disjunctive normal form, the antecedents of these rules can contain up to 30.000.000 disjuncts). A check of the full knowledge base of CADIAG-2 is feasible in principle but still to be done.

[3] Baaz's Δ is also used but it is a derivable operator in $\mathsf{G}^{\sim}$.

4. CADIAG-2 and probability theory

Our second approach towards a formalisation and interpretation of CADIAG-2 in this paper is based on probability theory. The approach, first presented in [Pic1], is motivated by the interpretation of the rules of the form $(\alpha \to \delta, d)$ in the KB as conditional probabilistic statements, where α is the conditioning event, δ is the uncertain event and d is the probability that δ is a correct diagnose for the patient given that α is present in the patient. This interpretation is also favoured by some literature about CADIAG-2 as the intended interpretation of the rules of the system, see, e.g., [Adl].

In order to formalise the inference process on probabilistic grounds and analyse its adequacy with probability theory we need also a suitable probabilistic interpretation of the graded propositions of the form (σ, t) taken as input by the system. For example, we can interpret the value t in (σ, t) as the degree of belief that a medical doctor has in the truth or presence of σ in the patient given the evidence supporting it. In this sense t is interpreted as a probability.

Formally, we will identify a graded statement of the form (σ, t) in the input of the system with a graded implication of the form $(\kappa \to \sigma, t)$, where κ represents the facts that support the presence of σ in the patient and t the probability that σ is present in the patient given κ.

The logic CadPL

We describe the system CadPL, aimed at formalising CADIAG-2's inference when restricted to the set of binary rules.

As above, let S and D contain the variables denoting symptoms and diseases, respectively. Furthermore, let $\kappa_1, \ldots, \kappa_n$, $n \geq 1$, be an additional set of variables that we shall call the *factual* variables; each of it refers to the actual—crisp—fact that gives rise to the assumption that a particular—possibly vague—symptom is present.

Note that in the present framework, the variables of all three sorts denote crisp facts. Let $\mathcal{P}$ the set of formulas built up from the variables in S, D and K by means of the Boolean connectives $\wedge, \vee, \sim$. For some $\alpha \in \mathcal{P}$, we write $\models \alpha$ to denote classical validity; for $\alpha, \beta \in \mathcal{P}$, we write $\alpha \models \beta$ to denote classical entailment and $\alpha \equiv \beta$ to denote classical equivalence.

Definition 4.1. A mapping $\omega \colon \mathcal{P} \to [0, 1]$ is called a *probability function* on $\mathcal{P}$ if the following two conditions hold for all $\alpha, \beta \in \mathcal{P}$:

- If $\models \alpha$ then $\omega(\alpha) = 1$.
- If $\models \neg(\alpha \wedge \beta)$, then $\omega(\alpha \vee \beta) = \omega(\alpha) + \omega(\beta)$.

From Definition 4.1 the standard properties of probability functions on propositional languages follow (see, e.g., [Par]).

We are now ready to define satisfiability in CadPL. The graded implication $(\alpha \to \beta, d)$ is *satisfied* by the probability function ω on $\mathcal{P}$ if $\omega(\alpha) > 0$ and

$$\frac{\omega(\alpha \wedge \beta)}{\omega(\alpha)} = d.$$

If there exists a probability function satisfying $(\alpha \to \beta, d)$ we say that $(\alpha \to \beta, d)$ is *satisfiable*.

We continue our description of CadPL defining the notion of a theory. We note that, in contrast with the fuzzy logic approach, here we need a distinction between graded implications representing the initial evidence about a patient and graded implications representing the system's rules.

Definition 4.2. A theory $\mathcal{T}$ of CadPL is a pair of the form (Φ, R) characterised as follows:

- Φ is a finite set of graded implications.
- $R = R^{\mathrm{c}} \cup R^{\mathrm{me}} \cup R^{\mathrm{ao}}$ is a collection of graded implications of type (c), (me) and (ao) respectively.

Φ is intended to represent the input of a run of the inference engine of CADIAG-2 which, as explained earlier, consists of a collection of graded implications of the form

$$(\kappa_1 \to \varphi_1, \eta_1), \ldots, (\kappa_n \to \varphi_n, \eta_n),$$

where $\kappa_1, \ldots$ are factual variables and $\varphi_1, \ldots$ are either variables or negated variables from S or D. The set $\{\kappa_1, ..., \kappa_n\} \subset \mathcal{P}$ constitutes the initial evidence about the patient, which is propagated along the inference process by the application of the rules of the system. Furthermore, R represents the binary rules in CADIAG2's knowledge base. For what follows, let $\mathcal{T} = (\Phi, R)$ be a theory of CadPL.

The system CadPL is defined by the following rules:

- *Reflexivity and valuation rules*

$$\text{(Ref)}\ \frac{(\kappa \to \alpha, d) \in \Phi}{\mathcal{T} \vdash (\kappa \to \alpha, d)} \qquad (Neg)\ \frac{\mathcal{T} \vdash (\alpha \to \beta, d)}{\mathcal{T} \vdash (\alpha \to \neg\beta, 1 - d)}$$

$$\text{(Eq)}\ \frac{\alpha \equiv \beta \quad \mathcal{T} \vdash (\gamma \to \alpha, d)}{\mathcal{T} \vdash (\gamma \to \beta, d)}$$

- *Manipulation rules*

$$\text{(Min I)}\ \frac{\mathcal{T} \vdash (\kappa \to \alpha, c) \quad (\alpha \to \beta, d) \in R^{\mathrm{c}}}{\mathcal{T} \vdash (\kappa \to \beta, c \barwedge d)}$$

$$\text{(Min II)}\ \frac{\mathcal{T} \vdash (\kappa \to \alpha, 1) \quad (\alpha \to \beta, 1) \in R^{\mathrm{me}} \cup R^{\mathrm{ao}}}{\mathcal{T} \vdash (\kappa \to \beta, 1)}$$

Within this frame, final outputs of the form (α, d) produced by the inference engine shall be interpreted as conditionals of the form $(\kappa_1 \wedge \ldots \wedge \kappa_n \to \alpha, d)$, that is, as the probability of α given all the medical evidence $\kappa_1, ..., \kappa_n$ available about the patient. In order to make such interpretation operative and formalise it we need to extend CadPL by introducing two new inference rules. The first of these rules formalises the maximisation process done by the system in order to yield as output the set of diagnoses along with the maximal value generated by it, with respect to the ordering set in Definition 2.3:

$$\text{(Conj)}\ \frac{\mathcal{T} \vdash (\kappa_{k_1} \wedge \ldots \wedge \kappa_{k_l} \to \alpha, c) \qquad \mathcal{T} \vdash (\kappa_{k_{l+1}} \wedge \ldots \wedge \kappa_{k_s} \to \alpha, d)}{\mathcal{T} \vdash (\kappa_{k_1} \wedge \ldots \wedge \kappa_{k_s} \to \alpha, e)}$$

for $\{c, d\} \neq \{0, 1\}$ and $e = c$ if $c \succcurlyeq d$ and $e = d$ if $d \succcurlyeq c$.

An additional rule is necessary to produce the desired outcome:

$$\text{(Exh)}\ \frac{\mathcal{T} \vdash (\kappa_{k_1} \wedge \ldots \wedge \kappa_{k_l} \to \alpha, c) \qquad \text{there is no } d \text{ such that } \mathcal{T} \vdash (\kappa \to \alpha, d)}{\mathcal{T} \vdash (\kappa_{k_1} \wedge \ldots \wedge \kappa_{k_l} \wedge \kappa \to \alpha, c)}$$

This last rule, which we call *Exh* as abbreviation of "*exhaustive*", simply states that, if κ is a piece of evidence that says nothing about the presence of α in the patient, that is, if κ and α are probabilistically independent, then the probability of α given $\kappa_1 \wedge \ldots \wedge \kappa_k$ does not change if in addition we consider the piece of evidence κ.

Theorem 4.3 ([Pic1])**.** *Let $\mathcal{T} = (\Phi, R)$ be a theory in* CadPL*, $\kappa_1, \ldots, \kappa_n$ the factual variables occurring in the graded implications in Φ, $\mathcal{T}'$ the theory of* CadL *that corresponds to $\mathcal{T}$ (i.e., with graded implications in Φ expressed by graded propositions) and α a proposition. $(\kappa_1 \wedge \ldots \wedge \kappa_n \to \alpha,\ d)$ follows maximally from $\mathcal{T}$ in* CadPL *if and only if $\mathcal{T}'$ proves $(\alpha,\ d)$ optimally in* CadL.

Let us apply the present framework to Example 2.4.

Example 4.4. The theory (1) of Example 2.4 translates to

$$(\kappa_1 \to \sigma_1,\ 1), \quad (\kappa_2 \to \sigma_2,\ 1), \quad (\sigma_1 \to \delta,\ 0.4) \quad (\sigma_2 \to \delta,\ 0.2), \tag{6}$$

where κ_1, κ_2 are factual variables, denoting the facts from which we conclude the presence of σ_1 and σ_2, respectively. By (Min I), we derive $(\kappa_1 \to \delta,\ 0.4)$ and $(\kappa_2 \to \delta,\ 0.2)$, and by (Conj), we get

$$(\kappa_1 \wedge \kappa_2 \to \delta,\ 0.4). \tag{7}$$

Thus we conclude that the probability of the presence of δ in the patient given the evidence $\kappa_1 \wedge \kappa_2$ is 0.4.

4.1. Discussion

The calculus CadPL and the concepts it builds on differ substantially from GZL and its underlying fuzzy logic approach. Thus our evaluation of CadPL comes to results that are very different from those of GZL.

Theorem 4.3 shows that the inferences in CadPL are equivalent to those of CadL. In contrast with GZL, CadPL is not stronger than CadL (and therefore of CADIAG-2). However CadPL takes only binary rules into account[4].

[4]Introducing the evaluation rules in CadPL would greatly modify the framework and, if still operative, it would be even more further away from probabilistic soundness (notice that except for the negation rule, the evaluation rules of CadL are not probabilistically sound).

As a consequence of being so close to (the binary fragment of) CadL, CadPL is not sound and complete with respect to probabilistic semantics. In particular, the rules (Min I) and (Conj) are not based on probability theory and (Exh) assumes probabilistic independence among propositions that may actually not be so.

In interpretive respects, CadPL scores naturally quite well. Again, numerical values are interpreted in a uniform way; all the degrees are interpreted probabilistically. In particular, the degrees of uncertainty are modelled in the most common and best accepted way. The interpretation of degrees of presence in a probabilistic framework involves, on the other hand, some artificiality.

The inference rules in CadPL are chosen in accordance with probability theory to the extent to which this is possible in order to formalise the inference of CADIAG-2. Thus, for the probabilistically sound rules, a justification is ensured. The remaining rules, for instance (Min I) or (Conj), which correspond to the (c) rule of CadL and to the maximisation process in CADIAG-2 respectively, are not inspired by probability theory but taken directly from CADIAG-2. In these cases, the situation is similar to the previous approach: a proper justification of the inferential mechanism of CADIAG-2 seems very hard in this framework, if possible at all.

The advantage of CadL to present traceable results is preserved. This might be surprising, as interpretability is a serious issue of probability based expert systems. The reason is, however, clear: The rules of CadPL are chosen in accordance with CadL, even where the probabilistic interpretation does not fit.

Summarising, the present approach manages the challenge to apply probability theory to an expert system based on weighted IF-THEN rules. Certain drawbacks are present but must be considered as unavoidable, provided that we aim at treating uncertainty within the best accepted formalism.

Rules Check

Possibly the main achievement in connection with the probabilistic framework concerns the satisfiability check of the *binary* fragment of CADIAG-2's knowledge base described in [KPP]. Such a satisfiability check is primarily based on a probabilistic interpretation of the system's rules. However, such interpretation is also in keeping with the *sigma-count* interpretation suggested in [AKS, AKSEG, AKSG] (see the formula (5) in Section 3.1).

In terms of the sigma-count interpretation, satisfiability of a certain set of binary rules in CADIAG-2 means the existence of a valuation v that yields the weights of all the rules in the set when calculated according to equation (5). It is proved in [KPP] that such a valuation v exists *if and only if* there exists a probability function that satisfies all the rules in the set under the natural probabilistic interpretation. Thus, a satisfiability check of sets of rules of this form based on a probabilistic interpretation of them is itself a satisfiability check with respect to the sigma-count interpretation.[5]

[5]It is worth noting that in our probabilistic check in [KPP] rules of type (me) and (ao) are expressed as they actually occur in CADIAG-2's knowledge base: a rule of type (me) of the form $(\alpha \rightarrow\sim \beta, 1)$ is expressed as $(\alpha \rightarrow \beta, 0)$ and a rule of type (ao) of the form $(\sim \alpha \rightarrow\sim \beta, 1)$ is expressed in its original form, as the (probabilistically) non-equivalent $(\beta \rightarrow \alpha, 1)$.

[KPP] presents a general methodology for a satisfiability check of sets of rules like those in the binary fragment of CADIAG-2's knowledge base and, in case of unsatisfiability, for the detection of all minimal unsatisfiable subsets (i.e, conflicts). Two main algorithms have been considered for this purpose: a PSAT algorithm and a conflict-finding algorithm. The PSAT algorithm (i.e., probabilistic satisfiability algorithm) employed in [KPP] is based on column generation techniques in linear programming and the conflict finding algorithm (i.e., the algorithm that, in connection with the previous one, searches the database in order to identify all conflicts in it) on the hitting set tree algorithm HST and other techniques that are well known in the field, see, e.g., [Rei]. In order to make the implementation of such algorithms feasible on large collections of rules (e.g., CADIAG-2's binary fragment) modularity techniques are introduced in [KPP] in order to split the knowledge base into smaller, feasible fragments.

The implementation of the methodology to (a slightly relaxed interpretation of)[6] CADIAG-2's binary rules detected the four types of conflicts below:

- *Type 1*, given by a collection of rules of the form

$$(\alpha \to \beta, d),\ (\alpha \to \gamma, c),\ (\beta \to \gamma, 1),$$

  with $d > c$. 420 conflicts of this type were found.

- *Type 2*, given by a set of rules of the form

$$(\alpha \to \beta, d),\ (\alpha \to \gamma, c),\ (\beta \to \gamma, 0),$$

  for $c + d > 1$. Of this type 5 conflicts were found.

- *Type 3*, given by a collection of rules of the form

$$(\alpha \to \gamma, c),\ (\beta \to \gamma, 1),\ (\alpha \to \beta, 1),$$

  for $c < 1$. A single conflict of this type was found.

- *Type 4*, given by a set of rules of the form

$$(\alpha \to \beta, c),\ (\alpha \to \gamma, d),\ (\alpha \to \vartheta, e)\ (\beta \to \gamma, 0),\ (\beta \to \vartheta, 1),\ (\gamma \to \vartheta, 1),$$

  for $e < c + d \leq 1$ and $c, d \leq e$. 269 conflicts of this type were found.

Notice that, given the reduction to satisfiability in classical logic of CADIAG-2's binary rules in the approach described in Section 3.1, the conflicts detected in relation to such approach are contained in the above list.

In addition to the consistency check and the identification of conflicts, inconsistency measures aimed at evaluating and quantifying the amount of inconsistency in CADIAG-2-like knowledge bases and related repair strategies are also considered in [KPP] itself and, in more detail, in [Pic2]. The approach to measuring inconsistency

[6] An interpretation that consists of the replacement of point values like d in $(\alpha \to \beta, d)$, whenever $0 < d < 1$, for the interval $[d - 0.01, d + 0.01]$—see [KPP] for more details.

in these papers is based on the consideration of *minimal* adjustments, with respect to suitable distance criteria, in the probability degrees of the corresponding propositions necessary to make the knowledge base consistent.

It turns out that the amount of inconsistency of the binary fragment of CADIAG-2's KB is only infinitesimal, despite the large number of conflicts that it contains. For instance a conflict of any of the types mentioned above can be *repaired* by replacing 0 (or 1) in any of its rules having this probability value by any other value strictly greater than 0 (or any other value strictly smaller than 1, respectively)—for more details see [KPP].

5. CADIAG-2 and possibilistic logic

The basic requirement that we set for the theoretical framework of a decision support system is a coherent interpretation of all involved notions. We recall that in case of the expert system CADIAG-2, the challenge is to treat different types of numerical degrees in an appropriate way, although all values are treated by the system in a uniform way.

In the approaches discussed so far, values are interpreted in a uniform way. In the framework based on fuzzy logic, all values are taken as compatibilities; in particular, the values attached to diagnoses are understood in this way. In the framework based on probability theory, all values are taken as probabilities; even the values attached to symptoms are interpreted in this way. The third approach, which we describe below, takes both aspects into account; it is thus the only approach which formalises vagueness and uncertainty in an independent way. The approach is based on a logic recently introduced in [ZeGo].

Let us outline how degrees are treated. The degrees of presence are handled within the framework of a fuzzy logic, similarly to the approach described in Section 3. Indeed, the fuzzy logic $\mathsf{RGL}_\sim$ is used: Gödel logic enriched with rational truth constants, the standard negation and the Δ operator [EGHN]. (We note that in $\mathsf{RGL}_\sim$, Δ is actually definable; we still include it in the language for its important role.) The implication is included as a connective in $\mathsf{RGL}_\sim$; for this reason $\mathsf{RGL}_\sim$ is stronger than GZL.

The aspect of uncertainty is not, like in the approach described in Section 4, treated probabilistically. Degrees of uncertainty are understood as degrees of necessity in the sense of Dubois and Prade's possibility theory [DuPr]. For a further example of the use of possibilistic logic in medical decision support, we may refer, e.g., to [BCPV].

The logic PGL

The logic PGL combines $\mathsf{RGL}_\sim$ with Possibilistic Logic, see [ZeGo] for details.

We start with two sets of variables: S and D. The variables $\sigma_1, \ldots \in S$ are called *many-valued* and are used to model the presence of symptoms. The variables $\delta_1, \ldots \in D$ are called *Boolean* and are used to model the presence of diseases. Formulas of PGL are split into two classes: Boolean formulas and N-formulas, where "N" refers to "necessity".

- The *Boolean formulas* are built up from the Boolean variables and the constants $\bot, \top$ by means of $\wedge$, $\vee$, and $\neg$. Boolean formulas are denoted by the lower case Greek letters α, β, γ.

- An atomic *N-formula* is a many-valued variable, a constant $\bar{r}$ for some $r \in [0,1] \cap \mathbb{Q}$, or of the form $\Box\alpha$, where α is a Boolean formula. A (general) *N-formula* is built up from atomic N-formulas by means of the binary operations $\wedge, \rightarrow$ and the unary operations $\sim, \Delta$. N-formulas are denoted by the lower case Greek letters ζ, η, ϑ.

A model for PGL, called a *PG-structure*, is a four-tuple

$$(v, W, e, \pi),$$

specified as follows:

- Recall that the many-valued variables S refer to the symptoms of a patient. v describes the state of a patient by assigning to each $\sigma \in S$ a value $v(\sigma) \in [0,1]$. $v(\sigma)$ is meant to be the degree to which the symptom applies to a patient.

- W is a non-empty set, called the *set of possible worlds*. Each $w \in W$ corresponds to a possible health state of the patient, specified by the presence or absense of diseases.

- Recall that the Boolean variables D refer to diseases. The possible health states of the patient is described by $e \in W \times D \rightarrow \{0,1\}$. For each $w \in W$ and $\delta \in D$, $e(w, \delta)$ is 1 if δ is present at w and it is 0 otherwise. Moreover, $e(w, \cdot)$ extends to a classical valuation of D.

- For each $w \in W$, $\pi(w)$ is the degree to which, in the sense of possibility theory, the agent believes w to be possible; that is, $\pi(w)$ is the degree to which the agent is inclined to assume that w can be the actual world. We require $\pi(w) = 1$ for at least one world w.

Let (v, W, e, π) be a PG-structure. In what follows, $\bar{\wedge}$, $\bar{\sim}$ denote again the minimum and standard negation on $[0,1]$; moreover, $\bar{\rightarrow}$ is the residuum associated to $\bar{\wedge}$, that is,

$$s \bar{\rightarrow} t = \begin{cases} 1 & \text{if } s \leq t, \\ t & \text{else;} \end{cases}$$

and $\bar{\Delta}$ is the evaluation of Δ on $[0,1]$ defined by (2).

Furthermore, for each Boolean formula α, let $[\alpha]$ denote the set of worlds in which α holds. Putting $P(A) = \max_{w \in A} \pi(w)$ for any $A \subseteq W$, P is a possibility measure, and N, defined by $N(A) = 1 - P(W \backslash A)$ for $A \subseteq W$, is the corresponding necessity measure.

For each N-formula ζ, its *truth value* $\|\zeta\|$ is recursively defined as follows:

(i) For a many-valued variable σ, we put $\|\sigma\| = v(\sigma)$

(ii) For a Boolean formulas α, we put $\|\Box\alpha\| = N([\alpha])$

(iii) For formulas ζ and η, we put $\|\zeta \wedge \eta\| = \|\zeta\| \mathbin{\bar{\wedge}} \|\eta\|$, $\|\zeta \rightarrow \eta\| = \|\zeta\| \mathbin{\bar{\rightarrow}} \|\eta\|$, and $\|{\sim}\zeta\| = \bar{\sim} \|\zeta\|$, furthermore $\|\Delta(\zeta)\| = \bar{\Delta}\, \|\zeta\|$, and $\|\bar{r}\| = r$ for each $r \in [0,1]_{\mathbb{Q}}$.

A formula ζ such that $\|\zeta\| = 1$ is called *satisfied* in (v, W, e, π).

The meaning of the truth values is as follows. σ being a symptom, $\|\sigma\|$ is the degree to which σ is present. δ being a disease, $\|\Box(\delta)\|$ is the degree of certainty that δ is present, in the sense of possibility theory. The truth degree of compound formulas are calculated in a truth-functional way, following the semantics of Gödel logic with the additional involutive negation, the operator Δ, and rational truth constants.

As shown in [ZeGo] a sound and complete Hilbert-style axiomatization for PGL consists of:

- the axioms and rules of classical propositional logic for Boolean formulas;
- the axioms of Gödel logic with rational constants, involutive negation, and Δ for N-formulas;
- the axioms

$$\sim\Box\bot, \quad \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta), \quad (\Box\alpha \wedge \Box\beta) \rightarrow \Box(\alpha \wedge \beta)$$

  for Boolean formulas α, β;
- modus ponens and the necessitation rule

$$\frac{\alpha \quad \alpha \rightarrow \beta}{\beta}, \quad \frac{\alpha}{\Box\alpha}$$

  for Boolean formulas α, β;
- modus ponens and the Δ-necessitation rule

$$\frac{\zeta \quad \zeta \rightarrow \eta}{\eta}, \quad \frac{\zeta}{\Delta\zeta}$$

  for N-formulas ζ, η.

The encoding in PGL of the rules of CADIAG-2 and of the initial information about a patient is easy. Let us first consider the input information. An input (σ, t) is translated into $(\sigma \rightarrow \bar{t}) \wedge (\bar{t} \rightarrow \sigma)$. We express in this way that the truth value of σ is exactly t. Furthermore, let δ denote a disease. The input (δ, d), where $d > 0$, reflects gradual uncertainty about δ and is accordingly translated into $\bar{d} \rightarrow \Box\delta$. This expresses that we are certain to the degree at least d about the presence of δ. Similarly, $(\delta, 0)$ is translated into $\Box{\sim}\delta$, which says that δ can be excluded.

To encode the rules, we have to distinguish subcases according to the entities the rules apply to.

For the rule (c) we distinguish *symptom-disease rules* (the antecedent is a possibly compound proposition, built up from symptoms and possibly also from diseases, and the consequent is a disease), *disease-disease rules* (the antecedent and the consequent are both diseases) and *symptom-symptom rules* (the antecedent and the consequent are both symptoms). Notice that weights of disease-disease rules and of symptom-symptom rules are always 1.

Let $(\sigma \to \sigma', 1)$ be a symptom-symptom rule. The corresponding N-formula is $\sigma \to \sigma'$. In this way we ensure, as intended, that the truth value of σ' is at least as large as the truth value of σ. A disease-disease rule $(\delta \to \delta', 1)$ is translated into $\Box\delta \to \Box\delta'$. In this case we express that the certainty about δ' is at least as large as the certainty about δ. Finally, consider the symptom-disease rule $(\sigma \to \delta, d)$. We have to express the degree to which are certain about η is at least the minimum of d and the truth degree of ζ, that is,

$$\|\sigma\| \bar{\wedge} d \leq N([\delta]). \tag{8}$$

Consequently, the translation is simply $\sigma \wedge \bar{d} \to \Box\delta$.

For the translation of a rule of type (me), we again have to distinguish cases. Rules of the form $(\sigma_1 \to {\sim}\sigma_2, 1)$, $(\sigma \to {\sim}\delta, 1)$, $(\delta \to {\sim}\sigma, 1)$, and $(\delta_1 \to {\sim}\delta_2, 1)$ are expressed by the formulas $\Delta\sigma_1 \to {\sim}\sigma_2$, $\Delta\sigma \to \Box\neg\delta$, $\Delta\Box\delta \to {\sim}\sigma$, and $\delta_1 \to \neg\delta_2$, respectively.

For rules of type (ao) we proceed similarly. Rules of the form $({\sim}\sigma_1 \to {\sim}\sigma_2, 1)$, $({\sim}\sigma \to {\sim}\delta, 1)$, $({\sim}\delta \to {\sim}\sigma, 1)$, and $({\sim}\delta_1 \to {\sim}\delta_2, 1)$ are translated into $\Delta{\sim}\sigma_1 \to {\sim}\sigma_2$, $\Delta{\sim}\sigma \to \Box\neg\delta$, $\Delta\Box\neg\delta \to {\sim}\sigma$, and $\neg\delta_1 \to \neg\delta_2$, respectively.

The mode of operation of CADIAG-2 is precisely reflected by PGL. Indeed, the truth value of compound N-formulas is determined in a compositional way, just like in CadL. Furthermore, the manipulation rules translate to inferences in PGL with the same effect.

Example 5.1. Again, we consider the inference of Example 2.4. The known facts are this time expressed by the following formulas:

$$\sigma_1, \quad \sigma_2, \quad \sigma_1 \wedge \overline{0.4} \to \Box\delta, \quad \sigma_2 \wedge \overline{0.2} \to \Box\delta. \tag{9}$$

The inference is particularly simple and results in

$$\overline{0.4} \to \Box\delta. \tag{10}$$

The conclusion is that the necessity degree of δ is at least 0.4; in other words, the degree to which we are inclined to say that δ is 0.4.

5.1. Discussion

Like in case of the other two approaches, our first concern about PGL is its strength compared to CadL. PGL extends a fuzzy logic; thus it is not surprising that this question answers similarly to the case of the fuzzy logic GZL. Namely PGL can reproduce the inference of CadL in its full extent. However, like in the case of GZL, the calculus is unintendedly stronger than required.

PGL is, like GZL and unlike CadPL, a calculus sound and complete with respect to a well-motivated semantics.

The model contains different features to take vagueness and uncertainty into account. For this reason, the present approach offers particular benefits in interpretive respects. PGL distinguishes between two different sorts of degrees and the aspects of vagueness and uncertainty are treated in different ways. Indeed, symptoms are formalised by many-valued variables. These variables are assigned the degrees of presence of the respective symptom, and formulas composed from many-valued variables are evaluated in a truth-functional way, interpreting the connectives in the way CADIAG-2 does. Furthermore, diseases are formalised by two-valued variables, corresponding to the fact that a disease is simply assumed to be present or not. Finally, a modality is used to express uncertainty about diseases or their logical combination. Uncertainty in turn is modelled in accordance with possibility theory. Accordingly, this is the way how the weights assigned to diseases after a run of CADIAG-2 are to be understood: as the degree to which an agent is inclined to find this disease a necessary consequence from the assumptions.

We may furthermore critically ask how the degrees of presence provided in the input are related to the degrees of certainty provided in the output. The crucial role is played by the rules (c), which are translated according to the condition (8). It follows that PGL processes values just in the same way as CadL. A full justification of the rule (c) remains open, as in the case of the other two approaches.

We finally ask if proofs are of value for the user of CADIAG-2, provided the latter is based on PGL rather than CadL. The situation is analogous to GZL, hence the answer is negative: results provided by the system cannot be easily traced back to the assumptions. In addition, as the only availlable calculus for PGL is Hilbert-style, derivations are also difficult to find.

Finally, PGL, and possibilistic logic in general, does not deal properly with the following situation. Assume that σ is a symptom that is fully present and that implies to some degrees $d > 0$ two mutually exclusive diseases δ_1 and δ_2. For instance, consider the following case, formulated in the language of CadL:

$$(\sigma,\, 1), \quad (\sigma \to \delta_1,\, 0.8), \quad (\sigma \to \delta_2,\, 0.7) \quad (\delta_1 \to {\sim}\delta_2,\, 1).$$

In PGL, we can derive $\overline{0.7} \to \Box(\delta_1 \wedge \delta_2)$ as well as $\Box\neg(\delta_1 \wedge \delta_2)$, resulting in the inconsistency $\overline{0.7} \to \bar{0}$.

We conclude that the present approach is based on a fuzzy logic, thus the major advantages and disadvantages of our first approach apply here as well. However, the present approach goes one step further than GZL. Weights assigned to diseases and rules are not treated as compatibilities, but as degrees of uncertainty; the main achievement is the successful combination of both aspects.

Rules Check

In contrast with the approaches based on fuzzy logic and probability theory, the one based on possibilistic logic does not seem to be suitable for checking the consistency of the rules for the formalized system. Indeed the definition of suitable SAT-solvers or theorem provers for PGL seem to be beyond the current state of the art, if possible at all (note here that the only availlable proof system for PGL in [ZeGo] is Hilbert-style, which is not usable for this purpose).

6. Conclusion

Although the interest among AI researchers in rule-based expert systems seems to be lesser today than some years ago, these systems are very popular among computer scientists and engineers working in various fields. Many such systems are in use and more are being built.

By processing graded inputs on the basis of weighted IF-THEN rules, CADIAG-2 and related systems are indeed very flexible: they can be easily tuned by the addition or deletion of rules or by a change of their weights. Moreover, despite of their ad-hoc design, their performance is notable. For example, the overall correctness of proposed diagnoses for CADIAG-2, compared with the actual diagnoses of physicians, is more than 80%, according to [AKSEG]. We note that similar results hold for MYCIN—the forefather of all rule-based medical expert systems; see, e.g., [BuSh].

As a common problem, we observe that CADIAG-2-like systems are not designed on the basis of clear principles. Consequently, the interpretation of the numbers dealt with—in particular the outcoming values—is open. As put in [DHN]: after propagating a patient's data through the rules and composing the contributions of the rules and attaching the result to a diagnosis, the user may ask: "How should I actually understand this number?"

This is the main point addressed in the present paper. Taking the system CADIAG-2 as a specific example, we discussed three mathematical models recently introduced by the authors in [CiVe, Pic1, ZeGo] to interpret the numbers and the inferential mechanism. We compiled the features of the three quite different approaches; we mentioned advantages and disadvantages and we focused on the question in which way numerical values are understood.

We now rise the question: for which reason one of the three frameworks should be chosen? Recall that our system is based on IF-THEN rules. IF-THEN rules may be understood in many ways; let us mention two:

(V) The weight $c \in [0, 1]$ of a rule contained in a medical IF-THEN rule is understood as a degree of compatibility. Then c is the degree to which the conclusion fits to the assumptions. We indicate to which degree to which known facts indicate that the unkown fact is true.

(P) The weight c of a rule is understood as a conditional probability. Then c is understood as a proportion in a long-term run, which can be determined by objective data or by an expert's estimation. In the ideal case, we are led to the probability of an unknown fact.

In case of option (V), we need what could be called a "logic of vagueness". The main obstacle for the design of such a logic is well-known: There is no canonical way to connect truth values that are understood as compatibilities. If some fact α fits to β to the degree d, and if α itself holds to the degree t, how should we determine to which degree β holds? Apparently, there are only two possibilities: performing experiments to test the actual estimation of subjects in such a case; or to test if the system performs well with a particular choice.

The fuzzy-logic based approach, discussed in Section 3, follows the lines of option (V). For the connection of degrees, it makes the choice in accordance with the original system CADIAG-2, which was tested to perform well: β is present to the degree $\min\{d, t\}$.

Approaches opting for some form of option (V) have been subject to criticism, e.g., in [DHN]. The complaints are untenable, however. The authors of [DHN] do accept the fact that degrees assigned to inputs are degrees of presence; and they expect results to reflect degrees of certainty. This is not inappropriate; but they inappropriately focus on a probabilistic interpretation of these degrees. If we start with compatibilities and process compatibilities, the result cannot be a probability—why should it be? What we get are compatibilities as well, which in turn can be *taken* as degrees of certainty, not probabilities, and just reflect to which degree the present situation provides hints to something unknown.

Option (P), in contrast, calls for a probabilistic framework. Design choices have to be made also in this case, but for other reasons. Namely, objective probabilities are not available in the numbers that would be needed for systems as large as CADIAG-2. Furthermore, values provided by CADIAG-2 being interpreted probabilistically, the rule weights alone do not allow significant inferences.

The probabilistic approach, discussed in Section 4, follows the ideas underlying option (P). The approach is based on the subjective interpretation of probabilities. Furthermore, to provide sufficient derivational strength without the need of additional data, a variant of the rule (c) is used, which is not based on probability theory but originates from fuzzy logic.

Both these points show that the conceptual gap between fuzzy-logic based approach and the probabilistic approaches is not as wide as it could seem at first sight. What makes the difference is the interpretation of the numerical degrees as probabilities rather than compatibilities. In this case, well-founded ways to process these values do exist, and are made use of. Furthermore, in the literature about CADIAG-2, rules are frequently interpreted according to (5); only the probabilistic interpretation processes the data accordingly.

Finally, an enhancement of the first approach is offered in the third approach. Degrees are degrees of compatibility as well, and the above discussion about fuzzy logic fully applies also here. However, the degrees of compatibility assigned to unkown facts are taken as degrees of certainty and further processed in the framework of a theory of uncertainty. As a framework, possibility theory is chosen rather than probability theory.

Besides providing suitable semantics for the system, the approaches based on fuzzy logic and probability theory are also useful to check the consistency of the system's rules. The checks were actually performed in [CiRu, KPP] and allowed the discovery of various errors in the knowledge representation of the rules of CADIAG-2. The check based on probability theory turned out to be more powerful (more than 600 conflict detected) and complete, that is the satisfiability of the set of rules implies the existence of a model with respect to the sigma-count interpretation. Moreover, it suggested some reparing strategies. However, the feasibility of this check for CADIAG-2 in particular is lost when also considering the compound rules. In contrast, the check based on fuzzy logic can in principle be applied also to compound rules, provided that suitable (counter)model generators for the considered fuzzy logic (Gödel logic with involutive negation, in case of CADIAG-2) are designed.

Though the three mathematical models described in this paper were introduced for CADIAG-2, they can be used to formalize related systems as well. For instance they apply to MONI (MONItoring of nosocomial infections) – a successful expert system recently developed under the supervision of K.-P. Adlassnig [KBBMA]. MONI is currently used to detect nosocomial infections in intensive care units in one of the largest hospitals in Europe, the Vienna General Hospital. MONI contains a data base of rules having the same structure as those in CADIAG-2 and a close inferential mechanism. Therefore considerations and results similar to those drown for CADIAG-2 can be inferred. Moreover the approaches based on fuzzy logic and probability theory also indicate how to perform consistency checks for the rules of MONI.

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